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1 Introduction

Chemotaxis is understood as a class of biological processes, where the movement of mobile species is influenced by chemical substances. In 1970, KELLER and SEGEL [13] proposed a model describing aggregation of cellular slime molds towards a higher concentration of a chemical substance. With $u = u(t,x)$ denoting the species density, and $v = v(t,x)$ a rescaled chemical substance density, the relevant mathematical problem consists of a system of partial differential equations which reads, after a suitable normalization [6],

$$\partial_t u(t,x) = \mathrm{div}_x\Big(\nabla_x u(t,x) - u(t,x)\nabla_x v(t,x)\Big), \tag{1.1}$$

$$\partial_t v(t,x) = \alpha\Delta v(t,x) - \beta v(t,x) + \gamma\Big(u(t,x) - 1\Big) \text{ for } t > 0,\ x \in \Omega, \tag{1.2}$$

supplemented with the Neumann boundary conditions

$$\nabla_x u(t,x)\cdot\mathbf{n}(x) = \nabla_x v(t,x)\cdot\mathbf{n}(x) = 0,\ t > 0,\ x \in \partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, and α, β, γ are positive constants.

Without loss of generality, we suppose $|\Omega| = 1$, and fix

$$\int_\Omega u(t,x)\ \mathrm{d}x = 1,\ \int_\Omega v(t,x)\ \mathrm{d}x = 0 \text{ for all } t > 0. \tag{1.4}$$

Various mathematical aspects of this model have been studied by many authors (see the comprehensive survey by HORSTMANN [10, 11]). In particular, the question of existence of solutions in various functional classes was studied by NAGAI et al. [18] (see also GAJEWSKI and ZACHARIAS [6], BILER [1]).

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As a matter of fact, there is a common belief that many solutions of (1.1 - 1.3) exhibit blow-up in a finite time but there are only a few rigorous results available (see HERRERO et al. [9, 8], and the related results by HORSTMANN and WINKLER [12], NAGAI et al. [17]). In this paper, we focus on the opposite situation, namely the long-time behaviour of global-in-time solutions. The main result reads as follows:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $|\Omega| = 1$. Assume that (u, v) is a classical solution to (1.1 - 1.4) defined on the time interval $(0, \infty)$ such that*

$$u(t, x) > 0 \text{ for all } t \in (0, \infty),\ x \in \Omega,$$

and

$$\limsup_{t \to \infty} \Big(\| u(t) \|_{L^\infty(\Omega)} + \| v(t) \|_{L^\infty(\Omega)} \Big) < \infty. \tag{1.5}$$

Then

$$u(t) \to U,\ v(t) \to V \text{ in } C^1(\overline{\Omega}) \text{ as } t \to \infty, \tag{1.6}$$

where (U, V) is a solution to the stationary problem

$$\left\{ \begin{array}{c} \mathrm{div}_x \Big(\nabla_x U(x) - U(x) \nabla_x V(x) \Big) = 0, \\ \\ \alpha \Delta V(x) - \beta V(x) + \gamma \Big(U(x) - 1 \Big) = 0 \text{ for } x \in \Omega, \end{array} \right\} \tag{1.7}$$

$$\nabla_x U(x) \cdot \mathbf{n}(x) = \nabla_x V(x) \cdot \mathbf{n}(x) = 0 \text{ for } x \in \partial\Omega, \tag{1.8}$$

$$\int_\Omega U(x)\ \mathrm{d}x = 1,\ \int_\Omega V(x)\ \mathrm{d}x = 0. \tag{1.9}$$

From the mathematical viewpoint, system (1.1 - 1.3) represents a truly "parabolic" problem, for which smoothness of solutions follows as soon as uniform *a priori* bounds are available. By the same token, the first component u of the solution stays positive on its existence interval (see BILER [1], YAGI [21]). As far as the uniform estimates are concerned, we refer to NAGAI et al. [18] or GAJEWSKI and ZACHARIAS [5]. Various generalizations with respect to the physical space dimension N as well as the so-called chemotactic sensitivity function $f = f(u)$ (the "diffusion" coefficient for v in (1.1)) are possible (see HORSTMANN and WINKLER [12]). Here, we concentrated on the mathematically "critical" case $N = 2$, $f(u) = u$. Note that there is no full agreement on the question whether the biologically relevant space dimension is $N = 2$ or $N = 3$.

It is well-known that problem (1.1 - 1.3) admits a Lyapunov function E, more specifically, the identity

$$\begin{aligned} &E[u(t_2), v(t_2)] - E[u(t_1), v(t_1)] = \\ &-\frac{1}{\gamma} \int_{t_1}^{t_2} \int_\Omega |\partial_t v|^2\ \mathrm{d}x\ \mathrm{d}t - \int_{t_1}^{t_2} \int_\Omega u |\nabla_x \log(u) - \nabla_x v|^2\ \mathrm{d}x\ \mathrm{d}t, \end{aligned} \tag{1.10}$$

with

$$E[u, v] \equiv \int_\Omega \Big(u \log(u) + \frac{\alpha}{2\gamma} |\nabla_x v|^2 + \frac{\beta}{2\gamma} |v|^2 - uv \Big)\ \mathrm{d}x, \tag{1.11}$$

holds for any $0 < t_1 \leq t_2$ (see [18, Lemma 3.3]). Accordingly, it is relatively easy to observe that the $\omega-$limit set of each bounded trajectory is contained in the set of stationary solutions satisfying (1.7 - 1.9).

On the other hand, however, the topology of the set of stationary solutions can be non-trivial, in particular, there may be a continuum of stationary solutions even when Ω is a disc (see SENBA and SUZUKI [19], HARADA et al. [7]). In such a situation, the question of convergence to one particular solution becomes rather delicate. To the best of our knowledge, there is only one universal method available to attack this kind of problems, namely the deep results of ŁOJASIEWICZ [15, 16] adapted to the context of parabolic problems by SIMON [20]. These results, despite their rather general character, should be viewed as a way of approach rather than a universal tool, each particular application requiring often non-trivial modifications of the existing theory. The relevant version used in this paper is based on the so-called non-smooth variant proved in [2] modified in the spirit of GAJEWSKI and GRIEPENTROG [4].

The paper is organized as follows. In Section 2, we review some basic facts concerning the solutions of the stationary problem. In particular, we shall show that the first component of all solutions belonging to a fixed $\omega-$limit set are bounded below away from zero. In Section 3, we derive some suitable *a priori* estimates of "bootstrap type", in particular, we shall show that the uniform estimate (1.5) give rise to compactness of trajectories in more regular spaces. Furthermore, it is shown that the $\omega-$limit set of each particular trajectory consists of the stationary solutions having the same "energy" E. In Section 4, we show a version of the Simon-Łojasiewicz theorem adapted to the present problem. Finally, the proof of Theorem 1.1 is completed in a routine manner in Section 5.

2 Stationary solutions

In this section, we review some basic properties of the solutions (U, V) to the stationary problem (1.7 - 1.9). More specifically, we focus on the class of regular solutions (U, V) to (1.7), (1.8), we require that

$$U,\ V \in C^2(\overline{\Omega}),\ U \geq 0, \tag{2.1}$$

in addition to (1.9).

To begin with, observe that U is bounded from below by a positive constant, i.e. there is $\underline{U} > 0$ such that

$$U(x) \geq \underline{U} > 0 \text{ for } x \in \overline{\Omega}. \tag{2.2}$$

Indeed, owing to the homogeneous Neumann boundary conditions (1.8), the first equation of (1.7) also reads $\nabla_x U = U \nabla_x V$ in Ω, whence $\log(U) - V$ is constant on each connected component of the open set $\{y \in \Omega : U(y) > 0\}$. By (1.9) there exists $x_0 \in \Omega$ such that $U(x_0) > 0$ and the previous property implies that there is $M \in \mathbb{R}$ such that $\log(U(x)) = V(x) + M$ for each x belonging to the same connected component $\Omega(x_0)$ of $\{y \in \Omega : U(y) > 0\}$ as x_0. Since V is bounded, we conclude that $U(x) \geq e^{M - \|V\|_{L^\infty(\Omega)}} > 0$ for $x \in \Omega(x_0)$, from which we readily deduce that $\Omega(x_0) = \Omega$ and the claim (2.2).

Furthermore, it follows from the classical elliptic regularity theory combined with a straightforward bootstrap argument that the class of classical solutions to problem (1.7), (1.8) satisfying (1.9) and (2.1) is closed in $C(\overline{\Omega})$.

3 A priori estimates

Using the approach of [18] and [5] we derive uniform bounds on global bounded solutions that may be of independent interest.

Proposition 3.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded regular domain with $|\Omega| = 1$, and $I \subset (0,\infty)$ a bounded open interval. Assume that (u,v) is a classical solution to the Keller-Segel system (1.1 - 1.4) in $I \times \Omega$ such that*

$$u(t,x) > 0 \text{ for all } t \in I,\ x \in \Omega, \tag{3.1}$$

$$\int_\Omega \Big\{ u(t)\log(u(t)) + |\nabla_x v(t)|^2 + |v(t)|^2 \Big\}\ \mathrm{d}x \le \overline{E} \text{ for all } t \in I, \tag{3.2}$$

and

$$\int_I \int_\Omega |\partial_t v|^2\ \mathrm{d}x\ \mathrm{d}t \le \overline{E}. \tag{3.3}$$

Then, for any $d > 0$, there exists a constant $K = K(\overline{E}, |I|, d)$ such that

$$\|u(t)\|_{C^2(\overline{\Omega})} + \|v(t)\|_{C^2(\overline{\Omega})} + \|\partial_t u(t)\|_{C(\overline{\Omega})} + \|\partial_t v(t)\|_{C(\overline{\Omega})} \le K(\overline{E}, |I|, d) \tag{3.4}$$

whenever $t \in I$, $\mathrm{dist}[t, \partial I] > 1/d$.

Proof: The proof will be performed in several steps.

Step 1:
Multiplying (1.1) by $\log(u+1)$, integrating by parts, and making use of the boundary conditions (1.3), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)\log(u+1)\ \mathrm{d}x + 4\int_\Omega |\nabla_x \sqrt{u+1}|^2\ \mathrm{d}x = \int_\Omega \frac{u}{u+1}\nabla_x u \cdot \nabla_x v\ \mathrm{d}x. \tag{3.5}$$

On the other hand, one can take the product of (1.2) with $u - \log(u+1)$ in order to deduce

$$\alpha \int_\Omega \frac{u}{u+1}\nabla_x u \cdot \nabla_x v\ \mathrm{d}x = -\int_\Omega \Big(\partial_t v + \beta v\Big)\Big(u - \log(u+1)\Big)\ \mathrm{d}x + \tag{3.6}$$

$$\gamma \int_\Omega \Big(u-1\Big)\Big(u - \log(u+1)\Big)\ \mathrm{d}x.$$

Thus combining (3.5), (3.6) with hypothesis (3.2) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)\log(u+1)\ \mathrm{d}x + 4\int_\Omega |\nabla_x \sqrt{u+1}|^2\ \mathrm{d}x \le \tag{3.7}$$

$$c_1\Big(\int_\Omega |u|^2\ \mathrm{d}x + \int_\Omega |\partial_t v|^2\ \mathrm{d}x\Big) + K_1(\overline{E}) \text{ in } I.$$

Now, by virtue of the Gagliardo-Nirenberg interpolation inequality and (1.4), we have for $\Lambda > 1$

$$\int_\Omega u^2\ \mathrm{d}x \le \int_\Omega \Big([\sqrt{u+1} - \Lambda]_+ + \Lambda\Big)^4\ \mathrm{d}x \le \tag{3.8}$$

$$8\| \ [\sqrt{u+1}-\Lambda]_+ \ \|^4_{L^4(\Omega)} + 8\Lambda^4 \leq$$
$$c_2\| \ [\sqrt{u+1}-\Lambda]_+ \ \|^2_{L^2(\Omega)}\| \ [\sqrt{u+1}-\Lambda]_+\|^2_{W^{1,2}(\Omega)} + 8\Lambda^4 \leq$$
$$c_3\Big[1+\Big(\int_\Omega [\sqrt{u+1}-\Lambda]^2_+ \ \mathrm{d}x\Big)\|\nabla_x\sqrt{u+1}\|^2_{L^2(\Omega)}\Big] + 8\Lambda^4,$$

where, in accordance with hypothesis (3.2), one can take $\Lambda = \Lambda(\overline{E})$ large enough such that

$$c_1c_3\int_\Omega [\sqrt{u(t)+1}-\Lambda]^2_+ \ \mathrm{d}x \leq 3 \text{ for any } t \in I.$$

Consequently, relations (3.7) together with (3.8) give rise to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)\log(u+1)\ \mathrm{d}x + \int_\Omega |\nabla_x\sqrt{u+1}|^2 \ \mathrm{d}x \leq c_1\int_\Omega |\partial_t v|^2 \ \mathrm{d}x + K_1(\overline{E}) \text{ in } I. \quad (3.9)$$

Integrating (3.9) over I and using hypotheses (3.2), (3.3), we conclude

$$\int_I \|(\sqrt{u+1})(t)\|^2_{W^{1,2}(\Omega)} \ \mathrm{d}t \leq K_2(\overline{E})(1+|I|). \quad (3.10)$$

Recalling (3.8) we further obtain

$$\int_I \|u(t)\|^2_{L^2(\Omega)} \ \mathrm{d}t \leq K_3(\overline{E})(1+|I|), \quad (3.11)$$

and, in view of (1.2) and (3.3),

$$\int_I \|\Delta v(t)\|^2_{L^2(\Omega)} \ \mathrm{d}t \leq K_4(\overline{E})(1+|I|). \quad (3.12)$$

Step 2:

Lemma 3.1 *For any $\varepsilon > 0$, there is a constant $K(\varepsilon, \overline{E})$ such that*

$$\|w\|^3_{L^3(\Omega)} \leq \varepsilon\|\nabla_x w\|^2_{L^2(\Omega)} + K(\varepsilon, \overline{E})$$

whenever $w \in W^{1,2}(\Omega)$, $w \geq 0$, and

$$\int_\Omega w\log(w) \ \mathrm{d}x \leq \overline{E}.$$

Proof: Similarly to above, we can use the Gagliardo-Nirenberg interpolation inequality and a Poincaré inequality to obtain, for $\Lambda > 0$,

$$\begin{aligned}\|w\|^3_{L^3(\Omega)} &\leq c_4\Big(\| \ [w-\Lambda]_+ \ \|^3_{L^3(\Omega)} + \Lambda^3\Big) \leq c_5\Big(\| \ [w-\Lambda]_+\|_{L^1(\Omega)}\|w\|^2_{W^{1,2}(\Omega)} + \Lambda^3\Big) \\ &\leq c_6\Big(\| \ [w-\Lambda]_+\|_{L^1(\Omega)}\|\nabla_x w\|^2_{L^2(\Omega)} + \| \ [w-\Lambda]_+\|^3_{L^1(\Omega)} + \Lambda^3\Big),\end{aligned}$$

where $\Lambda = \Lambda(\varepsilon, \overline{E})$ can be fixed in such a way that

$$c_6\| \ [w-\Lambda]_+\|_{L^1(\Omega)} < \varepsilon$$

by the superlinearity of the function $r \mapsto r\log(r)$ as $r \to \infty$. □

We now decompose the solution v to (1.2) as $v = v_1 + v_2$ where

$$\partial_t v_1 - \alpha\Delta v_1 + \beta v_1 = 0,\ v_1(\inf\{I\}) = v(\inf\{I\}),$$

$$\partial_t v_2 - \alpha\Delta v_2 + \beta v_2 = \gamma(u-1),\ v_2(\inf\{I\}) = 0,$$

both equations being supplemented with homogeneous Neumann boundary conditions. On the one hand, the $L^p - L^q$ estimates for the heat equation and (3.2) yield

$$\|\Delta v_1(t)\|_{L^3(\Omega)} \le c_7(d)\|v(\inf\{I\})\|_{L^2(\Omega)} \le c_8(d,\overline{E})$$

provided $t \in I \cap [\inf\{I\} + 1/(2d), \infty)$, while

$$\int_{t_1}^{t_2} \|\Delta v_2\|_{L^3(\Omega)}^3 \,\mathrm{d}t \le c_9(|I|)\left(1 + \int_{t_1}^{t_2} \|u\|_{L^3(\Omega)}^3 \,\mathrm{d}t\right)$$

for $(t_1,t_2) \in I \times I$, $t_1 < t_2$. On the other hand, by virtue of (3.11), there is $\tau \in (\inf\{I\} + 1/(2d), \inf\{I\} + 1/d)$ such that

$$\|u(\tau)\|_{L^2(\Omega)} \le K_5(\overline{E}, |I|, d).$$

Now, multiplying (1.1) by u and using the Young inequality, we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 \,\mathrm{d}x + \int_\Omega |\nabla_x u|^2 \,\mathrm{d}x &= -\frac{1}{2}\int_\Omega u^2\Delta v \,\mathrm{d}x \\ &\le 3\Big(\|u\|_{L^3(\Omega)}^3 + \|\Delta v\|_{L^3(\Omega)}^3\Big). \end{aligned} \tag{3.13}$$

We next integrate (3.13) over (τ, t_2) for $t_2 \in I \cap [\inf\{I\} + 1/d, \infty)$ and use the previous bounds on $u(\tau)$, Δv_1 and Δv_2 to conclude that

$$\frac{1}{2}\|u(t_2)\|_{L^2(\Omega)}^2 + \int_\tau^{t_2} \|\nabla_x u\|_{L^2(\Omega)}^2 \,\mathrm{d}t \le K_6(\overline{E}, |I|, d)\left(1 + \int_\tau^{t_2} \|u(t)\|_{L^3(\Omega)}^3 \,\mathrm{d}t\right).$$

A direct application of Lemma 3.1 yields a constant $K_7(\overline{E}, |I|, d)$ such that

$$\|u(t_2)\|_{L^2(\Omega)}^2 + \int_\tau^{t_2} \|\nabla_x u\|^2 \,\mathrm{d}t \le K_7(\overline{E}, |I|, d)$$

for $t_2 \in I \cap [\inf\{I\} + 1/d, \infty)$. Since $\tau < \inf\{I\} + 1/d$, we have thus proved that

$$\|u(t)\|_{L^2(\Omega)} \le K_8(\overline{E}, |I|, d) \text{ for all } t \in \left(\inf\{I\} + \frac{1}{d}, \sup\{I\} - \frac{1}{d}\right), \tag{3.14}$$

$$\int_{\inf\{I\}+1/d}^{\sup\{I\}-1/d} \|u\|_{W^{1,2}(\Omega)}^2 \,\mathrm{d}t \le K_8(\overline{E}, |I|, d). \tag{3.15}$$

Step 3:
Now, one can use estimate (3.15) in (1.2) to obtain

$$\int_{\inf\{I\}+1/d}^{\sup\{I\}-1/d} \Big(\|\partial_t v\|_{W^{1,2}(\Omega)}^2 + \|v\|_{W^{3,2}(\Omega)}^2\Big) \,\mathrm{d}t \le K_9(\overline{E}, |I|, d), \tag{3.16}$$

which yields, together with a simple interpolation argument [14, Théorème 3.1], that $v \in C([\inf\{I\} + 1/d, \sup\{I\} - 1/d]; W^{2,2}(\Omega))$. Consequently, there is $\mu \in (0,1)$ such that

$$\|v(t)\|_{C^\mu(\overline{\Omega})} \leq K_{10}(\overline{E}, |I|, d) \text{ for all } t \in \left[\inf\{I\} + \frac{1}{d}, \sup\{I\} - \frac{1}{d}\right], \tag{3.17}$$

and, for each $r \in [2, \infty)$,

$$\|\nabla_x v(t)\|_{L^r(\Omega)} \leq K_{11}(\overline{E}, |I|, d, r) \text{ for all } t \in \left[\inf\{I\} + \frac{1}{d}, \sup\{I\} - \frac{1}{d}\right]. \tag{3.18}$$

Next (1.1) also reads

$$\partial_t u - \Delta u = -u\Delta v - \nabla_x u \cdot \nabla_x v = g,$$

where, by virtue of (3.14 - 3.16) and (3.18),

$$\|g\|_{L^2(\inf\{I\}+1/d, \sup\{I\}-1/d; L^q(\Omega))} \leq K_{12}(\overline{E}, |I|, d, q) \text{ for any } q \in (1,2).$$

We then deduce from classical parabolic regularity that

$$\int_{\inf\{I\}+1/d}^{\sup\{I\}-1/d} \left\{ \|\partial_t u\|^2_{L^q(\Omega)} + \|u\|^2_{W^{2,q}(\Omega)} \right\} \mathrm{d}t \leq K_{13}(\overline{E}, |I|, d, q) \text{ for all } q \in (1,2). \tag{3.19}$$

Finally, a bootstrap argument yields

$$\|u(t)\|_{C^\mu(\overline{\Omega})} \leq K_{14}(\overline{E}, |I|, d) \text{ for any } \inf\{I\} + \frac{1}{d} \leq t \leq \sup\{I\} - \frac{1}{d}, \tag{3.20}$$

which, together with the standard regularity results for the heat equation, completes the proof.

□

Corollary 3.1 *Under the hypotheses of Theorem 1.1, we have*

$$\sup_{t>1} \left\{ \|u(t)\|_{C^2(\overline{\Omega})} + \|v(t)\|_{C^2(\overline{\Omega})} + \|\partial_t u(t)\|_{C(\overline{\Omega})} + \|\partial_t v(t)\|_{C(\overline{\Omega})} \right\} < \infty. \tag{3.21}$$

Finally, if (u, v) is a global classical solution to (1.1-1.4), we introduce the $\omega-$limit set

$$\omega[u,v] = \{(U,V) \mid \text{there exists } t_n \to \infty \text{ such that } u(t_n) \to U,\ v(t_n) \to V \text{ in } C^1(\overline{\Omega})\}. \tag{3.22}$$

Thanks to the identity (1.10) and Corollary 3.1, a classical dynamical systems argument (in the spirit of the proof of the LaSalle invariance principle) allows us to conclude that

$$\omega[u,v] \subset \{(U,V) \mid (U,V) \text{ solve (1.7), (1.8) and satisfy (1.9) }\}, \tag{3.23}$$

and there is $E_\infty \in \mathbb{R}$ such that

$$E[U,V] = E_\infty \equiv \inf_{t>0} E[u(t), v(t)] = \lim_{t\to\infty} E[u(t), v(t)], \tag{3.24}$$

the functional E being defined in (1.11). Moreover, by the analysis of Section 2 and Corollary 3.1, there is a constant $\underline{U}$ such that

$$\inf_{x\in\Omega} U(x) \geq \underline{U} > 0 \text{ whenever } (U,V) \in \omega[u,v]. \tag{3.25}$$

Owing to (3.22), (3.23) and (3.25), we may then assume without loss of generality that

$$\inf_{x\in\Omega} u(t,x) \geq \underline{U} > 0 \text{ for all } t \geq 0 \tag{3.26}$$

for a possibly smaller positive real number $\underline{U}$.

4 A version of the Łojasiewicz-Simon theorem

Consider the functional

$$F(\tilde{u},v) = \int_\Omega \Big[\frac{1}{2\gamma}\Big(\alpha |\nabla_x v|^2 + \beta v^2 \Big) + G(\tilde{u}) - \tilde{u} v \Big] \ \mathrm{d}x$$

defined on the space

$$X = \Big\{ (\tilde{u},v) \ \Big|\ v \in W^{1,2}(\Omega),\ \tilde{u} \in L^2(\Omega),\ \ \int_\Omega \tilde{u}\ \mathrm{d}x = 0 \Big\}.$$

We shall assume that

$$G \in C^2(\mathbb{R}),\ G(0) = 0; \tag{4.1}$$

$$0 < \underline{G} \leq G''(r) \leq \overline{G} \text{ for all } r \in \mathbb{R}; \tag{4.2}$$

$$G \text{ is real analytic on an open interval } (A,B). \tag{4.3}$$

It is easy to check that under these hypotheses F is continuously differentiable on X and

$$\partial F(\tilde{u},v)[g,h] = \int_\Omega \Big[\frac{1}{\gamma}\Big(\alpha \nabla_x v \cdot \nabla_x h + \beta v h \Big) - \tilde{u} h + G'(\tilde{u}) g - v g \Big] \ \mathrm{d}x \tag{4.4}$$

for all $[g,h] \in X$.

The proof of Theorem 1.1 is based on the following version of the celebrated Łojasiewicz-Simon result.

Proposition 4.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded regular domain with $|\Omega| = 1$. Assume that G complies with hypotheses (4.1 - 4.3). Furthermore, let $(\tilde{U},V) \in X$ be such that*

$$A < \operatorname*{ess\,inf}_\Omega \tilde{U} \leq \operatorname*{ess\,sup}_\Omega \tilde{U} < B. \tag{4.5}$$

Then there exist real numbers $\theta \in (0,1/2)$, $m > 0$, and $\varepsilon > 0$ (depending, in general, on $\tilde{U}$, V) such that

$$|F(\tilde{u},v) - F(\tilde{U},V)|^{1-\theta} \leq m \| \partial F(\tilde{u},v) \|_{X^*} \text{ whenever } \| (\tilde{u},v) - (\tilde{U},V) \|_X < \varepsilon. \tag{4.6}$$

Remark 4.1 *It is worth noting there are two principal differences between Proposition 4.1 and the standard applications of the Łojasiewicz-Simon theory available in the literature. First of all, as is well-known, the functional F is not twice continuously differentiable on X unless G is quadratic since the mapping $\tilde{u} \mapsto G''(\tilde{u})$ is not continuous on $L^2(\Omega)$ with values in $L^\infty(\Omega)$. Moreover, the gradient ∂F is of type "monotone operator + linear compact perturbation" while the usual form is "linear isomorphism + (non-linear) compact perturbation". Secondly, the differential operator associated to the gradient ∂F in the sense of distribution does not map the underlying space X into itself, more specifically, it does not conserve the (zero) mean of the first component. The former problem was solved via a non-smooth version of the Łojasiewicz-Simon theorem obtained in [2] while the latter obstacle was successfully attacked by GAJEWSKI and GRIEPENTROG [4]. The proof of Proposition 4.1 given below can be seen as a combination of both approaches.*

Remark 4.2 *As a matter of fact, we shall use Proposition 4.1 with $G(\tilde{u}) \approx (\tilde{u}+1)\log(\tilde{u}+1)$. In general, the function G can be viewed as a primitive of the chemotactic sensitivity function f considered in [12]. Accordingly, Theorem 1.1 holds in this case as well and clarifies the situation described in [12, Corollary 4.1].*

Proof:

Similarly to Proposition 3.1, the proof will be carried out by means of several steps.

Step 1:

Apparently, as the functional F is continuously differentiable on X, the result is trivial if $\partial F(\tilde{U}, V) \neq 0$. It is thus to consider the case where $(\tilde{U}, V)$ is a critical point of F.

Clearly, if $(\tilde{U}, V)$ is a critical point of F on X, we have $\partial F(\tilde{U}, V) = 0$ and the function V is a weak solution to the equation

$$-\alpha \Delta V + \beta V - \gamma \tilde{U} = 0 \text{ in } \mathcal{D}'(\Omega) \tag{4.7}$$

supplemented with homogeneous Neumann boundary conditions. By virtue of hypothesis (4.5) and standard elliptic regularity results, we have

$$V \in W^{2,p}(\Omega) \text{ for any } 1 \leq p < \infty, \ \nabla_x V \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

In addition, integration of (4.7) over Ω yields

$$\int_\Omega V(x) \ \mathrm{d}x = \frac{\gamma}{\beta} \int_\Omega \tilde{U}(x) \ \mathrm{d}x = 0.$$

On the other hand, by the same token,

$$G'(\tilde{U}) - V = \int_\Omega \Big(G'(\tilde{U}) - V \Big) \ \mathrm{d}x = \int_\Omega G'(\tilde{U}) \ \mathrm{d}x,$$

since $|\Omega| = 1$. Thus, in particular, $\tilde{U}$ is Hölder continuous together with its first derivatives, and, consequently, V is a classical solution of (4.7). As a matter of fact, as G is analytic on the range of $\tilde{U}$, both $\tilde{U}$ and V are smooth provided $\partial\Omega$ is regular.

Step 2:

Let $\{e_i\}_{i=0}^{\infty}$ be the orthonormal system of eigenfunctions of the operator

$$-\alpha\Delta e_i + \beta e_i = \lambda_i e_i,\ \nabla_x e_i \cdot \mathbf{n}|_{\partial\Omega} = 0,\ i = 0, 1, ...,$$

in $L^2(\Omega)$ with

$$e_0 \equiv 1,\ \lambda_0 = \beta.$$

Furthermore, for $N \geq 0$, let $P_N : L^2(\Omega) \to \text{span}\{e_0, ..., e_N\}$ be the corresponding system of orthogonal projections.

Now we claim that there are positive numbers $N > 0$ and $\Lambda > 0$ such that $\partial\Big[F(\tilde{u}, v) + \Lambda \int_\Omega \frac{1}{2}|P_n v|^2\ \mathrm{d}x\Big]$ is maximal monotone: more specifically, there are $0 < \underline{F} < \overline{F} < \infty$ such that

$$\underline{F}\|(\tilde{u}_1, v_1) - (\tilde{u}_2, v_2)\|_X^2 \leq$$

$$< \partial F(\tilde{u}_1, v_1) - \partial F(\tilde{u}_2, v_2), (\tilde{u}_1, v_1) - (\tilde{u}_2, v_2) >_{[X^*, X]} + \Lambda\|P_N(v_1 - v_2)\|_{L^2(\Omega)}^2 \leq \quad (4.8)$$

$$\overline{F}\|(\tilde{u}_1, v_1) - (\tilde{u}_2, v_2)\|_X^2 \text{ for any } (\tilde{u}_i, v_i) \in X,\ i = 1, 2.$$

To begin with, it is easy to see that the most right inequality in (4.8) follows directly from (4.2). By the same token,

$$< \partial F(\tilde{u}_1, v_1) - \partial F(\tilde{u}_2, v_2), (\tilde{u}_1, v_1) - (\tilde{u}_2, v_2) >= \quad (4.9)$$

$$\frac{\alpha}{\gamma}\|\nabla_x(v_1 - v_2)\|_{L^2(\Omega)}^2 + \frac{\beta}{\gamma}\|v_1 - v_2\|_{L^2(\Omega)}^2 + \underline{G}\|\tilde{u}_1 - \tilde{u}_2\|_{L^2(\Omega)}^2 - 2\int_\Omega (\tilde{u}_1 - \tilde{u}_2)(v_1 - v_2)\ \mathrm{d}x \geq$$

$$\frac{\alpha}{\gamma}\|\nabla_x(v_1 - v_2)\|_{L^2(\Omega)}^2 + \frac{\beta}{\gamma}\|v_1 - v_2\|_{L^2(\Omega)}^2 + \frac{\underline{G}}{2}\|\tilde{u}_1 - \tilde{u}_2\|_{L^2(\Omega)}^2 - \frac{2}{\underline{G}}\|v_1 - v_2\|_{L^2(\Omega)}^2.$$

On the other hand,

$$\|v_1 - v_2\|_{L^2(\Omega)}^2 \leq \|P_N(v_1 - v_2)\|_{L^2(\Omega)}^2 + \frac{1}{\underline{\lambda}_N}\|v_1 - v_2\|_{W^{1,2}(\Omega)}^2, \quad (4.10)$$

where

$$\underline{\lambda}_N = \min\{\lambda_i \mid i \geq N\} \to \infty \text{ as } N \to \infty.$$

Thus (4.9), (4.10) give rise to (4.8) provided $\Lambda > 2/\underline{G}$ and $N = N(\alpha, \beta, \gamma)$ is chosen large enough.

Consequently, we are allowed to conclude that the mapping

$$\partial F + \Lambda P_N^* : X \to X^*$$

is a Lipschitz homeomorphism together with its inverse $(\partial F + \Lambda P_N^*)^{-1}$. Here we have use the symbol $P_N^* : X \to X^*$ to denote the adjoint projection

$$< P_N^*(\tilde{u}, v), [g, h] >= \int_\Omega P_N v\ h\ \mathrm{d}x = \int_\Omega P_N v\ P_N h\ \mathrm{d}x$$

Step 3:
We set, formally,

$$\partial[\partial F + \Lambda P_N^*] : X \to \mathcal{L}[X, X^*], \quad (4.11)$$

$$< \partial[\partial F + \Lambda P_N^*](\tilde{u}, v)[g, h], (w, z) >=$$

$$\int_\Omega \Big[G''(\tilde{u})gw - hw\Big]\ \mathrm{d}x + \int_\Omega \Big[\frac{1}{\gamma}\Big(\alpha\nabla_x h\cdot\nabla_x z + \beta hz\Big) - gz + \Lambda P_N h\ z\Big]\ \mathrm{d}x.$$

Similarly to **Step 2**, making use of hypothesis (4.2), one can show that

$$\partial[\partial F + \Lambda P_N^*](\tilde{u}, v) \in \mathcal{L}[X, X^*] \tag{4.12}$$

is a linear isomorphism of X onto X^* for any fixed $(\tilde{u}, v) \in X$ and its norm is bounded from above by $\overline{F}$.

Consider the spaces

$$Y = C_{zm}(\overline{\Omega}) \times W_n^{2,2}(\Omega),\ Z = C_{zm}(\overline{\Omega}) \times L^2(\Omega),$$

where

$$W_n^{2,2}(\Omega) = \{v \in W^{2,2}(\Omega)\ |\ \nabla_x v\cdot\mathbf{n}|_{\partial\Omega} = 0\},\ C_{zm}(\overline{\Omega}) = \{\tilde{u} \in C(\overline{\Omega})\ |\ \int_\Omega \tilde{u}\ \mathrm{d}x = 0\},$$

together with a mapping $H : Y \to Z$, $H = \partial F|_Y$,

$$H(\tilde{u}, v) = \left(G'(\tilde{u}) - v - \int_\Omega (G'(\tilde{u}) - v)\ \mathrm{d}x, -\frac{\alpha}{\gamma}\Delta v + \frac{\beta}{\gamma}v - \tilde{u}\right)$$

Clearly, $Y \subset X$ while Z can be continuously embedded into X^* thanks to the standard relation

$Z \subset L^2(\Omega) \times L^2(\Omega) \approx L^2(\Omega)^* \times L^2(\Omega)^*$ via the Riesz representation theorem.

By virtue of our assumptions, there exists a neighbourhood $\mathcal{U}(\tilde{U}, V)$ of $(\tilde{U}, V)$ (defined in **Step 1**) in the space Y such that

$$H : \mathcal{U}(\tilde{U}, V) \to Z$$

is an analytic mapping. Consequently

$$[\partial F + \Lambda P_N^*]\Big|_Y = [H + \Lambda P_N^*] :\ (\tilde{u}, v) \in Y \mapsto H(\tilde{u}, v) + [0, \Lambda P_N v] \in Z$$

is also analytic in $\mathcal{U}(\tilde{U}, V)$.

Recalling (4.11), $\partial[H + \Lambda P_N^*] : Y \to \mathcal{L}(Y, Z)$ is given by

$$\partial[H + \Lambda P_N^*](\tilde{u}, v)[g, h] =$$

$$\Big(G''(\tilde{u})g - h - \int_\Omega (G''(\tilde{u})g - h)\ \mathrm{d}x, -\frac{\alpha}{\gamma}\Delta h + \frac{\beta}{\gamma}h + \Lambda P_N h - g\Big)$$

Now, it is easy to see that $\partial[H + \Lambda P_N^*]$ is an isomorphism of Y onto Z for any fixed $(\tilde{u}, v) \in Y$ (or even $(\tilde{u}, v) \in X$). Indeed, since $\partial[\partial F + \Lambda P_N^*](\tilde{u}, v)$ is an isomorphism of X onto X^*, given $(z_1, z_2) \in Z \subset X^*$, there is $(g, h) \in X$ such that $\partial[\partial F + \Lambda P_N^*](\tilde{u}, v)[g, h] = (z_1, z_2)$ and

$$\|(g, h)\|_X \le c_{10}\|(z_1, z_2)\|_{X^*} \le c_{11}\|(z_1, z_2)\|_Z, \tag{4.13}$$

the constants c_{10} and c_{11} depending neither on (g, h) nor on (z_1, z_2). In fact, (g, h) solves

$$G''(\tilde{u})g - h - \int_\Omega \Big(G''(\tilde{u})g - h\Big)\ \mathrm{d}x = z_1, \tag{4.14}$$

$$-\frac{\alpha}{\gamma}\Delta h + \frac{\beta}{\gamma}h + \Lambda P_N h = g + z_2,\ \nabla_x h \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{4.15}$$

and the solution to (4.14) is given by

$$g = \frac{1}{G''(\tilde{u})}\left[z_1 + h - \int_\Omega h\ \mathrm{d}x - \Big(\int_\Omega \frac{1}{G''(\tilde{u})}\ \mathrm{d}x\Big)^{-1}\int_\Omega \frac{1}{G''(\tilde{u})}\Big(z_1 + h - \int_\Omega h\ \mathrm{d}x\Big)\ \mathrm{d}x\right] \tag{4.16}$$

Clearly, thanks to (4.2) and (4.13),

$$\|g\|_{L^2(\Omega)} \le c_{12}\ \big(\|z_1\|_{L^2(\Omega)} + \|h\|_{L^2(\Omega)}\big) \le c_{13}\|(z_1, z_2)\|_Z,$$

and classical elliptic regularity results and (4.15) entail that

$$\|h\|_{W^{2,2}(\Omega)} \le c_{14}\ \big(\|g\|_{L^2(\Omega)} + \|z_2\|_{L^2(\Omega)}\big),$$

whence $\|h\|_{W^{2,2}(\Omega)} \le c_{15}\|(z_1, z_2)\|_Z$. Owing to the continuous embedding of $W^{2,2}(\Omega)$ in $C(\overline{\Omega})$, we use once more (4.2) and (4.16) to obtain that

$$\|g\|_{C(\overline{\Omega})} \le c_{16}\ \big(\|z_1\|_{C(\overline{\Omega})} + \|h\|_{C(\overline{\Omega})}\big) \le c_{17}\|(z_1, z_2)\|_Z,$$

and we end up with

$$\|(g, h)\|_Y \le c_{18}\|(g, h)\|_Z.$$

In particular, there exists an open neighbourhood $\mathcal{V}(0, \Lambda P_N V)$ of $(0, \Lambda P_N V)$ in Z (recall that we have assumed that $H(\tilde{U}, V) = \partial F(\tilde{U}, V) = 0$) such that the mapping

$$[H + \Lambda P_N^*] : \mathcal{U}(\tilde{U}, V) \to \mathcal{V}(0, \Lambda P_N V)$$

is an analytic homeomorphism of $\mathcal{U}(\tilde{U}, V) \subset Y$ onto $\mathcal{V}(0, \Lambda P_N V) \subset Z$.

Now, setting

$$E_N = \{0\} \times \mathrm{span}\{e_0, ..., e_N\} \subset Z,$$

one can see that the functional

$$F([H + \Lambda P_N^*]^{-1}) : \mathcal{V}(0, \Lambda P_N V) \cap E_N \to \mathbb{R}$$

is analytic on $\mathcal{V}(0, \Lambda P_N V) \cap E_N$. Thus the classical Łojasiewicz theorem [15, 16] ensures the existence of $\theta \in (0, 1/2)$, $m > 0$ and $\varepsilon > 0$ such that

$$\Big|F\Big([H + \Lambda P_N^*]^{-1}(0, \Lambda P_N v)\Big) - F\Big([H + \Lambda P_N^*]^{-1}(0, \Lambda P_N V)\Big)\Big|^{1-\theta} \le \tag{4.17}$$

$$m\Big\|\partial F\Big([H + \Lambda P_N^*]^{-1}(0, \Lambda P_N v)\Big)\partial\Big([H + \Lambda P_N^*]^{-1}\Big)(0, \Lambda P_N v)\Big\|^{1-\theta}_{\mathcal{L}(E_N, \mathbb{R})}$$

$$\text{for any } \|v - V\|_{W^{1,2}(\Omega)} < \varepsilon.$$

Step 4:

Now, we can write

$$\Big\|\partial F\Big([H + \Lambda P_N^*]^{-1}(0, \Lambda P_N v)\Big)\partial\Big([H + \Lambda P_N^*]^{-1}\Big)(0, \Lambda P_N v)\Big\|_{\mathcal{L}(E_N, \mathbb{R})} \le$$

$$\Big\|\partial F\Big([H + \Lambda P_N^*]^{-1}(0, \Lambda P_N v)\Big)\Big\|_{X^*}\Big\|\partial\Big([H + \Lambda P_N^*]^{-1}\Big)(0, \Lambda P_N v)\Big\|_{\mathcal{L}(E_N, X)} \le$$

$$c_{12}\Big\|\partial F\Big([H+\Lambda P_N^*]^{-1}(0,\Lambda P_N v)\Big)\Big\|_{X^*}$$

as $\partial[H+\Lambda P_N](\tilde{u},v)$ is a (linear) homeomorphism of X onto X^* for any $(\tilde{u},v)\in X$.

Furthermore,

$$\Big\|\partial F\Big([H+\Lambda P_N^*]^{-1}(0,\Lambda P_N v)\Big)\Big\|_{X^*} =$$

$$\Big\|\partial F\Big([H+\Lambda P_N^*]^{-1}(0,\Lambda P_N v)\Big) - \partial F(\tilde{u},v) + \partial F(\tilde{u},v)\Big\|_{X^*} \leq$$

$$\|\partial F(\tilde{u},v)\|_{X^*} + \Big\|\partial F\Big([H+\Lambda P_N^*]^{-1}(0,\Lambda P_N v)\Big) - \partial F(\tilde{u},v)\Big\|_{X^*},$$

where

$$\Big\|\partial F\Big([H+\Lambda P_N^*]^{-1}(0,\Lambda P_N v)\Big) - \partial F(\tilde{u},v)\Big\|_{X^*} =$$

$$\Big\|\partial F\Big([H+\Lambda P_N^*]^{-1}(0,\Lambda P_N v)\Big) - \partial F\Big([H+\Lambda P_N^*]^{-1}(H(\tilde{u},v)+\Lambda P_N v)\Big)\Big\|_{X^*}.$$

By virtue of (4.12), the mappings ∂F and $[H+\Lambda P_N^*]^{-1} = [\partial F + \Lambda P_N^*]^{-1}$ are globally Lipschitz on $\mathcal{L}(X,X^*)$, $\mathcal{L}(X^*,X)$, respectively; whence

$$\Big\|\partial F\Big([H+\Lambda P_N^*]^{-1}(0,\Lambda P_N v)\Big) - \partial F\Big([H+\Lambda P_N^*]^{-1}(H(\tilde{u},v)+\Lambda P_N v)\Big)\Big\|_{X^*} \leq \tag{4.18}$$

$$c_{13}\|\partial F(\tilde{u},v)\|_{X^*}.$$

Consequently, relation (4.17) reads

$$\Big|F\Big([H+\Lambda P_N^*]^{-1}(0,\Lambda P_N v)\Big) - F\Big([H+\Lambda P_N^*]^{-1}(0,\Lambda P_N V)\Big)\Big|^{1-\theta} \leq \tag{4.19}$$

$$m\|\partial F(\tilde{u},v)\|_{X^*} \text{ provided } \|v-V\|_{W^{1,2}(\Omega)} < \varepsilon.$$

Step 5:

Since $\partial F(\tilde{U},V)=0$, we have $[H+\Lambda P_N^*](\tilde{U},V) = (0,\Lambda P_N V)$ and we can write

$$F\Big([H+\Lambda P_N^*]^{-1}(0,\Lambda P_N v)\Big) - F\Big([H+\Lambda P_N^*]^{-1}(0,\Lambda P_N V)\Big) =$$

$$F\Big([H+\Lambda P_N]^{-1}(0,\Lambda P_N v)\Big) - F(\tilde{U},V).$$

Next, for $(\tilde{u},v)\in X$, we can use (4.12) to conclude

$$\Big|F\Big([\partial F+\Lambda P_N^*]^{-1}(0,\Lambda P_N v)\Big) - F\Big(\tilde{u},v\Big)\Big| \leq$$

$$\|\partial F(\tilde{u},v)\|_{X^*}\Big\|[\partial F+\Lambda P_N^*]^{-1}(0,\Lambda P_N v) - [\partial F+\Lambda P_N^*]^{-1}(\partial F(\tilde{u},v)+\Lambda P_N v)\Big\|_X +$$

$$c_{14}\Big\|[\partial F+\Lambda P_N]^{-1}(0,\Lambda P_N v) - [\partial F+\Lambda P_N]^{-1}(\partial F(\tilde{u},v)+\Lambda P_N v)\Big\|_X^2 \leq$$

$$c_{15}\|\partial F(\tilde{u},v)\|_{X^*}^2.$$

Thus we have shown, taking (4.19) into account, that

$$|F(\tilde{u},v)-F(\tilde{U},V)| \leq c_{16}\Big(\|\partial F(\tilde{u},v)\|_{X^*}^{1/(1-\theta)} + \|\partial F(\tilde{u},v)\|_{X^*}^2\Big) \text{ for } \|v-V\|_{W^{1,2}(\Omega)} < \varepsilon,$$

which yields the desired conclusion provided $\|\tilde{u}-\tilde{U}\|_{L^2(\Omega)}$ is small enough because of the Lipschitz continuity of F.

□

5 Proof of Theorem 1.1

We are now in a position to complete the proof of Theorem 1.1. We first recall that, for any stationary solution (U, V) belonging to the $\omega-$limit set $\omega[u, v]$, we have

$$E[U, V] = E_\infty = \lim_{t\to\infty} E[u(t), v(t)]$$

by (3.24), the functional E being defined by (1.11). Accordingly, letting $t_2 \to \infty$ in (1.10) gives rise to

$$\int_t^\infty \int_\Omega \Big(\frac{1}{\gamma}|\partial_t v|^2 + u|\nabla_x \log(u) - \nabla_x v|^2\Big)\, \mathrm{d}x\, \mathrm{d}s = E[u(t), v(t)] - E_\infty. \tag{5.1}$$

To comply with the notations of Proposition 4.1, we set

$$\tilde{u}(t, x) = u(t, x) - 1,\ \tilde{U}(x) = U(x) - 1,\ G(\tilde{u}) = (\tilde{u} + 1)\log(\tilde{u} + 1).$$

A direct application of Proposition 4.1 yields that, for every $(U, V) \in \omega[u, v]$, we have

$$E[u(t), v(t)] - E_\infty = F(\tilde{u}(t), v(t)) - F(\tilde{U}, V) \le m\|\partial F(\tilde{u}(t), v(t))\|_{X^*}^{1/(1-\theta)} \tag{5.2}$$

for some $m > 0$ and $\theta \in (0, 1/2)$ whenever t is such that

$$\|u(t) - U\|_{L^2(\Omega)} + \|v(t) - V\|_{W^{1,2}(\Omega)} < \varepsilon. \tag{5.3}$$

Note that, in accordance with (3.25) and (3.26), one can modify G outside the range of the trajectory $\overline{\{u(t)\}}_{t\ge 0}$, so that all hypotheses of Proposition 4.1 are satisfied. Note also that, in general, the quantities m, θ, and ε may depend on U and V.

Using equation (1.2) we get

$$\|\partial F(\tilde{u}, v)\|_{X^*} = \tag{5.4}$$

$$\frac{1}{\gamma}\|\partial_t v\|_{[W^{1,2}(\Omega)]^*} + \sup\Big\{\int_\Omega \Big(\log(u) + 1 - v\Big) g\, \mathrm{d}x \ \Big|\ \|g\|_{L^2(\Omega)} = 1,\ \int_\Omega g\, \mathrm{d}x = 0\Big\},$$

where

$$\sup\Big\{\int_\Omega \Big(\log(u) + 1 - v\Big) g\, \mathrm{d}x \ \Big|\ \|g\|_{L^2(\Omega)} = 1,\ \int_\Omega g\, \mathrm{d}x = 0\Big\} = \tag{5.5}$$

$$\Big\|\Big(\log(u) + 1 - v\Big) - \int_\Omega \Big(\log(u) + 1 - v\Big)\, \mathrm{d}x\Big\|_{L^2(\Omega)}.$$

Consequently, relations (5.1 - 5.5), together with the Poincaré inequality, give rise to

$$\int_t^\infty \int_\Omega \Big(\frac{1}{\gamma}|\partial_t v|^2 + u|\nabla_x \log(u) - \nabla_x v|^2\Big)\, \mathrm{d}x\, \mathrm{d}s \le \tag{5.6}$$

$$c\Big\{\int_\Omega \Big(\frac{1}{\gamma}|\partial_t v(t)|^2 + u(t)|\nabla_x \log(u(t)) - \nabla_x v(t)|^2\Big)\, \mathrm{d}x\Big\}^{1/(2(1-\theta))},\ c > 0,$$

for any time t for which (5.3) holds.

At this stage, we use an auxiliary result proved in [3, Lemma 7.1].

Lemma 5.1 *Let $Z \geq 0$ be a measurable function on $(0,\infty)$ such that*

$$Z \in L^2(0,\infty),\ \|Z\|_{L^2(0,\infty)} \leq Y.$$

Assume, moreover, that there exist $\alpha \in (1,2)$, $\eta > 0$, and an open set $M \subset (0,\infty)$ such that

$$\Big(\int_t^\infty Z(s)^2\ \mathrm{d}s\Big)^\alpha \leq \eta\ Z^2(t) \text{ for a.a. } t \in M.$$

Then $Z \in L^1(M)$ and there is a constant $c = c(\alpha,\eta,Y)$ independent of M such that

$$\int_M Z(s)\ \mathrm{d}s \leq c.$$

Now, fix $(U,V) \in \omega[u,v]$ and set

$$M = \bigcup_J \{J \mid J \text{ is an open interval on which (5.3) holds}\}.$$

Since $[U,V] \in \omega[u,v]$, M is non-empty. Combining (1.1), (5.6) with the conclusion of Lemma 5.1 we obtain

$$\int_M \|\partial_t v(t)\|_{L^2(\Omega)}\ \mathrm{d}t + \int_M \|\partial_t u(t)\|_{[W^{1,2}(\Omega)]^*} \leq \tag{5.7}$$

$$\int_M \|\partial_t v(t)\|_{L^2(\Omega)}\ \mathrm{d}t + \int_M \|\sqrt{u(t)}\nabla_x(\log(u) - v)(t)\|_{L^2(\Omega;\mathbb{R}^2)} = c([U,V]) < \infty.$$

Consequently, the uniform estimates established in Corollary 3.1 together with a simple interpolation argument yield the existence of a time $\tau > 0$ such that

$$\left.\begin{array}{c} \|u(t_1) - u(t_2)\|_{L^2(\Omega)} + \|v(t_1) - v(t_2)\|_{W^{1,2}(\Omega)} < \dfrac{\varepsilon}{3} \\ \text{whenever} \\ \|u(t) - U\|_{L^2(\Omega)} + \|v(t) - V\|_{W^{1,2}(\Omega)} < \varepsilon \text{ for all } t \in (t_1,t_2) \text{ where } \tau \leq t_1 < t_2. \end{array}\right\} \tag{5.8}$$

Moreover, as $(U,V) \in \omega[u,v]$, τ can be chosen so that

$$\|u(\tau) - U\|_{L^2(\Omega)} + \|v(\tau) - V\|_{W^{1,2}(\Omega)} < \frac{\varepsilon}{3}. \tag{5.9}$$

We claim that (5.8), (5.9) imply $[\tau,\infty) \subset M$. Indeed taking

$$\overline{t} = \inf\{t > \tau \mid \|u(t) - U\|_{L^2(\Omega)} + \|v(t) - V\|_{W^{1,2}(\Omega)} \geq \varepsilon\},$$

we have $\overline{t} > \tau$, and

$$\|u(\overline{t}) - U\|_{L^2(\Omega)} + \|v(\overline{t}) - V\|_{W^{1,2}(\Omega)} \geq \varepsilon \text{ if } \overline{t} \text{ is finite.} \tag{5.10}$$

On the other hand, by virtue of (5.8), (5.9),

$$\|u(t) - U\|_{L^2(\Omega)} + \|v(t) - V\|_{W^{1,2}(\Omega)} \leq$$

$$\|u(t) - u(\tau)\|_{L^2(\Omega)} + \|v(t) - v(\tau)\|_{W^{1,2}(\Omega)} + \|u(\tau) - U\|_{L^2(\Omega)} + \|v(\tau) - V\|_{W^{1,2}(\Omega)} <$$

$$\frac{2}{3}\varepsilon \text{ for all } \tau \leq t < \overline{t}$$

which, together with (5.10), yields $\overline{t} = \infty$.

Thus (5.7) yields convergence of $(u(t),v(t))$ to (U,V) as $t \to \infty$. Theorem 1.1 has been proved.

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