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# The classification of the virtually cyclic subgroups of the sphere braid groups

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# Abstract

Let  $n \geq 4$ , and let  $B_n(\mathbb{S}^2)$  denote the  $n$ -string braid group of the sphere. In [GG5], we showed that the isomorphism classes of the maximal finite subgroups of  $B_n(\mathbb{S}^2)$  are comprised of cyclic, dicyclic (or generalised quaternion) and binary polyhedral groups. In this paper, we study the infinite virtually cyclic groups of  $B_n(\mathbb{S}^2)$ , which are in some sense, its 'simplest' infinite subgroups. As well as helping to understand the structure of the group  $B_n(\mathbb{S}^2)$ , the knowledge of its virtually cyclic subgroups is a vital step in the calculation of the lower algebraic K-theory of the group ring of  $B_n(\mathbb{S}^2)$  over  $\mathbb{Z}$ , via the Farrell-Jones fibred isomorphism conjecture [GJM].

The main result of this manuscript is to classify, with a finite number of exceptions and up to isomorphism, the virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$ . As corollaries, we obtain the complete classification of the virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$  when  $n$  is either odd, or is even and sufficiently large. Using the close relationship between  $B_n(\mathbb{S}^2)$  and the mapping class group  $\mathcal{MCG}(\mathbb{S}^2, n)$  of the  $n$ -punctured sphere, another consequence is the classification (with a finite number of exceptions) of the isomorphism classes of the virtually cyclic subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$ .

The proof of the main theorem is divided into two parts: the reduction of a list of possible candidates for the virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$  obtained using a general result due to Epstein and Wall to an almost optimal family  $\mathbb{V}(n)$  of virtually cyclic groups; and the realisation of all but a finite number of elements of  $\mathbb{V}(n)$ . The first part makes use of a number of techniques, notably the study of the periodicity and the outer automorphism groups of the finite subgroups of  $B_n(\mathbb{S}^2)$ , and the analysis of the conjugacy classes of the finite order elements of  $B_n(\mathbb{S}^2)$ . In the second part, we construct subgroups of  $B_n(\mathbb{S}^2)$  isomorphic to the elements of  $\mathbb{V}(n)$  using mainly an algebraic point of view that is strongly inspired by geometric observations, as well as explicit geometric constructions in  $\mathcal{MCG}(\mathbb{S}^2, n)$  which we translate to  $B_n(\mathbb{S}^2)$ .

In order to classify the isomorphism classes of the virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$ , we obtain a number of results that we believe are interesting in their own right, notably the characterisation of the centralisers and normalisers of the maximal cyclic and dicyclic subgroups of  $B_n(\mathbb{S}^2)$ , a generalisation to  $B_n(\mathbb{S}^2)$  of a result due to Hodgkin for the mapping class group of the punctured sphere concerning conjugate powers of torsion elements, the study of the isomorphism classes of those virtually cyclic groups of  $B_n(\mathbb{S}^2)$  that appear as amalgamated products, as well as an alternative proof of a result due to [BCP, FZ] that the universal covering of the  $n^{\text{th}}$  configuration space of  $\mathbb{S}^2$ ,  $n \geq 3$ , has the homotopy type of  $\mathbb{S}^3$ .

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# Introduction and statement of the main results

The braid groups  $B_n$  of the plane were introduced by E. Artin in 1925 and further studied in 1947 [A1, A2]. They were later generalised by Fox to braid groups of arbitrary topological spaces via the following definition [FoN]. Let  $M$  be a compact, connected surface, and let  $n \in \mathbb{N}$ . We denote the set of all ordered  $n$ -tuples of distinct points of  $M$ , known as the  $n^{\text{th}}$  configuration space of  $M$ , by:

$$F_n(M) = \{(p_1, \dots, p_n) \mid p_i \in M \text{ and } p_i \neq p_j \text{ if } i \neq j\}.$$

Configuration spaces play an important rôle in several branches of mathematics and have been extensively studied, see [Bi, CG, FH, Hn] for example.

The symmetric group  $S_n$  on  $n$  letters acts freely on  $F_n(M)$  by permuting coordinates. The corresponding quotient space  $F_n(M)/S_n$  will be denoted by  $D_n(M)$ . The  $n^{\text{th}}$  pure braid group  $P_n(M)$  (respectively the  $n^{\text{th}}$  braid group  $B_n(M)$ ) is defined to be the fundamental group of  $F_n(M)$  (respectively of  $D_n(M)$ ).

Together with the real projective plane  $\mathbb{R}P^2$ , the braid groups of the 2-sphere  $\mathbb{S}^2$  are of particular interest, notably because they have non-trivial centre [GVB, GG1], and torsion elements [VB, Mu]. Indeed, Fadell and Van Buskirk showed that among the braid groups of compact, connected surfaces,  $B_n(\mathbb{S}^2)$  and  $B_n(\mathbb{R}P^2)$  are the only ones to have torsion [FVB, VB]. Let us recall briefly some of the properties of  $B_n(\mathbb{S}^2)$  [FVB, GVB, VB].

If  $\mathbb{D}^2 \subseteq \mathbb{S}^2$  is a topological disc, there is a homomorphism  $\iota: B_n \longrightarrow B_n(\mathbb{S}^2)$  induced by the inclusion. If  $\beta \in B_n$  then we shall denote its image  $\iota(\beta)$  simply by  $\beta$ . Then  $B_n(\mathbb{S}^2)$  is generated by  $\sigma_1, \dots, \sigma_{n-1}$  which are subject to the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2 \text{ and } 1 \leq i, j \leq n - 1 \quad (1)$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } 1 \leq i \leq n - 2, \text{ and} \quad (2)$$

$$\sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1 = 1. \quad (3)$$

Consequently,  $B_n(\mathbb{S}^2)$  is a quotient of  $B_n$ . The first three sphere braid groups are finite:  $B_1(\mathbb{S}^2)$  is trivial,  $B_2(\mathbb{S}^2)$  is cyclic of order 2, and  $B_3(\mathbb{S}^2)$  is a ZS-metacyclic group (a group whose Sylow subgroups, commutator subgroup and commutator quotient group are all cyclic) of order 12, isomorphic to the semi-direct product  $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$  of cyclic groups, the action being the non-trivial one. For  $n \geq 4$ ,  $B_n(\mathbb{S}^2)$  is infinite. The Abelianisation of

$B_n(\mathbb{S}^2)$  is isomorphic to the cyclic group  $\mathbb{Z}_{2(n-1)}$ . The kernel of the associated projection

$$\begin{cases} \xi: B_n(\mathbb{S}^2) \longrightarrow \mathbb{Z}_{2(n-1)} \\ \sigma_i \longmapsto \bar{1} \quad \text{for all } 1 \leq i \leq n-1 \end{cases} \quad (4)$$

is the commutator subgroup  $\Gamma_2(B_n(\mathbb{S}^2))$ . If  $w \in B_n(\mathbb{S}^2)$  then  $\xi(w)$  is the exponent sum (relative to the  $\sigma_i$ ) of  $w$  modulo  $2(n-1)$ . Further, we have a natural short exact sequence

$$1 \longrightarrow P_n(\mathbb{S}^2) \longrightarrow B_n(\mathbb{S}^2) \xrightarrow{\pi} S_n \longrightarrow 1, \quad (5)$$

$\pi$  being the homomorphism that sends  $\sigma_i$  to the transposition  $(i, i+1)$ .

Gillette and Van Buskirk showed that if  $n \geq 3$  and  $k \in \mathbb{N}$  then  $B_n(\mathbb{S}^2)$  has an element of order  $k$  if and only if  $k$  divides one of  $2n$ ,  $2(n-1)$  or  $2(n-2)$  [GVB]. The torsion elements of  $B_n(\mathbb{S}^2)$  and  $B_n(\mathbb{R}P^2)$  were later characterised by Murasugi:

**THEOREM 1** (Murasugi [Mu]). *Let  $n \geq 3$ . Then up to conjugacy, the torsion elements of  $B_n(\mathbb{S}^2)$  are precisely the powers of the following three elements:*

- (a)  $\alpha_0 = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}$  (of order  $2n$ ).
- (b)  $\alpha_1 = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2$  (of order  $2(n-1)$ ).
- (c)  $\alpha_2 = \sigma_1 \cdots \sigma_{n-3} \sigma_{n-2}^2$  (of order  $2(n-2)$ ).

So the maximal finite cyclic subgroups of  $B_n(\mathbb{S}^2)$  are isomorphic to  $\mathbb{Z}_{2n}$ ,  $\mathbb{Z}_{2(n-1)}$  or  $\mathbb{Z}_{2(n-2)}$ . In [GG3], we showed that  $B_n(\mathbb{S}^2)$  is generated by  $\alpha_0$  and  $\alpha_1$ . Let  $\Delta_n^2 = (\sigma_1 \cdots \sigma_{n-1})^n$  denote the so-called ‘full twist’ braid of  $B_n(\mathbb{S}^2)$ . If  $n \geq 3$ ,  $\Delta_n^2$  is the unique element of  $B_n(\mathbb{S}^2)$  of order 2, and it generates the centre of  $B_n(\mathbb{S}^2)$ . It is also the square of the ‘half twist’ element defined by:

$$\Delta_n = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2) \sigma_1. \quad (6)$$

It is well known that:

$$\Delta_n \sigma_i \Delta_n^{-1} = \sigma_{n-i} \quad \text{for all } i = 1, \dots, n-1. \quad (7)$$

The uniqueness of the element of order 2 in  $B_n(\mathbb{S}^2)$  implies that the three elements  $\alpha_0$ ,  $\alpha_1$  and  $\alpha_2$  are respectively  $n^{\text{th}}$ ,  $(n-1)^{\text{th}}$  and  $(n-2)^{\text{th}}$  roots of  $\Delta_n^2$ , and this yields the useful relation:

$$\Delta_n^2 = \alpha_i^{n-i} \quad \text{for all } i \in \{0, 1, 2\}. \quad (8)$$

In what follows, if  $m \geq 2$ ,  $\text{Dic}_{4m}$  will denote the *dicyclic group* of order  $4m$ . It admits a presentation of the form

$$\langle x, y \mid x^m = y^2, yxy^{-1} = x^{-1} \rangle. \quad (9)$$

If in addition  $m$  is a power of 2 then we will also refer to the dicyclic group of order  $4m$  as the *generalised quaternion group* of order  $4m$ , and denote it by  $\mathcal{Q}_{4m}$ . For example, if  $m = 2$  then we obtain the usual quaternion group  $\mathcal{Q}_8$  of order 8. Further,  $T^*$  (resp.  $O^*$ ,  $I^*$ ) will denote the *binary tetrahedral group* of order 24 (resp. the *binary octahedral*

group of order 48, the *binary icosahedral group* of order 120). We will refer collectively to  $T^*$ ,  $O^*$  and  $I^*$  as the *binary polyhedral groups*. More details on these groups may be found in [AM, Cox, CM, Wo], as well as in Section I.3 and the Appendix.

In order to understand better the structure of  $B_n(\mathbb{S}^2)$ , one may study (up to isomorphism) the finite subgroups of  $B_n(\mathbb{S}^2)$ . From Theorem 1, it is clear that the finite cyclic subgroups of  $B_n(\mathbb{S}^2)$  are isomorphic to the subgroups of  $\mathbb{Z}_{2(n-i)}$ , where  $i \in \{0, 1, 2\}$ . Motivated by a question of the realisation of  $\mathcal{Q}_8$  as a subgroup of  $B_n(\mathbb{S}^2)$  of R. Brown [ATD] in connection with the Dirac string trick [F, N], as well as the study of the case  $n = 4$  by J. G. Thompson [ThJ], we obtained partial results on the classification of the isomorphism classes of the finite subgroups of  $B_n(\mathbb{S}^2)$  in [GG4, GG6]. The complete classification was given in [GG5]:

**THEOREM 2 ([GG5]).** *Let  $n \geq 3$ . Up to isomorphism, the maximal finite subgroups of  $B_n(\mathbb{S}^2)$  are:*

- (a)  $\mathbb{Z}_{2(n-1)}$  if  $n \geq 5$ .
- (b)  $\text{Dic}_{4n}$ .
- (c)  $\text{Dic}_{4(n-2)}$  if  $n = 5$  or  $n \geq 7$ .
- (d)  $T^*$  if  $n \equiv 4 \pmod{6}$ .
- (e)  $O^*$  if  $n \equiv 0, 2 \pmod{6}$ .
- (f)  $I^*$  if  $n \equiv 0, 2, 12, 20 \pmod{30}$ .

**REMARKS 3.**

(a) By studying the subgroups of dicyclic and binary polyhedral groups, it is not difficult to show that any finite subgroup of  $B_n(\mathbb{S}^2)$  is cyclic, dicyclic or binary polyhedral (see Proposition 85).

(b) As we showed in [GG4, GG5], for  $i \in \{0, 2\}$ ,

$$\Delta_n \alpha'_i \Delta_n^{-1} = \alpha_i'^{-1}, \quad \text{where } \alpha_i' = \alpha_0 \alpha_i \alpha_0^{-1} = \alpha_0^{i/2} \alpha_i \alpha_0^{-i/2}, \quad (10)$$

and the dicyclic group of order  $4(n-i)$  is realised in terms of the generators of  $B_n(\mathbb{S}^2)$  by:

$$\langle \alpha_i', \Delta_n \rangle,$$

which we shall refer to hereafter as the *standard copy* of  $\text{Dic}_{4(n-i)}$  in  $B_n(\mathbb{S}^2)$ .

A key tool in the proof of Theorem 2 is the strong relationship due to Magnus of  $B_n(\mathbb{S}^2)$  with the mapping class group  $\mathcal{MCG}(\mathbb{S}^2, n)$  of the  $n$ -punctured sphere,  $n \geq 3$ , given by the short exact sequence [FM, MKS]:

$$1 \longrightarrow \langle \Delta_n^2 \rangle \longrightarrow B_n(\mathbb{S}^2) \xrightarrow{\varphi} \mathcal{MCG}(\mathbb{S}^2, n) \longrightarrow 1. \quad (11)$$

As we shall see, it will also play an important rôle in various parts of this paper, notably in the study of the centralisers and conjugacy classes of the finite order elements in Part I, as well as in some of the constructions in Part II. There is a short exact sequence for the mapping class group analogous to equation (5); the kernel of the homomorphism  $\mathcal{MCG}(\mathbb{S}^2, n) \longrightarrow S_n$  is the pure mapping class group  $\mathcal{PMCG}(\mathbb{S}^2, n)$ ,



which may also be seen as the image of  $P_n(\mathbb{S}^2)$  under  $\varphi$ . In particular, since for  $n \geq 4$ ,  $P_n(\mathbb{S}^2) \cong P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}) \times \mathbb{Z}_2$  [GG1], where the second factor is identified with  $\langle \Delta_n^2 \rangle$ , it follows from the restriction of equation (11) to  $P_n(\mathbb{S}^2)$  that  $\mathcal{PMCG}(\mathbb{S}^2, n) \cong P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ , in particular  $\mathcal{PMCG}(\mathbb{S}^2, n)$  is torsion free for all  $n \geq 4$ .

In this paper, we go a stage further by classifying (up to isomorphism) the virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$ . Recall that a group is said to be *virtually cyclic* if it contains a cyclic subgroup of finite index (see also Section I.1). It is clear from the definition that any finite subgroup is virtually cyclic, so in view of Theorem 2, it suffices to concentrate on the *infinite* virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$ , which are in some sense its ‘simplest’ infinite subgroups. The classification of the virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$  is an interesting problem in its own right. As well as helping us to understand better the structure of these braid groups, the results of this paper give rise to some  $K$ -theoretical applications. We remark that our work was partially motivated by a question of S. Millán-López and S. Prassidis concerning the calculation of the algebraic  $K$ -theory of the braid groups of  $\mathbb{S}^2$  and  $\mathbb{R}P^2$ . It was shown recently that the full and pure braid groups of these two surfaces satisfy the Fibred Isomorphism Conjecture of F. T. Farrell and L. E. Jones [BJL, JM1, JM2]. This implies that the algebraic  $K$ -theory groups of their group rings (over  $\mathbb{Z}$ ) may be computed by means of the algebraic  $K$ -theory groups of their virtually cyclic subgroups via the so-called ‘assembly maps’. More information on these topics may be found in [BLR, FJ, JP]. The main theorem of this paper, Theorem 5, is currently being applied to the calculation of the lower algebraic  $K$ -theory of  $\mathbb{Z}[B_n(\mathbb{S}^2)]$  [GJM], which generalises results of the thesis of Millán-López [JM3, ML] where she calculated the lower algebraic  $K$ -theory of the group rings of  $P_n(\mathbb{S}^2)$  and  $P_n(\mathbb{R}P^2)$  using our classification of the virtually cyclic subgroups of  $P_n(\mathbb{R}P^2)$  [GG8]. This application to  $K$ -theory thus provides us with additional reasons to find the virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$ .

As we observed previously, if  $n \leq 3$  then  $B_n(\mathbb{S}^2)$  is a known finite group, and so we shall suppose in this paper that  $n \geq 4$ . Our main result is Theorem 5, which yields the complete classification of the infinite virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$ , with a small number of exceptions, that we indicate below in Remark 6. Recall that by results of Epstein and Wall [Ep, Wa] (see also Theorem 17 in Section I.1), any infinite virtually cyclic group  $G$  is isomorphic to  $F \rtimes \mathbb{Z}$  or  $G_1 *_F G_2$ , where  $F$  is finite and  $[G_i : F] = 2$  for  $i \in \{1, 2\}$  (we shall say that  $G$  is of *Type I* or *Type II* respectively). Before stating Theorem 5, we define two families of virtually cyclic groups. If  $G$  is a group, let  $\text{Aut}(G)$  (resp.  $\text{Out}(G)$ ) denote the group of its automorphisms (resp. outer automorphisms).

DEFINITION 4. Let  $n \geq 4$ .

(1) Let  $\mathbb{V}_1(n)$  be the union of the following Type I virtually cyclic groups:

- (a)  $\mathbb{Z}_q \times \mathbb{Z}$ , where  $q$  is a strict divisor of  $2(n-i)$ ,  $i \in \{0, 1, 2\}$ , and  $q \neq n-i$  if  $n-i$  is odd.
- (b)  $\mathbb{Z}_q \rtimes_{\rho} \mathbb{Z}$ , where  $q \geq 3$  is a strict divisor of  $2(n-i)$ ,  $i \in \{0, 2\}$ ,  $q \neq n-i$  if  $n$  is odd, and  $\rho(1) \in \text{Aut}(\mathbb{Z}_q)$  is multiplication by  $-1$ .
- (c)  $\text{Dic}_{4m} \times \mathbb{Z}$ , where  $m \geq 3$  is a strict divisor of  $n-i$  and  $i \in \{0, 2\}$ .
- (d)  $\text{Dic}_{4m} \rtimes_{\nu} \mathbb{Z}$ , where  $m \geq 3$  divides  $n-i$ ,  $i \in \{0, 2\}$ ,  $(n-i)/m$  is even, and where

$\nu(1) \in \text{Aut}(\text{Dic}_{4m})$  is defined by:

$$\begin{cases} \nu(1)(x) = x \\ \nu(1)(y) = xy \end{cases} \quad (12)$$

for the presentation (9) of  $\text{Dic}_{4m}$ .

(e)  $\mathcal{Q}_8 \rtimes_{\theta} \mathbb{Z}$ , for  $n$  even and  $\theta \in \text{Hom}(\mathbb{Z}, \text{Aut}(\mathcal{Q}_8))$ , for the following actions:

(i)  $\theta(1) = \text{Id}$ .

(ii)  $\theta = \alpha$ , where  $\alpha(1) \in \text{Aut}(\mathcal{Q}_8)$  is given by  $\alpha(1)(i) = j$  and  $\alpha(1)(j) = k$ , where  $\mathcal{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ .

(iii)  $\theta = \beta$ , where  $\beta(1) \in \text{Aut}(\mathcal{Q}_8)$  is given by  $\beta(1)(i) = k$  and  $\beta(1)(j) = j^{-1}$ .

(f)  $T^* \times \mathbb{Z}$  for  $n$  even.

(g)  $T^* \rtimes_{\omega} \mathbb{Z}$  for  $n \equiv 0, 2 \pmod{6}$ , where  $\omega(1) \in \text{Aut}(T^*)$  is the automorphism defined as follows. Let  $T^*$  be given by the presentation [Wo, p. 198]:

$$\langle P, Q, X \mid X^3 = 1, P^2 = Q^2, PQP^{-1} = Q^{-1}, XPX^{-1} = Q, XQX^{-1} = PQ \rangle, \quad (13)$$

and let  $\omega(1) \in \text{Aut}(T^*)$  be defined by

$$\begin{cases} P \mapsto QP \\ Q \mapsto Q^{-1} \\ X \mapsto X^{-1}. \end{cases} \quad (14)$$

More details concerning this automorphism will be given in Section I.3.

(h)  $O^* \times \mathbb{Z}$  for  $n \equiv 0, 2 \pmod{6}$ .

(i)  $I^* \times \mathbb{Z}$  for  $n \equiv 0, 2, 12, 20 \pmod{30}$ .

(2) Let  $\mathbb{V}_2(n)$  be the union of the following Type II virtually cyclic groups:

(a)  $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \mathbb{Z}_{4q}$ , where  $q$  divides  $(n-i)/2$  for some  $i \in \{0, 1, 2\}$ .

(b)  $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \text{Dic}_{4q}$ , where  $q \geq 2$  divides  $(n-i)/2$  for some  $i \in \{0, 2\}$ .

(c)  $\text{Dic}_{4q} *_{\mathbb{Z}_{2q}} \text{Dic}_{4q}$ , where  $q \geq 2$  divides  $n-i$  strictly for some  $i \in \{0, 2\}$ .

(d)  $\text{Dic}_{4q} *_{\text{Dic}_{2q}} \text{Dic}_{4q}$ , where  $q \geq 4$  is even and divides  $n-i$  for some  $i \in \{0, 2\}$ .

(e)  $O^* *_{T^*} O^*$ , where  $n \equiv 0, 2 \pmod{6}$ .

Finally, let  $\mathbb{V}(n) = \mathbb{V}_1(n) \cup \mathbb{V}_2(n)$ . Unless indicated to the contrary, in what follows,  $\rho, \nu, \alpha, \beta$  and  $\omega$  will denote the actions defined in parts (1)(b), (d), (e)(ii), (e)(iii) and (g) respectively.

The main result of this paper is the following, which classifies (up to a finite number of exceptions), the infinite virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$ .

**THEOREM 5.** *Suppose that  $n \geq 4$ .*

- (1) *If  $G$  is an infinite virtually cyclic subgroup of  $B_n(\mathbb{S}^2)$  then  $G$  is isomorphic to an element of  $\mathbb{V}(n)$ .*
- (2) *Conversely, let  $G$  be an element of  $\mathbb{V}(n)$ . Assume that the following conditions hold:*
  - (a) *if  $G \cong \mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  then  $n \notin \{6, 10, 14\}$ .*
  - (b) *if  $G \cong T^* \times \mathbb{Z}$  then  $n \notin \{4, 6, 8, 10, 14\}$ .*
  - (c) *if  $G \cong O^* \times \mathbb{Z}$  or  $G \cong T^* \rtimes_{\omega} \mathbb{Z}$  then  $n \notin \{6, 8, 12, 14, 18, 20, 26\}$ .*
  - (d) *if  $G \cong I^* \times \mathbb{Z}$  then  $n \notin \{12, 20, 30, 32, 42, 50, 62\}$ .*
  - (e) *if  $G \cong O^* *_{T^*} O^*$  then  $n \notin \{6, 8, 12, 14, 18, 20, 24, 26, 30, 32, 38\}$ .*

*Then there exists a subgroup of  $B_n(\mathbb{S}^2)$  isomorphic to  $G$ .*

- (3) *Let  $G$  be isomorphic to  $T^* \times \mathbb{Z}$  (resp. to  $O^* \times \mathbb{Z}$ ) if  $n = 4$  (resp.  $n = 6$ ). Then  $B_n(\mathbb{S}^2)$  has no subgroup isomorphic to  $G$ .*

**REMARK 6.** Together with Theorem 2, Theorem 5 yields a complete classification of the virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$  with the exception of a small (finite) number of cases for which the problem of their existence is open. These cases are as follows:

- (a) Type I subgroups of  $B_n(\mathbb{S}^2)$  (see Propositions 62 and 66, as well as Remarks 64 and 67):
  - (i) the realisation of  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  as a subgroup of  $B_n(\mathbb{S}^2)$ , where  $n$  belongs to  $\{6, 10, 14\}$  and  $\alpha(1) \in \text{Aut}(\mathcal{Q}_8)$  is as in Definition 4(1)(e)(ii).
  - (ii) the realisation of  $T^* \times \mathbb{Z}$  as a subgroup of  $B_n(\mathbb{S}^2)$ , where  $n$  belongs to  $\{6, 8, 10, 14\}$ .
  - (iii) the realisation of  $T^* \rtimes_{\omega} \mathbb{Z}$  as a subgroup of  $B_n(\mathbb{S}^2)$ , where the action  $\omega$  is given by Definition 4(1)(g), and  $n \in \{6, 8, 12, 14, 18, 20, 26\}$ .
  - (iv) the realisation of  $O^* \times \mathbb{Z}$  as a subgroup of  $B_n(\mathbb{S}^2)$ , where  $n \in \{8, 12, 14, 18, 20, 26\}$ .
  - (v) the realisation of  $I^* \times \mathbb{Z}$  as a subgroup of  $B_n(\mathbb{S}^2)$ , where  $n \in \{12, 20, 30, 32, 42, 50, 62\}$ .
- (b) Type II subgroups of  $B_n(\mathbb{S}^2)$  (see Remark 72 and Proposition 73):
  - (i) for  $n \in \{6, 8, 12, 14, 18, 20, 24, 26, 30, 32, 38\}$ , the realisation of the group  $O^* *_{T^*} O^*$  as a subgroup of  $B_n(\mathbb{S}^2)$ .

Since the above open cases occur for even values of  $n$ , the complete classification of the infinite virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$  for all  $n \geq 5$  odd is an immediate consequence of Theorem 5.

**THEOREM 7.** *Let  $n \geq 5$  be odd. Then up to isomorphism, the following groups are the infinite virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$ .*

- (I) (a)  $\mathbb{Z}_m \rtimes_{\theta} \mathbb{Z}$ , where  $\theta(1) \in \{Id, -Id\}$ ,  $m$  is a strict divisor of  $2(n-i)$ , for  $i \in \{0, 2\}$ , and  $m \neq n-i$ .
- (b)  $\mathbb{Z}_m \times \mathbb{Z}$ , where  $m$  is a strict divisor of  $2(n-1)$ .
- (c)  $\text{Dic}_{4m} \times \mathbb{Z}$ , where  $m \geq 3$  is a strict divisor of  $n-i$  for  $i \in \{0, 2\}$ .
- (II) (a)  $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \mathbb{Z}_{4q}$ , where  $q$  divides  $(n-1)/2$ .
- (b)  $\text{Dic}_{4q} *_{\mathbb{Z}_{2q}} \text{Dic}_{4q}$ , where  $q \geq 2$  is a strict divisor of  $n-i$ , and  $i \in \{0, 2\}$

Most of this manuscript is devoted to proving Theorem 5, and is broadly divided into two parts, I and II, together with a short Appendix. The aim of Part I is to prove Theorem 5(1). In conjunction with Theorem 2, Theorem 17 gives rise to a family  $\mathcal{VC}$  of virtually cyclic groups, defined in Section I.1, with the property that any infinite virtually cyclic subgroup of  $B_n(\mathbb{S}^2)$  belongs to  $\mathcal{VC}$ . In that section, we shall discuss a number of properties pertaining to virtually cyclic groups. Proposition 26 describes the correspondence in general between the virtually cyclic subgroups of a group  $G$  possessing a unique element  $x$  of order 2 and its quotient  $G/\langle x \rangle$ . By the short exact sequence (11), this proposition applies immediately to  $B_n(\mathbb{S}^2)$  and  $\mathcal{MCG}(\mathbb{S}^2, n)$ , and will be used at various points, notably to obtain the classification of the virtually cyclic subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$  from that of  $B_n(\mathbb{S}^2)$ . Two other results of Section I.1 that will prove to be useful in Section II.8 are Proposition 20 which shows that almost all elements of  $\mathbb{V}_2(n)$  of the form  $G *_H G$  may be written as a semi-direct product  $\mathbb{Z} \rtimes G$ , and Proposition 27 which will be used to determine the number of isomorphism classes of the elements of  $\mathbb{V}_2(n)$ .

The principal difficulty in proving Theorem 5 is to decide which of the elements of  $\mathcal{VC}$  are indeed realised as subgroups of  $B_n(\mathbb{S}^2)$ . This is achieved in two stages, *reduction* and *realisation*. In the first stage, we reduce the subfamily of  $\mathcal{VC}$  of Type I groups in several ways. To this end, in Section I.2, we obtain a number of results of independent interest concerning structural aspects of  $B_n(\mathbb{S}^2)$ . The first of these is the calculation of the centraliser and normaliser of its maximal finite cyclic and dicyclic subgroups. Note that if  $i \in \{0, 1\}$ , the centraliser of  $\alpha_i$ , considered as an element of  $B_n$ , is equal to  $\langle \alpha_i \rangle$  [BDM, GW]. A similar equality holds in  $B_n(\mathbb{S}^2)$  and is obtained using equation (11) and the corresponding result for  $\mathcal{MCG}(\mathbb{S}^2, n)$  due to L. Hodgkin [Ho]:

PROPOSITION 8. *Let  $i \in \{0, 1, 2\}$ , and let  $n \geq 3$ .*

(a) *The centraliser of  $\langle \alpha_i \rangle$  in  $B_n(\mathbb{S}^2)$  is equal to  $\langle \alpha_i \rangle$ , unless  $i = 2$  and  $n = 3$ , in which case it is equal to  $B_3(\mathbb{S}^2)$ .*

(b) *The normaliser of  $\langle \alpha_i \rangle$  in  $B_n(\mathbb{S}^2)$  is equal to:*

$$\begin{cases} \langle \alpha_0, \Delta_n \rangle \cong \text{Dic}_{4n} & \text{if } i = 0 \\ \langle \alpha_2, \alpha_0^{-1} \Delta_n \alpha_0 \rangle \cong \text{Dic}_{4(n-2)} & \text{if } i = 2 \\ \langle \alpha_1 \rangle \cong \mathbb{Z}_{2(n-1)} & \text{if } i = 1, \end{cases}$$

*unless  $i = 2$  and  $n = 3$ , in which case it is equal to  $B_3(\mathbb{S}^2)$ .*

(c) *If  $i \in \{0, 2\}$ , the normaliser of the standard copy of  $\text{Dic}_{4(n-i)}$  in  $B_n(\mathbb{S}^2)$  is itself, except when  $i = 2$  and  $n = 4$ , in which case the normaliser is equal to  $\alpha_0^{-1} \sigma_1^{-1} \langle \alpha_0, \Delta_4 \rangle \sigma_1 \alpha_0$ , and is isomorphic to  $\mathcal{Q}_{16}$ .*

If  $F$  is a maximal dicyclic or finite cyclic subgroup of  $B_n(\mathbb{S}^2)$ , parts (a) and (b) imply immediately that  $B_n(\mathbb{S}^2)$  has no Type I subgroup of the form  $F \rtimes \mathbb{Z}$ .

The second reduction, given in Proposition 35 in Section I.3, will make use of the fact that if  $\theta: \mathbb{Z} \rightarrow \text{Aut}(F)$  is an action of  $\mathbb{Z}$  on the finite group  $F$ , the isomorphism class of the semi-direct product  $F \rtimes_{\theta} \mathbb{Z}$  depends only on the class of  $\theta(1)$  in  $\text{Out}(F)$ . Since we are interested in the realisation of isomorphism classes of virtually finite subgroups in

$B_n(\mathbb{S}^2)$ , it will thus be sufficient to study the Type I groups of the form  $F \rtimes_{\theta} \mathbb{Z}$ , where  $\theta(1)$  runs over a transversal of  $\text{Out}(F)$  in  $\text{Aut}(F)$ . To this end, in Section I.3, we recall the structure of  $\text{Out}(F)$  for the binary polyhedral groups. One could also carry out this analysis for the other finite subgroups of  $B_n(\mathbb{S}^2)$  given by Theorem 2, but the resulting conditions on  $\theta$  are weaker than those obtained from a generalisation of a second result of L. Hodgkin concerning the powers of  $\alpha_i$  that are conjugate in  $B_n(\mathbb{S}^2)$ . More precisely, in Section I.4, we prove the following proposition.

**PROPOSITION 9.** *Let  $n \geq 3$  and  $i \in \{0, 1, 2\}$ , and suppose that there exist  $r, m \in \mathbb{Z}$  such that  $\alpha_i^m$  and  $\alpha_i^r$  are conjugate in  $B_n(\mathbb{S}^2)$ .*

- (a) *If  $i = 1$  then  $\alpha_1^m = \alpha_1^r$ .*
- (b) *If  $i \in \{0, 2\}$  then  $\alpha_i^m = \alpha_i^{\pm r}$ .*

In particular, conjugate powers of the  $\alpha_i$  are either equal or inverse. So if  $F$  is a finite cyclic subgroup of  $B_n(\mathbb{S}^2)$  then by Theorem 1 the only possible actions of  $\mathbb{Z}$  on  $F$  are the trivial action and multiplication by  $-1$ . This also has consequences for the possible actions of  $\mathbb{Z}$  on dicyclic subgroups of  $B_n(\mathbb{S}^2)$ . As in Proposition 8, the proof of Proposition 9 will make use of a similar result for the mapping class group and the relation (11).

The final reduction, described in Section I.5.2, again affects the possible Type I subgroups that may occur, and is a manifestation of the periodicity (with least period 2 or 4) of the subgroups of  $B_n(\mathbb{S}^2)$  that was observed in [GG5] for the finite subgroups. The following proposition will be applied to rule out Type I subgroups of  $B_n(\mathbb{S}^2)$  isomorphic to  $F \rtimes_{\theta} \mathbb{Z}$  with non-trivial action  $\theta$ , where  $F$  is either  $O^*$  or  $I^*$  (one could also apply the result to the other possible finite groups  $F$ , but this is not necessary in our context in light of the consequences of Proposition 9 mentioned above). The following proposition may be found in [BCP, FZ], and may be compared with the analogous result for  $\mathbb{R}P^2$  [GG2, Proposition 6]. We shall give an alternative proof in Section I.5.1.

**PROPOSITION 10** ([BCP, FZ]).

- (a) *The space  $F_2(\mathbb{S}^2)$  (resp.  $D_2(\mathbb{S}^2)$ ) has the homotopy type of  $\mathbb{S}^2$  (resp. of  $\mathbb{R}P^2$ ). Hence the universal covering space of  $D_2(\mathbb{S}^2)$  is  $F_2(\mathbb{S}^2)$ .*
- (b) *If  $n \geq 3$ , the universal covering space of  $F_n(\mathbb{S}^2)$  or  $D_n(\mathbb{S}^2)$  has the homotopy type of the 3-sphere  $\mathbb{S}^3$ .*

Putting together these reductions will allow us to prove Theorem 5(1), first for the groups of Type I in Section I.6.1, and then for those of Type II in Section I.6.2. The structure of the finite subgroups of  $B_n(\mathbb{S}^2)$  imposes strong constraints on the possible Type II subgroups, and the proof in this case is more straightforward than that for Type I subgroups.

The second part of the paper, Part II, is devoted to the analysis of the realisation of the elements of  $\mathbb{V}_1(n) \amalg \mathbb{V}_2(n)$  as subgroups of  $B_n(\mathbb{S}^2)$  and to proving parts (2) and (3) of Theorem 5. With the exception of the values of  $n$  excluded by the statement of part (2), we prove the existence of the elements of  $\mathbb{V}(n)$  as subgroups of  $B_n(\mathbb{S}^2)$ , first those of Type I in Sections II.1–II.4, and then those of Type II in Section II.6. The results of these sections are gathered together in Proposition 68 (resp. Proposition 73) which

proves Theorem 5(2) for the subgroups of Type I (resp. Type II). The construction of the elements of  $\mathbb{V}(n)$  involving finite cyclic and dicyclic groups are largely algebraic, and will rely heavily on Lemma 51, as well as on Lemma 29 which describes the action by conjugation of the  $\alpha_i$  on the generators of  $B_n(\mathbb{S}^2)$ . In contrast, the realisation of the elements of  $\mathbb{V}(n)$  involving the binary polyhedral groups is geometric in nature, and occurs on the level of mapping class groups via the relation (11). The constraints involved in the constructions indicate why the realisation of such elements is an open problem for the values of  $n$  given in Remark 6. For  $n \in \{4, 6\}$ , in Proposition 62(d) we are also able to rule out the existence of the virtually cyclic groups given in Theorem 5(3).

In Section II.8, we discuss the isomorphism problem for the amalgamated products that occur as elements of  $\mathbb{V}_2(n)$ . It turns out that with one exception, abstractly there is only one way (up to isomorphism) to embed the amalgamating subgroup in each of the two factors. With the help of Proposition 27, we are able to prove the following result.

**PROPOSITION 11.** *For each of the amalgamated products given in Definition 4(2), abstractly there is exactly one isomorphism class, with the exception of  $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ , for which there are exactly two isomorphism classes.*

Note that Proposition 11 refers to abstract isomorphism classes, and does not depend on the fact that the amalgamated products occurring as elements of  $\mathbb{V}_2(n)$  are realised as subgroups of  $B_n(\mathbb{S}^2)$ . In the exceptional case, that of  $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ , abstractly there are two isomorphism classes defined by equations (75) and (77). In Corollary 76, we show that abstractly, all but one of the isomorphism classes of the elements of  $\mathbb{V}_2(n)$  of the form  $G *_H G$  may be written as a semi-direct product of  $\mathbb{Z}$  by  $G$ . In Propositions 77 and 78, if  $n \geq 4$  is even we show that one of these isomorphism classes is always realised as a subgroup of  $B_n(\mathbb{S}^2)$ , while the other isomorphism class is realised as a subgroup of  $B_n(\mathbb{S}^2)$  for all  $n \notin \{6, 14, 18, 26, 30, 38\}$ . It is an open question as to whether this second isomorphism class is realised as a subgroup of  $B_n(\mathbb{S}^2)$  for  $n \in \{6, 14, 18, 26, 30, 38\}$

In Section II.9, we deduce the classification of the virtually cyclic subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$  (with a finite number of exceptions). As we shall see, it will follow from Proposition 26 that the homomorphism  $\varphi$  of the short exact sequence (11) induces a correspondence that is one-to-one, with the exception of subgroups of  $B_n(\mathbb{S}^2)$  that are isomorphic to  $\mathbb{Z}_m \rtimes_{\theta} \mathbb{Z}$  or  $\mathbb{Z}_{2m} \rtimes_{\theta} \mathbb{Z}$  for  $m$  odd, which are sent to the same subgroup  $\mathbb{Z}_m \rtimes_{\theta'} \mathbb{Z}$  of  $\mathcal{MCG}(\mathbb{S}^2, n)$ , the action  $\theta'$  being given as in Proposition 12(b) below.

**PROPOSITION 12.** *Let  $n \geq 4$ , and let  $\varphi: B_n(\mathbb{S}^2) \rightarrow \mathcal{MCG}(\mathbb{S}^2, n)$  be the epimorphism given by equation (11).*

- (a) *Let  $H'$  be an infinite virtually cyclic subgroup of  $\mathcal{MCG}(\mathbb{S}^2, n)$  of Type I (resp. Type II). Then  $\varphi^{-1}(H')$  is a virtually cyclic subgroup of  $B_n(\mathbb{S}^2)$  of Type I (resp. Type II).*
- (b) *Let  $H$  be a Type I virtually cyclic subgroup of  $B_n(\mathbb{S}^2)$ , isomorphic to  $F \rtimes_{\theta} \mathbb{Z}$ , where  $F$  is a finite subgroup of  $B_n(\mathbb{S}^2)$  and  $\theta \in \text{Hom}(\mathbb{Z}, \text{Aut}(F))$ . Then  $\varphi(H) \cong \varphi(F) \rtimes_{\theta'} \mathbb{Z}$ , where  $\theta' \in \text{Hom}(\mathbb{Z}, \text{Aut}(F'))$  satisfies  $\theta'(1)(f') = \varphi(\theta(1)(f))$  for all  $f' \in F'$  and  $f \in F$  for which  $\varphi(f) = f'$ .*
- (c) *Let  $H$  be a Type II virtually cyclic subgroup of  $B_n(\mathbb{S}^2)$  isomorphic to  $G_1 *_F G_2$ , where  $G_1, G_2$  and  $F$  are finite subgroups of  $B_n(\mathbb{S}^2)$ , and  $F$  is an index 2 subgroup of  $G_1$  and  $G_2$ . Then  $\varphi(H) \cong \varphi(G_1) *_F \varphi(G_2)$ .*

Equation (11) and Definition 4 together imply that the following virtually cyclic groups are those that will appear in the classification of the virtually cyclic subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$ . If  $m \geq 2$ , let  $\text{Dih}_{2m}$  denote the *dihedral group* of order  $2m$ .

DEFINITION 13. Let  $n \geq 4$ .

(1) Let  $\tilde{\mathcal{V}}_1(n)$  be the union of the following Type I virtually cyclic groups:

- (a)  $\mathbb{Z}_q \times \mathbb{Z}$ , where  $q$  is a strict divisor of  $n - i$ ,  $i \in \{0, 1, 2\}$ .
- (b)  $\mathbb{Z}_q \rtimes_{\tilde{\rho}} \mathbb{Z}$ , where  $q \geq 3$  is a strict divisor of  $n - i$ ,  $i \in \{0, 2\}$ , and  $\tilde{\rho}(1) \in \text{Aut}(\mathbb{Z}_q)$  is multiplication by  $-1$ .
- (c)  $\text{Dih}_{2m} \times \mathbb{Z}$ , where  $m \geq 3$  is a strict divisor of  $n - i$  and  $i \in \{0, 2\}$ .
- (d)  $\text{Dih}_{2m} \rtimes_{\tilde{\nu}} \mathbb{Z}$ , where  $m \geq 3$  divides  $n - i$ ,  $i \in \{0, 2\}$ ,  $(n - i)/m$  is even, and where  $\tilde{\nu}(1) \in \text{Aut}(\text{Dih}_{2m})$  is defined by:

$$\begin{cases} \tilde{\nu}(1)(x) = x \\ \tilde{\nu}(1)(y) = xy \end{cases}$$

for the presentation of  $\text{Dih}_{2m}$  given by  $\langle x, y \mid x^m = y^2 = 1, yxy^{-1} = x^{-1} \rangle$ .

(e)  $(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes_{\tilde{\theta}} \mathbb{Z}$ , for  $n$  even and  $\tilde{\theta} \in \text{Hom}(\mathbb{Z}, \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ , for the following actions:

- (i)  $\tilde{\theta}(1) = \text{Id}$ .
- (ii)  $\tilde{\theta} = \tilde{\alpha}$ , where  $\tilde{\alpha}(1) \in \text{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$  is given by  $\tilde{\alpha}(1)((\bar{1}, \bar{0})) = (\bar{0}, \bar{1})$  and  $\tilde{\alpha}(1)((\bar{0}, \bar{1})) = (\bar{1}, \bar{1})$ .
- (iii)  $\tilde{\theta} = \tilde{\beta}$ , where  $\tilde{\beta}(1) \in \text{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$  is given by  $\tilde{\beta}(1)((\bar{1}, \bar{0})) = (\bar{1}, \bar{1})$  and  $\tilde{\beta}(1)((\bar{0}, \bar{1})) = (\bar{0}, \bar{1})$ .

(f)  $A_4 \times \mathbb{Z}$  for  $n$  even.

(g)  $A_4 \rtimes_{\tilde{\omega}} \mathbb{Z}$  for  $n \equiv 0, 2 \pmod{6}$ , where  $\tilde{\omega}(1) \in \text{Aut}(A_4)$  is the automorphism defined as follows. Let  $A_4 = (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_3$  where the action of  $\mathbb{Z}_3$  on  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$  permutes cyclically the three elements  $(\bar{1}, \bar{0})$ ,  $(\bar{0}, \bar{1})$  and  $(\bar{1}, \bar{1})$ , and let  $\tilde{X}$  be a generator of the  $\mathbb{Z}_3$ -factor. Then we define  $\tilde{\omega}(1) \in \text{Aut}(A_4)$  by:

$$\begin{cases} (\bar{1}, \bar{0}) \mapsto (\bar{1}, \bar{1}) \\ (\bar{0}, \bar{1}) \mapsto (\bar{0}, \bar{1}) \\ \tilde{X} \mapsto \tilde{X}^{-1}. \end{cases}$$

(h)  $S_4 \times \mathbb{Z}$  for  $n \equiv 0, 2 \pmod{6}$ .

(i)  $A_5 \times \mathbb{Z}$  for  $n \equiv 0, 2, 12, 20 \pmod{30}$ .

(2) Let  $\tilde{\mathcal{V}}_2(n)$  be the union of the following Type II virtually cyclic groups:

- (a)  $\mathbb{Z}_{2q} *_{\mathbb{Z}_q} \mathbb{Z}_{2q}$ , where  $q$  divides  $(n - i)/2$  for some  $i \in \{0, 1, 2\}$ .
- (b)  $\mathbb{Z}_{2q} *_{\mathbb{Z}_q} \text{Dih}_{2q}$ , where  $q \geq 2$  divides  $(n - i)/2$  for some  $i \in \{0, 2\}$ .
- (c)  $\text{Dih}_{2q} *_{\mathbb{Z}_q} \text{Dih}_{2q}$ , where  $q \geq 2$  divides  $n - i$  strictly for some  $i \in \{0, 2\}$ .
- (d)  $\text{Dih}_{2q} *_{\text{Dih}_q} \text{Dih}_{2q}$ , where  $q \geq 4$  is even and divides  $n - i$  for some  $i \in \{0, 2\}$ .
- (e)  $S_4 *_{A_4} S_4$ , where  $n \equiv 0, 2 \pmod{6}$ .

Finally, let  $\tilde{\mathcal{V}}(n) = \tilde{\mathcal{V}}_1(n) \cup \tilde{\mathcal{V}}_2(n)$ .

We thus obtain the classification of the virtually cyclic subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$  (with a finite number of exceptions)

**THEOREM 14.** *Let  $n \geq 4$ . Every infinite virtually cyclic subgroup of  $\mathcal{MCG}(\mathbb{S}^2, n)$  is the image under  $\varphi$  of an element of  $\mathbb{V}(n)$ , and so is an element of  $\tilde{\mathbb{V}}(n)$ . Conversely, if  $G$  is an element of  $\mathbb{V}(n)$  that satisfies the conditions of Theorem 5(2) then  $\varphi(G)$  is an infinite virtually cyclic subgroup of  $\mathcal{MCG}(\mathbb{S}^2, n)$ .*

In Proposition 81, we prove a result similar to that of Proposition 11 for the Type II subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$  that appear in Definition 13(2), namely that there is a single isomorphism class for such groups, with the exception of the amalgamated product  $\text{Dih}_8 *_{\text{Dih}_4} \text{Dih}_8$ , for which there are exactly two isomorphism classes. In an analogous manner to that of  $B_n(\mathbb{S}^2)$ , if  $n$  is even then Proposition 83 shows that each of these two classes is realised as a subgroup of  $\mathcal{MCG}(\mathbb{S}^2, n)$ , with the possible exception of the second isomorphism class when  $n$  belongs to  $\{6, 14, 18, 26, 30, 38\}$ .

As we mentioned previously, the real projective plane  $\mathbb{R}P^2$  is the only other surface whose braid groups have torsion. In light of the results of this paper, it is thus natural to consider the problem of the classification of the virtually cyclic subgroups of  $B_n(\mathbb{R}P^2)$  up to isomorphism. This is the subject of work in progress [GG10]. The first step, the classification of the finite subgroups of  $B_n(\mathbb{R}P^2)$ , was carried out in [GG9, Theorem 5]. As in this paper, the classification of the infinite virtually cyclic subgroups of  $B_n(\mathbb{R}P^2)$  is rather more difficult than in the finite case, but the combination of [GG9, Corollary 2], which shows that  $B_n(\mathbb{R}P^2)$  embeds in  $B_{2n}(\mathbb{S}^2)$ , with Theorem 5 should be helpful in this respect.

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# Part I

## Virtually cyclic groups: generalities, reduction and the mapping class group

In Part I, we start by recalling the definition of virtually cyclic groups and their characterisation due to Epstein and Wall. In Section 1, applying Theorem 2, in Proposition 22 we obtain a family  $\mathcal{VC}$  of virtually cyclic groups that are potential candidates to be subgroups of  $B_n(\mathbb{S}^2)$ . The initial aim is to whittle down  $\mathcal{VC}$  to the subfamily  $\mathbb{V}(n)$  of infinite virtually cyclic groups described in Definition 4 with the property that any infinite virtually cyclic subgroup of  $B_n(\mathbb{S}^2)$  is isomorphic to an element of  $\mathbb{V}(n)$ . In Section 1, we also prove a number of results concerning infinite virtually cyclic groups, in particular Proposition 26, which will be used in Part II to construct certain Type II subgroups, and to prove Theorem 14. Also of interest is Proposition 27, which will play an important rôle in Section II.8 in the study of the isomorphism classes of the Type II subgroups of  $B_n(\mathbb{S}^2)$ , notably in the proof of Proposition 11, which shows that there is just one isomorphism class of each such subgroup, with the exception of  $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ , for which there are two isomorphism classes. Another result that shall be applied in Section II.8 is Proposition 20 which implies that almost all elements of  $\mathbb{V}_2(n)$  of the form  $G *_H G$  may be written as semi-direct products  $\mathbb{Z} \rtimes G$ . In Section II.9 we will see that a similar result holds for the isomorphism classes of the Type II subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$ , the exceptional case being  $\text{Dih}_8 *_{\text{Dih}_4} \text{Dih}_8$ .

In Part I, we then study the elements of  $\mathcal{VC}$  of the form  $F \rtimes_{\theta} \mathbb{Z}$ , where  $F$  is one of the finite groups occurring as a finite subgroup of  $B_n(\mathbb{S}^2)$ . One of the main difficulties that we face initially is that in general there are many possible actions of  $\mathbb{Z}$  on  $F$ . However, as we shall see in Sections 2–5, a large number of these actions are incompatible with the structure of  $B_n(\mathbb{S}^2)$ . In Section 2, we prove Proposition 8, which will enable us to rule out the case where  $F$  is a maximal finite cyclic or dihedral group. In Section 3, we obtain a second reduction using the fact that the isomorphism class of  $F \rtimes_{\theta} \mathbb{Z}$  depends only on the outer automorphism induced by  $\theta(1)$  in  $\text{Out}(F)$ . Since we are primarily interested in the isomorphism classes of the virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$ , it follows that it suffices to consider automorphisms of  $F$  belonging to a transversal of  $\text{Out}(F)$  in  $\text{Aut}(F)$ . The subsequent study of the structure of  $\text{Out}(F)$ , where  $F$  is either  $\mathcal{Q}_8$  or one of the binary polyhedral groups, then narrows down the possible Type I subgroups of  $B_n(\mathbb{S}^2)$ . If  $F = T^*, O^*$  or  $I^*$  then  $\text{Out}(F) \cong \mathbb{Z}_2$ , so we have just two possible actions to

consider, the trivial one, and a non-trivial one, which we shall describe. In Section 4, we obtain in Proposition 9 an extension to  $B_n(\mathbb{S}^2)$  of a result of Hodgkin concerning the centralisers of finite order elements of  $\mathcal{MCG}(\mathbb{S}^2, n)$ . This allows us to reduce greatly the number of possible actions in the case where  $F$  is cyclic or dicyclic. In Section 5.1, in Proposition 10 we give an alternative proof of a result of [BCP, FZ] that says that if  $n \geq 3$ , the universal covering space of the  $n^{\text{th}}$  permuted configuration space  $D_n(\mathbb{S}^2)$  of  $\mathbb{S}^2$  has the homotopy type of  $\mathbb{S}^3$ . This fact will then be used in Section 5.2 to show in Lemma 41 that the non-trivial subgroups of  $B_n(\mathbb{S}^2)$  have cohomological period 2 or 4. The ensuing study of the cohomology of the groups of the form  $F \rtimes_{\theta} \mathbb{Z}$ , where  $F = O^*$  or  $I^*$ , will allow us to exclude the possibility of the non-trivial action in these cases. Putting together the analysis of Sections 2–5 will lead us to the proof of Theorem 5(1) for the Type I subgroups. In Section 6.2, we study the infinite virtually cyclic groups of the form  $G_1 *_F G_2$ , where  $F, G_1, G_2$  are finite and  $[G_i : F] = 2$  for  $i = 1, 2$ . Using the cohomological properties obtained in Section 5.2 and the relation with the groups of the form  $F \rtimes_{\theta} \mathbb{Z}$ , we show that any group of this form that is realised as a subgroup of  $B_n(\mathbb{S}^2)$  is isomorphic to an element of  $\mathbb{V}_2(n)$ . This will enable us to prove Theorem 5(1) in Section 6.2.

## 1 Virtually cyclic groups: generalities

We start by recalling the definition and Epstein and Wall's characterisation of virtually cyclic groups. We then proceed to prove some general results concerning these groups, notably Propositions 11 and 26, that will be used in Part II of the manuscript.

**DEFINITION 15.** A group is said to be *virtually cyclic* if it contains a cyclic subgroup of finite index.

**REMARKS 16.**

- (a) Every finite group is virtually cyclic.
- (b) Every infinite virtually cyclic group contains a normal subgroup of finite index.

The following criterion is well known; most of the first part is due to Epstein and Wall [Ep, Wa].

**THEOREM 17.** *Let  $G$  be a group. Then the following statements are equivalent.*

- (a)  $G$  is a group with two ends.
- (b)  $G$  is an infinite virtually cyclic group.
- (c)  $G$  has a finite normal subgroup  $F$  such that  $G/F$  is isomorphic to  $\mathbb{Z}$  or to the infinite dihedral group  $\mathbb{Z}_2 * \mathbb{Z}_2$ .

*Equivalently,  $G$  is of the form:*

- (i)  $F \rtimes_{\theta} \mathbb{Z}$  for some action  $\theta \in \text{Hom}(\mathbb{Z}, \text{Aut}(F))$ , or
- (ii)  $G_1 *_F G_2$ , where  $[G_i : F] = 2$  for  $i = 1, 2$ ,

*where  $G_1, G_2$  and  $F$  are finite groups.*

**DEFINITION 18.** An infinite virtually cyclic group will be said to be of *Type I* (resp. *Type II*) if it is of the form given by (i) (resp. by (ii)).

*Proof of Theorem 17.* The equivalence of parts (a) and (b) may be found in [Ep], and the implication (a) implies (c) is proved in [Wa]. So to prove the first part, it suffices to show that (c) implies (b). Suppose then that  $G$  has a finite normal subgroup  $F$  such that  $G/F$  is isomorphic to  $\mathbb{Z}$  or to  $\mathbb{Z}_2 * \mathbb{Z}_2$ . Clearly  $G$  is infinite. Assume first that  $G \cong F \rtimes_{\theta} \mathbb{Z}$ , where  $\theta \in \text{Hom}(\mathbb{Z}, \text{Aut}(F))$ , let  $k$  be the order of the automorphism  $\theta(1) \in \text{Aut}(F)$ , let  $s: G/F \rightarrow G$  be a section for the canonical projection  $p: G \rightarrow G/F$ , and let  $x$  be a generator of the infinite cyclic group  $G/F$ . Since  $\theta(x)(f) = s(x)fs(x^{-1})$  for all  $f \in F$ , it follows that the infinite cyclic subgroup  $\langle s(x^k) \rangle$  is central in  $G$ , and that there exists a commutative diagram of short exact sequences of the form:

$$\begin{array}{ccccccc}
 & & & 1 & & 1 & \\
 & & & \downarrow & & \downarrow & \\
 & & & \langle s(x^k) \rangle & \xrightarrow[p|_{\langle s(x^k) \rangle}]{\cong} & \langle x^k \rangle \cong k\mathbb{Z} & \\
 & & & \downarrow & & \downarrow & \\
 1 & \longrightarrow & F & \longrightarrow & G & \xrightarrow{p} & G/F \cong \mathbb{Z} \longrightarrow 1 \\
 & & \downarrow \varphi|_F \cong & & \downarrow \varphi & & \downarrow \hat{\varphi} \\
 1 & \longrightarrow & \text{Ker}(\hat{p}) & \longrightarrow & G/\langle s(x^k) \rangle & \xrightarrow[\hat{p}]{\dots\dots\dots} & \mathbb{Z}/k\mathbb{Z} \longrightarrow 1, \\
 & & & & \downarrow & & \downarrow \\
 & & & & 1 & & 1
 \end{array} \tag{15}$$

the left-hand vertical extension being central, where

$$\varphi: G \rightarrow G/\langle s(x^k) \rangle \text{ and } \hat{\varphi}: \mathbb{Z} \rightarrow \mathbb{Z}/k\mathbb{Z}$$

are the canonical projections, and  $\hat{p}: G/\langle s(x^k) \rangle \rightarrow \mathbb{Z}/k\mathbb{Z}$  is the epimorphism induced on the quotients. Since the restriction of  $p$  to  $\langle s(x^k) \rangle$  is an isomorphism, it follows that  $\varphi|_F: F \rightarrow \text{Ker}(\hat{p})$  is too. Thus  $G/\langle s(x^k) \rangle$  is of order  $k|F|$ . Since  $\langle s(x^k) \rangle$  is infinite cyclic, the left-hand vertical extension then implies that  $G$  is virtually cyclic.

Now suppose that  $G/F \cong \mathbb{Z}_2 * \mathbb{Z}_2$ . Then  $G/F \cong \mathbb{Z} \rtimes \mathbb{Z}_2$ , where the action of  $\mathbb{Z}_2$  on  $\mathbb{Z}$  is non trivial. So there exist a short exact sequence

$$1 \rightarrow F \rightarrow G \xrightarrow{p} \mathbb{Z} \rtimes \mathbb{Z}_2 \rightarrow 1$$

and a split extension

$$1 \rightarrow F \rightarrow \hat{G} \xrightarrow{p|_{\hat{G}}} \mathbb{Z} \rightarrow 1,$$

where  $\hat{G}$  is the inverse image of the  $\mathbb{Z}$ -factor of  $\mathbb{Z} \rtimes \mathbb{Z}_2$  under  $p$ . Let  $x$  be a generator of  $\mathbb{Z}$ , and let  $s: \mathbb{Z} \rightarrow \hat{G}$  be a section for  $p|_{\hat{G}}$ . Applying the argument of the previous paragraph to  $\hat{G} \cong F \rtimes \mathbb{Z}$ , there exists a central extension

$$1 \rightarrow \langle s(x^k) \rangle \rightarrow \hat{G} \rightarrow \hat{G}/\langle s(x^k) \rangle \rightarrow 1,$$

where  $k$  is the order of  $\text{Aut}(F)$ . Let  $m = |F|$ . We claim that  $\langle s(x^{mk}) \rangle$  is normal in  $G_1 *_F G_2$ . To see this, first note that  $\langle s(x^{mk}) \rangle \subset Z(\widehat{G})$ . Now let  $g \in G \setminus \widehat{G}$ . Then  $p(gs(x^k)g^{-1}) = x^{-k}$  since  $p(g)$  is sent to an element of the form  $(x^q, \bar{1})$  in  $\mathbb{Z} \rtimes \mathbb{Z}_2$ , where  $q \in \mathbb{Z}$ . Hence  $gs(x^k)g^{-1} = s(x^{-k})f$ , where  $f \in F$ . Since  $s(x^k) \in Z(\widehat{G})$ ,  $s(x^k)$  commutes with  $f$ , and so

$$gs(x^{mk})g^{-1} = (s(x^{-k})f)^m = s(x^{-mk}). \quad (16)$$

We thus have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & \langle s(x^{mk}) \rangle & \longrightarrow & \widehat{G} & \longrightarrow & \widehat{G} / \langle s(x^{mk}) \rangle \longrightarrow 1 \\
 & & \parallel & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \langle s(x^{mk}) \rangle & \longrightarrow & G & \longrightarrow & G / \langle s(x^{mk}) \rangle \longrightarrow 1. \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathbb{Z}_2 & \xlongequal{\quad} & \mathbb{Z}_2 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 1 & & 1
 \end{array} \quad (17)$$

An argument similar to that of the previous paragraph shows that  $|\widehat{G} / \langle s(x^{mk}) \rangle| = m^2k$ , and so  $|G / \langle s(x^{mk}) \rangle| = 2m^2k$ . Since  $\langle s(x^{mk}) \rangle \cong \mathbb{Z}$ , it follows from the second row of (17) that  $G$  is virtually cyclic. This shows that (c) implies (b), and thus completes the proof of the first part of the statement.

We now prove the second statement of the theorem. First note that in part (c), the fact that  $G/F$  is isomorphic to  $\mathbb{Z}$  is clearly equivalent to condition (i). Suppose then that condition (ii) holds. Since  $[G_i : F] = 2$  for  $i = 1, 2$ ,  $F$  is normal in  $G_i$ , so is normal in  $G = G_1 *_F G_2$ , and  $G/F \cong \mathbb{Z}_2 * \mathbb{Z}_2$ . Finally, suppose that  $G$  has a finite normal subgroup  $F$  such that  $G/F \cong \mathbb{Z}_2 * \mathbb{Z}_2$ . Let  $\Pi: G \rightarrow G/F$  denote the canonical projection. For  $i = 1, 2$ , let  $y_i \in G/F$  be such that  $G/F = \langle y_1, y_2 \mid y_1^2 = y_2^2 = 1 \rangle$ , and let  $G_i = \Pi^{-1}(\langle y_i \rangle)$ . Then the groups  $G_i$  are finite and each contain  $F$  as a subgroup of index 2. We can thus form the amalgamated product  $G_1 *_F G_2$ . So  $F$  is normal in  $G_1 *_F G_2$ , and the quotient  $(G_1 *_F G_2)/F$  is isomorphic to  $\mathbb{Z}_2 * \mathbb{Z}_2$ . By standard properties of amalgamated products, there exists a unique (surjective) homomorphism  $\varphi: G_1 *_F G_2 \rightarrow G$  that makes the following diagram of short exact sequences commutative:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & F & \longrightarrow & G_1 *_F G_2 & \xrightarrow{q} & (G_1 *_F G_2)/F \longrightarrow 1 \\
 & & \parallel & & \downarrow \varphi & & \downarrow \widehat{\varphi} \\
 1 & \longrightarrow & F & \longrightarrow & G & \xrightarrow{\Pi} & G/F \longrightarrow 1,
 \end{array}$$

$q$  being the canonical projection, and where  $\widehat{\varphi}$  is the induced homomorphism on the quotients. Now for  $i = 1, 2$ ,  $\varphi(g) = g$  for all  $g \in G_i$ , and so  $\widehat{\varphi}(q(x_i)) = y_i$ . In particular,

$\hat{\varphi}$  sends the  $\mathbb{Z}_2$ -factors of  $(G_1 *_F G_2)/F$  isomorphically onto those of  $G/F$ , and thus  $\hat{\varphi}$  is an isomorphism. It follows from the 5-Lemma that  $\varphi$  is also an isomorphism.  $\square$

The following result shows that the type of an infinite virtually cyclic group is determined by the (non) centrality of the extension given by Theorem 17(b).

PROPOSITION 19. *Let  $G$  be an infinite virtually cyclic group. Then  $G$  is of Type I (resp. of Type II) if and only if the extension*

$$1 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow F \longrightarrow 1 \quad (18)$$

*arising in the definition of virtually cyclic group is central (resp. is not central).*

*Proof.* In order to prove the proposition, we start by showing that if

$$1 \longrightarrow \mathbb{Z} \xrightarrow{\iota_j} G \longrightarrow F_j \longrightarrow 1 \quad \text{for } j = 1, 2,$$

are extensions of  $G$ , with  $F_j$  finite, then they are either both central or both non central. Note that the intersection  $\iota_1(\mathbb{Z}) \cap \iota_2(\mathbb{Z})$  is a normal subgroup of  $G$  of finite index, and so is infinite cyclic. Since an automorphism of  $\mathbb{Z}$  is completely determined by its restriction to the subgroup  $k\mathbb{Z} \subset \mathbb{Z}$  for any  $k \neq 0$ , (as the automorphism and its restriction are either both equal to  $\text{Id}$  or to  $-\text{Id}$ ), the two extensions are thus either both central or both non central.

To prove the necessity of the condition, consider the extension (18) given by the definition of virtually cyclic group. Assume first that  $G$  is of Type I. By the first part of Theorem 17, there exists a finite subgroup  $F'$  of  $G$  and  $\theta \in \text{Hom}(\mathbb{Z}, \text{Aut}(F'))$  such that  $F' \rtimes_{\theta} \mathbb{Z}$ . Using the notation of the first part of the proof of Theorem 17, as in the commutative diagram (15), we obtain a central extension

$$1 \longrightarrow \langle s(x^k) \rangle \longrightarrow G \longrightarrow G / \langle s(x^k) \rangle \longrightarrow 1.$$

Since  $\langle s(x^k) \rangle \cong \mathbb{Z}$ , it follows from the first paragraph that the extension (18) is central.

Now suppose that  $G$  is of Type II. From the proof of the first part of Theorem 17, from the commutative diagram (17) we obtain an extension

$$1 \longrightarrow \langle s(x^{mk}) \rangle \longrightarrow G \longrightarrow G / \langle s(x^{mk}) \rangle \longrightarrow 1,$$

where  $\langle s(x^{mk}) \rangle \cong \mathbb{Z}$ ,  $G / \langle s(x^{mk}) \rangle$  is finite. Equation (16) implies that this extension is non central. Using the first paragraph once more, it follows that the extension (18) is non central. This proves the necessity of the conditions.

Conversely, if the extension (18) is central (resp. non central) then from Theorem 17, it must be of Type I (resp. Type II) because as we saw in the two previous paragraphs, any group of Type I (resp. Type II) is the middle group of a central (resp. non central) extension. But by the first paragraph of this proof, this property is independent of the short exact sequence.  $\square$

The following proposition will be used in Section II.8 to give an alternative description of the elements of  $\mathbb{V}_2(n)$  as semi-direct products.

PROPOSITION 20. Let  $G_1$  and  $G_2$  be isomorphic groups, and consider the amalgamated product  $G = G_1 *_H G_2$  defined by

$$\begin{array}{ccccc}
 & & H_1 & \longrightarrow & G_1 \\
 & \nearrow^{i_1} & & & \searrow \\
 H & & & & G_1 *_H G_2 \\
 & \searrow_{i_2} & & & \nearrow \\
 & & H_2 & \longrightarrow & G_2
 \end{array}$$

where for  $j = 1, 2$ ,  $H_j$  is a subgroup of  $G_j$  of index 2 and  $i_j: H \rightarrow H_j$  is an embedding of the abstract group  $H$  in  $G_j$ , the remaining arrows being inclusions. Suppose that the isomorphism  $i_2 \circ i_1^{-1}: H_1 \rightarrow H_2$  extends to an isomorphism  $\iota: G_1 \rightarrow G_2$ . Then  $G \cong \mathbb{Z} \rtimes G_i$ , where the action is given by

$$g_i t g_i^{-1} = \begin{cases} t & \text{if } g_i \in H_i \\ t^{-1} & \text{if } g_i \in G_i \setminus H_i, \end{cases} \quad (19)$$

$t$  being a generator of the  $\mathbb{Z}$ -factor.

*Proof.* We start by constructing a homomorphism  $\alpha: G_1 *_H G_2 \rightarrow G_2$ . It suffices to define  $\alpha$  on the elements of  $G_1$  and  $G_2$ . Let

$$\alpha(x) = \begin{cases} \iota(x) & \text{if } x \in G_1 \\ x & \text{if } x \in G_2. \end{cases}$$

Then  $\alpha$  is well defined, since if  $h \in H$  then  $\alpha(i_1(h)) = \iota(i_1(h)) = i_2(h) = \alpha(i_2(h))$  since  $i_j(h) \in H_j$  for  $j \in \{1, 2\}$ . Hence we obtain a split short exact sequence:

$$1 \rightarrow \text{Ker}(\alpha) \rightarrow G_1 *_H G_2 \rightarrow G_2 \rightarrow 1, \quad (20)$$

where a section  $s: G_2 \rightarrow G_1 *_H G_2$  is just given by inclusion. It remains to show that  $\text{Ker}(\alpha) \cong \mathbb{Z}$ , and to determine the action.

Let  $p: G_1 *_H G_2 \rightarrow \mathbb{Z}_2 * \mathbb{Z}_2$  be the canonical projection of  $G_1 *_H G_2$  onto the quotient  $(G_1 *_H G_2)/H$ . If  $h \in H$  then  $\alpha(i_2(h)) = i_2(h)$ , so the lower left-hand square of the following diagram of short exact sequences is commutative:

$$\begin{array}{ccccccc}
 & & & 1 & & & 1 \\
 & & & \downarrow & & & \downarrow \\
 & & & \text{Ker}(\alpha) & & & \mathbb{Z} \\
 & & & \downarrow & & & \downarrow \\
 1 & \longrightarrow & H & \xrightarrow{i_2} & G_1 *_H G_2 & \xrightarrow{p} & \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow 1 \\
 & & \parallel & & \downarrow \alpha & & \downarrow \hat{\alpha} \\
 1 & \longrightarrow & H & \xrightarrow{i_2} & G_2 & \xrightarrow{p|_{G_2}} & \mathbb{Z}_2 \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

Thus  $\alpha$  induces a homomorphism  $\hat{\alpha}: \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$  that makes the lower right-hand square commute. Let  $i \in \{1, 2\}$ , and suppose that  $g_i \in G_i \setminus H_i$ . If  $i = 1$  then  $\alpha(g_1) = \iota(g_1) \in G_2 \setminus H_2$  because  $\iota$  is an isomorphism that sends  $H_1$  to  $H_2$ , while if  $i = 2$  then  $\alpha(g_2) = g_2 \in G_2 \setminus H_2$ . We conclude that  $p(\alpha(g_i)) = \bar{1}$ . Setting  $x_i = p(g_i)$ , the commutativity of the above diagram implies firstly that  $\hat{\alpha}(x_i) = \bar{1}$ , and hence  $\text{Ker}(\hat{\alpha}) = \langle x_1 x_2 \rangle \cong \mathbb{Z}$ , and secondly that the restriction

$$p|_{\text{Ker}(\alpha)} : \text{Ker}(\alpha) \longrightarrow \text{Ker}(\hat{\alpha})$$

is an isomorphism and that  $\text{Ker}(\alpha) = \langle g_1(\iota(g(1)))^{-1} \rangle \cong \mathbb{Z}$  for any  $g_1 \in G_1 \setminus H_1$ . Thus  $G_1 *_H G_2 \cong \mathbb{Z} \rtimes G_2$  by equation (20). Further, if  $g_2 \in G_2$  then

$$\begin{aligned} p\left(g_2(g_1(\iota(g(1)))^{-1})g_2^{-1}\right) &= p(g_2)x_1x_2p(g_1^{-1}) \\ &= \begin{cases} x_1x_2 & \text{if } g_2 \in i_2(H) = H_2 \\ x_2x_1x_2x_2^{-1} = x_2x_1 = (x_1x_2)^{-1} & \text{if } g_2 \in G_2 \setminus H_2. \end{cases} \end{aligned}$$

The action given by equation (19) then follows from the commutativity of the above diagram, where  $t$  is taken to be the element  $g_1(\iota(g(1)))^{-1}$ .  $\square$

We now turn our attention to the virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$ .

**DEFINITION 21.** Given  $n \geq 4$ , let  $\mathcal{VC}$  denote the family of virtually cyclic groups consisting of all groups of Type I and Type II whose factors  $F, G_1$  and  $G_2$ , as described by Theorem 17, are subgroups of  $\mathbb{Z}_{2(n-1)}, \text{Dic}_{4n}, \text{Dic}_{4(n-2)}, T^*, O^*$  or  $I^*$ .

The family  $\mathcal{VC}$  thus consists of the infinite virtually cyclic groups that are formed using the finite subgroups of  $B_n(\mathbb{S}^2)$ . The following proposition is an immediate consequence of Theorems 2 and 17.

**PROPOSITION 22.** *Let  $G$  be a virtually cyclic subgroup of  $B_n(\mathbb{S}^2)$ .*

- (a) *If  $G$  is finite then it is isomorphic to a subgroup of one of  $\mathbb{Z}_{2(n-1)}, \text{Dic}_{4n}, \text{Dic}_{4(n-2)}, T^*, O^*$  or  $I^*$ .*
- (b) *If  $G$  is infinite then it is isomorphic to an element of  $\mathcal{VC}$ .*

We recall the following general result from [GG8], which will prove to be very useful when it comes to constructing subgroups of  $B_n(\mathbb{S}^2)$  of Type II.

**PROPOSITION 23** ([GG8, Lemma 15]). *Let  $G = G_1 *_F G_2$  be a virtually cyclic group of Type II, and let  $\varphi: G_1 *_F G_2 \longrightarrow H$  be a homomorphism such that for  $i = 1, 2$ , the restriction of  $\varphi$  to  $G_i$  is injective. Then  $\varphi$  is injective if and only if  $\varphi(G)$  is infinite.*

**REMARK 24.** Proposition 23 will be applied in the following manner: we will be given finite subgroups  $\tilde{G}_1, \tilde{G}_2$  of  $B_n(\mathbb{S}^2)$  such that  $\tilde{F} = \tilde{G}_1 \cap \tilde{G}_2$  is of index two in both  $\tilde{G}_1$  and  $\tilde{G}_2$ . The aim will be to prove that the subgroup  $\langle \tilde{G}_1 \cup \tilde{G}_2 \rangle$  is the amalgamated product of  $\tilde{G}_1$  and  $\tilde{G}_2$  along  $\tilde{F}$ . It will suffice to show that  $\langle \tilde{G}_1 \cup \tilde{G}_2 \rangle$  is infinite. Suppose that this is indeed the case. Let  $G_1$  and  $G_2$  be abstract groups isomorphic respectively to  $\tilde{G}_1$  and  $\tilde{G}_2$  whose intersection is an index two subgroup  $F$ . We define a

map  $\varphi: G_1 *_F G_2 \longrightarrow \langle \tilde{G}_1 \cup \tilde{G}_2 \rangle$  that sends  $F$  onto  $\tilde{F}$  and  $G_i$  onto  $\tilde{G}_i$  isomorphically for  $i = 1, 2$ . Then  $\varphi$  is a surjective homomorphism, and by Proposition 23, is an isomorphism.

As an easy exercise, we may deduce the classification of the virtually cyclic subgroups of  $P_n(\mathbb{S}^2)$ . If  $n \leq 3$  then  $P_n(\mathbb{S}^2)$  is trivial if  $n \leq 2$ , and  $P_3(\mathbb{S}^2) \cong \mathbb{Z}_2$ . So suppose that  $n \geq 4$ . The only finite subgroups of  $P_n(\mathbb{S}^2)$  are  $\{e\}$  and  $\langle \Delta_n^2 \rangle$ , both of which are central.

**PROPOSITION 25.** *Let  $n \geq 4$ . The virtually cyclic subgroups of  $P_n(\mathbb{S}^2)$  are  $\{e\}$ ,  $\langle \Delta_n^2 \rangle$ ,  $\langle x \rangle \cong \mathbb{Z}$  and  $\langle \Delta_n^2, x \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}$ , where  $x$  is any element of  $P_n(\mathbb{S}^2) \setminus \{\Delta_n^2\}$ .*

*Proof.* Let  $G$  be an infinite virtually cyclic subgroup of  $P_n(\mathbb{S}^2)$ . The Type I subgroups are  $\mathbb{Z}$  and  $\mathbb{Z}_2 \times \mathbb{Z}$  (both are realised, by taking  $\langle x \rangle$  and  $\langle \Delta_n^2, x \rangle$  respectively, where  $x$  is any element of  $P_n(\mathbb{S}^2) \setminus \{\Delta_n^2\}$ ). As for the Type II subgroups, the only possibility is  $F = \{e\}$  and  $G_1 = G_2 = \langle \Delta_n^2 \rangle$ , but then  $G \not\cong \mathbb{Z}_2 * \mathbb{Z}_2$  since  $\langle \Delta_n^2 \rangle$  is the unique subgroup of  $P_n(\mathbb{S}^2)$  of order two.  $\square$

The following result will be used later on to show that there is an almost one-to-one correspondence between the virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$  and those of  $\mathcal{MCG}(\mathbb{S}^2, n)$ . This will also enable us to construct copies of  $T^* \times \mathbb{Z}$  (Proposition 62) and  $O^* *_T O^*$  (Proposition 71) in  $B_n(\mathbb{S}^2)$  for certain values of  $n$ , as well as to prove Theorem 14.

**PROPOSITION 26.** *Let  $G$  be a group that possesses a unique element  $x$  of order 2, let  $G' = G/\langle x \rangle$ , and let  $p: G \longrightarrow G'$  denote the canonical projection.*

(a) *Let  $H$  be a virtually cyclic subgroup of  $G$ .*

(i)  *$H' = p(H)$  is a virtually cyclic subgroup of  $G'$  of the same type (finite, of Type I or of Type II) as  $H$ .*

(ii) *Let  $H \cong F \rtimes_{\theta} \mathbb{Z}$ , where  $F$  is a finite subgroup of  $G$  and  $\theta \in \text{Hom}(\mathbb{Z}, \text{Aut}(F))$ . Then  $p(H) \cong p(F) \rtimes_{\theta'} \mathbb{Z}$ , where  $\theta' \in \text{Hom}(\mathbb{Z}, \text{Aut}(F'))$  is the action induced by  $\theta$ , and defined by  $\theta'(1)(f') = p(\theta(1)(f))$  for all  $f' \in F'$ , where  $f \in F$  satisfies  $p(f) = f'$ .*

(iii) *Let  $H \cong G_1 *_F G_2$ , where  $G_1, G_2$  are subgroups of  $H$ , and  $F = G_1 \cap G_2$  is of index 2 in  $G_1$  and  $G_2$ . Then  $p(H) \cong p(G_1) *_p(F) p(G_2)$ .*

(b) *Let  $H'$  be a virtually cyclic subgroup of  $G'$ .*

(i)  *$H = p^{-1}(H')$  is a virtually cyclic subgroup of  $G$  of the same type (finite, of Type I or of Type II) as  $H'$ .*

(ii) *If  $H' \cong G'_1 *_F G'_2$ , where  $G'_1, G'_2$  are subgroups of  $H'$ , and  $F' = G'_1 \cap G'_2$  is of index 2 in  $G'_1$  and  $G'_2$ , then  $H \cong p^{-1}(G'_1) *_p(F') p^{-1}(G'_2)$ .*

(c) *Let  $H_1$  and  $H_2$  be isomorphic subgroups of  $G$ . Then  $p(H_1)$  and  $p(H_2)$  are isomorphic subgroups of  $G'$ .*

*Proof.* First note that since  $x$  is the unique element of  $G$  of order 2, the subgroup  $\langle x \rangle$  is characteristic in  $G$ , in particular,  $x \in Z(G)$ . We start by proving parts (a)(i) and (b)(i). The result is clear if either  $H$  or  $H'$  is finite, so it suffices to consider the cases where they



are infinite. Before proving the statement in these cases, let us introduce some notation. Suppose that  $H$  (resp.  $H'$ ) is an infinite virtually cyclic subgroup of  $G$  (resp.  $G'$ ). Then by Theorem 17,  $H$  (resp.  $H'$ ) has a finite normal subgroup  $F$  (resp.  $F'$ ) such that  $H/F$  (resp.  $H'/F'$ ) is isomorphic to  $\mathbb{Z}$  if  $H$  (resp.  $H'$ ) is of Type I, and to  $\mathbb{Z}_2 * \mathbb{Z}_2$  if  $H$  (resp.  $H'$ ) is of Type II. Let  $H' = p(H)$  and  $F' = p(F)$  (resp.  $H = p^{-1}(H')$  and  $F = p^{-1}(F')$ ). So  $H'$  (resp.  $H$ ) is infinite, and  $F'$  (resp.  $F$ ) is finite. Further,  $p|_F: F \rightarrow F'$  and  $p|_H: H \rightarrow H'$  are surjective,  $F'$  (resp.  $F$ ) is normal in  $H'$  (resp.  $H$ ), and

$$\{e\} \subset \text{Ker}(p|_F) \subset \text{Ker}(p|_H) \subset \text{Ker}(p) = \langle x \rangle. \quad (21)$$

Then we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \text{Ker}(p|_F) & \longrightarrow & \text{Ker}(p|_H) & & \\
 & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & F & \longrightarrow & H & \xrightarrow{q} & H/F \longrightarrow 1 \\
 & & \downarrow p|_F & & \downarrow p|_H & & \downarrow \hat{p} \\
 1 & \longrightarrow & F' & \longrightarrow & H' & \xrightarrow{q'} & H'/F' \longrightarrow 1, \\
 & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 
 \end{array} \quad (22)$$

where  $q: H \rightarrow H/F$  and  $q': H' \rightarrow H'/F'$  are the canonical projections, the map

$$\hat{p}: H/F \rightarrow H'/F'$$

is the induced surjective homomorphism on the quotients and  $\text{Ker}(p|_F) \rightarrow \text{Ker}(p|_H)$  is inclusion. We claim that  $\text{Ker}(p|_F) = \text{Ker}(p|_H)$ . This being the case,  $\hat{p}$  is an isomorphism, and thus  $H$  and  $H'$  are virtually cyclic groups of the same type, which proves the proposition. If  $x \notin H$  then  $\text{Ker}(p|_F) = \text{Ker}(p|_H) = \{e\}$  trivially by equation (21). So assume that  $x \in H$ . To prove the claim, by equation (21), it suffices to show that  $x \in F$ . We separate the two cases corresponding to parts (a)(i) and (b)(i) of the statement.

(a)(i) If  $H$  is of Type I then  $H \cong F \times \mathbb{Z}$ , and so  $x \in F$  since  $x$  is of finite order. So suppose that  $H$  is of Type II. Then  $H \cong G_1 *_F G_2$ , where  $G_1, G_2$  are subgroups of  $H$  that contain  $F$  as a subgroup of index 2. By standard properties of amalgamated products,  $x$  belongs to a conjugate in  $H$  of one of the  $G_i$  because it is of finite order, and since  $x \in Z(G)$ , it belongs to one of the  $G_i$ , which shows that  $G_1$  and  $G_2$  are of (the same) even order. The fact that  $x$  is the unique element of  $G$  of order 2 implies that  $x \in G_1 \cap G_2 = F$  as required.

(b)(i) In this case,  $\text{Ker}(p|_F) = \text{Ker}(p|_H) = \text{Ker}(p) = \langle x \rangle$  by construction.

This proves the claim, and thus we obtain parts (a)(i) and (b)(i).

We now prove part (a)(ii). Let  $H$  be an infinite Type I subgroup of  $G$  and let  $F$  be a finite normal subgroup of  $H$  such that there exists a short exact sequence of the form

$$1 \longrightarrow F \longrightarrow H \xrightarrow{q} H/F \longrightarrow 1,$$

where  $H/F \cong \mathbb{Z}$ , and where  $q: H \rightarrow H/F$  is the canonical projection. By the previous paragraph, we thus have the commutative diagram (22),  $\hat{p}$  being an isomorphism. Let  $z$  be a generator of  $H/F$ , let  $s: H/F \rightarrow H$  be a section for  $q$  such that  $\theta(z)(f) = s(z).f.s(z^{-1})$  for all  $f \in F$ , where  $\theta \in \text{Hom}(H/F, \text{Aut}(F))$  is given. The commutativity of the diagram (22) implies that  $s' = p \circ s \circ \hat{p}^{-1}: H'/F' \rightarrow H'$  is a section for  $q'$ . Since  $x \in Z(G)$ , if  $x \in F$  then  $\theta(z)(x) = x$ , and so  $p$  induces a homomorphism  $\Phi: \text{Aut}(F) \rightarrow \text{Aut}(F')$  satisfying  $\Phi(\alpha)(p(f)) = p(\alpha(f))$  for all  $f \in F$  and  $\alpha \in \text{Aut}(F)$ . We thus obtain a homomorphism  $\theta': H'/F' \rightarrow \text{Aut}(F')$  defined by  $\theta' = \Phi \circ \theta \circ \hat{p}^{-1}$  that makes the following diagram commute:

$$\begin{array}{ccc} H/F & \xrightarrow{\theta} & \text{Aut}(F) \\ \downarrow \hat{p} & & \downarrow \Phi \\ H'/F' & \xrightarrow{\theta'} & \text{Aut}(F'). \end{array}$$

In particular, if  $f' \in F'$  and if  $f \in F$  is such that  $p(f) = f'$  then:

$$\begin{aligned} s'(z').f'.s'(z'^{-1}) &= p \circ s(z).p(f).p \circ s(z^{-1}) = p(s(z).f.s(z^{-1})) = p(\theta(z))(f) \\ &= \Phi \circ \theta(z)(f') = \theta'(z')(f'), \end{aligned}$$

and thus  $H' \cong F' \rtimes_{\theta'} \mathbb{Z}$ , where  $\theta' \in \text{Hom}(\mathbb{Z}, \text{Aut}(F'))$  is the homomorphism induced by  $\theta \in \text{Hom}(\mathbb{Z}, \text{Aut}(F))$  given by  $\theta'(1)(f') = p(\theta(1)(f))$  for all  $f' \in F'$ , where  $f \in F$  satisfies  $p(f) = f'$ , and where we write the generators of  $H/F$  and  $H'/F'$  as 1. This proves part (a)(ii).

We now prove part (a)(iii). Let  $H, G_1, G_2$  and  $F$  be as in the statement, and let  $H', G'_1, G'_2$  and  $F'$  be their respective images under  $p$ . Then  $H/F \cong \mathbb{Z}_2 * \mathbb{Z}_2$ , and once more we have the commutative diagram (22),  $\hat{p}$  being an isomorphism. By part (a)(i),  $H'$  is a Type II subgroup of  $G$ . Now  $F'$  is of index 2 in both  $G'_1$  and  $G'_2$ , and the inclusions  $F' \subset G'_i$  give rise to an amalgamated product  $G'_1 *_{F'} G'_2$  whose quotient by  $F'$  is isomorphic to  $\mathbb{Z}_2 * \mathbb{Z}_2$ . Since  $H = \langle G_1 \cup G_2 \rangle$  and  $p|_H: H \rightarrow H'$  is surjective, we have that  $H' = \langle G'_1 \cup G'_2 \rangle$ . By the universality property of amalgamated products, there exists a surjective homomorphism  $\alpha: G'_1 *_{F'} G'_2 \rightarrow H'$  satisfying  $\alpha(g'_i) = g'_i$  for all  $g'_i \in G'_i$ . We thus obtain the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & F' & \longrightarrow & G'_1 *_{F'} G'_2 & \longrightarrow & \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow 1 \\ & & \parallel & & \downarrow \alpha & & \downarrow \hat{\alpha} \\ 1 & \longrightarrow & F' & \longrightarrow & H' & \xrightarrow{q'} & H'/F' \longrightarrow 1, \end{array}$$

where  $\hat{\alpha}$  is the homomorphism induced on the quotients. The surjectivity of  $\alpha$  implies that of  $\hat{\alpha}$ . The finiteness of  $\mathbb{Z}_2$  implies that the free product  $\mathbb{Z}_2 * \mathbb{Z}_2$ , which is finitely

generated, is residually finite [Coh, Proposition 22]. It thus follows that  $H'/F' \cong \mathbb{Z}_2 * \mathbb{Z}_2$  is Hopfian [Coh, see the proof of the Corollary, page 12], so  $\hat{\alpha}$  is an isomorphism. Using the 5-Lemma, we see that  $\alpha$  is an isomorphism as required.

We now prove part (b)(ii). For  $i = 1, 2$ , let  $G_i = p^{-1}(G'_i)$ . Since  $H' \cong G'_1 *_F G'_2$  and  $\hat{p}$  is an isomorphism, we have that  $H'/F' \cong H/F \cong \mathbb{Z}_2 * \mathbb{Z}_2$ . Now  $F$  is a subgroup of both  $G_1$  and  $G_2$ , and the corresponding inclusions give rise to an amalgamated product  $G_1 *_F G_2$  whose quotient by  $F$  is isomorphic to  $\mathbb{Z}_2 * \mathbb{Z}_2$ . The equality  $H' = \langle G'_1 \cup G'_2 \rangle$  implies that  $H = \langle G_1 \cup G_2 \rangle$ , and it follows from the universality property of amalgamated products that there exists a (unique) surjective homomorphism  $\alpha: G_1 *_F G_2 \rightarrow H$  satisfying  $\alpha(g_i) = g_i$  for all  $g_i \in G_i$ . We thus have a commutative diagram of the form

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F & \longrightarrow & G_1 *_F G_2 & \longrightarrow & \mathbb{Z}_2 * \mathbb{Z}_2 & \longrightarrow & 1 \\ & & \parallel & & \downarrow \alpha & & \downarrow \hat{\alpha} & & \\ 1 & \longrightarrow & F & \longrightarrow & H & \xrightarrow{q} & H/F & \longrightarrow & 1, \end{array}$$

where  $\hat{\alpha}$  is the homomorphism induced by  $\alpha$  that makes the diagram commute, which is surjective because  $\alpha$  is, and is thus an isomorphism since  $\mathbb{Z}_2 * \mathbb{Z}_2$  is Hopfian. The 5-Lemma implies the result.

Finally, we prove part (c). Let  $\psi: H_1 \rightarrow H_2$  be an isomorphism between  $H_1$  and  $H_2$ . Since  $x$  is the unique element of  $G$  of order 2, then  $x \in H_1$  if and only if  $x \in H_2$ , and since  $\text{Ker}(p) = \langle x \rangle$ , we have  $\text{Ker}(p|_{H_1}) = \text{Ker}(p|_{H_2})$ . We thus have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \text{Ker}(p|_{H_1}) & \longrightarrow & H_1 & \xrightarrow{p} & p(H_1) & \longrightarrow & 1 \\ & & \parallel & & \downarrow \psi \cong & & \downarrow \hat{\psi} & & \\ 1 & \longrightarrow & \text{Ker}(p|_{H_2}) & \longrightarrow & H_2 & \xrightarrow{p} & p(H_2) & \longrightarrow & 1, \end{array}$$

where  $\hat{\psi}: p(H_1) \rightarrow p(H_2)$  is the surjective homomorphism induced by  $\psi$ . The 5-Lemma then implies that  $\hat{\psi}$  is an isomorphism.  $\square$

We thus obtain directly Proposition 12:

*Proof of Proposition 12.* Taking  $G = B_n(\mathbb{S}^2)$ ,  $G' = \mathcal{MCG}(\mathbb{S}^2, n)$  and  $\varphi$  as given in equation (11), and applying Proposition 26 yields the result.  $\square$

We finish this section with the following result that will be applied in Section II.8 to study the isomorphism classes of the elements of  $\mathbb{V}_2(n)$ . For  $k = 1, 2$ , let  $G_k, F$  be finite groups such that  $F$  is abstractly isomorphic to a subgroup of  $G_k$  of index 2, and let  $i_k, j_k: F \rightarrow G_k$  be pairs of embeddings. We can then form two amalgamated products,  $G_1 *_F G_2$  (with respect to the embeddings  $i_1, i_2$ ) and  $G_1 *_F' G_2$  (with respect to the embeddings  $j_1, j_2$ ). Suppose that for  $k = 1, 2$ , there exist automorphisms  $\theta_k: G_k \rightarrow G_k$  satisfying  $\theta_k \circ i_k = j_k$ .

**PROPOSITION 27.** *Under the above hypotheses, the two amalgamated products  $G_1 *_F G_2$  and  $G_1 *_F' G_2$  are isomorphic.*

*Proof.* The hypotheses imply the existence of the following commutative diagram:

$$\begin{array}{ccccc}
 & & G_1 & \xrightarrow{\theta_1} & G_1 & & & & \\
 & & \nearrow & & \searrow & & & & \\
 & i_1 & & & & & & & \\
 F & & & & & & & & \\
 & & \searrow & & \nearrow & & & & \\
 & & G_1 *_{F} G_2 & \xrightarrow{\quad} & G_1 *'_{F} G_2, & & & & \\
 & i_2 & & & & & & & \\
 & & \nearrow & & \searrow & & & & \\
 & & G_2 & \xrightarrow{\theta_2} & G_2 & & & & 
 \end{array}$$

where for  $l = 1, 2$ , the homomorphisms from  $G_l$  to  $G_1 *_{F} G_2$  and  $G_1 *'_{F} G_2$  are inclusions. By the universal property of amalgamated products, there exists a unique (surjective) homomorphism  $G_1 *_{F} G_2 \rightarrow G_1 *'_{F} G_2$ . We obtain the inverse of this homomorphism in a similar manner, by replacing  $i_1, i_2, \theta_1$  and  $\theta_2$  by  $j_1, j_2, \theta_1^{-1}$  and  $\theta_2^{-1}$  respectively and by exchanging the rôles of  $G_1 *_{F} G_2$  and  $G_1 *'_{F} G_2$ .  $\square$

## 2 Centralisers and normalisers of some maximal finite subgroups of $B_n(\mathbb{S}^2)$

Theorem 1 asserts that up to conjugacy, the maximal finite order cyclic subgroups of  $B_n(\mathbb{S}^2)$  are of the form  $\langle \alpha_i \rangle$  for  $i \in \{0, 1, 2\}$ . On the other hand, [GG5, Theorem 1.3 and Proposition 1.5(1)] implies that up to conjugacy, the maximal dicyclic subgroups of  $B_n(\mathbb{S}^2)$  are the standard dicyclic subgroups of Remark 3(b). In this section, we determine the centralisers and normalisers of these subgroups. In the cyclic case, our results mirror those for finite order elements of  $\mathcal{MCG}(\mathbb{S}^2, n)$ , and shall be used to construct the possible actions of  $\mathbb{Z}$  on cyclic and dicyclic subgroups of  $B_n(\mathbb{S}^2)$ .

We first prove the following proposition which states that an infinite subgroup of  $B_n(\mathbb{S}^2)$  cannot be formed solely of elements of finite order.

**PROPOSITION 28.** *Any infinite subgroup of  $B_n(\mathbb{S}^2)$  contains an element of infinite order. In particular, any subgroup of  $B_n(\mathbb{S}^2)$  consisting entirely of elements of finite order is itself finite.*

*Proof.* Let  $H$  be an infinite subgroup of  $B_n(\mathbb{S}^2)$ . Consider the following restriction of the short exact sequence (5):

$$1 \longrightarrow P_n(\mathbb{S}^2) \cap H \longrightarrow H \xrightarrow{\pi|_H} \pi(H) \longrightarrow 1,$$

where  $\pi(H)$  is a subgroup of  $S_n$ . If  $P_n(\mathbb{S}^2) \cap H$  is finite then it follows that  $H$  is finite, which contradicts the hypothesis. So  $P_n(\mathbb{S}^2) \cap H$  is infinite, but since the torsion of  $P_n(\mathbb{S}^2)$  is precisely  $\{e, \Delta_n^2\}$ ,  $H$  must contain an element of infinite order.  $\square$

The following lemma will play a fundamental rôle in the rest of the paper.

LEMMA 29. Let  $i \in \{0, 1, 2\}$ . Then:

$$\alpha_i^l \sigma_j \alpha_i^{-l} = \sigma_{j+l} \quad \text{for all } j, l \in \mathbb{N} \text{ satisfying } j+l \leq n-i-1, \quad (23)$$

$$\sigma_1 = \alpha_i^2 \sigma_{n-i-1} \alpha_i^{-2}. \quad (24)$$

Further, if  $0 \leq q \leq n$ , we have:

$$\alpha_0^q = (\sigma_1 \cdots \sigma_{q-1})^q \cdot \prod_{k=1}^q (\sigma_{q-k+1} \cdots \sigma_{n-k}). \quad (25)$$

REMARKS 30.

(a) An alternative formulation of equations (23) and (24) is that conjugation by  $\alpha_i$  permutes the  $n-i$  elements

$$\sigma_1, \dots, \sigma_{n-i-1}, \alpha_i \sigma_{n-i-1} \alpha_i^{-1}$$

cyclically.

(b) If  $0 \leq q \leq n$  then using equation (25),  $\alpha_0^q$  may be interpreted geometrically as a full twist on the first  $q$  strings, followed by the passage of these  $q$  strings over the remaining  $n-q$  strings (see Figure 1 for an example). If further  $q$  divides  $n$  then  $\alpha_0^q$  admits a block structure (see also Remark 39(b)).

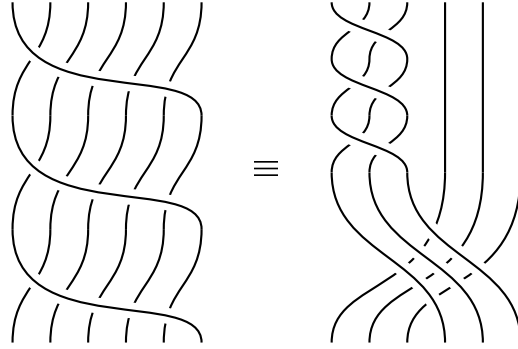


Figure 1: The braid  $\alpha_0^3$  in  $B_6(\mathbb{S}^2)$ , first in its usual form, and then in the form  $(\sigma_1 \sigma_2)^3 (\sigma_3 \sigma_4 \sigma_5) (\sigma_2 \sigma_3 \sigma_4) (\sigma_1 \sigma_2 \sigma_3)$  of equation (25).

*Proof of Lemma 29.* Let  $i \in \{0, 1, 2\}$ . We start by establishing equations (23) and (24). First note that  $\alpha_1 = \alpha_0 \sigma_{n-1}$  and  $\alpha_2 = \alpha_0 \sigma_{n-1}^{-1} \sigma_{n-2}$ , so if  $1 \leq j \leq n-i-2$ ,  $\alpha_i \sigma_j \alpha_i^{-1} = \alpha_0 \sigma_j \alpha_0^{-1} = \sigma_{j+1}$  using standard properties of  $\alpha_0$ . If further  $l \in \mathbb{N}$  and  $j+l \leq n-i-1$ ,  $\alpha_i^l \sigma_j \alpha_i^{-l} = \sigma_{j+l}$ , which proves equation (23). Since  $n-i-2 \geq 0$ , we obtain

$$\sigma_1 = \alpha_i^{n-i} \sigma_1 \alpha_i^{-(n-i)} = \alpha_i^2 \alpha_i^{n-i-2} \sigma_1 \alpha_i^{-(n-i-2)} \alpha_i^{-2} = \alpha_i^2 \sigma_{n-i-1} \alpha_i^{-2},$$

using equations (8) and (23), which proves equation (24). We now prove equation (25). Let us prove by induction that for all  $m \in \{0, \dots, q\}$ ,

$$\alpha_0^q = (\sigma_1 \cdots \sigma_{q-1})^m \alpha_0^{q-m} \cdot \prod_{k=1}^m (\sigma_{m-k+1} \cdots \sigma_{n-q+m-k}). \quad (26)$$

Clearly the equality holds if  $m = 0$ . So suppose that it is true for some  $m \in \{0, \dots, q-1\}$ . Then  $q - (m+1) \geq 0$ , and:

$$\begin{aligned}
\alpha_0^q &= (\sigma_1 \cdots \sigma_{q-1})^m \alpha_0^{q-m} \cdot \prod_{k=1}^m (\sigma_{m-k+1} \cdots \sigma_{n-q+m-k}) \\
&= (\sigma_1 \cdots \sigma_{q-1})^{m+1} \sigma_q \cdots \sigma_{n-1} \alpha_0^{q-(m+1)} \cdot \prod_{k=1}^m (\sigma_{m-k+1} \cdots \sigma_{n-q+m-k}) \\
&= (\sigma_1 \cdots \sigma_{q-1})^{m+1} \alpha_0^{q-(m+1)} \sigma_{m+1} \cdots \sigma_{n-q+m} \cdot \prod_{k=2}^{m+1} (\sigma_{m-k+2} \cdots \sigma_{n-q+m-k+1}) \\
&= (\sigma_1 \cdots \sigma_{q-1})^{m+1} \alpha_0^{q-(m+1)} \cdot \prod_{k=1}^{m+1} (\sigma_{(m+1)-k+1} \cdots \sigma_{n-q+(m+1)-k})
\end{aligned}$$

using equation (23), which gives equation (26). Taking  $m = q$  in that equation yields equation (25).  $\square$

As well as being of interest in its own right, the following result will prove to be useful when we come to discussing the possible Type I subgroups whose finite factor is cyclic. If  $H$  is a subgroup of a group  $G$  then we denote the centraliser (resp. normaliser) of  $H$  in  $G$  by  $Z_G(H)$  (resp.  $N_G(H)$ ).

**PROPOSITION 31.** *Let  $n \geq 4$ , and let  $i \in \{0, 1, 2\}$ . Then  $Z_{B_n(\mathbb{S}^2)}(\langle \alpha_i \rangle) = \langle \alpha_i \rangle$ .*

In order to prove Proposition 31, we first state a result due to L. Hodgkin concerning the centralisers of finite order elements in  $\mathcal{MCG}(\mathbb{S}^2, n)$ .

**PROPOSITION 32 ([Ho]).** *Let  $n \geq 3$ , let  $\gamma \in \mathcal{MCG}(\mathbb{S}^2, n)$  be an element of finite order  $r \geq 2$ , and let  $f$  be a rotation of  $\mathbb{S}^2$  by angle  $2\pi m/r$  about the axis passing through the poles which represents  $\gamma$ , where  $\gcd(m, r) = 1$ . Let  $\Lambda$  be the subgroup of the mapping class group of the quotient space  $\mathbb{S}^2/\langle f \rangle$  whose set of marked points is the union of the image of the  $n$  marked points under the quotient map  $\mathbb{S}^2 \rightarrow \mathbb{S}^2/\langle f \rangle$  with the two poles of  $\mathbb{S}^2/\langle f \rangle$ , and whose elements fix these two poles if  $r \neq 2$  or  $r$  divides  $n-1$ , and leaves the set of poles invariant if  $r = 2$  and  $r$  does not divide  $n-1$ . Then there is an exact sequence*

$$1 \longrightarrow \mathbb{Z}_r \longrightarrow Z_{\mathcal{MCG}(\mathbb{S}^2, n)}(\langle \gamma \rangle) \longrightarrow \Lambda \longrightarrow 1. \quad (27)$$

**REMARK 33.** Hodgkin's proof of the result is for elements of prime power order [Ho, Proposition 2.5], but one may check that it holds for any finite order element.

We now come to the proof of Proposition 31.

*Proof of Proposition 31.* Let  $z$  belong to  $Z_{B_n(\mathbb{S}^2)}(\langle \alpha_i \rangle)$ . We start by showing that either  $Z_{\mathcal{MCG}(\mathbb{S}^2, n)}(\langle a_i \rangle) = \langle a_i \rangle$ , or in the case  $n = 4, i = 2$ , the possibility that  $Z_{\mathcal{MCG}(\mathbb{S}^2, 4)}(\langle a_2 \rangle) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$  is also allowed. Consider the short exact sequence (11). Take  $m = 1$  and  $r = n - i \geq 2$  in the statement of Proposition 32. Up to conjugacy, we may suppose that  $\varphi(\alpha_i) = a_i$ , where we denote the mapping class of the rotation  $f$  of that proposition by

$a_i$ . Then  $\mathbb{S}^2/\langle f \rangle$  may be regarded as a sphere with three marked points, two of which are the poles, and the other marked point corresponds to the single orbit of  $r$  marked points in  $\mathbb{S}^2$ . Suppose first that  $r \neq 2$  or  $i = 1$  (in the latter case,  $r$  clearly divides  $n - 1$ ). Then  $\Lambda$  is the subgroup of the mapping class group of  $\mathbb{S}^2/\langle f \rangle$  whose elements fix each of the poles, as well as the remaining marked point. Hence  $\Lambda$  is the pure mapping class group of  $\mathbb{S}^2/\langle f \rangle$ , which is trivial. It follows from equation (27) that  $Z_{\mathcal{MCG}(\mathbb{S}^2, n)}(a_i)$  is cyclic of order  $r$ , and so is equal to  $\langle a_i \rangle$ . Now suppose that  $r = 2$  and  $i \neq 1$ . Since  $n \geq 3$ , we have that  $n = 4$  and  $i = 2$ . In this case,  $\Lambda$  is the subgroup of the mapping class group of  $\mathbb{S}^2/\langle f \rangle$  whose elements leave the set of poles invariant (and fix the remaining marked point), and so is isomorphic to  $\mathbb{Z}_2$ . By equation (27),  $Z_{\mathcal{MCG}(\mathbb{S}^2, n)}(\langle a_2 \rangle)$  is an extension of  $\mathbb{Z}_2$  by  $\mathbb{Z}_2$ , thus  $Z_{\mathcal{MCG}(\mathbb{S}^2, 4)}(\langle a_2 \rangle)$  is isomorphic to either  $\mathbb{Z}_4$  or  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ . In the former case, we obtain  $Z_{\mathcal{MCG}(\mathbb{S}^2, 4)}(\langle a_2 \rangle) = \langle a_2 \rangle$ .

We first consider the case where  $Z_{\mathcal{MCG}(\mathbb{S}^2, n)}(\langle a_i \rangle) = \langle a_i \rangle$  (so either  $r \neq 2$  or  $i = 1$ , or  $n = 4$ ,  $i = 2$  and  $Z_{\mathcal{MCG}(\mathbb{S}^2, 4)}(\langle a_2 \rangle) = \langle a_2 \rangle$ ). Now  $z' = \varphi(z)$  belongs to the centraliser of  $a_i$ , and so may be written in the form  $z' = a_i^t$ , where  $t \in \{0, \dots, n - i - 1\}$ . By equations (8) and (11),  $z = \alpha_i^t \Delta_n^{2\varepsilon} = \alpha_i^{t+\varepsilon(n-i)}$ , where  $\varepsilon \in \{0, 1\}$ , and hence  $z \in \langle \alpha_i \rangle$ . Since  $Z_{B_n(\mathbb{S}^2)}(\langle \alpha_i \rangle)$  clearly contains  $\langle \alpha_i \rangle$ , it follows that  $Z_{B_n(\mathbb{S}^2)}(\langle \alpha_i \rangle) = \langle \alpha_i \rangle$  as required.

Finally, suppose that  $n = 4$ ,  $i = 2$  and  $Z_{\mathcal{MCG}(\mathbb{S}^2, 4)}(\langle a_2 \rangle) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ . Since

$$\varphi(Z_{B_4(\mathbb{S}^2)}(\langle \alpha_2 \rangle)) \subset Z_{\mathcal{MCG}(\mathbb{S}^2, 4)}(\langle a_2 \rangle),$$

we have

$$\langle \alpha_2 \rangle \subset Z_{B_4(\mathbb{S}^2)}(\langle \alpha_2 \rangle) \subset \varphi^{-1}(Z_{\mathcal{MCG}(\mathbb{S}^2, 4)}(\langle a_2 \rangle)) \cong \mathcal{Q}_8$$

using equation (11). If  $Z_{B_4(\mathbb{S}^2)}(\langle \alpha_2 \rangle)$  is isomorphic to  $\mathcal{Q}_8$  then there exists at least one element of  $Z_{B_4(\mathbb{S}^2)}(\alpha_2)$  that does not commute with  $\alpha_2$ , which is a contradiction. So  $\langle \alpha_2 \rangle = Z_{B_4(\mathbb{S}^2)}(\langle \alpha_2 \rangle)$ , and this completes the proof of the proposition.  $\square$

REMARK 34. As we shall see in Section II.4.1, in general the binary polyhedral groups  $T^*$ ,  $O^*$  and  $I^*$  have infinite centraliser in  $B_n(\mathbb{S}^2)$ .

We now proceed with the proof of Proposition 8.

*Proof of Proposition 8.* We first deal with the case  $n = 3$  for both parts (a) and (b). We have  $\alpha_0 = \sigma_1\sigma_2$ ,  $\alpha_1 = \sigma_1\sigma_2^2$  and  $\alpha_2 = \sigma_1^2$ , which are of order 6, 4 and 2 respectively. In particular,  $\alpha_2 = \Delta_3^2$ , and so the centraliser of  $\alpha_2$  and the normaliser of  $\langle \alpha_2 \rangle$  are both equal to  $B_3(\mathbb{S}^2)$ . As for  $\alpha_0$  and  $\alpha_1$ , they may be taken to be the generators  $x$  and  $y$  of  $B_3(\mathbb{S}^2) \cong \text{Dic}_{12}$  appearing in equation (9). Indeed,  $\alpha_0^3 = \alpha_1^2 = \Delta_3^2$  by equation (8),  $\langle \alpha_0, \alpha_1 \rangle = B_3(\mathbb{S}^2)$  since  $\langle \alpha_0, \alpha_1 \rangle$  cannot be of order less than 12, and

$$\alpha_1\alpha_0\alpha_1^{-1} = \sigma_1\sigma_2^2\sigma_1\sigma_2\sigma_2^{-2}\sigma_1^{-1} = \sigma_1\sigma_2^2\sigma_1\sigma_2^{-1}\sigma_1^{-1} = \sigma_2^{-1}\sigma_1^{-1} \quad \text{by equation (3)}.$$

It follows from the presentation of equation (9) that  $\langle \alpha_0, \alpha_1 \rangle \cong \text{Dic}_{12}$ , and the rest of the statement follows using this presentation in the case  $m = 3$ .

We suppose henceforth that  $n \geq 4$ .

(a) This is the statement of Proposition 31.

(b) Let  $i \in \{0, 1, 2\}$ , let  $N = N_{B_n(\mathbb{S}^2)}(\langle \alpha_i \rangle)$ , and let  $x \in N$ . Then some power of  $x$  belongs to the centraliser of  $\langle \alpha_i \rangle$  in  $B_n(\mathbb{S}^2)$ , which is equal to  $\langle \alpha_i \rangle$  by Proposition 31. So  $x$  is of finite order, and thus  $N$  is finite by Proposition 28. Let

$$G = \begin{cases} \langle \alpha_0, \Delta_n \rangle \cong \text{Dic}_{4n} & \text{if } i = 0 \\ \langle \alpha_2, \alpha_0^{-1} \Delta_n \alpha_0 \rangle \cong \text{Dic}_{4(n-2)} & \text{if } i = 2 \\ \langle \alpha_1 \rangle \cong \mathbb{Z}_{2(n-1)} & \text{if } i = 1. \end{cases}$$

If  $i \in \{0, 2\}$  then  $G$  is conjugate to the standard copy of  $\text{Dic}_{4(n-i)}$ . Since  $[G : \langle \alpha_i \rangle] \in \{1, 2\}$ ,  $\langle \alpha_i \rangle$  is normal in  $G$ , and so  $N \supset G$ . If  $G = N$  then we are done. So suppose that  $G \neq N$ , and let  $M$  be a maximal finite subgroup containing  $N$ . Hence  $G$  is not maximal, and by Theorem 2, we are in one of the following cases:

(i)  $n = 4$  and  $i \in \{1, 2\}$ . If  $i = 1$  (resp.  $i = 2$ ) then  $G \cong \mathbb{Z}_6$  (resp.  $G \cong \mathcal{Q}_8$ ). Since  $G \subsetneq N \subset M$ , it follows from Theorem 2 and the subgroup structure of the finite maximal subgroups of  $B_4(\mathbb{S}^2)$  (see Proposition 85) that  $N = M$  and  $N \cong T^*$  (resp.  $N \cong T^*$  or  $N \cong \mathcal{Q}_{16}$ ). Now  $\langle \alpha_i \rangle$  is isomorphic to  $\mathbb{Z}_6$  (resp. to  $\mathbb{Z}_4$ ), but these subgroups are not normal in  $T^*$ , so  $N \not\cong T^*$ . Hence  $i = 2$  and  $N \cong \mathcal{Q}_{16}$ . Take  $N$  to have the presentation (9) with  $m = 4$ . Since  $\langle \alpha_2 \rangle$  is isomorphic to  $\mathbb{Z}_4$  and is normal in  $N$ , we have that  $\langle \alpha_2 \rangle = \langle x^2 \rangle$ . Hence  $\langle \alpha_2 \rangle \subsetneq \langle x \rangle$ , but this contradicts the fact that  $\langle \alpha_2 \rangle$  is maximal cyclic in  $B_4(\mathbb{S}^2)$  by Theorem 1.

(ii)  $n = 6$  and  $i = 2$ . Then  $\langle \alpha_2 \rangle \cong \mathbb{Z}_8$  and  $G \cong \mathcal{Q}_{16}$ . Since the maximal finite subgroups of  $B_6(\mathbb{S}^2)$  are isomorphic to  $\text{Dic}_{24}$ ,  $\mathbb{Z}_{10}$  or  $O^*$  by Theorem 2, and  $G \subsetneq N \subset M$ , it follows that  $N = M \cong O^*$ . However, this contradicts the fact that the copies of  $\mathbb{Z}_8$  in  $O^*$  are not normal by Proposition 85. This completes the proof of part (b).

(c) Let  $i \in \{0, 2\}$ , let  $G$  denote the standard copy of  $\text{Dic}_{4(n-i)}$ , and let  $N$  be the normaliser  $N_{B_n(\mathbb{S}^2)}(G)$  of  $G$  in  $B_n(\mathbb{S}^2)$ . If  $x \in N$  then some power of  $x$  centralises  $G$ , and so centralises its cyclic subgroup  $\langle \alpha_0 \alpha_i \alpha_0^{-1} \rangle$  of order  $2(n-i)$ . It follows from part (b) that  $N$  is finite. Since  $N \supset G$ , if  $G$  is maximal finite then  $G = N$ , and we are done. So suppose that  $G$  is not maximal, and let  $M$  be a finite maximal subgroup of  $B_n(\mathbb{S}^2)$  satisfying  $G \subsetneq N \subset M$ . Theorem 2 implies that  $n \in \{4, 6\}$  and  $i = 2$ .

Suppose first that  $n = 4$ , so  $G \cong \mathcal{Q}_8$ . Then  $M$  is isomorphic to  $T^*$  or  $\mathcal{Q}_{16}$  by Theorem 2, and  $N = M$  by Proposition 85. Suppose first that  $N \cong T^* \cong \mathcal{Q}_8 \rtimes \mathbb{Z}_3$ . Then  $G$  is the unique subgroup of  $M$  isomorphic to  $\mathcal{Q}_8$ . The form of the action of  $\mathbb{Z}_3$  on  $\mathcal{Q}_8$  implies that the elements of  $G$  of order 4 are pairwise conjugate. However, this is impossible since the permutations of the order 4 elements  $\alpha_2$  and  $\alpha_0^{-1} \Delta_4 \alpha_0$  of  $G$  have distinct cycle types. Thus  $N \cong \mathcal{Q}_{16}$ . By [GG5, Proposition 1.5 and Theorem 1.6], the standard copy  $\langle \alpha_0, \Delta_4 \rangle$  of  $\mathcal{Q}_{16}$  in  $B_4(\mathbb{S}^2)$ , which is a representative of the unique conjugacy class of subgroups isomorphic to  $\mathcal{Q}_{16}$ , contains representatives of the two conjugacy classes of  $\mathcal{Q}_8$ , from which it follows that there exists a subgroup  $K$  of  $B_4(\mathbb{S}^2)$  conjugate to  $\langle \alpha_0, \Delta_4 \rangle$  and containing  $G$ . Since  $[K : G] = 2$ ,  $G$  is normal in  $K$ , so  $K \subset N$ . The maximality of  $\mathcal{Q}_{16}$  as a finite subgroup of  $B_4(\mathbb{S}^2)$  implies that  $K = N \cong \mathcal{Q}_{16}$ . Furthermore, we claim that  $K = \alpha_0^{-1} \sigma_1^{-1} \langle \alpha_0, \Delta_4 \rangle \sigma_1 \alpha_0$ . Indeed,  $K$  has a subgroup isomorphic to  $\mathcal{Q}_8$  that is generated



by the following two elements:

$$\begin{aligned}\alpha_0^{-1}\sigma_1^{-1}\alpha_0^2\sigma_1\alpha_0 &= \alpha_0^{-1}\alpha_0^2\sigma_3^{-1}\sigma_1 = \alpha_0^{-1}\Delta_4\alpha_0 \\ \alpha_0^{-1}\sigma_1^{-1}\alpha_0\Delta_4\sigma_1\alpha_0 &= \alpha_0^{-1}\sigma_1^{-1}\alpha_0\sigma_3\Delta_4\alpha_0 = \alpha_0^{-1}\sigma_2\sigma_3^2\alpha_0 \cdot \alpha_0^{-1}\Delta_4\alpha_0 = \alpha_2 \cdot \alpha_0^{-1}\Delta_4\alpha_0,\end{aligned}$$

which are also generators of  $G$ . We have used relations (6) and (7), as well as Lemma 29 to obtain these equalities. This proves the claim, and completes the proof in the case  $n = 4$ .

Finally, suppose that  $n = 6$ , so  $G \cong \mathcal{Q}_{16}$ . If  $G \subsetneq N$  then as in part (b)(ii) it follows that  $N \cong O^*$ . But by Proposition 85, the copies of  $\mathcal{Q}_{16}$  in  $O^*$  are not normal, which yields a contradiction. We thus conclude that  $G = N$  as required.  $\square$

### 3 Reduction of isomorphism classes of $F \rtimes_{\theta} \mathbb{Z}$ via $\text{Out}(F)$

If  $F$  is a group, let  $\text{Inn}(F)$  denote the normal subgroup of inner automorphisms of the group  $\text{Aut}(F)$  of automorphisms of  $F$ , and recall that  $\text{Inn}(F) \cong F/Z(F)$ , where  $Z(F)$  denotes the centre of  $F$ . By Theorem 17, any Type I group is of the form  $F \rtimes_{\theta} \mathbb{Z}$  for some action  $\theta \in \text{Hom}(\mathbb{Z}, \text{Aut}(F))$ , where  $F$  is finite. The following proposition asserts that the isomorphism class of such a group depends only on the homomorphism  $\bar{\theta}: \mathbb{Z} \rightarrow \text{Out}(F)$  which is the composition of  $\theta$  with the canonical projection  $\text{Aut}(F) \rightarrow \text{Out}(F)$ .

PROPOSITION 35 ([AB, Chapter 1.2, Proposition 12]). *Let  $F$  be a finite group, and let*

$$\theta, \theta': \mathbb{Z} \rightarrow \text{Aut}(F)$$

*be homomorphisms such that  $\bar{\theta} = \bar{\theta}'$ . Then the groups  $F \rtimes_{\theta} \mathbb{Z}$  and  $F \rtimes_{\theta'} \mathbb{Z}$  are isomorphic.*

In order to help us determine (up to isomorphism) the possible Type I groups arising as subgroups of  $B_n(\mathbb{S}^2)$ , it will be appropriate at this juncture to describe  $\text{Out}(F)$ , where  $F$  is one of the finite subgroups  $\mathcal{Q}_8, T^*, O^*$  or  $I^*$  of  $B_n(\mathbb{S}^2)$ . By choosing a transversal in  $\text{Aut}(F)$  of  $\text{Out}(F)$ , from Proposition 35 we may obtain all possible isomorphism classes of the groups  $F \rtimes_{\theta} \mathbb{Z}$  (we shall always choose the identity as the representative of the trivial element of  $\text{Out}(F)$ ). It then follows directly from Proposition 22(b) that any Type I subgroup of  $B_n(\mathbb{S}^2)$  involving  $F$  is isomorphic to one of the groups belonging to this family. Cohomological considerations will then be applied in Section 5 to rule out those subgroups involving  $O^*$  and  $I^*$  for all but the trivial action. Note that we could carry out the study of  $\text{Out}(F)$  for the other finite subgroups of  $B_n(\mathbb{S}^2)$ , but in Section 4 we will obtain stronger conditions on the possible actions of  $\mathbb{Z}$  on  $F$  using Proposition 9.

(1)  $F = \mathcal{Q}_8$ : we have  $\text{Aut}(\mathcal{Q}_8) \cong S_4$  [AM, p. 149],  $Z(\mathcal{Q}_8) \cong \mathbb{Z}_2$  and  $\text{Inn}(\mathcal{Q}_8) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ . Therefore  $\text{Out}(\mathcal{Q}_8) \cong S_4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \cong S_3$ .

(2)  $F = T^*$ . Writing  $\mathcal{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ , it is well known that  $T^*$  is isomorphic to  $\mathcal{Q}_8 \rtimes \mathbb{Z}_3$  [AM, CM], where the action of  $\mathbb{Z}_3$  permutes cyclically the elements  $i, j, k$  of  $\mathcal{Q}_8$ . From [GoG3, Theorem 3.3], we have  $\text{Aut}(T^*) \cong S_4$ . Now  $Z(T^*) \cong \mathbb{Z}_2$ , so  $\text{Inn}(T^*) \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_3 \cong A_4$ , where the action permutes cyclically the three non-trivial elements

of  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ . Therefore  $\text{Out}(T^*) \cong \mathbb{Z}_2$ . Let  $T^*$  be given by the presentation (13). The non-trivial element of  $\text{Out}(T^*)$  is represented by the automorphism  $\omega(1)$  of  $T^*$  defined by equation (14). Indeed, if  $S \in \langle P, Q \rangle$  then any automorphism of  $T^*$  which sends  $X$  to  $SX^{-1}$  is not an inner automorphism. This follows since  $PXP^{-1} = PQ^{-1}X$ , and  $QXQ^{-1} = P^{-1}X$ , so any conjugate of  $X$  in  $T^*$  belongs to the coset  $\langle P, Q \rangle \cdot X$ , but on the other hand,  $SX^{-1}$  belongs to the coset  $\langle P, Q \rangle \cdot X^{-1}$  which is distinct from  $\langle P, Q \rangle \cdot X$ . As we shall see presently in case (3), the automorphism given by (14) is the restriction to  $T^*$  of conjugation by an element  $R \in O^* \setminus T^*$ .

(3)  $F = O^*$ : from [Wo, p. 198],  $O^*$  is generated by  $X, P, Q, R$  which are subject to the following relations:

$$\begin{cases} X^3 = 1, P^2 = Q^2 = R^2, PQP^{-1} = Q^{-1}, \\ XPX^{-1} = Q, XQX^{-1} = PQ, \\ RXR^{-1} = X^{-1}, RPR^{-1} = QP, RQR^{-1} = Q^{-1}. \end{cases} \quad (28)$$

Comparing the presentations given by equations (13) and (28), we see that  $O^*$  admits  $\langle P, Q, X \rangle \cong T^*$  as its index 2 subgroup. So we have the extensions [AM, page 150]:

$$1 \longrightarrow T^* \longrightarrow O^* \longrightarrow \mathbb{Z}_2 \longrightarrow 1,$$

and [GoG3, Proposition 4.1]:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Aut}(O^*) \longrightarrow \text{Aut}(T^*) \longrightarrow 1. \quad (29)$$

Now  $Z(O^*) \cong \mathbb{Z}_2$ , and so  $\text{Inn}(O^*) \cong O^*/Z(O^*) \cong S_4$ . From equation (29) and part (2) above,  $|\text{Aut}(O^*)| = 48$ , and thus  $\text{Out}(O^*) \cong \mathbb{Z}_2$ . The non-trivial element of  $\text{Out}(O^*)$  is represented by the following element of  $\text{Aut}(O^*)$ :

$$\begin{cases} P \mapsto P \\ Q \mapsto Q \\ X \mapsto X \\ R \mapsto R^{-1}. \end{cases}$$

To see this, suppose on the contrary that this automorphism arises as conjugation by some element  $S \in O^*$ . Since  $[O^* : T^*] = 2$  and  $R \notin T^*$ , there exists  $t \in T^*$  such that  $S = t$  or  $S = tR$ . If  $S = t$  then  $S$  commutes with all of the generators of  $T^*$ , hence belongs to the centre  $\langle P^2 \rangle$  of  $T^*$ . But  $Z(O^*) = \langle P^2 \rangle$ , so conjugation by  $S$  cannot send  $R$  to  $R^{-1}$  since  $R$  is of order 4. Thus  $S = tR$ , and so  $X = SXS^{-1} = tX^{-1}t^{-1}$ , but this implies that  $X$  and  $X^{-1}$  belong to the same conjugacy class in  $T^*$ , and as we saw in case (2) above, this is impossible. We conclude that the given automorphism is not an inner automorphism, so must represent the non-trivial element of  $\text{Out}(O^*)$ .

(4)  $F = I^*$ : we know that  $I^* \cong \text{SL}_2(\mathbb{F}_5)$  [AM, page 151],  $Z(I^*) \cong \mathbb{Z}_2$ ,  $\text{Inn}(I^*) \cong I^*/Z(I^*) \cong A_5$ ,  $\text{Aut}(I^*) \cong S_5$  [GoG3, see Theorem 2.1], and  $\text{Out}(I^*) \cong \mathbb{Z}_2$  (see [AM, page 151] or [GoG2, page 207]). The non-trivial element of  $\text{Out}(I^*)$  is represented by the automorphism of  $I^*$  which in terms of  $\text{SL}_2(\mathbb{F}_5)$  is conjugation by the matrix  $\begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix}$ , where  $w$  is a non square of  $\mathbb{F}_5$  [AM, page 152].

Let us come back to case (1) where  $F = \mathcal{Q}_8$ . Since  $\text{Out}(\mathcal{Q}_8) \cong S_3$ , *a priori*, we need to decide which of the six groups of the form  $\mathcal{Q}_8 \rtimes \mathbb{Z}$  are realised. We may however make a minor simplification as follows. Recall from Definition 4(e) that  $\alpha, \beta \in \text{Hom}(\mathbb{Z}, \text{Aut}(\mathcal{Q}_8))$  are such that  $\alpha(1)$  is the automorphism of  $\mathcal{Q}_8$  of order 3 that permutes  $i, j$  and  $k$  cyclically, and  $\beta(1)$  is the automorphism that sends  $i$  to  $k$  and  $j$  to  $j^{-1}$ . The following lemma shows that we may reduce further the number of isomorphism classes of  $\mathcal{Q}_8 \rtimes \mathbb{Z}$  from the six representatives of the elements of  $\text{Out}(\mathcal{Q}_8)$  to just three.

LEMMA 36. *Let  $H$  be of the form  $\mathcal{Q}_8 \rtimes_{\theta} \mathbb{Z}$ , where  $\theta \in \text{Hom}(\mathbb{Z}, \text{Aut}(\mathcal{Q}_8))$ . Then  $H$  is isomorphic to one of  $\mathcal{Q}_8 \rtimes \mathbb{Z}$ ,  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  and  $\mathcal{Q}_8 \rtimes_{\beta} \mathbb{Z}$ .*

*Proof.* Since  $\text{Out}(\mathcal{Q}_8) \cong S_3$ , there exists  $\gamma \in \{\text{Id}, \alpha, \alpha^2, \beta, \alpha \circ \beta, \alpha^2 \circ \beta\}$  such that  $H$  is isomorphic to  $\mathcal{Q}_8 \rtimes_{\gamma} \mathbb{Z}$  by Proposition 35. We claim that:

- (a)  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  and  $\mathcal{Q}_8 \rtimes_{\alpha^2} \mathbb{Z}$  are isomorphic.
- (b)  $\mathcal{Q}_8 \rtimes_{\beta} \mathbb{Z}$ ,  $\mathcal{Q}_8 \rtimes_{\alpha \circ \beta} \mathbb{Z}$  and  $\mathcal{Q}_8 \rtimes_{\alpha^2 \circ \beta} \mathbb{Z}$  are isomorphic.

To prove the claim, we define isomorphisms  $\varphi: \mathcal{Q}_8 \rtimes_{\theta} \mathbb{Z} \rightarrow \mathcal{Q}_8 \rtimes_{\theta'} \mathbb{Z}$ , where the actions  $\theta, \theta' \in \text{Hom}(\mathbb{Z}, \text{Aut}(\mathcal{Q}_8))$  run through the possible pairs given by (a) and (b). Let  $t$  (resp.  $t'$ ) denote the generator of the  $\mathbb{Z}$ -factor of  $\mathcal{Q}_8 \rtimes_{\theta} \mathbb{Z}$  (resp. of  $\mathcal{Q}_8 \rtimes_{\theta'} \mathbb{Z}$ ). Defining  $\varphi$  by:

- (i)  $i \mapsto i, j \mapsto k$  and  $t \mapsto kt'$  if  $\theta = \alpha$  and  $\theta' = \alpha^2$ ,
- (ii)  $i \mapsto k, k \mapsto j, j \mapsto i$  and  $t \mapsto jt'$  if  $\theta = \beta$  and  $\theta' = \alpha \circ \beta$ ,
- (iii)  $i \mapsto j, k \mapsto i, j \mapsto k$  and  $t \mapsto jt'$  if  $\theta = \beta$  and  $\theta' = \alpha^2 \circ \beta$ ,

we may check that  $\varphi$  gives rise to an isomorphism between each pair of groups. In particular, there are only three isomorphism classes of semi-direct products  $\mathcal{Q}_8 \rtimes_{\theta} \mathbb{Z}$ , namely those for which  $\theta(1) \in \{\text{Id}, \alpha(1), \beta(1)\}$ .  $\square$

REMARK 37. Since  $(\alpha(1))^3 = (\beta(1))^2 = \text{Id}_F$ , it will suffice to study the existence of semi-direct products of the form  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  and  $\mathcal{Q}_8 \rtimes_{\beta} \mathbb{Z}$ .

## 4 Reduction of isomorphism classes of $F \rtimes_{\theta} \mathbb{Z}$ via conjugacy classes

In this section, we use the relation between  $\text{MCG}(\mathbb{S}^2, n)$  and  $B_n(\mathbb{S}^2)$  given by equation (11) to prove Proposition 9. As a consequence, the only possible actions on cyclic groups that are realised as subgroups of  $B_n(\mathbb{S}^2)$  are the trivial action, and multiplication by  $-1$ . This will subsequently be used to rule out many Type I groups involving dicyclic factors.

In order to prove Proposition 9, we first state Proposition 38 whose statement, seemingly well known to the experts in the field, is related to a classical problem of Nielsen concerning the conjugacy problem in the mapping class group. The first proof we found in the literature is due to L. Hodgkin [Ho] (see also [McH] for related results).

PROPOSITION 38. Let  $n, r \geq 2$  be such that  $\mathcal{MCG}(\mathbb{S}^2, n)$  has elements of order  $r$ .

(a) Suppose that either  $r \geq 3$ , or  $r = 2$  and  $n$  is odd. Then there is a unique value of  $i \in \{0, 1, 2\}$  for which  $r$  divides  $n - i$ . Let  $f$  be a rotation of  $\mathbb{S}^2$  of angle  $2\pi m/r$ , where  $m \in \mathbb{N}$  and  $\gcd(m, r) = 1$ , and let  $\gamma \in \mathcal{MCG}(\mathbb{S}^2, n)$  denote the mapping class of  $f$ . Then any element  $\gamma' \in \mathcal{MCG}(\mathbb{S}^2, n)$  of order  $r$  is conjugate to  $\gamma$ . Further, any two distinct powers of  $\gamma$  are conjugate in  $\mathcal{MCG}(\mathbb{S}^2, n)$  if and only if the following conditions hold:

(i) they are inverse, and

(ii)  $i \in \{0, 2\}$ .

(b) If  $r = 2$  and  $n$  is even then  $r$  divides both  $n$  and  $n - 2$ , and so both the choices  $i = 0$  and  $i = 2$  are possible. In the first (resp. second) case, we obtain an element  $\gamma_0$  (resp.  $\gamma_2$ ) of order 2 that fixes none (resp. exactly two) of the  $n$  marked points of  $\mathbb{S}^2$ . Further, every element of  $\mathcal{MCG}(\mathbb{S}^2, n)$  of order 2 is conjugate to exactly one of  $\gamma_0$  or  $\gamma_2$ .

The proof of Proposition 38 may be deduced in a straightforward manner from that of [Ho, Proposition 2.1]. Before coming to the proof of Proposition 9, we first define some notation that shall also be used later in Sections II.4.1 and II.6.2. If  $X$  is an  $n$ -point subset of  $\mathbb{S}^2$ , let  $\text{Homeo}^+(\mathbb{S}^2, X)$  denote the set of orientation-preserving homeomorphisms that leave  $X$  invariant. There is a natural surjective homomorphism  $\Psi: \text{Homeo}^+(\mathbb{S}^2, X) \rightarrow \mathcal{MCG}(\mathbb{S}^2, n)$ , where  $\Psi(f) = [f]$  denotes the mapping class of the homeomorphism  $f \in \text{Homeo}^+(\mathbb{S}^2, X)$ .

*Proof of Proposition 9.* Let  $i \in \{0, 1, 2\}$ . Let  $1 \leq m, r \leq 2(n - i)$ , and suppose that  $\alpha_i^m$  and  $\alpha_i^r$  are conjugate powers of  $\alpha_i$  in  $B_n(\mathbb{S}^2)$ . Then there exists  $z \in B_n(\mathbb{S}^2)$  such that

$$z\alpha_i^m z^{-1} = \alpha_i^r. \quad (30)$$

Let  $\mu = \gcd(m, 2(n - i))$ , and set  $q = 2(n - i)/\mu$ . Then  $\alpha_i^m$  and  $\alpha_i^r$  are both of order  $q$ , and generate the same subgroup  $\langle \alpha_i^\mu \rangle$  of  $\langle \alpha_i \rangle$ . In particular, there exists  $1 \leq \tau < q$  with  $\gcd(\tau, q) = 1$  such that  $\alpha_i^{m\tau} = \alpha_i^\mu$ . Setting  $\xi = r\tau$  and raising equation (30) to the  $\tau^{\text{th}}$  power yields  $z\alpha_i^\mu z^{-1} = \alpha_i^\xi$ . Now  $\alpha_i^\mu$  and  $\alpha_i^\xi$  generate the same subgroup of  $\langle \alpha_i \rangle$ , so there exists  $1 \leq t < q$  with  $\gcd(t, q) = 1$  such that  $\alpha_i^\xi = \alpha_i^{t\mu}$ , and hence

$$z\alpha_i^\mu z^{-1} = \alpha_i^\xi = \alpha_i^{t\mu}. \quad (31)$$

We claim that it suffices to show that  $\alpha_i^\xi \in \{\alpha_i^\mu, \alpha_i^{-\mu}\}$ . Suppose for a moment that the claim holds. Since  $\gcd(\tau, q) = 1$ , there exist  $u, v \in \mathbb{Z}$  such that  $u\tau - vq = 1$ , and so

$$\alpha_i^{m\tau u} = \alpha_i^{m(1+vq)} = \alpha_i^m \cdot (\alpha_i^{mq})^v = \alpha_i^m \quad (32)$$

since  $\alpha_i^m$  is of order  $q$ . Similarly,

$$\alpha_i^{r\tau u} = \alpha_i^{r(1+vq)} = \alpha_i^r \cdot (\alpha_i^{rq})^v = \alpha_i^r \quad (33)$$

since  $\alpha_i^r$  is also of order  $q$ . But

$$\alpha_i^{m\tau u} = \alpha_i^{\mu u} = \alpha_i^{\pm \xi u} = \alpha_i^{\pm r\tau u}, \quad (34)$$

and putting together equations (32), (33) and (34), we obtain  $\alpha_i^m = \alpha_i^{\pm r}$ . As we shall see below, if  $i = 1$  then in fact  $\alpha_1^m = \alpha_1^r$ , which will prove the proposition in this case. We now proceed to prove the claim, separating the cases  $i = 1$  and  $i \in \{0, 2\}$ .

(i) Let  $i = 1$ . Projecting relation (31) onto the Abelianisation  $\mathbb{Z}_{2(n-1)}$  of  $B_n(\mathbb{S}^2)$ , we obtain  $n\mu \equiv n\mu t \pmod{2(n-1)}$ , in other words, there exists  $k \geq 0$  such that  $n\mu(t-1) = k \cdot 2(n-1)$ . Now  $n$  and  $n-1$  are coprime, so there exists  $l \geq 0$  such that  $\mu(t-1) = l(n-1)$  and  $2k = nl$ . But  $1 \leq t < q = 2(n-1)/\mu$ , thus  $\mu \leq \mu t < 2(n-1)$ , which implies that

$$0 \leq \mu(t-1) \leq 2(n-1) - \mu < 2(n-1),$$

and thus

$$0 \leq l(n-1) < 2(n-1).$$

It follows that  $l = 0$  or  $l = 1$ . If  $l = 1$  then  $n = 2k$ , so  $n$  is even. Further,  $t-1 = (n-1)/\mu = q/2$ , hence  $q$  is even. But  $\gcd(t, q) = 1$ , so  $t$  is odd, thus  $\mu(t-1) = n-1$  is even, and  $n$  is odd, a contradiction. We conclude that  $l = 0$ , so  $t = 1$ , and so  $\alpha_1^m = \alpha_1^{\xi}$ . As we saw above, this implies that  $\alpha_1^m = \alpha_1^r$ , which proves part (a) of the proposition.

(ii) Let  $i \in \{0, 2\}$ . Consider equation (31) and the short exact sequence (11). Let  $w = \varphi(z)$ , let  $a_i = \varphi(\alpha_i)$ , and let  $X$  be an  $n$ -point subset of  $\mathbb{S}^2$  consisting of  $n-i$  equally-spaced points on the equator, with the remaining  $i$  points distributed at the poles. Then  $wa_i^\mu w^{-1} = a_i^{\xi}$ , and we may suppose  $a_i$  to be represented by the homeomorphism  $f_i \in \text{Homeo}^+(\mathbb{S}^2, X)$  that is rigid rotation of  $\mathbb{S}^2$  of angle  $2\pi/(n-i)$ . It follows from Proposition 38 that  $a_i^\mu$  and  $a_i^{\xi}$  are either equal or are inverses, and since  $a_i$  is of order  $n-i$ ,  $\xi \equiv \pm\mu \pmod{n-i}$ , so  $\xi = \pm\mu + \delta(n-i)$ , where  $\delta \in \mathbb{Z}$ . If  $\delta$  is even then  $\alpha_i^{\xi} = \alpha_i^{\pm\mu}$  by equation (8), and as we saw above, this implies that  $\alpha_i^m = \alpha_i^{\pm r}$ , which proves part (b) of the proposition in this case. So assume that  $\delta$  is odd, in which case

$$z\alpha_i^\mu z^{-1} = \alpha_i^{\xi} = \alpha_i^{\pm\mu + \delta(n-i)} = \alpha_i^{\pm\mu} \Delta_n^2, \quad (35)$$

also using equation (8). Conjugating equation (35) by  $\alpha_0^{-i/2} \Delta_n \alpha_0^{i/2}$ , replacing  $z$  by the element  $\alpha_0^{-i/2} \Delta_n \alpha_0^{i/2} z$  and using equation (10) if necessary, we may suppose that

$$z\alpha_i^\mu z^{-1} = \alpha_i^\mu \Delta_n^2. \quad (36)$$

Notice however that since  $\Delta_n^2$  is central and of order 2, the relation

$$\alpha_i^{\xi} = \alpha_i^{\pm\mu} \Delta_n^2 \quad (37)$$

of equation (35) persists under this conjugation. Conjugating equation (36) by  $z^{-1}$  and multiplying by  $\Delta_n^2$  yields:

$$\alpha_i^\mu \Delta_n^2 = z^{-1} \alpha_i^\mu z. \quad (38)$$

The Abelianisation of equation (38) yields  $n(n-1) \equiv 0 \pmod{2(n-1)}$ , so  $n$  must be even for a solution to exist. In particular, if  $n$  is odd, there is no  $z \in B_n(\mathbb{S}^2)$  satisfying equation (38). So let  $n \geq 4$ , and suppose that equation (38) admits a solution  $z \in B_n(\mathbb{S}^2)$ .

If  $\mu \in \{n-i, 2(n-i)\}$  then  $\alpha_i^\mu \in \langle \Delta_n^2 \rangle$ , and this equation implies that  $\Delta_n^2 = \text{Id}$ , hence  $n \leq 2$ , which gives a contradiction. So  $\mu \notin \{n-i, 2(n-i)\}$ , and since  $\mu \mid 2(n-i)$ , we must have  $1 \leq \mu < n-i$ . Moreover,  $q = 2(n-i)/\mu$  cannot be odd, for if it were then  $\alpha_i^\mu \Delta_n^2$  would be of order  $2q$  because  $\alpha_i^\mu$  is of order  $q$  and  $\Delta_n^2$  is central. But this contradicts equation (38), so  $q$  is even, and hence  $\mu$  divides  $n-i$ . If  $(n-i)/\mu = 2$  then  $\mu = (n-i)/2$ , and

$$\alpha_i^\zeta = \alpha_i^{\pm \frac{(n-i)}{2} + (n-i)} = \alpha_i^{\mp (n-i)/2} = \alpha_i^{\mp \mu}$$

by equation (37), which proves the result in this case. Since  $\mu < n-i$ , we suppose henceforth that  $(n-i)/\mu \geq 3$ .

We first assume that  $i = 0$ , so  $\mu$  divides  $n$  and  $n/\mu \geq 3$ . Consider the image of equation (38) under the homomorphism  $\pi$  of equation (5). Then  $\pi(\alpha_0^\mu) = (n-\mu+1, n-2\mu+1, \dots, \mu+1, 1)(n-\mu+2, n-2\mu+2, \dots, \mu+2, 2) \cdots (n, n-\mu, \dots, 2\mu, \mu)$  consists of  $\mu$  disjoint  $n/\mu$ -cycles. For  $j = 1, \dots, \mu$ , the elements that appear in the  $j^{\text{th}}$  such cycle are of the form  $\mu \left( \frac{n}{\mu} - k \right) + j$ , where  $k = 1, \dots, n/\mu$ . Since  $\pi(\Delta_n^2)$  is trivial,  $\pi(z)$  commutes with  $\pi(\alpha_0^\mu)$ , and so  $\pi(z)$  permutes the  $\mu$   $n/\mu$ -cycles of  $\pi(\alpha_0^\mu)$ , and preserves the cyclic order of the elements within each cycle. In particular, if  $\pi(z)$  sends  $j$  to  $\mu \left( \frac{n}{\mu} - k' \right) + j'$ , where  $j' \in \{1, \dots, \mu\}$  and  $k' \in \{1, \dots, n/\mu\}$  then

$$\begin{aligned} \pi(z) \left( \mu \left( \frac{n}{\mu} - k \right) + j \right) &= \pi(\alpha_0^{k\mu}) \circ \pi(z)(j) = \pi(z) \circ \pi(\alpha_0^{k\mu})(j) \\ &= \pi(\alpha_0^{k\mu}) \left( \mu \left( \frac{n}{\mu} - k' \right) + j' \right) \\ &= \mu \left( \left( \frac{n}{\mu} - k \right) - k' \right) + j' \pmod{n}. \end{aligned} \quad (39)$$

To coincide with the convention that we use for braids, note that we compose permutations from left to right. Now let  $j = 1$ , and let  $j' \in \{1, \dots, \mu\}$  and  $k' \in \{1, \dots, n/\mu\}$  be such that  $\pi(z)(1) = \mu \left( \frac{n}{\mu} - k' \right) + j'$ . Set

$$\zeta = (\sigma_1 \cdots \sigma_{j'-1})(\sigma_{\mu+1} \cdots \sigma_{\mu+j'-1}) \cdots (\sigma_{n-\mu+1} \cdots \sigma_{n-\mu+j'-1}).$$

Since  $1 \leq j' \leq \mu$ , for  $k = 1, \dots, n/\mu$ , the blocks  $\sigma_{\mu(\frac{n}{\mu}-k)+1} \cdots \sigma_{\mu(\frac{n}{\mu}-k)+j'-1}$  commute pairwise. By equations (23) and (24),  $\zeta$  and  $\alpha_0^\mu$  commute, hence

$$\zeta z^{-1} \alpha_0^\mu z \zeta^{-1} = \alpha_0^\mu \Delta_n^2. \quad (40)$$

Now  $\pi(\zeta)(1) = j'$ , so for all  $k = 1, \dots, n/\mu$ ,

$$\begin{aligned} \pi(\zeta z^{-1}) \left( \mu \left( \frac{n}{\mu} - k \right) + 1 \right) &= \pi(z^{-1}) \left( \mu \left( \frac{n}{\mu} - k \right) + j' \right) \\ &= \mu \left( \frac{n}{\mu} - k + k' \right) + 1, \end{aligned}$$

by equation (39). Thus  $\zeta z^{-1}$  and  $\alpha_0^\mu$  belong to the subgroup  $B_{n/\mu, n-n/\mu}(\mathbb{S}^2)$  of  $B_n(\mathbb{S}^2)$  which here denotes the subgroup of those braids whose permutation leaves the set  $\{1, \mu + 1, \dots, n - \mu + 1\}$  invariant. Let  $z'$  denote the image of  $z\zeta^{-1}$  under the projection onto  $B_{n/\mu}(\mathbb{S}^2)$ . Since the kernel

$$B_{n-n/\mu}(\mathbb{S}^2 \setminus \{x_1, x_{\mu+1}, \dots, x_{n-\mu+1}\})$$

of the surjective homomorphism  $B_{n/\mu, n-n/\mu}(\mathbb{S}^2) \longrightarrow B_{n/\mu}(\mathbb{S}^2)$  is torsion free (this follows for example from [GG7, Proposition 2.5]), the element  $\alpha_0^\mu$ , which is of order  $q$ , is sent to an element  $\beta$  of  $B_{n/\mu}(\mathbb{S}^2)$  of order  $q$ , and  $\Delta_n^2$  is sent to  $\Delta_{n/\mu}^2$ , the unique element of  $B_{n/\mu}(\mathbb{S}^2)$  of order 2 (using equation (25), it is in fact possible to show that  $\beta$  is equal to the element  $\alpha_0$  of  $B_{n/\mu}(\mathbb{S}^2)$ , see Figure 2 for an example in the case  $n = 6$  and  $\mu = 2$ ). Now  $q = 2n/\mu \geq 6$ , so by Theorem 1, there exists  $z'' \in B_{n/\mu}(\mathbb{S}^2)$  and  $1 \leq k < q$ ,

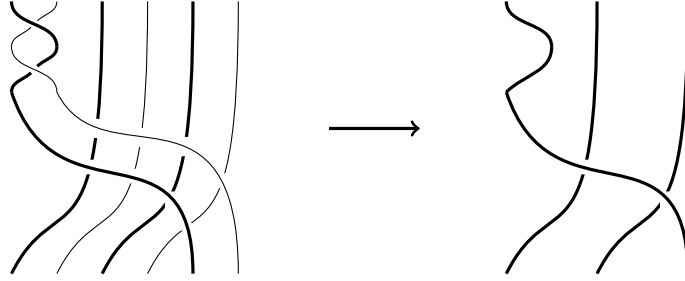


Figure 2: The element  $\alpha_0^2$  of  $B_{3,3}(\mathbb{S}^2)$  is sent to the element  $\alpha_0$  of  $B_3(\mathbb{S}^2)$  under the projection  $B_{3,3}(\mathbb{S}^2) \longrightarrow B_3(\mathbb{S}^2)$ .

$\gcd(k, q) = 1$ , such that  $\beta = z'' \alpha_0^k z''^{-1}$  ( $\alpha_0$  here being considered as the standard finite order element of  $B_{n/\mu}(\mathbb{S}^2)$ ). The image of equation (40) under this projection yields:

$$z'^{-1} z'' \alpha_0^k z''^{-1} z' = z'' \alpha_0^k z''^{-1} \Delta_{n/\mu}^2 \quad \text{in } B_{n/\mu}(\mathbb{S}^2),$$

so

$$z_1 \alpha_0^k z_1^{-1} = \alpha_0^k \Delta_{n/\mu}^2 \quad \text{in } B_{n/\mu}(\mathbb{S}^2). \quad (41)$$

where  $z_1 = z''^{-1} z'^{-1} z''$ . There exist  $\lambda_1, \lambda_2 \in \mathbb{Z}$  such that  $\lambda_1 k + \lambda_2 q = 1$ , so  $\alpha_0^{\lambda_1 k} = \alpha_0$  in  $B_{n/\mu}(\mathbb{S}^2)$ . Since  $q$  is even,  $\lambda_1$  is odd, and raising equation (41) to the  $\lambda_1$ <sup>th</sup> power yields

$$z_1 \alpha_0 z_1^{-1} = \alpha_0 \Delta_{n/\mu}^2 = \alpha_0^{1 + \frac{n}{\mu}} \in B_{n/\mu}(\mathbb{S}^2). \quad (42)$$

Hence  $z_1 \in N_{B_{n/\mu}(\mathbb{S}^2)}(\langle \alpha_0 \rangle)$ , and so by Proposition 8(b),  $z_1$  is an element of  $\langle \alpha_0, \Delta_{n/\mu} \rangle \cong \text{Dic}_{4n/\mu}$ , and  $z_1 = \alpha_0^\lambda \Delta_{n/\mu}^\varepsilon$ , where  $0 \leq \lambda < 2n/\mu$  and  $\varepsilon \in \{0, 1\}$ . Thus

$$z_1 \alpha_0 z_1^{-1} = \Delta_{n/\mu}^{-\varepsilon} \alpha_0 \Delta_{n/\mu}^\varepsilon = \begin{cases} \alpha_0 & \text{if } \varepsilon = 0 \\ \alpha_0^{-1} & \text{if } \varepsilon = 1. \end{cases} \quad (43)$$

Combining equations (42) and (43), we obtain  $\Delta_{n/\mu}^2 \in \{\text{Id}, \alpha_0^2\}$ . Now  $n/\mu \geq 3$ , so  $\alpha_0^2$  (resp.  $\Delta_{n/\mu}^2$ ) is of order  $n/\mu$  (resp. 2), which yields a contradiction.

Suppose finally that  $i = 2$ , so  $\mu \mid n - 2$  and  $(n - 2)/\mu \geq 3$ . Since  $n$  must be even in order that equation (38) possess a solution, these conditions imply that  $n \geq 6$ . Let  $t \in \{n - 1, n\}$ . Projecting equation (38) into  $S_n$  leads to the equality  $(\pi(\alpha_2^\mu) \circ \pi(z))(t) = (\pi(z) \circ \pi(\alpha_2^\mu))(t)$ , and this implies that

$$\pi(\alpha_2^\mu)(\pi(z)(t)) = \pi(z)(t),$$

so  $\pi(z)(t) \in \text{Fix}(\pi(\alpha_2^\mu))$ . Since  $1 \leq \mu < n - 2$ , we have  $\text{Fix}(\pi(\alpha_2^\mu)) = \{n - 1, n\}$ , and thus  $\pi(z)(t) \in \{n - 1, n\}$ . We conclude that  $z \in B_{n-2,2}(\mathbb{S}^2)$ ,  $B_{n-2,2}(\mathbb{S}^2)$  being the subgroup of  $B_n(\mathbb{S}^2)$  whose elements induce permutations that leave  $\{n - 1, n\}$  invariant. This permits us to project equation (38) onto  $B_{n-2}(\mathbb{S}^2)$  by forgetting the last two strings. It is clear that  $\alpha_2$  (as an element of  $B_{n-2,2}(\mathbb{S}^2)$ ) projects to  $\alpha_0$  (as an element of  $B_{n-2}(\mathbb{S}^2)$ ), and so  $\Delta_n^2 = \alpha_2^{n-2}$  (which is an element of  $B_{n-2,2}(\mathbb{S}^2)$ ) projects to  $\alpha_0^{n-2} = \Delta_{n-2}^2$  (as an element of  $B_{n-2}(\mathbb{S}^2)$ ) by equation (8). We thus obtain:

$$z'^{-1} \alpha_0^\mu z' = \alpha_0^\mu \Delta_{n-2}^2, \quad (44)$$

where  $z'$  is the image of  $z$  under this projection. But  $n - 2 \geq 4$ , and applying the analysis of the case  $i = 0$  to equation (44) yields a contradiction. This proves the result in the case  $i = 2$ , and thus completes the proof of the proposition.  $\square$

REMARKS 39.

(a) If  $i \in \{0, 2\}$  then the converse of Proposition 9(b) holds using the construction of the corresponding dicyclic groups of Remark 3(b).

(b) If  $\mu$  divides  $n - i$  where  $i \in \{0, 1, 2\}$ , the braid  $\alpha_i^\mu$  admits a block structure using arguments similar to those of the second part of Lemma 29. If  $q = (n - i)/\mu$  then  $\alpha_i^\mu$  may be thought of as a collection of  $q$  blocks, each comprised of  $\mu$  strings (see Figures 3 and 4 for examples where  $n - i = 12$  and  $\mu = 4$ , as well as Figure 1 for the case  $i = 0$ ,  $n = 6$  and  $\mu = 3$ ). The first block contains a full twist on its  $\mu$  strings, and passes over each of the remaining  $q - 1$  blocks. If  $i = 1$  (resp.  $i = 2$ ) then the last (resp. penultimate) string then wraps around this first block. If  $i = 2$  then there is an additional final vertical string. In terms of the Nielsen-Thurston classification of surface homeomorphisms applied to braid groups, these braids are reducible, and a set of reducing curves may be read off from these braid diagrams (see [BNG, GW] for more information).

One immediate consequence of Proposition 9 is that it allows us to narrow down the possible Type I subgroups of  $B_n(\mathbb{S}^2)$  involving cyclic or dicyclic factors, with the exception of  $Q_8$ .

COROLLARY 40. *Let  $G$  be a Type I subgroup of  $B_n(\mathbb{S}^2)$  of the form  $F \rtimes_\theta \mathbb{Z}$ .*

(a) *Suppose that  $F$  is cyclic.*

(i) *If  $|F|$  divides  $2(n - 1)$  then  $G \cong F \times \mathbb{Z}$ .*

(ii) *If  $|F|$  divides  $2(n - i)$ , where  $i \in \{0, 2\}$ , then either  $G \cong F \times \mathbb{Z}$  or  $G \cong F \rtimes_\rho \mathbb{Z}$ , where  $\rho$  is the action defined in Definition 4(b) (multiplication by  $-1$ ).*



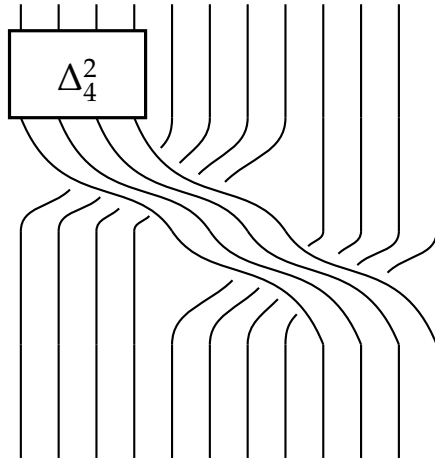


Figure 3: The braid  $\alpha_0^4$  in  $B_{12}(\mathbb{S}^2)$ .

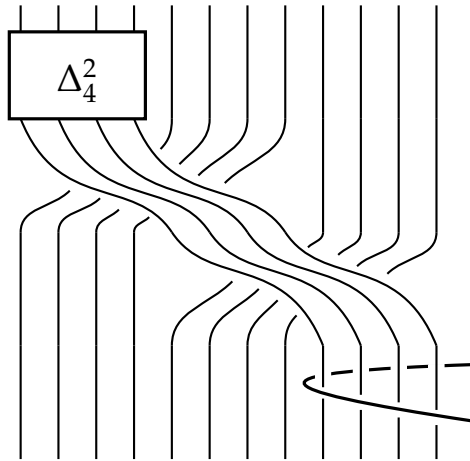


Figure 4: The braid  $\alpha_1^4$  in  $B_{13}(\mathbb{S}^2)$ . The braid  $\alpha_2^4$  in  $B_{14}(\mathbb{S}^2)$  is obtained by adding an extra vertical string on the right of this braid.

(b) Let  $m \geq 3$  divide  $n - i$ , where  $i \in \{0, 2\}$ , and let  $F$  be dicyclic of order  $4m$  with the presentation given by equation (9). Then either  $G \cong F \times \mathbb{Z}$  or  $G \cong F \rtimes_v \mathbb{Z}$ , where  $v$  is the action defined by equation (12).

*Proof.*

(a) Let  $G$  be a Type I subgroup of  $B_n(\mathbb{S}^2)$  of the form  $F \rtimes_{\theta} \mathbb{Z}$ , where  $F$  is cyclic. Up to conjugacy, we may suppose by Theorem 1 that there exist  $i \in \{0, 1, 2\}$  and  $1 \leq l \leq 2(n - i)$  such that  $l$  divides  $2(n - i)$ , and  $F = \langle \alpha_i^l \rangle$ , with  $|F| = 2(n - i)/l$ . There exists  $z \in B_n(\mathbb{S}^2)$  of infinite order such that the action  $\theta$  on  $F$  is realised by conjugation by  $z$ , so  $z\alpha_i^l z^{-1} = \alpha_i^{lm}$ , where  $\gcd(m, 2(n - i)/l) = 1$ . From Proposition 9,  $\alpha_i^{lm} = \alpha_i^l$  if  $i = 1$  and  $\alpha_i^{lm} \in \{\alpha_i^l, \alpha_i^{-l}\}$  if  $i \in \{0, 2\}$ , which implies the result.

(b) Let  $G$  be a Type I subgroup of  $B_n(\mathbb{S}^2)$  of the form  $F \rtimes_{\theta} \mathbb{Z}$ , where  $F \cong \text{Dic}_{4m}$  has the given presentation, and let the action  $\theta$  of  $\mathbb{Z}$  on  $F$  be realised by conjugation by  $z$ , where  $z \in B_n(\mathbb{S}^2)$  is of infinite order. Since  $m \geq 3$ ,  $\langle x \rangle$  is the unique cyclic subgroup of  $F$  of order  $2m$ , so is invariant under conjugation by  $z$ . By part (a)(ii),  $zxz^{-1} = \theta(1)(x) = x^{\varepsilon}$ , where  $\varepsilon \in \{1, -1\}$ . Further, the elements of  $F \setminus \langle x \rangle = \langle x \rangle y$  are permuted by the action, so  $zyz^{-1} = \theta(1)(y) = x^{2k+\delta}y$  for some  $k \in \{0, 1, \dots, m-1\}$  and  $\delta \in \{0, 1\}$ . If  $\varepsilon = 1$  (resp.  $\varepsilon = -1$ ) then consider the action  $\theta'$  defined by  $\theta'(1) = \iota \circ \theta(1)$ , where  $\iota \in \text{Inn}(F)$  is conjugation by  $x^{-k}$  (resp. by  $x^{k+\delta}y$ ). So  $\theta'(1)(x) = x$ , and  $\theta'(1)(y) = x^{\delta}y$ , which gives rise to the two possible actions given in the statement. Since the automorphisms  $\theta(1)$  and  $\theta'(1)$  of  $F$  differ by an inner automorphism, it follows from Proposition 35 that  $G$  and  $F \rtimes_{\theta'} \mathbb{Z}$  are isomorphic.  $\square$

## 5 Reduction of isomorphism classes of $F \rtimes_{\theta} \mathbb{Z}$ via periodicity

We now turn our attention to the Type I subgroups  $G$  of  $B_n(\mathbb{S}^2)$  of the form  $F \rtimes_{\theta} \mathbb{Z}$ , where  $F$  is equal to  $O^*$  or  $I^*$ . The arguments of Section 3 showed that there are two possible actions. The aim of this section is to rule out the non-trivial action in each case, which will imply that  $G$  is isomorphic to  $F \times \mathbb{Z}$ . This is achieved in two stages. First, in Section 5.1 we give an alternative proof of the fact that the homotopy type of the universal covering space of the configuration spaces  $F_n(\mathbb{S}^2)$  and  $D_n(\mathbb{S}^2)$  is that of  $\mathbb{S}^2$  if  $n \leq 2$ , and that of  $\mathbb{S}^3$  otherwise. This result appears to be an interesting fact in its own right, and mirrors that for the projective plane  $\mathbb{R}P^2$  [GG2]. As a consequence, in Lemma 41 we generalise the fact that any nontrivial finite subgroup of  $B_n(\mathbb{S}^2)$  is periodic of least period 2 or 4 [GG6] to its infinite subgroups. Secondly, if  $F \in \{O^*, I^*\}$ , in Proposition 44 we recall some facts concerning the cohomology of  $F$ . From this, it will follow in these cases that  $\theta(1)$  is an inner automorphism, and so by Proposition 35,  $F \rtimes_{\theta} \mathbb{Z}$  is isomorphic to  $F \times \mathbb{Z}$ .

### 5.1 The homotopy type of the configuration spaces $F_n(\mathbb{S}^2)$ and $D_n(\mathbb{S}^2)$

The purpose of this section is to describe the homotopy type of the universal covering space of  $F_n(\mathbb{S}^2)$  and  $D_n(\mathbb{S}^2)$ . For  $n = 1$ , we have  $F_1(\mathbb{S}^2) = D_1(\mathbb{S}^2) = \mathbb{S}^2$ , which is simply connected. So assume from now on that  $n \geq 2$ . We give an alternative proof of Proposition 10 which is due to [BCP, FZ].

*Proof of Proposition 10.* First observe that  $F_n(\mathbb{S}^2)$  and  $D_n(\mathbb{S}^2)$  have the same universal covering space because  $F_n(\mathbb{S}^2)$  is a finite  $n!$ -fold regular covering space of  $D_n(\mathbb{S}^2)$ .

(a) This was proved in [GG9, Lemma 8].

(b) Let  $n \geq 1$ . Consider the Fadell-Neuwirth fibration  $p_{n+1}: F_{n+1}(\mathbb{S}^2) \rightarrow F_n(\mathbb{S}^2)$  obtained by forgetting the last coordinate. The fibre over a point  $(x_1, \dots, x_n) \in F_n(\mathbb{S}^2)$  may be identified with  $F_1(\mathbb{S}^2 \setminus \{x_1, \dots, x_n\})$ . The related long exact sequence in homotopy is:

$$\begin{aligned} \dots \rightarrow \pi_{m+1}(F_{n+1}(\mathbb{S}^2)) \rightarrow \pi_{m+1}(F_n(\mathbb{S}^2)) \rightarrow \pi_m(F_1(\mathbb{S}^2 \setminus \{x_1, \dots, x_n\})) \rightarrow \\ \pi_m(F_{n+1}(\mathbb{S}^2)) \rightarrow \pi_m(F_n(\mathbb{S}^2)) \rightarrow \pi_{m-1}(F_1(\mathbb{S}^2 \setminus \{x_1, \dots, x_n\})) \rightarrow \dots \end{aligned}$$

The fact that  $F_1(\mathbb{S}^2 \setminus \{x_1, \dots, x_n\})$  is a  $K(\pi, 1)$ -space implies that the homomorphism  $\pi_m(F_{n+1}(\mathbb{S}^2)) \rightarrow \pi_m(F_n(\mathbb{S}^2))$  induced by  $p_{n+1}$  is an isomorphism for all  $m \geq 3$  and all  $n \geq 1$ . It remains to study the case  $m = 2$ .

First suppose that  $n = 3$ . From part (a),  $F_2(\mathbb{S}^2)$  has the homotopy type of  $\mathbb{S}^2$ , and  $\pi_2(F_3(\mathbb{S}^2)) = \{1\}$  and  $\pi_1(F_3(\mathbb{S}^2)) \cong \mathbb{Z}_2$  by [FVB]. Let  $\varphi: \mathbb{S}^3 \rightarrow F_3(\mathbb{S}^2)$  be such that  $p_2 \circ p_3 \circ \varphi$  is homotopic to the Hopf map  $\eta$  (such a  $\varphi$  exists because  $\pi_3(F_3(\mathbb{S}^2))$  is isomorphic to  $\pi_3(F_2(\mathbb{S}^2))$ ). We thus have the following diagram that commutes up to homotopy:

$$\begin{array}{ccc} \mathbb{S}^3 & \xrightarrow{\varphi} & F_3(\mathbb{S}^2) \\ \downarrow \eta & & \downarrow p_2 \circ p_3 \\ \mathbb{S}^2 & \xlongequal{\quad} & \mathbb{S}^2. \end{array}$$

If  $m \geq 3$ ,  $\eta$  induces an isomorphism  $\pi_m(\mathbb{S}^3) \rightarrow \pi_m(\mathbb{S}^2)$  and  $p_2 \circ p_3$  induces an isomorphism  $\pi_m(F_3(\mathbb{S}^2)) \rightarrow \pi_m(\mathbb{S}^2)$ . Since  $\pi_2(\mathbb{S}^3)$  and  $\pi_2(F_3(\mathbb{S}^2))$  are trivial, the commutativity of the above diagram implies that  $\varphi$  induces an isomorphism  $\pi_m(\mathbb{S}^3) \rightarrow \pi_m(F_3(\mathbb{S}^2))$  for all  $m \geq 2$ . Lifting to the corresponding universal covering spaces gives rise to an isomorphism  $\pi_m(\mathbb{S}^3) \rightarrow \pi_m(\widetilde{F_3(\mathbb{S}^2)})$  for all  $m \in \mathbb{N}$ ,  $\widetilde{F_3(\mathbb{S}^2)}$  being the universal covering space of  $F_3(\mathbb{S}^2)$ , and so by Whitehead's theorem,  $F_3(\mathbb{S}^2)$  has the homotopy type of  $\mathbb{S}^3$ .

Let  $n \geq 3$ . Then  $\pi_2(F_n(\mathbb{S}^2)) = \{1\}$  [FVB] and so the homomorphism

$$\pi_m(F_{n+1}(\mathbb{S}^2)) \rightarrow \pi_m(F_n(\mathbb{S}^2))$$

induced by  $p_{n+1}$  is an isomorphism for all  $m \geq 2$ . Lifting to the universal covering spaces and applying Whitehead's Theorem, part (a) and induction gives the result.  $\square$

## 5.2 A cohomological condition for the realisation of Type I virtually cyclic groups

In this section we apply Proposition 10 to derive a necessary cohomological condition for an abstract group to be realised as a subgroup of  $B_n(\mathbb{S}^2)$ . If  $F = O^*, I^*$ , this will allow us to rule out the possibility of  $F \rtimes_{\theta} \mathbb{Z}$  for the non-trivial action for each of these groups described in Section 3. Following [AS], we recall the definition of a periodic

group which extends the classical definition for finite groups. By Definition 2.1 and the definition given just before Corollary 2.10 in [AS], we say that a group  $G$  is *periodic* of period  $d \geq 1$  if there exist a non-negative integer  $r_0 \geq 0$  and a cohomology class  $u \in H^d(G, \mathbb{Z})$  such that the homomorphism  $H^r(G, A) \xrightarrow{\cup u} H^{r+d}(G, A)$  is an isomorphism for all  $r \geq r_0$  and for all local coefficient systems  $A$ . From [AS, Corollary 2.14], if a discrete group acts freely on a finite-dimensional CW-complex of dimension  $m$  whose homotopy type is that of the sphere  $\mathbb{S}^{d-1}$  then the group  $G$  is periodic. By a standard argument using the spectral sequence associated to the covering of the orbit space, it is not hard to see that  $d$  is a period, and that we can take  $r_0 = m + 1$ . An obvious consequence of the above is the following lemma.

LEMMA 41. *Let  $n \geq 3$ , and let  $G$  be a group abstractly isomorphic to a subgroup of  $B_n(\mathbb{S}^2)$ . Then there exists  $r_0 \geq 1$  such that  $H^r(G, \mathbb{Z}) \cong H^{r+4}(G, \mathbb{Z})$  for all  $r \geq r_0$ .*

*Proof.* Since the universal covering space  $\widetilde{D_n(\mathbb{S}^2)}$  of  $D_n(\mathbb{S}^2)$  is a finite-dimensional CW-complex, it is a homotopy 3–sphere by Proposition 10(a). Any subgroup of  $B_n(\mathbb{S}^2)$  acts freely on  $\widetilde{D_n(\mathbb{S}^2)}$ , and thus is periodic of period 4. Taking  $A = \mathbb{Z}$  yields the result.  $\square$

We now apply Lemma 41 to the Type I groups of the form  $F \rtimes_{\theta} \mathbb{Z}$ . If a group  $G$  acts on a module  $A$ , let  $A^G$  denote the submodule of  $A$  fixed by  $G$ , and let  $A_G$  denote the quotient of  $A$  by the submodule generated by  $\{a - ga \mid a \in A, g \in G\}$ .

LEMMA 42. *Let  $G = F \rtimes_{\theta} \mathbb{Z}$ , where  $F$  is a finite periodic group and  $\theta \in \text{Hom}(\mathbb{Z}, \text{Aut}(F))$ , and let  $\theta(1)^{(i)} : H^i(F, \mathbb{Z}) \rightarrow H^i(F, \mathbb{Z})$  be the induced automorphism on cohomology in dimension  $i$ . Then  $H^*(G, \mathbb{Z})$  is as follows:  $H^0(G, \mathbb{Z}) = \mathbb{Z}$ ,  $H^1(G, \mathbb{Z}) = \mathbb{Z}$ , and for all  $i \in \mathbb{N}$ ,  $H^{2i}(G, \mathbb{Z}) = H^{2i}(F, \mathbb{Z})^{\mathbb{Z}}$  and  $H^{2i+1}(G, \mathbb{Z}) = H^{2i}(F, \mathbb{Z})_{\mathbb{Z}}$  with respect to the  $\mathbb{Z}$ -module structure on  $H^{2i}(F, \mathbb{Z})$  induced by  $\theta$ .*

*Proof.* Consider the Lyndon-Hochschild-Serre spectral sequence associated with the short exact sequence

$$1 \longrightarrow F \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow 1.$$

The  $E_2$ -term of this spectral sequence, given by  $H^p(\mathbb{Z}, H^q(F, \mathbb{Z}))$ , vanishes if  $p \notin \{0, 1\}$  because the cohomological dimension of  $\mathbb{Z}$  is equal to one. So outside of the two vertical lines given by  $p = 0$  and  $p = 1$ , the terms vanish which implies that all differentials are necessarily trivial, and so the spectral sequence collapses. Further, since the cohomology of  $F$  in odd dimension vanishes, there is at most one non-trivial group  $E_2^{p,q}$  with  $p + q = r$  for each given  $r$ . Hence there is no extension problem from  $E_{\infty}$  to  $H^*(G)$ , and it suffices to compute the  $E_2$ -term. The result follows from the well-known description of the cohomology of  $\mathbb{Z}$  with coefficients in  $A$  (see [Br, Chapter III, Section 1, Example 1]).  $\square$

We now seek necessary conditions for the group  $G$  to have least period either 2 or 4. Let  $d$  be the least period of  $F$ . Then  $d$  is the least integer for which  $H^d(F, \mathbb{Z}) \cong \mathbb{Z}_{|F|}$ , and if  $H^{2i}(G, \mathbb{Z}) = H^{2i}(F, \mathbb{Z})^{\mathbb{Z}} \cong \mathbb{Z}_{|F|}$  then  $\theta(1)^{(2i)} = \text{Id}$ . So there exists  $k \in \mathbb{N}$  such that  $2i = kd$ . Let  $k_0$  be the least integer for which  $\theta(1)^{(k_0 d)} = \text{Id}$ . If  $G$  is periodic, its period is necessarily a multiple of  $k_0 d$ . In particular, if the least period of  $G$  is equal to either 2 or 4 then  $k_0 \in \{1, 2\}$  if  $d = 2$ , and  $k_0 = 1$  if  $d = 4$ .

REMARK 43. The additive structure of the cohomology of the virtually cyclic groups of Type I with integer coefficients was computed in detail in [Je] for the cases where  $F$  is one of the groups of the form  $\mathbb{Z}_a \rtimes \mathbb{Z}_b$  or  $\mathbb{Z}_a \rtimes (\mathbb{Z}_b \times \mathcal{Q}_{2^i})$ . This corresponds to the first two families of the classification of the finite periodic groups given by the Suzuki-Zassenhaus Theorem [AM, Theorem 6.15].

Based on Lemma 42 and the knowledge of the cohomology of finite periodic groups, we obtain the following result.

PROPOSITION 44. *Let  $F \in \{O^*, I^*\}$ , and let  $G \cong F \rtimes_{\theta} \mathbb{Z}$  be a Type I subgroup of  $B_n(\mathbb{S}^2)$ . Then  $\theta(1)$  is an inner automorphism of  $F$ .*

*Proof.* Suppose first that  $F \cong O^*$ . By [GoG1, p. 39], the group  $O^*$  has period 4, and the induced automorphism on  $H^4(O^*, \mathbb{Z}) \cong \mathbb{Z}_{48}$  is trivial if  $\theta(1)$  is an inner automorphism, and multiplication by 9 (so is non trivial) otherwise, thus the result follows.

Now suppose that  $F \cong I^*$ , which we interpret as  $\mathrm{SL}_2(\mathbb{F}_5)$ . The non-trivial element of  $\mathrm{Out}(I^*)$  is represented by the automorphism of  $I^*$  which is conjugation by the matrix  $\begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix}$ , where  $w$  is a non square of  $\mathbb{F}_5$  [AM, page 152]. From [GoG2, Proposition 1.5], the induced automorphism on the 5-primary component of the group  $H^4(I^*, \mathbb{Z}) \cong \mathbb{Z}_{120}$ , which is isomorphic to  $\mathbb{Z}_5$ , is multiplication by  $-1$ . For the trivial element of  $\mathrm{Out}(I^*)$ , the induced homomorphism is trivial and the result follows.  $\square$

## 6 Necessity of the conditions on $\mathbb{V}_1(n)$ and $\mathbb{V}_2(n)$

Let  $n \geq 4$ . In this section, we prove Theorem 5(1), which shows the necessity of the conditions on  $\mathbb{V}_1(n)$  and  $\mathbb{V}_2(n)$ . We start by considering the subgroups of  $B_n(\mathbb{S}^2)$  of Type I, and then go on to study those of Type II.

### 6.1 Necessity of the conditions on $\mathbb{V}_1(n)$

We gather together the results of the previous sections to prove the following proposition, which is the statement of Theorem 5(1) for the Type I subgroups of  $B_n(\mathbb{S}^2)$ .

PROPOSITION 45. *Let  $n \geq 4$ . Then every virtually cyclic subgroup of  $B_n(\mathbb{S}^2)$  of Type I is isomorphic to an element of  $\mathbb{V}_1(n)$ .*

Before proving Proposition 45, we state and prove the following result which shows that if  $F$  is a dicyclic subgroup of  $B_n(\mathbb{S}^2)$  then up to conjugacy, it may be taken to be a subgroup of one of the maximal dicyclic subgroups  $\mathrm{Dic}_{4(n-i)}$ ,  $i \in \{0, 2\}$ .

LEMMA 46. *Let  $n \geq 4$ , and let  $H$  be a subgroup of  $B_n(\mathbb{S}^2)$  isomorphic to  $\mathrm{Dic}_{4m}$ , where  $m \geq 2$ . Then there exists  $i \in \{0, 2\}$  such that  $H$  is conjugate to a subgroup of the standard maximal dicyclic subgroup  $\mathrm{Dic}_{4(n-i)}$  of Remark 3(b).*

REMARK 47. Under the hypotheses of Lemma 46, we have that  $m \mid n - i$ .

*Proof of Lemma 46.* Let  $H \cong \text{Dic}_{4m}$ , where  $m \geq 2$ . By [GG5, Proposition 1.5(2)],  $H$ , as an abstract finite group, is realised as a single conjugacy class in  $B_n(\mathbb{S}^2)$  with the exception that when  $n$  is even and  $m$  divides  $(n-i)/2$ ,  $i \in \{0, 2\}$ , there are exactly two conjugacy classes. Using the subgroup structure of dicyclic groups and the construction of [GG5, Theorem 1.6], it follows that  $H$  is conjugate to a subgroup of the one of the standard maximal dicyclic subgroups  $\text{Dic}_{4(n-i)}$  of  $B_n(\mathbb{S}^2)$ , where  $i \in \{0, 2\}$ .  $\square$

*Proof of Proposition 45.* Let  $G$  be a infinite virtually cyclic subgroup of  $B_n(\mathbb{S}^2)$  of Type I. Then  $G$  is of the form  $F \rtimes_{\theta} \mathbb{Z}$ , where  $F$  is a finite subgroup of  $B_n(\mathbb{S}^2)$ , and  $\theta(1) \in \text{Hom}(\mathbb{Z}, \text{Aut}(F))$ . We separate the discussion into two cases.

(a) Suppose that  $F$  is isomorphic to one of the binary polyhedral groups  $T^*, O^*, I^*$ . Then  $n$  must satisfy the conditions given in Theorem 2 for the existence of  $F$  as a subgroup of  $B_n(\mathbb{S}^2)$ . Applying Proposition 35, up to isomorphism, we may restrict ourselves to representative automorphisms  $\theta(1)$  of the elements of  $\text{Out}(F) \cong \mathbb{Z}_2$  given in Section 3. If  $\theta(1) = \text{Id}_F$  then  $G \cong F \times \mathbb{Z}$ , and these are the elements of  $\mathbb{V}_1(n)$  given by Definition 4(1)(f), (h) and (i) for the given values of  $n$ . So suppose that  $\theta(1)$  represents the nontrivial element of  $\text{Out}(F)$ . By Proposition 44,  $F \not\cong O^*, I^*$ , so  $F \cong T^*$ , and  $G$  is isomorphic to the element of  $\mathbb{V}_1(n)$  given by Definition 4(1)(g), the action  $\omega$  being that of equation (14). Since  $n$  must be even for the existence of  $T^*$ , it remains to show that  $n \equiv 0, 2 \pmod{6}$ . Suppose on the contrary that  $n = 6l + 4$ , where  $l \in \mathbb{N}$ , and suppose that  $T^* \rtimes_{\omega} \mathbb{Z}$  is realised as a subgroup  $L$  of  $B_n(\mathbb{S}^2)$ , with the  $T^*$ -factor (resp. the  $\mathbb{Z}$ -factor) realised as a subgroup  $H$  (resp.  $\langle z \rangle$ ) of  $B_n(\mathbb{S}^2)$ . Let (13) denote a presentation of  $H$ . By the definition of  $\omega$ , we have that  $\omega(1)(X) = X^{-1}$  by equation (14). On the other hand,  $X$  is of order 3, so up to conjugacy and inverses, it follows from Theorem 1 that  $X = \alpha_1^{2(n-1)/3} = \alpha_1^{4l+2}$ . Since the action  $\omega$  of  $\mathbb{Z}$  on  $H$  is realised by conjugation by  $z$ , we have  $\omega(1)(X) = zXz^{-1}$  in  $L$ , which implies that  $zXz^{-1} = X^{-1}$ . Abelianising this relation in  $\mathbb{Z}_{2(n-1)}$  yields  $\xi(X) = \xi(X^{-1})$ . However,

$$\xi(X) = \xi(\alpha_1^{4l+2}) = \overline{n(4l+2)} = \overline{(6l+4)(4l+2)} = \overline{4l+2}$$

in  $\mathbb{Z}_{12l+6}$ , so  $\xi(X) \neq \xi(X^{-1})$ , and we obtain a contradiction.

(b) Suppose that  $F$  is not isomorphic to any of the three binary polyhedral groups  $T^*, O^*, I^*$ . Proposition 85 implies that  $F$  is cyclic or dicyclic. If  $F$  is dicyclic, isomorphic to  $\text{Dic}_{4m}$  for some  $m \geq 2$ , then Lemma 46 implies that up to conjugation,  $F$  is a subgroup of one of the standard dicyclic groups  $\text{Dic}_{4(n-i)}$ ,  $i \in \{0, 2\}$ , and  $m \mid n-i$ . If  $m = 2$  then  $F \cong Q_8$  and  $n$  is even. Furthermore, by Lemma 36, up to an element of  $\text{Inn}(F)$ ,  $\theta(1) \in \{\text{Id}_{Q_8}, \alpha, \beta\}$ , so  $G$  is isomorphic to an element of  $\mathbb{V}_1(n)$  given by Definition 4(1)(e). If  $m \geq 3$  then Corollary 40(b) applies, and up to an element of  $\text{Inn}(F)$ , there are two cases to consider:

(i)  $\theta(1) = \text{Id}_F$ , in which case  $G \cong \text{Dic}_{4m} \times \mathbb{Z}$ . Since  $F$  admits a cyclic subgroup of order  $2m$ , the realisation of  $G$  implies that of  $\mathbb{Z}_{2m} \times \mathbb{Z}$ . If  $m = n-i$  then up to conjugacy, we may suppose by Theorem 1 that the cyclic factor is generated by  $\alpha_i$ , but this contradicts Proposition 31, and hence  $m < n-i$ . Thus  $G$  is an element of  $\mathbb{V}_1(n)$  given by Definition 4(1)(c).

(ii)  $G \cong \text{Dic}_{4m} \rtimes_v \mathbb{Z}$ , where  $v(1)$  is given by equation (12). Let  $F$  have the presentation given by equation (9). Abelianising the relation  $v(1)(y) = xy$  in  $B_n(\mathbb{S}^2)$  implies that the exponent sum of  $x$  is congruent to zero modulo  $2(n-1)$ . On the other hand,  $x$  is of order  $2m$ , so by Theorem 1 is conjugate to  $\alpha_i^{l(n-i)/m}$ , where  $\gcd(l, 2m) = 1$ . In particular,  $l$  is odd. Now the exponent sum of  $\alpha_i$  is congruent to  $n-1$  modulo  $2(n-1)$ , and since that of  $x$  is congruent to zero modulo  $2(n-1)$ , it follows that  $l(n-i)/m$  is even, and consequently  $(n-i)/m$  is even. Thus  $G$  is an element of  $\mathbb{V}_1(n)$  given by Definition 4(1)(d).

Finally, suppose that  $F$  is cyclic of order  $q$ , say. By Theorem 1 there exists  $i \in \{0, 1, 2\}$  such that  $q$  divides  $2(n-i)$ , and up to conjugacy,  $F = \langle \alpha_i^{2(n-i)/q} \rangle$ . Applying Corollary 40(a), we have that  $\theta(1) \in \{\text{Id}_F, -\text{Id}_F\}$  up to an element of  $\text{Inn}(F)$ . If  $\theta(1) = \text{Id}_F$  then  $G \cong F \times \mathbb{Z}$ . But  $F$  cannot be maximal cyclic, for then its centraliser would contain an element of infinite order, which contradicts Proposition 31, so  $q \neq 2(n-i)$ . Further, if  $n-i$  is odd then  $q \neq n-i$ , for then  $\langle \alpha_i^2 \Delta_n^2 \rangle = \langle \alpha_i \rangle$  would be of order  $2(n-i)$ , and its centraliser would contain an element of infinite order, which contradicts Proposition 31 once more. Hence  $G$  is isomorphic to an element of  $\mathbb{V}_1(n)$  given by Definition 4(1)(a). So suppose that  $\theta(1) = -\text{Id}_F$ . Then  $G \cong F \rtimes_\rho \mathbb{Z}$ , where  $\rho$  is the action by conjugation for which  $\rho(1)$  is multiplication by  $-1$ . By Corollary 40(a), we have  $i \in \{0, 2\}$ . Further, the subgroup of  $G$  isomorphic to  $F \rtimes_\rho 2\mathbb{Z}$  is abstractly isomorphic to  $F \times \mathbb{Z}$ , and so we conclude from the previous case that  $q \neq 2(n-i)$ , and that  $q \neq n-i$  if  $n$  is odd. Hence  $G$  is isomorphic to an element of  $\mathbb{V}_1(n)$  given by Definition 4(1)(b). This shows that any virtually cyclic subgroup of  $B_n(\mathbb{S}^2)$  is isomorphic to an element of the family  $\mathbb{V}_1(n)$  as required.  $\square$

## 6.2 Necessity of the conditions on $\mathbb{V}_2(n)$

We now prove Theorem 5(1) for the Type II subgroups of  $B_n(\mathbb{S}^2)$ .

**PROPOSITION 48.** *Let  $n \geq 4$ . Then every virtually cyclic subgroup of  $B_n(\mathbb{S}^2)$  of Type II is isomorphic to an element of  $\mathbb{V}_2(n)$ .*

**REMARK 49.** Combining Propositions 45 and 48 yields the proof of Theorem 5(1).

*Proof of Proposition 48.* Let  $G$  be an infinite virtually cyclic subgroup of  $B_n(\mathbb{S}^2)$  of Type II. Then  $G = G_1 *_F G_2$ , where  $F, G_1$  and  $G_2$  are finite subgroups of  $B_n(\mathbb{S}^2)$ , and  $F$  is of index 2 in  $G_j$ ,  $j = 1, 2$ . Suppose first that one of the  $G_j$ ,  $G_1$  say, is binary polyhedral. Then  $G_1 \cong O^*$  since  $T^*, I^*$  have no index 2 subgroup,  $F \cong T^*$  since  $T^*$  is the unique index 2 subgroup of  $O^*$ , and  $G_2 \cong O^*$  since  $O^*$  is the only finite subgroup of  $B_n(\mathbb{S}^2)$  to have  $T^*$  as an index 2 subgroup. Thus  $G \cong O^* *_F O^*$ , which is the element of  $\mathbb{V}_2(n)$  given by Definition 4(2)(e).

Assume now that the  $G_j$  are not binary polyhedral. By Remark 3(a), the  $G_j$  are cyclic or dicyclic, and since they possess an even index subgroup, they are of even order, so both contain the unique element  $\Delta_n^2$  of order 2. This implies that  $F = G_1 \cap G_2$  is of even order, so the  $G_j$  are in fact of order  $4q$  for some  $q \in \mathbb{N}$ .

Suppose that one of the  $G_j$ ,  $G_1$  say, is cyclic. Then  $G_1 \cong \mathbb{Z}_{4q}$  and  $F \cong \mathbb{Z}_{2q}$ . By Theorem 1, there exists  $i \in \{0, 1, 2\}$  such that  $4q \mid 2(n-i)$ , so  $q \mid (n-i)/2$ . If  $G_2 \cong \mathbb{Z}_{4q}$ ,

$G$  is isomorphic to the element of  $\mathbb{V}_2(n)$  given by Definition 4(2)(a). If  $G_2 \cong \text{Dic}_{4q}$  then  $q \geq 2$ , and there exists  $i' \in \{0, 2\}$  such that  $q \mid n - i'$  by Lemma 46. But  $n - i' = 2 \left( \frac{n-i}{2} \right) + (i - i')$ , so  $q \mid i - i'$ , and since  $q \geq 2$ , we must have  $i \in \{0, 2\}$ . In this case,  $G$  is isomorphic to the element of  $\mathbb{V}_2(n)$  given by Definition 4(2)(b).

Finally, suppose that  $G_1 \cong G_2 \cong \text{Dic}_{4q}$ , where  $q \geq 2$ . Then  $F \cong \mathbb{Z}_{2q}$  or  $F \cong \text{Dic}_{2q}$ , and there exists  $i \in \{0, 2\}$  such that  $q \mid n - i$  by Lemma 46. If  $F \cong \mathbb{Z}_{2q}$  then by standard properties of the amalgamated product  $G = G_1 *_F G_2$ ,  $G$  has an index 2 subgroup  $G'$  isomorphic to  $F \rtimes_{\theta} \mathbb{Z}$  for some  $\theta \in \text{Hom}(\mathbb{Z}, \text{Aut}(F))$ . Since  $F$  is cyclic,  $\theta(1) \in \{\text{Id}_F, -\text{Id}_F\}$  by Corollary 40(a)(ii), and hence the subgroup  $F \rtimes_{\theta} 2\mathbb{Z}$  is abstractly isomorphic to  $F \times \mathbb{Z}$ . It follows from Theorem 1 and Proposition 31 that  $q \neq n - i$ , and so  $G$  is isomorphic to the element of  $\mathbb{V}_2(n)$  given by Definition 4(2)(c). Now suppose that  $F \cong \text{Dic}_{2q}$ . Then  $q \geq 4$  is even, and hence  $G$  is isomorphic to the element of  $\mathbb{V}_2(n)$  given by Definition 4(2)(d).  $\square$

REMARK 50. The cohomological property of Section 5.2 used to define the family  $\mathbb{V}_1(n)$  appears to be important in this case. We do not know of an example of two finite periodic groups  $G_1, G_2$  of the same period  $d$  for which the amalgamated product  $G_1 *_F G_2$  does not have period  $d$ . The Mayer-Vietoris sequence [Br, Chapter II, Section 7, Corollary 7.7] suggests that such an example may not even exist.



## Part II

# Realisation of the elements of $\mathbb{V}_1(n)$ and $\mathbb{V}_2(n)$ in $B_n(\mathbb{S}^2)$

In this Part, we prove that with a small number of exceptions (those described in Remark 6), the isomorphism classes of  $\mathbb{V}(n)$  given in the statement of Theorem 5(2) are indeed realised as subgroups of  $B_n(\mathbb{S}^2)$ . For the realisation of the Type I groups, the cases  $F = \mathbb{Z}_q$ ,  $F = \text{Dic}_{4m}$  ( $m \geq 3$ ),  $F = \mathbb{Q}_8$  and  $F = T^*, O^*, I^*$  will be treated in Sections 1, 2, 3 and 4 respectively, and the results will be brought together in Section 5. The realisation of the Type II groups will be dealt with in Section 6, and this will enable us to prove Theorem 5(2) in Section 7. In the first three cases, the constructions are algebraic, but are heavily inspired by geometric considerations, and it may be helpful for the reader to draw some pictures. If  $F$  is binary polyhedral, the corresponding virtually cyclic groups will be obtained geometrically by considering certain multitwists in  $\mathcal{MCG}(\mathbb{S}^2, n)$ , and then lifting the corresponding mapping class to an element of  $B_n(\mathbb{S}^2)$  via equation (11). Theorem 5(3) will be proved in Section 4.1. In Section 8, we discuss the question of the number of isomorphism classes of the Type II virtually cyclic subgroups of  $B_n(\mathbb{S}^2)$ , which will enable us to prove Proposition 11. Finally, in Section 9, we apply Theorem 5 and Proposition 12 to the problem of the classification of the virtually cyclic subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$ , from which we will obtain directly Theorem 14.

## 1 Type I subgroups of $B_n(\mathbb{S}^2)$ of the form $F \rtimes \mathbb{Z}$ with $F$ cyclic

Let  $n \geq 4$ , and let  $F$  be a finite cyclic subgroup of  $B_n(\mathbb{S}^2)$ . In order to construct elements of  $\mathbb{V}_1(n)$  involving  $F$ , we require elements of  $B_n(\mathbb{S}^2)$  of infinite order whose action on  $F$  by conjugation is compatible with Proposition 9. Since these actions are given by multiplication by  $\pm 1$ , we will be interested in finding elements  $z \in B_n(\mathbb{S}^2)$  of infinite order for which  $zxz^{-1} = x^{\pm 1}$  for all  $x \in F$ . This comes down to studying the centraliser and normaliser of  $F$  in  $B_n(\mathbb{S}^2)$ . Note that by Theorem 1, there exist  $i \in \{0, 1, 2\}$  and  $0 \leq m < 2(n - i)$ ,  $m \mid 2(n - i)$ , such that  $F$  is conjugate to  $\langle \alpha_i^m \rangle$ . Since conjugate subgroups have conjugate centralisers and normalisers, we may suppose for our purposes that  $F = \langle \alpha_i^m \rangle$ .

## 1.1 Type I subgroups of the form $\mathbb{Z}_q \times \mathbb{Z}$

We first study the centralisers of powers of the  $\alpha_i$ ,  $i \in \{0, 1, 2\}$ , which will give rise to Type I subgroups of the form  $\mathbb{Z}_q \times \mathbb{Z}$ .

LEMMA 51. *Let  $n \geq 4$ , and let  $i \in \{0, 1, 2\}$ . Suppose that  $m \in \mathbb{N}$  divides  $2(n - i)$ , and let*

$$r = \begin{cases} m & \text{if } m \mid n - i \\ \frac{m}{2} & \text{if } m \nmid n - i. \end{cases}$$

Then:

- (a)  $r \mid n - i$ , and  $Z_{B_n(\mathbb{S}^2)}(\langle \alpha_i^r \rangle) = Z_{B_n(\mathbb{S}^2)}(\langle \alpha_i^m \rangle)$ .
- (b) If  $r = 1$  then  $Z_{B_n(\mathbb{S}^2)}(\langle \alpha_i^m \rangle) = \langle \alpha_i \rangle$ .
- (c) If  $r \geq 2$  then  $Z_{B_n(\mathbb{S}^2)}(\langle \alpha_i^m \rangle) \supset \langle \delta_{r,i} \rangle$ , where the element

$$\delta_{r,i} = \sigma_1 \sigma_{r+1} \cdots \sigma_{n-i-r+1} = \prod_{k=0}^{(n-i-r)/r} \sigma_{kr+1} \quad (45)$$

is of infinite order.

*Proof.*

(a) The statement clearly holds if  $m \mid n - i$ . So suppose that  $m \nmid n - i$ . Since  $qm = 2(n - i)$  for some  $q \in \mathbb{N}$ , we have  $q/2 = (n - i)/m$ . Thus  $q$  is odd,  $m$  is even,  $r = m/2$  is an integer and  $q = (n - i)/r$ , which proves the first part of the statement. For the second part, note first that  $Z_{B_n(\mathbb{S}^2)}(\langle \alpha_i^r \rangle) \subset Z_{B_n(\mathbb{S}^2)}(\langle \alpha_i^m \rangle)$ . Conversely, suppose that  $z \in B_n(\mathbb{S}^2)$  commutes with  $\alpha_i^m$ . Then  $z$  commutes with  $\alpha_i^m \Delta_n^2 = \alpha_i^{n+m-i}$  by equation (8). Further,  $\alpha_i^m$  is of order  $q$ , which is odd. Hence  $\alpha_i^m \Delta_n^2$  is of order  $2q$ . Since  $\alpha_i^m \Delta_n^2 \in \langle \alpha_i \rangle$  and  $|\langle \alpha_i^r \rangle| = 2q$ , we have  $\langle \alpha_i^m \Delta_n^2 \rangle = \langle \alpha_i^r \rangle$ , so  $z$  commutes with  $\alpha_i^r$ , and this completes the proof of part (a).

(b) If  $r = 1$  then  $Z_{B_n(\mathbb{S}^2)}(\langle \alpha_i^m \rangle) = Z_{B_n(\mathbb{S}^2)}(\langle \alpha_i \rangle) = \langle \alpha_i \rangle$  by part (a) and Proposition 8.

(c) Suppose that  $r \geq 2$ . We first show that  $\delta_{r,i}$  is of infinite order. Assume on the contrary that  $\delta_{r,i}$  is of finite order. By Theorem 1, there exist  $l \in \{0, 1, 2\}$  and  $0 \leq \mu < 2(n - i)$  such that  $\delta_{r,i}$  is conjugate to  $\alpha_i^\mu$ . Since  $r \geq 2$ , the permutation  $\pi(\delta_{r,i})$  consists of the product of  $s$  disjoint transpositions, plus  $n - 2s$  fixed points, where  $s = (n - i)/r$ . In particular,  $\delta_{r,i} \notin P_n(\mathbb{S}^2)$ , so  $\delta_{r,i} \neq \Delta_n^2$ ,  $\mu \neq n - l$  by equation (8), and  $\pi(\delta_{r,i})$  has exactly  $l$  fixed points. Suppose first that  $l \in \{0, 2\}$ . Since  $\zeta(\alpha_l) = n - 1$  in  $\mathbb{Z}_{2(n-1)}$ ,  $\zeta(\delta_{r,i}) = \zeta(\alpha_l^\mu) = \bar{s}$  belongs to the subgroup  $\langle n - 1 \rangle$ , so there exists  $\lambda \in \mathbb{N}$  such that  $\lambda(n - 1) = s$ . But

$$n - 1 \leq \lambda(n - 1) = s = (n - i)/r \leq n/2,$$

so  $n \leq 2$ , which yields a contradiction. Hence  $l = 1$ ,  $\pi(\delta_{r,i})$  has a single fixed point, thus  $1 = n - 2s = (rs + i) - 2s = s(r - 2) + i$ , and  $i \in \{0, 1\}$ . If  $i = 0$  then  $s = 1$  and  $r = n = 3$ , which gives rise to a contradiction. So  $i = 1$ ,  $r = 2$ ,  $n = 2s + 1$  (which implies that  $n \geq 5$ ) and  $\delta_{r,i} = \delta_{2,1} = \sigma_1 \sigma_3 \cdots \sigma_{n-2}$ . But  $\delta_{2,1}$  belongs to the subgroup  $B_{n-1,1}(\mathbb{S}^2)$  of  $n$ -string braids whose permutation fixes the element  $n$ . Under the projection  $B_{n-1,1}(\mathbb{S}^2) \rightarrow$

$B_{n-1}(\mathbb{S}^2)$  given geometrically by forgetting the last string,  $\delta_{2,1}$  is sent to the element  $\delta_{2,0}$  of  $B_{n-1}(\mathbb{S}^2)$ , which must then also be of finite order. However, using the fact that  $n - 1 \geq 4$ , the above discussion implies that the element  $\delta_{2,0}$  of  $B_{n-1}(\mathbb{S}^2)$  is of infinite order, hence the element  $\delta_{2,1}$  of  $B_n(\mathbb{S}^2)$  is also of infinite order.

It remains to prove that  $\delta_{r,i}$  commutes with  $\alpha_i^m$ . By part (a), it suffices to show that it commutes with  $\alpha_i^r$ . First note that the product in equation (45) is taken over  $k = 0, 1, \dots, s - 1$ . If  $0 \leq k \leq s - 2$ , we have

$$1 \leq r + (kr + 1) \leq r(s - 1) + 1 = n - i - (r - 1) \leq n - i - 1 \quad \text{since } r \geq 2,$$

and hence  $\alpha_i^r \sigma_{kr+1} \alpha_i^{-r} = \sigma_{(k+1)r+1}$  by equation (23). If  $k = s - 1$  then

$$\begin{aligned} \alpha_i^r \sigma_{(s-1)r+1} \alpha_i^{-r} &= \alpha_i^r \sigma_{n-i-(r-1)} \alpha_i^{-r} = \alpha_i^2 \sigma_{n-i-1} \alpha_i^{-2} \quad \text{by equation (23)} \\ &= \sigma_1 \quad \text{by equation (24)}. \end{aligned}$$

Since  $r \geq 2$ , the terms  $\sigma_{kr+1}$ ,  $0 \leq k \leq s - 1$ , commute pairwise, and so

$$\alpha_i^r \delta_{r,i} \alpha_i^{-r} = \alpha_i^r \left( \prod_{k=0}^{s-1} \sigma_{kr+1} \right) \alpha_i^{-r} = \left( \prod_{k=1}^{s-2} \sigma_{kr+1} \right) \sigma_1 = \delta_{r,i},$$

using the previous calculations. This completes the proof of the proposition.  $\square$

**PROPOSITION 52.** *Let  $n \geq 4$ , and let  $q \in \mathbb{N}$ . Then  $B_n(\mathbb{S}^2)$  possesses a subgroup isomorphic to  $\mathbb{Z} \times \mathbb{Z}_q$  if and only if there exists  $i \in \{0, 1, 2\}$  such that the following three conditions are satisfied:*

- (i)  $q$  divides  $2(n - i)$ .
- (ii)  $1 \leq q \leq n - i$ .
- (iii)  $q < n - i$  if  $n - i$  is odd.

*Proof.* The necessity of conditions (i)–(iii) was proved in Proposition 45. Conversely, suppose that there exists  $i \in \{0, 1, 2\}$  such that the conditions (i)–(iii) are satisfied. Then  $m = 2(n - i)/q$  is an integer greater than or equal to two. Consider the subgroup  $\langle \alpha_i^m \rangle$  of  $B_n(\mathbb{S}^2)$ , which is isomorphic to  $\mathbb{Z}_q$ . With the notation of Lemma 51:

- if  $m \mid n - i$  then  $r = m \geq 2$ .
- if  $m \nmid n - i$  then  $q$  is odd,  $m$  is even and  $r = m/2$ . If  $r = 1$  then  $m = 2$  and so  $q = n - i$ , but this contradicts condition (iii). Hence  $r \geq 2$ .

So by Lemma 51(c),  $\delta_{r,i} \in Z_{B_n(\mathbb{S}^2)}(\langle \alpha_i^m \rangle)$ , and thus the subgroup  $\langle \alpha_i^m, \delta_{r,i} \rangle$  of  $B_n(\mathbb{S}^2)$  is isomorphic to  $\mathbb{Z} \times \mathbb{Z}_q$  as required.  $\square$

## 1.2 Type I subgroups of the form $\mathbb{Z}_q \rtimes_{\rho} \mathbb{Z}$

In this section, we consider the realisation in  $B_n(\mathbb{S}^2)$  of Type I groups  $\mathbb{Z}_q \rtimes_{\rho} \mathbb{Z}$ , where  $\rho \in \text{Hom}(\mathbb{Z}, \text{Aut}(\mathbb{Z}_q))$ , and  $\rho(1)$  is multiplication by  $-1$ .

PROPOSITION 53. Let  $n \geq 4$ , and let  $q \in \mathbb{N}$ . Then  $B_n(\mathbb{S}^2)$  possesses a subgroup isomorphic to  $\mathbb{Z}_q \rtimes_{\theta} \mathbb{Z}$ , for some action  $\theta \in \text{Hom}(\mathbb{Z}, \text{Aut}(\mathbb{Z}_q))$ ,  $\theta(1) \neq \text{Id}_{\mathbb{Z}_q}$ , if and only if the following conditions are satisfied:

- (i)  $q$  divides  $2(n - i)$ , where  $i \in \{0, 2\}$ .
- (ii)  $3 \leq q \leq n - i$ , and  $q < n - i$  if  $n$  is odd.
- (iii)  $\theta(1) = \rho(1)$ .

*Proof.* Suppose first that  $B_n(\mathbb{S}^2)$  possesses a subgroup isomorphic to  $\mathbb{Z}_q \rtimes_{\theta} \mathbb{Z}$ , where  $\theta(1) \neq \text{Id}_{\mathbb{Z}_q}$ . Proposition 45 implies that conditions (i)–(iii) are satisfied (note that if  $q \in \{1, 2\}$  then  $\theta(1) = \text{Id}_{\mathbb{Z}_q}$ ).

Conversely, suppose that conditions (i)–(iii) are satisfied, and let  $m = 2(n - i)/q$ . From the proof of Proposition 52, and making use of the notation of Lemma 51, we know that  $r \geq 2$  and that  $\langle \alpha_i^m, \delta_{r,i} \rangle$  is isomorphic to  $\mathbb{Z}_q \times \mathbb{Z}$ . We will modify slightly the generator  $\delta_{r,i}$  of the  $\mathbb{Z}$ -factor in order to obtain an action on  $\alpha_i^m$  that is multiplication by  $-1$ . To achieve this, let  $\Delta'_n = \alpha_0^{-1} \Delta_n \alpha_0$ . Equation (10) implies that  $\alpha_i^{-1} = \Delta'_n \alpha_i \Delta_n^{-1}$ . Now let  $\delta'_{r,i} = \Delta'_n \delta_{r,i}$ . Since  $\delta_{r,i}$  commutes with  $\alpha_i^m$ , we have that  $\delta'_{r,i} \alpha_i^m \delta'^{-1}_{r,i} = \alpha_i^{-m}$ , which will give rise to the required action on  $\langle \alpha_i^m \rangle$ . We claim that  $\delta'_{r,i}$  is of infinite order. This being the case, the subgroup  $\langle \delta'_{r,i}, \alpha_i^m \rangle$  of  $B_n(\mathbb{S}^2)$  is isomorphic to  $\mathbb{Z}_q \rtimes_{\rho} \mathbb{Z}$ , where  $\rho(1) = -\text{Id}_{\mathbb{Z}_q}$ , which will prove that the conditions (i)–(iii) are sufficient. To prove the claim, first note that since  $\Delta_n^2$  is central and of order 2, it suffices to prove that  $\beta = \delta'^2_{r,i} \Delta_n^{-2}$  is of infinite order. Further:

$$\begin{aligned} \beta &= (\Delta'_n \delta_{r,i})^2 \Delta_n^{-2} = \alpha_0^{-1} \Delta_n \alpha_0 \delta_{r,i} \alpha_0^{-1} \Delta_n^{-1} \alpha_0 \delta_{r,i} \\ &= \alpha_0^{-2} \Delta_n \sigma_1 \sigma_{r+1} \cdots \sigma_{n-i-2r+1} \sigma_{n-i-r+1} \Delta_n^{-1} \alpha_0^2 \delta_{r,i} \text{ by equations (10) } (i = 0), \text{ and (45)} \\ &= \alpha_0^{-2} \sigma_{i+r-1} \sigma_{i+2r-1} \cdots \sigma_{n-r-1} \sigma_{n-1} \alpha_0^2 \delta_{r,i} \text{ using equation (7) and the fact that } r \geq 2 \\ &= \begin{cases} \alpha_0^{-2} \sigma_{i+r-1} \alpha_0^2 \cdot \sigma_{i+2r-3} \cdots \sigma_{n-r-3} \sigma_{n-3} \delta_{r,i} & \text{if } i + r \leq 3 \\ \sigma_{i+r-3} \sigma_{i+2r-3} \cdots \sigma_{n-r-3} \sigma_{n-3} \sigma_1 \sigma_{r+1} \cdots \sigma_{n-2r-i+1} \sigma_{n-r-i+1} & \text{if } i + r \geq 4, \end{cases} \end{aligned}$$

using equation (23). We distinguish these two cases:

(a)  $i + r \geq 4$ . Then  $n - (i + r) + 1 \leq n - 3$ , and the last two strings of  $\beta$  are vertical. If  $\beta$  were of finite order, it would have to be conjugate to a power of  $\alpha_2$  using Theorem 1 (observe that this is also the case if  $\beta$  is pure, since the only nontrivial torsion element of  $P_n(\mathbb{S}^2)$  is  $\Delta_n^2$ , which is a power of  $\alpha_2$  by equation (8)), and so its Abelianisation  $\zeta(\beta)$  would be congruent to 0 modulo  $n - 1$ . On the other hand,  $\zeta(\beta) = \zeta(\delta'^2_{r,i})$  is congruent to  $2(n - i)/r \pmod{2(n - 1)}$ . So there exists  $\lambda \in \mathbb{N}$  such that  $2(n - i)/r = \lambda(n - 1)$ . Hence  $\lambda r(n - 1) = 2(n - i) = 2(n - 1) + 2(1 - i)$ , and since  $1 - i \in \{1, -1\}$ , this implies that  $n - 1 \mid 2$ , which is impossible. So  $\beta$  is of infinite order.

(b)  $i + r \leq 3$ . Since  $r \geq 2$  and  $i \in \{0, 2\}$ , we must have  $i = 0$  and  $r \in \{2, 3\}$ . Suppose first that  $r = 3$ . Using equation (23), we obtain:

$$\begin{aligned} \beta &= \alpha_0^{-1} (\alpha_0^{-1} \sigma_2 \sigma_5 \cdots \sigma_{n-4} \sigma_{n-1} \alpha_0) (\alpha_0 \sigma_1 \sigma_4 \cdots \sigma_{n-5} \sigma_{n-2} \alpha_0^{-1}) \alpha_0 \\ &= \alpha_0^{-1} (\sigma_1 \sigma_2 \sigma_4 \sigma_5 \cdots \sigma_{n-5} \sigma_{n-4} \sigma_{n-2} \sigma_{n-1}) \alpha_0. \end{aligned}$$

Now  $3 \mid n$  by Lemma 51(a) and  $n \geq 4$ , so  $n \geq 6$ . Thus the projection of  $\alpha_0 \beta \alpha_0^{-1}$  onto the first six strings is the braid  $\beta' = \sigma_1 \sigma_2 \sigma_4 \sigma_5 \in B_6(\mathbb{S}^2)$ . If  $\beta'$  were of finite order, by Theorem 1, it would be conjugate in  $B_6(\mathbb{S}^2)$  to some power of  $\alpha_0 = \sigma_1 \cdots \sigma_5$  (because its permutation has no fixed point), so its exponent sum would be congruent to 5 modulo 10. But this is clearly not the case, and so  $\beta'$  and  $\beta$  are of infinite order in their respective groups. Now assume that  $r = 2$ . Then

$$\beta = \alpha_0^{-2} \sigma_1 \sigma_3 \cdots \sigma_{n-3} \sigma_{n-1} \alpha_0^2 \sigma_1 \sigma_3 \cdots \sigma_{n-3} \sigma_{n-1} = \sigma_1^2 \sigma_3^2 \cdots \sigma_{n-3}^2 \sigma_{n-1}^2$$

by equations (23) and (24). The projection of  $\beta$  onto  $B_4(\mathbb{S}^2)$  by forgetting all but the first four strings gives rise to the element  $\sigma_1^2 \sigma_3^2$  of  $P_4(\mathbb{S}^2)$ , which is equal to  $\delta_{2,0}^2$  by equation (45), and so is of infinite order by Lemma 51(c). This implies that  $\beta$  is also of infinite order.

So in both cases,  $\beta$  is of infinite order, and hence so is  $\delta'_{r,i}$ . This completes the proof of the claim, and thus that of the proposition.  $\square$

## 2 Type I subgroups of $B_n(\mathbb{S}^2)$ of the form $F \rtimes \mathbb{Z}$ with $F$ dicyclic, $F \neq Q_8$

Let  $n \geq 4$  and  $i \in \{0, 2\}$ . In this section, we consider the realisation in  $B_n(\mathbb{S}^2)$  of Type I subgroups of the form  $F \rtimes_{\theta} \mathbb{Z}$ , where  $F \cong \text{Dic}_{4s}$ ,  $s \geq 3$ . By Proposition 45, there are two possible actions of  $\mathbb{Z}$  on  $\text{Dic}_{4s}$  to be considered. The trivial action, given by Definition 4(c) will be analysed in Proposition 54, while the nontrivial action, given by Definition 4(d) will be studied in Proposition 56.

**PROPOSITION 54.** *Let  $n \geq 4$  and let  $s \geq 3$ . Then  $\text{Dic}_{4s} \times \mathbb{Z}$  is realised as a subgroup of  $B_n(\mathbb{S}^2)$  if and only if there exists  $i \in \{0, 2\}$  such that  $s$  divides  $n - i$  strictly.*

**REMARK 55.** In other words, if  $i \in \{0, 2\}$  and  $s \geq 3$  divides  $n - i$  then  $\text{Dic}_{4s} \times \mathbb{Z}$  is realised as a subgroup of  $B_n(\mathbb{S}^2)$  if and only if  $\text{Dic}_{4s}$  is non maximal. Further, the value of  $i \in \{0, 2\}$  is unique since  $s \geq 3$ .

*Proof of Proposition 54.* The necessity of the condition was shown in the proof of Proposition 45(b)(i). Conversely, suppose that  $i \in \{0, 2\}$ , let  $s \geq 3$  be such that  $s$  divides  $n - i$  strictly, so  $s \leq (n - i)/2$ . Set  $m = (n - i)/s \geq 2$ . Then  $2 \leq m \leq (n - i)/3$ . Consider the subgroup  $\langle \alpha_i^m, \rho^m \rangle$ , where

$$\begin{aligned} \rho &= (\sigma_1 \cdots \sigma_{m-1})(\sigma_{m+1} \cdots \sigma_{2m-1}) \cdots (\sigma_{(s-1)m+1} \cdots \sigma_{sm-1}) \\ &= \prod_{j=1}^s (\sigma_{(j-1)m+1} \cdots \sigma_{jm-1}). \end{aligned} \quad (46)$$

We claim that the bracketed terms of equation (46) are permuted cyclically under conjugation by  $\alpha_i^m$ . To prove the claim, first suppose that  $j \in \{1, \dots, s-1\}$ . Since  $jm - 1 + m = (j+1)m - 1 \leq sm - 1 = n - i - 1$ , it follows from equation (23) that

$$\alpha_i^m \left( \sigma_{(j-1)m+1} \cdots \sigma_{jm-1} \right) \alpha_i^{-m} = \sigma_{jm+1} \cdots \sigma_{(j+1)m-1}. \quad (47)$$

Now suppose that  $j = s$ . Then

$$\begin{aligned}
\alpha_i^m (\sigma_{(s-1)m+1} \cdots \sigma_{sm-1}) \alpha_i^{-m} &= \left( \prod_{k=1}^{m-1} \alpha_i^m \sigma_{n-i-m+k} \alpha_i^{-m} \right) \\
&= \left( \prod_{k=1}^{m-1} \alpha_i^{k+1} \sigma_{n-i-1} \alpha_i^{-(k+1)} \right) \\
&= \left( \prod_{k=1}^{m-1} \alpha_i^{k-1} \sigma_1 \alpha_i^{-(k-1)} \right) = \sigma_1 \cdots \sigma_{m-1}, \tag{48}
\end{aligned}$$

by equations (23) and equation (24). The claim then follows from equations (47) and (48). The fact that the bracketed terms of equation (46) commute pairwise implies that  $\alpha_i^m$  and  $\rho$  commute, and that  $\rho^m \in P_n(\mathbb{S}^2)$ . If  $\rho^m$  were of finite order then  $\rho^m \in \langle \Delta_n^2 \rangle$ , so  $\xi(\rho^m) \equiv n-1$  modulo  $2(n-1)$ , and the exponent sum of  $\rho^m$  would be congruent to 0 modulo  $n-1$ . On the other hand, the exponent sum of  $\rho^m$  modulo  $n-1$  is equal to:

$$\begin{aligned}
\overline{sm(m-1)} &= \overline{(n-i)(m-1)} \\
&= \overline{(n-1)(m-1)} + \overline{(1-i)(m-1)} = \overline{(1-i)(m-1)}. \tag{49}
\end{aligned}$$

Since  $1-i \in \{1, -1\}$ ,  $n-1$  would thus divide  $m-1$ , which is not possible because  $2 \leq m \leq (n-i)/3 < n$ . Thus  $\rho$  is of infinite order, and hence  $\langle \alpha_i^m, \rho^m \rangle \cong \mathbb{Z}_{2s} \times \mathbb{Z}$ .

Using the element  $\Delta_n$  and the subgroup  $\langle \alpha_i^m, \rho^m \rangle$ , we will now construct a subgroup isomorphic to  $\text{Dic}_{4s} \times \mathbb{Z}$ , which will complete the proof of the proposition. First note that for all  $1 \leq j_1 < j_2 \leq n-1$ , the relation

$$(\sigma_{j_1} \cdots \sigma_{j_2-1})^{j_2-j_1+1} = (\sigma_{j_2-1} \sigma_{j_2-2} \cdots \sigma_{j_1})^{j_2-j_1+1} \tag{50}$$

holds in  $B_n$  (cf. [MK, Chapter 2, Exercise 4.1]), and using the fact that  $B_{j_2-j_1}$  embeds in  $B_n$ , and so holds in  $B_n(\mathbb{S}^2)$ . Now

$$\begin{aligned}
\Delta_n \rho^m \Delta_n^{-1} &= \Delta_n \left( \prod_{j=1}^s (\sigma_{(j-1)m+1} \cdots \sigma_{jm-1})^m \right) \Delta_n^{-1} \\
&= \left( \prod_{j=1}^s (\sigma_{m(s-j+1)+i-1} \cdots \sigma_{m(s-j)+i+1})^m \right) \text{ by equation (7)} \\
&= \left( \prod_{j=1}^s (\sigma_{m(s-j)+1} \cdots \sigma_{m(s-j+1)-1})^m \right) \text{ by equation (50)} \\
&= \alpha_0^i \left( \prod_{j=1}^s (\sigma_{m(s-j)+1} \cdots \sigma_{m(s-j+1)-1})^m \right) \alpha_0^{-i} \text{ by equation (23)} \\
&= \alpha_0^i \left( \prod_{j'=1}^s (\sigma_{m(j'-1)+1} \cdots \sigma_{mj'-1})^m \right) \alpha_0^{-i} = \alpha_0^i \rho^m \alpha_0^{-i},
\end{aligned}$$

taking  $j' = s - j + 1$ , and using also the fact that the inner bracketed terms commute pairwise. It follows from equation (10) that  $\Delta_n$  commutes with the element  $\rho^{jm} = \alpha_0^{j/2} \rho^m \alpha_0^{-j/2}$ . Since  $\langle \alpha_i^{jm}, \Delta_n \rangle \cong \text{Dic}_{4s}$  where  $\alpha_i^{jm} = \alpha_0^{j/2} \alpha_i^m \alpha_0^{-j/2}$ , it follows that the group  $\langle \alpha_i^{jm}, \Delta_n, \rho^{jm} \rangle$  is isomorphic to  $\text{Dic}_{4s} \times \mathbb{Z}$  as required.  $\square$

We now turn our attention to the other possible action in  $B_n(\mathbb{S}^2)$  of  $\mathbb{Z}$  on the dicyclic subgroups.

**PROPOSITION 56.** *Let  $n \geq 4$  and  $s \geq 3$ , and consider the Type I group  $G = \text{Dic}_{4s} \rtimes_{\nu} \mathbb{Z}$ , where  $\nu$  is defined by equation (12). Then  $B_n(\mathbb{S}^2)$  possesses a subgroup isomorphic to  $G$  if and only if the following two conditions are satisfied:*

- (i)  $s$  divides  $n - i$  for some  $i \in \{0, 2\}$ , and
- (ii)  $(n - i)/s$  is even.

*Proof.* The necessity of the conditions was obtained in part (b)(ii) of the proof of Proposition 45. Conversely, suppose that conditions (i) and (ii) hold. Set  $m = (n - i)/s$ , and let  $\alpha'_i = \alpha_0 \alpha_i \alpha_0^{-1} = \alpha_0^{i/2} \alpha_i \alpha_0^{-i/2}$ . Since  $m/2 \in \mathbb{N}$  by condition (ii), we may consider the subgroup  $\langle \alpha_i^{m/2}, \Delta_n \rangle$  of  $B_n(\mathbb{S}^2)$  which is a dicyclic subgroup (of order  $8s$ ) of the standard copy of  $\text{Dic}_{4(n-i)}$ , and which contains the dicyclic subgroup  $\langle \alpha_i^m, \Delta_n \rangle$  of order  $4s$ . Taking  $x = \alpha_i^m$  and  $y = \Delta_n$ , the action by conjugation of  $\alpha_i^{m/2}$  on  $\langle x, y \rangle$  coincides with that given by  $\theta(1)$  in the statement of the proposition. From the proof of Proposition 54, the subgroup  $\langle \alpha_i^m, \Delta_n, \rho^{2m} \rangle$  is isomorphic to  $\text{Dic}_{4s} \times \mathbb{Z}$ ,  $\rho^{2m}$  being as defined in that proof. We claim that  $\alpha_i^{m/2} \rho^{2m}$  is of infinite order. This being the case, the subgroup  $\langle \alpha_i^m, \Delta_n, \alpha_i^{m/2} \rho^{2m} \rangle$  is isomorphic to  $\text{Dic}_{4s} \rtimes_{\theta} \mathbb{Z}$ , which will complete the proof of the proposition. To prove the claim, we suppose that  $\alpha_i^{m/2} \rho^{2m}$  is of finite order, and argue for a contradiction. Since  $\rho^{2m} \in P_n(\mathbb{S}^2)$ ,  $\pi(\alpha_i^{m/2} \rho^{2m}) = \pi(\alpha_i^{m/2})$ . Now  $\alpha_i^{m/2}$  is of order  $4s$ , and the cycle decompositions of  $\pi(\alpha_i^{m/2})$  and  $\pi(\alpha_i^{m/2} \rho^{2m})$  consist of  $m/2$   $2s$ -cycles (and  $i$  fixed elements). The fact that the finite order elements of  $P_n(\mathbb{S}^2)$  are the elements of  $\langle \Delta_n^2 \rangle$  then implies that  $\alpha_i^{m/2} \rho^{2m}$  is of order  $ks$ , where  $k \in \{2, 4\}$ . Now  $\alpha_i^{2m/k}$  also generates a subgroup of order  $ks$ , and since  $ks \geq 6$ , by [GG5, Proposition 1.5(2)], there is a single conjugacy class of such subgroups in  $B_n(\mathbb{S}^2)$ . So there exist  $\gamma \in B_n(\mathbb{S}^2)$  and  $\lambda \in \mathbb{N}$ , with  $\text{gcd}(\lambda, 2s) = 1$ , such that  $\alpha_i^{2m\lambda/k} = \gamma \alpha_i^{m/2} \rho^{2m} \gamma^{-1}$ . But  $\zeta(\alpha'_i) \equiv 0$  modulo  $n - 1$ , and so it follows that  $\zeta(\rho^{2m}) = \zeta(\rho^m) \equiv 0$  modulo  $n - 1$ . But using equation (49), we saw in the proof of Proposition 54 that this is not the case. This yields a contradiction, and proves the claim.  $\square$

### 3 Type I subgroups of $B_n(\mathbb{S}^2)$ of the form $\mathcal{Q}_8 \rtimes \mathbb{Z}$

The aim of this section is to prove the existence of Type I subgroups of  $B_n(\mathbb{S}^2)$  of the form  $\mathcal{Q}_8 \rtimes \mathbb{Z}$ . As we saw in Lemma 36, up to isomorphism it suffices to consider the two actions  $\alpha$  and  $\beta$  defined in Definition 4(1)(e), of order 3 and 2 respectively. We start by showing that the existence of the Type I subgroup  $T^* \times \mathbb{Z}$  (resp. of  $T^* \rtimes_{\omega} \mathbb{Z}$ , for the

nontrivial action  $\omega$  given by equation (14)) implies that of  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  (resp. of  $\mathcal{Q}_8 \rtimes_{\beta} \mathbb{Z}$ ). Using the results of Section II.4, this will imply the existence of  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  and  $\mathcal{Q}_8 \rtimes_{\beta} \mathbb{Z}$  as subgroups of  $B_n(\mathbb{S}^2)$  for most even values of  $n$ . In the second part of this section, we exhibit explicit algebraic constructions of  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  (resp.  $\mathcal{Q}_8 \rtimes_{\beta} \mathbb{Z}$ ) for all  $n \equiv 0 \pmod{4}$ ,  $n \geq 8$  (resp. all  $n \geq 4$  even).

PROPOSITION 57.

- (a) The group  $T^* \times \mathbb{Z}$  possesses a subgroup isomorphic to  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$ .  
(b) The group  $T^* \rtimes_{\omega} \mathbb{Z}$  for the action defined by equation (14) possesses a subgroup isomorphic to  $\mathcal{Q}_8 \rtimes_{\beta} \mathbb{Z}$ .

*Proof.* Consider  $T^* = \mathcal{Q}_8 \rtimes \mathbb{Z}_3$  given by the presentation (13).

- (a) Let  $G = T^* \times \mathbb{Z}$ , and let  $Z$  be the generator of the  $\mathbb{Z}$ -factor. Since  $X$  and  $Z$  commute, the group  $\langle XZ \rangle$  is of infinite order and its action on  $\mathcal{Q}_8$  by conjugation permutes cyclically the elements  $P, Q$  and  $PQ$  of  $\langle P, Q \rangle$ . Hence  $\langle P, Q, XZ \rangle \cong \mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$ , where  $\alpha$  is as defined in Definition 4(1)(e).  
(b) Let  $G = T^* \rtimes_{\omega} \mathbb{Z}$ , let  $Z$  be the generator of the  $\mathbb{Z}$ -factor. The action of  $Z$  on  $T^*$  by conjugation coincides with that of equation (14). The restriction of this action to  $\langle P, Q \rangle$  exchanges  $P$  and  $QP$ , and sends  $Q$  to  $Q^{-1}$ . Thus  $\langle P, Q, Z \rangle \cong \mathcal{Q}_8 \rtimes_{\beta} \mathbb{Z}$ , where  $\beta$  is as defined in Definition 4(1)(e).  $\square$

REMARK 58. The realisation of  $T^* \times \mathbb{Z}$  (resp.  $T^* \rtimes_{\omega} \mathbb{Z}$ ) as a subgroup of  $B_n(\mathbb{S}^2)$  for  $n$  even and satisfying  $n = 12$  or  $n \geq 16$  (resp.  $n \equiv 0, 2 \pmod{6}$  and satisfying  $n = 24$  or  $n \geq 30$ ) will follow from Propositions 62 and 66. Proposition 57 then implies the existence of  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  (resp.  $\mathcal{Q}_8 \rtimes_{\beta} \mathbb{Z}$ ) as a subgroup of  $B_n(\mathbb{S}^2)$  for these values of  $n$ .

We now turn our attention to the problem of the algebraic realisation of Type I subgroups of the form  $\mathcal{Q}_8 \rtimes \mathbb{Z}$ . In most cases, the existence of these subgroups follows by combining Proposition 57 with Propositions 62 and 66. As we shall see later, we will prove these two propositions using geometric constructions in  $\mathcal{MCG}(\mathbb{S}^2, n)$ . Before doing so, we exhibit explicit algebraic representations in terms of the standard generators of  $B_n(\mathbb{S}^2)$ , and in some cases, we obtain their existence for values of  $n$  that are not covered by these propositions. We start by defining certain elements that shall be used in the constructions, and in Lemma 59, we give some of their properties. Let  $n \geq 4$  be even, and let

$$\Omega_1 = \prod_{i=1}^{n/2-1} \sigma_1 \cdots \sigma_{n/2-i} \quad \text{and} \quad \Omega_2 = \prod_{i=1}^{n/2-1} \sigma_{n/2+1} \cdots \sigma_{n-i}. \quad (51)$$

Clearly  $\Omega_1$  and  $\Omega_2$  commute, and using equations (8) and (23), we see that

$$\alpha_0^{n/2} \Omega_1 \alpha_0^{-n/2} = \Omega_2 \quad \text{and} \quad \alpha_0^{n/2} \Omega_2 \alpha_0^{-n/2} = \Omega_1. \quad (52)$$

For  $i = 1, \dots, n/2$ , set

$$\rho_i = \sigma_i \cdots \sigma_{i+n/2-1}, \quad (53)$$



and

$$\rho = \rho_{n/2} \cdots \rho_1. \quad (54)$$

Geometrically,  $\Omega_1$  (resp.  $\Omega_2$ ) is the half twist on the first (resp. second)  $n/2$  strings, and  $\rho$  is the braid that passes the first  $n/2$  strings over the second  $n/2$  strings (see Figures 5, 6 and 7).

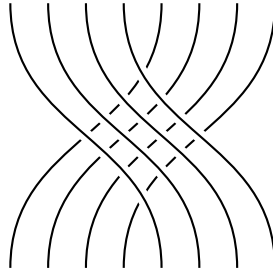


Figure 5: The braid  $\rho$  in  $B_8(\mathbb{S}^2)$ .

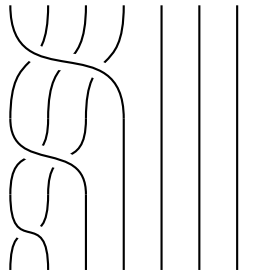


Figure 6: The braid  $\Omega_1$  in  $B_8(\mathbb{S}^2)$ .

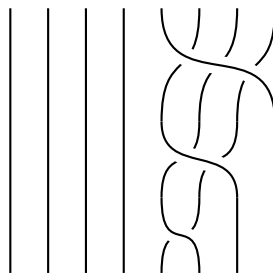


Figure 7: The braid  $\Omega_2$  in  $B_8(\mathbb{S}^2)$ .

LEMMA 59. *With the above notation, the following relations hold:*

- (a)  $\rho\Omega_1 = \Omega_2\rho$ .
- (b)  $\Delta_n = \Omega_1\Omega_2\rho$ .
- (c)  $\rho\Omega_2 = \Omega_1\rho$ .
- (d)  $\Omega_2 = \sigma_{n-1}(\sigma_{n-2}\sigma_{n-1}) \cdots (\sigma_{n/2+2} \cdots \sigma_{n-1})(\sigma_{n/2+1} \cdots \sigma_{n-1})$ .

- (e)  $\alpha_0^{n/2} = \Omega_1^2 \rho$ .  
(f)  $\Delta_n = \Omega_2 \alpha_0^{n/2} \Omega_2^{-1}$  and  $\alpha_0^{n/2} = \Omega_1 \Delta_n \Omega_1^{-1}$ .  
(g)  $\Delta_n^2 = \Omega_1^2 \Omega_2^{-2}$ .

*Proof.*

(a) First observe that

$$\begin{aligned}
\rho_1 \Omega_1 \rho_1^{-1} &= \sigma_1 \cdots \sigma_{n/2} \left( \prod_{i=1}^{n/2-1} \sigma_1 \cdots \sigma_{n/2-i} \right) \sigma_{n/2}^{-1} \cdots \sigma_1^{-1} \\
&= \sigma_1 \cdots \sigma_{n-1} \left( \prod_{i=1}^{n/2-1} \sigma_1 \cdots \sigma_{n/2-i} \right) \sigma_{n-1}^{-1} \cdots \sigma_1^{-1} \\
&= \alpha_0 \Omega_1 \alpha_0^{-1}.
\end{aligned} \tag{55}$$

For  $i = 1, \dots, n/2$ , we have

$$\begin{aligned}
\rho_i \alpha_0^{i-1} \Omega_1 \alpha_0^{-(i-1)} \rho_i^{-1} &= \alpha_0^{i-1} (\alpha_0^{-(i-1)} \rho_i \alpha_0^{i-1}) \Omega_1 (\alpha_0^{-(i-1)} \rho_i^{-1} \alpha_0^{i-1}) \alpha_0^{-(i-1)} \\
&= \alpha_0^{i-1} \rho_1 \Omega_1 \rho_1^{-1} \alpha_0^{-(i-1)} \quad \text{by equations (23) and (53)} \\
&= \alpha_0^i \Omega_1 \alpha_0^{-i} \quad \text{by equation (55)}.
\end{aligned}$$

By induction on  $i$ , equations (52) and (55), it follows that

$$\rho \Omega_1 \rho^{-1} = \rho_{n/2} \cdots \rho_1 \Omega_1 \rho_1^{-1} \cdots \rho_n^{-1} = \alpha_0^{n/2} \Omega_1 \alpha_0^{-n/2} = \Omega_2$$

as required.

(b) We have:

$$\begin{aligned}
\Delta_n &= \prod_{i=1}^{n-1} (\sigma_1 \cdots \sigma_{n-i}) = \prod_{i=1}^{n/2-1} (\sigma_1 \cdots \sigma_{n-i}) \prod_{i=n/2}^{n-1} (\sigma_1 \cdots \sigma_{n-i}) \quad \text{by equation (6)} \\
&= \prod_{i=1}^{n/2-1} (\sigma_1 \cdots \sigma_{n/2-i}) (\sigma_{n/2-i+1} \cdots \sigma_{n-i}) \prod_{i=0}^{n/2-1} (\sigma_1 \cdots \sigma_{n/2-i}) \\
&= \left( \prod_{i=1}^{n/2-1} (\sigma_1 \cdots \sigma_{n/2-i}) \prod_{i=1}^{n/2-1} (\sigma_{n/2-i+1} \cdots \sigma_{n-i}) \right) \rho_1 \prod_{i=1}^{n/2-1} (\sigma_1 \cdots \sigma_{n/2-i}) \\
&= \Omega_1 \left( \prod_{i=1}^{n/2-1} \rho_{n/2-i+1} \right) \rho_1 \Omega_1 \quad \text{by equations (51) and (53)} \\
&= \Omega_1 \rho \Omega_1 = \Omega_1 \Omega_2 \rho \quad \text{by equations (53) and (54), and part (a)}.
\end{aligned}$$

(c) Since  $\alpha_0^{n/2} \Delta_n \alpha_0^{-n/2} = \Delta_n^{-1}$  by equations (8) and (10), we have:

$$\alpha_0^{n/2} \rho \alpha_0^{-n/2} = \alpha_0^{n/2} \Omega_2^{-1} \Omega_1^{-1} \Delta_n \alpha_0^{-n/2} = \Omega_1^{-1} \Omega_2^{-1} \Delta_n^{-1} = \rho \Delta_n^2,$$

by part (b) and equation (52), using the fact that  $\Omega_1$  and  $\Omega_2$  commute. Conjugating the relation  $\rho \Omega_1 = \Omega_2 \rho$  of part (a) by  $\alpha_0^{n/2}$  and using equation (52) gives the result.

(d) For  $n/2 + 1 \leq i \leq j \leq n - 1$ , set  $\tau_{i,j} = \sigma_i \cdots \sigma_j$  (so  $\tau_{i,i} = \sigma_i$ ). For  $k = 1, \dots, n/2 - 1$ , set

$$\omega_k = \left( \prod_{i=n/2-k}^{n/2-1} \tau_{n-i,n-1} \right) \left( \prod_{i=n/2-k}^{n/2-1} \tau_{n/2+1,n/2+i}^{-1} \right).$$

Let  $\Omega'_2 = \sigma_{n-1}(\sigma_{n-2}\sigma_{n-1}) \cdots (\sigma_{n/2+2} \cdots \sigma_{n-1})(\sigma_{n/2+1} \cdots \sigma_{n-1})$ . Since

$$\begin{aligned} \omega_{n/2-1} &= \left( \prod_{i=1}^{n/2-1} \tau_{n-i,n-1} \right) \left( \prod_{i=1}^{n/2-1} \tau_{n/2+1,n/2+i}^{-1} \right) \\ &= \Omega'_2 \Omega_2^{-1} \quad \text{by equation (51),} \end{aligned}$$

it suffices to show that  $\omega_{n/2-1} = 1$ . To do so, we shall prove by induction that  $\omega_k = 1$  for all  $k = 1, \dots, n/2 - 1$ . If  $k = 1$  then  $\omega_1 = \tau_{n/2+1,n-1} \tau_{n/2+1,n-1}^{-1} = 1$ . So suppose that  $\omega_k = 1$  for some  $k = 1, \dots, n/2 - 2$ . First note that if  $i \leq l < j$ ,

$$\begin{aligned} \tau_{i,j} \tau_{i,l}^{-1} &= (\sigma_i \cdots \sigma_l \sigma_{l+1} \sigma_{l+2} \cdots \sigma_j) (\sigma_l^{-1} \cdots \sigma_i^{-1}) \\ &= (\sigma_i \cdots \sigma_l \sigma_{l+1} \sigma_l^{-1} \cdots \sigma_i^{-1}) (\sigma_{l+2} \cdots \sigma_j) \\ &= \tau_{i,l+1} \sigma_l^{-1} \cdots \sigma_i^{-1} \tau_{i,l+1}^{-1} \tau_{i,j} = \sigma_{l+1}^{-1} \cdots \sigma_{i+1}^{-1} \tau_{i,j} \\ &= \tau_{i+1,l+1}^{-1} \tau_{i,j}, \end{aligned} \tag{56}$$

using the fact that  $\tau_{i,l+1} \tau_m \tau_{i,l+1}^{-1} = \tau_{m+1}$  for all  $i \leq m \leq l$ . So

$$\begin{aligned} \omega_{k+1} &= \tau_{n/2+k+1,n-1} \left( \prod_{i=n/2-k}^{n/2-1} \tau_{n-i,n-1} \right) \tau_{n/2+1,n-k-1}^{-1} \left( \prod_{i=n/2-k}^{n/2-1} \tau_{n/2+1,n/2+i}^{-1} \right) \\ &= \tau_{n/2+k+1,n-1} \cdot \tau_{n/2+k,n-1} \cdots \tau_{n/2+1,n-1} \tau_{n/2+1,n-k-1}^{-1} \left( \prod_{i=n/2-k}^{n/2-1} \tau_{n/2+1,n/2+i}^{-1} \right) \\ &= \tau_{n/2+k+1,n-1} \cdot \tau_{n/2+k+1,n-1}^{-1} \left( \prod_{i=n/2-k}^{n/2-1} \tau_{n-i,n-1} \right) \left( \prod_{i=n/2-k}^{n/2-1} \tau_{n/2+1,n/2+i}^{-1} \right) \\ &= \omega_k = 1, \end{aligned}$$

where we have used equation (56)  $k$  times to obtain the first equality of the last line. The result follows by induction.

(e) Using parts (c) and (d), and equations (51), (53) and (54), we have:

$$\begin{aligned}
\alpha_0^{n/2} &= (\sigma_1 \cdots \sigma_{n-1})^{n/2} = \prod_{i=1}^{n/2} (\sigma_1 \cdots \sigma_{n/2-i} \sigma_{n/2-i+1} \cdots \sigma_{n-1}) \\
&= \prod_{i=1}^{n/2-1} (\sigma_1 \cdots \sigma_{n/2-i}) \prod_{i=1}^{n/2} (\sigma_{n/2-i+1} \cdots \sigma_{n-1}) \\
&= \Omega_1 \prod_{i=1}^{n/2} (\sigma_{n/2-i+1} \cdots \sigma_{n-i} \sigma_{n-i+1} \cdots \sigma_{n-1}) \\
&= \Omega_1 \prod_{i=1}^{n/2} (\sigma_{n/2-i+1} \cdots \sigma_{n-i}) \prod_{i=2}^{n/2} (\sigma_{n-i+1} \cdots \sigma_{n-1}) \\
&= \Omega_1 \rho \prod_{i=1}^{n/2-1} (\sigma_{n-i} \cdots \sigma_{n-1}) = \Omega_1 \rho \Omega_2 = \Omega_1^2 \rho.
\end{aligned}$$

(f) Applying successively parts (e), (c) and (b) and using the fact that  $\Omega_1$  and  $\Omega_2$  commute yields:

$$\Omega_2 \alpha_0^{n/2} \Omega_2^{-1} = \Omega_2 \Omega_1^2 \rho \Omega_2^{-1} = \Omega_2 \Omega_1^2 \Omega_1^{-1} \rho = \Omega_1 \Omega_2 \rho = \Delta_n.$$

Further, using parts (b), (a) and (e), we have:

$$\Omega_1 \Delta_n \Omega_1^{-1} = \Omega_1 \Omega_1 \Omega_2 \rho \Omega_1^{-1} = \Omega_1^2 \rho = \alpha_0^{n/2}.$$

(g) Using equations (52) and (8) and part (f), we have

$$\begin{aligned}
\Omega_1^2 \Omega_2^{-2} &= \Omega_1^2 \alpha_0^{n/2} \Omega_1^{-2} \alpha_0^{-n/2} = \Omega_1 \left( \Omega_1 \alpha_0^{n/2} \Omega_1^{-1} \right) \Omega_1^{-1} \alpha_0^{-n/2} \\
&= \Omega_1 \left( \alpha_0^{n/2} \Omega_2 \alpha_0^{-n/2} \cdot \alpha_0^{n/2} \cdot \alpha_0^{n/2} \Omega_2^{-1} \alpha_0^{-n/2} \right) \Omega_1^{-1} \alpha_0^{-n/2} \\
&= \Omega_1 \left( \alpha_0^{n/2} \Delta_n \alpha_0^{-n/2} \right) \Omega_1^{-1} \alpha_0^{-n/2} = \Omega_1 \Delta_n^{-1} \Omega_1^{-1} \alpha_0^{-n/2} \\
&= \alpha_0^{-n/2} \cdot \alpha_0^{-n/2} = \Delta_n^2. \quad \square
\end{aligned}$$

PROPOSITION 60. *With the notation defined above,*

(a)  $\langle \alpha_0^{n/2}, \Delta_n, \alpha_0^{n/4} \Omega_2 \rangle \cong \mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  for all  $n \equiv 0 \pmod{4}$ ,  $n \geq 8$ .

(b)  $\langle \alpha_0^{n/2}, \Delta_n, \Omega_1 \Delta_n \rangle \cong \mathcal{Q}_8 \rtimes_{\beta} \mathbb{Z}$  for all  $n \geq 4$  even.

*Proof.* Remark 3(b) implies that the subgroup  $\langle \alpha_0^{n/2}, \Delta_n \rangle$  of  $B_n(\mathbb{S}^2)$  is isomorphic to  $\mathcal{Q}_8$ . So to prove the proposition, we must study the action of the third generator in both cases on this subgroup. Let  $\Omega_1$  and  $\Omega_2$  be as defined in equation (51).

(a) Let  $n \equiv 0 \pmod{4}$  where  $n \geq 8$ , and let  $\nu = \alpha_0^{n/4} \Omega_2$ . We have:

$$\begin{aligned} \nu \alpha_0^{n/2} \nu^{-1} &= (\alpha_0^{n/4} \Omega_2) \alpha_0^{n/2} (\Omega_2^{-1} \alpha_0^{-n/4}) = \alpha_0^{n/4} \Delta_n \alpha_0^{-n/4} \quad \text{by Lemma 59(f)} \\ &= \alpha_0^{n/2} \Delta_n \quad \text{by equation (10),} \end{aligned} \tag{57}$$

$$\begin{aligned} \nu \alpha_0^{n/2} \Delta_n \nu^{-1} &= \alpha_0^{n/2} \Delta_n \cdot \alpha_0^{n/4} \Omega_2 \Delta_n \Omega_2^{-1} \alpha_0^{-n/4} \quad \text{by equation (57)} \\ &= \alpha_0^{n/2} \Delta_n \cdot \alpha_0^{3n/4} \alpha_0^{-n/2} \Omega_2^2 \alpha_0^{n/2} \Omega_2^{-2} \alpha_0^{-n/4} \quad \text{by Lemma 59(f)} \\ &= \alpha_0^{n/2} \Delta_n \cdot \alpha_0^{3n/4} \Omega_1^2 \Omega_2^{-2} \alpha_0^{-n/4} \quad \text{by equation (52)} \\ &= \alpha_0^{n/2} \Delta_n \cdot \alpha_0^{n/2} \Delta_n^2 \quad \text{by Lemma 59(g)} \\ &= \Delta_n^{-1} \quad \text{by equations (8) and (10), and} \end{aligned} \tag{58}$$

$$\nu \Delta_n^{-1} \nu^{-1} = \nu (\alpha_0^{n/2} \Delta_n)^{-1} \alpha_0^{n/2} \nu^{-1} \tag{59}$$

$$= \alpha_0^{n/2} \quad \text{by equations (8), (10), (57) and (58).} \tag{60}$$

Hence conjugation by  $\nu$  permutes cyclically the elements  $\alpha_0^{n/2}$ ,  $\alpha_0^{n/2} \Delta_n$  and  $\Delta_n^{-1}$ , and thus gives rise to the action  $\alpha$  on the copy  $\langle \alpha_0^{n/2}, \Delta_n \rangle$  of  $\mathcal{Q}_8$  in  $B_n(\mathbb{S}^2)$ . It remains to show that  $\nu$  is of infinite order. Its permutation is:

$$\begin{aligned} \pi(\nu) &= (1, 3n/4 + 1, n/2 + 1, n/4 + 1)(2, 3n/4 + 2, n/2 + 2, n/4 + 2) \cdots \\ &\quad (n/4, n, 3n/4, n/2)(n/2 + 1, n)(n/2 + 2, n - 1) \cdots (3n/4, 3n/4 + 1), \end{aligned}$$

and since  $n \geq 8$ , the cycle decomposition of  $\pi(\nu)$  contains the transposition  $(3n/4 + 1, n)$  and the 6-cycle  $(1, 3n/4, n/2, n/4, n/2 + 1, n/4 + 1)$ . By Theorem 1,  $\pi(\nu)$  cannot be the permutation of an element of  $B_n(\mathbb{S}^2)$  of finite order. This shows that  $\nu$  is of infinite order, and so  $\langle \alpha_0^{n/2}, \Delta_n, \alpha_0^{n/4} \Omega_2 \rangle \cong \mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$ .

(b) Let  $n \geq 4$  be even. Set  $\zeta = \Omega_1 \Delta_n$ . Then:

$$\zeta \Delta_n \zeta^{-1} = \Omega_1 \Delta_n \Omega_1^{-1} = \alpha_0^{n/2} \quad \text{by Lemma 59(f),}$$

$$\zeta \alpha_0^{n/2} \zeta^{-1} = \Omega_1 \Delta_n \alpha_0^{n/2} \Delta_n^{-1} \Omega_1^{-1} \alpha_0^{-n/2} \alpha_0^{n/2}$$

$$= \Delta_n^2 \Omega_1 \Omega_2^{-1} \alpha_0^{n/2} \quad \text{by equations (8), (10) and (52)}$$

$$= \Delta_n^2 \alpha_0^{n/2} \Delta_n^{-1} \alpha_0^{n/2} \quad \text{by Lemma 59(b) and (e), and the commutativity of } \Omega_1 \text{ and } \Omega_2$$

$$= \Delta_n \quad \text{by equations (8) and (10),}$$

$$\zeta \alpha_0^{n/2} \Delta_n \zeta^{-1} = \Delta_n \alpha_0^{n/2} = (\alpha_0^{n/2} \Delta_n)^{-1} \quad \text{by equation (8) and the above two relations.}$$

So conjugation by  $\zeta$  exchanges  $\Delta_n$  and  $\alpha_0^{n/2}$ , and sends  $\alpha_0^{n/2} \Delta_n$  to  $(\alpha_0^{n/2} \Delta_n)^{-1}$ , hence gives rise to the action  $\beta$  on the copy  $\langle \alpha_0^{n/2}, \Delta_n \rangle$  of  $\mathcal{Q}_8$  in  $B_n(\mathbb{S}^2)$ . It remains to show that  $\zeta$  is of infinite order. Suppose first that  $n \equiv 2 \pmod{4}$ . Then:

$$\begin{aligned} \pi(\zeta) &= (1, n/2)(2, n/2 - 1) \cdots ((n-2)/4, (n+6)/4). (1, n). \\ &\quad (2, n-1) \cdots (n/2, n/2 + 1). \end{aligned}$$

By Theorem 1,  $\pi(\zeta)$  cannot be the permutation of a finite-order element of  $B_n(\mathbb{S}^2)$  since its cycle decomposition contains the transposition  $((n+2)/4, (3n+2)/4)$  and the 4-cycle  $(1, n/2+1, n/2, n)$ . So suppose that  $n \equiv 0 \pmod{4}$ . Then:

$$\pi(\zeta) = (1, n/2)(2, n/2-1) \cdots (n/4, n/4+1). (1, n)(2, n-1) \cdots (n/2, n/2+1).$$

Hence the cycle decomposition of  $\pi(\zeta)$  consists of the 4-cycles of the form  $(j, n/2+j, n/2+1-j, n+1-j)$ , where  $1 \leq j \leq n/4$ , so if  $\zeta$  is of finite order then by Theorem 1, it is conjugate to a power of  $\alpha_0$ . Thus the Abelianisation of  $\zeta$  is congruent to 0 modulo  $n-1$ . On the other hand, the Abelianisation of  $\zeta = \Omega_1 \Delta_n$  is congruent to  $\frac{n}{4}(\frac{n}{2}-1) + \frac{n}{2}(n-1)$  modulo  $2(n-1)$ , which is congruent modulo  $n-1$  to  $\frac{n}{8}(n-2)$ . But  $\frac{n}{8}(n-2) \not\equiv 0$  modulo  $n-1$ , which gives a contradiction. So  $\zeta$  is of infinite order, and  $\langle \alpha_0^{n/2}, \Delta_n, \zeta \rangle \cong \mathcal{Q}_8 \rtimes_{\beta} \mathbb{Z}$  as required.  $\square$

REMARKS 61.

(a) If we take  $n = 4$  in the proof of Proposition 60(a) then  $\nu = \alpha_1$  is of order 6, and we obtain the subgroup  $\langle \alpha_0^2, \Delta_4, \alpha_0 \Omega_2 \rangle$  which is isomorphic to  $T^*$  (see [GG5, Remark 3.2]). However, a copy of  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  in  $B_4(\mathbb{S}^2)$  will be exhibited in the proof of Proposition 68. Combining this with Propositions 57, 60 and 62 will prove the existence of Type I subgroups of  $B_n(\mathbb{S}^2)$  of the form  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  for all  $n \geq 4$  even, with the exception of  $n \in \{6, 10, 14\}$ . Proposition 60(b) implies the existence of Type I subgroups of  $B_n(\mathbb{S}^2)$  of the form  $\mathcal{Q}_8 \rtimes_{\beta} \mathbb{Z}$  for all  $n \geq 4$  even.

(b) In the case where  $n \equiv 2 \pmod{4}$ , we do not know of an explicit algebraic representation of  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  similar to that of the construction of Proposition 60(a) in the case  $n \equiv 0 \pmod{4}$ . In order to obtain such a representation, note that by [GG5, Proposition 1.5 and Theorem 1.6], the standard copy  $\langle \alpha'_2, \Delta_n \rangle$  of  $\text{Dic}_{4(n-2)}$  exhibits both conjugacy classes of subgroups isomorphic to  $\mathcal{Q}_8$  in  $B_n(\mathbb{S}^2)$ . To construct a copy  $H$  of  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$ , the elements of the copy of  $\mathcal{Q}_8$  of order 4 must be conjugate, so  $H = \langle \alpha_2'^{(n-2)/2}, \alpha_2' \Delta_n \rangle$  (up to conjugacy). We then need to look for an element  $z$  of  $B_n(\mathbb{S}^2)$  of infinite order whose action by conjugacy on  $H$  permutes cyclically the elements  $\alpha_2'^{(n-2)/2}, \Delta_n$  and  $\alpha_2'^{(n-2)/2} \Delta_n$  of  $H$  (or perhaps their inverses). Propositions 57(a) and 62(a) imply the existence of  $z$ , but we have not been able to find explicitly such an element.

## 4 Type I subgroups of $B_n(\mathbb{S}^2)$ of the form $F \rtimes \mathbb{Z}$ with $F = T^*, O^*, I^*$

We now consider the problem of the existence of Type I subgroups of  $B_n(\mathbb{S}^2)$  of the form  $F \rtimes \mathbb{Z}$  with  $F = T^*, O^*, I^*$ . In the case where the product is direct, the question will be treated in Section 4.1. Proposition 45 asserts that the only nontrivial action occurs when  $F = T^*$ , in which case the action is that given by equation (14). This possibility will be dealt with in Section 4.2.

## 4.1 Type I subgroups of $B_n(\mathbb{S}^2)$ of the form $F \times \mathbb{Z}$ with $F = T^*, O^*, I^*$

In this section, we prove the following result.

PROPOSITION 62.

- (a) Suppose that  $n = 12$  or that  $n \geq 16$  is even. Then the group  $T^* \times \mathbb{Z}$  is realised as a subgroup of  $B_n(\mathbb{S}^2)$ .
- (b) Suppose that  $n = 24$  or that  $n \geq 30$  is congruent to 0 or 2 mod 6. Then the group  $O^* \times \mathbb{Z}$  is realised as a subgroup of  $B_n(\mathbb{S}^2)$ .
- (c) Suppose that  $n = 60$  or that  $n \geq 72$  is congruent to 0, 2, 12 or 20 mod 30. Then the group  $I^* \times \mathbb{Z}$  is realised as a subgroup of  $B_n(\mathbb{S}^2)$ .
- (d) The group  $T^* \times \mathbb{Z}$  (resp.  $O^* \times \mathbb{Z}$ ) is not realised as a subgroup of  $B_4(\mathbb{S}^2)$  (resp.  $B_6(\mathbb{S}^2)$ ).

REMARKS 63.

- (a) Since  $T^* \times \mathbb{Z}$  is not realised as a subgroup of  $B_4(\mathbb{S}^2)$ , neither is  $T^* \times_{\omega} \mathbb{Z}$ .
- (b) Theorem 5(3) follows immediately from Proposition 62(d).

REMARK 64. For the following values of  $n$  not covered by Proposition 62, the associated binary polyhedral group occurs as a subgroup of  $B_n(\mathbb{S}^2)$ , but it is an open question as to whether the given direct product is realised or not:

- (i)  $T^* \times \mathbb{Z}$ , for  $n \in \{6, 8, 10, 14\}$ ,
- (ii)  $O^* \times \mathbb{Z}$ , for  $n \in \{8, 12, 14, 18, 20, 26\}$ ,
- (iii)  $I^* \times \mathbb{Z}$ , for  $n \in \{12, 20, 30, 32, 42, 50, 62\}$ .

*Proof of Proposition 62.* We start by proving part (a). Suppose that  $n = 12$  or that  $n \geq 16$  is even. Set  $n = 6l + 4m$ , where  $l \geq 2$  and  $m \in \{0, 1, 2\}$ . Let  $\Delta$  be a regular tetrahedron, and let  $X \subset \Delta$  be an  $n$ -point subset invariant under the action of the group  $\Gamma \cong A_4$  of rotations of  $\Delta$ . We may suppose that each edge of  $\Delta$  contains  $l$  equally-spaced points in its interior. If  $m \geq 1$  then we place four points of  $X$  at the vertices of  $\Delta$ , and if  $m = 2$ , we add a further four points at the barycentres of the faces. We inscribe  $\Delta$  within the sphere  $\mathbb{S}^2$ , and from now on, the two shall be identified by radial projection without further comment.

Recall that  $\text{Homeo}^+(\mathbb{S}^2, X)$  and

$$\Psi: \text{Homeo}^+(\mathbb{S}^2, X) \longrightarrow \mathcal{MCG}(\mathbb{S}^2, n)$$

were defined in Section I.4. Now  $\Gamma$  is a subgroup of  $\text{Homeo}^+(\mathbb{S}^2, X)$  whose image  $\tilde{\Gamma} = \Psi(\Gamma)$  under  $\Psi$  is also isomorphic to  $A_4$ . Indeed,  $f \in \text{Homeo}^+(\mathbb{S}^2, X)$  belongs to  $\text{Ker}(\Psi)$  if and only if it is isotopic to the identity relative to  $X$ . Such an  $f$  would thus fix  $X$  pointwise, but the only element of  $\Gamma$  which achieves this is the identity. So the restriction of  $\Psi$  to  $\Gamma$  is injective.

Since  $\tilde{\Gamma} \cong A_4$ , the preimage  $\Lambda = \varphi^{-1}(\tilde{\Gamma})$  under the homomorphism  $\varphi$  of equation (11) is a copy of  $T^*$ . The aim is to prove the existence of an element  $v$  of infinite order belonging to the centraliser of  $\Lambda$  in  $B_n(\mathbb{S}^2)$ . We claim that it suffices to exhibit an element  $\tilde{z}$  of infinite order belonging to the centraliser of  $\tilde{\Gamma}$  in  $\mathcal{MCG}(\mathbb{S}^2, n)$ . Indeed, suppose such a  $\tilde{z}$  exists, and let  $z \in B_n(\mathbb{S}^2)$  be a preimage of  $\tilde{z}$  under  $\varphi$ . Clearly  $z$  is also

of infinite order. Let  $w \in \Lambda$ , and let  $\tilde{w} = \varphi(w) \in \tilde{\Gamma}$ . Then  $\varphi([w, z]) = [\tilde{w}, \tilde{z}] = 1$ , so  $[w, z] = \Delta_n^{2\varepsilon}$ , where  $\varepsilon \in \{0, 1\}$ . Thus  $wzw^{-1} = z\Delta_n^{2\varepsilon}$ , hence  $wz^2w^{-1} = z^2$  for all  $w \in \Lambda$ , and so we may take  $v = z^2$ . It follows that  $\langle \Lambda, v \rangle \cong T^* \times \mathbb{Z}$ .

To prove the existence of  $\tilde{z}$ , denote the edges of  $\Delta$  by  $e_1, \dots, e_6$ , and for  $j = 1, \dots, 6$ , let  $f_j \in \Gamma$  be such that  $f_j(e_1) = e_j$  (we choose  $f_1 = \text{Id}$ ). Let  $\mathcal{C}_1$  be a positively oriented simple closed curve containing the  $l$  points of  $X$  belonging to  $e_1$ , and let  $\mathcal{A}_1$  be a small annular neighbourhood of  $\mathcal{C}_1$ , chosen so that the orbit  $\mathcal{C}$  of  $\mathcal{C}_1$  (resp. the orbit  $\mathcal{A}$  of  $\mathcal{A}_1$ ) under the action of  $\Gamma$  consists of the six (disjoint) oriented simple closed curves  $\mathcal{C}_j = f_j(\mathcal{C}_1)$ ,  $j = 1, \dots, 6$  (resp. six pairwise-disjoint annuli  $\mathcal{A}_j = f_j(\mathcal{A}_1)$ ,  $j = 1, \dots, 6$ ) (see Figure 8). The orbits  $\mathcal{C}$  and  $\mathcal{A}$  are obviously invariant under this action. Each  $\mathcal{C}_i$  is associated with the edge  $e_i$  of  $\Delta$ , and bounds a disc containing the  $l$  points of  $X \cap e_i$ . Let  $T_1 \in \text{Homeo}^+(\mathbb{S}^2, X)$  be the (positive) Dehn twist along  $\mathcal{C}_1$  in  $\mathcal{A}_1$ , and set  $T_i = f_i \circ T_1 \circ f_i^{-1}$ . Then  $T_i$  is the (positive) Dehn twist along  $\mathcal{C}_i$  in  $\mathcal{A}_i$ . Since the  $\mathcal{A}_i$  are pairwise disjoint, the  $T_i$  commute pairwise.

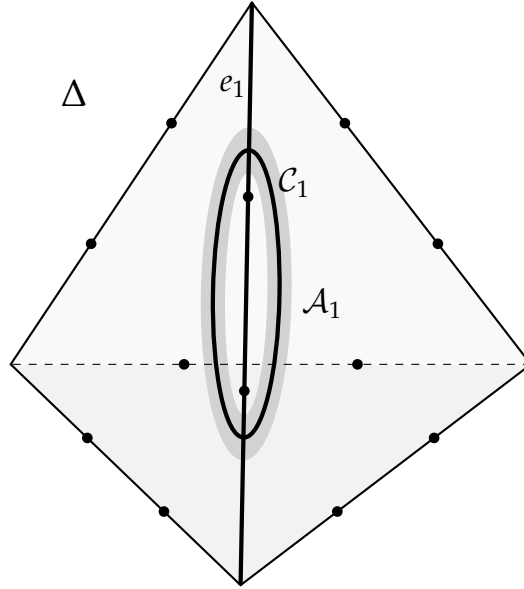


Figure 8: The geometric construction of  $\tilde{z}$  in  $\mathcal{MCG}(\mathbb{S}^2, 12)$ .

Set  $T = T_1 \circ \dots \circ T_6$ . Let us prove that  $T$  is an element of  $\text{Homeo}^+(\mathbb{S}^2, X)$  of infinite order belonging to the centraliser of  $\Gamma$ . To see this, let  $f \in \Gamma$ , and let  $x \in \Delta$ . First suppose that  $x \notin \bigcup_{i=1}^6 \mathcal{A}_i$ . Then  $f(x) \notin \bigcup_{i=1}^6 \mathcal{A}_i$  since  $\bigcup_{i=1}^6 \mathcal{A}_i$  is invariant under the action of  $\Gamma$ , and so  $f \circ T(x) = f(x) = T \circ f(x)$  as required. Now assume that  $x \in \mathcal{A}_j$  for some  $j = 1, \dots, 6$ . By relabelling the edges of  $\Delta$  if necessary, we may suppose that  $x \in \mathcal{A}_1$ . Let  $\langle g \rangle \cong \mathbb{Z}_2$  be the stabiliser of  $e_1$  in  $\Gamma$ . If we parametrise  $\mathcal{A}_1$  as  $[0, 1] \times \mathbb{S}^1$  then  $T_1$  is defined by  $T_1(t, s) = (t, se^{2\pi it})$ , and the restriction of  $g$  to  $\mathcal{A}_1$  is given by  $g(t, s) = (t, se^{\pi i})$ . A straightforward calculation shows that  $g \circ T_1 = T_1 \circ g$  on  $\mathcal{A}_1$ . By considering the action on the oriented edges of  $\Delta$ , it follows that there exist  $i \in \{1, \dots, 6\}$  and  $\varepsilon \in \{0, 1\}$  such that  $f = f_i \circ g^\varepsilon$ , so  $f(x) \in \mathcal{A}_i$ , and:

$$T \circ f(x) = T_i \circ f(x) = T_i \circ f_i \circ g^\varepsilon(x) = f_i \circ T_1 \circ g^\varepsilon(x) = f_i \circ g^\varepsilon \circ T_1(x) = f \circ T(x),$$



using the facts that the  $T_j$  commute pairwise, and that for  $j = 1, \dots, 6$ , the support of  $T_j$  is  $\mathcal{A}_j$ . This shows that  $T$  belongs to the centraliser of  $\Gamma$  in  $\text{Homeo}^+(\mathbb{S}^2, X)$ , and so  $\tilde{T} = \Psi(T)$  belongs to the centraliser of  $\tilde{\Gamma}$  in  $\mathcal{MCG}(\mathbb{S}^2, n)$ . It remains to show that  $\tilde{T}$  is of infinite order. This is a consequence of a generalisation of the intersection number formula for Dehn twists, see [FM, Propositions 3.2 and 3.4] for example. An alternative proof of this fact is as follows. Since  $\tilde{T}$  belongs to the pure mapping class group  $\mathcal{PMCG}(\mathbb{S}^2, n)$  of  $\mathbb{S}^2$  on  $n$  points, we may consider its image  $\hat{T}$  under the homomorphism  $\mathcal{PMCG}(\mathbb{S}^2, n) \rightarrow \mathcal{PMCG}(\mathbb{S}^2, 4)$ , obtained in an analogous manner to the Fadell-Neuwirth homomorphism by removing all but two pairs of points, one pair contained in the small disc bounded by  $\mathcal{C}_1$ , and another pair contained in that bounded by  $\mathcal{C}_2$ . Since  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are both positively oriented,  $\hat{T}$  is the image under  $\varphi$  of a pure braid, which choosing appropriate generators, may be written as  $\sigma_1^2 \sigma_3^2$ . We saw at the end of the proof of Proposition 53 that this element is of infinite order, and this implies that  $\tilde{T}$  is also of infinite order. We have thus shown that there exists an element  $\tilde{z} = \tilde{T}$  of infinite order belonging to the centraliser of  $\tilde{\Gamma}$  in  $\mathcal{MCG}(\mathbb{S}^2, n)$ , and this proves part (a).

Applying a similar construction for  $O^*$  (taking  $\Delta$  to be a cube) and for  $I^*$  (taking  $\Delta$  to be a dodecahedron) yields parts (b) and (c). Note that in the case of  $O^*$  (resp.  $I^*$ ), we have that  $n \equiv 0, 2 \pmod{6}$  (resp.  $n \equiv 0, 2, 12, 20 \pmod{30}$ ). We set  $n = 12l + 8m + 6r$  (resp.  $n = 30l + 20m + 12r$ ), where  $l \in \mathbb{N}$ , and  $m, r \in \{0, 1\}$  denote respectively the number of points of  $X$  placed at the vertices of  $\Delta$  and at the barycentre of the faces of  $\Delta$ . Since we require  $l \geq 2$  in the construction, the excluded values of  $n$  are 6, 8, 12, 14, 18, 20 and 26 (resp. 12, 20, 30, 32, 42, 50 and 62).

Finally, we prove part (d). Suppose that  $T^* \times \mathbb{Z}$  (resp.  $O^* \times \mathbb{Z}$ ) is realised as a subgroup  $K$  of  $B_4(\mathbb{S}^2)$  (resp.  $B_6(\mathbb{S}^2)$ ). Since  $T^*$  (resp.  $O^*$ ) possesses elements of order 6 (resp. 8) by Proposition 85,  $K$  contains a subgroup  $H$  isomorphic to  $\mathbb{Z}_6 \times \mathbb{Z}$  (resp.  $\mathbb{Z}_8 \times \mathbb{Z}$ ) whose finite factor is conjugate to  $\langle \alpha_1 \rangle$  (resp.  $\langle \alpha_2 \rangle$ ) by Theorem 1. But the existence of  $H$  then contradicts Proposition 31.  $\square$

As a consequence of Proposition 62(d), we obtain the following result which complements that of Proposition 8.

**COROLLARY 65.** *If  $n = 4$  (resp.  $n = 6$ ), let  $H$  be a subgroup of  $B_n(\mathbb{S}^2)$  isomorphic to  $T^*$  (resp.  $O^*$ ). Then the normaliser of  $H$  in  $B_n(\mathbb{S}^2)$  is  $H$  itself.*

*Proof.* In both cases,  $H$  is finite maximal by Theorem 2, so it suffices to prove that  $N = N_{B_n(\mathbb{S}^2)}(H)$  is finite. If  $x \in N$  then some power of  $x$  belongs to  $Z_{B_n(\mathbb{S}^2)}(H)$ , but by Proposition 62(d),  $x$  must be of finite order. Hence  $N$  is finite by Proposition 28.  $\square$

## 4.2 Realisation of $T^* \rtimes_{\omega} \mathbb{Z}$

We now consider the realisation of  $T^* \rtimes_{\omega} \mathbb{Z}$  as a subgroup of  $B_n(\mathbb{S}^2)$ , where  $\omega(1)$  is as defined in equation (14).

**PROPOSITION 66.** *If  $O^* \times \mathbb{Z}$  is realised as a subgroup of  $B_n(\mathbb{S}^2)$  then so is  $T^* \rtimes_{\omega} \mathbb{Z}$ .*

REMARK 67. Let  $n \notin \{6, 8, 12, 14, 18, 20, 26\}$ . Putting together the results of Propositions 62(b) and 66, we see that  $T^* \rtimes_{\omega} \mathbb{Z}$  is realised as a subgroup of  $B_n(\mathbb{S}^2)$  if and only if  $n \equiv 0, 2 \pmod{6}$ .

*Proof of Proposition 66.* Suppose that  $O^* \times \mathbb{Z}$  is realised as a subgroup  $L$  of  $B_n(\mathbb{S}^2)$ . Then there exist a subgroup  $K$  of  $B_n(\mathbb{S}^2)$  isomorphic to  $O^*$  and an element  $z \in B_n(\mathbb{S}^2)$  of infinite order such that  $z$  belongs to the centraliser of  $K$ . Let equation (28) denote a presentation of  $K$ , and let  $H = \langle P, Q, X \rangle$  denote the subgroup of  $K$  isomorphic to  $T^*$ , with the presentation of equation (13). Equations (14) and (28) imply that the restriction to  $H$  of conjugation by the element  $R$  of  $K$  represents the nontrivial element of  $\text{Out}(T^*)$ . But  $z$  commutes with  $R$ , so  $zR$  is of infinite order, and since  $z$  also belongs to the centraliser of  $H$ , it follows that  $\langle H, zR \rangle \cong T^* \rtimes_{\omega} \mathbb{Z}$ .  $\square$

## 5 Proof of the realisation of the elements of $\mathbb{V}_1(n)$ in $B_n(\mathbb{S}^2)$

In this section, we bring together the results of Sections II.1–4 to prove Proposition 68. This proposition will imply Theorem 5(2) for the Type I subgroups of  $B_n(\mathbb{S}^2)$ , namely the realisation of the virtually cyclic groups given by (1)(a)–(i) of Definition 4, with the exception of the values of  $n$  given in Remark 6(a) and not covered by Theorem 5(2)(a)–(d).

PROPOSITION 68. *Let  $n \geq 4$ . The following Type I virtually cyclic groups are realised as subgroups of  $B_n(\mathbb{S}^2)$ :*

- (a)  $\mathbb{Z}_q \times \mathbb{Z}$ , where  $q \mid 2(n-i)$  with  $i \in \{0, 1, 2\}$ ,  $1 \leq q \leq n-i$ , and  $q < n-i$  if  $n-i$  is odd.
- (b)  $\mathbb{Z}_q \rtimes_{\rho} \mathbb{Z}$ , where  $q \mid 2(n-i)$  with  $i \in \{0, 2\}$ ,  $3 \leq q \leq n-i$ ,  $q < n-i$  if  $n-i$  is odd, and  $\rho(1) \in \text{Aut}(\mathbb{Z}_q)$  is multiplication by  $-1$ .
- (c)  $\text{Dic}_{4m} \times \mathbb{Z}$ , where  $m \mid n-i$  with  $i \in \{0, 2\}$ , and  $3 \leq m \leq (n-i)/2$ .
- (d)  $\text{Dic}_{4m} \rtimes_{\nu} \mathbb{Z}$ , where  $m \mid n-i$  with  $i \in \{0, 2\}$ ,  $m \geq 3$ ,  $(n-i)/m$  is even, and  $\nu(1)$  is the automorphism of  $\text{Dic}_{4m}$  given by equation (12).
- (e) (i)  $\mathbb{Q}_8 \times \mathbb{Z}$  for all  $n$  even.  
(ii)  $\mathbb{Q}_8 \rtimes_{\alpha} \mathbb{Z}$ , for all  $n$  even,  $n \notin \{6, 10, 14\}$ , where  $\alpha(1) \in \text{Aut}(\mathbb{Q}_8)$  is given by  $\alpha(1)(i) = j$  and  $\alpha(1)(j) = k$ , and where  $\mathbb{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ .
- (iii)  $\mathbb{Q}_8 \rtimes_{\beta} \mathbb{Z}$  for all  $n$  even, where  $\beta(1) \in \text{Aut}(\mathbb{Q}_8)$  is given by  $\beta(1)(i) = k$  and  $\beta(1)(j) = j^{-1}$ .
- (f)  $T^* \times \mathbb{Z}$ , where  $n = 12$  or  $n \geq 16$  is even.
- (g)  $T^* \rtimes_{\omega} \mathbb{Z}$ , where  $n = 24$  or  $n \geq 30$  and  $n \equiv 0, 2 \pmod{6}$ , and  $\omega(1)$  is the automorphism of  $T^*$  given by equation (14).
- (h)  $O^* \times \mathbb{Z}$ , where  $n = 24$  or  $n \geq 30$  and  $n \equiv 0, 2 \pmod{6}$ .
- (i)  $I^* \times \mathbb{Z}$ , where  $n = 60$  or  $n \geq 72$  and  $n \equiv 0, 2, 12, 20 \pmod{30}$ .

*Proof.* Parts (a), (b), (c) and (d) are proved in Propositions 52, 53, 54 and 56 respectively. By Proposition 60(b),  $\mathbb{Q}_8 \rtimes_{\beta} \mathbb{Z}$  is realised as a subgroup of  $B_n(\mathbb{S}^2)$  for all  $n \geq 4$  even, and its subgroup generated by  $\mathbb{Q}_8$  and the square of the  $\mathbb{Z}$ -factor is abstractly isomorphic to  $\mathbb{Q}_8 \times \mathbb{Z}$ , which proves parts (e)(i) and (iii). We now consider the realisation of  $\mathbb{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  as a subgroup of  $B_n(\mathbb{S}^2)$ . Suppose first that  $n \equiv 0 \pmod{4}$ . If  $n \geq 8$  then the result follows from Proposition 60(a). So suppose that  $n = 4$ . By [GG6, Theorem 1.3(3)],  $B_4(\mathbb{S}^2)$

contains a copy of  $\mathcal{Q}_8$  generated by  $x = \sigma_3\sigma_1^{-1}$  and  $y = (\sigma_1^2\sigma_2\sigma_1^{-3})\sigma_3\sigma_1^{-1}(\sigma_1^3\sigma_2^{-1}\sigma_1^{-2})$ , and the element  $a = \sigma_1^2\sigma_2\sigma_1^{-3}$ , which is of infinite order, acts by conjugation on  $\langle x, y \rangle$  by sending  $x$  to  $y$  and  $y$  to  $xy$ . Hence the subgroup  $\langle x, y, a \rangle$  of  $B_4(\mathbb{S}^2)$  is isomorphic to  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  as required. Now suppose that  $n \equiv 2 \pmod{4}$ . If  $n \notin \{6, 10, 14\}$  then  $n \geq 18$ . So by Proposition 62(a),  $T^* \times \mathbb{Z}$  is realised as a subgroup of  $B_n(\mathbb{S}^2)$ , and we deduce from Proposition 57(a) that  $B_n(\mathbb{S}^2)$  contains a copy of  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$ , which proves part (e)(ii). Parts (f), (h) and (i) follow directly from Proposition 62(a)–(c). Finally, to prove part (g), if  $n = 24$  or  $n \geq 30$  and  $n \equiv 0, 2 \pmod{6}$  then  $O^* \times \mathbb{Z}$  is realised as a subgroup of  $B_n(\mathbb{S}^2)$  by Proposition 62(b), and so  $B_n(\mathbb{S}^2)$  contains a copy of  $T^* \rtimes_{\omega} \mathbb{Z}$  by Proposition 66.  $\square$

REMARK 69. In Proposition 68(e)(ii), we do not know whether the Type I group  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$  is realised as a subgroup of  $B_n(\mathbb{S}^2)$  for  $n \in \{6, 10, 14\}$ . In [GG5, Remark 3.3], we exhibited a copy  $\langle \gamma, \delta \rangle$  of  $T^*$  in  $B_6(\mathbb{S}^2)$ , where

$$\gamma = \sigma_5\sigma_4\sigma_1^{-1}\sigma_2^{-1} \text{ and } \delta = \sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_5\sigma_4\sigma_5\sigma_4\sigma_3$$

(note that there is a typing error in the original version, the expression for  $\delta$  there is missing the terms  $\sigma_5\sigma_4\sigma_5$ ). The action of conjugation by  $\gamma$  permutes cyclically the elements  $\gamma^i\delta\gamma^{-i}$ ,  $i = 0, 1, 2$ , and gives rise to the semi-direct product structure  $\mathcal{Q}_8 \rtimes \mathbb{Z}_3$  of  $T^*$ . In order to obtain a subgroup of  $B_6(\mathbb{S}^2)$  isomorphic to  $\mathcal{Q}_8 \rtimes_{\alpha} \mathbb{Z}$ , the proof of Proposition 57(a) shows that it suffices to exhibit an element  $z \in B_6(\mathbb{S}^2)$  of infinite order that commutes with  $\gamma$  and  $\delta$ , but up until now, we have not been able to find such a  $z$ .

## 6 Realisation of the elements of $\mathbb{V}_2(n)$ in $B_n(\mathbb{S}^2)$

We now turn our attention to the problem of the realisation in  $B_n(\mathbb{S}^2)$  of the virtually cyclic groups of Type II described in Definition 4(2). In Section 6.1, we consider those groups that contain a cyclic or dicyclic factor. In Section 6.2, we discuss the realisation of  $O^* *_{T^*} O^*$  in  $B_n(\mathbb{S}^2)$ .

### 6.1 Realisation of the elements of $\mathbb{V}_2(n)$ with cyclic or dicyclic factors

THEOREM 70. *For all  $n \geq 4$ , the following Type II virtually cyclic groups are realised as subgroups of  $B_n(\mathbb{S}^2)$ :*

- (a)  $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \mathbb{Z}_{4q}$ , where  $i \in \{0, 1, 2\}$  and  $q$  divides  $(n - i)/2$ .
- (b)  $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \text{Dic}_{4q}$ , where  $i \in \{0, 2\}$ ,  $q \geq 2$  and  $q$  divides  $(n - i)/2$ .
- (c)  $\text{Dic}_{4q} *_{\mathbb{Z}_{2q}} \text{Dic}_{4q}$ , where  $i \in \{0, 2\}$ ,  $q \geq 2$  and  $q$  divides  $n - i$  strictly.
- (d)  $\text{Dic}_{4q} *_{\text{Dic}_{2q}} \text{Dic}_{4q}$ , where  $i \in \{0, 2\}$ , and  $q \geq 4$  is an even divisor of  $n - i$ .

*Proof.* Let  $n \geq 4$ . First recall that if  $1 \leq j \leq n + 1$ , the kernel of the homomorphism  $P_{n+1}(\mathbb{S}^2) \rightarrow P_n(\mathbb{S}^2)$  defined geometrically by deleting the  $j^{\text{th}}$  string may be identified with the fundamental group

$$\pi_1 \left( \mathbb{S}^2 \setminus \{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}\}, x_j \right),$$

which is a free group of rank  $n - 1$  for which a presentation is given by

$$\langle A_{1,i}, \dots, A_{i-1,i}, A_{i,i+1}, \dots, A_{i,n} \mid A_{1,i} \cdots A_{i-1,i} A_{i,i+1} \cdots A_{i,n} = 1 \rangle, \quad (61)$$

and for which a basis is obtained by selecting any  $n - 1$  distinct elements of the set  $\{A_{1,j}, \dots, A_{j-1,j}, A_{j,j+1}, \dots, A_{j,n+1}\}$ , where for  $1 \leq i < j \leq n + 1$ ,

$$A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1} = \sigma_i^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^2 \sigma_{j-2} \cdots \sigma_i. \quad (62)$$

We consider the four cases of the statement of the theorem in turn.

(a) We first treat the case  $q = 1$ , and then go on to deal with the general case  $q \geq 2$ .

*1<sup>st</sup> case:  $q = 1$ .* We shall construct a subgroup of  $B_n(\mathbb{S}^2)$  isomorphic to  $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$ . Set  $\bar{i} = 2$  if  $n$  is even, and  $i = 1$  if  $n$  is odd. Then  $(n - i)$  is even, and the condition given in the statement is satisfied. Let  $v_1 = \alpha_i^{(n-i)/2}$ ,  $v_2 = \sigma_{n-i} v_1 \sigma_{n-i}^{-1}$ , and for  $j = 1, 2$ , let  $G_j = \langle v_j \rangle$ . Then  $|G_j| = 4$  by equation (8), and  $G_1 \cap G_2 \supset \langle \Delta_n^2 \rangle$  since  $\Delta_n^2$  is the unique element of  $B_n(\mathbb{S}^2)$  of order 2. Let  $H = \langle G_1 \cup G_2 \rangle$ . By Proposition 23, to prove that  $H \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$ , it suffices to show that  $H$  is of infinite order, or indeed that  $H$  contains an element of infinite order. Consider the element  $v_1 v_2$  of  $H$ . A straightforward calculation shows that:

$$\begin{aligned} \pi(v_1) &= \left(1, \frac{n-i}{2} + 1\right) \left(2, \frac{n-i}{2} + 2\right) \cdots \left(\frac{n-i}{2} - 1, n - i - 1\right) \left(\frac{n-i}{2}, n - i\right) \\ \pi(v_2) &= \left(1, \frac{n-i}{2} + 1\right) \left(2, \frac{n-i}{2} + 2\right) \cdots \left(\frac{n-i}{2} - 1, n - i - 1\right) \left(\frac{n-i}{2}, n - i + 1\right) \\ \pi(v_1 v_2) &= \left(\frac{n-i}{2}, n - i, n - i + 1\right), \end{aligned}$$

and thus  $\pi(v_1 v_2)$  consists of one 3-cycle and  $n - 3$  fixed points. So if  $n \geq 6$ , by Theorem 1,  $v_1 v_2$  is of infinite order, and this implies that  $H$  is infinite as required. It remains to treat the cases  $n = 4, 5$ . Suppose first that  $n = 4$ , and assume that  $H$  is finite. Then  $H$  is contained in a maximal finite subgroup  $K$  of  $B_4(\mathbb{S}^2)$ , where  $K$  is isomorphic to  $\mathcal{Q}_{16}$  or  $T^*$  by Theorem 2. Since  $\pi(v_1 v_2)$  is a 3-cycle and the set of torsion elements of  $P_n(\mathbb{S}^2)$  is  $\langle \Delta_n^2 \rangle$ ,  $v_1 v_2$  is of order 3 or 6, and so  $K \cong T^*$ . On the other hand, the elements of order 4 of  $T^* \cong \mathcal{Q}_8 \rtimes \mathbb{Z}_3$  all belong to its subgroup isomorphic to  $\mathcal{Q}_8$ , and so the product  $v_1 v_2$  of elements of order 4 is of order 1, 2 or 4. This yields a contradiction, so  $H$  is infinite in this case. Now suppose that  $n = 5$ . Using equations (8) and (23), as well as the fact that  $\alpha_1 = \alpha_0 \sigma_4$ , we obtain:

$$\begin{aligned} v_1 v_2 &= \alpha_1^2 \sigma_4 \alpha_1^2 \sigma_4^{-1} = \alpha_1^4 \alpha_1^{-2} \sigma_4 \alpha_1^2 \sigma_4^{-1} = \Delta_5^2 \alpha_1^{-1} \sigma_4^{-1} \alpha_0^{-1} \sigma_4 \alpha_0 \sigma_4 \alpha_1 \sigma_4^{-1} \\ &= \Delta_5^2 \sigma_4^{-1} \alpha_0^{-1} \sigma_4^{-1} \sigma_3 \sigma_4 \alpha_0 = \Delta_5^2 \sigma_4^{-1} \sigma_3^{-1} \sigma_2 \sigma_3 = \Delta_5^2 \sigma_2 (\sigma_4^{-1} \sigma_3) \sigma_2^{-1}. \end{aligned}$$

So to show that  $v_1 v_2$  is of infinite order, it suffices to prove that  $\sigma_4^{-1} \sigma_3$  is of infinite order. We have

$$(\sigma_4^{-1} \sigma_3)^3 = \sigma_4^{-2} \sigma_4 \sigma_3 \sigma_4^{-1} \sigma_3 \sigma_4^{-1} \sigma_3 = \sigma_4^{-2} \sigma_3^{-1} \sigma_4 \sigma_3^2 \sigma_4^{-1} \sigma_3 = \sigma_4^{-2} \sigma_3^{-2} \sigma_4^2 \sigma_3^2. \quad (63)$$

So  $(\sigma_4^{-1} \sigma_3)^3$  belongs to the free group  $\pi_1(\mathbb{S}^2 \setminus \{x_1, x_2, x_3, x_5\}, x_4)$ , and in terms of the basis  $\{A_{2,4}, A_{3,4}, A_{4,5}\}$  of the latter, may be written as the commutator  $[A_{4,5}^{-1}, A_{3,4}^{-1}]$ . It follows that  $(\sigma_4^{-1} \sigma_3)^3$  and  $H$  are of infinite order, and thus  $H \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4$  by Proposition 23.

2<sup>nd</sup> case:  $q \geq 2$ . We claim that it suffices to find distinct cyclic subgroups  $G_1, G_2$  of  $B_n(\mathbb{S}^2)$  of order  $4q$  for which  $G_1 \cap G_2$  contains a (cyclic) subgroup of order  $2q$ . To prove the claim, let  $G_1$  and  $G_2$  be subgroups of  $B_n(\mathbb{S}^2)$  satisfying these conditions, and suppose that  $H = \langle G_1 \cup G_2 \rangle$  is finite. Let  $K$  be a maximal finite subgroup of  $B_n(\mathbb{S}^2)$  containing  $H$ . Since  $G_1 \neq G_2$ ,  $K$  contains two distinct copies of  $\mathbb{Z}_{4q}$ , and so cannot be cyclic or dicyclic, nor by Proposition 85 can it be isomorphic to  $T^*$  or  $I^*$ , since  $q \geq 2$ . So suppose that  $K \cong O^*$ . Then  $q = 2$  by Proposition 85,  $G_j \cong \mathbb{Z}_8$ , where  $j = 1, 2$ , and  $G_1 \cap G_2 \cong \mathbb{Z}_4$  by hypothesis. Under the restriction of the homomorphism  $\varphi$  of equation (11),  $O^*$  is sent to  $S_4$ , the  $\varphi(G_j)$  are sent to subgroups of  $S_4$  generated by 4-cycles, and  $|\varphi(G_1 \cap G_2)| = 2$ . But in  $S_4$ , the intersection of two subgroups generated by 4-cycles cannot be of order 2, so  $K \not\cong O^*$ . We conclude that  $H$  is infinite, hence  $H \cong \mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \mathbb{Z}_{4q}$  by Proposition 23, which proves the claim.

We now exhibit subgroups  $G_1$  and  $G_2$  of  $B_n(\mathbb{S}^2)$  satisfying the properties of the claim. By hypothesis,  $m = (n - i)/2q \in \mathbb{N}$ . Let  $G_1 = \langle \alpha_i^m \rangle$  and  $G_2 = \zeta G_1 \zeta^{-1}$ , where

$$\begin{aligned} \zeta &= \delta_{2m,i} = \sigma_1 \sigma_{2m+1} \cdots \sigma_{m(2q-4)+1} \sigma_{m(2q-2)+1} \\ &= \sigma_1 \sigma_{2m+1} \cdots \sigma_{n-i-4m+1} \sigma_{n-i-2m+1}, \end{aligned}$$

using the notation of equation (45). Then  $G_j \cong \mathbb{Z}_{4q}$  for  $j = 1, 2$  by equation (8). Now  $\pi(\alpha_i^m)$  contains the  $2q$ -cycle  $(1, m(2q - 1) + 1, m(2q - 2) + 1, \dots, m + 1)$ , and so  $\pi(\alpha_i^{mk})(1) \in \{1, m + 1, \dots, m(2q - 1) + 1\}$  for all  $k \in \mathbb{N}$ . On the other hand,

$$\begin{aligned} \pi(\zeta \alpha_i^m \zeta^{-1})(1) &= \pi(\alpha_i^m \zeta^{-1})(2) \\ &= \pi(\zeta^{-1})(m(2q - 1) + 2) = m(2q - 1) + 2 \end{aligned}$$

(recall that as for braids, we compose permutations from left to right). Thus  $\zeta \alpha_i^m \zeta^{-1} \notin G_1$ , so  $G_2 \neq G_1$ . Taking the integer  $m$  of Lemma 51 to be  $2m$ , we have that  $2m$  divides  $n - i$  and so  $r = 2m \geq 2$ . By part (c) of that proposition,  $\zeta$  commutes with  $\alpha_i^{2m}$ . Thus  $G_1 \cap G_2 = \langle \alpha_i^{2m} \rangle \cong \mathbb{Z}_{2q}$ , and so  $G_1$  and  $G_2$  satisfy the hypotheses of the claim.

(b) Suppose that  $q \geq 2$  divides  $(n - i)/2$  for some  $i \in \{0, 2\}$ , so  $n$  is even. Set  $m = (n - i)/2q$ , and let

$$\zeta_i = \sigma_{1+\frac{i}{2}} \sigma_{1+2m+\frac{i}{2}} \cdots \sigma_{1+n-2m-\frac{i}{2}}, \quad (64)$$

Equation (23) implies that  $\zeta_i = \alpha_0^{i/2} \delta_{2m,i} \alpha_0^{-i/2}$ , where  $\delta_{2m,i}$  is as in equation (45). Taking the integer  $m$  of Lemma 51 to be  $2m$ , it follows from part (c) of that lemma that  $\delta_{2m,i}$  commutes with  $\alpha_i^{2m}$ , and thus  $\zeta_i$  commutes with  $\alpha_i'^{2m}$ , where  $\alpha_i'$  is given by equation (10). We analyse separately the two cases  $m = 1$  and  $m \geq 2$ .

1<sup>st</sup> case:  $m = 1$ . Then  $2q = n - i$ . Take  $G_1 = \langle \zeta_i \alpha_i' \zeta_i^{-1} \rangle$  and  $G_2 = \langle \alpha_i'^2, \alpha_i' \Delta_n \rangle$ , where  $\zeta_i$  is as defined above. Then  $G_1 \cong \mathbb{Z}_{2(n-i)} = \mathbb{Z}_{4q}$ , and  $G_2$  is one of the two dicyclic subgroups of order  $2(n - i)$  of the standard copy of  $\text{Dic}_{4(n-i)}$ , so  $G_2 \cong \text{Dic}_{4q}$  and  $G_1 \neq G_2$ . Since  $\zeta_i$  commutes with  $\alpha_i'^2$ , it follows that  $G_1 \cap G_2 = \langle \alpha_i'^2 \rangle \cong \mathbb{Z}_{2q}$ . Set  $H = \langle G_1 \cup G_2 \rangle$ . By Proposition 23, to see that  $H \cong \mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \text{Dic}_{4q}$ , it suffices to show that  $H$  contains an element of infinite order. Consider  $\eta = \zeta_i \alpha_i' \zeta_i^{-1} \cdot \alpha_i' \Delta_n \in H$ . Since  $\zeta_i$  commutes with  $\Delta_n$

by equation (7) as well as with  $\alpha_i'^2$ , and  $n - i$  is even, we have

$$\begin{aligned}\eta^{2(n-i)} &= (\xi_i \alpha_i' \xi_i^{-1} \alpha_i' \Delta_n \cdot \xi_i \alpha_i' \xi_i^{-1} \alpha_i' \Delta_n^{-1} \cdot \Delta_n^2)^{n-i} = (\xi_i \alpha_i' \xi_i^{-1} \alpha_i' \xi_i \alpha_i'^{-1} \xi_i^{-1} \alpha_i'^{-1} \Delta_n^2)^{n-i} \\ &= (\xi_i \alpha_i' \xi_i^{-1} \alpha_i' \cdot \alpha_i'^{-2})^{2(n-i)} = (\xi_i \alpha_i' \xi_i^{-1} \alpha_i')^{2(n-i)} \alpha_i'^{-4(n-i)} = \tilde{\eta}^{2(n-i)},\end{aligned}$$

using also equation (8), where

$$\tilde{\eta} = \xi_i \alpha_i' \xi_i^{-1} \alpha_i'. \quad (65)$$

So to prove that  $\eta$  is of infinite order, it suffices to show that  $\tilde{\eta}$  is of infinite order. Since  $\pi(\xi_i)$  is of order 2 and

$$\begin{aligned}\pi(\xi_i) &= \left(1 + \frac{i}{2}, 2 + \frac{i}{2}\right) \left(3 + \frac{i}{2}, 4 + \frac{i}{2}\right) \cdots \left(n - 1 - \frac{i}{2}, n - \frac{i}{2}\right) \\ \pi(\alpha_i') &= \left(n - \frac{i}{2}, n - 1 - \frac{i}{2}, \dots, 2 + \frac{i}{2}, 1 + \frac{i}{2}\right),\end{aligned}$$

we have

$$\pi(\tilde{\eta}) = (\pi(\xi_i \alpha_i'))^2 = \left(n - \frac{i}{2}, n - \frac{i}{2} - 2, \dots, 4 + \frac{i}{2}, 2 + \frac{i}{2}\right)^2,$$

and the cycle decomposition of  $\pi(\tilde{\eta})$  consists of two  $(n - i)/4$ -cycles (resp. one  $(n - i)/2$ -cycle) if  $n - i$  is divisible (resp. is not divisible) by 4, plus  $(n + i)/2$  fixed points. If either  $i = 0$  and  $n \geq 6$  or if  $i = 2$  and  $n \geq 8$  then the cycle decomposition of  $\pi(\tilde{\eta})$  contains a cycle of length at least two, plus at least three fixed points, and so  $\tilde{\eta}$  is of infinite order by Theorem 1. Let us deal with the three remaining cases, which are given by  $n = 4$  and  $i \in \{0, 2\}$ , and  $n = 6$  and  $i = 2$ .

(i)  $i = 0$  and  $n = 4$ . Using the presentation of equation (61), we have

$$\tilde{\eta} = \sigma_1 \sigma_3 \sigma_1 \sigma_2 \sigma_3 \sigma_3^{-1} \sigma_1^{-1} \sigma_1 \sigma_2 \sigma_3 = \sigma_1^2 \sigma_3 \sigma_2^2 \sigma_3 = A_{1,2} A_{2,4} A_{3,4} = A_{1,2} A_{1,4}^{-1},$$

which may be interpreted as an element of  $\pi_1(\mathbb{S}^2 \setminus \{x_2, x_3, x_4\}, x_1)$  for which a basis is  $\{A_{1,2}, A_{1,4}\}$ .

(ii)  $i = 2$  and  $n = 4$ . In this case,  $\tilde{\eta} = \sigma_2^2 \sigma_3^4 = A_{2,3} A_{3,4}^2$  belongs to the free group  $\pi_1(\mathbb{S}^2 \setminus \{x_1, x_2, x_4\}, x_3)$  for which a basis is  $\{A_{2,3}, A_{3,4}\}$ .

(iii)  $i = 2$  and  $n = 6$ . Then by equation (3),

$$\begin{aligned}\tilde{\eta} &= \sigma_2 \sigma_4 \cdot \sigma_2 \sigma_3 \sigma_4 \sigma_5^2 \cdot \sigma_4^{-1} \sigma_2^{-1} \cdot \sigma_2 \sigma_3 \sigma_4 \sigma_5^2 = \sigma_2 \sigma_4 \cdot \sigma_1^{-2} \sigma_2^{-1} \sigma_3^{-1} \sigma_4^{-1} \cdot \sigma_4^{-1} \sigma_3 \sigma_4 \sigma_5^2 \\ &= \sigma_2 \sigma_1^{-2} \sigma_2^{-1} \cdot \sigma_4 \sigma_3^{-1} \sigma_4^{-2} \sigma_3 \sigma_4 \sigma_5^2 = \sigma_2 \sigma_1^{-2} \sigma_2^{-1} \cdot \sigma_4^2 \sigma_3^{-2} \sigma_5^2 = A_{1,3}^{-1} A_{4,5} A_{3,4}^{-1} A_{5,6}.\end{aligned} \quad (66)$$

Projecting  $\tilde{\eta}$  onto  $P_4(\mathbb{S}^2)$  by forgetting the 4<sup>th</sup> and 5<sup>th</sup> strings yields  $A_{1,3}^{-1}$ , which is of infinite order.

In all three cases, we conclude that  $\tilde{\eta}$  is of infinite order, and this completes the proof of the case  $m = 1$ .

2<sup>nd</sup> case:  $m \geq 2$ . Let  $G_1 = \langle \xi_i \alpha_i'^m \xi_i^{-1} \rangle$  and  $G_2 = \langle \alpha_i'^{2m}, \Delta_n \rangle$ , where  $\xi_i$  is as defined in equation (64). Then  $G_1 \cong \mathbb{Z}_{4q}$  and  $G_2 \cong \text{Dic}_{4q}$ , so  $G_1 \neq G_2$ . Since  $\xi_i$  commutes with  $\alpha_i'^{2m}$ , we have  $G_1 \cap G_2 = \langle \alpha_i'^{2m} \rangle \cong \mathbb{Z}_{2q}$ . By Proposition 23, to prove that the group  $H = \langle G_1 \cup G_2 \rangle$  is isomorphic to  $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \text{Dic}_{4q}$ , it suffices to show that it contains an element

of infinite order. Consider the element  $\eta = \tilde{\zeta}_i \alpha_i'^m \tilde{\zeta}_i^{-1} \cdot \Delta_n$  of  $H$ . Using equation (10), we have that

$$\eta^2 = \tilde{\zeta}_i \alpha_i'^m \tilde{\zeta}_i^{-1} \Delta_n \cdot \tilde{\zeta}_i \alpha_i'^m \tilde{\zeta}_i^{-1} \Delta_n^{-1} \Delta_n^2 = \tilde{\zeta}_i \alpha_i'^m \tilde{\zeta}_i^{-1} \tilde{\zeta}_i' \alpha_i'^{-m} \tilde{\zeta}_i'^{-1} \Delta_n^2,$$

where

$$\tilde{\zeta}_i' = \Delta_n \tilde{\zeta}_i \Delta_n^{-1} = \sigma_{2m-1+\frac{i}{2}} \sigma_{4m-1+\frac{i}{2}} \cdots \sigma_{n-2m-1-\frac{i}{2}} \sigma_{n-1-\frac{i}{2}}. \quad (67)$$

All of the generators appearing in equations (64) and (67) commute pairwise, so  $\tilde{\zeta}_i$  commutes with  $\tilde{\zeta}_i'$ . Since  $\Delta_n^2$  is central and of order 2,  $\eta$  is of infinite order if and only if  $\eta^2 \Delta_n^{-2}$  is. We now distinguish three subcases.

1<sup>st</sup> subcase:  $m = 2$ . In this case,  $4q = n - i$ ,

$$\tilde{\zeta}_i = \sigma_{1+\frac{i}{2}} \sigma_{5+\frac{i}{2}} \cdots \sigma_{n-3-\frac{i}{2}} \quad \text{and} \quad \tilde{\zeta}_i' = \sigma_{3+\frac{i}{2}} \sigma_{7+\frac{i}{2}} \cdots \sigma_{n-5-\frac{i}{2}} \sigma_{n-1-\frac{i}{2}},$$

and hence  $\alpha_i'^2 \tilde{\zeta}_i \alpha_i'^{-2} = \tilde{\zeta}_i'$  by equation (23). Since  $\tilde{\zeta}_i$  commutes with  $\alpha_i'^4$ , this implies that  $\alpha_i'^2 \tilde{\zeta}_i' \alpha_i'^{-2} = \tilde{\zeta}_i$ , and thus

$$\eta^2 \Delta_n^{-2} = \tilde{\zeta}_i \alpha_i'^2 \tilde{\zeta}_i^{-1} \tilde{\zeta}_i' \alpha_i'^{-2} \tilde{\zeta}_i'^{-1} = \tilde{\zeta}_i \tilde{\zeta}_i'^{-1} \tilde{\zeta}_i \tilde{\zeta}_i'^{-1} = \tilde{\zeta}_i^2 \tilde{\zeta}_i'^{-2}.$$

Now  $n \geq 8$  since  $q \geq 2$ , and projecting  $\eta^2 \Delta_n^{-2}$  onto  $B_4(\mathbb{S}^2)$  by forgetting all but the  $\left(1 + \frac{i}{2}\right)^{\text{th}}$ ,  $\left(2 + \frac{i}{2}\right)^{\text{th}}$ ,  $\left(5 + \frac{i}{2}\right)^{\text{th}}$  and  $\left(6 + \frac{i}{2}\right)^{\text{th}}$  strings yields the braid  $\sigma_1^2 \sigma_3^2$  of  $P_4(\mathbb{S}^2)$ , which by equation (45) is the element  $\delta_{2,0}^2$ . But this element is of infinite order by Lemma 51(c), and we conclude that  $\eta$  is also of infinite order.

2<sup>nd</sup> subcase:  $m = 3$ . Since  $q \geq 2$ , we have  $n \geq 12 + i$  and  $\tilde{\zeta}_i = \sigma_{1+\frac{i}{2}} \sigma_{7+\frac{i}{2}} \cdots \sigma_{n-5-\frac{i}{2}}$ . So

$$\begin{aligned} \pi(\eta) &= \left(1 + \frac{i}{2}, 2 + \frac{i}{2}\right) \left(7 + \frac{i}{2}, 8 + \frac{i}{2}\right) \cdots \left(n - 5 - \frac{i}{2}, n - 4 - \frac{i}{2}\right) \cdot \\ &\quad \left(1 + \frac{i}{2}, n - 2 - \frac{i}{2}, n - 5 - \frac{i}{2}, \dots, 4 + \frac{i}{2}\right) \cdot \\ &\quad \left(2 + \frac{i}{2}, n - 1 - \frac{i}{2}, n - 4 - \frac{i}{2}, \dots, 5 + \frac{i}{2}\right) \cdot \\ &\quad \left(3 + \frac{i}{2}, n - \frac{i}{2}, n - 3 - \frac{i}{2}, \dots, 6 + \frac{i}{2}\right) \cdot \left(1 + \frac{i}{2}, 2 + \frac{i}{2}\right) \left(7 + \frac{i}{2}, 8 + \frac{i}{2}\right) \cdots \\ &\quad \left(n - 5 - \frac{i}{2}, n - 4 - \frac{i}{2}\right) \cdot (1, n) (2, n - 1) \cdots \left(\frac{n}{2}, \frac{n}{2} + 1\right). \end{aligned}$$

If  $i = 0$  (resp.  $i = 2$ ), the cycle decomposition of  $\pi(\eta)$  contains the two cycles  $(1, 2, 3)$  and  $(4, n - 1, 6, n - 2, 5, n)$  (resp.  $(1, n)$  and  $(2, 3, 4)$ ), and we deduce from Theorem 1 that  $\eta$  is of infinite order.

3<sup>rd</sup> subcase:  $m \geq 4$ . Since  $q \geq 2$ , we have  $n \geq 16$ . Using equations (23) and (24), we have that

$$\begin{aligned} \alpha_i'^m \sigma_{n-1-\frac{i}{2}} \alpha_i'^{-m} &= \alpha_0^{i/2} \alpha_i^m \alpha_0^{-i/2} \sigma_{n-1-\frac{i}{2}} \alpha_0^{i/2} \alpha_i^{-m} \alpha_0^{-i/2} = \alpha_0^{i/2} \alpha_i^m \sigma_{n-1-i} \alpha_i^{-m} \alpha_0^{-i/2} \\ &= \alpha_0^{i/2} \alpha_i^{m-2} \sigma_1 \alpha_i^{-(m-2)} \alpha_0^{-i/2} = \sigma_{m-1+\frac{i}{2}}, \end{aligned}$$

from which one may see that

$$\alpha_i'^m \zeta_i \alpha_i'^{-m} = \sigma_{m+1+\frac{i}{2}} \sigma_{3m+1+\frac{i}{2}} \cdots \sigma_{n-3m+1-\frac{i}{2}} \sigma_{n-m+1-\frac{i}{2}}, \quad \text{and} \quad (68)$$

$$\alpha_i'^m \zeta_i' \alpha_i'^{-m} = \sigma_{m-1+\frac{i}{2}} \sigma_{3m-1+\frac{i}{2}} \cdots \sigma_{n-3m-1-\frac{i}{2}} \sigma_{n-m-1-\frac{i}{2}}. \quad (69)$$

The terms in each of the expressions (64), (67), (68) and (69) commute pairwise, and since  $m \geq 4$ ,  $\zeta_i$ ,  $\zeta_i'$ ,  $\alpha_i'^m \zeta_i \alpha_i'^{-m}$  and  $\alpha_i'^m \zeta_i' \alpha_i'^{-m}$  also commute pairwise. So:

$$\begin{aligned} \eta^2 \Delta_n^{-2} &= \zeta_i \cdot \alpha_i'^m \zeta_i^{-1} \alpha_i'^{-m} \cdot \alpha_i'^m \zeta_i' \alpha_i'^{-m} \cdot \zeta_i'^{-1} \\ &= \sigma_{1+\frac{i}{2}} \sigma_{m-1+\frac{i}{2}} \sigma_{m+1+\frac{i}{2}}^{-1} \sigma_{2m-1+\frac{i}{2}}^{-1} \cdots \sigma_{n-2m+1-\frac{i}{2}} \sigma_{n-m-1-\frac{i}{2}} \sigma_{n-m+1-\frac{i}{2}}^{-1} \sigma_{n-1-\frac{i}{2}}^{-1} \end{aligned}$$

and all of the terms in this expression commute pairwise. Projecting  $\eta^2 \Delta_n^{-2}$  onto  $B_4(\mathbb{S}^2)$  by forgetting all but the strings numbered  $1 + \frac{i}{2}$ ,  $2 + \frac{i}{2}$ ,  $m - 1 + \frac{i}{2}$  and  $m + \frac{i}{2}$  yields the braid  $\sigma_1 \sigma_3$ , which we know to be of infinite order from the case  $m = 2$ . So  $\eta$  and  $H$  are also of infinite order. This completes the proof of the realisation of  $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \text{Dic}_{4q}$  as a subgroup of  $B_n(\mathbb{S}^2)$  for all  $q \geq 2$  dividing  $(n - i)/2$ .

(c) Let  $q \geq 2$  be a strict divisor of  $n - i$ , where  $i \in \{0, 2\}$ , and let  $m = (n - i)/q$ . Then  $m \geq 2$ . We distinguish the cases  $m = 2$  and  $m \geq 3$ .

1<sup>st</sup> case:  $m = 2$ . Then  $2q = n - i$ , and  $n$  is even. Let  $G_1 = \langle \alpha_i'^2, \alpha_i' \Delta_n \rangle$  and  $G_2 = \zeta_i G_1 \zeta_i^{-1}$ , where  $\zeta_i = \alpha_0^{i/2} \delta_{2,i} \alpha_0^{-i/2}$ . Then  $G_1, G_2 \cong \text{Dic}_{2(n-i)} = \text{Dic}_{4q}$ . As we saw in the case  $m = 1$  of part (b) above,  $\zeta_i$  commutes with  $\alpha_i'^2$ , and so  $G_1 \cap G_2 \supset F$ , where  $F = \langle \alpha_i'^2 \rangle \cong \mathbb{Z}_{2q}$ . Let  $H = \langle G_1 \cup G_2 \rangle$ . To complete the construction, it suffices once more by Proposition 23 to show that  $H$  contains an element of infinite order. Consider the following element of  $H$ :

$$\zeta_i \alpha_i' \Delta_n \zeta_i^{-1} \cdot \alpha_i' \Delta_n \cdot \alpha_i'^2 = \zeta_i \alpha_i' \zeta_i^{-1} \alpha_i' \Delta_n^2 = \tilde{\eta} \Delta_n^2,$$

using equation (10) and the fact that  $\Delta_n$  commutes with  $\zeta_i$ , and where  $\tilde{\eta}$  is as defined in equation (65). But we saw there that  $\tilde{\eta}$  is of infinite order, so  $\tilde{\eta} \Delta_n^2$  is too, and thus  $H$  is infinite.

2<sup>nd</sup> case:  $m \geq 3$ . Then  $n \geq 6 + i$ . Set  $G_1 = \langle \alpha_i'^m, \Delta_n \rangle$  and  $G_2 = \zeta_i G_1 \zeta_i^{-1}$ , where  $\zeta_i = \alpha_0^{i/2} \delta_{m,i} \alpha_0^{-i/2}$ , and since  $\delta_{m,i}$  commutes with  $\alpha_i'^m$  by Lemma 51(c),  $\zeta_i$  commutes with  $\alpha_i'^m$ . Thus  $G_2 = \langle \alpha_i'^m, \zeta_i \Delta_n \zeta_i^{-1} \rangle$ , and  $G_1 \cap G_2 \supset \langle \alpha_i'^m \rangle \cong \mathbb{Z}_{2q}$ . To complete the construction, it suffices to show that  $H = \langle G_1 \cup G_2 \rangle$  contains an element of infinite order. Consider the element  $[\zeta_i, \Delta_n] = \zeta_i \Delta_n \zeta_i^{-1} \cdot \Delta_n^{-1} \in H$ . Then:

$$\begin{aligned} [\zeta_i, \Delta_n] &= \zeta_i \cdot \Delta_n \zeta_i^{-1} \Delta_n^{-1} = \sigma_{1+\frac{i}{2}} \sigma_{m+1+\frac{i}{2}} \cdots \sigma_{n-m+1-\frac{i}{2}} \cdot \\ &\quad \sigma_{m-1+\frac{i}{2}}^{-1} \sigma_{2m-1+\frac{i}{2}}^{-1} \cdots \sigma_{n-m-1-\frac{i}{2}}^{-1} \sigma_{n-1-\frac{i}{2}}^{-1} \\ &= \left( \sigma_{1+\frac{i}{2}} \sigma_{m-1+\frac{i}{2}}^{-1} \right) \left( \sigma_{m+1+\frac{i}{2}} \sigma_{2m-1+\frac{i}{2}}^{-1} \right) \cdots \left( \sigma_{n-m+1-\frac{i}{2}} \sigma_{n-1-\frac{i}{2}}^{-1} \right), \end{aligned}$$

where the bracketed terms commute pairwise. If  $m = 3$  then after having projected  $[\zeta_i, \Delta_n]^3$  into  $P_4(\mathbb{S}^2)$  by forgetting all but the first four strings, we carry out a calculation



similar to that of equation (63). If  $m \geq 4$ , we project  $[\xi_i, \Delta_n]$  into  $B_4(\mathbb{S}^2)$  by forgetting all but the strings numbered  $1 + \frac{i}{2}$ ,  $2 + \frac{i}{2}$ ,  $m + 1 + \frac{i}{2}$  and  $m + 2 + \frac{i}{2}$ , which yields the braid  $\sigma_1\sigma_3$  of infinite order. In both cases, we conclude that  $[\xi_i, \Delta_n]$  is of infinite order. Thus  $H$  is of infinite order, and it follows from Proposition 23 that  $H \cong \text{Dic}_{4q} *_{\mathbb{Z}_{2q}} \text{Dic}_{4q}$ .

(d) Let  $q \geq 4$  be an even divisor of  $n - i$ , and set  $m = (n - i)/q$ ,  $G_1 = \langle \alpha_i^m, \Delta_n \rangle$  and  $G_2 = \lambda_i G_1 \lambda_i^{-1}$ , where

$$\lambda_i = \prod_{j=0}^{(q-2)/2} \sigma_{m(1+2j)+\frac{i}{2}} = \sigma_{m+\frac{i}{2}} \sigma_{3m+\frac{i}{2}} \cdots \sigma_{n-3m-\frac{i}{2}} \sigma_{n-m-\frac{i}{2}}. \quad (70)$$

Then both  $G_1$  and  $G_2$  are isomorphic to  $\text{Dic}_{4q}$ , and:

$$\begin{aligned} \Delta_n \lambda_i \Delta_n^{-1} &= \Delta_n \sigma_{m+\frac{i}{2}} \sigma_{3m+\frac{i}{2}} \cdots \sigma_{n-3m-\frac{i}{2}} \sigma_{n-m-\frac{i}{2}} \Delta_n^{-1} \\ &= \sigma_{n-m-\frac{i}{2}} \sigma_{n-3m-\frac{i}{2}} \cdots \sigma_{3m+\frac{i}{2}} \sigma_{m+\frac{i}{2}} = \lambda_i \end{aligned} \quad (71)$$

by equation (7). Further, by equations (23) and (24), we have

$$\begin{aligned} \alpha_i^{2m} \sigma_{n-m-\frac{i}{2}} \alpha_i'^{-2m} &= \alpha_0^{i/2} \alpha_i^{2m} \alpha_0^{-i/2} \sigma_{n-m-\frac{i}{2}} \alpha_0^{i/2} \alpha_i^{-2m} \alpha_0^{-i/2} = \alpha_0^{i/2} \alpha_i^{2m} \sigma_{n-m-i} \alpha_i^{-2m} \alpha_0^{-i/2} \\ &= \alpha_0^{i/2} \alpha_i^{m+1} \sigma_{n-i-1} \alpha_i^{-(m+1)} \alpha_0^{-i/2} = \alpha_0^{i/2} \alpha_i^{m-1} \sigma_1 \alpha_i^{-(m-1)} \alpha_0^{-i/2} = \sigma_{m+\frac{i}{2}}, \end{aligned}$$

and from this and equation (70) it follows that  $\lambda_i$  also commutes with  $\alpha_i^{2m}$ . This fact and equation (71) imply that  $G_1 \cap G_2 \supset \langle \alpha_i^{2m}, \Delta_n \rangle \cong \text{Dic}_{2q}$ . To complete the construction, it suffices to show that the subgroup  $H = \langle G_1 \cup G_2 \rangle$  is infinite, or equivalently, that it contains an element of infinite order. We consider the two cases  $m = 1$  and  $m \geq 2$  separately.

1<sup>st</sup> case:  $m = 1$ . Then  $q = n - i \geq 4$ ,  $n$  is even and  $G_1 \cong G_2 \cong \text{Dic}_{4(n-i)}$ . If the element  $\lambda_i \alpha_i' \lambda_i^{-1}$  of  $G_2$  belonged also to  $G_1$ , since it is of order  $2(n - i) \geq 8$ , it would be an element of the subgroup  $\langle \alpha_i' \rangle$  of  $G_1$ , and so  $\lambda_i$  would belong to the normaliser of  $\langle \alpha_i' \rangle$  in  $B_n(\mathbb{S}^2)$ . Proposition 8 then implies that  $\lambda_i$  is of finite order. However,  $\lambda_i = \alpha_0^{i/2} \delta_{2,i} \alpha_0^{-i/2}$  is of infinite order by Lemma 51(c), which yields a contradiction, and so we conclude that  $G_1 \neq G_2$ . If  $\text{Dic}_{4(n-i)}$  is maximal finite in  $B_n(\mathbb{S}^2)$  then  $H$  must then be infinite, which gives the result. So suppose that  $\text{Dic}_{4(n-i)}$  is not maximal. By Theorem 2, we have  $n = 6$  and  $i = 2$ , in which case  $\lambda_2 = \sigma_2 \sigma_4$  and  $\alpha_2' = \sigma_2 \sigma_3 \sigma_4 \sigma_5^2$ . Equation (66) implies that the element  $\lambda_2 \alpha_2' \lambda_2^{-1} \cdot \alpha_2'$  of  $H$  is of infinite order as required.

2<sup>nd</sup> case:  $m \geq 2$ . Consider the element  $\rho_i = \alpha_i^m \cdot \lambda_i \alpha_i'^{-m} \lambda_i^{-1}$  of  $H$ . Then

$$\begin{aligned} \rho_i &= \alpha_i'^{(m-1)} \cdot \alpha_i' \sigma_{m+\frac{i}{2}} \sigma_{3m+\frac{i}{2}} \cdots \sigma_{n-3m-\frac{i}{2}} \sigma_{n-m-\frac{i}{2}} \alpha_i'^{-1} \\ &= \alpha_i'^{-(m-1)} \sigma_{n-m-\frac{i}{2}}^{-1} \sigma_{n-3m-\frac{i}{2}}^{-1} \cdots \sigma_{3m+\frac{i}{2}}^{-1} \sigma_{m+\frac{i}{2}}^{-1} \alpha_i'^{(m-1)} \cdot \alpha_i'^{-(m-1)} \\ &= \alpha_i'^{(m-1)} \sigma_{m+1+\frac{i}{2}} \sigma_{3m+1+\frac{i}{2}} \cdots \sigma_{n-3m+1-\frac{i}{2}} \sigma_{n-m+1-\frac{i}{2}} \\ &= \alpha_i'^{-(m-1)} \sigma_{n-2m+1-\frac{i}{2}}^{-1} \sigma_{n-4m+1-\frac{i}{2}}^{-1} \cdots \sigma_{2m+1+\frac{i}{2}}^{-1} \sigma_{1+\frac{i}{2}}^{-1} \alpha_i'^{(m-1)} \\ &= \alpha_i'^{(m-1)} \sigma_{1+\frac{i}{2}}^{-1} \sigma_{m+1+\frac{i}{2}} \sigma_{2m+1+\frac{i}{2}}^{-1} \sigma_{3m+1+\frac{i}{2}} \cdots \sigma_{n-2m+1-\frac{i}{2}}^{-1} \sigma_{n-m+1-\frac{i}{2}} \alpha_i'^{-(m-1)}, \end{aligned}$$

using equations (23) and (24), the fact that  $m \geq 2$ , as well as the relations:

$$\begin{aligned}\alpha'_i \sigma_{n-m-\frac{i}{2}} \alpha_i'^{-1} &= \alpha_0^{i/2} \alpha_i \sigma_{n-m-i} \alpha_i^{-1} \alpha_0^{-i/2} \\ &= \alpha_0^{i/2} \sigma_{n-m-i+1} \alpha_0^{-i/2} \text{ by equation (23) as } n-m-i+1 \leq n-i-1 \\ &= \sigma_{n-m+1-\frac{i}{2}} \text{ by equation (23) since } n-m+1-\frac{i}{2} \leq n-1,\end{aligned}$$

and

$$\begin{aligned}\alpha_i'^{(m-1)} \sigma_{n-2m+1-\frac{i}{2}} \alpha_i'^{-(m-1)} &= \alpha_0^{i/2} \alpha_i^{m-1} \sigma_{n-2m+1-i} \alpha_i^{-(m-1)} \alpha_0^{-i/2} \\ &= \alpha_0^{i/2} \sigma_{n-m-i} \alpha_0^{-i/2} \text{ by equation (23) as } n-m-i \leq n-i-1 \\ &= \sigma_{n-m-\frac{i}{2}}.\end{aligned}$$

So  $\rho_i$  is conjugate to

$$\sigma_{1+\frac{i}{2}}^{-1} \sigma_{2m+1+\frac{i}{2}} \sigma_{2m+1+\frac{i}{2}}^{-1} \sigma_{3m+1+\frac{i}{2}} \cdots \sigma_{n-2m+1-\frac{i}{2}}^{-1} \sigma_{n-m+1-\frac{i}{2}},$$

which under the projection onto  $B_4(\mathbb{S}^2)$  that is obtained by forgetting all but the strings numbered  $1 + \frac{i}{2}$ ,  $2 + \frac{i}{2}$ ,  $2m + 1 + \frac{i}{2}$  and  $2m + 2 + \frac{i}{2}$  yields the element  $\sigma_1^{-1} \sigma_3^{-1}$ , which we know to be of infinite order in  $B_4(\mathbb{S}^2)$ , so  $H$  is also of infinite order. This completes the proof of the realisation of  $\text{Dic}_{4q} *_{\text{Dic}_{2q}} \text{Dic}_{4q}$  as a subgroup of  $B_n(\mathbb{S}^2)$ , as well as that of Theorem 70.  $\square$

## 6.2 Realisation of $O^* *_{T^*} O^*$ in $B_n(\mathbb{S}^2)$

For the realisation of the Type II subgroups of  $B_n(\mathbb{S}^2)$  described in Definition 4(2), there is just one outstanding case not covered by Theorem 70 to be dealt with, that of  $O^* *_{T^*} O^*$ . We start by making some general comments. From Theorem 2, there is no finite subgroup of  $B_n(\mathbb{S}^2)$  that contains two copies of  $O^*$ . In particular, any subgroup of  $B_n(\mathbb{S}^2)$  generated by two distinct copies  $G_1, G_2$  of  $O^*$  is necessarily infinite. If further  $G_1 \cap G_2 \cong T^*$  then it follows from Proposition 23 that  $\langle G_1 \cup G_2 \rangle \cong O^* *_{T^*} O^*$ , from which we also obtain a subgroup isomorphic to  $T^* \rtimes \mathbb{Z}$  for one of the two actions of  $\mathbb{Z}$  on  $T^*$  of Definition 4(1)(f) and (g). Notice also that in this case, [GG5, Proposition 1.5] implies that  $G_1$  and  $G_2$  are conjugate by an element that belongs to the normaliser of  $G_1 \cap G_2$  since  $O^*$  contains a unique subgroup isomorphic to  $T^*$ . Conversely, if  $\zeta$  is an element of  $B_n(\mathbb{S}^2)$  that belongs to the normaliser of a subgroup  $K$  of  $B_n(\mathbb{S}^2)$  isomorphic to  $T^*$ , and if  $n \not\equiv 4 \pmod{6}$  then  $K$  is contained in a subgroup  $G_1$  of  $B_n(\mathbb{S}^2)$  isomorphic to  $O^*$  by Theorem 2. Either  $\zeta G_1 \zeta^{-1} = G_1$ , in which case  $\zeta$  belongs to the normaliser of  $G_1$ , or else  $G_1 \neq \zeta G_1 \zeta^{-1}$ , in which case  $\langle G_1 \cup \zeta G_1 \zeta^{-1} \rangle \cong O^* *_{T^*} O^*$  in light of the above remarks.

We now prove the realisation of  $O^* *_{T^*} O^*$  in  $B_n(\mathbb{S}^2)$  in the following cases.

**PROPOSITION 71.** *Let  $n \equiv 0, 2 \pmod{6}$ , and suppose that  $n = 36$  or  $n \geq 42$ . Then  $B_n(\mathbb{S}^2)$  possesses a subgroup that is isomorphic to  $O^* *_{T^*} O^*$ .*

REMARK 72. It follows from Theorem 5(2)(e) and Proposition 71 that the condition given in Definition 4(2)(e) for the existence of  $O^* *_T O^*$  as a subgroup of  $B_n(\mathbb{S}^2)$  is necessary and sufficient, unless  $n$  belongs to  $\{6, 8, 12, 14, 18, 20, 24, 26, 30, 32, 38\}$ . For these values of  $n$ , which are those of Remark 6(i), it is an open question as to whether  $O^* *_T O^*$  is realised as a subgroup of  $B_n(\mathbb{S}^2)$ .

*Proof of Proposition 71.* In order to obtain a subgroup of  $B_n(\mathbb{S}^2)$  that is isomorphic to  $O^* *_T O^*$ , we shall construct a copy of  $S_4 *_A S_4$  in  $\mathcal{MCG}(\mathbb{S}^2, n)$ , and then take its inverse image by the homomorphism  $\varphi$  of equation (11). Let  $n \equiv 0, 2 \pmod 6$ , and set  $n = 12m + 6\varepsilon_1 + 8\varepsilon_2$ , where  $m \in \mathbb{N}$  and  $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$ . Since  $n = 36$  or  $n \geq 42$ , we have that  $m \geq 3$ . We use the notation of the proof of Proposition 62, taking  $\Delta$  to be a cube with  $m$  (resp.  $\varepsilon_1, \varepsilon_2$ ) marked points lying on each edge (resp. at the centre of each face, at each vertex). As in that proof, we consider the group of rotations  $\Gamma \cong S_4$  of  $\Delta$  to be a subgroup of  $\text{Homeo}^+(\mathbb{S}^2, X)$ , and we set  $\tilde{\Gamma} = \Psi(\Gamma)$ , which is a subgroup of  $\mathcal{MCG}(\mathbb{S}^2, n)$  isomorphic to  $S_4$ . Choose an edge  $e$  of  $\Delta$ , fix an orientation of  $e$ , and denote the marked points lying on  $e$  by  $p_1, \dots, p_m$ ; these points are numbered coherently with the orientation of  $e$  (see Figure 9). Let  $h$  be the unique element of  $\Gamma$  different from the identity and fixing  $e$  setwise (so  $h$  reverses the orientation of  $e$ ).

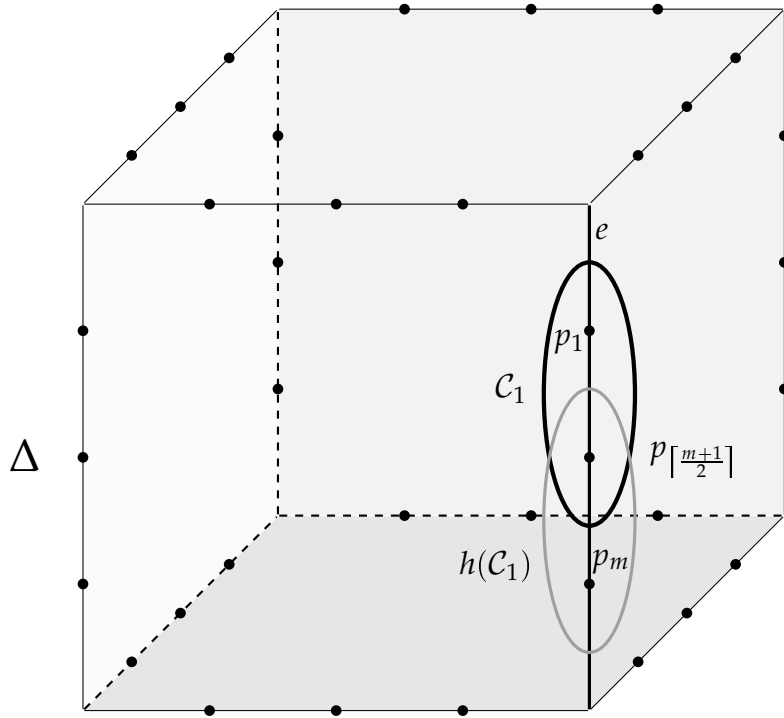


Figure 9: The construction of  $S_4 *_A S_4$  in  $B_m(\mathbb{S}^2)$ ,  $m = 36$ .

The group  $\Gamma$  possesses a unique subgroup  $\Omega = \{f_1, \dots, f_{12}\}$  isomorphic to  $A_4$ , where we take  $f_1 = \text{Id}$ . For  $i = 1, \dots, 12$ , let  $e_i = f_i(e)$ , whose orientation is that induced by  $e = e_1$ . For any two edges  $e'$  and  $e''$  of  $\Delta$ , there are precisely two elements of  $\Gamma$  that send  $e'$  to  $e''$  (as non-oriented edges). One of these elements respects the orientation, and belongs to  $\Omega$ , and the other reverses the orientation, and belongs to  $\Gamma \setminus \Omega$ . Thus

$h \in \Gamma \setminus \Omega$  and  $\Gamma = \Omega \coprod h\Omega$  since  $[\Gamma : \Omega] = 2$ . Let  $\mathcal{C}_1$  be a simple closed curve bounding a disc that is a small neighbourhood of the subsegment  $[p_1, p_{\lfloor \frac{m+1}{2} \rfloor}]$  of the edge  $e$ . Let  $g_1$  be the positive Dehn twist along  $\mathcal{C}_1$ . For  $i = 1, \dots, 12$ , let  $g_i = f_i \circ g_1 \circ f_i^{-1}$  (resp.  $g'_i = f_i \circ h \circ g_1 \circ h^{-1} \circ f_i^{-1}$ ) be the positive Dehn twist along the simple closed curve  $f_i(\mathcal{C}_1)$  (resp.  $f_i \circ h(\mathcal{C}_1)$ ). Since the stabiliser of the edge  $e$  in  $\Gamma$  is  $\langle h \rangle$ , which is isomorphic to  $\mathbb{Z}_2$ , the condition on  $\mathcal{C}_1$  implies that the  $f_i(\mathcal{C}_1)$  (resp. the  $f_i \circ h(\mathcal{C}_1)$ ) are pairwise disjoint, and that  $f_i(\mathcal{C}_1)$  and  $f_j \circ h(\mathcal{C}_1)$  are disjoint if  $i \neq j$ . We conclude that the  $g_i$  (resp. the  $g'_i$ ) commute pairwise, and that  $g_i$  and  $g'_j$  commute if  $i \neq j$ .

Let  $g = g_1 \circ \dots \circ g_{12}$ . If  $j \in \{1, \dots, 12\}$ , conjugation of  $g$  by  $f_j$  permutes the  $g_i$ , which commute pairwise, so  $g$  and  $f_j$  commute, and thus  $g$  belongs to the centraliser of  $\Omega$ . Let  $\Gamma' = g\Gamma g^{-1}$ . By construction,  $\Gamma \cong \Gamma' \cong S_4$  and  $\Gamma \cap \Gamma' \supset \Omega$ . Let  $\tilde{\Gamma} = \Psi(\Gamma)$ ,  $\tilde{\Gamma}' = \Psi(\Gamma')$  and  $\tilde{\Omega} = \Psi(\Omega)$ . Then  $\tilde{\Gamma} \cong \tilde{\Gamma}' \cong S_4$ ,  $\tilde{\Omega} \subset \tilde{\Gamma} \cap \tilde{\Gamma}'$  and  $\tilde{\Omega} \cong A_4$ . Let us show that  $H = \langle \tilde{\Gamma} \cup \tilde{\Gamma}' \rangle \cong S_4 *_{A_4} S_4$ . To do so, by Proposition 23 it suffices to prove that  $H$  is infinite, and in light of the maximality of  $S_4$  as a finite subgroup of  $\mathcal{MCG}(\mathbb{S}^2, n)$  [St], this comes down to showing that  $\tilde{\Gamma} \neq \tilde{\Gamma}'$ . It is enough to prove that  $\tilde{h}' \notin \tilde{\Gamma}$ , where  $\tilde{h}' = \Psi(h') \in \tilde{\Gamma}'$ , and  $h' = ghg^{-1}$ . To achieve this, suppose on the contrary that  $\tilde{h}' \in \tilde{\Gamma}$ . Let  $\tilde{g} = \Psi(g)$  and  $\tilde{h} = \Psi(h)$ . Since  $\tilde{h} \in \tilde{\Gamma}$ , we have  $\tilde{h}'\tilde{h}^{-1} = \tilde{\gamma} \in \tilde{\Gamma}$ , where  $\tilde{\gamma} = \Psi(\gamma)$  and  $\gamma = [g, h]$ . On the other hand,  $g$  is the product of Dehn twists, so  $\tilde{g} \in \mathcal{PMCG}(\mathbb{S}^2, n)$ , thus  $\tilde{\gamma} \in \mathcal{PMCG}(\mathbb{S}^2, n)$  by normality of  $\mathcal{PMCG}(\mathbb{S}^2, n)$  in  $\mathcal{MCG}(\mathbb{S}^2, n)$ , which implies that  $\tilde{\gamma} \in \tilde{\Gamma} \cap \mathcal{PMCG}(\mathbb{S}^2, n)$ . As we mentioned in the Introduction,  $\mathcal{PMCG}(\mathbb{S}^2, n)$  is torsion free, and the finiteness of  $\tilde{\Gamma}$  forces  $\tilde{\gamma} = 1$ . In particular, if  $\alpha: \mathcal{PMCG}(\mathbb{S}^2, n) \rightarrow \mathcal{PMCG}(\mathbb{S}^2, 4)$  is the projection induced by forgetting all of the marked points with the exception of  $p_1, p_{\lfloor \frac{m+1}{2} \rfloor}, p_m$  and  $f_2(p_1)$  (for example) then  $\alpha(\tilde{\gamma}) = 1$ .

In order to reach a contradiction, we now analyse  $\gamma$  more closely. Since  $\Omega \triangleleft \Gamma$ , there exists a permutation  $\lambda \in S_{12}$  satisfying  $\lambda(1) = 1$  such that for all  $i \in \{1, \dots, 12\}$ ,  $f_i \circ h = h \circ f_{\lambda(i)}$ . Then

$$h \circ g_{\lambda(i)} \circ h^{-1} = h \circ f_{\lambda(i)} \circ g_1 \circ f_{\lambda(i)}^{-1} \circ h^{-1} = f_i \circ h \circ g_1 \circ h^{-1} \circ f_i^{-1} = g'_i. \quad (72)$$

Hence

$$\begin{aligned} \gamma &= g \cdot h g^{-1} h^{-1} = g_1 \circ \dots \circ g_{12} \circ \left( h \left( g_{12}^{-1} \circ \dots \circ g_1^{-1} \right) h^{-1} \right) \\ &= g_1 \circ \dots \circ g_{12} \circ \left( h \left( g_{\lambda(1)}^{-1} \circ \dots \circ g_{\lambda(12)}^{-1} \right) h^{-1} \right) \quad \text{since the } g_j \text{ commute pairwise} \\ &= g_1 \circ \dots \circ g_{12} \circ g_1'^{-1} \circ \dots \circ g_{12}'^{-1} \quad \text{by equation (72)} \\ &= (g_1 \circ g_1'^{-1}) \circ \dots \circ (g_{12} \circ g_{12}'^{-1}) \quad \text{by the commutativity relations on } g_i \text{ and } g'_j. \end{aligned}$$

Now for  $i = 1, \dots, 12$ , the Dehn twists  $g_i$  and  $g'_i$  are along curves contained in a small neighbourhood of the subsegment  $[f_i(p_1), f_i(p_m)]$  of the edge  $e_i$  of  $\Delta$ , and since the homomorphism  $\alpha$  forgets all of the marked points lying outside  $e$  with the exception of  $f_2(p_1)$ , we see that  $\alpha \circ \Psi(g_i)$  and  $\alpha \circ \Psi(g'_i)$  are trivial for all  $i = 2, \dots, 12$ . In particular,  $\alpha(\tilde{\gamma}) = \alpha \circ \Psi(\gamma) = (\alpha \circ \Psi(g_1)) \circ (\alpha \circ \Psi(g_1'^{-1}))$ . Taking the four marked points of

$\mathcal{PMCG}(\mathbb{S}^2, 4)$  in the given order,  $\alpha \circ \Psi(g_1)$  (resp.  $\alpha \circ \Psi(g_1'^{-1})$ ) is a positive (resp. negative) Dehn twist along a curve that bounds a disc containing the first two (resp. the second and third) points, and so a preimage of  $\alpha(\tilde{\gamma})$  in  $P_4(\mathbb{S}^2)$  under the homomorphism  $\varphi$  of equation (11) is given by  $A_{1,2}A_{2,3}^{-1}$ , where the  $A_{i,j}$  are defined by equation (62). But this element belongs to the free subgroup  $\pi_1(\mathbb{S}^2 \setminus \{x_1, x_3, x_4\}, x_2)$  of  $P_4(\mathbb{S}^2)$  of rank two, for which a basis is  $\{A_{1,2}, A_{2,3}\}$ . Thus  $A_{1,2}A_{2,3}^{-1}$  is an element of  $P_4(\mathbb{S}^2)$  of infinite order, and taking into account equation (11), we deduce that  $\alpha(\tilde{\gamma}) = \varphi(A_{1,2}A_{2,3}^{-1})$  is also of infinite order, which contradicts the conclusion of the previous paragraph. Thus  $\tilde{h}' \notin \tilde{\Gamma}$ , and from the above arguments, we see that  $H \cong S_4 *_{A_4} S_4$ . Taking  $G = B_n(\mathbb{S}^2)$ ,  $x = \Delta_n^2$  and  $p = \varphi$ ,  $\varphi$  being as in the short exact sequence (11), in Proposition 26(b)(ii), we deduce that  $\varphi^{-1}(\tilde{\Gamma} *_{\tilde{\Omega}} \tilde{\Gamma}')$  is an infinite virtually cyclic subgroup of  $B_n(\mathbb{S}^2)$  of Type II isomorphic to  $\varphi^{-1}(\tilde{\Gamma}) *_{\varphi^{-1}(\tilde{\Omega})} \varphi^{-1}(\tilde{\Gamma}')$ . But  $\varphi^{-1}(\tilde{\Gamma}) \cong \varphi^{-1}(\tilde{\Gamma}') \cong O^*$  and  $\varphi^{-1}(\tilde{\Gamma}) \cap \varphi^{-1}(\tilde{\Gamma}') = \varphi^{-1}(\tilde{\Omega}) \cong T^*$ , so this subgroup is indeed isomorphic to  $O^* *_{T^*} O^*$ , which proves the proposition.  $\square$

## 7 Proof of the realisation of elements of $\mathbb{V}_2(n)$ in $B_n(\mathbb{S}^2)$

In this section, we bring together the results of Section 6 in order to prove Proposition 73. This will enable us to complete the proof of Theorem 5.

**PROPOSITION 73.** *Let  $n \geq 4$ . The following Type II virtually cyclic groups are realised as subgroups of  $B_n(\mathbb{S}^2)$ :*

- (a)  $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \mathbb{Z}_{4q}$ , where  $q$  divides  $(n - i)/2$  for some  $i \in \{0, 1, 2\}$ .
- (b)  $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \text{Dic}_{4q}$ , where  $q \geq 2$  divides  $(n - i)/2$  for some  $i \in \{0, 2\}$ .
- (c)  $\text{Dic}_{4q} *_{\mathbb{Z}_{2q}} \text{Dic}_{4q}$ , where  $q \geq 2$  divides  $n - i$  strictly for some  $i \in \{0, 2\}$ .
- (d)  $\text{Dic}_{4q} *_{\text{Dic}_{2q}} \text{Dic}_{4q}$ , where  $q \geq 4$  is even and divides  $n - i$  for some  $i \in \{0, 2\}$ .
- (e)  $O^* *_{T^*} O^*$ , where  $n \equiv 0, 2 \pmod{6}$  and  $n = 36$  or  $n \geq 42$ .

*Proof.* Parts (a)–(d) follow directly from Theorem 70, while part (e) follows from Proposition 71.  $\square$

*Proof of Theorem 5.* Theorem 5(1) was proved in Propositions 45 and 48 for virtually cyclic subgroups of Types I and II respectively. Theorem 5(2) was proved in Propositions 68 and 73 for virtually cyclic subgroups of Types I and II respectively. Finally, as we mentioned in Remarks 63(b), the proof of Theorem 5(3) is an immediate consequence of Proposition 62(d).  $\square$

## 8 Isomorphism classes of virtually cyclic subgroups of $B_n(\mathbb{S}^2)$ of Type II

By Theorem 5, we know which elements of  $\mathbb{V}_2(n)$  are realised as subgroups of  $B_n(\mathbb{S}^2)$ . Such subgroups are of one of the following forms:

- (a)  $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \mathbb{Z}_{4q}$ , where  $q \in \mathbb{N}$ .
- (b)  $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \text{Dic}_{4q}$ , where  $q \geq 2$ .
- (c)  $\text{Dic}_{4q} *_{\mathbb{Z}_{2q}} \text{Dic}_{4q}$ , where  $q \geq 2$ .
- (d)  $\text{Dic}_{4q} *_{\text{Dic}_{2q}} \text{Dic}_{4q}$ , where  $q \geq 4$  is even.
- (e)  $O^* *_{T^*} O^*$ .

There are of course additional constraints on  $q$  imposed by the value of  $n$ . The aim of this section is to study the isomorphism classes of these amalgamated products. As we shall see in Proposition 11, there is a single such class, with the exception of  $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ , for which there are two possible classes. In Corollary 76, we will also show that with one exception (that occurs for one of the two isomorphism classes  $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ ), each of the above amalgamated products of the form  $G *_H G$  is isomorphic to a semi-direct product  $\mathbb{Z} \rtimes G$ . We stress that Proposition 11 and Corollary 76 are consequences of the groups considered abstractly, and do not depend on the fact that they are realised as subgroups of  $B_n(\mathbb{S}^2)$ .

Let  $G$  be a group and  $H$  a normal subgroup. Let  $\text{Aut}_H(G)$  denote the subgroup of  $\text{Aut}(G)$  whose elements induce an automorphism of  $H$ . In some cases (if  $H$  is characteristic, for example), the two groups  $\text{Aut}(G)$  and  $\text{Aut}_H(G)$  coincide. We will concentrate our attention on the cases where  $G$  is either cyclic of order a multiple of 4, dicyclic, or equal to  $O^*$ . These are precisely the groups that appear as factors in the above list.

LEMMA 74.

- (a) Let  $G$  be isomorphic to  $\mathbb{Z}_{4q}$ ,  $q \geq 1$ , or to  $\text{Dic}_{4q}$ ,  $q \geq 3$ . Then  $G$  possesses a unique subgroup  $H$  that is isomorphic to  $\mathbb{Z}_{2q}$ , which is characteristic. Further, the homomorphism  $\text{Aut}(G) \rightarrow \text{Aut}(H)$  given by restriction is surjective.
- (b) Let  $G$  be isomorphic to  $O^*$ . Then  $G$  possesses a unique subgroup  $H$  isomorphic to  $T^*$ , which is characteristic. Further, the homomorphism  $\text{Aut}(G) \rightarrow \text{Aut}(H)$  given by restriction is surjective.
- (c) Let  $G$  be isomorphic to  $\mathcal{Q}_8$ . Then  $G$  possesses three subgroups  $H_1, H_2, H_3$  that are isomorphic to  $\mathbb{Z}_4$ . Further, there is an automorphism of  $\mathcal{Q}_8$  that sends  $H_i$  to  $H_j$  for all  $i, j = 1, 2, 3$ . For  $i = 1, 2, 3$ , the homomorphism  $\text{Aut}_{H_i}(G) \rightarrow \text{Aut}(H_i)$  given by restriction is surjective.
- (d) Let  $G$  be isomorphic to  $\text{Dic}_{4q}$ , where  $q \geq 4$  is even. Then  $G$  possesses two subgroups  $H_1, H_2$  that are isomorphic to  $\text{Dic}_{2q}$ , and there exists an automorphism of  $G$  that sends  $H_1$  to  $H_2$ . Further, if  $q \geq 6$ , for  $i = 1, 2$ , the homomorphism  $\text{Aut}_{H_i}(G) \rightarrow \text{Aut}(H_i)$  given by restriction is surjective.

*Proof.*

(a) If  $G$  is cyclic then the uniqueness of  $H$  is clear. Now let  $G \cong \text{Dic}_{4q}$ ,  $q \geq 3$ . If  $q$  is even (resp. odd) then  $G$  possesses three subgroups (resp. one subgroup) of index 2 because the Abelianisation of  $\text{Dic}_{4q}$  is isomorphic to  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$  (resp.  $\mathbb{Z}_2$ ), and exactly one is isomorphic to  $\mathbb{Z}_{2q}$ . In both the cyclic and dicyclic cases, the uniqueness of  $H$  implies that it is characteristic. The surjectivity of the given homomorphism  $\text{Aut}(G) \rightarrow \text{Aut}(H)$  is a consequence of the isomorphisms  $\text{Aut}(\mathbb{Z}_{4q}) \cong \mathbb{Z}_{4q}^\times$ , the group of units of  $\mathbb{Z}_{4q}$ , and  $\text{Aut}(\text{Dic}_{4q}) \cong \mathbb{Z}_{2q} \rtimes \mathbb{Z}_{2q}^\times$  (if  $\text{Dic}_{4q}$  is described by the presentation (9) then the elements of  $\text{Aut}(\text{Dic}_{4q})$  are given by automorphisms of the form  $x \mapsto x^i, y \mapsto x^j y$ , where

$1 \leq i \leq 2q - 1$  is coprime with  $2q$ , and  $1 \leq j \leq 2q$ , see [GoG3, Example 1.4] for more details).

(b) Let  $G \cong O^*$  be given by the presentation (28), and let  $H = \langle P, Q, X \rangle \cong T^*$ . Then  $G/H \cong \mathbb{Z}_2$  is the Abelianisation of  $G$ , generated by the  $H$ -coset of  $R$ , and so  $\Gamma_2(G) \cong H$ . If  $K$  is a subgroup of  $G$  isomorphic to  $T^*$  then the canonical projection  $G \rightarrow G/K$  factors through the canonical projection  $G \rightarrow G/H$ , from which it follows that  $G$  possesses a unique subgroup isomorphic to  $T^*$ . The surjectivity of  $\text{Aut}(G) \rightarrow \text{Aut}(H)$  was proved in [GoG3, Proposition 4.1].

(c) The first part is clear. Note that the automorphism  $\alpha(1)$  of  $\mathcal{Q}_8$  given in Definition 4(e)(ii) may be used to permute the  $H_i$ . If  $i \in \{1, 2, 3\}$  then the non-trivial element of  $\text{Aut}(H_i) \cong \mathbb{Z}_2$  is the restriction to  $H_i$  of conjugation on  $G$  by any element of  $G \setminus H_i$ .

(d) Let  $G$  be isomorphic to  $\text{Dic}_{4q}$ , where  $q \geq 4$  is even, and let  $G$  have the presentation given by equation (9). From part (a),  $G$  possesses exactly two subgroups isomorphic to  $\text{Dic}_{2q}$ ,  $H_k = \langle x^2, x^{k-1}y \rangle$  for  $k = 1, 2$ . The automorphism of  $G$  given by  $x \mapsto x$  and  $y \mapsto xy$  sends  $H_1$  to  $H_2$ . Suppose further that  $q \geq 6$ , and let  $f \in \text{Aut}(H_k)$ . Using the description of  $\text{Aut}(\text{Dic}_{2q})$  given in the proof of part (a), there exist  $1 \leq i \leq q - 1$ ,  $\gcd(i, q) = 1$ , and  $1 \leq j \leq q$  such that  $f(x^2) = x^{2i}$  and  $f(x^{k-1}y) = x^{2j} \cdot x^{k-1}y$ . Since  $q$  is even,  $i$  is odd, so  $\gcd(i, 2q) = 1$ , and  $f$  is the restriction to  $H$  of the automorphism  $x \mapsto x^i$ ,  $y \mapsto x^{(1-i)(k-1)+2j}y$  of  $G$ . Hence the homomorphism  $\text{Aut}_{H_i}(G) \rightarrow \text{Aut}(H_i)$  is surjective.  $\square$

REMARKS 75.

(a) In the case  $q = 4$  of Lemma 74(d),  $\text{Aut}(\mathcal{Q}_{16}) \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_4^\times$ , while  $\text{Aut}(\mathcal{Q}_8) \cong S_4$ , so the homomorphism  $\text{Aut}_{\mathcal{Q}_8}(\mathcal{Q}_{16}) \rightarrow \text{Aut}(\mathcal{Q}_8)$  clearly cannot be surjective.

(b) Note that Proposition 11 depends only on the amalgamated products considered in an abstract sense, and does not use the fact that the groups are realised as subgroups of  $B_n(\mathbb{S}^2)$ .

We now come to the proof of Proposition 11.

*Proof of Proposition 11.* First suppose that  $G_1 *_F G_2$  is one of the amalgamated products (a)–(e) appearing in the above list, with the exception of the group  $\mathcal{Q}_{16} *_F \mathcal{Q}_8$ . Then for  $k = 1, 2$ , there exist embeddings  $i_k: F \rightarrow G_k$  that give rise to the amalgamated product  $G_1 *_F G_2$ . Suppose that there exists another amalgamated product  $G_1 *_F' G_2$  involving the same groups, and for  $k = 1, 2$ , let  $j_k: F \rightarrow G_k$  be the associated embeddings. Let  $i_k^{-1}: i_k(F) \rightarrow F$  denote the inverse of the restriction  $i_k: F \rightarrow i_k(F)$ . Then  $j_k \circ i_k^{-1}: i_k(F) \rightarrow j_k(F)$  is an isomorphism of subgroups of  $G_k$  isomorphic to  $F$ , and so by Lemma 74, there exists  $\rho_k \in \text{Aut}(G_k)$  whose restriction to  $j_k(F)$  is sent to  $i_k(F)$ , in other words, the upper left hand ‘square’ of the diagram given in Figure 10 commutes, where all of the arrows from  $i_k(F)$  and  $j_k(F)$  to  $G_k$  are inclusions. Thus  $\rho_k \circ j_k \circ i_k^{-1}$  is an automorphism of  $i_k(F)$ , and so once more by Lemma 74, there exists  $\lambda_k \in \text{Aut}(G_k)$  whose restriction to  $i_k(F)$  is equal to  $\rho_k \circ j_k \circ i_k^{-1}$ , in other words, the lower ‘square’ of the diagram commutes. Hence  $\rho_k^{-1} \circ \lambda_k \in \text{Aut}(G_k)$ , and the restriction of this automorphism to  $i_k(F)$  yields the isomorphism  $j_k \circ i_k^{-1}: i_k(F) \rightarrow j_k(F)$ . Taking  $\theta = \rho_k^{-1} \circ \lambda_k$  and applying Proposition 27, we see that  $G_1 *_F' G_2 \cong G_1 *_F G_2$ , which gives the result in this case.

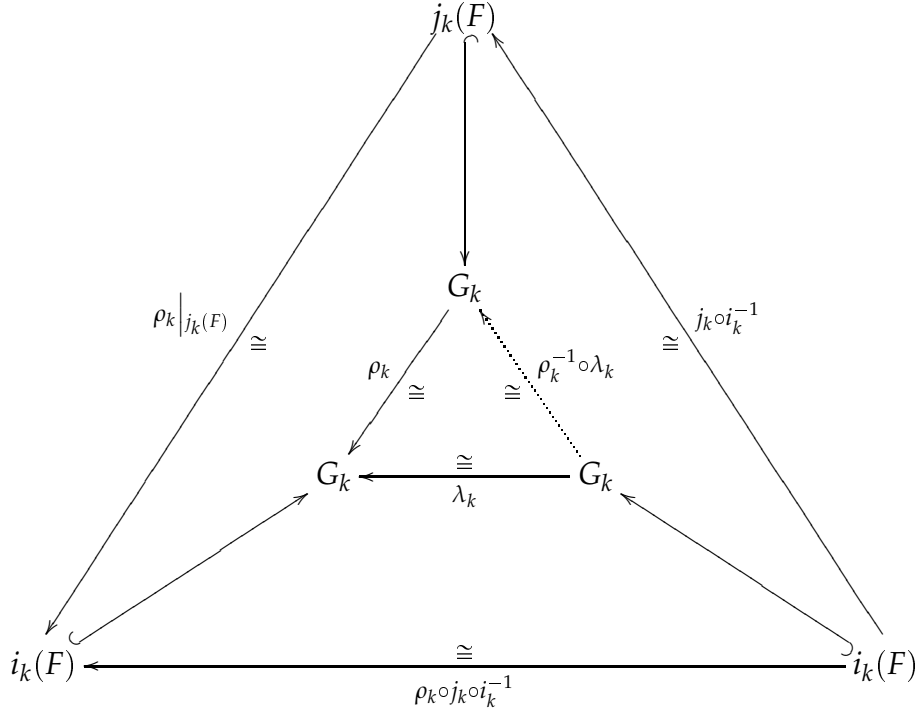


Figure 10: The commutative diagram involving the embeddings  $i_k$  and  $j_k$ .

We now turn to the exceptional case of  $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ . We have already seen that  $\mathcal{Q}_{16}$  possesses two subgroups isomorphic to  $\mathcal{Q}_8$ , and that there exists an automorphism of  $\mathcal{Q}_{16}$  that sends one subgroup into the other. Applying Proposition 27 in a manner similar to that of the previous paragraph, it thus suffices to restrict our attention to one of these subgroups. It remains to understand the amalgamated products obtained by considering all possible embeddings of  $\mathcal{Q}_8$  whose image in each of the two copies of  $\mathcal{Q}_{16}$  is fixed. So let us consider the two copies of  $\mathcal{Q}_{16}$  of the form

$$G_1 = \langle x, y \mid x^4 = y^2, yxy^{-1} = x^{-1} \rangle \quad \text{and} \quad G_2 = \langle a, b \mid a^4 = b^2, bab^{-1} = a^{-1} \rangle$$

respectively, and let  $H_1 = \langle x^2, y \rangle$  and  $H_2 = \langle a^2, b \rangle$  be their respective fixed subgroups isomorphic to  $\mathcal{Q}_8$ . Let  $F = \langle P, Q \mid P^2 = Q^2, QPQ^{-1} = P^{-1} \rangle$  be an abstract copy of  $\mathcal{Q}_8$ . Up to isomorphism, every amalgamated product of  $G_1$  and  $G_2$  along  $F$  is obtained via an isomorphism between  $H_1$  and  $H_2$ . This leads to twenty-four possibilities that we identify with the elements of  $\text{Aut}(F) \cong S_4$  without further comment (see case (1) of Section I.3). Let  $\delta: F \rightarrow H_1$  be a fixed isomorphism, which we shall take to be defined by  $\delta(P) = x^2$  and  $\delta(Q) = y$ . Suppose that  $\varphi, \varphi': H_1 \rightarrow H_2$  are isomorphisms that differ by the inner automorphism  $\iota_h$  of  $H_2$ , where  $h \in H_2$ , and let  $G_1 *_{\varphi} G_2$  and  $G_1 *'_{\varphi'} G_2$  denote the respective amalgamated products. Then  $\varphi' = \iota_h \circ \varphi$ , and we have



the following commutative diagram:

$$\begin{array}{ccccc}
& & G_1 & \xlongequal{\quad\quad\quad} & G_1 \\
& \swarrow & \delta \searrow & & \swarrow \\
G_1 *'_F G_2 & & F & & G_1 *_F G_2 \\
& \nwarrow & \varphi' \circ \delta \swarrow & & \nwarrow \\
& & G_2 & \xlongequal{\quad\quad\quad} & G_2 \\
& & \iota_h \longleftarrow & & 
\end{array} \quad (73)$$

where we also denote the extension of  $\iota_h$  to  $G_2$  by  $\iota_h$ . Taking  $\theta_1 = \text{Id}_{G_1}$  and  $\theta_2 = \iota_h$  in Proposition 27 leads to the conclusion that  $G_1 *_F G_2 \cong G_1 *'_F G_2$  if  $\varphi$  and  $\varphi'$ , considered as elements of  $\text{Aut}(F)$ , project to the same element of  $\text{Out}(F)$ . So it suffices to consider the six following coset representatives of  $\text{Out}(F)$  in  $\text{Aut}(F)$  (recall from case (1) of Section I.3, that  $\text{Out}(\mathcal{Q}_8) \cong S_3$ ):

$$\begin{aligned}
\varphi_1: x^2 &\mapsto a^2, y \mapsto b, x^2y \mapsto a^2b, & \varphi_2: x^2 &\mapsto b, y \mapsto a^2b, x^2y \mapsto a^2, \\
\varphi_3: x^2 &\mapsto a^2b, y \mapsto a^2, x^2y \mapsto b, & \varphi_4: x^2 &\mapsto a^2, y \mapsto a^2b, x^2y \mapsto b^{-1}, \\
\varphi_5: x^2 &\mapsto a^2b, y \mapsto b^{-1}, x^2y \mapsto a^2, & \varphi_6: x^2 &\mapsto b, y \mapsto a^{-2}, x^2y \mapsto a^2b.
\end{aligned} \quad (74)$$

Let  $\psi \in \text{Aut}(G_2)$  be defined by  $a \mapsto a, b \mapsto a^2b$ . Then  $\varphi_4 = \psi \circ \varphi_1$  (resp.  $\varphi_5 = \psi \circ \varphi_2$ ). Consider the above diagram (73), and replace  $\varphi$  by  $\varphi_1$  (resp.  $\varphi_2$ ),  $\varphi'$  by  $\varphi_4$  (resp.  $\varphi_5$ ), and  $\iota_h$  by the automorphism of  $G_2$  given by  $a \mapsto a, b \mapsto a^2b$ . Applying Proposition 27 implies that the two automorphisms  $\varphi_1$  (resp.  $\varphi_2$ ) and  $\varphi_4$  (resp.  $\varphi_5$ ) give rise to isomorphic amalgamated products. If  $\psi' \in \text{Aut}(G_2)$  is defined by  $a \mapsto a^{-1}, b \mapsto a^2b$ , then  $\varphi_6 = \psi' \circ \varphi_3$ , and a similar argument shows that  $\varphi_3$  and  $\varphi_6$  also give rise to isomorphic amalgamated products. Now let  $G_1 *_F G_2$  and  $G_1 *'_F G_2$  be the amalgamated products associated with  $\varphi_2$  and  $\varphi_3$  respectively. Let  $\delta': F \rightarrow H_1$  be the isomorphism defined by  $\delta'(P) = x^{-2}$  and  $\delta'(Q) = x^2y$ , let  $\theta_1 \in \text{Aut}(G_1)$  be defined by  $x \mapsto x^{-1}, y \mapsto x^2y$ , and let  $\theta_2 \in \text{Aut}(G_2)$  be defined by  $a \mapsto a, b \mapsto a^2b^{-1}$ . Then the following diagram commutes:

$$\begin{array}{ccccc}
& & G_1 & \xlongequal{\quad\quad\quad} & G_1 \\
& \swarrow & \delta' \searrow & & \swarrow \\
G_1 *'_F G_2 & & F & & G_1 *_F G_2 \\
& \nwarrow & \varphi_3 \circ \delta' \swarrow & & \nwarrow \\
& & G_2 & \xlongequal{\quad\quad\quad} & G_2 \\
& & \theta_2 \longleftarrow & & 
\end{array}$$

So  $\varphi_2$  and  $\varphi_3$  give rise to isomorphic amalgamated products by Proposition 27. We conclude that there are at most two non-isomorphic amalgamated products of the form  $K_i = G_1 *_F G_2$ , defined by the automorphism  $\varphi_i$ , where  $i \in \{1, 2\}$ .

To complete the proof, we now prove that  $K_1 \not\cong K_2$ . We start by showing that  $K_1 \cong \mathbb{Z} \rtimes \mathcal{Q}_{16}$ , where the action shall be defined presently. By definition,

$$K_1 = \langle x, y, a, b \mid x^4 = y^2, a^4 = b^2, yxy^{-1} = x^{-1}, bab^{-1} = a^{-1}, x^2 = a^2, y = b \rangle. \quad (75)$$

Let  $N$  be the infinite cyclic subgroup of  $K_1$  generated by  $t = xa^{-1}$ . Using the presentation (75), one may check that

$$vtv^{-1} = \begin{cases} t^{-1} & \text{if } v \in \{x, a\} \\ t & \text{if } v \in \{y, b\}, \end{cases}$$

so  $N$  is normal in  $K_1$ . A presentation of the quotient  $K_1/N$  is obtained by adjoining the relation  $x = a$  to that of  $K_1$ , from which it follows that

$$K_1/N = \langle a, b \mid a^4 = b^2, bab^{-1} = a^{-1} \rangle \cong \mathcal{Q}_{16}.$$

Considered as a subgroup of  $K_1$ ,  $G_2 = \langle a, b \rangle$  is isomorphic to  $\mathcal{Q}_{16}$ , which implies that the short exact sequence

$$1 \longrightarrow N \longrightarrow K_1 \longrightarrow K_1/N \longrightarrow 1$$

splits, and so  $K_1 \cong \mathbb{Z} \rtimes \mathcal{Q}_{16}$ . The action of  $K_1/N$  on  $N$  is defined as follows:

$$wtw^{-1} = \begin{cases} t^{-1} & \text{if } w \in G_2 \setminus \langle a^2, b \rangle \\ t & \text{if } w \in \langle a^2, b \rangle. \end{cases} \quad (76)$$

To see that  $K_1 \not\cong K_2$ , let us suppose on the contrary that  $K_1 \cong K_2$  and argue for a contradiction. By definition,

$$K_2 = \langle x, y, a, b \mid x^4 = y^2, a^4 = b^2, yxy^{-1} = x^{-1}, bab^{-1} = a^{-1}, x^2 = b, y = a^2b \rangle. \quad (77)$$

From this presentation, we obtain:

$$\begin{aligned} ax \cdot x^2 \cdot x^{-1}a^{-1} &= ax^2a^{-1} = aba^{-1} = a^2b = y, \\ ax \cdot y \cdot x^{-1}a^{-1} &= ax^2ya^{-1} = aba^2ba^{-1} = a^2 = x^2y \\ ax \cdot x^2y \cdot x^{-1}a^{-1} &= yx^2y = x^2. \end{aligned}$$

So  $K_2$  possesses a copy  $\langle x^2, y \rangle$  of  $\mathcal{Q}_8$  and an element  $ax$  such that conjugation by  $ax$  permutes the subgroups  $\langle x^2 \rangle$ ,  $\langle y \rangle$  and  $\langle x^2y \rangle$  cyclically. Since  $K_1 \cong K_2$  by hypothesis,  $K_1$  thus possesses a subgroup  $H$  isomorphic to  $\mathcal{Q}_8$  and an element  $z$  (of infinite order) such that  $zLz^{-1} \neq L$  for every subgroup  $L$  of  $H$  of order 4. We take  $K_1$  to be described by the semi-direct product  $\mathbb{Z} \rtimes G_2$ , where the action is given by equation (76). In particular,  $K_1 = \langle a, b, t \rangle$ , and there exist  $s, \lambda, \mu \in \mathbb{Z}$  such that  $z = t^s a^\lambda b^\mu$ . Consider the projection  $p: \mathbb{Z} \rtimes G_2 \longrightarrow G_2$  onto the second factor. Since  $\text{Ker}(p) = \mathbb{Z}$  is torsion free,  $p(H)$  is isomorphic to  $\mathcal{Q}_8$ , and thus must be equal to one of the two subgroups of  $G_2$  isomorphic to  $\mathcal{Q}_8$ . These two subgroups both contain  $a^2$ , so there exists  $u \in H$  of order 4 such that  $p(u) = a^2$ . Now  $p(a^2) = a^2$ , hence there exists  $m \in \mathbb{Z}$  such that  $u = t^m a^2$ . But  $t$  commutes with  $a^2$  by equation (76), and since  $u$  and  $a^2$  are of finite order, and  $t$  is of infinite order, it follows that  $m = 0$ ,  $u = a^2$  and:

$$zuz^{-1} = t^s a^\lambda b^\mu a^2 b^{-\mu} a^{-\lambda} t^{-s} = t^s a^\lambda a^{2\varepsilon} a^{-\lambda} t^{-s} = a^{2\varepsilon},$$

where  $\varepsilon$  is equal to 1 (resp.  $-1$ ) if  $\mu$  is even (resp. odd), and so  $z\langle u \rangle z^{-1} = \langle u \rangle$ . This contradicts the fact that  $zLz^{-1} \neq L$  for every subgroup  $L$  of  $H$  of order 4, and completes the proof of the fact that  $K_1 \not\cong K_2$ .  $\square$

Combining Proposition 20 and Lemma 74 yields an alternative description of most of the amalgamated products of the form  $G *_H G$  appearing in Proposition 73 as semi-direct products of  $\mathbb{Z}$  by  $G$ .

**COROLLARY 76.** *Let  $\Gamma = G *_H G$  be an amalgamated product, where  $G$  and  $H$  satisfy one of the following conditions:*

- (a)  $G$  is isomorphic to  $\mathbb{Z}_{4q}$  or  $\text{Dic}_{4q}$  and  $H$  is isomorphic to  $\mathbb{Z}_{2q}$ .
- (b)  $G$  is isomorphic to  $\text{Dic}_{4q}$ ,  $q \geq 6$  is even and  $H$  is isomorphic to  $\text{Dic}_{2q}$ .
- (c)  $q = 4$ ,  $G \cong \mathcal{Q}_{16}$ ,  $H \cong \mathcal{Q}_8$  and  $\Gamma$  is isomorphic to  $K_1$ .
- (d)  $G$  is isomorphic to  $O^*$  and  $H$  is isomorphic to  $T^*$ .

Then  $\Gamma \cong \mathbb{Z} \rtimes G$ , where

$$gtg^{-1} = \begin{cases} t & \text{if } g \in H \\ t^{-1} & \text{if } g \in G \setminus H, \end{cases}$$

$t$  being a generator of the  $\mathbb{Z}$ -factor.

*Proof.* First let  $G$  and  $H$  satisfy one of the conditions (a), (b) or (d). If  $i_1, i_2$  are the embeddings of  $H$  into each of the  $G$ -factors of  $\Gamma$  then  $i_2 \circ i_1^{-1}$  is an automorphism of  $H$  that extends to an automorphism of  $G$  by Lemma 74. The result then follows from Proposition 20. Now suppose that  $G$  and  $H$  satisfy condition (c). Since  $\Gamma$  is isomorphic to  $K_1$ , using the presentation (75), we see that the isomorphism  $\langle x^2, y \rangle \rightarrow \langle a^2, b \rangle$  of the amalgamating subgroup of  $K_2$  isomorphic to  $\mathcal{Q}_8$  that given by  $x^2 \mapsto a^2$  and  $y \mapsto b$  extends to an isomorphism  $\langle x, y \rangle \rightarrow \langle a, b \rangle$  of the factors that are isomorphic to  $\mathcal{Q}_{16}$ , where the extension is given by  $x \mapsto a$  and  $y \mapsto b$ . Once more, Proposition 20 yields the result.  $\square$

The following two results will imply the existence of subgroups of  $B_n(\mathbb{S}^2)$  isomorphic to  $K_1$  and  $K_2$  for all but a finite number of even values of  $n$ . The first proposition holds in general, while the second makes use of the structure of  $B_n(\mathbb{S}^2)$ .

**PROPOSITION 77.** *Let  $G$  be a group that is isomorphic to  $O^* *_T O^*$ . Then  $G$  possesses a subgroup that is isomorphic to  $K_2$ .*

*Proof.* Suppose that  $G$  is isomorphic to  $O^* *_T O^*$ . Let  $G_1, G_2$  be subgroups of  $G$  isomorphic to  $O^*$  such that  $F = G_1 \cap G_2 \cong T^*$  and  $G = \langle G_1 \cup G_2 \rangle \cong O^* *_T O^*$ . Let  $Q$  be the unique subgroup of  $F \cong \mathcal{Q}_8 \times \mathbb{Z}_3$  that is isomorphic to  $\mathcal{Q}_8$ . By Lemma 74(b),  $F$  is the unique subgroup of  $G_i$  isomorphic to  $T^*$  for  $i = 1, 2$ . From the proof of Proposition 85(b), if  $i \in \{1, 2\}$ , the Sylow 2-subgroups of  $G_i$  consist of three conjugate subgroups isomorphic to  $\mathcal{Q}_{16}$  that contain  $Q$ . Let  $H_1$  be one of the Sylow 2-subgroups of  $G_1$  with presentation

$$H_1 = \langle a, b \mid a^4 = b^2, bab^{-1} = a^{-1} \rangle.$$

Since the subgroups of  $H_1$  isomorphic to  $\mathcal{Q}_8$  are of the form  $\langle a^2, a^\varepsilon b \rangle$ ,  $\varepsilon \in \{0, 1\}$ , by replacing  $b$  by  $ab$  if necessary in the presentation of  $H_1$ , we may suppose that  $Q = \langle a^2, b \rangle$ . Now let  $H_2$  be a subgroup of  $G_2$  that is isomorphic to  $\mathcal{Q}_{16}$ . Since for  $i \in \{1, 2\}$ ,  $H_i \not\subset F$ , it follows that  $H_1 \cap H_2 = Q$  and that  $H_i$  contains elements of  $G_i \setminus F$ , whence  $H =$

$\langle H_1 \cup H_2 \rangle \cong \mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ . The proof of Proposition 11 implies that  $H$  is isomorphic to one of  $K_1$  and  $K_2$ . If  $H \cong K_2$  then we are done. So suppose that  $H \cong K_1$ . Then by equation (75), there exist generators  $x, y$  of  $H_2$  such that  $x^4 = y^2$ ,  $yx y^{-1} = x^{-1}$ ,  $x^2 = a^2$ ,  $y = b$  and  $Q = \langle x^2, y \rangle$ . Since  $Q$  is the unique subgroup of  $F$  that is isomorphic to  $\mathcal{Q}_8$ , there exists  $t \in F$  such that  $tx^2t^{-1} = y$  and  $tyt^{-1} = x^2y$  corresponding to the action of  $\mathbb{Z}_3$  on  $\mathcal{Q}_8$ . Now  $F \subset G_2$ , so  $tG_2t^{-1} = G_2$ . Let  $H'_2 = tH_2t^{-1} \subset G_2$ . Then  $x' = txt^{-1}$  and  $y' = tyt^{-1}$  are generators of  $H'_2$  satisfying  $x'^4 = y'^2$  and  $y'x'y'^{-1} = x'^{-1}$ . Now  $x'^2 = tx^2t^{-1} = y = b$  and  $y' = tyt^{-1} = x^2y = a^2b$ , and since  $H_1 \cap H'_2 = Q$ , it follows from equation (77) that  $\langle H_1 \cup H'_2 \rangle \cong K_2$  as required.  $\square$

PROPOSITION 78. Let  $n \geq 4$  be even.

(a) There exists a subgroup of  $B_n(\mathbb{S}^2)$  isomorphic to  $K_1$ .

(b) Suppose that either  $n \equiv 0 \pmod{4}$  or  $n \equiv 10 \pmod{12}$ . There exists a subgroup of  $B_n(\mathbb{S}^2)$  isomorphic to  $K_2$ .

REMARK 79. Let  $n \geq 4$  be even. Propositions 71, 77 and 78 imply that  $B_n(\mathbb{S}^2)$  possesses subgroups isomorphic to  $K_1$  and  $K_2$  with the possible exception of  $K_2$  when  $n \in \{6, 14, 18, 26, 30, 38\}$ .

*Proof of Proposition 78.*

(a) Suppose that  $n \geq 4$  is even, let  $i \in \{0, 2\}$  be such that  $4 \mid n - i$ , and let  $m = (n - i)/4$ . In the construction of  $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$  in  $B_n(\mathbb{S}^2)$  given in part (d) of the proof of Theorem 70, we have that  $G_1 = \langle x, y \rangle$  and  $G_2 = \langle a, b \rangle$ , where  $x = \alpha_i^m$ ,  $a = \lambda_i \alpha_i^m \lambda_i^{-1}$  and  $y = b = \Delta_n$ , where  $\lambda_i = \sigma_{m+\frac{i}{2}} \sigma_{3m+\frac{i}{2}}$ . Since  $\lambda_i$  commutes with  $\alpha_i^{2m}$ , we have also that  $x^2 = a^2$ . So  $\langle G_1 \cup G_2 \rangle$  is isomorphic to  $K_1$  by equation (75).

(b) We consider the two cases separately.

(1)  $n \equiv 0 \pmod{4}$ . Set  $G_1 = \langle a, b \rangle$ , where  $a = \alpha_0^{n/4}$  and  $b = \Delta_n$ , let  $G_2 = \nu G_1 \nu^{-1}$ , where  $\nu = \alpha_0^{n/4} \Omega_2$  is as in the proof of Proposition 60(a), and let  $x = \nu a \nu^{-1}$  and  $y = \nu b^{-1} \nu^{-1}$  be generators of  $G_2$ . Then  $G_1 \cong G_2 \cong \mathcal{Q}_{16}$ , and  $F = \langle a^2, b \rangle$  is isomorphic to  $\mathcal{Q}_8$ . Since  $\nu F \nu^{-1} = F$  by Proposition 60(a), it follows that  $G_1 \cap G_2 \supset F$ . Suppose that  $x \in G_1$ . Since  $x$  is of order 8, there exists  $j \in \{1, 3, 5, 7\}$  such that  $x = a^j$ , and so  $x^2 \in \{\alpha_0^{n/2}, \alpha_0^{-n/2}\}$ . On the other hand, using Lemma 59(f) and equation (10), we have:

$$x^2 = \alpha_0^{n/4} \Omega_2 \alpha_0^{n/2} \Omega_2^{-1} \alpha_0^{-n/4} = \alpha_0^{n/4} \Delta_n \alpha_0^{-n/4} = \alpha_0^{n/2} \Delta_n = a^2 b, \quad (78)$$

and so  $x^2$  does not belong to  $\{\alpha_0^{n/2}, \alpha_0^{-n/2}\}$ , which gives a contradiction. We thus conclude that  $x \notin G_1$ , and so  $G_1 \cap G_2 = F$ . Let  $K = \langle G_1 \cup G_2 \rangle$ . By equation (60), we have

$$y = \nu b^{-1} \nu^{-1} = \nu \Delta_n^{-1} \nu^{-1} = \alpha_0^{n/2} = a^2. \quad (79)$$

Equations (78) and (79) correspond to the automorphism  $\varphi_3$  of equation (74), which using the proof of Proposition 11, will imply that  $K \cong K_2$  provided that  $K$  is indeed isomorphic to  $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ . By Proposition 23, it thus suffices to show that  $K$  is infinite. To see this, we consider the following three cases.

(i)  $n = 4$ . Since the maximal finite subgroups of  $B_4(\mathbb{S}^2)$  are isomorphic to  $Q_{16}$  and  $T^*$  by Theorem 2, the fact that  $G_1 \neq G_2$  implies that  $K$  is infinite.

(ii)  $n = 8$ . Let  $\gamma = a^{-2}vav^{-1}a \in K$ . Then

$$\begin{aligned}\gamma &= \alpha_0^{-2}\Omega_2\alpha_0^2\Omega_2^{-1} = \sigma_3\sigma_4\sigma_5\sigma_3\sigma_4\sigma_3\sigma_5^{-1}\sigma_6^{-1}\sigma_5^{-1}\sigma_7^{-1}\sigma_6^{-1}\sigma_5^{-1} \\ &= \sigma_5(\sigma_3\sigma_4\sigma_5\sigma_3\sigma_4\sigma_5^{-1}\sigma_6^{-1}\sigma_5^{-1}\sigma_7^{-1}\sigma_6^{-1})\sigma_5^{-1} \\ &= \sigma_5(\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6^{-1}\sigma_5^{-1}\sigma_7^{-1}\sigma_6^{-1})\sigma_5^{-1}.\end{aligned}$$

by equations (23) and (51). The braid  $\hat{\gamma} = \sigma_4\sigma_3\sigma_5\sigma_4\sigma_6^{-1}\sigma_5^{-1}\sigma_7^{-1}\sigma_6^{-1}$  is conjugate to  $\gamma$ , and its geometric representation is given in Figure 11. Forgetting the 2<sup>nd</sup>, 4<sup>th</sup>, 6<sup>th</sup> and 8<sup>th</sup>

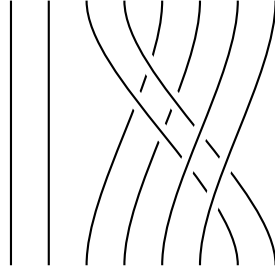


Figure 11: The braid  $\hat{\gamma}$  in  $B_8(\mathbb{S}^2)$ .

strings of  $\hat{\gamma}$  yields the braid  $\sigma_2\sigma_3^{-1}$  of  $B_4(\mathbb{S}^2)$ , which may be seen to be of infinite order using an argument similar to that of equation (63). Hence  $\hat{\gamma}$  and  $\gamma$  are of infinite order, and so  $K$  is infinite.

(iii)  $n \geq 12$ . Consider the element  $\gamma' = vav \cdot a^{-1} = \alpha_0^{n/4}\Omega_2\alpha_0^{n/4}\Omega_2^{-1}\alpha_0^{-n/2} \in K$ , and let  $n/2 + 1 \leq t \leq 3n/4$ . Then  $\pi(\alpha_0^{n/4})(t) = t - n/4$ ,  $\pi(\Omega_2)(t - n/4) = t - n/4$  since  $t - n/4 \leq n/2$ ,  $\pi(\alpha_0^{n/4})(t - n/4) = t - n/2$ , and  $\pi(\Omega_2^{-1}\alpha_0^{-n/2})(t - n/2) = t$ , so  $\pi(\gamma')(t) = t$ . Further,  $\pi(\gamma')(1) = n$ . Thus the cycle decomposition of  $\pi(\gamma')$  has at least  $n/4 \geq 3$  fixed points, and at least one non-trivial cycle. Theorem 1 then implies that  $\gamma'$  is of infinite order, and so  $K$  is infinite.

(2)  $n \equiv 10 \pmod{12}$ . Then 4 divides  $n - 2$ , and we may write  $n - 2 = 2^r s$ , where  $r \geq 2$  and  $s \in \mathbb{N}$  is odd. Since  $n$  is even,  $B_n(\mathbb{S}^2)$  possesses a subgroup  $L$  isomorphic to  $T^* \cong Q_8 \rtimes \mathbb{Z}_3$  by Theorem 2. The fact that the action by conjugation of the generator of  $\mathbb{Z}_3$  permutes cyclically the elements  $i, j$  and  $k$  of the subgroup  $Q$  of  $L$  isomorphic to  $Q_8$  implies that these elements are pairwise conjugate in  $L$ . On the other hand, by [GG5, Proposition 1.5],  $\langle \alpha_2^{n/2}, \Delta_n \rangle$  represents the unique conjugacy class of the group  $Q_{2^{r+2}}$  in  $B_n(\mathbb{S}^2)$ , and it possesses two subgroups  $\langle \alpha_2^{n/2^r}, \Delta_n \rangle$  and  $\langle \alpha_2^{n/2^r}, \alpha_2^{n/2^r} \Delta_n \rangle$  isomorphic to  $Q_{2^{r+1}}$  that contain respectively  $\Gamma_0 = \langle \alpha_2^{2^{r-1}s}, \Delta_n \rangle$  and  $\Gamma_1 = \langle \alpha_2^{2^{r-1}s}, \alpha_2^{n/2^r} \Delta_n \rangle$  which are subgroups isomorphic to  $Q_8$ . The fact that  $n/2$  and  $s$  are odd implies that  $\pi(\Delta_n) = \pi(\alpha_2^{n/2}) = n - 1 \pmod{2(n - 1)}$ . In particular,  $\alpha_2^{2^{r-1}s}$  and  $\Delta_n$  are not conjugate, so  $\Gamma_0$  is neither conjugate to  $\Gamma_1$  nor to  $Q$ , and since  $B_n(\mathbb{S}^2)$  possesses two conjugacy classes of subgroups isomorphic to  $Q_8$  [GG5, Proposition 1.5], we deduce that  $\Gamma_1$  and  $Q$  are conjugate. Set  $G_1 = \langle a, b \rangle$ , where  $a = \alpha_2^{2^{r-2}s}$  and  $b = \alpha_2^{n/2^r} \Delta_n$ . Then  $G_1$  is isomorphic to

$\mathcal{Q}_{16}$ , and it contains  $\Gamma_1 = \langle a^2, b \rangle$ . Since  $B_n(\mathbb{S}^2)$  possesses two isomorphism classes of subgroups isomorphic to  $\mathcal{Q}_8$ , we deduce that  $\Gamma_1$  and  $Q$  are conjugate, and using the fact that  $Q$  is a subgroup of  $L$ , there exists an element  $z \in B_n(\mathbb{S}^2)$  conjugate to an element of  $L \setminus Q$  for which  $za^2z^{-1} = b$ ,  $z bz^{-1} = a^2b$  and  $za^2bz^{-1} = a^2$ . Let  $G_2 = zG_1z^{-1}$ . From the action by conjugation of  $z$  on  $\Gamma_1$ , we have that  $G_1 \cap G_2 \supset \Gamma_1$ . Suppose that  $G_1 = G_2$ . Then  $zaz^{-1} \in G_2$ , which is of order 8, would be equal to an element of order 8 of  $G_1$ , and so would be of the form  $a^j$ ,  $j \in \{1, 3, 5, 7\}$ . Thus  $za^2z^{-1} \in \{a^2, a^{-2}\}$ , which is not possible. This implies that  $G_1 \neq G_2$ , and thus  $G_1 \cap G_2 = \Gamma_1$ . Let  $K = \langle G_1 \cup G_2 \rangle$ . To see that  $K \cong \mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ , by Proposition 23 it suffices to prove that  $K$  is infinite. Suppose on the contrary that  $K$  is finite, and let  $M$  be a finite maximal subgroup of  $B_n(\mathbb{S}^2)$  that contains  $K$ . Since  $K$  contains copies of  $\mathcal{Q}_{16}$ ,  $M$  cannot be cyclic, nor can it be isomorphic to  $T^*$  or  $I^*$  by Proposition 85. By the hypothesis on  $n$ ,  $O^*$  is not realised as a subgroup of  $B_n(\mathbb{S}^2)$  by Theorem 2, so  $M \not\cong O^*$ , and thus  $M \cong \text{Dic}_{4(n-2)}$ . Let  $u \in M$  be an element of order  $2(n-2)$ . Since  $G_1$  and  $G_2$  are subgroups of  $M$  isomorphic to  $\mathcal{Q}_{16}$ , they both contain the unique cyclic subgroup  $\langle u^{(n-2)/4} \rangle$  of  $M$  of order 8, but this contradicts the fact that  $G_1 \cap G_2 = \Gamma_1$ . So  $M \not\cong \text{Dic}_{4(n-2)}$ , and thus  $K$  is infinite by Theorem 2. Proposition 23 then implies that  $K \cong \mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ . It remains to show that  $K \cong K_2$ . This may be seen as follows. To see this, let  $x = zaz^{-1}$  and  $y = zbz^{-1}$  be generators of  $G_2$ . Then  $x^4 = y^2$ ,  $yx y^{-1} = x^{-1}$ ,  $x^2 = za^2z^{-1} = za^2z^{-1} = b$  and  $y = zbz^{-1} = a^2b$ . Equation (77) implies that  $K \cong K_2$  as required.  $\square$

## 9 Classification of the virtually cyclic subgroups of the mapping class group $\mathcal{MCG}(\mathbb{S}^2, n)$

We apply Theorem 5 and Proposition 12 to deduce Theorem 14, which up to a finite number of exceptions, yields the classification of the virtually cyclic subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$ .

*Proof of Theorem 14.* Let  $n \geq 4$ . The homomorphism  $\varphi$  of the short exact sequence (11) satisfies the hypothesis of Proposition 12 with  $x = \Delta_n^2$ . Theorem 5 and Proposition 12 then imply the result, using the fact that if a finite subgroup  $F$  of  $B_n(\mathbb{S}^2)$  is isomorphic to  $\mathbb{Z}_q$  (resp.  $\text{Dic}_{4m}$ ,  $\mathcal{Q}_8$ ,  $T^*$ ,  $O^*$ ,  $I^*$ ) then  $\varphi(F)$  is isomorphic to  $\mathbb{Z}_{q/2}$  if  $q$  is even and to  $\mathbb{Z}_q$  if  $q$  is odd (resp. is isomorphic to  $\text{Dih}_{2m}$ ,  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ ,  $A_4$ ,  $S_4$ ,  $A_5$ ). Note that the only cases where the conditions given in Definition 4 on the order  $q$  of  $F$  differ from those on the order  $q'$  of  $\varphi(F)$  given by Definition 13 is when  $F$  is cyclic, and correspond to cases (a) and (b) of these definitions. To see that one does indeed obtain the given conditions in parts (a) and (b) of Definition 13, suppose that  $q$  satisfies the corresponding conditions given in parts (a) and (b) of Definition 4. In particular,  $q$  is a strict divisor of  $2(n-i)$ , and  $q \neq n-i$  if  $n-i$  is odd. If  $q$  is even then  $q' = q/2$ , and  $q'$  is a strict divisor of  $n-i$ . So suppose that  $q$  is odd, in which case  $q' = q$  and  $q'$  divides  $n-i$ . Clearly, if  $n-i$  is even then  $q' \neq n-i$ . On the other hand, if  $n-i$  is odd then  $q \neq n-i$ . In both cases it follows once more that  $q'$  is a strict divisor of  $n-i$ , which yields the condition on the order of the finite cyclic factor in parts (a) and (b) of Definition 13.  $\square$

One may ask a similar question to that of Section II.8 concerning the isomorphism classes of the amalgamated products that are realised as subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$ . From Definition 13 and Theorem 14, these subgroups are of the form:

- (a)  $\mathbb{Z}_{2q} *_{\mathbb{Z}_q} \mathbb{Z}_{2q}$ , where  $q$  divides  $(n - i)/2$  for some  $i \in \{0, 1, 2\}$ .
- (b)  $\mathbb{Z}_{2q} *_{\mathbb{Z}_q} \text{Dih}_{2q}$ , where  $q \geq 2$  divides  $(n - i)/2$  for some  $i \in \{0, 2\}$ .
- (c)  $\text{Dih}_{2q} *_{\mathbb{Z}_q} \text{Dih}_{2q}$ , where  $q \geq 2$  divides  $n - i$  strictly for some  $i \in \{0, 2\}$ .
- (d)  $\text{Dih}_{2q} *_{\text{Dih}_q} \text{Dih}_{2q}$ , where  $q \geq 4$  is even and divides  $n - i$  for some  $i \in \{0, 2\}$ .
- (e)  $S_4 *_{A_4} S_4$ , where  $n \equiv 0, 2 \pmod{6}$ .

A key element in the analysis of the isomorphism classes of the amalgamated products that are realised as subgroups of  $B_n(\mathbb{S}^2)$  was the use of Lemma 74. This may be generalised as follows to the groups that appear as factors in the above list.

LEMMA 80. *Let  $G'$  be a group isomorphic to  $\mathbb{Z}_{2q}$ ,  $q \geq 1$  (resp. to  $\text{Dih}_{2q}$ ,  $q \geq 2$ , to  $S_4$ ), let  $G$  be a group isomorphic to  $\mathbb{Z}_{4q}$ ,  $q \geq 1$  (resp. to  $\text{Dic}_{4q}$ ,  $q \geq 2$ , to  $O^*$ ), and let  $\varphi: G \rightarrow G'$  be the canonical homomorphism, where we identify  $G'$  with the quotient of  $G$  by its unique subgroup  $K$  of order 2. Let  $H'$  be a subgroup of  $G'$  of index 2, non isomorphic to  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$  if  $G' \cong \text{Dih}_8$ . Then the homomorphism  $\text{Aut}_{H'}(G') \rightarrow \text{Aut}(H')$  given by restriction is surjective.*

*Proof.* Let  $H = \varphi^{-1}(H')$ . Then  $H$  is of index 2 in  $G$ , and if  $G \cong \mathcal{Q}_{16}$  then  $H \not\cong \mathcal{Q}_8$ . Let  $\alpha' \in \text{Aut}(H')$ . We must show that there exists an automorphism of  $G'$  that leaves  $H'$  invariant, and whose restriction to  $H'$  is equal to  $\alpha'$ . Note that:

- (a) if  $G' \cong \mathbb{Z}_{2q}$ ,  $q \geq 1$ , then  $G \cong \mathbb{Z}_{4q}$ ,  $H' \cong \mathbb{Z}_q$  and  $H \cong \mathbb{Z}_{2q}$ .
- (b) if  $G' \cong \text{Dih}_{2q}$ ,  $q \geq 2$  then  $G \cong \text{Dic}_{4q}$ . If  $q$  is odd then  $H' \cong \mathbb{Z}_q$  and  $H \cong \mathbb{Z}_{2q}$ . If  $q$  is even then  $H'$  is isomorphic to  $\mathbb{Z}_q$  or to  $\text{Dih}_q$ , and  $H$  is isomorphic to  $\mathbb{Z}_{2q}$  or to  $\text{Dic}_{2q}$  respectively.
- (c) if  $G \cong O^*$  then  $G' \cong S_4$ ,  $H' \cong A_4$  and  $H \cong T^*$ .

The kernel of  $\varphi|_H: H \rightarrow H'$  is that of  $\varphi$ , equal to  $K$ . Since  $K$  is characteristic in  $G$  (resp.  $H$ ), for each automorphism  $f \in \text{Aut}(G)$  (resp.  $f \in \text{Aut}(H)$ ), there exists a unique automorphism  $f' \in \text{Aut}(G')$  (resp.  $f' \in \text{Aut}(H')$ ) such that  $\varphi \circ f = f' \circ \varphi$ , and the correspondence  $f \mapsto f'$  gives rise to a homomorphism  $\tilde{\Phi}: \text{Aut}(G) \rightarrow \text{Aut}(G')$  (resp.  $\Phi: \text{Aut}(H) \rightarrow \text{Aut}(H')$ ) satisfying  $\tilde{\Phi}(f) \circ \varphi = \varphi \circ f$  (resp.  $\Phi(f) \circ \varphi = \varphi \circ f$ ).

Let  $r$  and  $r'$  denote the restriction homomorphisms  $\text{Aut}_H(G) \rightarrow \text{Aut}(H)$  and  $\text{Aut}_{H'}(G') \rightarrow \text{Aut}(H')$  respectively, and let  $\iota$  and  $\iota'$  denote the inclusions  $\text{Aut}_H(G) \rightarrow \text{Aut}(G)$  and  $\text{Aut}_{H'}(G') \rightarrow \text{Aut}(G')$  respectively. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
 \text{Aut}(G) & \xleftarrow{\iota} & \text{Aut}_H(G) & \xrightarrow{r} & \text{Aut}(H) \\
 \downarrow \tilde{\Phi} & & \downarrow \tilde{\Phi}|_{\text{Aut}_H(G)} & & \downarrow \Phi \\
 \text{Aut}(G') & \xleftarrow{\iota'} & \text{Aut}_{H'}(G') & \xrightarrow{r'} & \text{Aut}(H')
 \end{array} \tag{80}$$

Note that the restriction of  $\tilde{\Phi}$  to  $\text{Aut}_H(G)$  is well defined. Indeed, let  $f \in \text{Aut}_H(G)$ , and let  $h' \in H'$ . Then there exists  $h \in H$  such that  $\varphi(h) = h'$ , and

$$\Phi(f)(h') = \Phi(f) \circ \varphi(h) = \varphi \circ f(h) \in H, \quad \text{since } f(h) \in H.$$

We claim that for the groups  $H, H'$  described in (a)–(c) above,  $\Phi$  is surjective. If  $H \cong \mathbb{Z}_{2q}$ , which covers case (a) above and part of case (b),  $\text{Aut}(H) \cong \mathbb{Z}_{2q}^\times$ ,  $\text{Aut}(H') \cong \mathbb{Z}_q^\times$ , and if  $\alpha' \in \text{Aut}(H')$  is given by multiplication by  $j$ , where  $1 \leq j \leq q-1$ ,  $\gcd(j, q) = 1$ , then  $\Phi(\alpha) = \alpha'$ , where  $\alpha \in \text{Aut}(H)$  is given by multiplication by  $j + \varepsilon q$ , where  $\varepsilon = 0$  if  $j$  is odd, and  $\varepsilon = 1$  if  $j$  is even. Let us now consider the remaining part of case (b) where  $q$  is even,  $H \cong \text{Dic}_{2q}$  and  $H' \cong \text{Dih}_q$ . Let  $H$  admit the presentation

$$H = \langle x, y \mid x^{q/2} = y^2, yxy^{-1} = x^{-1} \rangle,$$

and let  $\bar{x} = \varphi(x)$  and  $\bar{y} = \varphi(y)$ , so that

$$H' = \langle \bar{x}, \bar{y} \mid \bar{x}^{q/2} = \bar{y}^2 = 1, \bar{y}\bar{x}\bar{y}^{-1} = \bar{x}^{-1} \rangle.$$

Any automorphism  $\alpha'$  of  $H'$  is given by  $\bar{x} \mapsto \bar{x}^j, \bar{y} \mapsto \bar{x}^k \bar{y}$ , where  $1 \leq j \leq q/2 - 1$ ,  $\gcd(j, q/2) = 1$  and  $0 \leq k \leq q/2 - 1$ . The presentation of  $H$  implies that the map  $\alpha: H \rightarrow H$  given by  $x \mapsto x^{j+\varepsilon q/2}, y \mapsto x^k y$ , where  $\varepsilon = 0$  if  $j$  is odd, and  $\varepsilon = 1$  if  $j$  is even, is an automorphism. Further,  $\Phi(\alpha)(\bar{x}) = \varphi(\alpha(x)) = \bar{x}^j = \alpha'(\bar{x})$ , and  $\Phi(\alpha)(\bar{y}) = \varphi(\alpha(y)) = \bar{x}^k \bar{y} = \alpha'(\bar{y})$ , which proves the surjectivity of  $\Phi$  in this case. Finally, in case (c), the result is a consequence of [GoG3, Theorem 3.3]

It just remains to show that  $r'$  is surjective. By the commutative diagram (80), this follows from the surjectivity of  $\Phi$ , and that of  $r$ , which is a consequence of Lemma 74.  $\square$

In principle, if we are given finite groups  $H, G_1, G_2$ , where  $H$  is an index 2 subgroup of both  $G_1$  and  $G_2$ , there may be various non-isomorphic amalgamated products of the form  $G_1 *_H G_2$ . As for  $B_n(\mathbb{S}^2)$ , such a situation occurs exceptionally in  $\mathcal{MCG}(\mathbb{S}^2, n)$ , and we obtain a similar result to that of Proposition 11 for the virtually cyclic subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$  of Type II.

PROPOSITION 81. *Let  $n \geq 4$  be even.*

- (a) *Let  $H'_1, H'_2$  be subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$  that are both isomorphic to one of the amalgamated products given in (a)–(e) above, with the exception of  $\text{Dih}_8 *_{\text{Dih}_4} \text{Dih}_8$ . Then  $H'_1 \cong H'_2$ .*  
(b) *Let  $H'$  be a subgroup of  $\mathcal{MCG}(\mathbb{S}^2, n)$  that is isomorphic to an amalgamated product of the form  $\text{Dih}_8 *_{\text{Dih}_4} \text{Dih}_8$ . Then  $H'$  is isomorphic to exactly one of the following two groups:*

$$K'_1 = \langle x, y, a, b \mid x^4 = y^2 = a^4 = b^2 = 1, yxy^{-1} = x^{-1}, bab^{-1} = a^{-1}, x^2 = a^2, y = b \rangle, \quad (81)$$

and

$$K'_2 = \langle x, y, a, b \mid x^4 = y^2 = a^4 = b^2 = 1, yxy^{-1} = x^{-1}, bab^{-1} = a^{-1}, x^2 = b, y = a^2 b \rangle. \quad (82)$$

REMARK 82. One may mimic the proof of Proposition 11 to obtain an analogous result for the amalgamated products given in (a)–(e) above, that is, abstractly there is a single isomorphism class, with the exception of  $\text{Dih}_8 *_{\text{Dih}_4} \text{Dih}_8$ , for which there are



two isomorphism classes, for which  $K'_1$  and  $K'_2$  are representatives. However, using Proposition 11, we shall give an alternative proof in the case that interests us, where the groups in question are realised as subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$ .

*Proof of Proposition 81.* Consider one of the amalgamated products given in the list (a)–(e) above, and suppose that  $n \geq 4$  is such that this amalgamated product is realised as a subgroup  $G'_1 *_F G'_2$  of  $\mathcal{MCG}(\mathbb{S}^2, n)$ , where  $G'_1, G'_2$  and  $F'$  are finite subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$ , and  $[G'_i: F'] = 2$  for  $i = 1, 2$ . Taking  $\bar{G} = B_n(\mathbb{S}^2)$ ,  $G' = \mathcal{MCG}(\mathbb{S}^2, n)$ ,  $x = \Delta_n^2$  and  $p = \varphi$  in the statement of Proposition 26, where  $\varphi$  is the homomorphism of equation (11), we have that  $\varphi^{-1}(G'_1) *_\varphi^{-1}(F') \varphi^{-1}(G'_2)$  is a subgroup of  $B_n(\mathbb{S}^2)$  by part (b)(ii) of that proposition. We claim that the number of isomorphism classes of subgroups of  $B_n(\mathbb{S}^2)$  that are isomorphic to an amalgamated product of the form  $\varphi^{-1}(G'_1) *_\varphi^{-1}(F') \varphi^{-1}(G'_2)$  (which are the amalgamated products (a)–(e) that appear at the beginning of Section II.8) is greater than or equal to the number of isomorphism classes of subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$  that are isomorphic to an amalgamated product of the form  $G'_1 *_F G'_2$ . To prove the claim, let  $H'_1, H'_2$  be subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$  that may be written in the form  $G'_1 *_F G'_2$ , and for  $i = 1, 2$ , let  $H_i = \varphi^{-1}(H'_i)$ . From above,  $H_1$  and  $H_2$  are subgroups of  $B_n(\mathbb{S}^2)$  that may be written in the form  $\varphi^{-1}(G'_1) *_\varphi^{-1}(F') \varphi^{-1}(G'_2)$ . If they are isomorphic then  $H'_1 = p(H_1)$  and  $H'_2 = p(H_2)$  are isomorphic by Proposition 26(c), which proves the claim. If  $G'_1 *_F G'_2 \not\cong \text{Dih}_8 *_{\text{Dih}_4} \text{Dih}_8$  then  $\varphi^{-1}(G'_1) *_\varphi^{-1}(F') \varphi^{-1}(G'_2) \not\cong \mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ , and combining the claim with Proposition 11 implies that  $\mathcal{MCG}(\mathbb{S}^2, n)$  possesses a single isomorphism class of subgroups that are isomorphic to amalgamated products of the form  $G'_1 *_F G'_2$ , which proves part (a) of the proposition. Similarly, if  $G'_1 *_F G'_2 \cong \text{Dih}_8 *_{\text{Dih}_4} \text{Dih}_8$  then  $\varphi^{-1}(G'_1) *_\varphi^{-1}(F') \varphi^{-1}(G'_2) \cong \mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$ , and  $\mathcal{MCG}(\mathbb{S}^2, n)$  possesses at most two isomorphism class of subgroups that are isomorphic to amalgamated products of the form  $\text{Dih}_8 *_{\text{Dih}_4} \text{Dih}_8$ , and these isomorphism classes are represented by subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$  that are isomorphic to  $K'_1$  and  $K'_2$ . To complete the proof of part (b) of the proposition, it thus suffices to show that  $K'_1 \not\cong K'_2$ . Taking  $K'_1$  and  $K'_2$  to be presented by equations (81) and (82) respectively, following the proof of Proposition 11 from equation (75) onwards, and letting  $N'$  be the infinite cyclic subgroup of  $K'_1$  generated by  $t = xa^{-1}$ , we see that  $N'$  is normal in  $K'_1$ ,

$$K'_1/N' = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle \cong \text{Dih}_8,$$

and  $K'_1 \cong \langle t \rangle \rtimes \text{Dih}_8$ , where the action of  $K'_1/N'$  on  $N'$  is given by equation (76),  $G_2$  being in this case the subgroup  $\langle a, b \rangle$  of  $K'_1$ . The rest of the proof of Proposition 11 then goes through, where  $\mathcal{Q}_8$  is replaced by  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ , and the subgroups  $L$  of  $H$  are now of order 2. We conclude that  $K'_1 \not\cong K'_2$  as required.  $\square$

We thus obtain the following result on the existence of subgroups of  $\mathcal{MCG}(\mathbb{S}^2, n)$  isomorphic to  $K'_1$  and  $K'_2$ .

PROPOSITION 83. *Let  $n \geq 4$  be even.*

(a) *There exists a subgroup of  $\mathcal{MCG}(\mathbb{S}^2, n)$  isomorphic to  $K'_1$ .*

(b) *Suppose that either  $n \equiv 0 \pmod{4}$  or  $n \equiv 10 \pmod{12}$  and  $n \notin \{6, 14, 18, 26, 30, 38\}$ . There exists a subgroup of  $\mathcal{MCG}(\mathbb{S}^2, n)$  isomorphic to  $K'_2$ .*

*Proof.* Let  $n$  be even, and let  $\varphi$  be the homomorphism of equation (11). If  $n \geq 4$  (resp.  $n \not\equiv 4 \pmod{12}$  and  $n \notin \{6, 14, 18, 26, 30, 38\}$ ) then Proposition 78 and Remark 79 imply that  $B_n(\mathbb{S}^2)$  possesses a subgroup  $H$  that is isomorphic to  $K_1$  (resp.  $K_2$ ). The presentations of  $K_1$  and  $K'_1$  (resp.  $K_2$  and  $K'_2$ ) given by equations (75) and (81) (resp. equations (77) and (82)) imply that  $\varphi(H)$  is isomorphic to  $K'_1$  (resp. to  $K'_2$ ).  $\square$

REMARK 84. As in the case of  $B_n(\mathbb{S}^2)$ , we do not know whether  $\mathcal{MCG}(\mathbb{S}^2, n)$  possesses a subgroup isomorphic to  $K'_2$  if  $n \in \{6, 14, 18, 26, 30, 38\}$ .

# Appendix: The subgroups of the binary polyhedral groups

In this Appendix, we derive the structure of the subgroups of the binary polyhedral groups  $T^*$ ,  $O^*$ ,  $I^*$  that we refer to in the main body of the manuscript. More information on these groups may be found in [AM, Cox, CM, ThC].

PROPOSITION 85.

- (a) The proper subgroups of the binary tetrahedral group  $T^*$  are  $\{e\}$ ,  $\mathbb{Z}_2$ ,  $\mathbb{Z}_3$ ,  $\mathbb{Z}_4$ ,  $\mathbb{Z}_6$  and  $\mathcal{Q}_8$ . Its maximal subgroups are isomorphic to  $\mathbb{Z}_6$  or  $\mathcal{Q}_8$ , its maximal cyclic subgroups are isomorphic to  $\mathbb{Z}_4$  or  $\mathbb{Z}_6$ , and its non-trivial normal subgroups are isomorphic to  $\mathbb{Z}_2$  or  $\mathcal{Q}_8$ .
- (b) The proper subgroups of the binary octahedral group  $O^*$  are isomorphic to  $\{e\}$ ,  $\mathbb{Z}_2$ ,  $\mathbb{Z}_3$ ,  $\mathbb{Z}_4$ ,  $\mathbb{Z}_6$ ,  $\mathbb{Z}_8$ ,  $\mathcal{Q}_8$ ,  $\text{Dic}_{12}$ ,  $\mathcal{Q}_{16}$  or  $T^*$ . Its maximal subgroups are isomorphic to  $\text{Dic}_{12}$ ,  $\mathcal{Q}_{16}$  or  $T^*$ , its maximal cyclic subgroups are isomorphic to  $\mathbb{Z}_4$ ,  $\mathbb{Z}_6$  or  $\mathbb{Z}_8$ , and its non-trivial normal subgroups are isomorphic to  $\mathbb{Z}_2$ ,  $\mathcal{Q}_8$  or  $T^*$ .
- (c) The proper subgroups of the binary icosahedral group  $I^*$  are isomorphic to  $\{e\}$ ,  $\mathbb{Z}_2$ ,  $\mathbb{Z}_3$ ,  $\mathbb{Z}_4$ ,  $\mathbb{Z}_5$ ,  $\mathbb{Z}_6$ ,  $\mathcal{Q}_8$ ,  $\mathbb{Z}_{10}$ ,  $\text{Dic}_{12}$ ,  $\text{Dic}_{20}$  or  $T^*$ , its maximal subgroups are isomorphic to  $\text{Dic}_{12}$ ,  $\text{Dic}_{20}$  or  $T^*$ , its maximal cyclic subgroups are isomorphic to  $\mathbb{Z}_4$ ,  $\mathbb{Z}_6$  or  $\mathbb{Z}_{10}$ , and it has a unique non-trivial normal subgroup, isomorphic to  $\mathbb{Z}_2$ .

*Proof.* Recall first that if  $G$  is a binary polyhedral group, it is periodic [AM] and has a unique element of order 2 that generates  $Z(G)$ . By periodicity, the group  $G$  satisfies the  $p^2$ -condition (if  $p$  is prime and divides the order of  $G$  then  $G$  has no subgroup isomorphic to  $\mathbb{Z}_p \times \mathbb{Z}_p$ ), which implies that every Sylow  $p$ -subgroup of  $G$  is cyclic or generalised quaternion, as well as the  $2p$ -condition (each subgroup of order  $2p$  is cyclic).

(a) Consider first the binary tetrahedral group  $T^*$ . It is isomorphic to  $\mathcal{Q}_8 \rtimes \mathbb{Z}_3$ . Using the presentation given by equation (13), one may check that  $T^* \setminus \mathcal{Q}_8$  consists of the eight elements of

$$\left\{ S^{-j} X^j \mid j \in \{-1, 1\} \text{ and } S \in \{1, P, Q, PQ\} \right\},$$

and of the eight elements of order 6 which are obtained from those of order 3 by multiplying by the unique (central) element  $P^2$  of order 2. The proper non-trivial subgroups of  $T^*$  are isomorphic to  $\mathbb{Z}_2$ ,  $\mathbb{Z}_3$ ,  $\mathbb{Z}_4$ ,  $\mathbb{Z}_6$  and  $\mathcal{Q}_8$ . The fact that  $T^*$  has a unique element of order 2 rules out the existence of subgroups isomorphic to  $S_3$ . Since  $\mathcal{Q}_8$  is a Sylow 2-subgroup of  $T^*$ ,  $\mathbb{Z}_8$  cannot be a subgroup of  $T^*$ . Further, since  $T^*/Z(T^*) \cong A_4$ , the quotient by  $Z(T^*)$  of any order 12 subgroup of  $T^*$  would be a subgroup of  $A_4$  of order 6,

which is impossible. Also, any copy of  $\mathbb{Z}_3$  (resp.  $\mathbb{Z}_4$ ) is contained in a copy of  $\mathbb{Z}_6$  (resp.  $\mathcal{Q}_8$ ). The maximal subgroups of  $T^*$  are thus isomorphic to  $\mathbb{Z}_6$  or  $\mathcal{Q}_8$ , and its maximal cyclic subgroups are isomorphic to  $\mathbb{Z}_4$  or  $\mathbb{Z}_6$ . Among these possible subgroups, it is straightforward to check that the normal non-trivial subgroups are those isomorphic to  $\mathbb{Z}_2$  or  $\mathcal{Q}_8$ .

(b) Consider the binary octahedral group  $O^*$ , with presentation given by equation (28). Recall from Lemma 74(b) that  $\langle P, Q, X \rangle$  is the unique subgroup of  $O^*$  isomorphic to  $T^*$ . The twenty-four elements of  $O^* \setminus T^*$  are comprised of twelve elements of order 4 and twelve of order 8. Under the canonical projection onto  $O^*/Z(O^*) \cong S_4$ , these elements are sent to the six transpositions and the six 4-cycles of  $S_4$  respectively. The squares of the elements of order 8 are the elements of  $T^*$  of order 4. Consequently, the elements of  $O^* \setminus T^*$  of order 4 generate maximal cyclic subgroups. Thus  $O^*$  has three subgroups isomorphic to  $\mathbb{Z}_8$ . The Sylow 2-subgroups are copies of  $\mathcal{Q}_{16}$ , and since each copy of  $\mathbb{Z}_8$  is contained in a copy of  $\mathcal{Q}_{16}$  and each copy of  $\mathcal{Q}_{16}$  contains a unique copy of  $\mathbb{Z}_8$ , it follows from Sylow's Theorems that  $O^*$  possesses exactly three (maximal and non-normal) copies of  $\mathcal{Q}_{16}$ , and that the subgroups of  $O^*$  of order 8 are isomorphic to  $\mathbb{Z}_8$  or  $\mathcal{Q}_8$ .

It remains to determine the subgroups of order 12. Under the projection onto the quotient  $O^*/Z(O^*)$ , such a subgroup would be sent to a subgroup of  $S_4$  of order 6, so is the inverse image under this projection of a copy of  $S_3$ , isomorphic to  $\text{Dic}_{12}$ . It is not normal because the subgroups of  $S_4$  isomorphic to  $S_3$  are not normal. Further it cannot be a subgroup of  $\langle P, Q, X \rangle$  since projection onto  $O^*/Z(O^*)$  would imply that the image of  $\langle P, Q, X \rangle$ , which is isomorphic to  $A_4$ , would have a subgroup of order 6, which is impossible. We thus obtain the isomorphism classes of the subgroups of  $O^*$  given in the statement, as well as the isomorphism classes of the maximal and maximal cyclic subgroups.

We now determine the normal subgroups of  $O^*$ . As we already mentioned, the subgroups of  $O^*$  isomorphic to  $\text{Dic}_{12}$  or  $\mathcal{Q}_{16}$  are not normal, and the fact that each of the three cyclic subgroups of order 8 belongs to a single copy of  $\mathcal{Q}_{16}$  implies that these subgroups are not normal in  $O^*$ . Clearly  $Z(O^*) \cong \mathbb{Z}_2$  and  $\langle P, Q, X \rangle \cong T^*$  are normal in  $O^*$ . Since  $T^*$  is normal in  $O^*$  and possesses a unique copy  $\langle P, Q \rangle$  of  $\mathcal{Q}_8$ , this copy of  $\mathcal{Q}_8$  is normal in  $O^*$ . The subgroups isomorphic to  $\mathbb{Z}_3$  or  $\mathbb{Z}_6$  are not normal because they are contained in  $\langle P, Q, X \rangle$  and are not normal there. The same is true for the subgroups isomorphic to  $\mathbb{Z}_4$  and lying in  $\langle P, Q, X \rangle$ . Finally, under the canonical projection onto  $O^*/Z(O^*)$ , any subgroup of order 4 generated by an element  $O^* \setminus T^*$  is sent to subgroup of  $S_4$  generated by a transposition, so cannot be normal in  $O^*$ . This yields the list of isomorphism classes of normal subgroups of  $O^*$ .

(c) Finally, consider the binary icosahedral group  $I^*$  of order 120. It is well known that  $I^*$  admits the presentation  $\langle S, T \mid (ST)^2 = S^3 = T^5 \rangle$ , is isomorphic to the group  $\text{SL}_2(\mathbb{F}_5)$ , and  $I^*/Z(I^*) \cong A_5$ . The group  $I^*$  has thirty elements of order 4 (which project to the fifteen elements of  $A_5$  of order 2), twenty elements each of order 3 and 6 (which project to the twenty 3-cycles of  $A_5$ ), and twenty-four elements each of order 5 and 10 (which project to the twenty-four 5-cycles of  $A_5$ ). Its proper subgroups of order less than or equal to 10 are  $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_6, \mathcal{Q}_8$  and  $\mathbb{Z}_{10}$ . The only difficulty here is the case of order 8 subgroups:  $I^*$  has no element of order 8 since under the projection onto

$I^*/Z(I^*)$ , such an element would project onto an element of  $A_5$  of order 4, which is not possible. Since  $I^*$  possesses a unique element of order 2, the Sylow 2-subgroups of  $I^*$ , which are of order 8, are isomorphic to  $Q_8$ . Any subgroup of order 15 or 30 (resp. 60) would project to a subgroup of  $A_5$  of order 15 (resp. 30), which is not possible either. Note that  $I^*$  has no element of order 12 (resp. 20) since such an element would project to one of order 6 (resp. 10) in  $A_5$ . Since  $I^*$  has a unique element of order 2, any subgroup of order 12 (resp. 20) must thus be isomorphic to  $\text{Dic}_{12}$  (resp.  $\text{Dic}_{20}$ ) using the classification of the groups of these orders up to isomorphism. Such a subgroup exists by taking the inverse image of the projection of any subgroup of  $A_5$  isomorphic to  $\text{Dih}_6$  (resp.  $\text{Dih}_{10}$ ).

Any subgroup of  $I^*$  of order 24 projects to a subgroup of  $A_5$  of order 12, which must be a copy of  $A_4$ . Hence any subgroup of  $A_5$  of order 12, which is isomorphic to  $A_4$ , lifts to a subgroup of  $I^*$  isomorphic to  $T^*$ . So any subgroup of  $I^*$  of order 24 is isomorphic to  $T^*$ , and such a subgroup exists.

Let  $G$  be a subgroup of  $I^*$  of order 40, and let  $G'$  be its projection in  $I^*/Z(I^*)$ . Then  $G'$  is of order 20, and the Sylow 5-subgroup  $K$  of  $G'$  is normal. Now  $G'$  has no element of order 4 since  $I^*$  has no element of order 8, so  $G'/K \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ . We thus have a short exact sequence:

$$1 \longrightarrow \mathbb{Z}_5 \longrightarrow G' \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1$$

which splits since the kernel and the quotient have coprime orders [McL, Theorem 10.5]. Since  $\text{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_4$ , the action of any non-trivial element of  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$  on  $\mathbb{Z}_5$  must be multiplication by  $-1$  (it could not be the identity, for otherwise  $A_5$  would have an element of order 10, which is impossible), but this is not compatible with the structure of  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ . Hence  $I^*$  has no subgroup of order 40. We thus obtain the list of subgroups of  $I^*$  given in the statement. The cyclic subgroups of order 3 and 5 of  $I^*$  are contained in the cyclic subgroups of order 6 and 10 respectively obtained by multiplying a generator by the central element of order 2. Thus the maximal cyclic subgroups of  $I^*$  are isomorphic to  $\mathbb{Z}_4$ ,  $\mathbb{Z}_6$  or  $\mathbb{Z}_{10}$ .

We now consider the maximal subgroups. Clearly, any subgroup of  $I^*$  isomorphic to  $\text{Dic}_{12}$  or  $T^*$  is maximal. Further, since  $T^*$  has no subgroup of order 12, any subgroup of  $I^*$  isomorphic to  $\text{Dic}_{12}$  is also maximal. The subgroups of  $I^*$  isomorphic to  $Q_8$  are its Sylow 2-subgroups, so are conjugate, and since one of these subgroups is contained in a copy of  $T^*$ , the same is true for any such subgroup. Thus the subgroups of  $I^*$  isomorphic to  $Q_8$  are not maximal. Replacing  $Q_8$  by  $\mathbb{Z}_3$  (resp.  $Q_8$  by  $\mathbb{Z}_5$  and  $T^*$  by  $\text{Dic}_{20}$ ) and applying a similar argument shows that the subgroups of  $I^*$  isomorphic to  $\mathbb{Z}_6$  (resp.  $\mathbb{Z}_{10}$ ) are not maximal either. This yields the list of the isomorphism classes of the maximal subgroups of  $I^*$  given in the statement. Finally, since  $A_5$  is simple, the only non-trivial normal subgroup of  $I^*$  is its unique subgroup of order 2.  $\square$

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