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# CONVERGENCE TO STEADY STATES FOR A ONE-DIMENSIONAL VISCOUS HAMILTON-JACOBI EQUATION WITH DIRICHLET BOUNDARY CONDITIONS

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## Abstract

The convergence to steady states of solutions to the one-dimensional viscous Hamilton-Jacobi equation  $\partial_t u - \partial_x^2 u = |\partial_x u|^p$ ,  $(t, x) \in (0, \infty) \times (-1, 1)$  with homogeneous Dirichlet boundary conditions is investigated. For that purpose, a Liapunov functional is constructed by the approach of Zelenyak (1968). Instantaneous extinction of  $\partial_x u$  on a subinterval of  $(-1, 1)$  is also shown for suitable initial data.

MSC 2000: 35B40, 35K55, 37B25

Key words: diffusive Hamilton-Jacobi equation, convergence, gradient extinction, Liapunov functional

## 1 Introduction

Non-negative solutions to the one-dimensional viscous Hamilton-Jacobi equation

$$\partial_t u - \partial_x^2 u = a |\partial_x u|^p, \quad (t, x) \in (0, \infty) \times (-1, 1), \quad (1)$$

$$u(t, \pm 1) = 0, \quad t \in (0, \infty), \quad (2)$$

$$u(0) = u_0 \geq 0, \quad x \in (-1, 1), \quad (3)$$

exhibits a rich variety of qualitative behaviours, according to the sign of  $a \in \{-1, 1\}$  and the values of  $p \in (0, \infty)$ . Indeed, on the one hand, extinction in finite time (that is, there is  $T_\star > 0$  such that  $u(t) \equiv 0$  for  $t \geq T_\star$ ) occurs for  $a = -1$  and  $p \in (0, 1)$ , while  $u(t)$  converges exponentially fast to zero as  $t \rightarrow \infty$  if  $a = -1$  and  $p \geq 1$  [5]. On the other hand, if  $a = 1$  and  $p > 2$ , finite time gradient blow-up takes place for suitably large initial data [13] while convergence to zero of  $u(t)$  as  $t \rightarrow \infty$  still holds true for global solutions [2, 14]. In addition, all solutions are global for  $a = 1$  and  $p \in [1, 2]$  and converge to zero as  $t \rightarrow \infty$  [5, 14].

The case  $a = 1$  and  $p \in (0, 1)$  offers an interesting novelty and is the subject of the present paper. Indeed, in contrast to the previous cases, the initial-boundary value problem (1)-(3) has a one parameter family  $(U_\vartheta)_{\vartheta \in [0, 1]}$  of steady states when  $a = 1$  and  $p \in (0, 1)$  with  $U_1 \equiv 0$  and  $U_\vartheta$  is not constant if  $\vartheta \in [0, 1)$ . These steady states play an important role in the dynamics of solutions to (1)-(3): indeed, we will prove that any solution  $u$  to (1)-(3) converges as  $t \rightarrow \infty$  towards a steady state, which is non-trivial if, for instance, the initial datum  $u_0$  is non-negative with a positive maximum. In addition, an interesting feature of  $U_\vartheta$  for  $\vartheta \in (0, 1)$  is that they are constant on a subinterval of  $(-1, 1)$ . This property is of course related to the fact that  $p$  ranges in  $(0, 1)$  and is reminiscent of the finite time extinction phenomenon already alluded to for non-negative solutions when  $a = -1$  and  $p \in (0, 1)$ . It is then natural to wonder whether the nonlinear term  $|\partial_x u|^p$  may induce a similar singular behaviour on the dynamics of  $u$ . More precisely, for a particular class of non-negative initial data, we will show that the gradient  $\partial_x u$  vanishes identically on  $[T_\star, \infty) \times I$  for some  $T_\star > 0$  and some subinterval  $I$  of  $(-1, 1)$ . Let us point out here that, for non-negative initial data, extinction in finite time cannot occur when  $a = 1$  and  $p \in (0, 1)$ , for the comparison principle warrants that  $u$  is bounded from below by the solution to the linear heat equation with the same initial and boundary data.

From now on, we thus assume that

$$a = 1 \quad \text{and} \quad p \in (0, 1), \quad (4)$$

and

$$u_0 \in Y := \{w \in \mathcal{C}^1([-1, 1]), \quad w(\pm 1) = 0\}. \quad (5)$$

It then follows from [3, Theorem 3.1 & Proposition 4.1] that the initial-boundary value problem (1)-(3) has a unique classical solution

$$u \in \mathcal{C}([0, \infty) \times [-1, 1]) \cap \mathcal{C}^{2,1}((0, \infty) \times (-1, 1))$$

satisfying

$$\min_{[-1,1]} u_0 \leq u(t, x) \leq \max_{[-1,1]} u_0, \quad (t, x) \in [0, \infty) \times [-1, 1]. \quad (6)$$

In addition, setting

$$M(t) := \max_{x \in [-1,1]} u(t, x) \geq 0, \quad (7)$$

the comparison principle ensures that  $t \mapsto M(t)$  is a non-increasing function of time and we put

$$M_\infty := \lim_{t \rightarrow \infty} M(t) \in \left[ 0, \max_{[-1,1]} u_0 \right]. \quad (8)$$

Let us recall at this point that classical solutions to (1)-(3) enjoy the comparison principle: this fact may be proved by standard arguments as in, e.g., [8, Theorem 4].

**Remark 1** *The initial-boundary value problem (1)-(3) is actually well-posed in a larger space than  $Y$ , which depends on  $p$ , and we refer to [3] for a more detailed account. Still, the solutions constructed in [3] belong to  $Y$  for any positive time. Since we are interested here in the large time behaviour, the assumption (5) that  $u_0 \in Y$  is thus not restrictive.*

For further use, we also introduce the following notations:

$$\alpha := \frac{2-p}{1-p} \quad \text{and} \quad \mathcal{M}_0 := \frac{(1-p)^\alpha}{2-p}. \quad (9)$$

We may now state our main result.

**Theorem 2** *Consider  $u_0 \in Y$  and denote by  $u$  the corresponding classical solution to (1)-(3). Then  $M_\infty \in [0, \mathcal{M}_0]$  and there is a non-negative stationary solution  $u_s$  to (1)-(2) such that*

$$\lim_{t \rightarrow \infty} \|u(t) - u_s\|_\infty = 0. \quad (10)$$

*Furthermore,  $u_s \not\equiv 0$  and  $M_\infty > 0$  if*

$$\int_{-1}^1 u_0(x) \cos\left(\frac{\pi x}{2}\right) dx > 0. \quad (11)$$

It readily follows from the second assertion of Theorem 2 that the set of non-trivial and non-negative steady states to (1)-(2) attracts all solutions to (1)-(3) starting from a non-negative initial datum  $u_0 \not\equiv 0$ . Observe however that the set of non-trivial and non-negative steady states to (1)-(2) also attracts sign-changing solutions  $u$  to (1)-(3) since there are sign-changing initial data fulfilling (11).

The proof of Theorem 2 requires several steps and is performed as follows: we first identify the stationary solutions to (1)-(2) in Section 2 and use them together with comparison arguments to establish that, if  $u_0 \in Y$  is non-negative with  $u_0 \not\equiv 0$ , then  $M_\infty > 0$  and  $\{u(t); t \geq 0\}$  is bounded in  $\mathcal{C}^1([-1, 1])$  (Section 3). In Section 4, we employ the technique of Zelenyak [15] to construct a Liapunov functional for non-negative solutions to (1)-(3). Let us mention here that this technique has also been used recently for related problems in [2, 12]. For non-negative initial data convergence towards a steady state then follows from the results of Section 3 and Section 4 by a LaSalle invariance principle argument. The large time behaviour of sign-changing initial data is next deduced from that of non-negative solutions after observing that the negative part of any solution to (1)-(3) vanishes in a finite time (Section 6).

**Remark 3** *A further outcome of Theorem 2 is that the large behaviour of solutions to (1) on a bounded interval is more complex for homogeneous Dirichlet boundary conditions than for periodic and homogeneous Neumann boundary conditions. Indeed, for the latter boundary conditions, it follows from [4, 6] that there are  $T_\star > 0$  and  $m_\star \in \mathbb{R}$  such that  $u(t) \equiv m_\star$  for  $t \geq T_\star$  whatever the signs of  $a$  and  $u_0$  are.*

In Section 7, we prove the extinction in finite time of  $\partial_x u$  on a subinterval of  $(-1, 1)$  for a specific class of initial data. More precisely, we have the following result:

**Theorem 4** *Assume further that there are  $m_0 \in (0, \mathcal{M}_0)$  and  $\varepsilon > 0$  such that*

$$m_0 - \mathcal{M}_0 |x|^\alpha + \varepsilon |x|^{1+\alpha} \leq u_0(x) \leq m_0, \quad x \in [-1, 1]. \quad (12)$$

*Then, for each  $t \in (0, \infty)$ , there is  $X(t) \in (0, 1)$  such that*

$$u(t, x) = m_0 \quad \text{for } x \in (-X(t), X(t)).$$

Furthermore, if

$$\delta_0 := 1 - \left( \frac{m_0}{\mathcal{M}_0} \right)^{1/\alpha} \in (0, 1), \quad (13)$$

and  $\delta \in (0, \delta_0)$ , there exists  $T(\delta) > 0$  such that

$$u(t, x) = m_0 \quad \text{for} \quad (t, x) \in [T(\delta), \infty) \times [-\delta, \delta].$$

An example of initial datum in  $Y$  fulfilling (12) is the following:  $u_0(x) = \mathcal{M}_0 - \varepsilon - \mathcal{M}_0 |x|^\alpha + \varepsilon |x|^\beta$  for  $x \in [-1, 1]$ , where  $\beta \in (\alpha, \alpha + 1]$  and  $\varepsilon \in (0, \alpha \mathcal{M}_0 / \beta)$ .

The second assertion of Theorem 4 shows that  $\partial_x u$  vanishes identically after some time on a subinterval of  $[-1, 1]$ , a phenomenon which one could call *finite time incomplete extinction* in comparison to what occurs for periodic or homogeneous Neumann boundary conditions. But the first assertion of Theorem 4 reveals that the extinction mechanism is somewhat stronger since, even if  $\partial_x u_0(x)$  vanishes only for  $x = 0$ ,  $\partial_x u$  vanishes instantaneously on a subinterval of  $[-1, 1]$  with positive measure.

Another consequence of Theorem 4 and (6) is that  $\|u(t)\|_\infty = m_0$  for every  $t \geq 0$ . Therefore, for an initial datum  $u_0$  in  $Y$  satisfying (12), the corresponding solution  $u$  to (1)-(3) does not obey the strong maximum principle.

The proof of Theorem 4 relies on comparison arguments with travelling wave solutions to (1) and is similar to that of [7, Theorem 9], some care being needed to cope with the boundary conditions.

**Notations.** Throughout the paper, we denote by  $r_+ := \max\{r, 0\}$  the positive part of the real number  $r$ . For  $r \in \mathbb{R}$  and  $s \in \mathbb{R}$ , we put  $r \vee s := \max\{r, s\}$  and  $r \wedge s := \min\{r, s\}$ . Also, for  $q \in [1, \infty]$ ,  $\|\cdot\|_q$  denotes the  $L^q(-1, 1)$ -norm.

## 2 Non-negative steady states

In this section, we look for non-negative stationary solutions to (1), (2), that is, non-negative functions  $U \in \mathcal{C}^2([-1, 1])$  such that

$$\frac{d^2 U}{dx^2} + \left| \frac{dU}{dx} \right|^p = 0, \quad x \in (-1, 1), \quad (14)$$

$$U(\pm 1) = 0. \quad (15)$$

**Proposition 5** *Let  $U \in \mathcal{C}^2([-1, 1])$  be a non-negative solution to (14), (15). Then there is  $\vartheta \in [0, 1]$  such that  $U = U_\vartheta$ , where*

$$U_\vartheta(x) := \mathcal{M}_0 \left[ (1 - \vartheta)^\alpha - (|x| - \vartheta)_+^\alpha \right], \quad x \in [-1, 1].$$

Observe that  $U_\vartheta$  is constant on  $[-\vartheta, \vartheta]$  for each  $\vartheta \in (0, 1)$  and that  $U_1 \equiv 0$ .

**Proof.** Let  $U \in \mathcal{C}^2([-1, 1])$  be a non-negative solution to (14), (15). Then  $U$  is concave by (14) and we infer from the non-negativity of  $U$  and the boundary conditions (15) that  $dU/dx(-1) \geq 0$  and  $dU/dx(1) \leq 0$ .

If  $dU/dx(-1) = 0$ , the concavity of  $U$  entails that  $U$  is a non-increasing function in  $(-1, 1)$ . Consequently,  $U \equiv 0 = U_1$  to comply with the boundary conditions (15).

Similarly, if  $dU/dx(1) = 0$ , it follows from the concavity of  $U$  that  $U$  is non-decreasing on  $(-1, 1)$ , whence  $U \equiv 0 = U_1$  by (15).

We finally consider the case where  $dU/dx(-1) > 0$  and  $dU/dx(1) < 0$  and put

$$\begin{aligned} x_I &:= \sup \{X \in (-1, 1) \text{ such that } dU/dx(x) > 0 \text{ on } [-1, X)\}, \\ x_S &:= \inf \{X \in (-1, 1) \text{ such that } dU/dx(x) < 0 \text{ on } (X, 1]\}. \end{aligned}$$

Owing to the continuity of  $dU/dx$ , we have  $-1 < x_I \leq x_S < 1$  and  $dU/dx(x) = 0$  for  $x \in [x_I, x_S]$  by the concavity of  $U$ . Direct integration of (14) then entails that there are two constants  $A$  and  $B$  such that

$$\left| \frac{dU}{dx}(x) \right|^{-p} \frac{dU}{dx}(x) + (1 - p)x = \begin{cases} A & \text{if } x \in (x_S, 1], \\ B & \text{if } x \in [-1, x_I]. \end{cases} \quad (16)$$

Since  $p \in (0, 1)$  and  $dU/dx$  vanishes for  $x \in \{x_I, x_S\}$ , we may let  $x \rightarrow x_I$  and  $x \rightarrow x_S$  in (16) to deduce that  $A = (1 - p)x_S$  and  $B = (1 - p)x_I$ . We next integrate (16) to obtain that there are two constants  $C_I$  and  $C_S$  such that

$$U(x) = \begin{cases} C_S - \mathcal{M}_0 (x - x_S)^\alpha & \text{if } x \in (x_S, 1], \\ C_I - \mathcal{M}_0 (x_I - x)^\alpha & \text{if } x \in [-1, x_I]. \end{cases}$$

Requiring the boundary conditions (15) to be fulfilled provides the values of  $C_I$  and  $C_S$ , whence

$$U(x) = \begin{cases} \mathcal{M}_0 (1 - x_S)^\alpha - \mathcal{M}_0 (x - x_S)^\alpha & \text{if } x \in (x_S, 1], \\ \mathcal{M}_0 (x_I + 1)^\alpha - \mathcal{M}_0 (x_I - x)^\alpha & \text{if } x \in [-1, x_I]. \end{cases}$$

Now, since  $dU/dx$  vanishes for  $x \in [x_I, x_S]$ , we shall have  $U(x_S) = U(x_I)$ , which implies that  $1 - x_S = x_I + 1$ , whence  $x_S = -x_I$ . Thus, necessarily,  $x_S \in [0, 1]$ , from which the equality  $U = U_{x_S}$  readily follows.  $\square$

It is worth mentioning that  $\|U_\vartheta\|_\infty \leq \mathcal{M}_0$  for each  $\vartheta \in [0, 1]$ . Combining this property with the convergence to a steady state to be proved in Section 5, we will conclude that  $M_\infty \leq \mathcal{M}_0$ .

**Remark 6** *Proposition 5 shows in particular that there is non-uniqueness of classical solutions to (14), (15). A similar construction is performed in [1, 11] for the boundary-value problem*

$$-\Delta u = |\nabla u|^p \text{ in } B(0, 1), \quad u = 0 \text{ on } \partial B(0, 1),$$

where  $B(0, 1)$  denotes the open unit ball of  $\mathbb{R}^N$ ,  $N > 1$ , to establish the non-uniqueness of weak solutions for  $p > N/(N - 1)$ .

### 3 Some properties of $\{u(t) ; t \geq 0\}$

Introducing the positive cone  $Y_+ := \{w \in Y \text{ such that } w \geq 0\}$  of  $Y$ , we first prove that  $M_\infty > 0$  for  $u_0 \in Y_+$ ,  $u_0 \not\equiv 0$ , by constructing suitable subsolutions to (1)-(3) with the help of  $U_0$ .

**Lemma 7** *Let  $u_0 \in Y_+$  and denote by  $u$  the corresponding classical solution to (1)-(3). If  $u_0 \not\equiv 0$ , we have  $M_\infty > 0$ .*

**Proof.** Since  $u_0 \not\equiv 0$ , there are  $x_0 \in (-1, 1)$ ,  $\delta \in (0, 1)$  and  $m > 0$  such that  $(x_0 - \delta, x_0 + \delta) \subset (-1, 1)$  and

$$u_0(x) \geq m \quad \text{for } x \in (x_0 - \delta, x_0 + \delta). \quad (17)$$

We put  $x_1 := (x_0 - 1) \vee (-1)$ ,  $x_2 := (x_0 + 1) \wedge 1$ ,  $J := [x_1, x_2]$ ,

$$\lambda := 1 \wedge \frac{m}{\mathcal{M}_0 - U_0(\delta)},$$

and  $v(x) := \lambda (U_0(x - x_0) - U_0(\delta))$  for  $x \in J$ .

On the one hand, it follows from (1) and (14) that

$$\partial_t v - \partial_x^2 v - |\partial_x v|^p = (\lambda - \lambda^p) |\partial_x U_0(\cdot - x_0)|^p \leq 0 = \partial_t u - \partial_x^2 u - |\partial_x u|^p$$



on  $[0, \infty) \times J$ . On the other hand, the non-negativity of  $u_0$  and the maximum principle entail the non-negativity of  $u$  which then warrants that

$$\begin{aligned} v(x_1) &\leq v(x_0 - \delta) = 0 \leq u(t, x_1), \\ v(x_2) &\leq v(x_0 + \delta) = 0 \leq u(t, x_2), \end{aligned}$$

while the choice of  $\lambda$  entails that

$$\begin{aligned} v(x) &\leq \lambda (\mathcal{M}_0 - U_0(\delta)) \leq m \leq u_0(x) \quad \text{for } x \in (x_0 - \delta, x_0 + \delta), \\ v(x) &\leq v(x_0 \pm \delta) = 0 \leq u_0(x) \quad \text{for } x \in J \setminus (x_0 - \delta, x_0 + \delta). \end{aligned}$$

We then infer from the comparison principle that  $u(t, x) \geq v(x)$  for  $(t, x) \in [0, \infty) \times J$ . In particular,  $M(t) = \|u(t)\|_\infty \geq u(t, x_0) \geq v(x_0) = \lambda (\mathcal{M}_0 - U_0(\delta))$  for each  $t \geq 0$ , whence  $M_\infty \geq \lambda (\mathcal{M}_0 - U_0(\delta)) > 0$ .  $\square$

We now turn to the global boundedness of the trajectory  $\{u(t); t \geq 0\}$  in  $\mathcal{C}^1([-1, 1])$ .

**Lemma 8** *Let  $u_0 \in Y_+$  and denote by  $u$  the corresponding classical solution to (1)-(3). There is a constant  $\Lambda > 0$  depending only on  $\|u_0\|_{W^{1,\infty}(-1,1)}$  and  $p$  such that*

$$\|u(t)\|_{W^{1,\infty}(-1,1)} \leq \Lambda \quad \text{for } t \geq 0. \quad (18)$$

**Proof.** We first recall that  $\{u(t); t \geq 0\}$  is bounded in  $L^\infty(-1, 1)$  by (6) and we are left with the proof that  $\{\partial_x u(t); t \geq 0\}$  is bounded in  $L^\infty(-1, 1)$ . For that purpose, we choose  $\lambda > 1$  such that

$$\lambda \geq \left[ \left( \frac{2}{1-p} \right)^{1/(1-p)} \|\partial_x u_0\|_\infty \right] \vee \left[ \frac{\|u_0\|_\infty}{(1-2^{-\alpha}) \mathcal{M}_0} \right]. \quad (19)$$

Putting  $v := \lambda U_0$ , we first notice that the condition  $\lambda > 1$  ensures that

$$\partial_t v - \partial_x^2 v - |\partial_x v|^p = (\lambda - \lambda^p) |\partial_x U_0|^p \geq 0 \quad \text{in } (0, \infty) \times (-1, 1),$$

while  $v(\pm 1) = u(t, \pm 1) = 0$  for each  $t \geq 0$ . Next, on the one hand, it follows from (19) and the monotonicity properties of  $U_0$  that, if  $x \in (-1/2, 1/2)$ , we have

$$v(x) = \lambda U_0(x) \geq \lambda U_0(1/2) = \lambda \mathcal{M}_0 (1 - 2^{-\alpha}) \geq \|u_0\|_\infty \geq u_0(x).$$

On the other hand, if  $x \in [1/2, 1]$ , we have by (19) that

$$\begin{aligned} v(x) &= \lambda (U_0(x) - U_0(1)) = \lambda \int_x^1 \left| \frac{dU_0}{dx}(y) \right| dy = \alpha \lambda \mathcal{M}_0 \int_x^1 y^{1/(1-p)} dy \\ &\geq \alpha \lambda \mathcal{M}_0 \int_x^1 2^{-1/(1-p)} dy \geq \int_x^1 \|\partial_x u_0\|_\infty dy \geq \int_x^1 |\partial_x u_0(y)| dy \\ &\geq u_0(x). \end{aligned}$$

A similar computation shows that  $v(x) \geq u_0(x)$  also holds true for  $x \in [-1, -1/2]$ . Therefore,  $v \geq u_0$  in  $[-1, 1]$  and the previous analysis allows us to apply the comparison principle and conclude that  $u(t, x) \leq v(x)$  for  $(t, x) \in [0, \infty) \times [-1, 1]$ . In particular, if  $t \geq 0$  and  $x \in (0, 1)$ , we have

$$\frac{u(t, x) - u(t, 1)}{x - 1} = \frac{u(t, x)}{x - 1} \geq \frac{v(x)}{x - 1} = \frac{v(x) - v(1)}{x - 1}.$$

Letting  $x \rightarrow 1$ , we deduce that  $\partial_x u(t, 1) \geq \partial_x v(1) = -\lambda (1 - p)^{1/(1-p)}$ . Since  $u_0 \geq 0$ , the comparison principle ensures that  $u(t, x) \geq 0 = u(t, 1)$  for  $x \in (0, 1)$ , so that we also have  $\partial_x u(t, 1) \leq 0$ . Arguing in a similar way for  $x = -1$ , we end up with

$$|\partial_x u(t, \pm 1)| \leq \lambda (1 - p)^{1/(1-p)} \quad \text{for } t \geq 0. \quad (20)$$

We now put  $k := \|\partial_x u_0\|_\infty \vee \lambda (1 - p)^{1/(1-p)}$ ,  $z := \partial_x u$  and  $\mathcal{R} := \{(t, x) \in (0, \infty) \times (-1, 1), z(t, x) \neq 0\}$ . In the neighbourhood of each point  $(t_0, x_0) \in \mathcal{R}$ , the function  $|\partial_x u|^p$  is smooth, and classical parabolic regularity theory implies that  $z$  is  $\mathcal{C}^{1,2}$  in a neighbourhood of  $(t_0, x_0)$  and satisfies

$$\partial_t z(t, x) - \partial_x^2 z(t, x) = p |z(t, x)|^{p-2} z(t, x) \partial_x z(t, x).$$

Since  $\{(t, x) \in (0, \infty) \times (-1, 1), z(t, x) > k\} \subset \mathcal{R}$ , we deduce from the previous identity and (20) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(z - k)_+\|_2^2 &= [(z - k)_+ \partial_x z]_{x=-1}^{x=1} - \int_{-1}^1 |\partial_x (z - k)_+|^2 dx \\ &+ \left[ \left( \frac{p}{p+1} z - k \right) |z|^p \frac{(z - k)_+}{|z - k|} \right]_{x=-1}^{x=1} \\ &= - \int_{-1}^1 |\partial_x (z - k)_+|^2 dx, \end{aligned}$$

whence

$$\|(z(t) - k)_+\|_2^2 \leq \|(z(0) - k)_+\|_2^2 = 0,$$

the last equality being true thanks to the choice of  $k$ . Consequently,  $\partial_x u(t, x) = z(t, x) \leq k$  in  $[0, \infty) \times [-1, 1]$ . By a similar argument, we also establish that  $\partial_x u(t, x) = z(t, x) \geq -k$  in  $[0, \infty) \times [-1, 1]$ . Therefore,

$$|\partial_x u(t, x)| \leq \|\partial_x u_0\|_\infty \vee \lambda (1 - p)^{1/(1-p)}$$

for  $(t, x) \in [0, \infty) \times [-1, 1]$ , which completes the proof of Lemma 8.  $\square$

## 4 A Liapunov functional

We now construct a Liapunov functional for non-negative solutions to (1)-(3) with the help of the technique developed by Zelenyak [15]. Let  $u_0 \in Y_+$  and denote by  $u$  the corresponding classical solution to (1)-(3) which is also non-negative by the maximum principle. We look for a pair of functions  $\Phi$  and  $\varrho \geq 0$  such that

$$\frac{d}{dt} \int_{-1}^1 \Phi(u, \partial_x u) dx = \int_{-1}^1 \varrho(u, \partial_x u) |\partial_t u|^2 dx. \quad (21)$$

Since  $\partial_t u(t, \pm 1) = 0$  by (2), the first term of the right-hand side of the above equality also reads

$$\begin{aligned} & \frac{d}{dt} \int_{-1}^1 \Phi(u, \partial_x u) dx \\ &= \int_{-1}^1 [\partial_1 \Phi(u, \partial_x u) \partial_t u + \partial_2 \Phi(u, \partial_x u) \partial_x \partial_t u] dx \\ &= \int_{-1}^1 [\partial_1 \Phi(u, \partial_x u) - \partial_1 \partial_2 \Phi(u, \partial_x u) \partial_x u - \partial_2^2 \Phi(u, \partial_x u) \partial_x^2 u] \partial_t u dx, \end{aligned}$$

and it is then natural to require that

$$\begin{aligned} & [\partial_1 \Phi(u, \partial_x u) - \partial_1 \partial_2 \Phi(u, \partial_x u) \partial_x u - \partial_2^2 \Phi(u, \partial_x u) \partial_x^2 u] \\ &= \varrho(u, \partial_x u) \partial_t u \\ &= \varrho(u, \partial_x u) (|\partial_x u|^p + \partial_x^2 u) \end{aligned}$$

for (21) to hold true. Following [15], we realize that a sufficient condition for the previous equality to be valid is

$$\partial_1 \Phi(u, \partial_x u) - \partial_1 \partial_2 \Phi(u, \partial_x u) \partial_x u = \varrho(u, \partial_x u) |\partial_x u|^p, \quad (22)$$

$$-\partial_2^2 \Phi(u, \partial_x u) = \varrho(u, \partial_x u). \quad (23)$$

Performing the computations as in [15], we see that the functions

$$\Phi(u, \partial_x u) := u - \frac{|\partial_x u|^{2-p}}{(2-p)(1-p)} \quad \text{and} \quad \varrho(u, \partial_x u) := |\partial_x u|^{-p}$$

solve the differential system (22), (23). However,  $\varrho$  is singular when  $\partial_x u$  vanishes and it is not clear how to give a meaning to (21) for such a choice of functions  $\Phi$  and  $\varrho$ . Nevertheless, we have the following weaker result which turns out to be sufficient for our purposes.

**Proposition 9** *For each  $t > 0$  and  $\delta \in (0, 1]$ , we have*

$$\frac{d}{dt} \int_{-1}^1 \left( \frac{|\partial_x u(t, x)|^{2-p}}{(2-p)(1-p)} - u(t, x) \right) dx + \int_{-1}^1 \frac{|\partial_t u|^2}{(|\partial_x u|^2 + \delta^2)^{p/2}} dx \leq 0. \quad (24)$$

**Proof.** We fix  $\delta \in (0, 1]$  and define  $\psi_\varepsilon$  by

$$\psi_\varepsilon(0) = \psi'_\varepsilon(0) = 0 \quad \text{and} \quad \psi''_\varepsilon(r) = (|r| \vee \varepsilon)^{-p}, \quad r \in \mathbb{R}$$

for  $\varepsilon \in (0, \delta)$ . We infer from (1) and (2) that

$$\begin{aligned} & \frac{d}{dt} \int_{-1}^1 [\psi_\varepsilon(\partial_x u) - u] dx \\ &= \int_{-1}^1 [\psi'_\varepsilon(\partial_x u) \partial_x \partial_t u - \partial_t u] dx \\ &= [\psi'_\varepsilon(\partial_x u) \partial_t u]_{x=-1}^{x=1} - \int_{-1}^1 [\psi''_\varepsilon(\partial_x u) \partial_x^2 u + 1] \partial_t u dx \\ &= - \int_{-1}^1 \psi''_\varepsilon(\partial_x u) (\partial_x^2 u + (|\partial_x u| \vee \varepsilon)^p) \partial_t u dx \\ &= - \int_{-1}^1 \psi''_\varepsilon(\partial_x u) (\partial_t u + (|\partial_x u| \vee \varepsilon)^p - |\partial_x u|^p) \partial_t u dx \\ &= - \int_{-1}^1 \psi''_\varepsilon(\partial_x u) |\partial_t u|^2 dx - \int_{-1}^1 \left( 1 - \frac{|\partial_x u|^p}{\varepsilon^p} \right)_+ \partial_t u dx. \end{aligned}$$

On the one hand, since  $\varepsilon \in (0, \delta)$ , we have

$$|\partial_x u| \vee \varepsilon \leq (|\partial_x u|^2 + \delta^2)^{1/2},$$

so that

$$\int_{-1}^1 \psi_\varepsilon''(\partial_x u) |\partial_t u|^2 dx \geq \int_{-1}^1 \frac{|\partial_t u|^2}{(|\partial_x u|^2 + \delta^2)^{p/2}} dx.$$

On the other hand, introducing

$$\xi(r) := \begin{cases} r - \frac{|r|^p r}{(p+1)\varepsilon^p} & \text{if } |r| \leq \varepsilon, \\ \frac{p\varepsilon}{p+1} \frac{r}{|r|} & \text{if } |r| \geq \varepsilon, \end{cases}$$

we have  $\xi'(r) = (1 - |r|^p/\varepsilon^p)_+$  and  $|\xi(r)| \leq \varepsilon$ . Consequently, thanks to (1),

$$\begin{aligned} \left| \int_{-1}^1 \left(1 - \frac{|\partial_x u|^p}{\varepsilon^p}\right)_+ \partial_t u dx \right| &\leq \left| \int_{-1}^1 \left(1 - \frac{|\partial_x u|^p}{\varepsilon^p}\right)_+ \partial_x^2 u dx \right| \\ &\quad + \varepsilon^p \int_{-1}^1 \left(1 - \frac{|\partial_x u|^p}{\varepsilon^p}\right)_+ dx \\ &\leq \left| \int_{-1}^1 \partial_x \xi(\partial_x u) dx \right| + 2\varepsilon^p \\ &\leq |\xi(\partial_x u(t, 1))| + |\xi(\partial_x u(t, -1))| + 2\varepsilon^p \\ &\leq 4\varepsilon^p. \end{aligned}$$

Consequently, for each  $\varepsilon \in (0, \delta)$ , we have

$$\frac{d}{dt} \int_{-1}^1 [\psi_\varepsilon(\partial_x u) - u] dx + \int_{-1}^1 \frac{|\partial_t u|^2}{(|\partial_x u|^2 + \delta^2)^{p/2}} dx \leq 4\varepsilon^p. \quad (25)$$

It remains to pass to the limit in (25) as  $\varepsilon \rightarrow 0$ . For that purpose, we notice that

$$\left| \psi_\varepsilon'(r) - \frac{|r|^{-p} r}{1-p} \right| \leq \frac{p}{1-p} \varepsilon^{1-p}$$

for  $r \in \mathbb{R}$ , so that  $(\psi_\varepsilon)$  converges uniformly towards  $r \mapsto |r|^{2-p}/((2-p)(1-p))$  on compact subsets of  $\mathbb{R}$ . Recalling that  $\partial_x u(t)$  belongs to  $L^\infty(-1, 1)$  by Lemma 8, we may let  $\varepsilon \rightarrow 0$  in (25) and obtain (24).  $\square$

**Remark 10** *It turns out that, at least formally, the functional*

$$w \mapsto \int_{-1}^1 \left( \frac{|\partial_x w(x)|^{2-p}}{(2-p)(1-p)} - w(x) \right) dx$$

*is also a Liapunov functional for (1)-(3) when  $p \in (1, 2)$ , while*

$$w \mapsto \int_{-1}^1 (|\partial_x w(x)| \ln(|\partial_x w(x)|) - |\partial_x w(x)| - w(x)) dx$$

*is a Liapunov functional for (1)-(3) when  $p = 1$ . For  $p > 2$ , (1)-(3) still have Liapunov functionals but of a different kind [2].*

**Corollary 11** *We have*

$$\int_0^\infty \int_{-1}^1 |\partial_t u(t, x)|^2 dx dt < \infty. \quad (26)$$

**Proof.** Let  $T > 0$ . We integrate (24) with  $\delta = 1$  over  $(0, T)$  and use (18) and the non-negativity of  $u$  to obtain

$$\begin{aligned} \int_0^T \int_{-1}^1 \frac{|\partial_t u(t, x)|^2}{(1 + \Lambda^2)^{p/2}} dx dt &\leq \int_0^T \int_{-1}^1 \frac{|\partial_t u(t, x)|^2}{(|\partial_x u(t, x)|^2 + 1)^{p/2}} dx dt \\ &\leq \int_{-1}^1 \left( \frac{|\partial_x u(0, x)|^{2-p}}{(2-p)(1-p)} - u(0, x) \right) dx \\ &\quad - \int_{-1}^1 \left( \frac{|\partial_x u(T, x)|^{2-p}}{(2-p)(1-p)} - u(T, x) \right) dx \\ &\leq \frac{2 \|\partial_x u_0\|_\infty^{2-p}}{(2-p)(1-p)} + \int_{-1}^1 u(T, x) dx \\ &\leq \frac{2 \|\partial_x u_0\|_\infty^{2-p}}{(2-p)(1-p)} + 2 \Lambda, \end{aligned}$$

whence (26), for the right-hand side of the above inequality does not depend on  $T > 0$ .  $\square$

## 5 Convergence to steady states

**Proof of Theorem 2: non-negative initial data.** Let  $u_0 \in Y_+$ ,  $u_0 \not\equiv 0$ , and denote by  $u$  the corresponding classical solution to (1)-(3). We consider an increasing sequence  $(t_n)_{n \geq 1}$  of positive real numbers such that  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$  and define a sequence of functions  $(u_n)_{n \geq 1}$  by  $u_n(t, x) := u(t_n + t, x)$  for  $(t, x) \in [0, 1] \times [-1, 1]$  and  $n \geq 1$ . We next denote by  $g_n$  the solution to

$$\partial_t g_n - \partial_x^2 g_n = 0, \quad (t, x) \in (0, 1) \times (-1, 1), \quad (27)$$

$$g_n(t, \pm 1) = 0, \quad t \in (0, 1), \quad (28)$$

$$g_n(0) = u_n(0) = u(t_n), \quad x \in (-1, 1), \quad (29)$$

and put  $h_n = u_n - g_n$ . Then  $h_n$  is a solution to

$$\partial_t h_n - \partial_x^2 h_n = |\partial_x u_n|^p, \quad (t, x) \in (0, 1) \times (-1, 1), \quad (30)$$

$$h_n(t, \pm 1) = 0, \quad t \in (0, 1), \quad (31)$$

$$h_n(0) = 0, \quad x \in (-1, 1). \quad (32)$$

On the one hand, owing to Lemma 8, the sequence  $(|\partial_x u_n|^p)$  is bounded in  $L^q((0, 1) \times (-1, 1))$  for every  $q \in (1, \infty)$ . Since  $h_n$  is a solution to (30)-(32), we infer from [10, Theorem IV.9.1] that  $(h_n)$  is bounded in  $\{w \in L^q(0, 1; W^{2,q}(-1, 1)), \partial_t w \in L^q((0, 1) \times (-1, 1))\}$  for every  $q \in (1, \infty)$ . We may then use [10, Lemma II.3.3] with  $q = 4$  to deduce that there is  $\beta \in (0, 1)$  such that  $(h_n)$  and  $(\partial_x h_n)$  are bounded in  $\mathcal{C}^{\beta/2, \beta}([0, 1] \times [-1, 1])$ . This last property together with the Arzelà-Ascoli theorem entail that  $(h_n)$  and  $(\partial_x h_n)$  are relatively compact in  $\mathcal{C}([0, 1] \times [-1, 1])$ . On the other hand, it follows from Lemma 8 and classical regularity properties of the heat equation that  $(g_n)$  is relatively compact in  $\mathcal{C}([0, 1] \times [-1, 1])$ , while  $(\partial_x g_n)$  is relatively compact in  $\mathcal{C}([\tau, 1] \times [-1, 1])$  for each  $\tau \in (0, 1)$ . Consequently, there are a subsequence of  $(u_n)$  (not relabeled) and  $U \in \mathcal{C}([0, 1] \times [-1, 1])$  such that  $\partial_x U \in \mathcal{C}((0, 1) \times [-1, 1])$  and

$$u_n \longrightarrow U \quad \text{in } \mathcal{C}([0, 1] \times [-1, 1]), \quad (33)$$

$$\partial_x u_n \longrightarrow \partial_x U \quad \text{in } \mathcal{C}([\tau, 1] \times [-1, 1])$$

for every  $\tau \in (0, 1)$ .

Now, since  $(u_n)$  satisfies (1), (2), a straightforward consequence of (33) is that

$$\partial_t U - \partial_x^2 U = |\partial_x U|^p \quad \text{in } \mathcal{D}'((0, 1) \times (-1, 1)). \quad (34)$$

Furthermore, it follows from Corollary 11 that

$$\lim_{n \rightarrow \infty} \int_0^1 \int_{-1}^1 |\partial_t u_n|^2 dx dt = \lim_{n \rightarrow \infty} \int_{t_n}^{1+t_n} \int_{-1}^1 |\partial_t u|^2 dx dt = 0.$$

By a weak lower semicontinuity argument, we infer from (33) and the previous identity that  $\partial_t U = 0$ . Then  $U$  does not depend on time and thus belongs to  $\mathcal{C}^1([-1, 1])$ . Furthermore, recalling (34), we conclude that  $\partial_x^2 U + |\partial_x U|^p = 0$  in  $\mathcal{D}'(-1, 1)$ . The already established regularity of  $U$  implies that  $U \in \mathcal{C}^2([-1, 1])$  and solves (14), (15). Consequently, by Proposition 5, there exists  $\vartheta \in [0, 1]$  such that  $U = U_\vartheta$  and  $(u_n(0)) = (u(t_n))$  converges towards  $U_\vartheta$  in  $\mathcal{C}([-1, 1])$  as  $n \rightarrow \infty$  by (33). In particular, recalling that  $M(t)$  is defined by (7), we have

$$\mathcal{M}_0 (1 - \vartheta)^\alpha = \|U_\vartheta\|_\infty = \lim_{n \rightarrow \infty} \|u(t_n)\|_\infty = \lim_{n \rightarrow \infty} M(t_n) = M_\infty,$$

whence  $M_\infty \leq \mathcal{M}_0$  and

$$\vartheta = 1 - \left( \frac{M_\infty}{\mathcal{M}_0} \right)^{1/\alpha}. \quad (35)$$

Since this identity determines  $\vartheta$  in a unique way, we deduce that the set of cluster points of  $\{u(t); t \geq 0\}$  is reduced to a single point  $\{U_\vartheta\}$  with  $\vartheta$  given by (35). The set  $\{u(t); t \geq 0\}$  being relatively compact in  $\mathcal{C}([-1, 1])$  by Lemma 8 and the Arzelà-Ascoli theorem, we finally conclude that  $\|u(t) - U_\vartheta\|_\infty \rightarrow 0$  as  $t \rightarrow \infty$ , whence (10). In addition, since  $u_0 \not\equiv 0$ , Lemma 7 guarantees that  $\vartheta < 1$ , so that  $U_\vartheta$  is indeed a non-trivial steady state to (1)-(3). We have thus proved that,

$$\begin{aligned} &\text{if } u_0 \in Y_+, u_0 \not\equiv 0, \text{ then } M_\infty > 0 \text{ and there is } \vartheta \in [0, 1) \\ &\text{such that } \|u(t) - U_\vartheta\|_\infty \rightarrow 0 \text{ as } t \rightarrow \infty, \end{aligned} \quad (36)$$

and Theorem 2 holds true for non-negative initial data.  $\square$

## 6 Sign-changing solutions

We now show that the family  $(U_\vartheta)_{\vartheta \in [0, 1]}$  of non-negative steady states to (1)-(2) constructed in Proposition 5 also describes the large time behaviour of sign-changing solutions to (1)-(3). For that purpose, we first establish that any solution to (1)-(3) becomes non-negative after a finite time.



**Lemma 12** Consider  $u_0 \in Y$  and denote by  $u$  the corresponding classical solution to (1)-(3). Then there is  $T_\star > 0$  such that  $u(t, x) \geq 0$  for  $(t, x) \in [T_\star, \infty) \times [-1, 1]$ . Moreover, if  $u_0 \leq 0$ , then  $u(t, x) = 0$  for  $(t, x) \in [T_\star, \infty) \times [-1, 1]$ .

**Proof.** We put  $\tilde{u}_0(x) = 0 \wedge u_0(x)$  for  $x \in [-1, 1]$  and  $\tilde{u}_0(x) = 0$  for  $x \in \mathbb{R} \setminus [-1, 1]$ . Since  $\tilde{u}_0$  is a non-positive, bounded and continuous function in  $\mathbb{R}$ , we infer from [8, Theorem 3] that there is a unique classical solution  $\tilde{u} \in \mathcal{C}([0, \infty) \times \mathbb{R}) \cap \mathcal{C}^{1,2}((0, \infty) \times \mathbb{R})$  to the Cauchy problem

$$\partial_t \tilde{u} - \partial_x^2 \tilde{u} = a |\partial_x \tilde{u}|^p, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \quad (37)$$

$$\tilde{u}(0) = \tilde{u}_0, \quad x \in \mathbb{R}. \quad (38)$$

Furthermore,  $\tilde{u}$  is non-positive in  $(0, \infty) \times \mathbb{R}$  and is thus clearly a subsolution to (1)-(3) since  $\tilde{u}_0 \leq u_0$ . The comparison principle then entails that

$$\tilde{u}(t, x) \leq u(t, x) \quad \text{for } (t, x) \in [0, \infty) \times [-1, 1].$$

But, since  $\tilde{u}_0$  is a non-positive, bounded and continuous function with compact support in  $\mathbb{R}$ , it follows from [6, 7] that  $\tilde{u}$  enjoys the property of finite time extinction, that is, there is  $T_\star > 0$  such that

$$\tilde{u}(t, x) = 0 \quad \text{for } (t, x) \in [T_\star, \infty) \times \mathbb{R}.$$

Combining these two facts yield the first assertion of Lemma 12. Next, if  $u_0 \leq 0$ , we have also  $u \leq 0$  in  $[0, \infty) \times [-1, 1]$  by (6) and  $u$  thus identically vanishes in  $[T_\star, \infty) \times [-1, 1]$ .  $\square$

**Proof of Theorem 2: sign-changing initial data.** By Lemma 12, there is  $T_\star > 0$  such that  $u(T_\star, x) \geq 0$  for  $x \in [-1, 1]$ . Then either  $u(T_\star) \equiv 0$  and thus  $u(t) \equiv 0$  for  $t \geq T_\star$ , and  $u(t)$  converges towards  $U_1$  as  $t \rightarrow \infty$ . Or  $u(T_\star) \not\equiv 0$  and we infer from (36) that there is  $\vartheta \in [0, 1)$  such that  $u(t + T_\star)$  converges towards  $U_\vartheta$  as  $t \rightarrow \infty$ , which completes the proof of the first statement of Theorem 2.

Assume next that  $u_0$  fulfils (11). Putting  $\varphi_1(x) := \cos(\pi x/2)$  for  $x \in [-1, 1]$  and  $\lambda_1 := \pi^2/4$ , we recall that  $-d^2\varphi_1/dx^2 = \lambda_1\varphi_1$  in  $(-1, 1)$  with  $\varphi_1(\pm 1) = 0$ . We infer from (1), (11) and the non-negativity of  $\varphi_1$  and  $|\partial_x u|^p$  that

$$\int_{-1}^1 u(t, x) \varphi_1(x) dx \geq e^{-\lambda_1 t} \int_{-1}^1 u_0(x) \varphi_1(x) dx > 0$$

for  $t \geq 0$ . In particular, with the previous notations, we have  $u(T_\star) \geq 0$  with

$$\int_{-1}^1 u(T_\star, x) \varphi_1(x) dx > 0,$$

which, together with the positivity of  $\varphi_1$  on  $(-1, 1)$ , ensures that  $u(T_\star)$  is non-negative with  $u(T_\star) \not\equiv 0$ . Arguing as before, we infer from (36) that there is  $\vartheta \in [0, 1)$  such that  $u(t)$  converges towards  $U_\vartheta$  as  $t \rightarrow \infty$ , which completes the proof of the second statement of Theorem 2.  $\square$

## 7 Partial extinction of $\partial_x u$ in finite time

Before proceeding with the proof of Theorem 4, we recall that, if  $\sigma \in (0, \infty)$  and  $\mu \in \mathbb{R}$ , the function  $(t, x) \mapsto \mu + W_\sigma(x - \sigma t)$  is a travelling wave solution to  $\partial_t w - \partial_x^2 w = |\partial_x w|^p$  in  $(0, \infty) \times \mathbb{R}$  (see, e.g., [9, Chapter 13]), where

$$W_\sigma(\xi) := -\sigma^{-1/(1-p)} \int_0^\xi (1 - e^{-\sigma(1-p)\eta})_+^{1/(1-p)} d\eta, \quad \xi \in \mathbb{R}. \quad (39)$$

Introducing  $W_0(\xi) = -\mathcal{M}_0 \xi_+^\alpha$  for  $\xi \in \mathbb{R}$ , we claim that

$$0 \leq W_\sigma(\xi) - W_0(\xi) \leq \sigma \kappa_p \xi_+^{1+\alpha}, \quad \xi \in \mathbb{R}, \quad (40)$$

with  $\kappa_p := (1-p)^\alpha / (2(3-2p))$ . Indeed, introducing  $\zeta(r) := (r-1+e^{-r})/r^2$  and  $\zeta_1(r) := r\zeta(r)$  for  $r \geq 0$ , we have for  $\xi \geq 0$

$$W_\sigma(\xi) - W_0(\xi) = \int_0^\xi ((1-p)\eta)^{1/(1-p)} \left\{ 1 - (1 - \zeta_1(\sigma(1-p)\eta))^{1/(1-p)} \right\} d\eta.$$

We deduce from the elementary inequalities  $0 \leq \zeta_1(r) \leq 1$  for  $r \geq 0$  and

$$(1-r)^{1/(1-p)} \geq 1 - \frac{r}{1-p}, \quad r \in [0, 1],$$

that  $W_\sigma(\xi) - W_0(\xi) \geq 0$  and

$$W_\sigma(\xi) - W_0(\xi) \leq \int_0^\xi ((1-p)\eta)^{1/(1-p)} \frac{\zeta_1(\sigma(1-p)\eta)}{1-p} d\eta.$$

We next use the fact that  $\zeta(r) \leq 1/2$  for  $r \geq 0$  to complete the proof of (40).

**Proof of Theorem 4.** As already mentioned, the proof is similar to that of [7, Theorem 9], the main difference being due to the boundary conditions. We nevertheless reproduce the whole argument here for the sake of completeness. We first observe that (12) implies that  $u_0(x) \geq m_0 - \mathcal{M}_0 + U_0(x)$  for  $x \in [-1, 1]$  and that  $m_0 - \mathcal{M}_0 + U_0$  is a subsolution to (1) with  $m_0 - \mathcal{M}_0 + U_0(\pm 1) \leq 0$ . We then infer from the comparison principle and (6) that

$$m_0 - \mathcal{M}_0 + U_0(x) \leq u(t, x) \leq m_0 \quad \text{for } (t, x) \in [0, \infty) \times [-1, 1]. \quad (41)$$

In particular,

$$u(t, 0) = m_0 \quad \text{for } t \in [0, \infty). \quad (42)$$

We now consider  $\sigma \in (0, \varepsilon/\kappa_p)$  and put  $w_\sigma(t, x) = m_0 + W_\sigma(x - \sigma t)$  for  $(t, x) \in [0, \infty) \times \mathbb{R}$  (recall that  $\varepsilon$  and  $m_0$  are both defined in (12)). We readily have that

$$\partial_t w_\sigma - \partial_x^2 w_\sigma - |\partial_x w_\sigma|^p = 0 = \partial_t u - \partial_x^2 u - |\partial_x u|^p \quad \text{in } (0, \infty) \times (0, 1) \quad (43)$$

with

$$w_\sigma(t, 0) = m_0 = u(t, 0), \quad t \geq 0, \quad (44)$$

by (39) and (42). In addition, we infer from (12), (40) and the choice of  $\sigma$  that, for  $x \in [0, 1]$ ,

$$\begin{aligned} w_\sigma(0, x) &= m_0 + W_\sigma(x) = m_0 + W_0(x) + W_\sigma(x) - W_0(x) \\ &\leq m_0 - \mathcal{M}_0 x^\alpha + \sigma \kappa_p x^{1+\alpha} \leq m_0 - \mathcal{M}_0 x^\alpha + \varepsilon x^{1+\alpha} \\ &\leq u_0(x). \end{aligned} \quad (45)$$

Finally, if  $\delta \in (0, \delta_0)$  and  $t \in [0, \delta/\sigma]$ , it follows from (40) that

$$\begin{aligned} w_\sigma(t, 1) &= m_0 + W_\sigma(1 - \sigma t) \\ &= m_0 + W_0(1 - \sigma t) + W_\sigma(1 - \sigma t) - W_0(1 - \sigma t) \\ &\leq m_0 - \mathcal{M}_0 (1 - \sigma t)^\alpha + \sigma \kappa_p (1 - \sigma t)^{1+\alpha} \\ &\leq \mathcal{M}_0 ((1 - \delta_0)^\alpha - (1 - \delta)^\alpha) + \sigma \kappa_p \\ &\leq 0 \end{aligned} \quad (46)$$

as soon as  $\sigma$  is sufficiently small. Owing to (43), (44), (45) and (46), there is  $\sigma_\delta$  depending only on  $p$ ,  $m_0$ ,  $\varepsilon$  and  $\delta$  such that, if  $\sigma \in (0, \sigma_\delta)$ , we may apply the comparison principle on  $[0, \delta/\sigma] \times [0, 1]$  to deduce that

$$w_\sigma(t, x) \leq u(t, x), \quad (t, x) \in [0, \delta/\sigma] \times [0, 1]. \quad (47)$$

Recalling (41), we conclude from (47) that, if  $\sigma \in (0, \sigma_\delta)$ ,

$$u(t, x) = m_0 \quad \text{for } t \in [0, \delta/\sigma] \quad \text{and } x \in [0, \sigma t]. \quad (48)$$

A first consequence of (47) is that, if  $t > 0$ , we may find  $\sigma$  small enough such that  $\sigma \in (0, \sigma_\delta)$  and  $t \in [0, \delta/\sigma]$ . It then follows from (48) that  $u(t, x) = m_0$  for  $x \in [0, X(t)]$  with  $X(t) := \sigma t$ .

As a second consequence of (47), we note that, if  $t \geq T(\delta) := \delta/\sigma_\delta$ , there is  $\sigma \in (0, \sigma_\delta)$  such that  $t = \delta/\sigma$ . Then  $u(t, x) = m_0$  for  $x \in [0, \delta]$  by (48).

To complete the proof of Theorem 4, it suffices to notice that  $v : (t, x) \mapsto u(t, -x)$  also solves (1)-(2) with initial datum  $x \mapsto u_0(-x)$  which satisfies (12). Then,  $v$  also enjoys the above two properties from which we deduce that we have also  $u(t, x) = m_0$  for  $x \in [-X(t), 0]$  for every  $t > 0$  and  $u(t, x) = m_0$  for  $x \in [-\delta, 0]$  for  $t \geq T(\delta)$ , thus completing the proof of Theorem 4.  $\square$

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