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**NON-ISOTHERMAL SMOLUCHOWSKI-POISSON EQUATIONS
AS A SINGULAR LIMIT OF THE
NAVIER-STOKES-FOURIER-POISSON SYSTEM**

EDUARD FEIREISL AND PHILIPPE LAURENÇOT

ABSTRACT. The convergence of weak solutions to the compressible Navier-Stokes-Fourier-Poisson system with a friction term is studied in the high friction limit, the pressure law including that corresponding to Fermi-Dirac particles. The limit is shown to be a weak solution of a non-isothermal Smoluchowski-Poisson system with a time-dependent and spatially homogeneous temperature determined by the conservation of the total energy.

1. INTRODUCTION

There are many equations and systems studied in mathematical fluid mechanics that can be obtained, mostly formally, as singular limits of the complete system of equations describing the motion of a general, compressible, viscous, and heat conducting fluid. The best known examples are the geostrophic system arising in meteorology, various models of the turbulence phenomena considered as low Reynolds number limits of a viscous flow, and the classical Navier-Stokes system describing the motion of an incompressible fluid that can be viewed as a low Mach number limit of the full Navier-Stokes-Fourier system (see, for instance, the monograph of ZEYTOUNIAN [17]). Problems of this type are characterized by multiple space and time scales, where a careful asymptotic analysis is not only of significant theoretical interest but proved to be an efficient tool in numerical experiments (see KLEIN et al. [14]).

Pursuing this strategy we consider the *Smoluchowski equation*

$$\partial_t \varrho + \operatorname{div}_x \mathbf{J} = 0, \tag{1.1}$$

$$\mathbf{J} = -\nabla_x p_F(\varrho, \vartheta) - \varrho \nabla_x \Phi, \tag{1.2}$$

where the density $\varrho = \varrho(t, x) \geq 0$, and the current $\mathbf{J} = \mathbf{J}(t, x)$ are functions of the time $t \in (0, T)$ and the spatial position $x \in \Omega \subset \mathbb{R}^3$, satisfying the conservative boundary conditions

$$\mathbf{J} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.3}$$

The scalar potential $\Phi = \Phi(t, x)$ obeys the *Poisson equation*

$$\Delta \Phi = \mathbf{1}_\Omega \varrho \text{ in } (0, T) \times \mathbb{R}^3, \tag{1.4}$$

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while the absolute temperature $\vartheta = \vartheta(t) > 0$ is a spatially homogeneous function determined through the total energy balance relation

$$\int_{\Omega} \left(\varrho e_F(\varrho, \vartheta) + \frac{1}{2} \varrho \Phi \right) dx = E_0 \text{ for } t \in (0, T). \quad (1.5)$$

Given the initial distribution of the density,

$$\varrho(0, x) = \varrho_0(x) \text{ for } x \in \Omega, \quad (1.6)$$

the quantities $\vartheta_0 = \vartheta(0)$ and E_0 are interrelated through

$$\int_{\Omega} \left(\varrho_0 e_F(\varrho_0, \vartheta_0) + \frac{1}{2} \varrho_0 \Phi_0 \right) dx = E_0 \quad (1.7)$$

with $\Delta \Phi_0 = \mathbf{1}_{\Omega} \varrho_0$ to be satisfied in \mathbb{R}^3 .

The pressure p_F and the (specific) internal energy e_F obey the perfect gas state equation

$$p_F(\varrho, \vartheta) = \frac{2}{3} \varrho e_F(\varrho, \vartheta), \quad (1.8)$$

supplemented with Gibbs' relation

$$\vartheta D s_F(\varrho, \vartheta) = D e_F(\varrho, \vartheta) + p_F(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right), \quad (1.9)$$

where the symbol s_F stands for the specific entropy and $D = (\partial_{\varrho}, \partial_{\vartheta})$. It is easy to see that, necessarily,

$$p_F(\varrho, \vartheta) = \vartheta^{5/2} P_F \left(\frac{\varrho}{\vartheta^{3/2}} \right) \text{ for a certain function } P_F : [0, \infty) \rightarrow \mathbb{R}. \quad (1.10)$$

As far as we know, a system similar to (1.1 - 1.5) has been introduced in [6] where it is derived by formal asymptotic expansions from a Vlasov-Fokker-Planck-Poisson kinetic equation modelling the statistical mechanics of collisionless stellar systems. The derivation performed in [6] actually involves two steps: first, taking the moments of order zero, one and two of the solutions to the kinetic equation and using a closure method yield a Euler-Poisson system [6, Eqns. (5.10)-(5.12)]. A high friction limit then leads to non-isothermal Smoluchowski-Poisson equations similar to (1.1 - 1.5) (still at a formal level), the pressure P_F being either $P_F(Z) = Z$ or $P_F(Z) = (2/3) \left(I_{3/2} \circ I_{1/2}^{-1} \right) (Z)$, where I_{α} denotes the Fermi integral

$$I_{\alpha}(Z) = \int_0^{\infty} \frac{r^{\alpha}}{1 + Z e^r} dr, \quad \alpha > -1,$$

and I_{α}^{-1} its inverse function. This approach has further been developed in [3] in a more general setting, allowing for other pressure laws p_F .

The purpose of this work is then an attempt to give a rigorous proof of the second step, the high friction limit, the starting point being not the Euler-Poisson system but the Navier-Stokes-Fourier-Poisson system. Also, the friction term introduced below is simpler than the one arising from [6].

More precisely, as an hydrodynamics counterpart to (1.1 - 1.5), we consider the *Navier-Stokes-Fourier-Poisson system*:

$$\varepsilon \partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \quad (1.11)$$

$$\varepsilon \partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \operatorname{div}_x \mathbb{S} - \frac{1}{\varepsilon} \varrho \mathbf{u} - \varrho \nabla_x \Phi, \quad (1.12)$$

$$\Delta \Phi = \mathbf{1}_\Omega \varrho, \quad (1.13)$$

$$\begin{aligned} \varepsilon \partial_t(\varrho s) + \operatorname{div}_x(\varrho s \mathbf{u}) + \operatorname{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) \\ \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} + \frac{1}{\varepsilon} \varrho |\mathbf{u}|^2 \right), \end{aligned} \quad (1.14)$$

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e + \frac{1}{2} \varrho \Phi \right) dx = E_{0,\varepsilon}, \quad (1.15)$$

supplemented with the conservative boundary conditions:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad (\mathbb{S}\mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0, \quad \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0. \quad (1.16)$$

Here, the viscous (deviatoric) stress tensor \mathbb{S} obeys *Newton's rheological law*:

$$\mathbb{S} = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right), \quad (1.17)$$

the shear viscosity coefficient μ being a continuously differentiable function of the absolute temperature ϑ .

Similarly, the heat flux \mathbf{q} is determined by *Fourier's law*:

$$\mathbf{q} = -\kappa(\vartheta) \nabla_x \vartheta, \quad \kappa(\vartheta) = \varepsilon \vartheta^3 + \kappa_F(\vartheta), \quad (1.18)$$

with the heat conductivity coefficient $\kappa_F \in C^1([0, \infty))$.

Finally, we assume that the pressure p , the internal energy e , and the entropy s are continuously differentiable for $\varrho, \vartheta > 0$ and satisfy

$$p(\varrho, \vartheta) = \frac{\varepsilon}{3} \vartheta^4 + p_F(\varrho, \vartheta), \quad e(\varrho, \vartheta) = \varepsilon \frac{\vartheta^4}{\varrho} + e_F(\varrho, \vartheta), \quad (1.19)$$

$$s(\varrho, \vartheta) = \frac{4\varepsilon}{3} \frac{\vartheta^3}{\varrho} + s_F(\varrho, \vartheta),$$

where p_F , e_F , and s_F are the same as in (1.8 - 1.10).

The system (1.11 - 1.15) can be viewed as a simple model of a self-gravitating fluid subjected to high temperature radiation effects expressed through the ε -dependent quantities appearing in the constitutive relations (1.18), (1.19) (see [12]). Furthermore, the parameter ε scaling the time derivatives corresponds to a (small) value of the Strouhal number while the quantity $\varrho \mathbf{u}/\varepsilon$ in the momentum equation (1.12) can be interpreted as a "friction" term due to the surrounding medium at rest.

The main goal of the present paper is to show that the Smoluchowski-Poisson system (1.1 - 1.5) can be obtained as the asymptotic limit for $\varepsilon \rightarrow 0$ of (1.11 - 1.19). Explicitly, we claim the following result.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary. Assume that p , e , and s are given by (1.19), where p_F , e_F , and s_F obey (1.9), (1.10), with $p_F \in C^1([0, \infty))$ such that*

$$P_F(0) = 0, \quad P'_F(Z) > 0 \text{ for all } Z \geq 0, \quad \lim_{Z \rightarrow \infty} \frac{P'_F(Z)}{Z^{2/3}} = a > 0, \quad (1.20)$$

$$0 < \frac{Q_F(Z)}{Z} < C_v \text{ with } Q_F(Z) = \frac{5}{3} P_F(Z) - P'_F(Z) Z \text{ for all } Z > 0.$$

We also require that

$$\text{either } s_F \text{ is bounded from below or } \inf_{Z>0} \frac{Q_F(Z)}{Z} > 0. \quad (1.21)$$

Furthermore, suppose that μ and κ_F belong to $C^1([0, \infty))$ and satisfy

$$\begin{aligned} 0 < \underline{\mu}\vartheta \leq \mu(\vartheta) \leq \bar{\mu}(1 + \vartheta) \text{ for all } \vartheta > 0, \\ 0 < \underline{\kappa}(1 + \vartheta) \leq \kappa_F(\vartheta) \leq \bar{\kappa}(1 + \vartheta) \text{ for all } \vartheta > 0. \end{aligned} \quad (1.22)$$

For $\varepsilon \in (0, 1)$ let $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon)$ be a variational solution to (1.11 - 1.16) in the sense of Definition 2.2 below satisfying, in addition,

$$\int_{\Omega} \varrho_\varepsilon(0) \, dx = M > 0, \quad \sup_{\varepsilon>0} E_{0,\varepsilon} < \infty, \quad \text{ess lim inf}_{t \rightarrow 0^+} \int_{\Omega} \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \, dx \geq S_0, \quad (1.23)$$

with M, S_0 independent of ε , and there are a non-negative function $\tilde{\varrho} \in L^1(\Omega)$ and a positive function $\tilde{\vartheta} \in L^1(\Omega)$ such that

$$S_0 > \int_{\Omega} \tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta}) \, dx. \quad (1.24)$$

Then, passing to a subsequence if necessary, we have

$$\begin{aligned} \varrho_\varepsilon &\rightharpoonup \varrho \text{ in } C([0, T]; L_{\text{weak}}^{5/3}(\Omega)) \cap L^1((0, T) \times \Omega), \\ \mathbf{J}_\varepsilon = \frac{1}{\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon &\rightharpoonup \mathbf{J} \text{ weakly in } L^{3/2}(0, T; L^{90/77}(\Omega; \mathbb{R}^3)), \\ \vartheta_\varepsilon &\rightarrow \vartheta \text{ in } L^p((0, T) \times \Omega), \quad \nabla_x \vartheta_\varepsilon \rightarrow 0 \text{ in } L^p((0, T) \times \Omega; \mathbb{R}^3) \text{ for any } 1 \leq p < 2, \\ E_{0,\varepsilon} &\rightarrow E_0, \end{aligned}$$

where the limit quantities $\varrho, \mathbf{J}, \vartheta$, and E_0 represent a weak (distributional) solution of the Smoluchowski-Poisson system (1.1 - 1.5) (cf. Definition 2.1 below). In addition, $p_F(\varrho, \vartheta) \in L^{15/13}(0, T; W^{1,15/13}(\Omega))$ and both ϑ and $1/\vartheta$ belong to $L^\infty(0, T)$.

Remark 1.2. The last stipulation in (1.20) is equivalent, conformably with (1.9), to the requirement of strict positivity and boundedness of the specific heat at constant volume

$$c_v(\varrho, \vartheta) = \frac{\partial e_F(\varrho, \vartheta)}{\partial \vartheta}.$$

The choices $P_F(Z) = Z + Z^{5/3}$ or $P_F(Z) = (2/3) \left(I_{3/2} \circ I_{1/2}^{-1} \right) (Z)$ fulfil this assumption and (1.21). Indeed, concerning the latter, the positivity of Q_F follows from [2, Lemma 5.3] while we have $Q_F(Z) \sim cZ^{1/3}$ as $Z \rightarrow \infty$ by the well-known Sommerfeld representation of the Fermi integrals with α half an odd integer (see, e.g., [7, Eqs. (6)-(7)]). This property implies the boundedness of $Z \mapsto Q_F(Z)/Z$ and that s_F is bounded from below. For $P_F(Z) = Z + Z^{5/3}$, we have $Q_F(Z) = 2Z/3$ which clearly fulfils the second requirement of (1.21).

The existence of global-in-time solutions to system (1.11 - 1.16) (with the no-slip boundary conditions imposed on the velocity field \mathbf{u}) was established in [12, Theorem 2.4] (see also [11] for the necessary modifications to accommodate the growth conditions (1.22)). Note that, in accordance with the general philosophy discussed in [11], the energy balance equation has been substituted with the entropy

inequality (1.14) together with the total energy *conservation* principle expressed through (1.15).

Besides proving the convergence as $\varepsilon \rightarrow 0$, Theorem 1.1 also provides the existence of a weak solution to the non-isothermal Smoluchowski-Poisson equations (1.1-1.6) for pressure laws satisfying (1.20) and for non-negative initial data $\varrho_0 \in L^{5/3}(\Omega)$ with an arbitrary large mass M and total energy E_0 complying with the relation (1.7). To our knowledge this existence result is also new, a related existence result having been obtained in [16] for the non-isothermal Smoluchowski-Poisson equations derived in [6] for initial data ϱ_0 with a sufficiently small mass. Actually, it is mainly the non-isothermal Smoluchowski-Poisson equations with $p_F(\varrho, \vartheta) = \varrho\vartheta$ which has been studied recently [1, 4, 9]: in that particular case, global existence of solutions is known to hold true for initial data with small mass while finite time blow-up occurs for initial data with large mass. Such a phenomenon does not take place for pressure laws satisfying (1.20). At a formal level, these results have been extended to pressure laws $p_F(\varrho, \vartheta) = \vartheta\varrho^\gamma$, $\gamma \in (0, \infty)$, in [5].

The rest of the paper is devoted to the proof of Theorem 1.1. After some preliminary material discussed in Section 2, we establish uniform bounds on the family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$, independent of the parameter $\varepsilon > 0$ (see Section 3).

In Section 4, having identified all available estimates, we pass to the limit in the field equations (1.11 - 1.16) in order to obtain (1.1), (1.4).

Sections 5 and 6 are devoted to the proof of the strong (pointwise) convergence of $\{\varrho_\varepsilon\}_{\varepsilon>0}$, $\{\vartheta_\varepsilon\}_{\varepsilon>0}$, respectively, necessary to establish the constitutive relation (1.2). This is the most delicate part of the proof because of insufficient uniform bounds on the sequence $\{\varrho_\varepsilon\}_{\varepsilon>0}$. For the system (1.11 - 1.16), the “standard” way to deal with this problem is to introduce the renormalized continuity equation

$$\varepsilon \partial_t b(\varrho) + \operatorname{div}_x(b(\varrho)\mathbf{u}) + (b'(\varrho)\varrho - b(\varrho))\operatorname{div}_x \mathbf{u} = 0, \quad (1.25)$$

replacing $\varrho \approx b(\varrho)$, where b is a suitable bounded function. In the present setting, however, the time derivative $\partial_t b(\varrho)$ expressed through (1.25) contains a *singular term*

$$\frac{1}{\varepsilon} (b'(\varrho)\varrho - b(\varrho))\operatorname{div}_x \mathbf{u},$$

which is not (known to be) bounded uniformly with respect to ε . Consequently, instead of using (1.25), our approach is based on the concept of *oscillation defect measure* introduced in the existence theory developed in [11].

The proof of Theorem 1.1 is completed in Section 7.

2. VARIATIONAL FORMULATION

In this section, we collect some preliminary material concerning the concept of variational (distributional) solutions to problems (1.1 - 1.5) and (1.11 - 1.16). We start with the Smoluchowski-Poisson system.

Definition 2.1. *We shall say that ϱ , \mathbf{J} , Φ , and ϑ represent a variational solution to problem (1.1 - 1.5) if*

- $\varrho \geq 0$, $\varrho \in L^1((0, T) \times \Omega)$, $\mathbf{J} \in L^1((0, T) \times \Omega; \mathbb{R}^3)$, and

$$\int_0^T \int_\Omega (\varrho \partial_t \varphi + \mathbf{J} \cdot \nabla_x \varphi) \, dx \, dt = 0$$

for any $\varphi \in \mathcal{D}((0, T) \times \bar{\Omega})$;

- the potential $\Phi = \Delta^{-1}[\varrho \mathbf{1}_\Omega]$ is determined through the integral identity

$$\Phi(t, x) = -\frac{1}{4\pi} \int_{\Omega} \frac{\varrho(t, y)}{|x - y|} dy \text{ for a.a. } (t, x) \in (0, T) \times \Omega;$$

- the temperature $\vartheta = \vartheta(t)$ is a non-negative spatially homogeneous function such that $\vartheta \in L^1(0, T)$, $p_F(\varrho, \vartheta) \in L^1((0, T) \times \Omega)$, $\varrho \nabla_x \Phi \in L^1((0, T) \times \Omega; \mathbb{R}^3)$, and

$$\int_0^T \int_{\Omega} \mathbf{J} \cdot \varphi \, dx \, dt = \int_0^T \int_{\Omega} \left(p_F(\varrho, \vartheta) \operatorname{div}_x \varphi - \varrho \nabla_x \Phi \cdot \varphi \right) \, dx \, dt$$

for any $\varphi \in \mathcal{D}((0, T) \times \Omega; \mathbb{R}^3)$;

•

$$\int_{\Omega} \left(\varrho e_F(\varrho, \vartheta) + \frac{1}{2} \varrho \Phi \right) \, dx = E_0 \text{ for a.a. } t \in (0, T).$$

In a similar way, the variational solutions to the Navier-Stokes-Fourier-Poisson system are defined as follows.

Definition 2.2. We shall say that ϱ , \mathbf{u} , and ϑ represent a variational solution to problem (1.11 - 1.16) if

- the density $\varrho \geq 0$, $\varrho \in L^1((0, T) \times \Omega)$ and the velocity $\mathbf{u} \in L^1(0, T; W^{1,1}(\Omega; \mathbb{R}^3))$ are such that $\varrho \mathbf{u} \in L^1((0, T) \times \Omega; \mathbb{R}^3)$ and satisfy the renormalized continuity equation expressed through the integral identity

$$\int_0^T \int_{\Omega} \left(\varepsilon \varrho B(\varrho) \partial_t \varphi + \varrho B(\varrho) \mathbf{u} \cdot \nabla_x \varphi - b(\varrho) (\operatorname{div}_x \mathbf{u}) \varphi \right) \, dx \, dt = 0 \quad (2.1)$$

to hold for any $\varphi \in \mathcal{D}((0, T) \times \bar{\Omega})$, $b \in BC([0, \infty))$, where

$$B(\varrho) = B(0) + \int_1^{\varrho} \frac{b(z)}{z^2} \, dz; \quad (2.2)$$

- $\varrho \mathbf{u} \otimes \mathbf{u} \in L^1((0, T) \times \Omega; \mathbb{R}^{3 \times 3})$, the pressure $p = p(\varrho, \vartheta)$ is given by (1.19), $p \in L^1((0, T) \times \Omega)$, the viscous stress \mathbb{S} satisfies (1.17), $\mathbb{S} \in L^1((0, T) \times \Omega; \mathbb{R}^{3 \times 3})$, $\varrho \nabla_x \Phi \in L^1((0, T) \times \Omega; \mathbb{R}^3)$, and

$$\begin{aligned} & \int_0^T \int_{\Omega} \left(\varepsilon \varrho \mathbf{u} \cdot \partial_t \varphi + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \varphi + p \operatorname{div}_x \varphi \right) \, dx \, dt \\ &= \int_0^T \int_{\Omega} \left(\mathbb{S} : \nabla_x \varphi + \frac{1}{\varepsilon} \varrho \mathbf{u} \cdot \varphi + \varrho \nabla_x \Phi \cdot \varphi \right) \, dx \, dt \end{aligned} \quad (2.3)$$

for any $\varphi \in \mathcal{D}((0, T) \times \bar{\Omega}; \mathbb{R}^3)$ satisfying $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$;

•

$$\Phi(t, x) = -\frac{1}{4\pi} \int_{\Omega} \frac{\varrho(t, y)}{|x - y|} dy \text{ for a.a. } (t, x) \in (0, T) \times \Omega; \quad (2.4)$$

- $\vartheta > 0$ a.e. in $(0, T) \times \Omega$, the specific entropy $s = s(\varrho, \vartheta)$ is given by (1.19), $\varrho s \in L^1((0, T) \times \Omega)$, $\varrho s \mathbf{u} \in L^1((0, T) \times \Omega; \mathbb{R}^3)$, $\vartheta \in L^1(0, T; W^{1,1}(\Omega))$, \mathbf{q} is determined by (1.18), $\mathbf{q}/\vartheta \in L^1((0, T) \times \Omega; \mathbb{R}^3)$,

$$(\mathbb{S} : \nabla_x \mathbf{u} + (1/\varepsilon) \varrho |\mathbf{u}|^2 - (\mathbf{q} \cdot \nabla_x \vartheta)/\vartheta) / \vartheta \in L^1((0, T) \times \Omega),$$

and

$$\int_0^T \int_{\Omega} \left(\varepsilon \varrho s \partial_t \varphi + \varrho s \mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q} \cdot \nabla_x \varphi}{\vartheta} \right) dx dt \leq \quad (2.5)$$

$$\int_0^T \int_{\Omega} \frac{1}{\vartheta} \left(\frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} - \mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\varepsilon} \varrho |\mathbf{u}|^2 \right) \varphi dx dt$$

for any $\varphi \in \mathcal{D}((0, T) \times \overline{\Omega})$, $\varphi \geq 0$;

- the specific internal energy $e = e(\varrho, \vartheta)$ is given by (1.19), $\varrho e \in L^1((0, T) \times \Omega)$, and the total energy balance

$$\int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e + \frac{1}{2} \varrho \Phi \right) dx = E_{0, \varepsilon} \text{ holds for a.a. } t \in (0, T); \quad (2.6)$$

- the impermeability boundary conditions $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$ hold in the sense that

$$\int_{\Omega} \operatorname{div}_x(\varphi \mathbf{u}) dx = 0 \text{ a.a. on } (0, T) \quad (2.7)$$

for any $\varphi \in \mathcal{D}(\overline{\Omega})$.

3. UNIFORM ESTIMATES

The proof of Theorem 1.1 is based on uniform estimates on the sequence of variational solutions $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon > 0}$ to be derived in this section. These represent a direct consequence of the underlying physical principles, namely the conservation of the mass, the momentum, and the total energy. In addition, a useful piece of information is obtained from the dissipativity properties of the Navier-Stokes-Fourier system expressed through the production term in the entropy balance (2.5).

3.1. Total mass conservation. Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon > 0}$ be a family of variational solutions satisfying the hypotheses of Theorem 1.1. It follows directly from (2.1) that the total mass

$$M = \int_{\Omega} \varrho_\varepsilon dx \quad (3.1)$$

is a constant of motion. As the densities ϱ_ε are non-negative, we get immediately that

$$\{\varrho_\varepsilon\}_{\varepsilon > 0} \text{ is bounded in } L^\infty(0, T; L^1(\Omega)). \quad (3.2)$$

Furthermore, the standard elliptic estimates applied to (2.4) give rise to

$$\|\Phi_\varepsilon(t)\|_{L^{5/2}(\Omega)} \leq c \|\varrho_\varepsilon(t)\|_{L^1(\Omega)} \text{ for } t \in [0, T]; \quad (3.3)$$

whence, in accordance with (3.2),

$$\{\Phi_\varepsilon\}_{\varepsilon > 0} \text{ is bounded in } L^\infty(0, T; L^{5/2}(\Omega)). \quad (3.4)$$

3.2. Energy estimates. By virtue of hypotheses (1.8), (1.10), (1.19), the internal energy density can be written in the form

$$\varrho e(\varrho, \vartheta) = \varepsilon \vartheta^4 + \frac{3}{2} \vartheta^{5/2} P_F\left(\frac{\varrho}{\vartheta^{3/2}}\right),$$

where, in accordance with (1.20), there are $C > c > 0$ such that

$$c(Z + Z^{5/3}) \leq P_F(Z) \leq C(Z + Z^{5/3}) \text{ for all } Z \geq 0. \quad (3.5)$$

Consequently,

$$\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) \geq \varepsilon \vartheta_\varepsilon^4 + c \left(\varrho_\varepsilon \vartheta_\varepsilon + \varrho_\varepsilon^{5/3} \right).$$

Taking estimate (3.4) into account, we can use the total energy balance (2.6) together with the hypothesis (1.23) in order to conclude that

$$\{\varrho_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; L^{5/3}(\Omega)), \quad (3.6)$$

$$\{\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)), \quad (3.7)$$

$$\{\varepsilon^{1/4} \vartheta_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; L^4(\Omega)), \quad (3.8)$$

$$\{\varrho_\varepsilon \vartheta_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; L^1(\Omega)), \quad (3.9)$$

and

$$E_{0,\varepsilon} \rightarrow E_0$$

passing to a subsequence as the case may be.

3.3. Dissipation estimates. Gibb's relation (1.9), (1.8), and (1.10) yield

$$\begin{aligned} \frac{\partial s_F}{\partial \varrho}(\varrho, \vartheta) &= -\frac{1}{\varrho^2} \frac{\partial p_F}{\partial \vartheta}(\varrho, \vartheta) = -\frac{2}{3\varrho} \frac{\partial e_F}{\partial \vartheta}(\varrho, \vartheta) = -\frac{3\vartheta^{3/2}}{2\varrho^2} Q_F\left(\frac{\varrho}{\vartheta^{3/2}}\right), \\ \frac{\partial s_F}{\partial \vartheta}(\varrho, \vartheta) &= \frac{1}{\vartheta} \frac{\partial e_F}{\partial \vartheta}(\varrho, \vartheta) = \frac{9\vartheta^{1/2}}{4\varrho} Q_F\left(\frac{\varrho}{\vartheta^{3/2}}\right), \end{aligned}$$

the function Q_F being defined in (1.20). Owing to (1.19) we may write

$$s(\varrho, \vartheta) = \varepsilon \frac{4}{3} \frac{\vartheta^3}{\varrho} + s_F(1, 1) + \int_1^\varrho \frac{\partial s_F(z, \vartheta)}{\partial \varrho} dz + \int_1^\vartheta \frac{\partial s_F(1, z)}{\partial \vartheta} dz$$

and use (1.20) to obtain

$$\begin{aligned} s(\varrho, \vartheta) &\leq \varepsilon \frac{4}{3} \frac{\vartheta^3}{\varrho} + s_F(1, 1) + \int_{\min\{\varrho, 1\}}^1 \left| \frac{\partial s_F(z, \vartheta)}{\partial \varrho} \right| dz + \int_1^{\max\{\vartheta, 1\}} \frac{\partial s_F(1, z)}{\partial \vartheta} dz \\ &\leq \varepsilon \frac{4}{3} \frac{\vartheta^3}{\varrho} + s_F(1, 1) - \frac{3C_v}{2} \log(\min\{\varrho, 1\}) + \frac{9C_v}{4} \log(\max\{\vartheta, 1\}) \\ &\leq \varepsilon \frac{4}{3} \frac{\vartheta^3}{\varrho} + s_F(1, 1) + \frac{3C_v}{2} |\log(\varrho)| + \frac{9C_v}{4} [\log(\vartheta)]^+, \end{aligned} \quad (3.10)$$

where $[r]^+ = \max\{r, 0\}$ denotes the positive part of the real number r . We then infer from the energy estimates (3.2), (3.6), (3.8), and (3.9) that

$$\int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) dx \leq c \text{ for a.a. } t \in (0, T). \quad (3.11)$$

Consequently, choosing a suitable spatially homogeneous test function in the entropy inequality (2.5) and utilizing the hypothesis (1.23) together with the estimate (3.11), we get a uniform bound on the entropy production rate:

$$\int_0^T \int_\Omega \frac{1}{\vartheta_\varepsilon} \left(\mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon + \frac{\kappa(\vartheta_\varepsilon) |\nabla_x \vartheta_\varepsilon|^2}{\vartheta_\varepsilon} + \frac{1}{\varepsilon} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \right) dx dt \leq \varepsilon c.$$

In particular, by virtue of hypotheses (1.18) and (1.22), we obtain

$$\left\{ \frac{1}{\varepsilon} \sqrt{\left(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon} \right)} \mathbf{u}_\varepsilon \right\}_{\varepsilon>0} \text{ is bounded in } L^2((0, T) \times \Omega; \mathbb{R}^3), \quad (3.12)$$

together with

$$\left\{ \frac{1}{\sqrt{\varepsilon}} \left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x \mathbf{u}_\varepsilon^t - \frac{2}{3} \operatorname{div}_x \mathbf{u}_\varepsilon \mathbb{I} \right) \right\}_{\varepsilon>0} \text{ is bounded in } L^2((0, T) \times \Omega; \mathbb{R}_{sym}^{3 \times 3}), \quad (3.13)$$

$$\left\{ \nabla_x \vartheta_\varepsilon^{3/2} \right\}_{\varepsilon>0} \text{ is bounded in } L^2((0, T) \times \Omega; \mathbb{R}^3), \quad (3.14)$$

and

$$\left\{ \frac{1}{\sqrt{\varepsilon}} \nabla_x \sqrt{\vartheta_\varepsilon} \right\}_{\varepsilon>0} \text{ and } \left\{ \frac{1}{\sqrt{\varepsilon}} \nabla_x \log(\vartheta_\varepsilon) \right\}_{\varepsilon>0} \text{ are bounded in } L^2((0, T) \times \Omega; \mathbb{R}^3). \quad (3.15)$$

In order to continue, we shall need the following “weighted” version of the Poincaré inequality [10, Lemma 3.2]:

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded regular domain and M and K be two positive real numbers. Assume that ϱ is a non-negative function such that*

$$0 < M = \int_{\Omega} \varrho \, dx \text{ and } \int_{\Omega} \varrho^{5/3} \, dx \leq K. \quad (3.16)$$

Then there exists a constant $c = c(M, K, p)$ such that

$$\left\| w - \frac{1}{M} \int_{\Omega} \varrho w \, dx \right\|_{L^p(\Omega)} \leq c(M, K, p) \|\nabla_x w\|_{L^p(\Omega; \mathbb{R}^3)}$$

for any $w \in W^{1,p}(\Omega)$ if $p > 15/11$.

Proof: Assuming the contrary there are a sequence $\{w_n\}_{n=1}^\infty \subset W^{1,p}(\Omega)$ and a sequence $\{\varrho_n\}_{n=1}^\infty \subset L^{5/3}(\Omega)$ such that

$$\left\| w_n - \frac{1}{M} \int_{\Omega} \varrho_n w_n \, dx \right\|_{L^p(\Omega)} \geq n \|\nabla_x w_n\|_{L^p(\Omega; \mathbb{R}^3)} > 0,$$

where ϱ_n satisfies (3.16) for each $n \geq 1$.

Consequently, setting

$$z_n = \left(w_n - \frac{1}{M} \int_{\Omega} \varrho_n w_n \, dx \right) \left\| w_n - \frac{1}{M} \int_{\Omega} \varrho_n w_n \, dx \right\|_{L^p(\Omega)}^{-1}$$

we readily get

$$z_n \rightarrow z \text{ in } W^{1,p}(\Omega),$$

where z is a constant function, specifically, $z = |\Omega|^{-1/p}$.

Furthermore, as $p > 15/11$, the Sobolev space $W^{1,p}(\Omega)$ is compactly imbedded into $L^{5/2}(\Omega)$; therefore we can assume

$$\varrho_n \rightharpoonup \varrho \text{ weakly in } L^{5/3}(\Omega), \int_{\Omega} \varrho \, dx = M; \quad z_n \rightarrow z \text{ strongly in } L^{5/2}(\Omega),$$

and, consequently,

$$0 = \int_{\Omega} \varrho_n z_n \, dx \rightarrow \int_{\Omega} \varrho z \, dx = M |\Omega|^{-1/p},$$

whence a contradiction. □

In a similar way, one can establish a more standard result:

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded regular domain and M , K and Λ be positive real numbers with $\Lambda \in (0, 1]$. Furthermore, assume that ϱ is a non-negative function such that*

$$0 < M = \int_{\Omega} \varrho \, dx \text{ and } \int_{\Omega} \varrho^{5/3} \, dx \leq K.$$

Then there exists a constant $c = c(M, K)$ such that

$$\begin{aligned} \|\mathbf{w}\|_{W^{1,2}(\Omega; \mathbb{R}^3)} &\leq c(M, K) \left(\int_{\Omega} \varrho |\mathbf{w}|^{\Lambda} dx \right)^{1/\Lambda} \\ &\quad + c(M, K) \left\| \left(\nabla_x \mathbf{w} + \nabla_x \mathbf{w}^t - \frac{2}{3} \operatorname{div}_x \mathbf{w} \mathbb{I} \right) \right\|_{L^2(\Omega; \mathbb{R}_{sym}^{3 \times 3})} \end{aligned}$$

for any $\mathbf{w} \in W^{1,2}(\Omega; \mathbb{R}^3)$. In particular,

$$\|w\|_{W^{1,2}(\Omega)} \leq c(M, K) \left[\left(\int_{\Omega} \varrho |w|^{\Lambda} dx \right)^{1/\Lambda} + \|\nabla_x w\|_{L^2(\Omega; \mathbb{R}^3)} \right]$$

for any $w \in W^{1,2}(\Omega)$.

Now, Lemma 3.2 with $\Lambda = 2/3$ together with estimates (3.1), (3.6), (3.9), (3.14), give rise to

$$\{\vartheta_{\varepsilon}^{3/2}\}_{\varepsilon>0} \text{ is bounded in } L^2(0, T; W^{1,2}(\Omega)).$$

Next, since

$$\int_{\Omega} \varrho_{\varepsilon} \vartheta_{\varepsilon}^{1/2} dx \leq \left(\int_{\Omega} \varrho_{\varepsilon} \vartheta_{\varepsilon} dx \right)^{1/2} \left(\int_{\Omega} \varrho_{\varepsilon} dx \right) \leq c$$

by (3.2) and (3.9), another use of Lemma 3.2 with $\Lambda = 1$ and (3.15) entail that

$$\{\vartheta_{\varepsilon}^{1/2}\}_{\varepsilon>0} \text{ is bounded in } L^2(0, T; W^{1,2}(\Omega)).$$

By a simple interpolation argument we end up with

$$\{\vartheta_{\varepsilon}^{\alpha/2}\}_{\varepsilon>0} \text{ is bounded in } L^2(0, T; W^{1,2}(\Omega)) \text{ for any } 1 \leq \alpha \leq 3. \quad (3.17)$$

In particular, it readily follows from (3.17) (with $\alpha = 3$) and the embedding of $W^{1,2}(\Omega)$ in $L^6(\Omega)$ that

$$\{\vartheta_{\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^3(0, T; L^9(\Omega)). \quad (3.18)$$

Moreover, writing

$$\frac{1}{\varepsilon} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} = \frac{1}{\varepsilon} \sqrt{\frac{\varrho_{\varepsilon}}{\vartheta_{\varepsilon}}} \mathbf{u}_{\varepsilon} \sqrt{\varrho_{\varepsilon} \vartheta_{\varepsilon}}$$

one can use (3.9), (3.12) to obtain that

$$\left\{ \frac{1}{\varepsilon} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \right\}_{\varepsilon>0} \text{ is bounded in } L^2(0, T; L^1(\Omega; \mathbb{R}^3)) \quad (3.19)$$

while (3.6), (3.12), and (3.17) (with $\alpha = 3$) yield

$$\left\{ \frac{1}{\varepsilon} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \right\}_{\varepsilon>0} \text{ is bounded in } L^{3/2}(0, T; L^{90/77}(\Omega; \mathbb{R}^3)). \quad (3.20)$$

Finally, with (3.13), (3.19) at hand, another application of Lemma 3.2 (with $\Lambda = 1$) gives rise to

$$\left\{ \frac{1}{\sqrt{\varepsilon}} \mathbf{u}_{\varepsilon} \right\}_{\varepsilon>0} \text{ is bounded in } L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)). \quad (3.21)$$

3.4. Pressure estimates. The pressure estimates can be deduced formally “computing” the pressure p in the momentum equation (2.3) and using the energy estimates established above. More precisely, consider the operator \mathcal{A} defined by

$$\mathcal{A}[w] = \nabla_x \Delta_N^{-1} \left[w - \frac{1}{|\Omega|} \int_{\Omega} w \, dx \right]$$

where Δ_N denotes the Laplace operator supplemented with homogeneous Neumann boundary conditions on $\partial\Omega$. For $g \in C^1([0, \infty))$ such that g and $z \mapsto zg'(z)$ are bounded, the function $g(\varrho_\varepsilon)$ satisfies the renormalized equation (2.1). It then follows from [8, Theorem II.1] that, if $\eta_\delta = \eta_\delta(x)$ is a family of regularizing kernels, then $\eta_\delta * g(\varrho_\varepsilon)$ solves

$$\varepsilon \partial_t (\eta_\delta * g(\varrho_\varepsilon)) + \operatorname{div}_x (\eta_\delta * g(\varrho_\varepsilon) \mathbf{u}_\varepsilon) + \eta_\delta * \left((g'(\varrho_\varepsilon) \varrho_\varepsilon - g(\varrho_\varepsilon)) \operatorname{div}_x \mathbf{u}_\varepsilon \right) = r_{\varepsilon, \delta}$$

in $\mathcal{D}'((0, T) \times \Omega)$, where

$$r_{\varepsilon, \delta} \rightarrow 0 \text{ in } L^2((0, T) \times \Omega) \text{ as } \delta \rightarrow 0.$$

Note that, in accordance with Definition 2.2, equation (2.1) holds in $\mathcal{D}'((0, T) \times \mathbb{R}^3)$ provided $\varrho_\varepsilon = 1_\Omega \varrho_\varepsilon$ and the velocity \mathbf{u}_ε is extended to a function belonging to $L^2(0, T; W^{1,2}(\mathbb{R}^3; \mathbb{R}^3))$.

We next take $\varphi(t, x) = \psi(t) \mathcal{A}[\eta_\delta * g(\varrho_\varepsilon)]$ as test function in (2.3) where $\psi \geq 0$ belongs to $\mathcal{D}(0, T)$ and use the previous equation for $\eta_\delta * g(\varrho_\varepsilon)$ to obtain

$$\begin{aligned} & \int_0^T \int_{\Omega} \varepsilon \partial_t \psi \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{A}[\eta_\delta * g(\varrho_\varepsilon)] \, dx dt && (= F_1(\varepsilon, \delta)) \\ & + \int_0^T \int_{\Omega} \psi \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{A} \left[r_{\varepsilon, \delta} - \operatorname{div}_x (\eta_\delta * g(\varrho_\varepsilon) \mathbf{u}_\varepsilon) \right] \, dx dt && (= F_2(\varepsilon, \delta)) \\ & + \int_0^T \int_{\Omega} \psi \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{A} \left[\eta_\delta * \left(g(\varrho_\varepsilon) - g'(\varrho_\varepsilon) \varrho_\varepsilon \right) \operatorname{div}_x \mathbf{u}_\varepsilon \right] \, dx dt && (= F_3(\varepsilon, \delta)) \\ & + \int_0^T \int_{\Omega} \psi \varrho_\varepsilon [\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon] : \nabla_x \mathcal{A}[\eta_\delta * g(\varrho_\varepsilon)] \, dx dt && (= F_4(\varepsilon, \delta)) \quad (3.22) \\ & + \int_0^T \int_{\Omega} \psi p(\varrho_\varepsilon, \vartheta_\varepsilon) \left[\eta_\delta * g(\varrho_\varepsilon) - \frac{1}{|\Omega|} \int_{\Omega} \eta_\delta * g(\varrho_\varepsilon) \, dx \right] \, dx dt && (= F_5(\varepsilon, \delta)) \\ & = \int_0^T \int_{\Omega} \psi \mathbb{S}_\varepsilon : \nabla_x \mathcal{A}[\eta_\delta * g(\varrho_\varepsilon)] \, dx dt && (= F_6(\varepsilon, \delta)) \\ & + \int_0^T \int_{\Omega} \psi \left[\frac{1}{\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon + \varrho_\varepsilon \nabla_x \Phi_\varepsilon \right] \cdot \mathcal{A}[\eta_\delta * g(\varrho_\varepsilon)] \, dx dt. && (= F_7(\varepsilon, \delta)) \end{aligned}$$

We now assume further that g is such that

$$0 \leq g(\varrho) + |\varrho g'(\varrho)| \leq C \varrho^{1/9}, \quad \varrho \geq 0, \quad (3.23)$$

for some $C > 0$. We infer from (3.6) and (3.23) that

$$\sup_{[0, T]} \left\{ \|\eta_\delta * g(\varrho_\varepsilon)\|_{L^{15}(\Omega)} + \|\eta_\delta * (g(\varrho_\varepsilon) - \varrho_\varepsilon g'(\varrho_\varepsilon))\|_{L^{15}(\Omega)} \right\} \leq c. \quad (3.24)$$

Now, by (3.6), (3.7), (3.24), and classical elliptic estimates, we have

$$\begin{aligned}
& |F_1(\varepsilon, \delta)| \\
& \leq \varepsilon \int_0^T |\partial_t \psi| \|\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon\|_{L^2(\Omega; \mathbb{R}^3)} \|\sqrt{\varrho_\varepsilon}\|_{L^{10/3}(\Omega)} \|\mathcal{A}[\eta_\delta * g(\varrho_\varepsilon)]\|_{L^5(\Omega; \mathbb{R}^3)} dt \\
& \leq c\varepsilon \|\partial_t \psi\|_{L^1(0, T)}.
\end{aligned}$$

It next follows from (3.6), (3.21), (3.24), and classical elliptic estimates that

$$\begin{aligned}
& |F_2(\varepsilon, \delta)| \\
& \leq \|\psi\|_{L^\infty(0, T)} \int_0^T \|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^{30/23}(\Omega; \mathbb{R}^3)} \|\mathcal{A}[r_{\varepsilon, \delta}]\|_{L^{30/7}(\Omega; \mathbb{R}^3)} dt \\
& + \|\psi\|_{L^\infty(0, T)} \int_0^T \|\mathcal{A}[\operatorname{div}_x(\eta_\delta * g(\varrho_\varepsilon) \mathbf{u}_\varepsilon)]\|_{L^{30/7}(\Omega; \mathbb{R}^3)} dt \\
& \leq \|\psi\|_{L^\infty(0, T)} \int_0^T \|\varrho_\varepsilon\|_{L^{5/3}(\Omega)} \|\mathbf{u}_\varepsilon\|_{L^6(\Omega; \mathbb{R}^3)} \|r_{\varepsilon, \delta}\|_{L^2(\Omega)} dt \\
& + \|\psi\|_{L^\infty(0, T)} \int_0^T \|\varrho_\varepsilon\|_{L^{5/3}(\Omega)} \|\mathbf{u}_\varepsilon\|_{L^6(\Omega; \mathbb{R}^3)} \|\eta_\delta * g(\varrho_\varepsilon) \mathbf{u}_\varepsilon\|_{L^{30/7}(\Omega; \mathbb{R}^3)} dt \\
& \leq c\sqrt{\varepsilon} \|\psi\|_{L^\infty(0, T)} \int_0^T \left\| \frac{\mathbf{u}_\varepsilon}{\sqrt{\varepsilon}} \right\|_{W^{1,2}(\Omega; \mathbb{R}^3)} \left[\|r_{\varepsilon, \delta}\|_{L^2(\Omega)} + \|\mathbf{u}_\varepsilon\|_{L^6(\Omega; \mathbb{R}^3)} \right] dt \\
& \leq c\sqrt{\varepsilon} \|\psi\|_{L^\infty(0, T)} \left[\|r_{\varepsilon, \delta}\|_{L^2((0, T) \times \Omega)} + \sqrt{\varepsilon} \right].
\end{aligned}$$

Similar arguments also yield that

$$|F_3(\varepsilon, \delta)| \leq c\sqrt{\varepsilon} \|\psi\|_{L^\infty(0, T)}.$$

Using once more (3.6), (3.21) and (3.24), we obtain

$$\begin{aligned}
& |F_4(\varepsilon, \delta)| \\
& \leq \|\psi\|_{L^\infty(0, T)} \int_0^T \|\varrho_\varepsilon\|_{L^{5/3}(\Omega)} \|\mathbf{u}_\varepsilon\|_{L^6(\Omega; \mathbb{R}^3)}^2 \|\mathcal{A}[\eta_\delta * g(\varrho_\varepsilon) \mathbf{u}_\varepsilon]\|_{L^{15}(\Omega; \mathbb{R}^3)} dt \\
& \leq c\varepsilon \|\psi\|_{L^\infty(0, T)}.
\end{aligned}$$

We next infer from (1.22), (3.13) and (3.18) that

$$\|\mathbb{S}_\varepsilon\|_{L^{6/5}(0, T; L^{18/11}(\Omega; \mathbb{R}_{sym}^{3 \times 3}))} \leq c\sqrt{\varepsilon}. \quad (3.25)$$

Consequently, thanks to (3.24) and classical elliptic estimates,

$$\begin{aligned}
|F_6(\varepsilon, \delta)| & \leq \|\psi\|_{L^\infty(0, T)} \int_0^T \|\mathbb{S}_\varepsilon\|_{L^{18/11}(\Omega; \mathbb{R}_{sym}^{3 \times 3})} \|\eta_\delta * g(\varrho_\varepsilon)\|_{L^{18/7}(\Omega; \mathbb{R}^3)} dt \\
& \leq c\sqrt{\varepsilon} \|\psi\|_{L^\infty(0, T)}.
\end{aligned}$$

Finally, by (3.4), (3.6), (3.19), (3.24), the Calderon-Zygmund inequality and the embedding of $W^{2,5/3}(\Omega)$ in $W^{1,15/4}(\Omega)$, we have

$$\begin{aligned}
& |F_7(\varepsilon, \delta)| \\
& \leq \|\psi\|_{L^\infty(0,T)} \int_0^T \left\| \frac{1}{\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \right\|_{L^1(\Omega; \mathbb{R}^3)} \|\mathcal{A}[\eta_\delta * g(\varrho_\varepsilon)]\|_{L^\infty(\Omega; \mathbb{R}^3)} dt \\
& + \|\psi\|_{L^\infty(0,T)} \int_0^T \|\varrho_\varepsilon\|_{L^{5/3}(\Omega)} \|\nabla_x \Phi_\varepsilon\|_{L^{5/2}(\Omega; \mathbb{R}^3)} \|\mathcal{A}[\eta_\delta * g(\varrho_\varepsilon)]\|_{L^\infty(\Omega; \mathbb{R}^3)} dt \\
& \leq \|\psi\|_{L^\infty(0,T)} \int_0^T \left\| \frac{1}{\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \right\|_{L^1(\Omega; \mathbb{R}^3)} \|\mathcal{A}[\eta_\delta * g(\varrho_\varepsilon)]\|_{W^{1,15}(\Omega; \mathbb{R}^3)} dt \\
& + \|\psi\|_{L^\infty(0,T)} \int_0^T \|\Phi_\varepsilon\|_{W^{2,5/3}(\Omega)} \|\mathcal{A}[\eta_\delta * g(\varrho_\varepsilon)]\|_{W^{1,15}(\Omega; \mathbb{R}^3)} dt \\
& \leq c \|\psi\|_{L^\infty(0,T)},
\end{aligned}$$

while (3.5), (3.8), (3.18), and (3.24) ensure that

$$\begin{aligned}
& \left| \int_0^T \int_\Omega \psi p(\varrho_\varepsilon, \vartheta_\varepsilon) \left[\frac{1}{|\Omega|} \int_\Omega \eta_\delta * g(\varrho_\varepsilon) dx \right] dx dt \right| \\
& \leq c \|\psi\|_{L^\infty(0,T)} \int_0^T \int_\Omega p(\varrho_\varepsilon, \vartheta_\varepsilon) dx dt \\
& \leq c \|\psi\|_{L^\infty(0,T)}.
\end{aligned}$$

Collecting these information, one can deduce from (3.22) that

$$\begin{aligned}
& \int_0^T \int_\Omega \psi p(\varrho_\varepsilon, \vartheta_\varepsilon) \eta_\delta * g(\varrho_\varepsilon) dx \\
& \leq c \left(\|\psi\|_{L^\infty(0,T)} + \|\partial_t \psi\|_{L^1(0,T)} \right) \left(1 + \|r_{\varepsilon, \delta}\|_{L^2((0,T) \times \Omega)} \right),
\end{aligned}$$

provided g satisfies (3.23), where the bound is independent of both ε and δ . We may then let $\delta \rightarrow 0$, $\psi \rightarrow \mathbf{1}_{[0,T]}$ and $g(\varrho) \rightarrow \varrho^{1/9}$ to conclude, in view of (3.5), that

$$\{\varrho_\varepsilon\}_{\varepsilon > 0} \text{ is bounded in } L^{16/9}((0,T) \times \Omega). \quad (3.26)$$

Remark 3.3. *Similar estimates were established locally in Ω by LIONS [15] and extended to the whole domain Ω with the no-slip boundary conditions for \mathbf{u} in [13]. In both cases, $g(\varrho) \approx \varrho^\theta$, with $\theta = (2\gamma/3) - 1$ provided $\varrho \in L^\infty(0,T; L^\gamma(\Omega))$. Thus the value $\theta = 1/9$ corresponds to $\gamma = 5/3$ in agreement with (3.6).*

4. THE SINGULAR LIMIT

With the estimates obtained in Section 3, it is easy to pass to the limit for $\varepsilon \rightarrow 0$ in the system of equations (1.11 - 1.13). After a straightforward manipulation, we deduce from (1.11), (3.20) that

$$\varrho_\varepsilon \rightarrow \varrho \text{ in } C([0,T]; L_{weak}^{5/3}(\Omega)), \quad (4.1)$$

$$\mathbf{J}_\varepsilon = \frac{1}{\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \rightharpoonup \mathbf{J} \text{ weakly in } L^{3/2}(0,T; L^{90/77}(\Omega; \mathbb{R}^3)). \quad (4.2)$$

We may thus pass to the limit as $\varepsilon \rightarrow 0$ in (1.11) (or (2.1) with $B = 1$) to obtain

$$\int_0^T \int_{\Omega} (\varrho \partial_t \varphi + \mathbf{J} \cdot \nabla_x \varphi) \, dx \, dt = 0 \quad (4.3)$$

for any $\varphi \in \mathcal{D}((0, T) \times \overline{\Omega})$.

Furthermore, a simple interpolation argument yields

$$\|\varepsilon^{1/4} \vartheta_\varepsilon\|_{L^{17/3}(\Omega)}^{17/3} \leq c \varepsilon^{3/4} \|\vartheta_\varepsilon\|_{L^9(\Omega)}^3 \|\varepsilon^{1/4} \vartheta_\varepsilon\|_{L^4(\Omega)}^{8/3}; \quad (4.4)$$

therefore, (3.8) and (3.18) give rise to

$$\varepsilon \vartheta_\varepsilon^4 \rightarrow 0 \text{ in } L^{17/3}((0, T) \times \Omega). \quad (4.5)$$

It also follows from (3.5), (3.18), and (3.26) that

$$p_F(\varrho_\varepsilon, \vartheta_\varepsilon) \rightarrow \overline{p_F(\varrho, \vartheta)} \text{ weakly in } L^{48/43}((0, T) \times \Omega). \quad (4.6)$$

Next, on the one hand, by (3.6) and classical elliptic estimates,

$$\{\nabla_x \Phi_\varepsilon\}_{\varepsilon > 0} \text{ is bounded in } L^\infty(0, T; L^{15/4}(\Omega; \mathbb{R}^3)) \cap L^\infty(0, T; W^{1,5/3}(\Omega; \mathbb{R}^3)).$$

On the other hand, we infer from (1.11), (3.6) and (3.20) that $\{\varrho_\varepsilon\}_{\varepsilon > 0}$ is bounded in $L^\infty(0, T; L^{5/3}(\Omega))$ while

$$\{\partial_t \varrho_\varepsilon\}_{\varepsilon > 0} \text{ is bounded in } L^{3/2}(0, T; W^{-1,90/77}(\Omega)). \quad (4.7)$$

We are then in a position to apply [15, Lemma 5.1] to conclude that

$$\varrho_\varepsilon \nabla_x \Phi_\varepsilon \rightharpoonup \varrho \nabla_x \Phi \text{ weakly in } L^{15/13}((0, T) \times \Omega; \mathbb{R}^3). \quad (4.8)$$

We can now pass to the limit as $\varepsilon \rightarrow 0$ in (1.12) (or (2.3)) and (1.13) (or (2.4)) with the help of (3.6), (3.21), (3.25), (4.6), and (4.8) and obtain

$$\int_0^T \int_{\Omega} \overline{p_F(\varrho, \vartheta)} \operatorname{div}_x \varphi \, dx \, dt = \int_0^T \int_{\Omega} (\mathbf{J} + \varrho \nabla_x \Phi) \cdot \varphi \, dx \, dt \quad (4.9)$$

for any $\varphi \in \mathcal{D}((0, T) \times \overline{\Omega}; \mathbb{R}^3)$, $\nabla_x \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, and

$$\Delta \Phi = \mathbf{1}_\Omega \varrho \text{ in } (0, T) \times \mathbb{R}^3. \quad (4.10)$$

In particular, relation (4.9) implies that

$$\overline{p_F(\varrho, \vartheta)} \in L^{15/13}(0, T; W^{1,15/13}(\Omega)),$$

and

$$\mathbf{J} = -\nabla_x \overline{p_F(\varrho, \vartheta)} - \varrho \nabla_x \Phi. \quad (4.11)$$

Consequently, in order to complete the proof of Theorem 1.1, it is enough to show the strong (pointwise) convergence of the sequences $\{\varrho_\varepsilon\}_{\varepsilon > 0}$ and $\{\vartheta_\varepsilon\}_{\varepsilon > 0}$.

In the remaining sections, we use the following notation: if $f : [0, \infty)^2 \rightarrow \mathbb{R}$ is a function such that the sequence $\{f_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon)\}_{\varepsilon > 0}$ is weakly relatively compact in $L^1((0, T) \times \Omega)$, we denote by $\overline{f(\varrho, \vartheta)}$ its weak limit (after possible extraction of a subsequence).

5. STRONG CONVERGENCE OF THE DENSITY

As already pointed out in Section 1, the strong convergence of $\{\varrho_\varepsilon\}_{\varepsilon>0}$ represents a rather delicate issue mainly because of the fact that, strangely enough, the renormalized equation (2.1) contains a singular term.

Let $g \in C^1([0, \infty))$ be a bounded function such that $z \mapsto zg'(z)$ is also bounded. Owing to the analysis of Section 3.4, $F_i(\varepsilon, \delta) \rightarrow 0$ as $\delta, \varepsilon \rightarrow 0$ for $i \in \{1, 2, 3, 4, 6\}$ and we may let first $\delta \rightarrow 0$ and then $\varepsilon \rightarrow 0$ in (3.22) to obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} p_F(\varrho_\varepsilon, \vartheta_\varepsilon) \left(g(\varrho_\varepsilon) - \frac{1}{|\Omega|} \int_{\Omega} g(\varrho_\varepsilon) dx \right) dx dt = & \quad (5.1) \\ \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} (\mathbf{J}_\varepsilon + \varrho_\varepsilon \nabla_x \Phi_\varepsilon) \cdot \mathcal{A}[g(\varrho_\varepsilon)] dx dt. \end{aligned}$$

Owing to (2.4), (4.7) and the boundedness of g , classical elliptic estimates ensure that $\{\mathcal{A}[g(\varrho_\varepsilon)]\}_{\varepsilon>0}$ is bounded in $L^\infty(0, T; W^{1, \infty}(\Omega))$ and $\{\partial_t \nabla_x \Phi_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^{3/2}(0, T; W^{-1, 90/77}(\Omega; \mathbb{R}^3))$. Applying [15, Lemma 5.1] we get

$$\nabla_x \Phi_\varepsilon \cdot \mathcal{A}[g(\varrho_\varepsilon)] \rightharpoonup \nabla_x \Phi \cdot \mathcal{A}[\overline{g(\varrho)}] \quad \text{weakly in } L^{15/4}((0, T) \times \Omega; \mathbb{R}^3).$$

Next, in accordance with the standard elliptic estimates, we have

$$\left\{ \nabla_x \Phi_\varepsilon \cdot \mathcal{A}[g(\varrho_\varepsilon)] \right\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; W^{1, 5/3}(\Omega)),$$

which, together with (3.6) and (4.7) allow us to use again [15, Lemma 5.1] in order to obtain

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \Phi_\varepsilon \cdot \mathcal{A}[g(\varrho_\varepsilon)] dx dt = \int_0^T \int_{\Omega} \varrho \nabla_x \Phi \cdot \mathcal{A}[\overline{g(\varrho)}] dx dt. \quad (5.2)$$

Now, taking $\varphi = \mathcal{A}[\overline{g(\varrho)}]$ in (4.9), and making use of (5.1), (5.2), we get

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} \left[p_F(\varrho_\varepsilon, \vartheta_\varepsilon) g(\varrho_\varepsilon) - \overline{p_F(\varrho, \vartheta)} \overline{g(\varrho)} \right] dx dt \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|\Omega|} \int_0^T \left[\int_{\Omega} p_F(\varrho_\varepsilon, \vartheta_\varepsilon) dx \int_{\Omega} g(\varrho_\varepsilon) dx - \int_{\Omega} \overline{p_F(\varrho, \vartheta)} dx \int_{\Omega} \overline{g(\varrho)} dx \right] dt \\ &+ \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} \mathbf{J}_\varepsilon \cdot \mathcal{A}[g(\varrho_\varepsilon)] dx dt - \int_0^T \int_{\Omega} \mathbf{J} \cdot \mathcal{A}[\overline{g(\varrho)}] dx dt \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|\Omega|} \int_0^T \int_{\Omega} p_F(\varrho_\varepsilon, \vartheta_\varepsilon) dx \int_{\Omega} \left[g(\varrho_\varepsilon) - \overline{g(\varrho)} \right] dx dt \\ &+ \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} \mathbf{J}_\varepsilon \cdot \mathcal{A} \left[g(\varrho_\varepsilon) - \overline{g(\varrho)} \right] dx dt. \end{aligned} \quad (5.3)$$

At this stage, we introduce the cut-off functions $T_k \in C^\infty(\mathbb{R})$, $k \geq 1$, satisfying

$$\begin{cases} T_k(z) = z \text{ for } 0 \leq z \leq k, \\ T_k \text{ is concave and strictly increasing for } z \in [0, \infty), \\ T_k(z) \leq 2k \text{ for all } z \in [0, \infty), \\ T_k(z) = -T_k(-z) \text{ for all } z \in (-\infty, 0), \end{cases} \quad (5.4)$$

together with the quantities (defect measures)

$$\omega_k = \limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} |T_k(\varrho_\varepsilon) - T_k(\varrho)|^{8/3} dx dt, \quad k \geq 1$$

(cf. [10, Chapter 6]). It is easy to check that

$$\|T_k(z) - z\|_{L^q(\Omega)} \leq \frac{1}{k^{(p-q)/q}} \|z\|_{L^p(\Omega)}^{p/q} \quad \text{for any } 1 \leq q < p < \infty.$$

In particular,

$$\begin{aligned} \left| \int_{\Omega} \left(\overline{T_k(\varrho)} - \varrho \right) dx \right| &\leq \left\| \overline{T_k(\varrho)} - \varrho \right\|_{L^1(\Omega)} \leq \liminf_{\varepsilon \rightarrow 0} \|T_k(\varrho_\varepsilon) - \varrho_\varepsilon\|_{L^1(\Omega)} \\ &\leq \frac{1}{k^{2/3}} \sup_{\varepsilon > 0} \left\{ \|\varrho_\varepsilon\|_{L^{5/3}(\Omega)}^{5/3} \right\}. \end{aligned}$$

Consequently, observing that

$$\int_{\Omega} \left[T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right] dx = \int_{\Omega} \left[T_k(\varrho_\varepsilon) - \varrho_\varepsilon + \varrho - \overline{T_k(\varrho)} \right] dx$$

by (3.1) and (4.1), one can use (5.3) (with $g = T_k$) to obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} \left[p_F(\varrho_\varepsilon, \vartheta_\varepsilon) T_k(\varrho_\varepsilon) - \overline{p_F(\varrho, \vartheta)} \overline{T_k(\varrho)} \right] dx dt &\leq \quad (5.5) \\ \frac{c}{k^{2/3}} + \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} \mathbf{J}_\varepsilon \cdot \mathcal{A} \left[T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right] dx dt. \end{aligned}$$

where c is independent of k .

Now, we write

$$\int_0^T \int_{\Omega} \mathbf{J}_\varepsilon \cdot \mathcal{A} \left[T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right] dx dt = G_1(\varepsilon, k, m) + G_2(\varepsilon, k, m)$$

with

$$G_1(\varepsilon, k, m) = \int_0^T \int_{\Omega} \left(\mathbf{J}_\varepsilon - \mathbf{T}_m(\mathbf{J}_\varepsilon) \right) \cdot \mathcal{A} \left[T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right] dx dt,$$

$$G_2(\varepsilon, k, m) = \int_0^T \int_{\Omega} \mathbf{T}_m(\mathbf{J}_\varepsilon) \cdot \mathcal{A} \left[T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right] dx dt,$$

where we have set

$$\mathbf{T}_m(\mathbf{J})^i = T_m(J^i), \quad i = 1, 2, 3.$$

In accordance with the standard Sobolev imbedding theorem, we have

$$\begin{aligned} |G_1(\varepsilon, k, m)| &\leq c \int_0^T \|\mathbf{J}_\varepsilon - \mathbf{T}_m(\mathbf{J}_\varepsilon)\|_{L^{24/23}(\Omega; \mathbb{R}^3)} \left\| \mathcal{A} \left[T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right] \right\|_{W^{1,8/3}(\Omega; \mathbb{R}^3)} dt \\ &\leq c \int_0^T \|\mathbf{J}_\varepsilon - \mathbf{T}_m(\mathbf{J}_\varepsilon)\|_{L^{24/23}(\Omega; \mathbb{R}^3)} \left\| T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right\|_{L^{8/3}(\Omega)} dt; \end{aligned}$$

whence

$$\limsup_{\varepsilon \rightarrow 0} |G_1(\varepsilon, k, m)| \leq \sup_{\varepsilon > 0} \|\mathbf{J}_\varepsilon - \mathbf{T}_m(\mathbf{J}_\varepsilon)\|_{L^{8/5}(0, T; L^{24/23}(\Omega; \mathbb{R}^3))} \omega_k^{3/8}.$$

Next, the standard interpolation argument can be used to show

$$\|\mathbf{J}_\varepsilon - \mathbf{T}_m(\mathbf{J}_\varepsilon)\|_{L^{24/23}(\Omega; \mathbb{R}^3)} \leq c \|\mathbf{J}_\varepsilon - \mathbf{T}_m(\mathbf{J}_\varepsilon)\|_{L^1(\Omega; \mathbb{R}^3)}^{2/3} \|\mathbf{J}_\varepsilon - \mathbf{T}_m(\mathbf{J}_\varepsilon)\|_{L^{8/7}(\Omega; \mathbb{R}^3)}^{1/3};$$

therefore, by virtue of the uniform estimates (3.19) and (3.20),

$$\begin{aligned} & \|\mathbf{J}_\varepsilon - \mathbf{T}_m(\mathbf{J}_\varepsilon)\|_{L^{8/5}(0,T;L^{24/23}(\Omega;\mathbb{R}^3))} \leq \\ & c\|\mathbf{J}_\varepsilon - \mathbf{T}_m(\mathbf{J}_\varepsilon)\|_{L^2(0,T;L^1(\Omega;\mathbb{R}^3))}^{2/3}\|\mathbf{J}_\varepsilon - \mathbf{T}_m(\mathbf{J}_\varepsilon)\|_{L^{16/11}(0,T;L^{8/7}(\Omega;\mathbb{R}^3))}^{1/3} \leq \\ & c\frac{1}{m^{1/132}}\|\mathbf{J}_\varepsilon\|_{L^{90/77}((0,T)\times\Omega;\mathbb{R}^3)}^{1/3} \leq cm^{-1/132} \end{aligned}$$

from which we conclude that

$$\limsup_{\varepsilon \rightarrow 0} |G_1(\varepsilon, k, m)| \leq cm^{-1/132}\omega_k^{3/8}. \quad (5.6)$$

Furthermore, by virtue of (4.1),

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} G_2(\varepsilon, k, m) = \\ & \limsup_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega \mathbf{T}_m(\mathbf{J}_\varepsilon) \cdot \mathcal{A}[T_k(\varrho_\varepsilon) - \varrho_\varepsilon] \, dx \, dt + \int_0^T \int_\Omega \overline{\mathbf{T}_m(\mathbf{J})} \cdot \mathcal{A}[\varrho - \overline{T_k(\varrho)}] \, dx \, dt \leq \\ & m \sup_{\varepsilon > 0} \sup_{t \in (0,T)} \{ \|T_k(\varrho_\varepsilon)(t) - \varrho_\varepsilon(t)\|_{L^1(\Omega)} \} \leq c\frac{m}{k^{2/3}}; \end{aligned}$$

Thanks to (5.6) and the previous estimate, (5.5) transforms to

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega \left[p_F(\varrho_\varepsilon, \vartheta_\varepsilon) T_k(\varrho_\varepsilon) - \overline{p_F(\varrho, \vartheta)} \overline{T_k(\varrho)} \right] \, dx \, dt \leq \\ & c\left((1+m)k^{-2/3} + m^{-1/132}\omega_k^{3/8} \right) \text{ for any } k, m \geq 1. \end{aligned} \quad (5.7)$$

Next, using (1.20), one can write

$$p_F(\varrho, \vartheta) = a_0\varrho^{5/3} + q(\varrho, \vartheta), \quad (5.8)$$

for some $a_0 \in (0, a)$ where

$$\frac{\partial q(\varrho, \vartheta)}{\partial \varrho} \geq 0 \text{ for all } \varrho, \vartheta > 0. \quad (5.9)$$

Now, proceeding as in the proof of [10, Proposition 6.2] and using the concavity of T_k and the convexity of $z \mapsto z^{5/3}$, one can check that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega \left(\varrho_\varepsilon^{5/3} T_k(\varrho_\varepsilon) - \overline{\varrho_\varepsilon^{5/3}} \overline{T_k(\varrho_\varepsilon)} \right) \, dx \, dt \geq \omega_k; \quad (5.10)$$

whence (5.7) reduces to

$$\begin{aligned} & a_0\omega_k + \lim_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega \left[q(\varrho_\varepsilon, \vartheta_\varepsilon) T_k(\varrho_\varepsilon) - \overline{q(\varrho, \vartheta)} \overline{T_k(\varrho)} \right] \, dx \, dt \leq \\ & c\left((1+m)k^{-2/3} + m^{-1/132}\omega_k^{3/8} \right) \text{ for any } k, m \geq 1. \end{aligned} \quad (5.11)$$

We next set

$$h_k = T_k^{-1} \left(\overline{T_k(\varrho)} \right).$$

By virtue of (5.9), q is non-decreasing in ϱ , and, consequently,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega \left[q(\varrho_\varepsilon, \vartheta_\varepsilon) T_k(\varrho_\varepsilon) - \overline{q(\varrho, \vartheta)} \overline{T_k(\varrho)} \right] \, dx \, dt \geq \\ & \liminf_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega \left[q\left(T_k^{-1}(T_k(\varrho_\varepsilon)), \vartheta_\varepsilon \right) - q\left(T_k^{-1}\left(\overline{T_k(\varrho)} \right), \vartheta_\varepsilon \right) \right] \left[T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right] \, dx \, dt + \end{aligned}$$

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} q\left(T_k^{-1}\left(\overline{T_k(\varrho)}\right), \vartheta_{\varepsilon}\right) \left[T_k(\varrho_{\varepsilon}) - \overline{T_k(\varrho)}\right] dx dt &\geq \\ \liminf_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} q\left(h_k, \vartheta_{\varepsilon}\right) \left[T_k(\varrho_{\varepsilon}) - \overline{T_k(\varrho)}\right] dx dt. &\quad (5.12) \end{aligned}$$

Invoking estimates (3.15) and (3.18), we have

$$\|\nabla_x \vartheta_{\varepsilon}\|_{L^{3/2}((0,T) \times \Omega; \mathbb{R}^3)} = 2\sqrt{\varepsilon} \left\| \sqrt{\vartheta_{\varepsilon}} \frac{\nabla_x \sqrt{\vartheta_{\varepsilon}}}{\sqrt{\varepsilon}} \right\|_{L^{3/2}((0,T) \times \Omega; \mathbb{R}^3)} \leq c\sqrt{\varepsilon};$$

whence, by virtue of (3.17),

$$\nabla_x \vartheta_{\varepsilon} \rightarrow 0 \text{ in } L^p((0, T) \times \Omega; \mathbb{R}^3) \text{ for any } 1 \leq p < 2. \quad (5.13)$$

Introducing

$$\chi_{\varepsilon} = \chi_{\varepsilon}(t) = \frac{1}{M} \int_{\Omega} \varrho_{\varepsilon} \vartheta_{\varepsilon} dx, \quad (5.14)$$

and recalling Lemma 3.1 we deduce from (3.6), (3.18), and (5.13) that

$$\left(\vartheta_{\varepsilon} - \chi_{\varepsilon}\right) \rightarrow 0 \text{ in } L^p((0, T) \times \Omega) \text{ for any } 1 \leq p < 3, \quad (5.15)$$

and

$$\chi_{\varepsilon} \rightharpoonup \chi \text{ weakly } -(*) \text{ in } L^{\infty}(0, T). \quad (5.16)$$

In addition, in accordance with (5.8) and hypothesis (1.20), we have $0 \leq \partial_{\vartheta} q(\varrho, \vartheta) = \partial_{\vartheta} p_F(\varrho, \vartheta) \leq 3C_v \varrho/2$, so that

$$\left|q\left(h_k, \vartheta_{\varepsilon}\right) - q\left(h_k, \chi_{\varepsilon}\right)\right| \leq ch_k |\vartheta_{\varepsilon} - \chi_{\varepsilon}| \leq c\varrho |\vartheta_{\varepsilon} - \chi_{\varepsilon}|,$$

the inequality $h_k \leq \varrho$ being a consequence of the concavity of T_k (guaranteeing that $\overline{T_k(\varrho)} \leq T_k(\varrho)$) and the monotonicity of T_k^{-1} . Integrating the previous inequality over $(0, T) \times \Omega$ and using (3.6) and (5.15) lead us to

$$\begin{aligned} &\left\|q\left(h_k, \vartheta_{\varepsilon}\right) - q\left(h_k, \chi_{\varepsilon}\right)\right\|_{L^1((0,T) \times \Omega)} \\ &\leq \frac{3}{2}C_v \|\varrho\|_{L^{\infty}(0,T; L^{5/3}(\Omega))} \|\vartheta_{\varepsilon} - \chi_{\varepsilon}\|_{L^{5/2}((0,T) \times \Omega)} \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

This makes possible to rewrite (5.12) as

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} \left[q(\varrho_{\varepsilon}, \vartheta_{\varepsilon}) T_k(\varrho_{\varepsilon}) - \overline{q(\varrho, \vartheta)} \overline{T_k(\varrho)}\right] dx dt \geq \quad (5.17)$$

$$\liminf_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} q\left(h_k, \chi_{\varepsilon}\right) \left(T_k(\varrho_{\varepsilon}) - \overline{T_k(\varrho)}\right) dx dt.$$

We also note at this point that the above mentioned inequality $h_k \leq \varrho$, (3.6), (3.26) and (4.1) imply that

$$\|h_k\|_{L^{\infty}(0,T; L^{5/3}(\Omega))} + \|h_k\|_{L^{16/9}((0,T) \times \Omega)} \leq c \text{ uniformly in } k \geq 1. \quad (5.18)$$

Now, we write

$$\begin{aligned} &\liminf_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} q\left(h_k, \chi_{\varepsilon}\right) \left(T_k(\varrho_{\varepsilon}) - \overline{T_k(\varrho)}\right) dx dt \geq \\ &\liminf_{\varepsilon \rightarrow 0} \int \int_{\{h_k \geq m\}} \left(q\left(h_k, \chi_{\varepsilon}\right) - q\left(h_k, \chi\right)\right) \left(T_k(\varrho_{\varepsilon}) - \overline{T_k(\varrho)}\right) dx dt + \end{aligned}$$

$$\liminf_{\varepsilon \rightarrow 0} \int \int_{\{0 \leq h_k \leq m\}} q(h_k, \chi_\varepsilon) \left(T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right) dx dt.$$

Similarly to above, we have

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \int \int_{\{h_k \geq m\}} \left| q(h_k, \chi_\varepsilon) - q(h_k, \chi) \right| \left| T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right| dx dt \leq \\ & c \limsup_{\varepsilon \rightarrow 0} \int \int_{\{h_k \geq m\}} h_k \left| T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right| dx dt \leq \\ & c \sup_{k \geq 1} \left\{ \|\mathbf{1}_{\{h_k \geq m\}} h_k\|_{L^{8/5}((0,T) \times \Omega)} \right\} \omega_k^{3/8} \leq cm^{-1/9} \omega_k^{3/8}, \end{aligned}$$

where we have used estimates (5.18). Consequently,

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega q(h_k, \chi_\varepsilon) \left(T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right) dx dt \geq \quad (5.19) \\ & -cm^{-1/9} \omega_k^{3/8} + \liminf_{\varepsilon \rightarrow 0} \int \int_{\{0 \leq h_k \leq m\}} q(h_k, \chi_\varepsilon) \left(T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right) dx dt. \end{aligned}$$

Finally, for each pair (m, k) , we fix $H_{m,k} \in \mathcal{D}((0, T) \times \Omega)$ such that

$$\|H_{m,k} - h_k\|_{L^1((0,T) \times \Omega)} \leq \frac{1}{m^{2/3} k^2}.$$

We can then estimate

$$\begin{aligned} & \left| \int \int_{\{0 \leq h_k \leq m\}} q(h_k, \chi_\varepsilon) \left(T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right) dx dt \right| = \\ & \left| \int \int_{\{0 \leq h_k \leq m\}} q(T_m(h_k), \chi_\varepsilon) \left(T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right) dx dt \right| \leq \\ & 4k \sup_{\varepsilon > 0} \int_0^T \int_\Omega \left| q(T_m(h_k), \chi_\varepsilon) - q(T_m(H_{m,k}), \chi_\varepsilon) \right| dx dt + \\ & \left| \int \int_{\{0 \leq h_k \leq m\}} q(T_m(H_{m,k}), \chi_\varepsilon) \left(T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right) dx dt \right|. \end{aligned}$$

As for the former expression, we infer from (1.20) and (5.8) that

$$\begin{aligned} & 4k \sup_{\varepsilon > 0} \int_0^T \int_\Omega \left| q(T_m(h_k), \chi_\varepsilon) - q(T_m(H_{m,k}), \chi_\varepsilon) \right| dx dt \leq \quad (5.20) \\ & ck(1 + m^{2/3}) \|h_k - H_{m,k}\|_{L^1((0,T) \times \Omega)} \leq \frac{c}{k}, \end{aligned}$$

thanks to the choice $H_{m,k}$; while the latter can be treated as

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \left| \int \int_{\{0 \leq h_k \leq m\}} q(T_m(H_{m,k}), \chi_\varepsilon) \left(T_k(\varrho_\varepsilon) - \overline{T_k(\varrho)} \right) dx dt \right| \leq \quad (5.21) \\ & \limsup_{\varepsilon \rightarrow 0} \left| \int \int_{\{0 \leq h_k \leq m\}} q(T_m(H_{m,k}), \chi_\varepsilon) \left(T_k(\varrho_\varepsilon) - \varrho_\varepsilon + \varrho - \overline{T_k(\varrho)} \right) dx dt \right| + \\ & \limsup_{\varepsilon \rightarrow 0} \left| \int \int_{\{0 \leq h_k \leq m\}} q(T_m(H_{m,k}), \chi_\varepsilon) (\varrho_\varepsilon - \varrho) dx dt \right| \leq \\ & c(1 + m^{5/3}) \sup_{\varepsilon \rightarrow 0} \left\{ \|T_k(\varrho_\varepsilon) - \varrho_\varepsilon\|_{L^1((0,T) \times \Omega)} + \left\| \varrho - \overline{T_k(\varrho)} \right\|_{L^1((0,T) \times \Omega)} \right\} + \\ & \sup_{\varepsilon > 0} \left\{ \|q(T_m(H_{m,k}), \chi_\varepsilon)\|_{L^1(0,T; W^{1,2}(\Omega))} \right\} \limsup_{\varepsilon \rightarrow 0} \|\varrho_\varepsilon - \varrho\|_{L^\infty(0,T; W^{1,2}(\Omega)^*)} \leq c \frac{m^{5/3}}{k^{2/3}}. \end{aligned}$$

Thus relations (5.19 - 5.21) allow us to conclude that

$$\liminf_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} q(h_k, \chi_{\varepsilon}) (T_k(\varrho_{\varepsilon}) - \overline{T_k(\varrho)}) \, dx \, dt \geq -c \left(m^{-1/9} \omega_k^{3/8} + \frac{1}{k} + \frac{m^{5/3}}{k^{2/3}} \right).$$

Inserting the above lower bound in (5.17), we deduce that

$$a_0 \omega_k \leq c \left(k^{-1} + k^{-2/3} + m^{-1/132} \omega_k^{3/8} + m^{-1/9} \omega_k^{3/8} + \frac{m + m^{5/3}}{k^{2/3}} \right) \text{ for any } m, k;$$

whence

$$\limsup_{k \rightarrow \infty} \left(\limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} |T_k(\varrho_{\varepsilon}) - T_k(\varrho)|^{8/3} \, dx \, dt \right) = 0,$$

which yields the desired conclusion

$$\varrho_{\varepsilon} \rightarrow \varrho \text{ (strongly) in } L^1((0, T) \times \Omega). \quad (5.22)$$

6. STRONG CONVERGENCE OF THE TEMPERATURE

Up to now, we have collected the following information concerning the sequence $\{\vartheta_{\varepsilon}\}_{\varepsilon > 0}$:

- by virtue of (5.13),

$$\nabla_x \vartheta_{\varepsilon} \rightarrow 0 \text{ in } L^p((0, T) \times \Omega; \mathbb{R}^3) \text{ for any } 1 \leq p < 2; \quad (6.1)$$

- the limit temperature distribution is spatially homogeneous, specifically, by virtue of (5.14), (5.15), (5.16), and (6.1),

$$\vartheta_{\varepsilon} = (\vartheta_{\varepsilon} - \chi_{\varepsilon}) + \chi_{\varepsilon}, \quad \chi_{\varepsilon} = \frac{1}{M} \int_{\Omega} \varrho_{\varepsilon} \vartheta_{\varepsilon} \, dx, \quad (6.2)$$

and

$$\begin{cases} (\vartheta_{\varepsilon} - \chi_{\varepsilon}) \rightarrow 0 \text{ strongly in } L^p(0, T; W^{1,p}(\Omega)) \text{ for any } 1 \leq p < 2, \\ (\vartheta_{\varepsilon} - \chi_{\varepsilon}) \rightharpoonup 0 \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)), \end{cases} \quad (6.3)$$

$$\chi_{\varepsilon} \rightharpoonup \chi = \vartheta \text{ weakly - (*) in } L^{\infty}(0, T); \quad (6.4)$$

- the radiation component of the internal energy vanishes asymptotically (see (4.5)),

$$\varepsilon \vartheta_{\varepsilon}^4 \rightarrow 0 \text{ in } L^{17/3}((0, T) \times \Omega). \quad (6.5)$$

In order to go on, we establish a positivity property for the temperature. More specifically, we claim that there is $\varepsilon_0 > 0$

$$\chi_{\varepsilon} = \frac{1}{M} \int_{\Omega} \varrho_{\varepsilon} \vartheta_{\varepsilon} \, dx \geq \eta > 0 \text{ for a.a. } t \in (0, T) \text{ and } \varepsilon \in (0, \varepsilon_0). \quad (6.6)$$

For that purpose we adapt the proof of the positivity property (4.8) in [11] and first observe that, by virtue of Gibb's relation (1.9), there is a function S_F such that $s_F(\varrho, \vartheta) = S_F(\varrho \vartheta^{-3/2})$. Owing to the monotonicity of s_F with respect to the temperature, we may define

$$s_F(\varrho, 0+) = \lim_{\vartheta \rightarrow 0} s_F(\varrho, \vartheta) = \inf_{\vartheta > 0} \{s_F(\varrho, \vartheta)\} \in [-\infty, \infty).$$

The scaling invariance of s_F then implies that $s_F(\varrho, 0+)$ does not depend on ϱ . We therefore have either $s_F(\varrho, 0+) = -\infty$ for all $\varrho \geq 0$ or $s_F(\varrho, 0+) = \ell \in \mathbb{R}$ for all

$\varrho \geq 0$. Since s_F is defined up to an additive constant, we may assume that $\ell = 0$ in the latter case and $S_F(1) = 0$ in the former. We thus have either

$$s_F(\varrho, 0+) = \lim_{\vartheta \rightarrow 0} s_F(\varrho, \vartheta) = -\infty \quad \text{for all } \varrho \geq 0, \quad (6.7)$$

or

$$s_F(\varrho, 0+) = \lim_{\vartheta \rightarrow 0} s_F(\varrho, \vartheta) = 0 \quad \text{for all } \varrho \geq 0. \quad (6.8)$$

The next crucial observation is that, in accordance with (2.5), the total entropy

$$S_\varepsilon : t \mapsto \int_{\Omega} \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \, dx \text{ is non-decreasing in } (0, T)$$

and bounded from above and from below by (1.23) and (3.11). Therefore, after possibly extracting a subsequence, we deduce from Helly's selection principle that there is $S \in L^\infty(0, T)$ such that

$$S_\varepsilon \rightarrow S \text{ in } L^1(0, T), \quad (6.9)$$

and $S_\varepsilon(t) \geq S_0$ for a.a. $t \in (0, T)$ and $\varepsilon > 0$. It next follows from (1.24) and (3.8) that

$$\int_{\Omega} \varrho_\varepsilon s_F(\varrho_\varepsilon, \vartheta_\varepsilon) \, dx \geq \tilde{S} = \int_{\Omega} \tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta}) \, dx \quad (6.10)$$

for a.a. $t \in (0, T)$ and $\varepsilon \in (0, \varepsilon_0)$ with ε_0 small enough. Now the claim (6.6) is a straightforward consequence of (3.1), (3.6), (3.9), (6.10) and the following lemma.

Lemma 6.1. *Consider $R_0 > 0$. There exists a positive constant $m > 0$ depending only on M , R_0 and \tilde{S} such that, if ϱ is a non-negative function in $L^{5/3}(\Omega)$ and $\vartheta : \Omega \rightarrow (0, \infty)$ is a positive and measurable function satisfying*

$$\begin{cases} \|\varrho\|_{L^{5/3}(\Omega)}^{5/3} + \|\varrho\vartheta\|_{L^1(\Omega)} \leq R_0, \\ \int_{\Omega} \varrho \, dx = M \quad \text{and} \quad \int_{\Omega} \varrho s_F(\varrho, \vartheta) \, dx \geq \tilde{S}, \end{cases} \quad (6.11)$$

then

$$\int_{\Omega} \varrho\vartheta \, dx \geq m > 0.$$

In order not to delay further the proof of the strong convergence of $\{\vartheta_\varepsilon\}_{\varepsilon>0}$, we postpone the proof of Lemma 6.1 to the end of the section.

Owing to (6.4), the positivity property (6.6) is also enjoyed by $\vartheta = \chi$ and thus

$$\vartheta \geq \eta > 0 \quad \text{for a.a. } t \in (0, T). \quad (6.12)$$

The next consequence of (6.9), along with the strong convergence of the densities established in (5.22) and (6.3), (6.4), is that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} \left(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta) \right) (\vartheta_\varepsilon - \vartheta) \, dx \, dt = 0. \quad (6.13)$$

Indeed one has

$$\int_0^T \int_{\Omega} \left(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta) \right) (\vartheta_\varepsilon - \vartheta) \, dx \, dt = H_1^\varepsilon + H_2^\varepsilon + H_3^\varepsilon,$$

with

$$H_1^\varepsilon = \int_0^T \int_{\Omega} \left(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta) \right) (\vartheta_\varepsilon - \chi_\varepsilon) \, dx \, dt$$

and

$$H_2^\varepsilon = \int_0^T (\chi_\varepsilon - \chi) S_\varepsilon \, dt, \quad H_3^\varepsilon = \int_0^T (\chi_\varepsilon - \chi) \left(\int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta) \, dx \right) dt.$$

It readily follows from (5.4) and (6.9) that $H_2^\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Similarly, we deduce from (5.22) and the positivity (6.12) of ϑ that $\{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta)\}_{\varepsilon>0}$ converges towards $\varrho s(\varrho, \vartheta)$ in $L^1((0, T) \times \Omega)$ as $\varepsilon \rightarrow 0$, whence $H_3^\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ by (5.4). Concerning H_1^ε , we first observe that, if s is bounded from below, the convergence of H_1^ε to zero as $\varepsilon \rightarrow 0$ is a straightforward consequence of (3.6), (3.8), (3.10), (3.18) and (6.3). However, some difficulties arise when s_F is not bounded below as we do not have a uniform positive lower bound for ϑ_ε . To remedy to this fact, we are going to use the extra assumption (1.24) and thus consider the case where $Q_F(Z)/Z \geq \alpha > 0$ for all $Z > 0$. Then

$$s_F(\varrho, \vartheta) \leq -\frac{3\alpha}{2} \log \varrho + \frac{9\alpha}{4} \log \vartheta \quad \text{if } \varrho \geq \vartheta^{3/2},$$

and we infer from (3.6), (3.8), (3.10), and (3.18) that

$$\begin{aligned} S_0 \leq S_\varepsilon &= \frac{4}{3}\varepsilon \int_\Omega \vartheta_\varepsilon^3 \, dx + \int_{\{\varrho_\varepsilon \geq \vartheta_\varepsilon^{3/2}\}} \left(\frac{9\alpha}{4} \varrho_\varepsilon \log \vartheta_\varepsilon - \frac{3\alpha}{2} \varrho_\varepsilon \log \varrho_\varepsilon \right) \, dx \\ &+ \int_{\{\varrho_\varepsilon < \vartheta_\varepsilon^{3/2}\}} \left(s_F(1, 1) \varrho_\varepsilon + \frac{3C_v}{2} \varrho_\varepsilon |\log(\varrho_\varepsilon)| + \frac{9C_v}{4} \varrho_\varepsilon [\log(\vartheta_\varepsilon)]^+ \right) \, dx \\ &\leq \int_\Omega \left(\frac{9\alpha}{4} \varrho_\varepsilon \log \vartheta_\varepsilon - \frac{3\alpha}{2} \varrho_\varepsilon \log \varrho_\varepsilon \right) \, dx + c \int_\Omega \left(\varrho_\varepsilon + \varrho_\varepsilon^{5/3} + \varrho_\varepsilon \vartheta_\varepsilon \right) \, dx \\ &\leq \frac{9\alpha}{4} \int_\Omega \varrho_\varepsilon \log \vartheta_\varepsilon \, dx + c. \end{aligned}$$

Combining the previous bound with (3.9) allows us to deduce that

$$\int_\Omega \varrho_\varepsilon |\log \vartheta_\varepsilon| \, dx \leq c.$$

Thanks to (3.15), we may now apply Lemma 3.1 to conclude that

$$\{\log \vartheta_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(0, T; W^{1,2}(\Omega)). \quad (6.14)$$

Using again (3.10) and the assumption (1.24), we have

$$\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \leq s_F(1, 1) \varrho_\varepsilon + \frac{3C_v}{2} \varrho_\varepsilon |\log(\varrho_\varepsilon)| + \frac{9C_v}{4} \varrho_\varepsilon [\log(\vartheta_\varepsilon)]^+$$

and

$$\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \geq -c (\varrho_\varepsilon |\log(\varrho_\varepsilon)| + \varrho_\varepsilon |\log(\vartheta_\varepsilon)|),$$

and (3.6), (3.8), (3.18) and (6.14) warrant that

$$\{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\}_{\varepsilon>0} \text{ is bounded in } L^2(0, T; L^{30/23}(\Omega)).$$

This last property, (3.18), and (6.3) finally allow us to conclude that $H_1^\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ and complete the proof of (6.13).

Moreover, writing

$$s(\varrho, \vartheta) = \frac{4\varepsilon}{3\varrho} \vartheta^3 + s_F(\varrho, \vartheta),$$

we get

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega \left(\varrho_\varepsilon s_F(\varrho_\varepsilon, \vartheta_\varepsilon) - \varrho_\varepsilon s_F(\varrho_\varepsilon, \vartheta) \right) (\vartheta_\varepsilon - \vartheta) \, dx \, dt = 0. \quad (6.15)$$

Now, for $\delta \in (0, 1)$ and $K > 1/\eta$ (the constant η being defined in (6.6)), we have

$$\int \int_{\{\varrho > \delta\}} |\vartheta_\varepsilon - \vartheta|^2 \, dx \, dt \leq I_\varepsilon^1(K) + I_\varepsilon^2 + I_\varepsilon^3(K) + I_\varepsilon^4(K),$$

where

$$\begin{aligned} I_\varepsilon^1(K) &= \int \int_{\{\varrho_\varepsilon > K; \vartheta_\varepsilon > K\}} |\vartheta_\varepsilon - \vartheta|^2 \, dx \, dt, \\ I_\varepsilon^2 &= \int \int_{\{\varrho_\varepsilon < \delta; \varrho > \delta\}} |\vartheta_\varepsilon - \vartheta|^2 \, dx \, dt, \\ I_\varepsilon^3(K) &= \int \int_{\{\delta \leq \varrho_\varepsilon \leq K; 1/K \leq \vartheta_\varepsilon \leq K\}} |\vartheta_\varepsilon - \vartheta|^2 \, dx \, dt, \end{aligned}$$

and

$$I_\varepsilon^4(K) = \int \int_{\{\vartheta_\varepsilon \leq 1/K\}} |\vartheta_\varepsilon - \vartheta|^2 \, dx \, dt.$$

By (3.6) and (3.18) we have

$$\lim_{K \rightarrow \infty} \sup_{\varepsilon > 0} \{I_\varepsilon^1(K)\} = 0.$$

Moreover, as $\varrho_\varepsilon \rightarrow \varrho$ a.e. in $(0, T) \times \Omega$ by (5.22),

$$\lim_{\varepsilon \rightarrow 0} I_\varepsilon^2 = 0.$$

Similarly, as $(\vartheta_\varepsilon - \chi_\varepsilon) \rightarrow 0$ a.e. in $(0, T) \times \Omega$ by (6.3) and $\chi_\varepsilon \geq \eta$ a.e. in $(0, T)$ by (6.6),

$$\lim_{\varepsilon \rightarrow 0} I_\varepsilon^4(K) = 0 \quad \text{as soon as } \frac{1}{K} \leq \eta.$$

Finally, if $\varrho_\varepsilon \in [\delta, K]$ and $\vartheta_\varepsilon \in [1/K, K]$, we have

$$(\varrho_\varepsilon s_F(\varrho_\varepsilon, \vartheta_\varepsilon) - \varrho_\varepsilon s_F(\varrho_\varepsilon, \vartheta)) (\vartheta_\varepsilon - \vartheta) \geq \delta \left\{ \min_{[\delta, K] \times [1/K, K]} s_F \right\} (\vartheta_\varepsilon - \vartheta)^2$$

and the strict positivity of s_F in $[\delta, K] \times [1/K, K]$ and (6.15) entail that

$$\lim_{\varepsilon \rightarrow 0} I_\varepsilon^3(K) = 0 \quad \text{for any fixed } K.$$

Consequently, for $K > 1/\eta$, we have

$$\limsup_{\varepsilon \rightarrow 0} \int \int_{\{\varrho > \delta\}} |\vartheta_\varepsilon - \vartheta|^2 \, dx \, dt \leq \sup_{\varepsilon > 0} \{I_\varepsilon^1(K)\}.$$

Letting $K \rightarrow \infty$ leads us to

$$\vartheta_\varepsilon \rightarrow \vartheta \text{ in } L^2(\{\varrho > \delta\}) \text{ for any } \delta > 0,$$

which, combined with (6.3), (6.4) yields

$$\vartheta_\varepsilon \rightarrow \vartheta \text{ in } L^p(0, T; W^{1,p}(\Omega)) \text{ for any } 1 \leq p < 2. \quad (6.16)$$

As a consequence of (4.6), (5.22), and (6.16), we have

$$\overline{p_F(\varrho, \vartheta)} = p_F(\varrho, \vartheta).$$

Proof of Lemma 6.1: Assume for contradiction that the conclusion of the lemma is false. Then there are a sequence of non-negative functions $\{\varrho_n\}_{n \geq 1}$ in

$L^{5/3}(\Omega)$ and a sequence of positive and measurable functions $\{\vartheta_n\}_{n \geq 1}$ satisfying (6.11) and

$$\lim_{n \rightarrow \infty} \int_{\Omega} \varrho_n \vartheta_n \, dx = 0.$$

Given $j \geq 1$ and $k \geq 1$, we infer from (3.10) that

$$\begin{aligned} & \int_{\{\varrho_n \leq 1/k\}} \varrho_n s_F(\varrho_n, \vartheta_n) \, dx \\ & \leq \int_{\{\varrho_n \leq 1/k\}} \left\{ s_F(1, 1) + \frac{3C_v}{2} |\log(\varrho_n)| + \frac{9C_v}{4} [\log(\vartheta_n)]^+ \right\} \varrho_n \, dx \\ & \leq c \left(\frac{1}{k} + \frac{\log k}{k} + \int_{\Omega} \varrho_n \vartheta_n \, dx \right), \end{aligned}$$

so that

$$\limsup_{n \rightarrow \infty} \int_{\{\varrho_n \leq 1/k\}} \varrho_n s_F(\varrho_n, \vartheta_n) \, dx \leq c \left(\frac{1}{k} + \frac{\log k}{k} \right) \leq c k^{-1/2}.$$

Similarly, by (3.10) and (6.11) we have

$$\begin{aligned} & \int_{\{\varrho_n \geq k\}} \varrho_n s_F(\varrho_n, \vartheta_n) \, dx \\ & \leq \int_{\{\varrho_n \geq k\}} \left\{ s_F(1, 1) \frac{\varrho_n^{2/3}}{k^{2/3}} + \frac{3C_v}{2} \frac{\varrho_n^{2/3}}{k^{1/3}} + \frac{9C_v}{4} [\log(\vartheta_n)]^+ \right\} \varrho_n \, dx \\ & \leq c \left(\frac{1}{k^{1/3}} + \int_{\Omega} \varrho_n \vartheta_n \, dx \right), \end{aligned}$$

and thus

$$\limsup_{n \rightarrow \infty} \int_{\{\varrho_n \geq k\}} \varrho_n s_F(\varrho_n, \vartheta_n) \, dx \leq c k^{-1/3}.$$

Using once more (3.10) and (6.11) we obtain

$$\begin{aligned} & \int_{\{1/k < \varrho_n < k; \vartheta_n \geq 1/(jk^{2/3})\}} \varrho_n s_F(\varrho_n, \vartheta_n) \, dx \\ & \leq \left(k s_F(1, 1) + \frac{3C_v}{2} k \log k \right) \int_{\{\varrho_n \vartheta_n \geq 1/(jk^{5/3})\}} dx + \frac{9C_v}{4} \int_{\Omega} \varrho_n \vartheta_n \, dx \\ & \leq c \left(j k^{8/3} + j k^{8/3} \log k + 1 \right) \int_{\Omega} \varrho_n \vartheta_n \, dx, \end{aligned}$$

whence

$$\limsup_{n \rightarrow \infty} \int_{\{1/k < \varrho_n < k; \vartheta_n \geq 1/(jk^{2/3})\}} \varrho_n s_F(\varrho_n, \vartheta_n) \, dx \leq 0.$$

Finally, introducing $\varrho_{n,k} = \varrho_n \mathbf{1}_{[1/k, k]}(\varrho_n)$ and using the monotonicity of s_F , we get

$$\begin{aligned} \int_{\{1/k < \varrho_n < k; \vartheta_n < 1/(jk^{2/3})\}} \varrho_n s_F(\varrho_n, \vartheta_n) \, dx & \leq \int_{\{\vartheta_n < 1/(jk^{2/3})\}} \varrho_{n,k} s_F\left(\frac{1}{k}, \frac{1}{jk^{2/3}}\right) \, dx \\ & \leq s_F\left(1, \frac{1}{j}\right) \int_{\{\varrho_n \vartheta_n < k^{1/3}/j\}} \varrho_{n,k} \, dx \\ & \leq s_F\left(1, \frac{1}{j}\right) M. \end{aligned}$$

Summarizing, we have shown that

$$\limsup_{n \rightarrow \infty} \int_{\Omega} \varrho_n s_F(\varrho_n, \vartheta_n) \, dx \leq ck^{-1/3} + s_F\left(1, \frac{1}{j}\right) M$$

for all $k \geq 1$ and $j \geq 1$. Letting $k \rightarrow \infty$ and using (6.11) we end up with

$$\tilde{S} \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \varrho_n s_F(\varrho_n, \vartheta_n) \, dx \leq \limsup_{n \rightarrow \infty} \int_{\Omega} \varrho_n s_F(\varrho_n, \vartheta_n) \, dx \leq s_F\left(1, \frac{1}{j}\right) M.$$

At this stage, we have either (6.7) or (6.8). On the one hand, if s_F fulfils (6.7) we may let $j \rightarrow \infty$ and find that $\tilde{S} = -\infty$ and a contradiction. On the other hand, if (6.8) holds true, we obtain that $\tilde{S} = 0$ by passing to the limit as $j \rightarrow \infty$. The strict monotonicity of s_F with respect to the temperature then implies that $\tilde{\vartheta} = 0$. But this contradicts the positivity of $\tilde{\vartheta}$ and completes the proof. \square

7. CONCLUSION

Summarizing the above considerations we infer that the limit quantities ϱ , \mathbf{J} solve the problem

$$\partial_t \varrho + \operatorname{div}_x \mathbf{J} = 0, \quad (7.1)$$

$$\varrho(0) = \varrho_0, \quad \mathbf{J} \cdot \mathbf{n}|_{\partial\Omega} = 0 \quad (7.2)$$

in the sense of distributions (cf. Definition 2.1).

Moreover, the current \mathbf{J} is given by the constitutive equation

$$\mathbf{J} = -\nabla_x p_F(\varrho, \vartheta) - \varrho \nabla_x \Phi, \quad (7.3)$$

where

$$\Delta \Phi = \mathbf{1}_{\Omega} \varrho \text{ in } (0, T) \times \mathbb{R}^3. \quad (7.4)$$

The temperature ϑ is a spatially homogeneous function such that

$$E_0 = \int_{\Omega} \left(\varrho e_F(\varrho, \vartheta) + \frac{1}{2} \varrho \Phi \right) \, dx \text{ a.e. in } (0, T). \quad (7.5)$$

In addition, we have shown that the total entropy

$$S : t \mapsto \int_{\Omega} \varrho s_F(\varrho, \vartheta) \, dx \quad (7.6)$$

is a non-decreasing function of time satisfying

$$\operatorname{ess\,lim\,inf}_{t \rightarrow 0^+} S(t) \geq S_0. \quad (7.7)$$

Since ϑ and $1/\vartheta$ belong to $L^\infty(0, T)$ by virtue of (5.16) and (6.12), Theorem 1.1 has been proved.

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