



HAL
open science

A quantitative version of the Morse lemma and ideal boundary fixing quasiisometries

Vladimir Shchur

► **To cite this version:**

Vladimir Shchur. A quantitative version of the Morse lemma and ideal boundary fixing quasiisometries. 2011. hal-00636904v1

HAL Id: hal-00636904

<https://hal.science/hal-00636904v1>

Preprint submitted on 28 Oct 2011 (v1), last revised 30 Mar 2012 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A QUANTITATIVE VERSION OF THE MORSE LEMMA AND IDEAL BOUNDARY FIXING QUASIISOMETRIES

VLADIMIR SHCHUR

ABSTRACT. The article is devoted to a proof of the optimal upper-bound for Morse Lemma, its "anti"-version and their applications. Roughly speaking, Morse Lemma states that in a hyperbolic metric space, a λ -quasi-geodesic γ sits in a λ^2 -neighborhood of every geodesic σ with same endpoints. Anti-Morse Lemma states that σ sits in a $\log \lambda$ -neighborhood of γ . Applications include the displacement of points under quasi-isometries fixing the ideal boundary.

1. INTRODUCTION

The aim of the article is to prove the optimal upper bound for Morse lemma which tells that a quasi-geodesic can not go too far from a geodesic connecting its ends.

Theorem. *In a δ -hyperbolic space E consider a (λ, c) -quasigeodesic γ . Let σ be a geodesic segment, connecting its ends. Then γ lies in H -vicinity of σ , where*

$$(1) \quad H = \lambda^2 (A_1 c + A_2 \delta),$$

where A_1 and A_2 are two universal constants.

This result is an optimal one, that is there exists an example of a quasi-geodesic with such a value of H , see Section 6.

Morse lemma is an important brick in the geometry of hyperbolic spaces. For example it is used to prove that hyperbolicity is invariant under quasi-isometries between geodesic spaces [1] (see Chapter 5.2, Theorem 12): let E and F be δ_1 - and δ_2 -hyperbolic geodesic spaces. If there exists a (λ, c) -quasiisometry between these two spaces, then

$$\delta_1 \leq 8\lambda(2H + 4\delta_2 + c).$$

Hyperbolic metric spaces have recently appeared in discrete mathematics and computer science (see for example [2]). δ -hyperbolicity turns out to be more appropriate than other notions of approximation by trees (e.g. tree-width) used previously. This motivates our search for optimal bounds in cornerstones of hyperbolic group theory like Morse Lemma.

Another motivation comes from Gromov's quasi-isometry classification problem for groups [3]. When two groups are shown not to be quasi-isometric, one would like to give a quantitative measurement of this fact like a lower bound on the distortion of maps between balls

2010 *Mathematics Subject Classification.* 20F65, 20F67.

Key words and phrases. Morse lemma, quasi-isometries, quasi-geodesics, hyperbolic spaces, hyperbolic groups.

in these groups (we thank Itai Benjamini for pointing out to us this issue). We expect our optimal bound in Morse Lemma to be instrumental in proving such lower bounds. As an indication, we shall show that the center of a ball in a tree cannot be moved very far by a self quasi-isometry.

Proposition 1. *Let O be the center of a ball in a d -regular metric tree ($d \geq 3$) of radius R . Consider a self (λ, c) -quasi-isometry f of this ball. Then for any image of the center O*

$$d(f(O), O) \leq \min\{R, H + c + \lambda(c + 1)\}.$$

As for a tree $\delta = 0$ we have $d(f(O), O) \leq 2A_1\lambda^2c$ for large enough λ . See Section 6 for the proof of this proposition.

We shall give an example of a (λ, c) -quasiisometry of a ball in a d -regular tree which moves the center a distance λc away. We are currently unable to fill the gap between λc and λ^2c .

Here is a second illustration: in certain hyperbolic metric spaces, self-quasiisometries fixing the ideal boundary move points a bounded distance away. Direct application of the Morse Lemma yields a bound of $H \sim \lambda^2c$, whereas the examples we know achieve merely λc . For this problem we are able to fill the gap partially. Our argument relies on the following theorem, which we call Anti-Morse Lemma.

Theorem. *In a δ -hyperbolic metric space let γ be a (λ, c) -quasi-geodesic and σ a geodesic, connecting the end-points of γ . Assume that 4δ is much less than $\log \lambda$. Then σ lies in the $A_3(c + \delta) \log \lambda$ -neighbourhood of γ , where A_3 is some constant.*

In Section 9 we shall define the class of geodesically rich hyperbolic spaces (it contains all Gromov hyperbolic groups) for which we can prove

Theorem. *Let X be a geodesically rich δ -hyperbolic metric space, let f be a (λ, c) -self-quasiisometry fixing the boundary ∂X . Then for any point O of X the displacement $d(O, f(O))$ does not exceed $(A_4 + c)\lambda \log \lambda$ where A_4 is a constant depending on the space X .*

First we discuss the geometry of hyperbolic spaces, prove a lemma on exponential contraction of lengths of curves with projections on geodesics. Then we discuss the invariance of Δ -length of geodesics by quasi-isometries. Using these results we prove the quantitative version of the Morse and Anti-Morse lemmas. We define the class of geodesically rich spaces give for this class an estimation of a displacement of points by the ideal boundary-fixing self-quasi-isometries. Finally, we show that this class includes all Gromov-hyperbolic groups.

2. ON THE GEOMETRY OF δ -HYPERBOLIC SPACES

Let E be a metric space with metric d . We shall also write $|x - y|$ for the distance $d(x, y)$ between two points x and y of the space E . There exists several equivalent definitions of hyperbolic metric spaces. First we present the most general definition, given by M. Gromov [4], [1], but for our aims another one will be more convenient.

Definition 1. *Gromov's product of two points x and y at point z is*

$$(x, y)_z = \frac{1}{2} (|x - z| + |y - z| - |x - y|).$$

Definition 2. *A metric space E with metric d is called δ -hyperbolic if for every four points p, x, y, z*

$$(x, z)_p \geq \min \{(x, y)_p, (y, z)_p\} - \delta.$$

Definition 3. *A geodesic triangle xyz is called δ -thin if the distance from any point p of xy to the union of xz and yz does not exceed δ :*

$$d(p, xz \cup yz) \leq \delta.$$

For a complete geodesic space it can be easily shown that Definition 2 is equivalent to the following (in fact, to be precise we should demand $(\frac{1}{2}\delta)$ -thin triangles in the following definition):

Proposition 2. *A geodesic metric space E is δ -hyperbolic if and only if every geodesic triangle is $\frac{1}{2}\delta$ -thin.*

According to M. Bonk and O. Schramm [5], every δ -hyperbolic metric space embeds isometrically into a complete δ -hyperbolic geodesic metric space. So without loss of generality, we shall consider only complete δ -hyperbolic geodesic spaces in the sequel. Given points a and b , ab denotes an arbitrary geodesic segment from a to b .

Definition 4. *In a metric space, a perpendicular from a point to a curve (in particular, a geodesic) is a shortest path from this point to the curve.*

Of course, perpendiculars are not necessarily unique.

Lemma 1. *In a geodesic δ -hyperbolic space, consider a point b and a geodesic σ at distance R from b . Let ba be a perpendicular from b to σ , where a lies on σ . Let c be a point of σ at distance $R + 2\Delta$ from b . Then $|a - c| \leq 2\Delta + 4\delta$*

Proof. The triangle abc is δ -thin by the definition of δ -hyperbolic space, hence, there exists a point t on σ such that $d(t, ba) \leq \delta$ and $d(a, bc) \leq \delta$. Let t_1 and t_2 be projections of t on ba and bc respectively. R is the minimal distance from b to the points of σ . So $R = |b - a| \leq |b - t_1| + |t_1 - t| \leq |b - t_1| + \delta$, $R \leq |b - t_2| + |t_2 - t| \leq |b - t_2| + \delta$. Hence, $|a - t_1| \leq \delta$ and $|c - t_2| \leq 2\Delta + \delta$. By the triangle inequality, we obtain $|a - c| \leq |a - t_1| + |t_1 - t| + |t - t_2| + |t_2 - c| \leq 2\Delta + 4\delta$. \square

Remark 1. *In particular, all the orthogonal projections of a point to a geodesic lie in a segment of length 4δ .*

Lemma 2. *In a δ -hyperbolic space, take two points b and d such that $|b - d| = \Delta$. Let σ be a geodesic, let a and c be orthogonal projections of b and d respectively on σ . Assume that $|a - b| > 3\Delta + 6\delta$, and that d is farther than b from σ . Consider two points x_1 and x_4 on ab and cd , equidistant from σ , such that $2\Delta + 5\delta < d(x_1, \sigma) = d(x_4, \sigma) < |a - b| - (\Delta + 2\delta)$. Then $|x_1 - x_4| \leq 4\delta$ and $|a - c| \leq 8\delta$.*

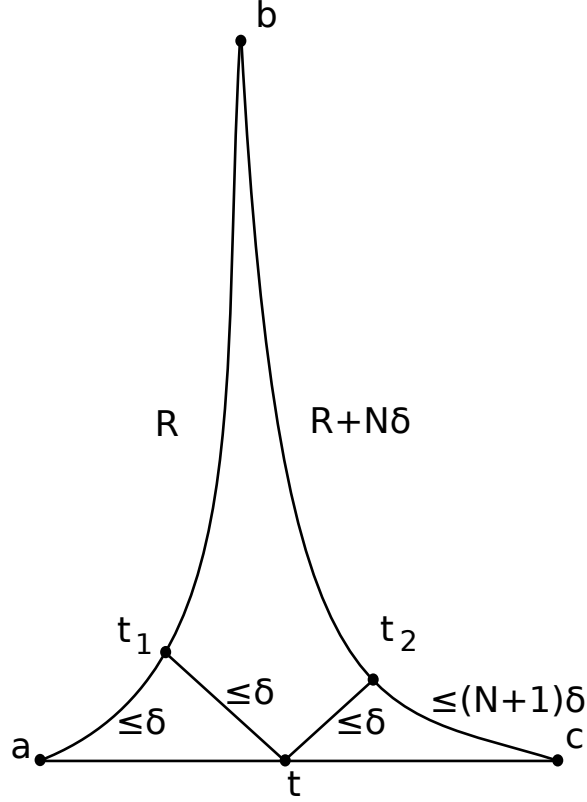


FIGURE 1. Illustration for Lemma 1.

Proof. By the triangle inequality and since cd is a perpendicular to σ , $|c-d| \leq |a-b| + |b-d|$, whence $|b-c| \leq |c-d| + |b-d| \leq |a-b| + 2|b-d|$. By Lemma 1, $|a-c| \leq 2\Delta + 4\delta$. The triangle abc is δ -thin, $|a-x_1| > |a-c| + \delta$, thus by the triangle inequality, $d(x_1, ac) > \delta$, hence, $d(x_1, bc) \leq \delta$. Denote by x_2 the point of bc nearest to x_1 . As the triangle bcd is also δ -thin and $|b-x_2| \geq |b-x_1| - |x_1-x_2| \geq \Delta + \delta$, there exists a point x_3 on cd such that $|x_3-x_2| \leq \delta$. From the triangle cx_1x_3 it follows that $|x_3-c| \geq |x_1-c| - 2\delta \geq |x_1-a| - 2\delta$. On the other hand as x_3c is a perpendicular to σ , $|x_3-c| \leq |x_3-x_1| + |x_1-a|$. Now, $|a-x_1| = |c-x_4|$, hence $|x_4-x_3| \leq 2\delta$. Finally we get the statement of the Lemma: $|x_1-x_4| \leq 4\delta$.

By the triangle inequality and as x_1 and x_4 are equidistant from σ : $|x_1-c| \leq |c-x_4| + |x_4-x_1| \leq |a-x_1| + 4\delta$. So using Lemma 1, we conclude that $|a-c| \leq 8\delta$. \square

Lemma 3. *Let σ be a geodesic segment, let a be a point not lying on σ and let c be a projection of a on σ . Take an arbitrary point b of σ . Then the distance from c to the projection d of b onto ac is at most 2δ .*

Proof. By assumption, bd minimizes the distance from any its points to ac and as the triangle bcd is δ -thin, there exists a point e of bd such that $d(e, ac) = |e-d| \leq \delta$ and

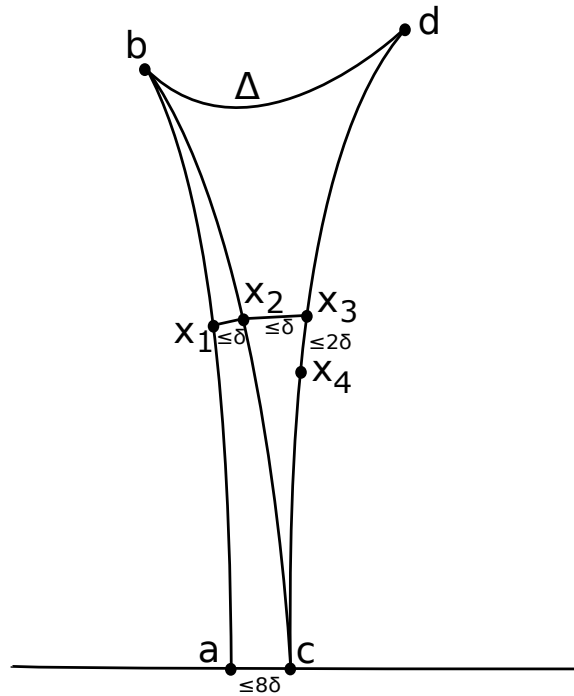


FIGURE 2. Illustration for Lemma 2.

$d(e, bc) \leq \delta$. As ac is a perpendicular to σ , $|a - c| \leq |a - d| + |d - e| + d(e, bc) \leq |a - d| + 2\delta$. Hence $|c - d| \leq 2\delta$. \square

Lemma 4. *As in the previous lemma, let σ be a geodesic segment, let a be a point not lying on σ , let c be a projection of a on σ and let b be some point of σ . Denote by d a point on ac at distance δ from c and by e a point of bc at distance 3δ from c . Then*

- *the distances from d and e to ab do not exceed δ , the distance from c to ab does not exceed 2δ ;*
- *the length of ab differs from the sum of the lengths of the two other sides at most by 8δ ,*

$$|a - c| + |b - c| - 2\delta \leq |a - b| \leq |a - c| + |b - c| + 8\delta.$$

Proof. The triangle abc is δ -thin. So, evidently, $d(d, ab) \leq \delta$ (the distance from a point of ac to ab is a continuous function). Take a point x of bc such that $d(x, ca) \leq \delta$. Using the previous Lemma we get $|b - x| + d(x, ca) \geq |b - c| - 2\delta$, hence $|c - x| \leq d(x, ca) + 2\delta \leq 3\delta$.

Now denote projections of d and e onto ab by d_1 and e_1 respectively. Then by the triangle inequality we have:

- $|a - d| - \delta \leq |a - d_1| \leq |a - d| + \delta$;
- $|b - e| - \delta \leq |b - e_1| \leq |b - e| + \delta$;
- $0 \leq |d_1 - e_1| \leq |d_1 - d| + |d - c| + |c - e| + |e - e_1| \leq 6\delta$.

Combining all these inequalities, we get the second point of our statement. \square

Lemma 5. *Let σ be a geodesic and let a and b be two points not lying on σ . Assume that a and b have a common projection c on σ . Let d be a point of σ , denote by c_1 the projection of d on ab . Then*

$$|d - c| \leq |d - c_1| + 6\delta.$$

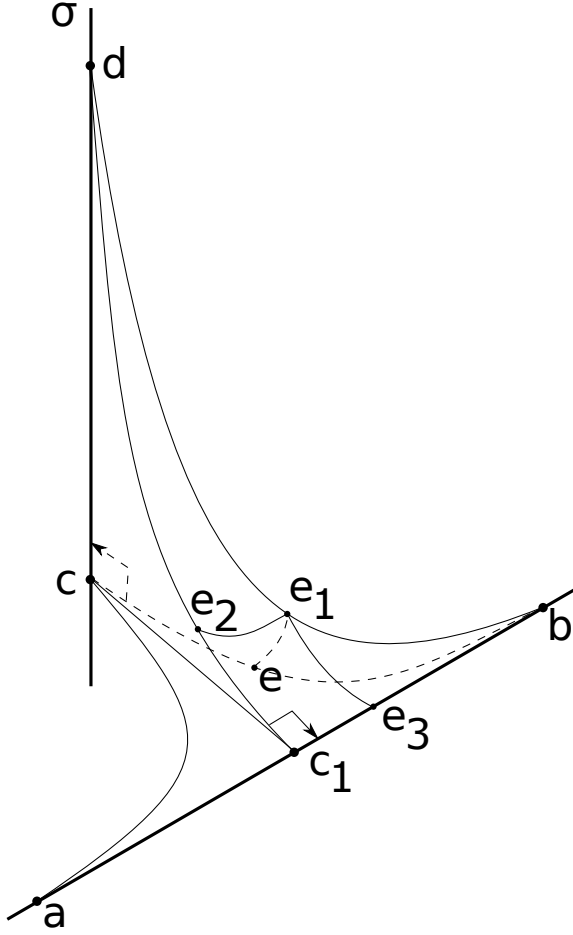


FIGURE 3. Illustration for Lemma 5.

Remark 2. *Lemma 5 deals with a geodesic segment. For the complete geodesic passing through a and b , the statement is not true as seen from Fig. 4.*

Proof. Take a point e on bc such that $|c - e| = \delta$. Consider the triangle bcd . bc is a perpendicular to dc , hence $d(e, bd) \leq \delta$. Denote by e_1 a projection of e on bd . Let e_2 and

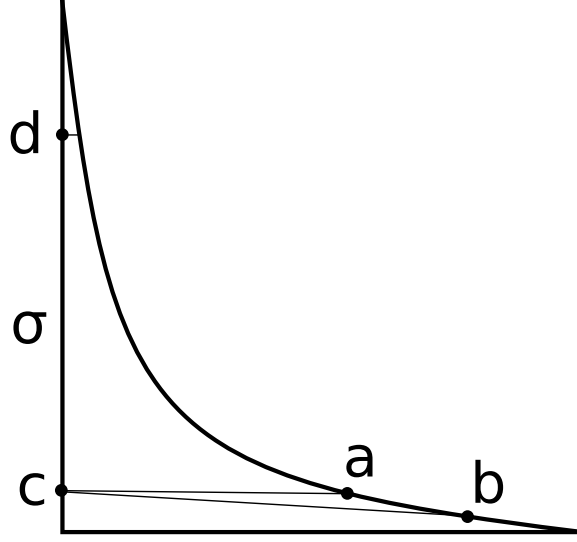


FIGURE 4. Illustration for Remark 2.

e_3 be projections of e_1 on the geodesic segments dc_1 and bc_1 respectively. As the triangle dbc_1 is δ -thin, either $|e_1 - e_2| \leq \delta$ or $|e_1 - e_3| \leq \delta$.

I. If $d(e_1, e_2) \leq \delta$, then $|d - c| \leq |c - e| + |e - e_1| + |e_1 - e_2| + |e_2 - d| \leq |d - c_1| + 3\delta$.

II. If $d(e_1, e_2) > \delta$, the path cee_3 has length at most 3δ . Apply the same arguments to ad (we assume that it is possible, otherwise we could apply the first case to it). We get points g, g_1, g_3 and the length of the path cgg_3 is also at most 3δ . If none of these paths intersects cc_1 , then its length does not exceed 6δ (it follows from consideration of the triangle ce_3g_3).

□

Lemma 6. *Let E be a δ -hyperbolic metric space, abc a triangle. Then the diameter of the set S of points of the side ab , such that distance to bc and ac does not exceed $2d$, is not greater than $C(d + \delta)$ where C is a constant.*

Proof. Let x be a point of ab such that $d(x, bc) \leq \delta$ and $d(x, ac) \leq \delta$, y be a point of ab such that $d(y, bc) \leq d$ and $d(y, ac) < d$. Without loss of generality, assume that $y \in (a, x)$. As the triangle abc is δ -thin one of these two distances is not greater than δ . First, assume that $d(y, ac) \leq \delta$.

Let x', y' be the points of ac such that $d(xx'), d(yy') \leq \delta$. Denote by t, t', s, s' the projections of x, x', y, y' on bc respectively. As $x't'$ is a perpendicular to bc then $|x' - t'| \leq |x' - x| + |x - t| \leq 2\delta$ and hence, $|t - t'| \leq 4\delta$. If y and y' are far enough from bc , that is if $d \geq 9\delta$, by Lemma 2 we have that $|s - s'| \leq 6\delta$. Otherwise we can give a rough estimation by triangle inequality $|s - s'| \leq |s - y| + |y - y'| + |y' - s'| \leq 19\delta$. So, in any case $|s - s'| \leq 19\delta$. Consider two cases.

If s lies on the segment $[b, t']$, then by several applications of triangle inequality we have $|b - y| \leq |b - s| + |s - y| \leq |b - t'| + |s - y| \leq |b - x| + |x - t| + |t - t'| + |s - y| \leq |b - x| + 5\delta + d$.

And as $|b - y| = |b - x| + |x - y|$, we obtain that $|x - y| \leq 5\delta + d$.

The same arguments we apply if $s \in [t', c]$. Just notice that we can change y by y' and t with t' with the errors less than δ and 19δ respectively.

$$|c - y'| \leq |c - s'| + |s' - y'| \leq |c - s'| + |s' - y'| \leq |c - s| + 19\delta + |s - y| + \delta \leq |c - t'| + 20\delta + d$$

Now, as $|c - t'| \leq |c - x'| + |x' - t'| \leq |c - x'| + 2\delta$ we get

$$|c - x'| + |x' - y'| = |c - y'| \leq |c - x'| + 22\delta + d.$$

Finally, $|x - y| \leq |y - y'| + |y' - x'| + |x - x'| \leq 24\delta + d$.

The case of $d(y, bc) \leq \delta$ is treated in the same way just by interchanging d and δ . \square

3. QUASI-GEODESICS AND Δ -LENGTH

Definition 5. A map $f : E \rightarrow F$ between metric spaces is a (λ, c) -quasi-isometry if for any two points x, y of E

$$\frac{1}{\lambda}|x - y|_E - c \leq |f(x) - f(y)|_F \leq \lambda|x - y|_E + c.$$

Definition 6. A (λ, c) -quasi-geodesic in F is a (λ, c) -quasi-isometry from a real interval $I = [0, l]$ to F .

Let $\gamma : I \rightarrow F$ be a curve. Assume that the interval $I = [x_0, x_n]$ of length $|I| = l$ gives the parametrization of the quasi-geodesic γ . Take a subdivision $T_n = (x_0, x_1, \dots, x_n)$, denote $\gamma(x_i)$ by y_i , $i = 0, 1, \dots, n$. The mesh of T_n is $d(T_n) = \min_{0 < i \leq n} |y_i - y_{i-1}|$.

Definition 7 (Δ -length). Let $\gamma : I \rightarrow F$ be a curve. The value $L_\Delta(\gamma) = \sup_{T_n: d(T_n) \geq \Delta} \sum_{i=1}^n |y_i - y_{i-1}|$ is called the Δ -length of the quasi-geodesic γ .

Note that for a geodesic, the values of Δ -length and classical length are the same.

Lemma 7. Let $\gamma : I \rightarrow F$ be a (λ, c) -quasi-geodesic. For $\Delta \geq 2c$,

$$L_\Delta(\gamma) \leq 2\lambda l.$$

Proof. By the definition of Δ -length, $\Delta \leq |y_i - y_{i-1}| \leq \lambda|x_i - x_{i-1}| + c$, so as $\Delta \geq 2c$, we get $|x_i - x_{i-1}| \geq \frac{\Delta - c}{\lambda} \geq \frac{c}{\lambda}$.

Now by definition of quasi-geodesic (and quasi-isometry in particular) we have:

$$\sup_{T_n} \sum_i |y_i - y_{i-1}| \leq \sup_{T_n} \sum_i (\lambda|x_i - x_{i-1}| + c) \leq \sup_{T_n} \sum_i 2\lambda|x_i - x_{i-1}| = 2\lambda l.$$

For the last equality we used that for every subdivision of the interval I the sum of $|x_i - x_{i-1}|$ is exactly equal to the length of I . \square

Lemma 8. *Let $\gamma : I \rightarrow F$ be a (λ, c) -quasi-geodesic. Let $R \geq c$ be the distance between the ends of γ , let $\Delta \geq 2c$. Then the Δ -length of γ does not exceed $L_\Delta(\gamma) \leq 4\lambda^2 R$.*

Proof. By the definition of quasi-isometry,

$$\frac{l}{\lambda} - c \leq R \leq \lambda l + c.$$

Hence $l \leq \lambda(R + c)$. And by Lemma 7,

$$L_\Delta(\gamma) \leq 2\lambda^2(R + c).$$

In particular, for $R \geq c$,

$$L_\Delta(\gamma) \leq 4\lambda^2 R.$$

□

The next Lemma allows us to replace arbitrary quasi-geodesics with continuous ones.

Lemma 9. *Take a (λ, c) -quasi-geodesic γ and some $\Delta \geq c$. Let $T = t_0, t_1, \dots, t_n \subset \gamma$ be the set of points on γ such that T gives the value of Δ -length L_Δ .*

- (1) *Then the curve $\tilde{\gamma}$ which consists of geodesic segments $[t_i, t_{i+1}]$, $i = 0, 1, \dots, n-1$ is a $(\lambda, 12\Delta + 3c)$ -geodesic with (classical) length being equal to L_Δ .*
- (2) *Let y, y' be points of $\tilde{\gamma}$ such that $d(y, y') \geq 6\Delta + c$. Let $\tilde{\gamma}_0$ be the part of $\tilde{\gamma}$ between y and y' . Then the (classical) length of $\tilde{\gamma}_0$ is not greater than*

$$L(\tilde{\gamma}_0) \leq 4\lambda^2(R + 6\Delta).$$

Proof. First, note that for every $i = 0, 1, \dots, n-1$ the length of the interval $[[t_i, t_{i+1}]] \leq 3\Delta$. Indeed, if $[[t_i, t_{i+1}]] > 3\Delta$ we can add a point t'_i to the partition T . As the gaps on a quasi-geodesic cannot be bigger than c such a point exists.

Assume that γ is parametrized by an interval I , $t_i^{-1} \in I$ are parameters of t_i , $i = 0, 1, \dots, n$ (see Fig. 5). Let $[t_i^{-1}, t_{i+1}^{-1}]$ be the affine parametrization of geodesic segments $[t_i, t_{i+1}]$. Then for the points of the same segment the conditions of being a $(\lambda, 4c)$ -geodesic are satisfied automatically.

To simplify notations, denote two different intervals of $\tilde{\gamma}$ by $[x_1, x_2]$ and $[x_3, x_4]$, their parameters by $[z_1, z_2]$ and $[z_3, z_4]$. Take two points $y_1 \in [x_1, x_2]$ and $y_2 \in [x_3, x_4]$, w_1 and w_2 are their parameters. By the triangle inequality and by definition of a quasi-isometry:

$$|y_1 - y_2| \leq |x_2 - x_3| + |y_1 - x_2| + |y_2 - x_3| \leq |x_2 - x_3| + 6\Delta \leq \lambda|z_2 - z_3| + c + 6\Delta.$$

In the same manner we get the lower bound

$$|y_1 - y_2| \geq |x_2 - x_3| - |y_1 - x_2| - |y_2 - x_3| \geq |x_2 - x_3| - 6\Delta \geq \frac{1}{\lambda}|z_2 - z_3| - c - 6\Delta.$$

By the definition of quasi-isometry, $|z_k - z_{k+1}| \leq \lambda(|x_k - x_{k+1}| + c) \leq \lambda(3\Delta + c)$ with $k = 1, 3$. So,

$$|w_1 - w_2| - 2\lambda(3\Delta + c) \leq |z_2 - z_3| \leq |w_1 - w_2|.$$

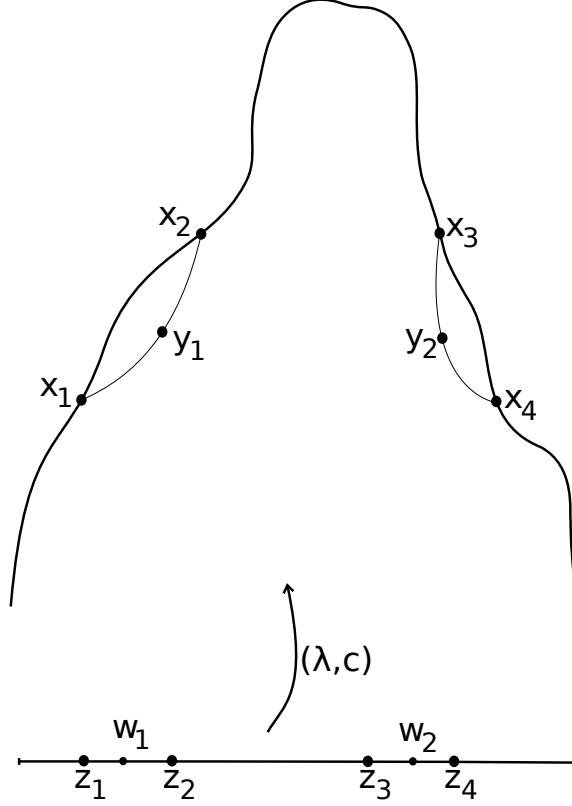


FIGURE 5. Construction of continuous arc $\tilde{\gamma}$ from quasi-geodesic γ .

Hence,

$$\frac{1}{\lambda}|w_1 - w_2| - \frac{2\lambda(3\Delta + c)}{\lambda} - 6\Delta - c \leq |y_1 - y_2| \leq \lambda|w_1 - w_2| + 6\Delta + c.$$

So, $\tilde{\gamma}$ is a quasi-geodesic with constants λ and $12\Delta + 3c$ and the first statement of the Lemma is proved.

To prove the second point we have just to notice that if $|y_1 - y_2| \geq 6\Delta + c$ then by triangle inequality $c \leq |x_1 - x_4| \leq |y_1 - y_2| + 6\Delta$. The lefthand inequality lets us apply Lemma 8 to the part γ_0 between x_1 and x_4 of the initial quasi-geodesic γ and to get the upper bound we use the right part,

$$L(\tilde{\gamma}_0) \leq L_\Delta(\gamma_0) \leq 4\lambda^2(R + 6\Delta).$$

□

4. EXPONENTIAL CONTRACTION

Lemma 10 (Exponential contraction). *Take some $\Delta > 0$. In a geodesic δ -hyperbolic space E , let γ be a connected curve lying at distance not less than $R \geq \Delta + 58\delta$ from a geodesic σ .*

Let L_Δ be the Δ -length of γ . Denote $r = \lfloor \frac{R-\Delta-58\delta}{19\delta} \rfloor 19\delta$. Then the length of the projection of γ on σ is not greater than

$$\max \left(\frac{4\delta}{\Delta} e^{-\frac{K}{\delta}r} (L_\Delta + \Delta), 8\delta \right).$$

In other words, it is not greater than

- $\frac{4\delta}{\Delta} e^{-\frac{K}{\delta}r} (L_\Delta + \Delta)$ if $R \leq \Delta + 58\delta + \frac{\delta}{K} \log \frac{L_\Delta + \Delta}{2\Delta}$,
- 8δ otherwise.

Here the constant $K = \frac{\log 2}{19}$ depends only on the space E .

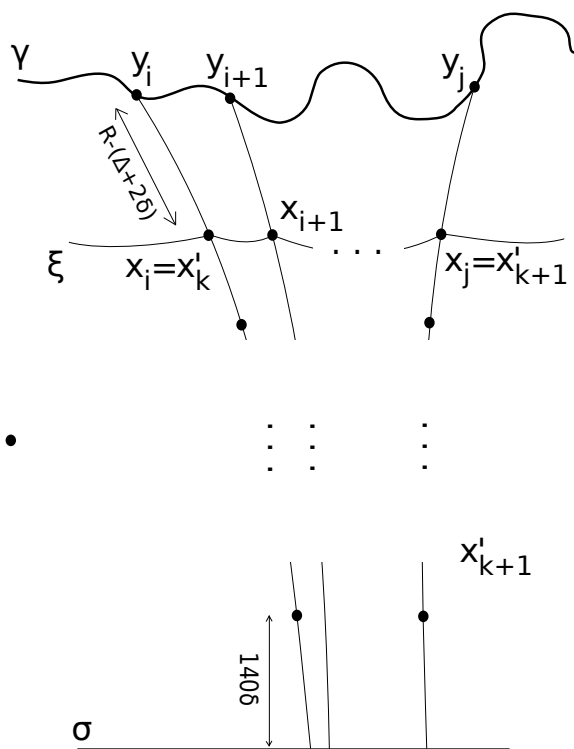


FIGURE 6. Exponential contraction of the length of a curve γ under projection on a geodesic σ .

Proof. Let y_0, y_1, \dots, y_n be points of γ such that $|y_i - y_{i-1}| = \Delta$ for $i = 1, 2, \dots, n - 1$, $|y_n - y_{n-1}| \leq \Delta$ and y_0 and y_n are the ends of γ . Let y_k be the point of this set which is closest to σ . Take a perpendicular from y_k to σ and a point x_k on it with $|y_k - x_k| = \Delta + 3\delta$. Now on the perpendiculars from all other points y_i we take points x_i such that

$d(x_i, \sigma) = d(x_k, \sigma)$. By Lemma 2, $|x_i - x_{i-1}| \leq 4\delta$ for $i = 1, 2, \dots, n$. So,

$$\sum_{i=1}^n |x_i - x_{i-1}| \leq n4\delta \leq n\Delta \frac{4\delta}{\Delta} \leq \frac{4\delta}{\Delta} (L_\Delta + \Delta).$$

Let $\bar{x}_0 = x_0$, $\bar{x}_{n^1} = x_n$ and select points $\bar{x}_i \in \{x_1, x_2, \dots, x_{n-1}\}$ such that $8\delta \leq |x_i - x_{i-1}| \leq 16\delta$. For each $i = 0, 1, \dots, n^1$, pick a perpendicular from \bar{x}_i to σ , move \bar{x}_i along it towards σ for a distance $16\delta + 3\delta = 19\delta$ and get x_i^1 . By Lemma 2, $|x_i^1 - x_{i-1}^1| \leq 4\delta$ and

$$\sum_{i=1}^{n^1} |x_i^1 - x_{i-1}^1| \leq n^1 4\delta \leq \frac{1}{2} \sum_{i=1}^{n^1} |\bar{x}_i - \bar{x}_{i-1}| \leq \frac{1}{2} \sum_{i=1}^n |x_i - x_{i-1}| \leq \frac{1}{2} \frac{4\delta}{\Delta} (L_\Delta + \Delta).$$

We can continue such a process while the distance from the set of points $\{x_i^m, i = 0, 1, \dots, n^m\}$ to σ is not less than 19δ and $|x_0^m - x_{n^m}^m| \geq 8\delta$. After k steps we shall have

$$\sum_{i=1}^{n^k} |x_i^k - x_{i-1}^k| \leq \frac{1}{2^k} \frac{4\delta}{\Delta} (L_\Delta + \Delta) = \frac{4\delta}{\Delta} e^{-\frac{\log 2}{19\delta}(19\delta k)} (L_\Delta + \Delta).$$

Denote $19\delta k$ by r and $\frac{\log 2}{19}$ by K . We need $8\delta \leq \frac{4\delta}{\Delta} e^{-Kr} (L_\Delta + \Delta)$, hence $r \leq \frac{\delta}{K} \log \frac{L_\Delta + \Delta}{2\Delta}$. Now, if at some step m the distance between projections of the end points $|x_0^m - x_{n^m}^m|$ is not less than 8δ we use Lemma 2 to do the last projection onto σ and its length does not exceed 8δ . Else, we have to do the last descent to the distance 55δ , using Lemma 2 (estimation for the projection onto a geodesic with $\Delta = 16\delta$ gives the necessary distance from the set of points to the geodesic be greater than $3 * 16\delta + 6\delta = 54\delta$) and intervals of length not less than 8δ contract to intervals of length not more than δ , so at the last step, we have a contraction factor of 1. □

5. QUANTITATIVE VERSION OF THE MORSE LEMMA

Now we are ready to prove our main result. In a δ -hyperbolic space E any (λ, c) -quasi-geodesic γ lies in an H -neighbourhood of a geodesic σ , connecting its ends, where the constant H depends only on the space E (in particular, on the constant δ) and the quasiisometry constants λ and c .

5.1. Attempts. To motivate our method, let us describe a sequence of arguments yielding sharper and sharper estimates.

Let us start with the proof in [1], Chapter 5.1, Theorem 6 and Lemma 8. [1] gets the upper bound $H \leq \lambda^8 c^2 \delta$ (up to universal constants, factors of order $\log_2(\lambda c \delta)$). The first weak step of this proof is changing a (λ, c) -quasi-geodesic by a discrete (λ', c) -quasi-geodesic γ' parametrized by an interval $[1, 2, \dots, l]$ of integers, where $\lambda' \sim \lambda^2 c$. For a suitable $R \sim \lambda^2$ take an arc $x_u x_v$ of γ' . Introduce a partition of the arc $x_u x_v$: $x_u, x_{u+N}, x_{u+2N}, \dots, x_v$ for some well-chosen $N \sim \lambda'$. The approximation of a δ -hyperbolic space by a tree (see [1] Chapter 2.2, Theorem 12.ii) is used to give an estimation of the form $|y_{u+iN} - y'_{u+(i+1)N}| \leq c' \sim \log \lambda'$. By triangle inequality $|x_u - x_v| \leq |x_u - y_u| + |y_u - y_{u+N}| + \dots + |y_v - x_v| \leq$

$2(R + \lambda') + (\frac{|u-v|}{N} + 1)c'$. On the other hand $\frac{|u-v|}{\lambda'} \leq |x_u - x_v|$. Combining these two inequalities we get an estimation for $|u - v|$ and, hence, for a distance from any point of the arc $x_u x_v$ to the point x_u . The second weak step in this argument lies in the estimation of the length of projections which can be improved significantly.

To improve [1]'s bound, we can use the Exponential Contraction Lemma 10 and Lemma 8 which do not require discretization and give much more precise estimation for a length of a projection. One can then take $R = \log \lambda$ and by a similar triangle inequality get $H \leq O(\lambda^2 \log \lambda)$.

Next we shall give a sketch of the proof of a stronger result (but not yet an optimal one): $H \leq O(\lambda^2 \log^* \lambda)$ where $\log^* \lambda$ is the minimal number n of logarithms such that $\underbrace{\log \dots \log \lambda}_n \leq 1$.

The previous argument is used as an initial step. It allows us to assume that the endpoints x, x' of γ satisfy $|x - x'| \leq O(\log \lambda)$.

Then comes an iterative step. One proves that if xx' is an arc on γ and $|x - x'| = d_1$, then there exist two points y, y' at distance at most $C_2(c, \delta)\lambda^2$ from a geodesic σ_1 connecting x and x' such that $d_2 := |y - y'| \leq C_3(c, \delta) \log d_1$. Indeed, pick a point z of the arc xx' which is farthest from σ_1 . Let σ' be a perpendicular from z to σ_1 . If all points of arc xx' (on either side of z) whose projection to σ' lies at distance $\leq \lambda^2$ of σ_1 are at distance $\geq \log d_1$ from σ' , Lemma 10 implies that the length of the arc is $\gg \lambda^2 \log d_1$, contradicting quasigeodesic assumption. So there are points y and y' which are close to σ' . One can arrange that their projections to σ' are close to each other, yielding $|y - y'| \leq \log d_1$.

We apply this relation several times starting with $d_1 = C_1(c, \delta) \log \lambda$ until $d_i \leq 1$ for some $i = \log^* \lambda$.

So, to improve the upper-bound of H we used two key ideas: Exponential Contraction and consideration of a projection of γ onto a different geodesic σ' .

5.2. Proof of the Morse Lemma. The same ideas will be used in the proof of the quantitative version of the Morse Lemma but we should do it more accurately.

Theorem 1. *In a δ -hyperbolic space E consider a (λ, c) -quasi-geodesic γ . Let σ be a geodesic segment connecting its ends. Then γ lies in H -neighborhood of σ , where*

$$(2) \quad H = 4\lambda^2 \left(78c + \left(78 + \frac{133}{\log 2} e^{\frac{157 \log 2}{28}} \right) \delta \right).$$

Remark 3. *It is easy to give an example where $H = \frac{\lambda^2 c}{2}$ (see Section 6.2).*

Indeed, in a tree, a path that goes back and forth along a geodesic segment of length $\lambda^2 c$ is a (λ, c) -quasi-geodesic, for details see Section 6.

Proof. Applying Lemma 9 to the quasi-geodesic γ with $\Delta = 2c$, we obtain a continuous $(\lambda, 27c)$ -quasi-geodesic $\tilde{\gamma}$. By Lemma 8, γ lies in the $4\lambda^2 \cdot 6c = 24\lambda^2 c$ -neighborhood of $\tilde{\gamma}$. **From this moment we shall speak only about the $(\lambda, 27c)$ -quasi-geodesic $\tilde{\gamma}$. To save notations, we redenote it by γ , and let $\tilde{c} = 27c$.** The classical length of the part

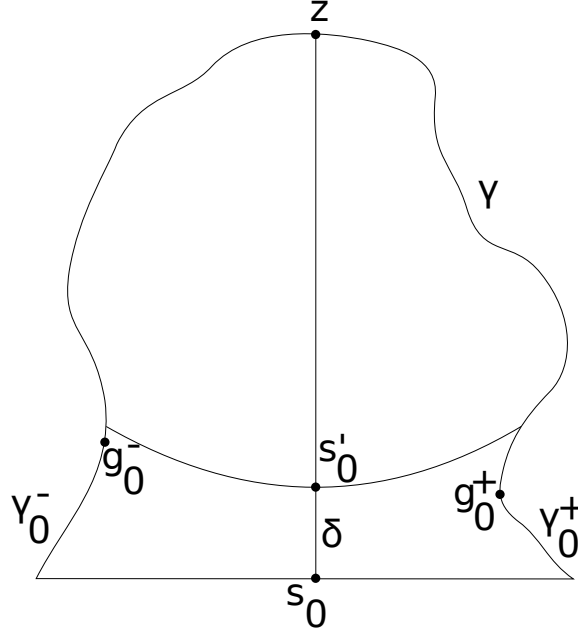


FIGURE 7. Illustration of proof of Theorem 1

of this quasi-geodesic between two points at distance R from each other is not greater than $4\lambda^2(R + \tilde{c})$.

Introduce the following construction for the subdivision of the quasi-geodesic γ . Denote the point of our quasi-geodesic which is farthest from σ by z . Let $\sigma_0 = \sigma$ be the geodesic connecting the ends of γ . Let σ'_0 be the geodesic minimizing the distance between z and σ_0 (as σ_0 is a geodesic segment σ'_0 is not necessarily perpendicular to the complete geodesic carrying σ_0). Let s_0 denote the point of intersection of σ_0 and σ'_0 . Let s'_0 be the point of σ'_0 such that the length of the segment $[s_0, s'_0]$ is equal to δ . Consider the set of points of γ whose projections onto σ'_0 belong to the segment $[s_0, s'_0]$. The point z separates this set into two subsets γ_0^+ and γ_0^- .

Let d_0^\pm denote the minimal distance of points of γ_0^\pm to σ'_0 . Introduce also the following notations:

- $d_0 = d_0^+ + d_0^- + \delta$;
- γ_1 is a connected component of $\gamma \setminus (\gamma_0^+ \cup \gamma_0^-)$ containing z , it is also a quasi-geodesic with the same constants and properties as γ ;
- σ_1 is a geodesic, connecting the ends of the subquasi-geodesic γ_1 ;
- L_1 is the length of γ_1 .

Applying the same idea to the curve γ_1 , the same point z and the geodesic σ_1 we obtain the geodesic σ'_1 , parts γ_1^\pm of the quasi-geodesic and distances d_1^\pm . We have $l(\sigma'_0) \leq l(\sigma'_1) + \delta + 6\delta$. To show it apply Lemma 5 assuming $c = s'_0$, $d = z$ and a and b are the end

points of γ_1 . Continuing the process we obtain a subdivision of γ by γ_i^\pm and two families of geodesics σ_i and σ'_i . Finally for some n we shall obtain $d_n \leq \tilde{c} + \delta + 77\delta = 78\delta + \tilde{c}$. See Figure 7.

L_i is the length of the subcurve γ_{i-1} , which is also a quasi-geodesic. So, by construction, $l(\sigma'_n) \leq L_n \leq 4(d_n + \tilde{c})\lambda^2$. Hence

$$l(\sigma'_0) \leq \sum_{i=1}^n 7\delta + 4(78\delta + 2\tilde{c})\lambda^2.$$

Our goal is to prove that for large enough λ the sum $\sum d_i$ does not exceed $C\lambda^2$, where C is a constant, depending only on \tilde{c} and δ .

As for a segment, the value of classical length is not less then the value of Δ' -length, we get by Lemma 10 (with $\Delta' = \delta$) and as $\left\lfloor \frac{(d_{i+1}^\pm - \delta - 58\delta)}{19\delta} \right\rfloor 19\delta \geq d_{i+1}^\pm - 78\delta$ that

$$l(\gamma_i^+ \cup \gamma_i^-) \geq \delta \frac{\delta}{4\delta} \max \left(e^{\frac{K}{\delta}(d_{i+1}^+ - 78\delta)}, e^{\frac{K}{\delta}(d_{i+1}^- - 78\delta)} \right) \geq \frac{\delta}{4} e^{\frac{K}{2\delta}(d_{i+1} - \delta - 156\delta)}.$$

On the other hand $l(\gamma_i^+ \cup \gamma_i^-) = L_i - L_{i+1}$. Hence, after denoting the expression $\frac{\delta}{4} e^{-\frac{157K}{2}}$ by C_0 ,

$$(3) \quad C_0 e^{\frac{K}{2\delta}d_{i+1}} \leq L_i - L_{i+1}.$$

Let g_i^\pm be a point of γ_i^\pm which minimizes distance to σ'_i . The part of the quasi-geodesic γ between g_i^+ and g_i^- is also a quasi-geodesic with the same constants and properties. By the triangle inequality, $|g_i^- - g_i^+| < d_i^+ + d_i^- + \delta$. So, by construction (see the beginning of the proof) and as $d_i \geq 78\delta$,

$$(4) \quad L_i \leq 4\lambda^2(d_i + \tilde{c}) \leq 8\lambda^2d_i.$$

The function e^{-d} is decreasing. Thus, as $d_i \geq \frac{4}{\lambda^2}L_i$, we get

$$\frac{K}{2\delta}d_i e^{-\frac{K}{2\delta}d_i} \leq \frac{K}{2\delta} \frac{4}{\lambda^2}L_i e^{-\frac{K}{2\delta} \frac{4}{\lambda^2}L_i}.$$

Now we are ready to estimate n :

$$n = \sum_{i=1}^n 1 = \frac{1}{C_0} \sum_{i=1}^n e^{-\frac{K}{2\delta}d_i} C_0 e^{\frac{K}{2\delta}d_i} \leq \frac{1}{C_0} \frac{\lambda^2\delta}{4K} \sum_{i=1}^n e^{-\frac{K}{2\delta} \frac{8}{\lambda^2}L_i} \frac{4K}{\lambda^2\delta} (L_{i-1} - L_i).$$

Denoting by $X_i = \frac{4K}{\lambda^2\delta}L_i$,

$$\sum_{i=1}^n i \leq \frac{\lambda^2\delta}{4C_0K} \sum_{i=1}^n e^{-X_i} (X_{i-1} - X_i),$$

and as the function e^{-X} is decreasing for $X \geq 0$ we can use the estimation

$$\sum_{i=1}^n e^{-X_i} (X_{i-1} - X_i) \leq \int_0^\infty e^{-X} dX = -e^{-x}|_0^\infty = 1.$$

Summarizing all the facts and returning to the initial notations and remembering that $K = \frac{\log 2}{19}$, finally we obtain the claimed result

$$H = 4\lambda^2 \left(78c + \left(78 + \frac{133}{\log 2} e^{\frac{157 \log 2}{38}} \right) \delta \right).$$

□

6. EXAMPLES

6.1. Proof of Proposition 1.

Proposition. *Let O be a center of the ball in a d -regular metric tree ($d \geq 3$) of radius R . Consider self (λ, c) -quasi-isometry f of this ball. Then for any image $f(O)$ of center O*

$$d(f(O), O) \leq \min\{R, H + c + \lambda(c + 1)\}.$$

Proof. Call a *branch* every connected component of a ball with deleted center O . Call the points which are sent to the branch containing the image of the center $f(O)$ green points, all other points of T will be called red.

We are going to show that there exist two red points r_1 and r_2 such that the distance from the geodesic (r_1, r_2) to the center O is not greater than $r = c + 1$. By Definition 5 a c -neighbourhood of every point of the border should contain a point of the image. We need to have at least $(d-1)d^{R-c-1}$ red points near the border (we exclude the green part).

The number of points in each connected component of complement of the ball of radius r is less than d^{R-r} . So, if $r \gg c$ then one component contains insufficient number of points to cover the border of B . Hence, there exists two points r_1 and r_2 in different components of T , which means that the geodesic $r_1 r_2$ passes at distance less than r from the center O and the quasi-geodesic $f([r_1, r_2])$ passes at distance $\lambda r + c$ from $f(O)$ and lies at H -neighbourhood of the geodesic $f(r_1)f(r_2)$. As every path from $f(O)$ to $f(r_1)f(r_2)$ goes through O we conclude that $d(O, f(O)) < H + c + \lambda r$. We have just to choose some good value for r . The simple calculation of the number of points in a mentioned component gives an estimation $1 + d + d^2 + \dots + d^{R-r} \leq \frac{1}{\ln d} d^{R-r+1}$. For $r = c + 1$ we have that $\frac{1}{\ln d} d^{R-r+1} \leq (d-1)d^{R-c-1}$ what finishes the proof.

□

6.2. Optimality of Theorem 1. Let us give an example of a (λ, c) -quasi-geodesic γ in a tree with $H = \frac{\lambda^2 c}{2}$. Indeed we take a real interval $[a, b]$ of length $\frac{\lambda^2 c}{2}$ which is a subtree. To parametrize γ we use an interval $I = [u, v]$ of length λc . Define γ as follows,

- $\gamma(u) = \gamma(v) = a$,
- for the midpoint w of I we put $\gamma(w) = b$,
- for any $x \in [a, b]$ let $D = \min\{|u - x|, |v - x|\}$ and $|a - \gamma(x)| = \lambda D$.

It is easy to verify that γ is a well-defined quasi-geodesic. On the half intervals $[u, w]$ and $[w, v]$ γ just stretches out the distances by λ . Now take any two points $x \in [u, w]$ and $y \in [w, v]$. Assume $|u - x| \leq |v - y|$, evidently $|x - y| = |u - v| - |u - x| - |v - y|$.

I. The lower bound for distance between $\gamma(x)$ and $\gamma(y)$ is given by

$$\frac{1}{\lambda}(|u - v| - |u - x| - |v - y|) - c \leq 0 \leq |\gamma(x) - \gamma(y)|.$$

II. For the upper bound,

$$\lambda(|u - v| - |u - x| - |v - y|) + c - (|a - \gamma(y)| - |a - \gamma(x)|) = \lambda(|u - v| - |u - x| - |v - y|) + c - \lambda(|v - y| - |u - x|) = \lambda^2 c - 2\lambda|v - y| + c \geq c \geq 0.$$

6.3. Achieving displacement λc . Now we are going to describe a self-quasi-isometry f of a ball B in a tree which moves the center O at distance $\lambda c/2$. Assume that the radius of B is greater than λc . Notice, that for any two points inside the ball B_1 with a center in O and a radius λc their images can be just the same point. Let the quasi-isometry f to fix the boundary of B_1 , let the distance $|O - f(O)| = \lambda c/2$. The segment $[O, f(O)]$ is sent in the only point $f(O)$. For any point a of ∂B_1 denote a projection of a on $[O, f(O)]$ by a' and we assume that the interval $[a, a']$ is linearly stretched out and sent to the interval $[a, f(O)]$. Such a map f assigns to any point only one image. It is easy to verify, that f is a quasi-isometry as the distances between points can be diminished up to 0 and are not increased more than λ times.

7. ANTI-MORSE LEMMA

We have already proved, that any quasi-geodesic γ in a hyperbolic space lies at distance not more than $\lambda^2(c + \delta)$ from a geodesic segment σ , connecting its ends. This estimation cannot be improved. But the curious thing is that the mentioned above geodesic lies in $\log \lambda$ -neighbourhood of the quasi-geodesic! So, we can say that any quasi-geodesic is $\log \lambda$ -quasi-convex. In some particular spaces this upper-bound can be improved: for example in a tree any quasi-geodesic is c -quasi-convex.

The proof of this theorem will be based on

- the Lemma 10 on exponential contraction to prove that at the distance $\log \lambda$ from the geodesic σ is at most $\lambda^2 \log \lambda$;
- the analog of the Lemma 10 on exponential contraction to prove that the length of a circle of radius R is at least e^R (up to some constants).

Lemma 11. *Let X be a hyperbolic metric space, γ - a (λ, c) -quasi-geodesic, σ - the geodesic connecting the ends of γ . Take an arc (y_u, y_v) of γ such that no point of this arc lies at distance less than $C_1 \log \lambda + C_2$ from σ and y_u and y_v are the closest to σ points of the arc. Then the length of the projection of arc (y_u, y_v) on σ does not exceed $\max(8\delta, C_3 \log \lambda)$ (with some well-choosen constants C_1, C_2, C_3 depending linearly on c).*

Proof. By the definition of a quasi-geodesic we get

$$\frac{|u - v|}{\lambda} - c \leq |y_u - y_v| \leq \lambda|u - v| + c.$$

On the other hand

$$|y_u - y_v| \leq |y_u - y'_u| + |y'_u - y'_v| + |y'_v - y_v|,$$

where y'_u and y'_v are the projections of y_u and y_v on σ .

Adjust constants c_1 and c_2 so that $c_1 \log \lambda + c_2 = \frac{19\delta^2}{K} \log \frac{8\delta\lambda^4}{\Delta} + \Delta + 58\delta$ with $\Delta = 2c$ (such a choice allows us to apply Lemma 8). Apply the Lemma on exponential contraction (we assume that the length of the arc is rather big to use the estimation with an exponential factor and not to treat the evident case of 8δ for the length of the projection). Denote the Δ -length of an arc (y_u, y_v) by $l(y_u, y_v)$.

$$|y'_u - y'_v| \leq l(y_u, y_v) e^{-\frac{K}{\delta}(r - \Delta - 58\delta)} = \frac{1}{2\lambda^4} l(y_u, y_v).$$

Combining all these inequalities and using Lemma 8 we get

$$\begin{aligned} \frac{|u - v|}{\lambda} - c &\leq |y_u - y_v| \leq \frac{8}{K} \log \sqrt[4]{2}\lambda + \frac{1}{8\lambda^4} l(y_u, y_v) \\ &\leq \frac{8}{K} \log \sqrt[4]{2}\lambda + 4\lambda^2 \frac{1}{8\lambda^4} |y_u - y_v| \\ &\leq \frac{8}{K} \log \sqrt[4]{2}\lambda + \frac{1}{2\lambda^2} (\lambda|u - v| + c). \end{aligned}$$

So, finally we conclude that $|y_u - y_v| \leq C_3 \lambda^2 \log \lambda$, hence $l(y_u, y_v) \leq C_3 \lambda^4 \log \lambda$ and, finally, the length of the projection of the arc (y_u, y_v) of γ does not exceed $\max(8\delta, C_3 \log \lambda)$. \square

Theorem 2. *In a δ -hyperbolic metric space let γ be a (λ, c) -quasi-geodesic and σ a geodesic, connecting the end-points of γ . Assume that 4δ is much less than $\log \lambda$. Then σ lies in $(c + \delta) \log \lambda$ -neighbourhood of γ .*

Proof. The proof of the theorem follows directly from the previous Lemma. As we have already proven for every point z' of σ there exists a point z of γ such that the projection of z on σ lies at distance not more than several times $(c + \delta)$ from z' . So, for simplicity we assume, that for any point of σ there exists a point of γ projecting onto this point.

If the distance between z and z' is less than $C_1 \log \lambda$ for some constant $C_1 = C_1(c, \delta)$ (the value of C_1 can be found from the previous lemma) then the statement is already proved. If not, take an arc of γ containing the point z such that the end-points y_u and y_v of γ lies at distance $C_1 \log \lambda$ from σ and these points are points of this arc which are the closest to σ . So, by the previous Lemma we get that the length of the projection (which includes z) of the arc (y_u, y_v) does not exceed $C_4 \log \lambda$. Hence, the distance from z to y_u (and y_v) is not greater than $(C_1 + C_4) \log \lambda$. \square

8. GEODESICALLY RICH SPACES

Definition 8. *Call a metric space X geodesically rich if there exist constants r_0, r_1, r_2, r_3, r_4*

- *for every pair of points p and q with $d(p, q) \geq r_0$ there exists a geodesic γ such that $d(p, \gamma) < r_1$ and $|d(q, \gamma) - d(q, p)| < r_2$;*
- *for any geodesic γ and any point p in X there exists a geodesic γ' , passing in a r_3 -neighbourhood of the point p and such that the distances from p to γ and from γ' to γ differ by not more than r_4 .*

Example 1. *A line and a ray are not geodesically rich. Both of them satisfy the second, but not the first condition of the definition.*

Example 2. *Non-elementary hyperbolic groups are geodesically rich. We will prove this fact later.*

Any δ -hyperbolic metric space H can be isometrically embedded in a geodesically-rich δ -hyperbolic metric space G (with the same constant of hyperbolicity). Take a 3-regular tree with a root (T, O) , assume $G = H \times T$, and set the metric in a similar manner as in a real tree:

- the distance between points in the subspace (H, O) equals the distance between corresponding point in H ;
- the distance between other points equals the sum of three distances: from the points to their projections on (H, O) and the distance between the projections in (H, O) .

It is easy to show that the space G is δ -hyperbolic and geodesically rich. But such a procedure completely changes the ideal boundary of the space. So we ask another question:

Question 1. *Let H be a δ -hyperbolic metric space. Is it possible to embed isometrically H in a geodesically rich δ -hyperbolic metric space G with an isomorphic boundary?*

Lemma 12. *Let G be a non-elementary hyperbolic group. Then there exist constants c_1, c_2 , such that for any point p and any geodesic γ in G such that distance between them is not less than c_1 , there exists a geodesic γ' with a point q minimizing (up to constant times δ) the distance to γ and at the distance at most c_2 from p .*

Proof. By symmetry we can assume that p is the unity of the group G . We supply the ideal boundary $G(\infty)$ by a visual distance. As G is a non-elementary group, its ideal boundary $G(\infty)$ has at least three points (hence, infinitely many points).

First we are going to prove by contradiction that there exists such ε that for every pair of points ξ, η of $G(\infty)$ the union of the two balls with centers in these points and radii ε does not cover the whole ideal boundary. On the contrary, assume that there exist two sequences of points ξ_n and η_n such that the union of $B(\xi_n, \frac{1}{n})$ and $B(\eta_n, \frac{1}{n})$ includes $G(\infty)$. By compactness, we can assume that $\xi_n \rightarrow \xi$ and $\eta_n \rightarrow \eta$ and we get that $G(\infty)$ lies in the union of $B(\xi, \frac{2}{n})$ and $B(\eta, \frac{2}{n})$, hence the ideal boundary consists only of two points ξ and η which contradicts the assumption that G is non-elementary.

Let c_1 be such a constant that if geodesic γ lies at distance at least c_1 from the point p then the visual distance between its end-points (at infinity) is less than $\varepsilon/2$. Now take two

points ξ and η of $G(\infty)$ outside of $\varepsilon/4$ -neighbourhood of $\gamma(\infty)$ such that distance between ξ and η is greater than ε . Such a choice is possible due to previous argument. Let γ' be a geodesic with ends ξ and η . Hence, $d(p, \gamma') < c_1$. Lemma 13 completes the proof. \square

Lemma 13. *Let X be a δ -hyperbolic space. Then for every $\varepsilon > 0$ there exist constants c_1, c_2 such that for every geodesics γ, γ' and every point p such that $d(p, \gamma) < c_1$ and visual distance between the ends $\gamma(\infty)$ and $\gamma'(\infty) \geq \varepsilon$ there exists a point q on γ minimizing the distance to γ' up to some constant times δ and such that $d(q, p) \leq c_2$.*

Proof. By Lemma 15 we can change the point p with its projection p' on the geodesic γ . Let a' and b' be the projections on γ of the end $a = \gamma'(-\infty)$ and the point b of γ' which minimizes the distance from γ' to γ .

Consider two sequences x_n and y_n of points on aa' and $a'\gamma(+\infty)$ respectively such that $\lim_{n \rightarrow \infty} x_n = a$ and $\lim_{n \rightarrow \infty} y_n = \gamma(+\infty)$. Denote the projections of x_n by a'_n , evidently $a'_n \rightarrow a'$ with $n \rightarrow \infty$. By the definition of Gromov's product, $(x|y)_{p'} = \lim_{n \rightarrow \infty} (x_n|y_n)_{p'}$. Now using Lemma 4 we will estimate $(x_n|y_n)_{p'}$:

$$(x_n|y_n)_{p'} = \frac{1}{2}(|p' - x_n| + |p' - y_n| - |x_n - y_n|) \leq \frac{1}{2}(|p' - a'_n| + |a'_n - x_n| + 8\delta + |p' - y_n| - |a'_n - x_n| - |a'_n - y_n| + 2\delta).$$

Now, if p' lies between a' and b' then $(x_n|y_n)_{p'} \leq 5\delta$, if not (we assume that p' is closer to a' , that is the order of points on γ is p', a', b'), $(x_n|y_n)_{p'} \leq |p' - a'| + 5\delta$.

So, to finish the proof we have now to prove that the point a' lies not far from ab . Apply Lemma 4 one more time to the triangle $aa'b'$: the distance $d(a', ab')$ does not exceed 2δ . So, as the triangle abb' is δ -thin, then the distance from a' to ab or bb' is not greater than 3δ . In the first case the statement is proven immediately. In the second case we have to notice, that bb' is a perpendicular to ab' , hence $d(a'b') \leq 2d(a', bb') \leq 6\delta$. So, in this case a' is near the projection of the closest to ab' point of ab what finishes the proof. \square

Lemma 14. *Let G be a non-elementary hyperbolic group. Then there exist constants c_0, c_1, c_2 such that for every two points p, q in the group G with $d(p, q) > r_0$ there exists a geodesic γ such that $d(p, \gamma) \leq r_1$ and $|d(p, q) - d(q, \gamma)| \leq r_2$.*

Proof. First we assume that p is the unity of the group. We argue by contradiction: suppose the statement not to be true, there exists a sequence of points q_n such that $|q_n - p| \rightarrow \infty$ with $n \rightarrow \infty$ and all pairs p and q_n do not satisfy the conditions of the Lemma. We suppose ξ be a limit point of this sequence. As in the proof of Lemma 12 we supply the border of a group with a visual metric. And the same arguments provide that there exist $\varepsilon > 0$ and points η and η' on the ideal boundary $G(\infty)$ such that pairwise visual distances between ξ, η and η' are greater than ε . We are going to show that the geodesic γ with ends η and η' satisfies the conditions of the statement of Lemma which leads to the contradiction.

In the sequel we shall write ξ, η and η' but we will assume that we consider three sequences of points converging to the corresponding points of the ideal boundary. The triangle $p\eta\eta'$ is δ -thin. Take a point s of $\eta\eta'$ such that $d(s, p\eta) \leq \delta$ and $d(s, p\eta') \leq \delta$.

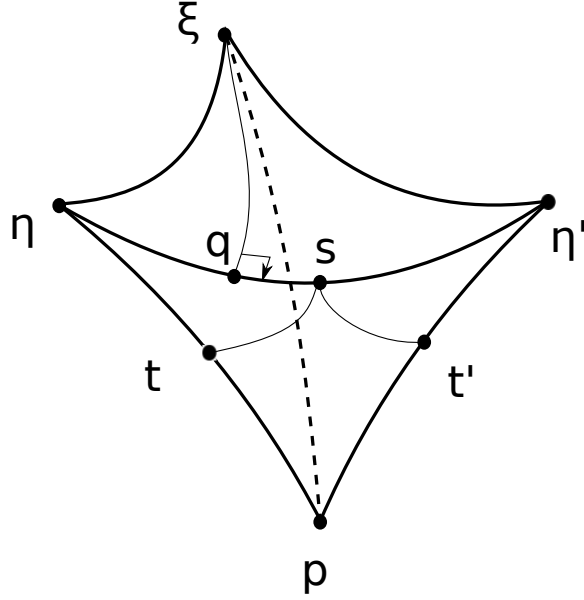


FIGURE 8. Illustration for Lemma 14.

Denote by t and t' projections of s onto $p\eta$ and $p\eta'$ respectively. By the triangle inequality

$$|\eta - t| + |\eta' - t'| - 2\delta \leq |\eta - \eta'| \leq |\eta - t| + |\eta' - t'| + 2\delta.$$

And by the hypothesis

$$\text{visdist}_p(\eta, \eta') = e^{-(\eta|\eta')_p} > \epsilon.$$

Hence,

$$|p - \eta| + |p - \eta'| - |\eta - \eta'| < 2\epsilon_0,$$

where $\epsilon_0 = -\ln \epsilon$

Combining two inequalities we obtain that $|p - t| + |p - t'| \leq 2(\epsilon_0 + \delta)$ and $d(p, \eta\eta') \leq 2\epsilon_0 + 3\delta$. The same arguments applied to the triangles $p\eta\xi$ and $p\eta'\xi$ show that the distance from the point p to the geodesics $\eta\xi$ and $\eta'\xi$ also does not exceed $2\epsilon_0 + 3\delta$. Denote the projections of p onto $\eta\eta'$, $\eta\xi$ and $\eta'\xi$ by p_1, p_2 and p_3 respectively, the projection of ξ onto $\eta\eta'$ by q . Simply by triangle inequality $|p_1 - p_2| \leq |p_1 - p| + |p - p_2| \leq 2(2\epsilon_0 + 3\delta)$. Applying Lemma 4 to the triangles $q\xi\eta$ and $q\xi\eta'$ we get that the point q lies not farther than 2δ from both $\eta\xi$ and $\eta'\xi$. So, both p_1 and q lie at bounded distances from $\eta\xi$ and $\eta'\xi$ and we can apply Lemma 6 which gives us that p_1 and q lie near each other, at a distance of order $\epsilon_0 + \delta$. \square

Lemma 15. *Let X be a δ -hyperbolic space, ξ and η - two points of ideal boundary ∂X , p and p' - two points at distance D from each other. Then visual distances between ξ and η from points p and p' satisfy the following inequality:*

$$d_{p'}(\xi, \eta) \leq e^D d_p(\xi, \eta).$$

Proof. By definition, Gromov's product of x and y in p is

$$(x|y)_p = \frac{1}{2}(|p-x| + |p-y| - |x-y|).$$

We have the same equality for x, y and p' . Hence,

$$|(x|y)_{p'} - (x|y)_p| = \left| \frac{1}{2}(|p'-x| + |p'-y| - |p-x| - |p-y|) \right| \leq |p-p'|.$$

The last inequality follows from the triangle inequality. So, by definition of visual metric:

$$d_{p'}(\xi, \eta) = e^{(\xi|\eta)_{p'}} \leq e^{(\xi|\eta)_p + |p-p'|} = e^D d_p(\xi, \eta).$$

□

9. IDEAL BOUNDARY FIXING QUASI-ISOMETRIES

In this section we are going to give some estimations on the displacement of points in geodesically rich spaces under the ideal boundary fixing quasi-isometries. For the instance we do not know if these results are optimal or not.

Remark 4. *Let X be some metric space satisfying the first condition of being geodesically rich. Let $f : X \rightarrow X$ be a (λ, c) -self-quasiisometry fixing the boundary ∂X . Then for sufficiently big λ for any point $O \in X$ the distance between this point and its image $f(O)$ does not exceed $H(\lambda, c, \delta) + r_2$ where the constant C_1 depends only on the space X .*

Proof. For any point O if $d(O, f(O)) < r_0$ then for sufficiently big λ $r_1 \leq H(\lambda, c, \delta)$. Otherwise let γ be a geodesic such that $d(O, \gamma) \leq r_1$ and $d(f(O), \gamma) > d(O, f(O)) - r_2$. As $f(\gamma)$ is a quasigeodesic with the same as γ end-points, then the quasi-geodesic lies near γ : $f(\gamma) \subset U_H(\gamma)$. Combining all the arguments we obtain:

$$d(O, f(O)) \leq d(f(O), \gamma) + r_2 \leq H + r_2.$$

□

We do not know if it is possible to improve this upper-bound in the general case. But for the case of geodesically rich space we improve the boundary from λ^2 to $\lambda \log \lambda$.

Theorem 3. *Let X be a (r_1, r_2) -geodesically rich δ -hyperbolic metric space, f be a (λ, c) -self-quasiisometry fixing a boundary ∂X . Then for any point O of X the displacement $d(O, f(O))$ does not exceed $\max(r_0, \lambda(r_3 + c + c_1 \log \lambda) + r_1 + r_2 + r_4)$.*

Proof. As f fixes the boundary of X and by Anti-Morse Lemma, a $(c_1 \log \lambda)$ -neighbourhood (where $c_1 = c + \delta$) of an image $f(\sigma)$ of any geodesic σ includes σ : $\sigma \subset V_{c_1 \log \lambda}(f(\sigma))$. All the constants r_0, r_1, r_2, r_3, r_4 are the same constants as in the definition of a geodesically rich space. Take any point O in X . Assume that $d(O, f(O)) \geq r_0$ because otherwise there is nothing to prove. There exists a geodesic γ such that $d(\gamma, O) \leq r_1$ and $|d(O, f(O)) - d(f(O), \gamma)| \leq r_2$ and there also exists a geodesic γ' such that $f(O)$ lies in r_3 -neighbourhood of γ' and such that $f(O)$ is (up to r_4) the point of γ' which is closest to γ .

Since $\gamma' \subset V_{c_1 \log \lambda}(f(\gamma'))$, there exists a point O' of γ' such that $|f(O') - f(O)| \leq r_3 + c_1 \log \lambda$. Now, $d(f(O), \gamma) \leq d(O', \gamma) + r_4 \leq |O' - O| + r_1 + r_4$ and by definition of

quasi-isometry $|O' - O| \leq \lambda(|f(O') - f(O)| + c) \leq \lambda(r_3 + c + c_1 \log \lambda)$. Hence, $d(f(O), \gamma) \leq \lambda(r_3 + c + c_1 \log \lambda) + r_1 + r_4$. Finally we conclude that $d(O, f(O)) \leq d(f(O), \gamma) + r_2 \leq \lambda(r_3 + c + c_1 \log \lambda) + r_1 + r_2 + r_4$. \square

10. ACKNOWLEDGMENT

I am thankful to Prof. P. Pansu for advising me through the all steps of the research done. This work was supported by NSH-8462.2010.1.

REFERENCES

- [1] E. Ghys and P. de la Harpe eds., *Les groupes hyperboliques d'après M. Gromov*, Progress in Mathematics Vol. 83 Birkhuser (1990)
- [2] V. Chepoi, F. Dragan, B. Estellon, M. Habib, Y. Vaxes, *Diameters, centers, and approximating trees of delta-hyperbolic geodesic spaces and graphs*, Symposium on Computational Geometry, SoCG'2008, pp.59-68
- [3] M. Gromov, *Infinite groups as geometric objects*, Proc. Int. Congress Math. Warsaw 1983 1 (1984) pp. 385-392
- [4] M. Gromov, *Hyperbolic Groups*, Essays in group theory (S.M. Gersten ed.), MSRI Series 8 (1987) pp. 75-263.
- [5] M. Bonk, O. Schramm *Embeddings of Gromov Hyperbolic Spaces*, Geom. funct. anal. vol. 10 (2000) pp. 266-306

UNIVERSITÉ PARIS-SUD 11, F-91405 ORSAY CEDEX

ÉCOLE NORMALE SUPÉRIEURE, 45 RUE D'ULM F-75230 PARIS CEDEX 05

VLSHCHUR@GMAIL.COM