Motion planning for flat systems using positive B-splines: An LMI approach

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Abstract

In this paper, the motion planning problem is studied for nonlinear differentially flat systems using B-splines parametrization of the flat output history. In order to satisfy the constraints continuously in time, the motion planning problem is transformed into a B-splines positivity problem. The latter problem is formulated as a convex semidefinite programming problem by means of a non-negative piecewise polynomial functions description based on sum of squares decomposition. The contribution of the paper is thus a one-step design procedure for motion planning that satisfies constraints continuously in time where usual B-spline and collocation techniques need post-analysis. Finally, an example of flexible link manipulator motion is presented to illustrate the overall approach.

Key words: Motion planning, Nonlinear differentially flat systems, B-splines positivity, LMI.

1 Introduction

In this paper, the motion planning problem is considered for a subset of nonlinear systems referred to as differentially flat systems. For such systems, introduced first in [8], there exists a minimal set of particular outputs (the so-called flat outputs) that characterize all the state space motions and the corresponding input history. Such systems arise in several relevant domains of engineering control (see [14] and the references therein). Contrary to classical methods for solving optimal control problem [2, 3], solving the motion planning problem through flatness avoids integrating the differential equations of the dynamics. In fact, since the flat output is the state of a trivial system [9], the above optimal control problem boils down to find the best flat output motion that lies into the subspace defined by the path constraints expressed in terms of flat outputs and their derivatives and passes through a given set of points. In this context, a classical and tractable methodology relies on B-splines based collocation [15, 7, 13]. However, this technique involves time sampling: there is no guarantee on constraints satisfaction between collocation points. This may lead to critical issues that need to be detected by an appropriate post-analysis. Thus, the B-splines collocation requires an interactive procedure between the trajectory synthesis and a specific post-analysis.

In this paper, our goal is to design flat system trajectories that guarantee constraints continuously in time and without a post-analysis. Henrion and Lasserre tackled this problem in the case of linear systems in [11]. They proved that motion planning under constraints can be recast as the inclusion of a univariate polynomial in a linear semi-algebraic subset. By using results on positive polynomials [16], the late problem turns out to be a linear matrix inequalities (LMI) problem. The main result of this paper is obtained in two steps. First, We extend the results from [16] to positive piecewise polynomials. This result is then used, in conjunction with B-splines parametrization, to provide a new motion planning methodology that allows to design a trajectory fulfilling the constraints continuously in time. In this works, we choose to use the B-splines basis to represent the squared piecewise polynomials rather than the monomial representation and to develop a B-spline positivity concept instead of classical SOS methods used in [11] for the following reasons: (i) B-splines provide a minimal basis for piecewise polynomials, (ii) B-splines basis contains explicitely information about continuity class at breakpoints contrary to others representations.

In section 2, the optimal path planning problem for flat systems is described. Then, our contribution is detailed in two steps. First, in section 3.1 and 3.2, the results on positive polynomials [16] are extended to positive piecewise polynomials by characterizing them over the cone of positive semidefinite matrices using the sum of squares formalism. Subsequently, in section 3.3, the constrained

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B-splines optimization is formulated as a convex optimization problem over linear matrix inequalities (LMI) for which efficient programming (SDP) solvers are available. In section 4, an example illustrates the methodology.

2 Problem Statement

2.1 Differential flatness

Before adressing the motion planning problem for flat systems, let us briefly recall the differential flatness concept. Differential flatness, or shortly flatness, has been introduced by Fliess et al. [8] in 1992. Consider a nonlinear system: $\dot{x} = f(x, u)$, where x is the n-component state vector and u the m-component control with $m \leq n$.

Definition 1 *The nonlinear system is differentially flat if there exists an m-dimensional vector z, so-called flat outputs, such that:*

$$z(t) = \Phi\left(x(t), u(t), \dot{u}(t), \ldots, u^{(\eta)}(t)\right), \text{ and} \tag{1}$$

$$\begin{cases} x(t) = & \Psi_x\left(z(t), \dot{z}(t), \ldots, z^{(\beta-1)}(t)\right), \\ u(t) = & \Psi_u\left(z(t), \dot{z}(t), \ldots, z^{(\beta)}(t)\right), \end{cases} \tag{2}$$

where Ψ_x and Ψ_u are smooth functions, $z_i^{(k)}(t)$ denoting the k^{th} order time derivative of the i^{th} component of $z(t)$ and $\eta, \beta \in \mathbb{N}$.

Flatness is also defined as a Lie-Bäcklund equivalence between a nonlinear system and a trivial system in [9]. As z represents the state of the equivalent trivial system, the m-components of z are differentially independent. Indeed, a $\overline{z}$-space of dimension n_z is considered with the coordinates $\overline{z} = \{z, \dot{z}, \ddot{z}, \ldots, z^{(p)}\}$ with $p \in \mathbb{N}$ where any curve is equivalent to a trajectory of the corresponding nonlinear system.

2.2 Optimal path planning for the flat system

Using the flatness properties in line with the works [15, 13], the generation of an optimal and constrained trajectory, $t \mapsto (x^*(t), u^*(t))$, for nonlinear flat system is equivalent to the following integration-free optimization problem:

$$\min_{\overline{z}} J(\overline{z}) \text{ subject to: } \overline{z}(t_i) = \overline{z}_i,\ \overline{z}(t_f) = \overline{z}_f,\ \overline{z}(t) \in S, \tag{3}$$

where $\overline{z}$ is flat space coordinate, J is an objective function, and the subset S, so-called the feasible region, is such that $S = \{\overline{z} |\, \gamma(\overline{z}(t)) \leq 0\}$, where the inequality $\gamma(\overline{z}(t))$ expresses the path and actuators constraints. We assume that S is a polytopic subset. If it is not the case, it will be replaced by a polytopic inner convex approximation (see [7] for details on the approximation technique and [13] for the study of approximation conservatism).

Let us describe the flat output $t \mapsto z(t)$ components by univariate piecewise polynomials. Piecewise polynomials have been historically used in solving optimal control problem due to their ability to handle multiple convex or non-convex constraints from especially dynamics transcription [1, 2, 10]. Indeed, $\overline{z}(t)$ admits the k^{th} order B-splines basis $\{B_{j,k}\}$ (see [5, Chap. IX]) as,

$$\begin{aligned} z_i(t) = &\ \textstyle\sum_{j=1}^{n_B} C_{i,j}.B_{j,k}(t), \qquad i = 1, \ldots, m \\ z_i^{(p)}(t) = &\ \textstyle\sum_{j=1}^{n_B} C_{i,j}.B_{j,k}^{(p)}(t), \qquad i = 1, \ldots, m. \end{aligned} \tag{4}$$

with $C = (C_{1,1}, \ldots, C_{1,n_B}, \ldots, C_{m,1}, \ldots, C_{m,n_B})^T$ as the coordinates of $\overline{z}(t)$ in the B-splines basis $B(t)$. In the sequel, the control points C become the decision variables of the flat optimal control problem (3). It comes the parametrized optimal flat path planning:

$$\begin{array}{c} \min\limits_{C} J(\overline{z}(C)) \\ \text{subject to:} \end{array} \begin{cases} \overline{z}(t_i, C) = \overline{z}_i, \\ \overline{z}(t_f, C) = \overline{z}_f, \ . \\ \overline{z}(C) \in S. \end{cases} \tag{5}$$

The purpose of section 3 is to set constraint $\overline{z}(C) \in S$ as an inclusion problem of $\overline{z}(t)$ trajectory within the intersection of several half-spaces. In fact, it will be shown that positioning the trajectory $\overline{z}(t)$ in a half-space is equivalent to evaluate the sign of the piecewise polynomial gap function, $\kappa(t)$, between $\overline{z}(t)$ and the hyperplane boundary. In subsections 3.1 and 3.2, the positivity of a piecewise polynomial function in terms of its coordinates in a B-splines basis $v(t)$ is defined through the sums of squares formalism. Then, in subsection 3.3, we formulate the $\kappa(t)$-positivity in terms of the coefficients C using the LMI condition developped in section 3.2 in order to solve (5).

3 Motion planning as B-splines positivity problem

3.1 B-splines representation of squared piecewise polynomials

Inspired by the seminal work of Nesterov [16], let us describe here the decomposition in sums of squares of the non-negative piecewise polynomial functions (PP). First, the k_v-order PP functions will be described as a weighted sum of squared k_w-order PP functions.

Let $\mathbb{P}_{k_w,\xi,\nu_w}$ be the linear subspace defined by the collection of the k_w-order piecewise polynomial functions of $\mathbb{P}_{k_w,\xi}$ defined with breakpoints $\xi = \{\xi_1, \cdots, \xi_{l+1}\}$ whose first ν_{w_i} derivatives are continuous at ξ_i (i.e. that are $C^{\nu_{w_i}}$ at ξ_i with $\nu_w = \{\nu_{w_1}, \cdots, \nu_{w_{l+1}}\}$); see [5, Chap. VIII] for more detailed definitions. Then, by virtue of the Curry-Schoenberg theorem [4], $\mathbb{P}_{k_w,\xi,\nu_w}$ admits a B-splines basis $w(t) = (B_{1,k_w}, \ldots, B_{n_w,k_w})^T =$

$(w_1(t), \ldots, w_{n_w}(t))^T$.
The function subspace of sums of squared functions from $\mathbb{P}_{k_w,\xi,\nu_w}$ is described with $N \in \mathbb{N}$ by:

$$\mathcal{F}(w^2) = \{P(t) = \sum_{i=1}^{N} d_i \tau_i^2(t) | \tau_i(t) \in \mathbb{P}_{k_w,\xi,\nu_w}, d_i \in \mathbb{R}\}.$$

By construction, the description of $\mathcal{F}(w^2)$ depends only on the properties of the set of the squared basis functions:

$$w^2 = \{w_i(t)w_j(t),\ i,j = 1,\ldots,n_w\}. \quad (6)$$

It can be proved that $\mathcal{F}(w^2)$ is included $\mathbb{P}_{k_v,\xi,\nu_v}$ with $k_v = 2k_w - 1$, $\nu_v = \nu_w$ and ξ as defined above. Using the Curry-Schoenberg theorem [4] it follows that $\mathcal{F}(w^2)$ admits a k_v-order B-splines basis $v(t)$ such that:

$$\text{for all } P(t) \in \mathcal{F}(w^2), \quad P(t) = \sum_{i=1}^{n_v} \mu_i v_i(t). \quad (7)$$

Let us now introduce the operators Λ and Λ^* that determine the coordinates of a element of $\mathcal{F}(w^2)$ in the B-splines basis $v(t)$. These definitions will be useful in section 3.2.

Definition 2 (Operator Λ) *Let Λ a linear operator such that $\mathbb{P}^{n_v}_{k_v,\xi,\nu_v} \to \mathbb{P}^{n_w \times n_w}_{k_v,\xi,\nu_v} : v(t) \mapsto \Lambda(v) \equiv w(t)w(t)^T$ be defined by:*

$$\begin{aligned}\Lambda(v) &= [w_i.w_j],\ i,j = 1,\ldots,n_w \\ &= \begin{bmatrix} \lambda_{1,1}^T v(t) & \ldots & \lambda_{1,n_w}^T v(t) \\ \vdots & \ddots & \vdots \\ \lambda_{n_w,1}^T v(t) & \ldots & \lambda_{n_w,n_w}^T v(t) \end{bmatrix}, \lambda_{i,j} \in \mathbb{R}^n\end{aligned} \quad (8)$$

In the following, $\langle \cdot, \cdot \rangle$ denotes the bilinear form:

$$\mathbb{R}^{\alpha\times\beta} \times \mathbb{R}^{\alpha\times\beta} \to \mathbb{R},\ A, B \mapsto \langle A, B \rangle = \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} a_{i,j} b_{i,j}. \quad (9)$$

Definition 3 (Operator Λ^*) *The linear dual operator $\Lambda^* : \mathbb{R}^{n_w \times n_w} \to \mathbb{R}^{n_v}, Y \mapsto \Lambda^*(Y)$ is defined such that:*

$$\langle Y, \Lambda(v(t)) \rangle \equiv \langle \Lambda^*(Y), v(t) \rangle, \quad \forall Y \in \mathbb{R}^{n_w \times n_w},\ v(t) \in \mathbb{R}^{n_v}. \quad (10)$$

Recalling that $\langle Y, \Lambda(v(t)) \rangle = \sum_{i,j=1}^{n_w} Y_{i,j} \sum_{l=1}^{n_v} \lambda_{i,j,l} v_l(t)$, the operator Λ^* is defined as follow:

$$\Lambda^*(Y) = (\Lambda_1^*(Y_{i,j}, \lambda_{i,j}), \ldots, \Lambda_{n_v}^*(Y_{i,j}, \lambda_{i,j}))^T, \quad (11)$$

where Λ_i^* are scalar functions linear in $Y_{i,j}$.

The latter definition describes the coordinates in the B-splines basis $v(t)$ of a $\mathcal{F}(w^2)$ PP function. In fact, the matrix Y represents the weighting coefficients on $w^2(t)$ and $\Lambda^*(Y)$ denotes the coordinates in the B-splines basis $v(t)$.

3.2 Piecewise polynomials positivity

A piecewise polynomial has two representations: the B-splines and the sums of squares. The sums of squared PP function representation is convenient since its positiveness only depends on the semi definite positiveness of the weighting matrix Y. Then, through the operators defined above, we are able to describe the set of the coefficients μ on basis $v(t)$ that define positive PP function. In fact, this set is shown to be a linear image of the cone of the positive semidefinite matrices.

Theorem 4 *Let μ be an element of the closed, pointed and convex cone K defined by:*

$$\mathsf{K} = \{\mu \in \mathbb{R}^{n_v} \ :\ \mu = \Lambda^*(Y),\ Y \succeq 0\}. \quad (12)$$

where $Y \succeq 0$ define the semidefinite positiveness of Y.eigenvalues non negative. Each element μ of K describes a positive semidefinite polynomial on the basis $v(t)$ so that

$$P(t) = \sum_{i=1}^{n_v} \mu_i v_i(t) \geq 0. \quad (13)$$

Proof First, Let $P = \mu^T v$ with $\mu = \Lambda^*(Y)$ and $Y \succeq 0$. Then, P can be written as:

$$\begin{aligned} P(t) &= \langle \Lambda^*(Y), v(t) \rangle = \langle Y, \Lambda(v(t)) \rangle \\ &= \langle Y, w(t)w(t)^T \rangle. \end{aligned} \quad (14)$$

Considering $\langle Iw(t), w(t) \rangle = \langle I, w(t)w(t)^T \rangle$ with I the identity matrix, the previous relation becomes:

$$\begin{aligned} P(t) &= \langle Yw(t), w(t) \rangle = (Yw(t))^T w(t) \\ &= w(t)^T Y^T w(t) \geq 0. \end{aligned} \quad (15)$$

Second, if $P(t) \geq 0$, then there exits a set of vectors $\gamma_i \in \mathbb{R}^{n_w},\ i = 1, \ldots N$ such that:

$$\begin{aligned} P(t) &= \sum_{i=1}^{N} \langle \gamma_i, w(t) \rangle^2 = \left\langle \sum_{i=1}^{N} \gamma_i \gamma_i^T, w(t)w(t)^T \right\rangle \\ &= \left\langle \sum_{i=1}^{N} \gamma_i \gamma_i^T, \Lambda(v(t)) \right\rangle = \left\langle \Lambda^* \left(\sum_{i=1}^{N} \gamma_i \gamma_i^T \right), v(t) \right\rangle. \end{aligned} \quad (16)$$

Thus, taking $Y = \sum_{i=1}^{N} \gamma_i \gamma_i^T \succeq 0$ and $\mu = \Lambda^*(Y)$ allows us to establish (13).

Remark *For the sake of conciseness only the odd order case, $k_v = 2k_w - 1$, has been exposed here. But the even order case $k_v = 2k_w$ is very similar to the odd case: from the beginning, one need to replace the basis function w_i by $\bar{w}_i = \sqrt{\bar{\tau}(t)}w_i$ with $\bar{\tau}(t) = t - \xi_1 + 1$ strictly positive on $[\xi_1, \xi_{l+1}]$. In this way, operator Λ^* can be redefined in order to describe the non negative even order PP function: $P(t) = \sum_{i=1}^{n_v} \mu_i v_i(t) = \bar{\tau}(t) \sum_{i=1}^{N} \tau_i^2(t) \geq 0$*

3.3 Path planning problem as a positivity problem

The next step of the contribution consists of setting problem (5) as an SDP problem. This result, mainly based on theorem 4, is the description of the PP trajectory inclusion into a polytope as a B-spline positivity problem and consequently as an LMI problem.

Let $O_{\bar{z}}$ be the finite dimensional flat output space with the following coordinates:

$$\overline{z} = (z_1, \ldots, z_m, \dot{z}_1, \ldots, \dot{z}_m, z_1^{(r)}, \ldots, z_m^{(r)})^T.$$

Recall that the flat trajectories $[t_0,\, t_f] \to \mathbb{R}^{n_{\overline{z}}},\ t \mapsto \overline{z}(t)$ are parametrized on k-order B-splines basis $\{B_k\}$ (see equation (4)).

Let the feasible region S be an intersection of n_c half-spaces of $O_{\bar{z}}$ and H_i be the i^{th} half-space described by its Cartesian coordinates:

$$H_i = \{\overline{z} \in \mathbb{R}^{n_{\overline{z}}} |\, a_i^T \overline{z} \leq b_i\}, \tag{17}$$

where $a_i \in \mathbb{R}^{n_{\overline{z}}}$ and $b_i \in \mathbb{R}$ with $i = 1, \ldots, n_c$. We note that$\overline{z}(t)$ belongs to the half-space H_i if and only if

$$a_i^T \overline{z}(t) \leq b_i \tag{18}$$

Theorem 5 *Solving the path planning problem defined by* (5)*, is equivalent to solving the following SDP problem:*

$$\min_C J(\overline{z}(C))$$
$$\textit{subject to:} \begin{cases} \alpha_i C - b_i = \Lambda^*(Y_i) \\ Y_i \succeq 0 \\ \Theta C = \theta \end{cases} \qquad ,\forall i = 1, \ldots, n_c. \tag{19}$$

with the objective function assumed to be linear in $\overline{z}$ and so in $C = (C_{1,1}, \ldots, C_{1,n}, C_{2,1}, \ldots, C_{2,n}, C_{3,1}, \ldots, C_{3,n})$, the control points.

Λ^* is the dual operator defined by (10)*. $\alpha_i \in \mathbb{R}^{n_v \times N_C}$ are linear matrix functions of a_i, with a_i and b_i associated to the i^{th} half-space H_i (cf. equation (17)). The equality constraint $\Theta C = \theta$ represents the initial and final conditions. The proof is detailled in appendix A.*

4 Example

To illustrate the developed methodology, the motion planning problem of a flexible link manipulator system [6] will be solved, subject to path constraints and a cost criterion. The dynamics is described by the following equations:

$$I_1 \ddot{q}_1 + MgL \sin q_1 + k(q_1 - q_2) = 0, \tag{20}$$
$$I_2 \ddot{q}_2 - k(q_1 - q_2) = u, \tag{21}$$

where $x = [q_1,\, \dot{q}_1,\, q_2,\, \dot{q}_2]^T$ is the state vector and w the input. For this example, the following values (in S.I. units) are considered: $L = 0.1,\ M = 0.1,\ I_1 = 1,\ I_2 = 1,\ k = 1,\ g = 9.8$. The goal is to steer the system from initial state $q(t_0) = [0.8, 0, 0.67, 0]$ to final state $q(t_f) = [-0.8, 0, -0.67, 0]$ at final time $t_f = 5.35s$. The operating constraints are $-\frac{\pi}{3} \leq q_1 \leq \frac{\pi}{3}$, $-\frac{\pi}{4} \leq q_2 \leq \frac{\pi}{4}$, $-\frac{\pi}{12} \leq q_2 - q_1 \leq \frac{\pi}{12}$. A flat output for this system has been described in [6]: $z = q_1$. Thus, the flat parametrization of the state x is described by:

$$x = \phi(z, \dot{z}, \ddot{z}, z^{(3)})$$
$$= \left(z,\ \dot{z},\ \tfrac{I_1}{k}\ddot{z} + \tfrac{MgL}{k}\sin z + z,\ \tfrac{I_1}{k}z^{(3)} + \tfrac{MgL}{k}\dot{z}\cos z + \dot{z}\right)^T \tag{22}$$

Using the flat parametrization (22), the path planning problem is translated in the following flat optimal control problem described by:

$$\begin{array}{c} \min\limits_{\overline{z}} J(\overline{z}) \\ \text{subject to:} \end{array} \begin{cases} \overline{z}(t_i) = \overline{z}_i,\ \overline{z}(t_f) = \overline{z}_f \\ -\frac{\pi}{3} \leq z \leq \frac{\pi}{3}, \\ -\frac{\pi}{4} \leq \frac{I_1}{k}\ddot{z} + \frac{MgL}{k}\sin z + z \leq \frac{\pi}{4}, \\ -\frac{\pi}{16} \leq \frac{I_1}{k}\ddot{z} + \frac{MgL}{k}\sin z \leq \frac{\pi}{16}. \end{cases} \tag{23}$$

Note that the operating constraints depend only on z and $\ddot{z}$. Thus, the flat output space $O_{\bar{z}}$ has the following coordinates $\overline{z} = \{z,\ \ddot{z}\}$. We note that S is non convex. However as we said earlier, it can be replaced by a polytopic inner approximation S_p calculated with Faiz' techniques [7]. The half spaces representation of $S_p = \{\overline{z} | H\overline{z} \leq K\}$ is given by the matrices

$$H = \begin{bmatrix} -0.086 & 0.731 & 0.086 & -0.731 \\ -0.996 & 0.683 & 0.996 & -0.683 \end{bmatrix}^T$$

and $K = [0.192\ 0.527\ 0.192\ 0.527]$.

After B-splines parametrization, the motion planning problem can be represented as

$$\begin{array}{c} \min\limits_{C} J(C) \\ \text{subject to:} \end{array} \begin{cases} \overline{z}(t_i, C) = \overline{z}_i,\ \overline{z}(t_f, C) = \overline{z}_f, \\ \overline{z}(t) \in S_p. \end{cases} \tag{24}$$

The trajectory $t \mapsto z(t)$ is a 5^{th} order PP function defined on the sequence of equidistant knots $\xi = \{\xi_1, \ldots, \xi_{10}\}$. Indeed $z(t)$ as element of $\mathbb{P}_{k,\xi,\nu}$ admits $B(t)$ as B-splines basis. The continuity parameter

B-splines basis	order	ν_i	
		$i = 1, 10$	$i = 2 - 9$
$B(t)$	5	0	4
$v(t)$	5	0	2
$w(t)$	3	0	2

Table 1
B-splines basis parameters

vector ν are given in table 1. The dimension of the basis $B(t)$ is $n = 14$.

In order to define Λ^*, we need to characterize the gap function $\kappa(t)$. Since the higher derivation order involved in $O_{\bar{z}}$ is two, $\kappa(t)$ belongs to $\mathbb{P}_{k,\xi,\nu\ominus 2}$ (see appendix A) and thus it admits a B-splines basis $v(t)$ of dimension $n_v = 32$. Then, we calculate the corresponding basis $w(t)$. Its dimension is $n_w = 12$. The operator Λ^* is deduced from $v(t)$, $w(t)$ and definition 3.

Coefficient matrices α_i of problem (19) are then calculated for each half-space of polytope S_p (see (A.3)). The problem equivalent to (24) is finally set using Yalmip [12]:

$$\begin{array}{l} \min_C J(C) \\ \text{subject to:} \end{array} \begin{cases} \alpha_1 C - b_1 = \Lambda^*(Y_1),\ Y_1 \succeq 0, \\ \vdots \\ \alpha_4 C - b_4 = \Lambda^*(Y_4),\ Y_4 \succeq 0, \\ \Theta C = \theta \end{cases} \tag{25}$$

where $C \in \mathbb{R}^{3n}$, $\alpha_i \in \mathbb{R}^{n_v \times 3n}$. For the sake if simplicity, the cost is linear here, $J(C) = \ddot{z}(t_f, C)$. But one can choose any convex function $J_{LMI}(C)$ that is LMI representable. In this case, an additional decision variable τ is minimized instead of $J_{LMI}(C)$ such that $\tau \geq J_{LMI}(C)$. Problem (25) is solved using SDPT3 [17] in $0.29\,s$ with a 2.26GHz Intel Core 2 processor. The obtained trajectory is given in figure 1. Figure 1 shows that trajectory $\overline{z}$ is clearly included in S_p as expected.

For sake of comparison, we solve problem (5) by means of flatness and collocation methods described in [13, 15, 7]. Recall that the constraints are checked in a finite number of time called collocation points. The method is programmed and solved with the linear solver Matlab `linprog` considering 10 equidistant collocation points. Although an admissible solution for the collocation problem is obtained quickly ($0.06\,s$), the trajectory does not respect the constraints between collocation points ♯2 and ♯3 then between ♯8 and ♯9 (see figure 1). Then, with the above collocation methods, an iterative process is needed to re-distribute the sequence or increase the number the collocation points, conversely to the one-run method exposed in this paper.

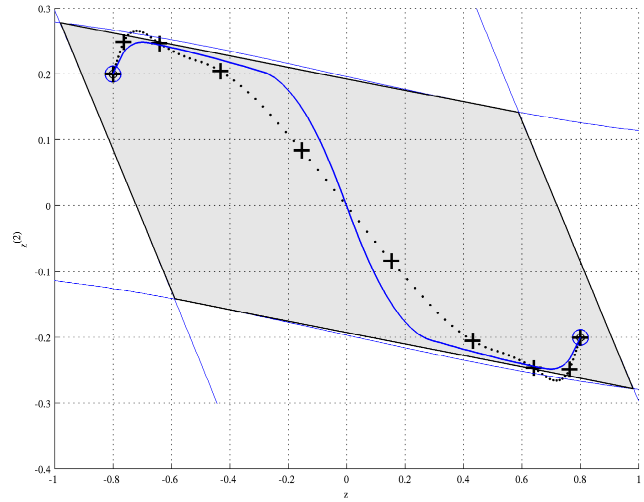

Fig. 1. Trajectories $\overline{z}(t)$ obtained by SDP (plain line) and by collocation (dotted line), the collocation points are the crosses, thin lines give actual constraints, and the polytopic approximation is the shaded zone.

5 Concluding remarks

This paper considers motion planning for differentially flat systems. As opposed to most works on direct methods for optimal control problem reported in the literature, the developed methodology provides a new framework for satisfying constraints continuously in time. In this methodology, a crucial point is the approximation of non convex feasible regions since the accuracy of the results depends on its conservatism. Despite the method used above, the computation of convex (especially LMI representable) inner approximations remains an open research area.

A Proof of theorem 5

To apply the positivity theorem, the inequality $a_i^T \overline{z}(t) \geq b_i$ must be expressed in a B-splines basis (we flipped the inequality (18) without loss of generality). Thus, by using (4), equation (18) is equivalent to

$$\begin{aligned} &\sum_{j=1}^{n_B} (a_{i,1} B_{j,k}(t) + \cdots + a_{i,(r-1)m+1} B_{j,k}^{(r)}(t)) C_{1,j} \ldots \\ &\quad + (a_{i,2} B_{j,k}(t) + \cdots + a_{i,(r-1)m+2} B_{j,k}^{(r)}(t)) C_{2,j} + \ldots \\ &\quad + (a_{i,m} B_{j,k}(t) + \cdots + a_{i,rm} B_{j,k}^{(r)}(t)) C_{m,j} \geq b_i. \end{aligned} \tag{A.1}$$

In inequality (A.1), the PP function is composed of a $\mathbb{P}_{k,\xi,\nu}$ piecewise polynomial and its r first derivatives. Considering that for a B-spline $B_{j,k} \in \mathbb{P}_{k,\xi,\nu}$, one has $\dot{B}_{j,k} \in \mathbb{P}_{k-1,\xi,\nu\ominus 1}, \ldots, B_{j,k}^{(r)} \in \mathbb{P}_{k-r,\xi,\nu\ominus r}$ where the operator $\ominus$ is defined by $\nu \ominus r = (\max\{\nu_1 - r, 0\}, \dots, \max\{\nu_{l+1} - r, 0\})$, with $r \leq \nu_i$ for $i = 2, \dots, l$. Then the sum (A.1), representing the gap, belongs to $\mathbb{P}_{k,\xi,\nu\ominus r}$ and, by virtue of the Curry-Schoenberg theorem [4], admits the B-splines representation basis $v(t)$ such that (A.1) becomes:

$$\sum_{i=1}^{n_v} \left(\sum_{j=1}^{n_B} \alpha_{1,i,j} C_{1,j} + \dots + \alpha_{m,i,j} C_{m,j} \right) v_{i,k}(t) \geq \dots$$
$$b = b \sum_{i=1}^{n_v} v_{i,k}(t). \quad \text{(A.2)}$$

with $\alpha_{p,j} \in \mathbb{R}^{n_v}$ $p = 1, \dots, m$ and $j = 1, \dots, n$. Vectors $\alpha_{l,i,j}$ are identified using the following equalities system:

$$\begin{cases} E_1(\sum_{i=1}^{n_v} \alpha_{l,i,j} v_{i,k}(t)) = & E_1((a_{i,l} B_{j,k}(t) + \dots \\ & + a_{i,(r-1)m+l} B_{j,k}^{(r)}(t))), \\ & \vdots \\ E_p(\sum_{i=1}^{n_v} \alpha_{l,i,j} v_{i,k}(t)) = & E_p((a_{i,l} B_{j,k}(t) + \dots \\ & + a_{i,(r-1)m+l} B_{j,k}^{(r)}(t))), \end{cases} \quad \text{(A.3)}$$

with $l = 1, \dots, m$ and $j = 1, \dots, n$.
$E_p(f) = \int_0^{t_f} x^p f(x)\, dx$ denotes the p^{th} order moment of the function f. The index p is chosen such that equation (A.3) leads to a square linear matrix equality to obtain $\alpha_{l,i,j}$. Thus, inequality (18) is equivalent to the following positivity problem:

$$\kappa(t) = \sum_{i=1}^{n_v} \kappa_i v_{i,k}(t) \geq 0 \quad \text{(A.4)}$$

where $\kappa_i = \sum_{j=1}^{n_B} (\alpha_{1,i,j} C_{1,j} + \dots + \alpha_{m,i,j} C_{m,j}) - b$.
Then, determining the operators Λ and Λ^* is needed to recast the positivity problem (A.4) into an LMI problem by using theorem 4. These operators are built with a basis $u(t)$ satisfying the following inequality: $\nu \ominus r < \frac{k+1}{2}$ if k is odd, or $\nu \ominus r < \frac{k}{2}$ if k is even. So, theorem 4 gives conditions on the κ coefficients so that inequality (A) holds:

$$\begin{cases} \kappa = \Lambda^*(Y), & Y \succeq 0, \\ \kappa = \alpha C - b. \end{cases} \quad \text{(A.5)}$$

Hence,

$$\alpha C - b = \Lambda^*(Y),\ Y \succeq 0. \quad \text{(A.6)}$$

Finally, the inclusion of a trajectory $t \mapsto \overline{z}(t)$ to the intersection of n_c half-spaces is written as the conjunction of the n_c membership problem defined in the theorem 5 i.e.

$$\begin{cases} \alpha_1 C - b_1 = \Lambda^*(Y_1), & Y_1 \succeq 0, \\ \vdots \\ \alpha_{n_c} C - b_{n_c} = \Lambda^*(Y_{n_c}), & Y_{n_c} \succeq 0. \end{cases} \quad \text{(A.7)}$$

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