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Efficient trading strategies in financial markets with proportional transaction costs

Luciano CAMPI * Elyès JOUINI † Vincent PORTE ‡

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Abstract

In this article, we characterize efficient portfolios, i.e. portfolios which are optimal for at least one rational agent, in a very general financial market model with proportional transaction costs. In our setting, transaction costs may be random, time-dependent, have jumps and the preferences of the agents are modeled by multivariate expected utility functions. Thanks to the dual formulation of expected multivariate utility maximization problem established in Campi and Owen [3], we provide a complete characterization of efficient portfolios, generalizing earlier results of Dybvig [10] and Jouini and Kallal [16]. We basically show that a portfolio is efficient if and only if it is cyclically anticomonotonic with respect to at least one consistent price system. Finally, we introduce the notion of utility price of a given contingent claim as the minimal amount of a given initial portfolio allowing any agent to reach the claim by trading in the market, and give a dual representation of it.

Keywords: Cyclic anticomonotonicity, utility maximization, proportional transaction costs, duality, utility price.

MSC Classification (2000): 91B16, 91B28.

1 Introduction

In this paper we characterize efficient portfolios in a general multivariate financial market with transaction costs as in [28, 4, 3], where agents can trade in finitely many risky assets (e.g. foreign currency) facing transaction costs at each trading. Such transaction costs are proportional, time dependent, random and they may have jumps.

An efficient portfolio is a portfolio which is optimal for at least one agent, in the sense that it solves an expected utility maximization problem from terminal wealth of agents with preferences described by *multivariate*, concave, strictly increasing (with respect to $\mathbb{R}^d_+$-preorder) utility functions. The choice of multivariate utility functions reflects the idea that the agents will not necessarily liquidate their positions to a single numeraire at the final date (which is realistic, in particular, on a currency market). This is coherent with the recent papers [1, 3] dealing with optimal investment problem under frictions. Moreover, it allows us to rely upon the duality methods developed therein.

A crucial ingredient of our results is the notion of (multivariate) cyclic anticomonotonicity. Cyclic anticomonotonicity is one possible extension to a multidimensional setting of the well-known notion of anticomonotonicy between two random variables (see the complete survey by Puccetti and Scarsini [24] and the references therein). Other multivariate extension with a more variational flavour have been recently studied and applied to vector risk measure and, more specifically, to optimal risk sharing (see, e.g., [5, 11, 12, 27]). Roughly speaking, two random variables are anticomonotone when they move in opposite directions.

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Our main result (Theorem 3.2 in this paper) states essentially that a portfolio X_0 is efficient if and only if there exists a consistent price system Z such that X_0 and Z are cyclically anticomotonic. This theorem considerably generalizes previous results in Dybvig [9, 10] and in Jouini and Kallal [17]. Moreover, following Jouini and Kallal [17], we compute for every contingent claim a measure of inefficiency that does not refer to any specific utility functions, using the anticomonotonicity property.

The notion of portfolio's efficiency goes back to a couple of articles by Dybvig [9, 10], where he studies those strategies that are chosen by at least one rational agent in a simple frictionless complete market with finitely many and equiprobable states of the world. In this context, the no-arbitrage condition is equivalent to the existence of a unique linear pricing rule and – this is Dybvig's result – a consumption bundle is efficient if and only if it provides at least as much consumption in cheaper states of the world (according to the unique risk-neutral pricing rule). Relying on this characterization, Dybvig introduces the notion of distributional price as the minimal price to pay for getting in exchange a contingent claim with a given distribution. Then he quantifies the inefficiency size of any contingent claims, focusing on some practical example such as stop-loss strategies.

In a market with frictions (e.g. with transaction costs), things are different. First, the pricing rule is not linear anymore. Nonetheless, Jouini and Kallal [16] show that the pricing rule is sublinear and can be viewed as the supremum over a set of linear pricing rules. Furthermore, in [17], they prove that Dybvig's characterization of efficient claims still holds under the following form : a portfolio is efficient if and only if it is anticomonotonic with respect to at least one pricing rule. Moreover, they introduce the notion of "utility price", defined as the minimal price to obtain a contingent claim preferred by *all* rational agents, allowing them to measure the inefficiency of a given contingent claim.

We continue here the research started in those papers and analyze further the notions of efficiency and utility price in a frictional model, which is the natural generalization of [16] and has been developed and studied by several authors. We refers to the recent book by Kabanov and Safarian [21] for an exhaustive treatment. Because of their generality, the results obtained in this paper are rather abstract. The study of the efficiency of specific trading strategies in more concrete models (e.g. Black-Scholes model with constant proportional transaction costs) is interesting on its own and it is postponed to future research.

This paper is organized as follows: In Section 2, we describe the financial model, recalling in particular the superhedging theorem proved in [4] and some definitions and properties of vector finitely additive measures, that we will be used in the proofs of our main results. Section 3 contains the main result of the paper on characterizing efficient trading strategies. Finally, Section 4 is devoted to quantifying the inefficency of a given portfolio through the notion of utility price.

Notation. Throughout the paper, we will frequently use the following notation:

- On the space $\mathbb{R}^d$, we will set $\|x\| := \max_i |x^i|$ and denote xy or, equivalently, $\langle x, y\rangle$ the canonical scalar product of x and y. Given two vectors x, y we will use the notation $x \geq y$ (resp. $x > y$) for $x - y \in \mathbb{R}^d_+$ (resp. $x - y \in \operatorname{int} \mathbb{R}^d_+$).
- For a concave (utility) function U defined on $\mathbb{R}^d_+$, we will denote ∂U the sub-differential of the function $-U$, that is:
$$\partial U(x) = \left\{ y \in \mathbb{R}^d \mid U(z) \leq U(x) + \langle y, z - x\rangle \quad \forall z \in \mathbb{R}^d \right\}.$$
- For a set A we denote $\mathrm{cl}(A)$ its closure and $\mathrm{int}(A)$ its interior, with respect to a given topology.
- c.d.f. will stand for cumulative distribution function.
- For a random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote $\operatorname{ess\,sup} X$ and $\operatorname{ess\,inf} X$ the essential supremum and, respectively, the essential infimum, of the random variable X, i.e.
$$\operatorname{ess\,sup} X := \inf\{a \in \mathbb{R} : \mathbb{P}(X > a) = 0\}, \quad \inf \emptyset := +\infty,$$
$$\operatorname{ess\,inf} X := \sup\{a \in \mathbb{R} : \mathbb{P}(X < a) = 0\}, \quad \sup \emptyset := -\infty.$$

- We denote $(X)^- = \max(-X, 0)$ the negative part of the random variable X.
- Given two random vectors X and Y, we will write $X \sim Y$ when they have the same law and we will denote by $\mathcal{L}(X)$ the set of all random vectors (defined on the same probability space) that have the same law as X.

2 The financial market

2.1 Assets and trading strategies

Let us recall the basic features of the transaction costs model as formalized in [4] (see also [28]). In such a model, all agents can trade in d assets according to a random and time varying bid-ask matrix. A $d \times d$ matrix $\Pi = (\pi^{ij})_{1 \leq i,j \leq d}$ is called a *bid-ask matrix* if (i) $\pi^{ij} > 0$ for every $1 \leq i, j \leq d$, (ii) $\pi^{ii} = 1$ for every $1 \leq i \leq d$, and (iii) $\pi^{ij} \leq \pi^{ik}\pi^{kj}$ for every $1 \leq i, j, k \leq d$.

Given a bid-ask matrix Π, the *solvency cone* $K(\Pi)$ is defined as the convex polyhedral cone in $\mathbb{R}^d$ generated by the canonical basis vectors e^i, $1 \leq i \leq d$ of $\mathbb{R}^d$, and the vectors $\pi^{ij}e^i - e^j$, $1 \leq i, j \leq d$. The convex cone $-K(\Pi)$ should be intepreted as those portfolios available at price zero. The (positive) polar cone of $K(\Pi)$ is defined by

$$K^*(\Pi) = \left\{ w \in \mathbb{R}^d \,:\, \langle v, w \rangle \geq 0, \forall v \in K(\Pi) \right\}.$$

Next, we introduce randomness and time in our model. Let $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be an **atomless**[†] filtered probability space satisfying the usual conditions and supporting all processes appearing in this paper. An adapted, càdlàg process $(\Pi_t)_{t\in[0,T]}$ taking values in the set of bid-ask matrices will be called a *bid-ask process*. A bid-ask process $(\Pi_t)_{t\in[0,T]}$ will now be fixed, and we drop it from the notation by writing K_τ (resp. K^*_τ) instead of $K(\Pi_\tau)$ (resp. $K^*(\Pi_\tau)$) for a stopping time τ.

Moreover, given any two vectors x, y, we will write $x \succeq_t y$ (resp. $x \succeq^*_t y$) whenever $x - y \in K_t$ (resp. $x - y \in K^*_t$).

In accordance with the framework developed in [4] we make the following technical assumption throughout the paper. The assumption is equivalent to disallowing a final trade at time T, but it can be relaxed via a slight modification of the model (see [4, Remark 4.2]). For this reason, we shall not explicitly mention the assumption anywhere.

Assumption 2.1 *$\mathcal{F}_{T-} = \mathcal{F}_T$ and $\Pi_{T-} = \Pi_T$ a.s.*

Definition 2.1 *An adapted, $\mathbb{R}^d_+ \setminus \{0\}$-valued, càdlàg martingale $Z = (Z_t)_{t\in[0,T]}$ is called a* consistent price process *for the bid-ask process $(\Pi_t)_{t\in[0,T]}$ if $Z_t \in K^*_t$ a.s. for every $t \in [0, T]$. Moreover, Z will be called a* strictly consistent price process *if it satisfies the following additional condition: For every $[0, T] \cup \{\infty\}$-valued stopping time τ, $Z_\tau \in \operatorname{int}(K^*_\tau)$ a.s. on $\{\tau < \infty\}$, and for every predictable $[0, T] \cup \{\infty\}$-valued stopping time σ, $Z_{\sigma-} \in \operatorname{int}(K^*_{\sigma-})$ a.s. on $\{\sigma < \infty\}$. The set of all (strictly) consistent price processes will be denoted by $\mathcal{Z}$ ($\mathcal{Z}^s$).*

The following assumption, which is used extensively in [4], will also hold throughout the paper.

Assumption 2.2 (SCPS) *Existence of a strictly consistent price system: $\mathcal{Z}^s \neq \emptyset$.*

This assumption is intimately related to the absence of arbitrage in continuous time financial markets with proportional transaction costs (see also [16, 13]).

Definition 2.2 *Suppose that $(\Pi_t)_{t\in[0,T]}$ is a bid-ask process such that Assumption 2.2 holds true. An $\mathbb{R}^d$-valued process $V = (V_t)_{t\in[0,T]}$ is called a self-financing portfolio process for the bid-ask process $(\Pi_t)_{t\in[0,T]}$ if it satisfies the following properties:*

[†] I.e., such a space supports a uniform random variable U on $(0, 1)$.

(i) *It is predictable and a.e. path has finite variation (not necessarily right-continuous).*

(ii) *For every pair of stopping times $0 \leq \sigma \leq \tau \leq T$, we have*

$$V_\tau - V_\sigma \in -\overline{\text{conv}}\left(\bigcup_{\sigma \leq t < \tau} K_t, 0\right) \quad a.s.$$

A self-financing portfolio process V is called admissible *if it satisfies the additional property*

(i) *There is a constant $a > 0$ such that $V_T + a\mathbf{1} \in K_T$ a.s. and $\langle V_\tau + a\mathbf{1}, Z^s_\tau \rangle \geq 0$ a.s. for all $[0,T]$-valued stopping times τ and for every strictly consistent price process $Z^s \in \mathcal{Z}^s$. Here, $\mathbf{1} \in \mathbb{R}^d$ denotes the vector whose entries are all equal to 1.*

Let $\mathcal{A}^x$ denote the set of all admissible, self-financing portfolio processes with initial endowment $x \in \mathbb{R}^d$, and let

$$\mathcal{A}^x_T := \{V_T \,:\, V \in \mathcal{A}^x\}$$

be the set of all contingent claims attainable at time T with initial endowment x. Note that $\mathcal{A}^x_T = x + \mathcal{A}^0_T$ for all $x \in \mathbb{R}^d$. Moreover, we denote $\mathcal{A} := \cup_{x \in \mathbb{R}^d} \mathcal{A}^x$ the set of all admissible strategies.

In such a market with transactions costs, the value of a portfolio V is not equivalent to its liquidation value at the final date $\langle V_T, S_T \rangle$, where S is some price process measured in terms of some given numéraire, and it is restrictive to assume that utility functions are only functions of the liquidation value. It is more relevant to consider each agent endowed with a utility function $U \in \mathcal{U}$, where $\mathcal{U}$ is the set of functions $U : \mathbb{R}^d \to [-\infty, \infty)$ supported on the non-negative orthant $\mathbb{R}^d_+$, i.e. the closure of its effective domain $\operatorname{dom} U := \{x \in \mathbb{R}^d : U(x) > -\infty\}$ is $\mathbb{R}^d_+$, and satisfying the following conditions :

- U is upper semi-continuous
- $\operatorname{int} \mathbb{R}^d_+ \subset \operatorname{dom} U$.
- U is strictly $\mathbb{R}^d_+$-increasing, i.e. $U(x_1) > U(x_2)$ whenever $x_1 > x_2$, i.e. $x_1 - x_2 \in \operatorname{int} \mathbb{R}^d_+$.
- U is concave on the interior of the positive orthant, i.e. $\operatorname{int} \mathbb{R}^d_+ = (0, \infty)^d$.

The case of an agent willing to liquidate his portfolio into a given subset of the risky assets could be included in the picture (see Remark 3.2 for more details).

Example 2.1 Examples of utility functions belonging to the class $\mathcal{U}$ are $U(x) = \sum_{i=1}^d U_i(x_i)$ with U_i one-dimensional HARA utility function, and $U(x) = \prod_{i=1}^d x_i^{\gamma_i}$ with $\gamma_i > 0$ and $\sum_i \gamma_i < 1$.

Each agent chooses an optimal strategy, depending on his preferences (i.e. on the utility function U) and on his initial portfolio x. This optimal strategy is chosen in order to maximize the expected utility of the terminal holdings vector.

Definition 2.3 *An admissible strategy $V \in \mathcal{A}$, with an initial portfolio $x \in \mathbb{R}^d$, is said to be efficient if and only if there exists a utility function $U \in \mathcal{U}$ such that $V_T \in \mathcal{A}^x_T$ is solution of:*

$$u(x) = \sup_{X \in \mathcal{A}^x_T} \mathbb{E}\left[U(X)\right]. \tag{2.1}$$

We say that a contingent claim X is efficient if it is the terminal value of an efficient strategies V, i.e. $X = V_T$ for some efficient strategy V.

For the convenience of the reader we present now a reformulation of [4, Theorem 4.1], giving a dual characterization of super-replicable contingent claims. We will use it to derive a dual formula for the amount of a given portfolio x necessary to hedge a contingent claim X.

For some positive contingent claim $X \in L^0(\mathbb{R}^d_+, \mathcal{F}_T)$, let:

$$\Gamma(X) := \left\{ x \in \mathbb{R}^d \mid V_T \succeq_T X \text{ for some } V \in \mathcal{A}^x \right\}. \tag{2.2}$$

The set $\Gamma(X)$ is the set of initial portfolios allowing to construct a strategy which hedges the contingent claim X. Before stating the super-hedging theorem we need to define one more set

$$\mathcal{D} := \{m \in \mathrm{ba}(\mathbb{R}^d) : m(X) \leq 0 \quad \forall X \in \mathcal{C}\},$$

where $\mathcal{C} := \mathcal{A}^0_T \cap L^\infty(\mathbb{R}^d)$. With these definitions we can state the following result :

Theorem 2.1 (Super-replication) *Let $x \in \mathbb{R}^d$ and let X be an $\mathcal{F}_T$-measurable, $\mathbb{R}^d_+$-valued random variable. Under Assumption 2.2, the following are equivalent :*

(i) $X \in \mathcal{A}^x_T$;

(ii) $\mathbb{E}\left[\langle X, Z_T\rangle\right] \leq \langle x, Z_0\rangle$, *for all* $Z \in \mathcal{Z}^s$;

(iii) $m^c(X) \leq \langle x, m^c(\Omega)\rangle$ *for all* $m \in \mathcal{D}$.

Proof. The equivalence between (i) and (ii) is a straightforward consequence of [4, Theorem 4.1]. Moreover, since to each $Z^s \in \mathcal{Z}^s$ we can associate a countably additive measure $m \in \mathrm{ba}(\mathbb{R}^d_+)$ as follows $\frac{dm}{d\mathbb{P}} := Z^s_T$, we clearly have that (iii) implies (ii). To complete the proof it suffices to show that (i) implies (iii). Consider the contingent claim X and decompose it as $X = \tilde{X} + x$ where $\tilde{X} \in \mathcal{A}^0_T$. Since $\tilde{X}$ is not necessarily in $L^\infty(\mathbb{R}^d)$, we need to consider the sequence $\tilde{X}^n := \tilde{X} \wedge (\mathbf{1}n)$, where the maximum is taken componentwise, i.e. with respect to the preorder induced by $\mathbb{R}^d_+$. The sequence clearly belongs to $\mathcal{C}$ and converges a.s. towards $\tilde{X}$. For any $n \geq 1$ and $m \in \mathcal{D}$, by definition of the set $\mathcal{D}$ we have $m^c(\tilde{X}^n) \leq 0$ so that passing to the limit yields

$$m^c(X) = m^c(\tilde{X}) + \langle x, m^c(\Omega)\rangle \leq \langle x, m^c(\Omega)\rangle.$$

The proof is now complete. ■

2.2 Euclidean vector measures

A function m from a field $\mathcal{F}$ of subsets of a set Ω to a Banach space $\mathcal{X}$ is called a finitely additive vector measure, or simply a vector measure if $m(A_1 \cup A_2) = m(A_1) + m(A_2)$, whenever A_1 and A_2 are disjoint members of $\mathcal{F}$. In this paper, we will be concerned with the special case where $\mathcal{X} = \mathbb{R}^d$; we refer to the associated vector measure as a "Euclidean vector measure", or simply a "Euclidean measure".

Let us recall a few definitions from the classical, one-dimensional setting. The *total variation* of a (finitely additive) measure $m : \mathcal{F} \to \mathbb{R}$ is the function $|m| : \mathcal{F} \to [0, \infty]$ defined by

$$|m|(A) := \sup \sum_{j=1}^{n} |m(A_j)|,$$

where the supremum is taken over all finite sequences $(A_j)_{j=1}^n$ of disjoint sets in $\mathcal{F}$ with $A_j \subseteq A$. A measure m is said to have *bounded total variation* if $|m|(\Omega) < \infty$. A measure m is said to be *bounded* if $\sup\{|m(A)| : A \in \mathcal{F}\} < \infty$. It is straightforward to show that

$$\sup\{|m(A)| : A \in \mathcal{F}\} \leq |m|(\Omega) \leq 2\sup\{|m(A)| : A \in \mathcal{F}\},$$

hence a measure is bounded if and only if it has bounded total variation. A measure m is said to be *purely finitely additive* if $0 \leq \mu \leq |m|$ and μ is countably additive imply that $\mu = 0$. A measure m is said to be *(weakly) absolutely continuous* with respect to $\mathbb{P}$ if $m(A) = 0$ whenever $A \in \mathcal{F}$ and $\mathbb{P}(A) = 0$.

We turn now to the d-dimensional case. A Euclidean measure m can be decomposed into its one-dimensional coordinate measures $m_i : \mathcal{F} \to \mathbb{R}$ by defining $m_i(A) := \langle e^i, m(A) \rangle$, where e^i is the i-th canonical basis vector of $\mathbb{R}^d$. In this way, $m(A) = (m_1(A), \ldots, m_d(A))$ for every $A \in \mathcal{F}$. We shall say that a Euclidean measure m is *bounded*, *purely finitely additive* or *(weakly) absolutely continuous* with respect to $\mathbb{P}$ if each of its coordinate measures is bounded, purely finitely additive or (weakly) absolutely continuous with respect to $\mathbb{P}$.

Let $\mathrm{ba}(\mathbb{R}^d) = \mathrm{ba}(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$ denote the vector space of bounded Euclidean measures $m : \mathcal{F}_T \to \mathbb{R}^d$, which are (weakly) absolutely continuous with respect to $\mathbb{P}$. Let $\mathrm{ca}(\mathbb{R}^d)$ the subspace of countably additive members of $\mathrm{ba}(\mathbb{R}^D)$. Equipped with the norm

$$\|m\|_{\mathrm{ba}(\mathbb{R}^d)} := \sum_{i=1}^{d} |m_i|(\Omega),$$

the spaces $\mathrm{ba}(\mathbb{R}^d)$ and $\mathrm{ca}(\mathbb{R}^d)$ are Banach spaces.

Let $\mathrm{ba}(\mathbb{R}^d_+)$ denote the convex cone of $\mathbb{R}^d_+$-valued measures within $\mathrm{ba}(\mathbb{R}^d)$. We recall the following fundamental *Yosida-Hewitt decomposition* : Given any $m \in \mathrm{ba}(\mathbb{R}^d)$ there exists a unique decomposition $m = m^c + m^p$ where $m^c \in \mathrm{ca}(\mathbb{R}^d)$ and m^p is purely finitely additive. If $m \in \mathrm{ba}(\mathbb{R}^d_+)$ then $m^c, m^p \in \mathrm{ba}(\mathbb{R}^d_+)$.

We shall see now that elements of $\mathrm{ba}(\mathbb{R}^d)$ play a natural role as linear functionals on spaces of (essentially) bounded $\mathbb{R}^d$-valued random variables. First, some more notation: Let $L^0(\mathbb{R}^d) = L^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$ denote the space of $\mathbb{R}^d$-valued random variables (identified under the equivalence relation of a.s. equality). Given $X \in L^0(\mathbb{R}^d)$ we define the coordinate random variables $X^i \in L^0(\mathbb{R})$ for $i = 1, \ldots, D$ by $X^i := \langle X, e^i \rangle$, so that $X = (X^1, \ldots, X^d)$. Let $L^1(\mathbb{R}^d)$ denote the subspace of $L^0(\mathbb{R}^d)$ consisting of those random variables X for which $\|X\|_1 := \mathbb{E}\left[\sum_i |X^i|\right] < \infty$. Let $L^\infty(\mathbb{R}^d)$ denote the subspace of $L^0(\mathbb{R}^d)$ consisting of those random variables X for which $\|X\|_\infty := \operatorname{ess\,sup}\left\{\max_i |X^i|\right\} < \infty$. Finally, let $L^\infty(\mathbb{R}^d)^*$ denote the dual space of $(L^\infty(\mathbb{R}^d), \|.\|_\infty)$.

We now define the map $\Psi : \mathrm{ba}(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)^*$ by

$$\big(\Psi(m)\big)(X) := \int_\Omega \langle X, \mathrm{d}m \rangle := \sum_{i=1}^{d} \int_\Omega X^i \mathrm{d}m_i, \tag{2.3}$$

where $(m_1, \ldots, m_d)$ is the coordinate-wise representation of m. We also define the map $\Phi : \mathrm{ca}(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ by $\Phi(m) := \left(\frac{\mathrm{d}m_1}{\mathrm{d}\mathbb{P}}, \ldots, \frac{\mathrm{d}m_d}{\mathrm{d}\mathbb{P}}\right)$, where $\frac{\mathrm{d}m_i}{\mathrm{d}\mathbb{P}}$ is the Radon-Nikodým derivative of the i-th coordinate measure. Finally, we define the isometric embedding $i : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)^*$ by $\big(i(Y)\big)(X) := \mathbb{E}\left[\langle X, Y \rangle\right]$. We recall the following two facts, whose proofs can be found in [3]:

- The maps Ψ and Φ are isometric isomorphisms. Furthermore, $i \circ \Phi = \Psi|_{\mathrm{ca}(\mathbb{R}^d)}$.
- $(\mathrm{ba}(\mathbb{R}^d), \|.\|_{\mathrm{ba}(\mathbb{R}^d)})$ has a $\sigma(\mathrm{ba}(\mathbb{R}^d), L^\infty(\mathbb{R}^d))$-compact unit ball (Alaoglu's theorem).

For the remainder of the paper, we shall overload our notation as follows: Given $m \in \mathrm{ba}(\mathbb{R}^d)$ and $X \in L^\infty(\mathbb{R}^d)$, we write $m(X)$ as an abbreviation of $\big(\Psi(m)\big)(X)$, and we define $\frac{\mathrm{d}m}{\mathrm{d}\mathbb{P}} := \left(\frac{\mathrm{d}m_1}{\mathrm{d}\mathbb{P}}, \ldots, \frac{\mathrm{d}m_d}{\mathrm{d}\mathbb{P}}\right) = \Phi(m)$.

Given $x \in \mathbb{R}^d$ and $A \in \mathcal{F}_T$ it follows from equation (2.3) that $m(x\chi_A) = \langle x, m(A) \rangle$, where χ_A denotes the indicator random variable of A. In the special case where $A = \Omega$, we have $m(x) = \langle x, m(\Omega) \rangle$.

Let $L^0(\mathbb{R}^d_+)$ and $L^\infty(\mathbb{R}^d_+)$ denote respectively the convex cones of random variables in $L^0(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ which are $\mathbb{R}^d_+$-valued a.s. Note that if $m \in \mathrm{ba}(\mathbb{R}^d_+)$ and $X \in L^\infty(\mathbb{R}^d_+)$ then $m(X) \geq 0$ (see [25, Theorem 4.4.13]). This observation allows us to extend the definition of $m(X)$ to cover the case where $m \in \mathrm{ba}(\mathbb{R}^d_+)$ and $X \in L^0(\mathbb{R}^d_+)$ by setting

$$m(X) := \lim_{n \uparrow \infty} m\left(X \wedge (n\mathbf{1})\right), \tag{2.4}$$

where $\mathbf{1} \in \mathbb{R}^d$ denotes the vector whose entries are all equal to 1, and $(x_1, \dots, x_d) \wedge (y_1, \dots, y_d) := (x_1 \wedge y_1, \dots, x_d \wedge y_d)$. It is trivial that (2.4) is consistent with the definition of $m(X)$ for $X \in L^\infty(\mathbb{R}^d)$. It follows that given $m_1, m_2 \in \mathrm{ba}(\mathbb{R}^d_+)$, $\lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0$ and $X_1, X_2 \in L^0(\mathbb{R}^d_+)$, we have

$$(\lambda_1 m_1 + \lambda_2 m_2)(\mu_1 X_1 + \mu_2 X_2) = \lambda_1 \mu_1 m_1(X_1) + \lambda_1 \mu_2 m_1(X_2) + \lambda_2 \mu_1 m_2(X_1) + \lambda_2 \mu_2 m_2(X_2).$$

Note that the property $i \circ \Phi = \Psi|_{\mathrm{ca}(\mathbb{R}^d)}$ means that given $m \in \mathrm{ca}(\mathbb{R}^d)$ and $X \in L^\infty(\mathbb{R}^d)$ we have $m(X) = \mathbb{E}\left[\langle X, \frac{\mathrm{d}m}{\mathrm{d}\mathbb{P}} \rangle\right]$. It is easy to show that this property is also true under the extended definition (2.4). For more details on such measures and their application to multivariate utility maximization, we refer to the paper [3] and the references therein.

3 Characterization of efficient trading strategies

As in Jouini and Kallal [17], the basic idea is to characterize efficient strategies using a dual characterization of expected utility maximization problems. In a continuous time setting, duality is more complex to handle. Such optimization problems have been studied in a very general frictionless market model by Kramkov and Schachermayer [21]. In a multidimensional case, the problem was studied first by Deelstra and al. [6]. However, these articles focus on some suitable hypothesis on the utility function, as the one on the asymptotic elasticity, in order to have a solution for the primal problem. In our setting, we are not interested in minimal conditions guaranteeing existence since existence of optimal portfolios is part of the definition of efficient portfolios. Thus we do not need to impose regularity conditions on the utility functions. That's why we will use instead recent results established in Campi and Owen [3], which seem to be more suitable for our purposes.

In the following subsection, we introduce the notion of *cyclic anticomonotonicity*, which turns out to be relevant in our setting. Then, we present our principal result, which characterizes efficient contingent claims via cyclic anticomonotonicity.

3.1 Cyclic anticomonotonicity

In one-dimensional setting, two random variables X, Y are anticomonotonic if they vary in opposite directions. The precise definition goes as follows.

Definition 3.1 *Two univariate random variables X and Y defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are anticomonotonic if there exists $A \in \mathcal{F}$, with $\mathbb{P}(A) = 1$, such that:*

$$(X(\omega) - X(\omega'))\,(Y(\omega) - Y(\omega')) \leq 0 \text{ for every } (\omega, \omega') \in A \times A. \tag{3.1}$$

In the multidimensional case, there exists several extension depending on which properties of the one-dimensional notion one wants to keep (see the survey [24]). In our case, it turns out that the "good" extension is the so-called *cyclic anticomotonicity*.

Definition 3.2 *Two d-dimensional random vectors X and Y defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are said to be* cyclically anticomonotonic *if and only there exists $A \in \mathcal{F}$ with $\mathbb{P}(A) = 1$ such that for every $p \geq 2$ and $(\omega_1, \omega_2, ..., \omega_p) \in A^p$, we have:*

$$\sum_{i=1}^{p} \langle X(\omega_i), Y(\omega_i) - Y(\omega_{i+1}) \rangle \leq 0. \tag{3.2}$$

where we set $\omega_{p+1} = \omega_1$.

Rockafellar([26], Theorem 24.9) shows that the multivalued maps belonging to the subdifferential of some concave function are characterized by the cyclic anticomonotonicity. This is the property that makes cyclic

anticomonotonicity the good multivariate extension in our setting. It will be clear from the proof of Theorem 3.2.

The concept of anticomonotonicity and cyclic anticomonotonicity are in fact equivalent in the one dimensional setting, i.e. for random variables. However, this is not the case in the multidimensional framework (see Rockafellar [26], Section 24, in particular the discussion following the proof of Theorem 24.9, and the survey by Puccetti and Scarsini [24], where this notion of multivariate (anti)comonotonicity is called c-monotonicity).
Note that one could have defined cyclic anticomonotonicity using the notion of product probability spaces. These two concepts are in fact equivalent. This is the content of next proposition, whose proof is provided in the Appendix.

Proposition 3.1 *Let X, Y be two random vectors on the space $(\Omega, \mathcal{F}, \mathbb{P})$. For each $p \in \mathbb{N}^*$, define $\mathbb{P}^{\otimes p}$ as the usual product probability on the product space $(\Omega^{\otimes p}, \mathcal{F}^{\otimes p})$. The cyclic anticomonotonicity between X and Y is equivalent to:*

$$\mathbb{P}^{\otimes p}\left[\sum_{i=1}^{p}\langle X(\omega_i), Y(\omega_i) - Y(\omega_{i+1})\rangle \leq 0\right] = 1 \tag{3.3}$$

for every $(\omega_1, \ldots, \omega_p) \in \Omega^p$ and for all $p \geq 2$, where we set $\omega_{p+1} = \omega_1$.

An important corollary of this proposition gives us a useful criterion to check whether two random vectors X, Y are cyclically anticomonotonic:

Corollary 3.1 *Let $d \in \mathbb{N}^*$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose X and Y are two d-dimensional random vectors such that there aren't any positive number $\varepsilon > 0$ and any finitely many sets $\Omega_i \in \mathcal{F}$ with $\mathbb{P}(\Omega_i) > 0$, $i = 1, \ldots, p$, such that*

$$\sum_{i=1}^{p}\langle Y(\omega_i), X(\omega_i) - X(\omega_{i+1})\rangle \geq \varepsilon, \quad \forall(\omega_1, ..., \omega_p) \in \Omega_1 \times ... \times \Omega_p, \tag{3.4}$$

where we set $\omega_{p+1} = \omega_1$. Then X and Y are cyclically anticomonotonic.

In the one dimensional case, for two fixed distributions F_X and F_Y on discrete probability spaces with equiprobable states or on atomless probability spaces, we can find random variables X distributed as F_X and Y distributed as F_Y such that X and Y are anticomonotonic (see Hardy and al. [15]). The next result proves that this result is extendable to the multidimensional case with the notion of cyclic anticomonotoncity. Its proof is provided in the Appendix. Recall that $\mathcal{L}(X)$ denotes the set of all random vectors (defined on the same probability space as X) that have the same law as X.

Proposition 3.2 *Let $X_0, Y \in L^0(\mathbb{R}^d_+)$.* Assume that the c.d.f. of X_0 is continuous. *There exists a random vector $\tilde{X}_0 \in \mathcal{L}(X_0)$, such that $\tilde{X}_0$ and Y are cyclically anticomonotonic. Furthermore, $\tilde{X}_0$ satisfies*

$$\mathbb{E}[\tilde{X}_0 Y] = \min\left\{\mathbb{E}\left[XY\right] : X \in \mathcal{L}(X_0)\right\}. \tag{3.5}$$

3.2 Characterization of efficient strategies

We turn now to the problem of characterizing efficient portfolios, i.e. admissible portfolios solving

$$u(x) = \sup_{X \in \mathcal{A}^x} \mathbb{E}\left[U(X_T)\right] \tag{3.6}$$

for some utility function $U \in \mathcal{U}$ and some initial portfolio $x \in \mathbb{R}^d$.

We will make use of the duality result that has been established in Campi and Owen [3]. To do that, we recall once more that $\mathcal{D}$ denotes the dual cone of $\mathcal{C} = \mathcal{A}^0_T \cap L^\infty(\mathbb{R}^d)$, i.e.

$$\mathcal{D} := (-\mathcal{C})^* = \{m \in \mathrm{ba}(\mathbb{R}^d) : m(X) \leq 0 \text{ for all } X \in \mathcal{C}\}.$$

Note that since $-L^\infty(\mathbb{R}^d_+) \subseteq \mathcal{C}$, we have $\mathcal{D} \subseteq \mathrm{ba}(\mathbb{R}^d_+)$. Moreover, we denote U^* the conjugate function of U, i.e.

$$U^*(y) := \sup_{x\in\mathbb{R}^d} \{U(x) - \langle x, y\rangle\}, \quad y \in \mathbb{R}^d.$$

For reader's convenience, we collect in the next proposition the results in [3] that we are going to use in the proofs. The next proposition states in particular that there is no duality gap between primal and a (suitably defined) dual problem provided the initial portfolio x does not lie on the boundary of $\mathrm{dom}(u)$, and that the dual problem has a solution whenever x lies in the interior of $\mathrm{dom}(u)$.

Proposition 3.1 (Duality) *Let U be a multivariate utility function in $\mathcal{U}$. The following hold:*

(i) *u is a utility function (recall the definition above) such that*

$$\mathrm{cl}(\mathrm{dom}(u)) = -\{x \in \mathbb{R}^d : x \in \mathcal{A}^0_T\}$$

(ii) *If $x \in \mathrm{int}(\mathrm{dom}(u))$ then*

$$u(x) = \min_{m\in\mathcal{D}} \left\{ \mathbb{E}\left[U^*\left(\frac{dm^c}{d\mathbb{P}}\right)\right] + m(x)\right\} \in \mathbb{R}. \tag{3.7}$$

Proof. It is a combination of Proposition 3.1, Lemma 3.3 and Proposition 3.5 in [3]. ■

Using this result, we can provide a characterization of efficient strategies based on the notion of cyclic anticomonotonicity introduced before. This is one of our main results and it is the content of the next theorem. Let $\mathcal{I} := \mathrm{int}(-\mathcal{A}^0_T \cap \mathbb{R}^d)$. We know that for any initial portfolio in $\mathcal{I}$ the corresponding expected utility maximization problem is well-defined. Since $U = -\infty$ outside $\mathbb{R}^d_+$, we will consider without loss of generality only positive contingent claims $X_0 \in L^0(\mathbb{R}^d_+)$.

Theorem 3.2 (Efficient contingent claims). *Let $x_0 \in \mathcal{I}$. A $\mathbb{R}^d_+$-valued contingent claim X_0 is efficient for the initial portfolio x_0 if and only if there exists a finitely additive measure $m_0 = m^c_0 + m^p_0 \in \mathcal{D}$, such that:*

(i) $Y_0 := \frac{dm^c_0}{d\mathbb{P}} \in \mathrm{int}\,\mathbb{R}^d_+$ *a.s. ;*

(ii) $m^c_0(X_0) = \mathbb{E}[X_0Y_0] = m_0(x_0)$*;*

(iii) *the random vectors X_0 and $Y_0 = \frac{dm^c_0}{d\mathbb{P}}$ are cyclically anticomonotonic;*

(iv) *the following properties hold :*

$$\mathrm{ess\,sup}\,\|Y_0\| = \infty \quad \Rightarrow \quad \mathrm{ess\,inf}\,\|X_0\| = 0, \tag{3.8}$$

$$\mathrm{ess\,sup}\,\|X_0\| < \infty \quad \Rightarrow \quad \mathrm{ess\,inf}\,\|Y_0\| > 0. \tag{3.9}$$

Proof. **Efficiency $\Rightarrow$ properties (i)-(iv).** Let X_0 be an $\mathbb{R}^d_+$-valued efficient contingent claim for a utility function $U \in \mathcal{U}$ and an initial portfolio x_0. Since $x_0 \in \mathrm{int}(\mathrm{dom}(u))$, u being the value function corresponding to U, there exists a vector-valued finitely additive measure $m_0 = m^c_0 + m^p_0 \in \mathcal{D}$ such that

$$u(x_0) = \mathbb{E}[U(X_0)] = \mathbb{E}\left[U^*\left(\frac{dm^c_0}{d\mathbb{P}}\right)\right] + m_0(x_0), \tag{3.10}$$

where the second equality is due to (3.7). On the other hand we also have

$$\begin{aligned} u(x_0) &= \mathbb{E}\left[U(X_0) - \left\langle X_0, \frac{dm^c_0}{d\mathbb{P}}\right\rangle + \left\langle X_0, \frac{dm^c_0}{d\mathbb{P}}\right\rangle\right] \\ &\le \mathbb{E}\left[U^*\left(\frac{dm^c_0}{d\mathbb{P}}\right)\right] + \mathbb{E}\left[\left\langle X_0, \frac{dm^c_0}{d\mathbb{P}}\right\rangle\right] \\ &\le \mathbb{E}\left[U^*\left(\frac{dm^c_0}{d\mathbb{P}}\right)\right] + m_0(x_0). \end{aligned}$$

Because of (3.10), all those inequalities are in fact equalities, so that in particular we have $m_0^c(X_0) = m_0(x_0)$, which is property (ii).

Property (i) is in Proposition 3.9 in Campi and Owen [3].

We now prove property (iii), i.e. that $\frac{dm_0^c}{d\mathbb{P}}$ and X_0 are cyclically anticomonotonic. First, set $Y_0 := \frac{dm_0^c}{d\mathbb{P}}$ and notice that by the definition of the dual function U^* we have

$$U(X) - XY_0 - U^*(Y_0) \leq 0$$

for all $\mathbb{R}_+^d$-valued contingent claim X, and that by optimality $\mathbb{E}[U(X_0) - X_0Y_0 - U^*(Y_0)] = 0$. That implies $U(X_0) - X_0Y_0 - U^*(Y_0) = 0$ a.s., which is equivalent to Y_0 belonging a.s. to the subdifferential $\partial U(X_0)$ (Theorem 23.5(d) in Rockafellar [26]), which is in turn equivalent to cyclic anticomonotonicity (Theorem 24.8 in Rockafellar [26] as well as Theorem 4.7 in [24]).

We conclude this part of the proof showing the two properties in (iv). We prove first property (3.8). Suppose first there exist integers $i, j \in \{1, ..., d\}$ such that $\operatorname{ess\,sup} Y_0^i = +\infty$ and $\operatorname{ess\,inf} X_0^j > 0$, and let's work toward a contradiction. There exists a real number $\varepsilon > 0$, such that $\varepsilon\mathbf{1} \in \operatorname{dom} U$ and $X_0^j > 2\varepsilon$, a.s., which implies that $\|X_0\|\mathbf{1} \geq X_0^j\mathbf{1} \geq 2\varepsilon\mathbf{1}$, where $\|x\| := \max_i |x_i|$. Since $Y_0 \in \partial U(X_0) \subset \partial U(\|X_0\|\mathbf{1})$ a.s., we have:

$$U(\varepsilon\mathbf{1}) - U(\|X_0\|\mathbf{1}) \leq U(\varepsilon\mathbf{1}) - U(X_0) \leq \langle Y_0, \varepsilon - X_0\rangle \leq \langle Y_0, \varepsilon - 2\varepsilon\rangle = -\langle Y_0, \varepsilon\rangle. \tag{3.11}$$

Since $\operatorname{ess\,sup} Y_0^i = +\infty$, we can choose (using a diagonal procedure) an $\mathbb{R}_+^d$-increasing sequence $(Y_0(\omega_n))_{n\geq 1}$ such that $\lim_{n\to+\infty} -\langle Y_0(\omega_n), \varepsilon\rangle = -\infty$. Moreover, by monotonicity of the function U, we have $U(\|X_0\|\mathbf{1}) \geq U(X_0) > -\infty$ a.s. (since X_0 is efficient, thus a maximizer). By letting n tend to $+\infty$, the RHS in (3.11) tends to $-\infty$, while its LHS is finite (since ε lies in the domain of U), which is a contradiction.

Now, we prove property (3.9), i.e. $\operatorname{ess\,sup} \|X_0\| < \infty \Rightarrow \operatorname{ess\,inf} \|Y_0\| > 0$. Indeed if $\operatorname{ess\,sup} \|X_0\| < \infty$ one can choose two points x', x'' with $x' - x'' \in \operatorname{int} \mathbb{R}_+^d$ and $x'' \geq X_0$ almost surely. Since $Y_0 \in \partial U(X_0)$, we deduce that

$$U(x') - U(x'') \leq U(x') - U(X_0) \leq \langle Y_0, x' - X_0\rangle \leq \langle Y_0, x'\rangle.$$

Assume, by contradiction, that $\operatorname{ess\,inf} \|Y_0\| > 0$, so that we can find a sequence $(Y_0(\omega_n))_{n\geq 1}$ converging compentwise to 0. Thus, $\langle Y_0(\omega_n), x'\rangle$ converges to 0 as well, yielding that $U(x') = U(x'')$, which contradicts the strict monotonicity of the utility function U.

Properties (i)-(iv) $\Rightarrow$ efficiency. Let X_0 and $m_0 = m_0^c + m_0^p$ satisfy properties (i)-(iv) of this theorem with $Y_0 := \frac{dm_0^c}{d\mathbb{P}}$. Let A be one of the measurable sets with probability one given in the definition of cyclic anticomonotonicity. We fix an $\omega_0 \in A$ and, motivated by the proof of Theorem 24.8 in [26], we consider the function U on $\mathbb{R}_+^d$ by:

$$U(x) := \inf \left\{\langle x - X_0(\omega_p), Y_0(\omega_p)\rangle + \cdots + \langle X_0(\omega_1) - X_0(\omega_0), Y_0(\omega_0)\rangle\right\}, \quad x \in \mathbb{R}_+^d$$

where the infimum is taken over all finite sets $(\omega_0, \omega_1, \ldots, \omega_p)$ (p arbitrary) of elements of A. Moreover, we set $U(x) = -\infty$ for $x \notin \mathbb{R}_+^d$, so that we have in particular that $\operatorname{dom}(U) \subset \mathbb{R}_+^d$.

We will prove that U belongs to $\mathcal{U}$ and that X_0 is efficient for U along the following steps. First, we prove in (1) that U is a proper closed concave function. Then for each $\omega \in A$, we note that $Y_0(\omega) \in \partial U\left(X_0(\omega)\right)$, which corresponds to property (2) below, and deduce in (3) that U is nondecreasing with respect to $\succeq$. Finally, establishing in (4) that $\operatorname{cl}(\operatorname{dom}(U)) = \mathbb{R}_+^d$ and in (5) that U is strictly increasing, we deduce in (6) that U belongs to $\mathcal{U}$ and X_0 is efficient for U.

(1) Since U is an infimum of a collection of affine functions, U is a closed concave function. Moreover, by construction we have $U(X_0(\omega_0)) = 0$ by cyclic anticomonotonicity of X_0 and Y_0 and hence U is proper.

(2) Let $\omega \in A$, it is enough to show that for any $\alpha > U\left(X_0(\omega)\right)$, and any $z \in \mathbb{R}^d$, the following property holds:

$$U(z) < \alpha + \langle Y_0(\omega), z - X_0(\omega)\rangle.$$

By definition of U, there exists some ω_i, $i = 1, \dots, p$ such that:

$$\alpha > \langle Y_0(\omega_p), X_0(\omega) - X_0(\omega_p)\rangle + \cdots + \langle Y_0(\omega_0), X_0(\omega_1) - X_0(\omega_0)\rangle.$$

By definition of U and setting $\omega_{p+1} = \omega$, we deduce:

$$U(z) \leq \langle Y_0(\omega_p), z - X_0(\omega_{p+1})\rangle + \cdots + \langle Y_0(\omega_0), X_0(\omega_1) - X_0(\omega_0)\rangle < \alpha + \langle Y_0(\omega_p), z - X_0(\omega)\rangle$$

and this proves that $Y_0(\omega) \in \partial U(X_0(\omega))$.

(3) We prove that $U(x') \geq U(x)$ as soon as $x' \geq x$, i.e. $x' - x \in \mathbb{R}^d_+$. Indeed, using U's definition, we can choose $\varepsilon > 0$ and $(\omega_0, \omega_1, \dots, \omega_p)$ such that

$$\langle x' - X_0(\omega_p), Y_0(\omega_p)\rangle + \cdots + \langle X_0(\omega_1) - X_0(\omega_0), Y_0(\omega_0)\rangle \leq U(x') + \varepsilon.$$

Using the definition of U once more, we have

$$\begin{aligned} U(x) &\leq \langle x - X_0(\omega_p), Y_0(\omega_p)\rangle + \cdots + \langle X_0(\omega_1) - X_0(\omega_0), Y(\omega_0)\rangle \\ &\leq U(x') + \varepsilon + \langle x - x', Y_0(\omega_p)\rangle \end{aligned}$$

and since $x' - x \in \mathbb{R}^d_+$ and $Y(\omega_p) \in \operatorname{int}\mathbb{R}^d_+$, we have $U(x) \leq U(x') + \varepsilon$. Being $\varepsilon > 0$ arbitrary, we can conclude that U is increasing for the preorder induced by $\mathbb{R}^d_+$.

(4) First, notice that U is finite on each $X_0(\omega)$ for $\omega \in A$.[‡] Consider first the case where $\operatorname{ess\,inf}\|X_0\| = 0$ and take some $x \in \operatorname{int}\mathbb{R}^d_+$. We can choose a $\omega \in A$ such that $X_0(\omega) \leq x$, and deduce that $U(X_0(\omega)) \leq U(x)$, i.e. $x \in \operatorname{dom}(U)$ so implying that $\operatorname{int}\mathbb{R}^d_+ \subset \operatorname{dom} U$. Since $\operatorname{dom} U \subset \mathbb{R}^d_+$ (by definition of U), we have that $\operatorname{cl}(\operatorname{dom} U)) = \mathbb{R}^d_+$. If, on the contrary, one has $\operatorname{ess\,inf}\|X_0\| > 0$, then, by property (iv) in the statement we have $\operatorname{ess\,sup}\|Y_0\| < \infty$. Assume that for some $x \in \operatorname{int}\mathbb{R}^d_+$, $U(x) = -\infty$, and choose a sequence $x_n \in \operatorname{dom} U$ such that $u_n := U(x_n)$ goes to $-\infty$ as $n \to +\infty$. Proceeding as in (2) above, for all $n \geq 1$ there exists some finite sequence $(\omega_i)_{i=1}^p$ (depending on the chosen n) such that

$$u_n > \langle Y_0(\omega_p), z_n - X_0(\omega_p)\rangle + \cdots + \langle Y_0(\omega_0), X_0(\omega_1) - X_0(\omega_0)\rangle.$$

By definition of U ad setting $\omega_{p+1} = \omega_0$, we have

$$\begin{aligned} U(X_0(\omega_0)) &\leq \langle Y_0(\omega_p), X_0(\omega_0) - X_0(\omega_{p+1})\rangle + \cdots + \langle Y_0(\omega_0), X_0(\omega_1) - X_0(\omega_0)\rangle \\ &\leq u_n + \langle Y_0(\omega_p), X_0(\omega_0) - z_n\rangle \\ &\leq u_n + M\|X_0(\omega_0) - z_n\|, \end{aligned}$$

where $M \geq \operatorname{ess\,sup}\|Y_0\|$. This leads to a contradiction since the RHS above goes to $-\infty$ as $n \to +\infty$ while the LHS stays finite.

(5) Let's take x and x' such that $x' > x$, i.e. $x' - x \in \operatorname{int}\mathbb{R}^d_+$. First, suppose that $X_0 \leq x'$ almost surely. In particular $\operatorname{ess\,sup}\|X_0\| < \infty$, and from property (iv) we have $\operatorname{ess\,inf}|Y_0^j| > 0$ for some j. As in (3), we can choose $\varepsilon > 0$ and $\omega \in A$ such that :

$$U(x) \leq U(x') + \varepsilon + \langle x - x', Y(\omega)\rangle.$$

Therefore, $\langle x' - x, Y_0(\omega)\rangle \geq (x'^j - x^j)\operatorname{ess\,inf} Y_0^j > 0$ a.s. so that $U(x) < U(x')$. Now, suppose that there exists $\omega \in A$ such that $X_0(\omega) \not\leq x'$. We can nonetheless choose $x' \geq x'' \geq x$ with $x'' = \langle \alpha, X_0(\omega) - x'\rangle$ for a suitable $\alpha \in \mathbb{R}^d$. Thus, we have

$$U(x'') \geq U(x) + \langle Y_0(\omega), x'' - x\rangle.$$

But $Y_0(\omega) \in \operatorname{int}\mathbb{R}^d_+$ (property (i) of this theorem) and consequently $\langle Y_0(\omega_0), x'' - x\rangle > 0$. We can conclude that $U(x') \geq U(x'') > U(x)$.

[‡]Since $Y_0(\omega) \in \partial U(X_0(\omega))$ for all $\omega \in A$, we have $\partial U(X_0(\omega)) \neq \emptyset$. Thus, the fact that $\operatorname{dom}(\partial U) \subset \operatorname{dom}(U)$ implies that $X_0(\omega) \in \operatorname{dom}(\partial U) \subset \operatorname{dom}(U)$, so that $U(X_0(\omega)) > -\infty$.

(6) Let $X \in \mathcal{A}_T^{x_0}$. Since, $Y_0 \in \partial U(X_0)$ almost surely, we have:

$$\mathbb{E}\left[U(X)\right] - \mathbb{E}\left[U(X_0)\right] \leq \mathbb{E}\left[Y_0(X - X_0)\right] \leq 0.$$

Therefore X_0 is efficient for U and x_0. ■

Remark 3.1 A careful inspection of the proof of the previous theorem reveals that the assumption of upper semicontinuity in the definition of $\mathcal{U}$ can be replaced by the minimal requirement of measurability.

Remark 3.2 In many papers on utility maximization under transaction costs (see, e.g., [2, 6, 19]), the agent liquidates his terminal portfolio to the first asset. As in the two papers [1, 3], one could include the slightly more general case of agents willing to liquidate their final portfolios to the first k assets with $k \leq d$. It suffices to consider utility functions of the form $U(x_1, \ldots, x_d) = U_k(x_1, \ldots, x_k)$. In this case, the characterization can be proved performing the same arguments with respect to the first k coordinates and using the facts that, in this case, an optimal portfolio X_0 is zero everywhere but in the first k component (which has been proved in [3]). Of course, the property of cyclic anticomonotonicity would hold between the first k coordinates of X_0 and the first k coordinates of at least one $Y = dm^c/d\mathbb{P}$ with $m = m^c + m^p \in \mathcal{D}$.

4 Utility price and inefficiency size

In this section, we study the quality of an admissible strategy starting from an initial portfolio $x_0 \in \mathcal{I}$ and leading to a final positive *not necessarily efficient* gain $X_0 \in \mathcal{A}_T^{x_0}$, whatever the preferences of the agent are. We would like to quantify the possible inefficiency of that strategy.

Using the duality result in Theorem 2.1 (super-replication theorem), we can evaluate as a first step the amount of portfolio x_0 needed to hedge the contingent claim X_0. To do that, consider a contingent claim X_0 and an initial nonzero portfolio $x_0 \in \mathbb{R}^d \setminus \{0\}$. We denote $\pi(X_0, x_0)$ the amount of portfolio x_0 allowing an agent to (super-)hedge X_0. The next lemma contains a dual characterization of it. Before that, we need to give one more definition. Let $\mathcal{D}^\perp(x_0)$ denote the subset of $\mathcal{D}$ given by

$$\mathcal{D}^\perp(x_0) := \{m \in \mathcal{D} : m^c(\Omega) \in K_0^*(x_0)\}, \tag{4.1}$$

where

$$K_0^*(x_0) := \left\{ z \in K_0^* : z = \frac{x_0}{\|x_0\|^2} + z^\perp, \langle z^\perp, x_0 \rangle = 0 \right\} \tag{4.2}$$

when x_0 is nonzero, otherwise $K_0^*(0) := K_0^*$. Notice that $K_0^*(x_0)$ is a closed convex non empty set and that whenever $x_0 \in \mathbb{R}_+^d$ we have $K_0^* = \mathrm{cone}(K_0^*(x_0))$.

Lemma 1 *The smallest amount of portfolio x_0 needed to hedge the contingent claim X_0 is given by*

$$\pi(X_0, x_0) = \sup_{m \in \mathcal{D}^\perp(x_0)} m^c(X_0). \tag{4.3}$$

As a consequence, $X_0 \in \mathcal{A}_T^{x_0}$ if and only if

$$\sup_{m \in \mathcal{D}^\perp(x_0)} m^c(X_0) \leq 1. \tag{4.4}$$

Proof. Let $\lambda > 0$ be a certain amount of portfolio x_0. Using Theorem (2.1), we have that X_0 can be hedged by the initial holdings vector λx_0, if and only if:

$$m^c(X_0) \leq \lambda \langle x_0, m^c(\Omega) \rangle \text{ for every } m \in \mathcal{D}. \tag{4.5}$$

We claim that the latter condition is equivalent to the following

$$m^c(X_0) \leq \lambda \langle x_0, m^c(\Omega)\rangle \text{ for every } m \in \mathcal{D}^\perp(x_0). \tag{4.6}$$

Clearly (4.5) implies (4.6). Assume (4.6) and let m be any measure in $\mathcal{D}$. Since $m^c(\Omega) \in K_0^*$ and $K_0^* = \text{cone}(K_0^*(x_0))$, we have $m^c(\Omega) = \beta z_0$ where $\beta \geq 0$ and $z_0 \in K_0^*(x_0)$. If $\beta = 0$ there is nothing to prove. Consider the case $\beta > 0$ and define the measure $m_\beta := m/\beta \in \mathcal{D}$. Moreover, $m_\beta \in \mathcal{D}^\perp(x_0)$ by construction. The claim is proved. To conclude the proof, notice that whenever $m \in \mathcal{D}^\perp(x_0)$ one has $m^c(\Omega) = x_0/\|x_0\|^2 + z^\perp$ with $\langle z^\perp, x_0\rangle = 0$, so that $\langle x_0, m^c(\Omega)\rangle = 1$. Thus, X_0 can be hedged by the initial portfolio λx_0 if and only if

$$\sup_{m \in \mathcal{D}^\perp(x_0)} m^c(X_0) \leq \lambda$$

which ends the proof. ■

Remark 4.1 Using property (ii) of Theorem 2.1 instead of (iii), we can obtain the following representation as well

$$\pi(X_0, x_0) = \sup_{Z \in \mathcal{Z}^\perp(x_0)} \mathbb{E}\left[Z_T X_0\right],$$

where $\mathcal{Z}^\perp(x_0) = \{Z \in \mathcal{Z} : Z_0 x_0 \in K_0^*(x_0)\}$.

We want to define a "universal" measure of portfolios' efficiency, in the sense that it does not depend on the preferences of the agents. We propose the following one, which generalizes the notion of "utility price" considered in, e.g., Jouini and Kallal [17]. Let us first introduce the set $\mathcal{B}^U(X_0)$ of all contingent claims which are better than X_0 for an agent with the utility function U (the notation $\mathcal{B}$ clearly standing for 'better'), i.e.

$$\mathcal{B}^U(X_0) = \left\{X \in L^0(\mathbb{R}^d_+) : \mathbb{E}\left[U(X)\right] \geq \mathbb{E}\left[U(X_0)\right]\right\}. \tag{4.7}$$

Notice that if $X' \sim X_0$ then $\mathcal{B}^U(X') = \mathcal{B}^U(X_0)$ for all utility functions $U \in \mathcal{U}$.

Definition 4.1 *Let $X_0 \in \mathcal{A}_T^{x_0}$ be an attainable contingent claim with $x_0 \in \mathcal{I}$. The* utility price *of X_0 with respect to the initial portfolio x_0, denoted $P^{\mathcal{U}}(X_0, x_0)$, is defined as the minimum percentage of x_0 needed for* any agent *to fund an admissible strategy in $\mathcal{A}$ giving at least the same expected utility as X_0, i.e.*

$$P^{\mathcal{U}}(X_0, x_0) := \sup_{U \in \mathcal{U}} \inf_{X \in \mathcal{B}^U(X_0)} \pi(X, x_0). \tag{4.8}$$

The inefficiency size *is defined as $I^{\mathcal{U}}(X_0, x_0) := 1 - P^{\mathcal{U}}(X_0, x_0) \in [0, 1]$.*

Remark 4.2 Notice that if X_0 is efficient for an initial portfolio x_0 and some utility function $U \in \mathcal{U}$, we have $\pi(X_0, x_0) = 1$ for all $X \in \mathcal{B}^U(X_0)$, implying that $P^{\mathcal{U}}(X_0, x_0) = 1$. Indeed, assume that $\pi(X_0, x_0) < 1$ for some $X \in \mathcal{B}^U(X_0)$. Since X_0 is efficient for some x_0 and $U \in \mathcal{U}$, we have that X_0 is a maximizer for an agent having utility function U and an initial portfolio x_0. Moreover, by definition of $\pi(X_0, x_0)$, the initial portfolio $\pi(X_0, x_0)x_0 < x_0$ leads to X as well. In other terms, the initial wealth x_0 may lead to the terminal portfolio $X_0 + (1 - \pi(X_0, x_0))x_0 > X$. Since U is strictly increasing, this contradicts the fact that X_0 is a maximizer. Thus, when X_0 is efficient its utility price $P^{\mathcal{U}}(X_0, x_0) = 1$. This justifies the use of $I^{\mathcal{U}}(X_0, x_0) = 1 - P^{\mathcal{U}}(X_0, x_0)$ as a measure of the possible inefficiency of a given terminal portfolio X_0.

It is natural in this framework to introduce the following set

$$\mathcal{B}(X_0) := \bigcap_{U \in \mathcal{U}} \mathcal{B}^U(X_0) = \left\{X \in L^0(\mathbb{R}^d_+) : \mathbb{E}\left[U(X)\right] \geq \mathbb{E}\left[U(X_0)\right], \forall U \in \mathcal{U}\right\}, \tag{4.9}$$

to describe the set of all (positive) contingent claims which are better than X_0 for all those agents whose preferences belong to $\mathcal{U}$.

We can now state the main result of this section, giving in particular in (ii) a dual representation of the utility price. For a subset A of $L^0(\mathbb{R}^d_+)$, we denote by $\overline{\text{conv}}(A)$ the closure (for the a.s. convergence) of the convex hull generated by A.

Theorem 4.1 *Let $x_0 \in \mathcal{I}$ be an initial portfolio and let X_0 be a contingent claim in $\mathcal{A}_T^{x_0}$. Moreover, assume that the c.d.f. of X_0 is continuous. Thus, the utility price of X_0 satisfies the following properties.*

(i) *There exists a contingent claim $\tilde{X}_0 \in \mathcal{B}(X_0)$ such that*

$$P^{\mathcal{U}}(X_0, x_0) = \min_{X \in \mathcal{B}(X_0)} \pi(X, x_0) = \pi(\tilde{X}_0, x_0). \tag{4.10}$$

Furthermore, $\tilde{X}_0 \in \overline{\text{conv}}\mathcal{L}(X_0)$.

(ii) *The utility price of X_0 with respect to the initial portfolio x_0 can be also computed as follows:*

$$P^{\mathcal{U}}(X_0, x_0) = \sup_{m \in \mathcal{D}^{\perp}(x_0)} P(X_0, m) \tag{4.11}$$

where

$$\begin{aligned} P(X_0, m) &= \min_{X \in \mathcal{B}(X_0)} m^c(X) && (4.12) \\ &= m^c(\tilde{X}_0^m) && (4.13) \end{aligned}$$

for some random vector $\tilde{X}_0^m \in \mathcal{L}(X_0)$ such that $\tilde{X}_0^m$ and $\frac{dm^c}{d\mathbb{P}}$ are cyclically anticomonotonic.

Before proving this theorem in the next section, let us comment the results obtained above and give some explanations. A first consequence of the theorem above is the existence of a contingent claim $\tilde{X}_0$ giving at least the same expected utility as the contingent claim X_0 for all utility functions $U \in \mathcal{U}$. However, since it belongs to the convex hull of all random variables having the same law as X_0, that contingent claim is not necessarily distributed as X_0, as in the case of complete and frictionless markets (see Dybvig [9]). This implies in particular that for some utility function $U \in \mathcal{U}$, expected utility of $\tilde{X}_0$ may be strictly bigger than the expected utility of X_0. This phenomenon has already been observed in Jouini and Kallal [17] in a less general frictional setting.

4.1 Proof of Theorem 4.1

We will prove Theorem 4.1 using the following preliminary lemmas.

Lemma 4.1 *Let $X_0 \in \mathcal{A}_T^{x_0}$ with $x_0 \in \mathcal{I}$ and let $m = m^c + m^s \in \mathcal{D}^{\perp}(x_0)$ with $Y := \frac{dm^c}{d\mathbb{P}} \in \operatorname{int} \mathbb{R}_+^d$ a.s.. Then there exists $\tilde{X}_0 \sim X_0$ such that $\tilde{X}_0$ and Y are cyclically anticomonotonic and*

$$\sup_{U \in \mathcal{U}} \inf_{X \in \mathcal{B}^U(X_0)} \mathbb{E}[YX] = \min_{X \in \mathcal{B}(X_0)} \mathbb{E}[YX] = \mathbb{E}\left[Y\tilde{X}_0\right]. \tag{4.14}$$

Proof. First note that we obviously have $\mathcal{B}(X_0) \subset \mathcal{B}^U(X_0)$ for all $U \in \mathcal{U}$, so that

$$\sup_{U \in \mathcal{U}} \inf_{X \in \mathcal{B}^U(X_0)} \mathbb{E}[XY] \leq \inf_{X \in \mathcal{B}(X_0)} \mathbb{E}[XY].$$

We are now going to prove the converse inequality and that the infimum in the RHS above is attained. By Proposition 3.2, we can choose $\tilde{X}_0 \sim X_0$ such that $\tilde{X}_0$ and Y are cyclically anticomonotonic and $\tilde{X}_0$ satisfies

$$\mathbb{E}[\tilde{X}_0 Y] = \min\{\mathbb{E}[YX] : X \in \mathcal{L}(X_0)\} =: \lambda_0,$$

so that the infimum in (4.14) is attained. Notice that $\lambda_0 \leq 1$ since $m \in \mathcal{D}^{\perp}(x_0)$ and $X_0 \in \mathcal{A}_T^{x_0}$, so that $m^c(X_0) = \mathbb{E}[YX_0] \leq 1$. At this point, we would be tempted to follow the second part of the proof of Theorem 3.2 to construct a utility function $U \in \mathcal{U}$ such that $\tilde{X}_0$ solves

$$\sup\{\mathbb{E}[U(X)] : X \in \mathcal{A}_T^{x_0}, \mathbb{E}[XY] \leq \lambda_0\}.$$

However, it may happen that $\tilde{X}_0$ and Y do not satisfy the condition (iv) of Theorem 3.2, i.e.

$$\operatorname{ess\,sup}\|Y\| = \infty \Rightarrow \operatorname{ess\,inf}\|\tilde{X}_0\| = 0, \quad \operatorname{ess\,inf}\|Y\| = 0 \Rightarrow \operatorname{ess\,sup}\|\tilde{X}_0\| = \infty. \tag{4.15}$$

Nevertheless, for any $\varepsilon > 0$ and for all $i = 1, \dots, d$, we can exhibit a random vector $\tilde{X}_\varepsilon$ satisfying the following properties

(i) $\tilde{X}_\varepsilon$ and Y are cyclically anticomonotonic and satisfy the properties (4.15) above;

(ii) $\mathbb{E}[Y\tilde{X}_0] - \varepsilon_0 \leq \mathbb{E}[Y\tilde{X}_\varepsilon] \leq \mathbb{E}[Y\tilde{X}_0]$, for some $0 < \varepsilon_0 \leq \varepsilon$.

In order to obtain such random variables, one may proceed as follows: for any $\varepsilon > 0$ and any $y \in \mathbb{R}^d_+$, define

$$\tilde{X}^i_{\varepsilon,y} := e^{-\varepsilon Y^i}\tilde{X}^i_0 \mathbf{1}_{\{\|Y\|<y\}} + \left(\tilde{X}_0 + \frac{\varepsilon}{Y^i}\right)\mathbf{1}_{\{\|Y\|\geq y\}}, \quad i = 1, \dots, d.$$

Consider now the function $y \mapsto \psi(y) := \mathbb{E}[Y(\tilde{X}_{\varepsilon,y} - \tilde{X}_0)]$ and notice that $\psi(y) \to \varepsilon > 0$ as $y \downarrow 0$, while $\psi(y)$ tends to a strictly negative value as $y \to +\infty$. Thus, one can choose $y = y(\varepsilon_0)$ such that property (ii) above is satisfied. It is then easy to check that, by construction, properties (i) above is fulfilled as well. Let us denote $\tilde{X}_\varepsilon := \tilde{X}_{\varepsilon,y(\varepsilon_0)}$.

Thus, even though $\tilde{X}_\varepsilon$ might not be attainable, we can nonetheless reproduce step-by-step the second part of the proof of Theorem 3.2, and find a utility function $U_\varepsilon \in \mathcal{U}$ such that $\tilde{X}_\varepsilon$ solves

$$\sup\left\{\mathbb{E}\left[U_\varepsilon(X)\right] : X \in \mathcal{A}^{x_0}_T, \mathbb{E}\left[XY\right] \leq \mathbb{E}[\tilde{X}_\varepsilon Y]\right\}.$$

We first deduce that

$$\sup_{U\in\mathcal{U}} \inf_{X\in\mathcal{B}^U(\tilde{X}_\varepsilon)} \mathbb{E}\left[XY\right] \geq \inf_{X\in\mathcal{B}^{U_\varepsilon}(\tilde{X}_\varepsilon)} \mathbb{E}\left[XY\right] = \mathbb{E}[Y\tilde{X}_\varepsilon].$$

Moreover, $\tilde{X}_0$ is a contingent claim preferred by each agent to $\tilde{X}_\varepsilon$, i.e. $\mathbb{E}[U(\tilde{X}_0)] \geq \mathbb{E}[U(\tilde{X}_\varepsilon)]$ for all $U \in \mathcal{U}$. Indeed, let $U \in \mathcal{U}$. Since $Y \in \partial U(\tilde{X}_0)$ (recall that Y and $\tilde{X}_0$ are cyclically anticomonotonic), we have

$$\mathbb{E}\left[U(\tilde{X}_\varepsilon)\right] - \mathbb{E}\left[U(\tilde{X}_0)\right] \leq \mathbb{E}\left[\langle Y, \tilde{X}_\varepsilon - \tilde{X}_0\rangle\right] \leq 0,$$

by the RHS in property (ii) above. Therefore $\mathcal{B}^U(\tilde{X}_0) \subset \mathcal{B}^U(\tilde{X}_\varepsilon)$ for all $U \in \mathcal{U}$, yielding

$$\sup_{U\in\mathcal{U}} \inf_{X\in\mathcal{B}^U(\tilde{X}_0)} \mathbb{E}\left[XY\right] \geq \sup_{U\in\mathcal{U}} \inf_{X\in\mathcal{B}^U(\tilde{X}_\varepsilon)} \mathbb{E}\left[XY\right] = \mathbb{E}\left[Y\tilde{X}_\varepsilon\right] \geq \mathbb{E}\left[Y\tilde{X}_0\right] - \varepsilon_0,$$

where the last inequality is due to the LHS in property (ii) above. Since $\mathcal{B}^U(\tilde{X}_0) = \mathcal{B}^U(X_0)$, for all $U \in \mathcal{U}$, we finally obtain

$$\sup_{U\in\mathcal{U}} \inf_{X\in\mathcal{B}^U(X_0)} \mathbb{E}\left[XY\right] = \sup_{U\in\mathcal{U}} \inf_{X\in\mathcal{B}^U(\tilde{X}_0)} \mathbb{E}\left[XY\right] \geq \min_{X\in\mathcal{B}(X_0)} \mathbb{E}\left[YX\right] - \varepsilon_0.$$

Letting ε (and so ε_0) tend to zero ends the proof. ■

Lemma 4.2 *Let $X_0 \in \mathcal{A}^{x_0}_T$ with $x_0 \in \mathcal{I}$. Then we have*

$$\inf_{X\in\mathcal{B}(X_0)} \pi(X, x_0) = \sup_{m\in\mathcal{D}^\perp(x_0)} \inf_{X\in\mathcal{B}(X_0)} m^c(X). \tag{4.16}$$

Proof. Let

$$\mathcal{D}^\perp_n(x_0) := \mathcal{D}^\perp(x_0) \cap \{m \in \mathrm{ba}(\mathbb{R}^d) : \|m\| \leq n\}, \quad n \geq 1.$$

The set $\mathcal{D}^\perp_n(x_0)$ is a nonempty, convex, $\sigma(\mathrm{ba}(\mathbb{R}^d), L^\infty(\mathbb{R}^d))$-compact subset of $\mathrm{ba}(\mathbb{R}^d)$ (by Alaoglu's theorem). On the other hand the set $\mathcal{B}(X_0)$ is nonempty and convex as well. Furthermore, $m \mapsto m^c(X)$ is weak*

continuous on the set $\mathcal{D}_n^{\perp}(x_0)$ and $X \mapsto m^c(X)$ is a linear and so convex function. Thus we can apply the minmax theorem as in, e.g., [29] (Theorem 2.10.2), yielding

$$\sup_{m\in\mathcal{D}_n^{\perp}(x_0)} \inf_{X\in\mathcal{B}(X_0)} m^c(X) = \inf_{X\in\mathcal{B}(X_0)} \sup_{m\in\mathcal{D}_n^{\perp}(x_0)} m^c(X). \tag{4.17}$$

Furthermore, since $\mathcal{D}_n^{\perp}(x_0) \subset \mathcal{D}^{\perp}(x_0)$ for all $n \geq 1$, it is easy to see that

$$\lim_{n\to+\infty} \sup_{m\in\mathcal{D}_n^{\perp}(x_0)} \inf_{X\in\mathcal{B}(X_0)} m^c(X) \leq \sup_{m\in\mathcal{D}^{\perp}(x_0)} \inf_{X\in\mathcal{B}(X_0)} m^c(X). \tag{4.18}$$

Consider now the function $\pi_{x_0}^n$, defined by

$$\pi_{x_0}^n(X) := \sup_{m\in\mathcal{D}_n^{\perp}(x_0)} m^c(X).$$

Using (4.17) and (4.18), we obtain therefore that

$$\lim_{n\to+\infty} \inf_{X\in\mathcal{B}(X_0)} \pi_{x_0}^n(X) \leq \sup_{m\in\mathcal{D}^{\perp}(x_0)} \inf_{X\in\mathcal{B}(X_0)} m^c(X). \tag{4.19}$$

Now, we study the convergence of $\inf_{X\in\mathcal{B}(X_0)} \pi_{x_0}^n(X)$ as n goes to infinity. We can find a minimizing sequence X_n in $\mathcal{B}(X_0)$ (and so $\mathbb{R}_+^d$-valued) such that

$$\lim_{n\to+\infty} \inf_{X\in\mathcal{B}(X)} \pi_{x_0}^n(X) = \lim_{n\to+\infty} \pi_{x_0}^n(X_n).$$

Since X_n is a sequence in $\mathcal{A}_T^{x_0}$ with values in $\mathbb{R}_+^d$, it is also positive for the preorder induced by the random cone K_T. Thus we can apply Lemma 3.2 in Campi and Schachermayer [4] implying that the sequence X_n is bounded in $L^1(\mathbb{Q})$ for some probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{F}_T$. Thanks to Komlòs Theorem (see, e.g., Theorem 5.2 in [20] or [14]), we can find a sequence $\widehat{X}_n \in \text{conv}\{X_n, X_{n+1}, ...\}$ still in $\mathcal{B}(X_0)$ (which is convex) such that $\widehat{X}_n \to \widehat{X}$ a.s. under $\mathbb{Q}$ as well as under $\mathbb{P}$, being the two measures equivalent. To prove that $\widehat{X}$ belongs to $\mathcal{B}(X_0)$ we use the upper semicontinuity of each utility functions $U \in \mathcal{U}$ getting that $U(\widehat{X}) \geq \limsup_n U(\widehat{X}_n)$. We take expectation under $\mathbb{P}$ and we obtain $\mathbb{E}[U(\widehat{X})] \geq \mathbb{E}[\limsup_n U(\widehat{X}_n)]$. Fatou's lemma yields $\mathbb{E}[U(\widehat{X})] \geq \limsup_n \mathbb{E}[U(\widehat{X}_n)]$. Finally, the concavity of U gets $\mathbb{E}[U(\widehat{X}_n)] \geq \mathbb{E}[U(X_0)]$ and so $\mathbb{E}[U(\widehat{X})] \geq \mathbb{E}[U(X_0)]$ for all $U \in \mathcal{U}$, so that $\widehat{X} \in \mathcal{B}(X_0)$. Moreover, since $\mathcal{A}_T^{x_0}$ is Fatou closed (Theorem 3.5 in [4]), we also have $\widehat{X} \in \mathcal{A}_T^{x_0}$.

We then obtain:

$$\pi_{x_0}^n(\widehat{X}_n) \leq \sup_{m\geq n} \pi_{x_0}^n(X_m) \leq \sup_{m\geq n} \pi_{x_0}^m(X_m).$$

Thus, we have

$$\inf_{k\geq n} \sup_{m\in\mathcal{D}_n^{\perp}(x_0)} m^c(\widehat{X}_k) \geq \sup_{m\in\mathcal{D}_n^{\perp}(x_0)} \inf_{k\geq n} m^c(\widehat{X}_k) \geq \sup_{Y\in\mathcal{D}_n^{\perp}(x_0)} \mathbb{E}\left[\inf_{k\geq n} \frac{dm^c}{d\mathbb{P}} X_k\right].$$

and, therefore by the theorem of monotone convergence we have[§]

[§]Let $m \in \mathcal{D}_n^{\perp}(x_0)$, $n \geq 1$. One has

$$m^c(\inf_{k\geq n} \widehat{X}^k) \leq m^c(\widehat{X}^n) \leq \sum_{i\geq n} \alpha_i m^c(X^i) \leq m^c(X^n).$$

Taking supremum over $\mathcal{D}_n^{\perp}(x_0)$, one has

$$\pi^n(X^n) \geq \sup_{m\in\mathcal{D}_n^{\perp}(x_0)} m^c\left(\inf_{k\geq n} \widehat{X}^k\right).$$

Take the limit and conclude.

$$\lim_{n\to+\infty} \pi^n_{x_0}(YX_n) \geq \lim_{n\to+\infty} m^c\left(\inf_{k\geq n} \widehat{X}_k\right) = \sup_{m\in\mathcal{D}^\perp(x_0)} m^c(\widehat{X}).$$

Since $\widehat{X} \in \mathcal{B}(X_0)$, by (4.19) we have that

$$\inf_{X\in\mathcal{B}(X_0)} \sup_{m\in\mathcal{D}^\perp(x_0)} m^c(X) \leq \sup_{m\in\mathcal{D}^\perp(x_0)} m^c(\widehat{X}),$$

so getting

$$\inf_{X\in\mathcal{B}(X_0)} \sup_{m\in\mathcal{D}^\perp(x_0)} m^c(X) \leq \sup_{m\in\mathcal{D}^\perp(x_0)} \inf_{X\in\mathcal{B}(X_0)} m^c(X).$$

The other inequality is straightforward and the duality result (4.16) is proved. ■

Now, we can conclude the proof of Theorem 4.1 using the previous two lemmas. First, observe that using (4.16) we have

$$P^{\mathcal{U}}(X_0, x_0) = \sup_{U\in\mathcal{U}} \inf_{X\in\mathcal{B}^U(X_0)} \pi(X, x_0) = \sup_{U\in\mathcal{U}} \sup_{m\in\mathcal{D}^\perp(x_0)} \inf_{X\in\mathcal{B}^U(X_0)} m^c(X).$$

Therefore

$$P^{\mathcal{U}}(X_0, x_0) \geq \sup_{m\in\mathcal{D}^{\perp,*}(x_0)} \sup_{U\in\mathcal{U}} \inf_{X\in\mathcal{B}^U(X_0)} m^c(X),$$

where $\mathcal{D}^\perp_+(x_0) = \{m \in \mathcal{D}^\perp(x_0) : \frac{dm^c}{d\mathbb{P}} \in \operatorname{int} \mathbb{R}^d_+\}$. From (4.14), we have

$$\sup_{U\in\mathcal{U}} \inf_{X\in\mathcal{B}^U(X_0)} m^c(X) = \min_{X\in\mathcal{B}(X_0)} m^c(X)$$

for $m \in \mathcal{D}^\perp(x_0)$ and therefore

$$P^{\mathcal{U}}(X_0, x_0) \geq \sup_{m\in\mathcal{D}^\perp_+(x_0)} \min_{X\in\mathcal{B}(X_0)} m^c(X).$$

Moreover, let $m' \in \mathcal{D}^\perp_+(x_0)$, $m \in \mathcal{D}^\perp(x_0)$, and $m_\lambda = \lambda m + (1-\lambda)m'$ for $\lambda \in [0,1)$. We have $m_\lambda \in \mathcal{D}^\perp_+(x_0)$ and for all $X \in \mathcal{B}(X_0)$, $\lim_{\lambda\uparrow 1} m^c_\lambda(X) = m^c(X)$. We deduce that:

$$\sup_{m\in\mathcal{D}^\perp(x_0)} \min_{X\in\mathcal{B}(X_0)} m^c(X) = \sup_{m\in\mathcal{D}^\perp_+(x_0)} \min_{X\in\mathcal{B}(X_0)} m^c(X).$$

Now, we can use equality (4.16) to obtain

$$P^{\mathcal{U}}(X_0, x_0) \geq \inf_{X\in\mathcal{B}(X_0)} \sup_{m\in\mathcal{D}^\perp(x_0)} m^c(X).$$

The reverse inequality is straightforward and we conclude that

$$\begin{aligned} P^{\mathcal{U}}(X_0, x_0) &= \inf_{X\in\mathcal{B}(X_0)} \sup_{m\in\mathcal{D}^\perp(x_0)} m^c(X) \\ &= \inf_{X\in\mathcal{B}(X_0)} \pi(X, x_0) \\ &= \min\left\{\pi(X, x_0) : X \in \overline{\operatorname{conv}}\mathcal{L}(X_0)\right\}, \end{aligned}$$

i.e. we just proved property (i) in this theorem (recall that the closure above refers to a.s. convergence). Finally, we can deduce from (4.1) that

$$P^{\mathcal{U}}(X_0, x_0) = \sup_{m\in\mathcal{D}^\perp(x_0)} \min_{X\in\mathcal{B}(X_0)} m^c(X)$$

which gives property (ii) after applying (4.16) once more. The proof of the theorem is now complete.

A Appendix

This appendix collects the proofs of some technical results that have been used throughout the paper.

Proof of Proposition 3.1. It is straightforward that cyclic anticomonotonicity implies (3.3). Conversely, assume that X and Y satisfy property (3.3). To obtain that X and Y are cyclically anticomonotonic, it suffices to prove that there exist two random vectors X', Y' with $X' = X$ and $Y' = Y$ a.s., such that X' and Y' are cyclically anticomonotonic. Denote $X = (X^1, \dots, X^d)$ and $Y = (Y^1, \dots, Y^d)$, and consider the product set for $n \in \mathbb{N}^*$:

$$\mathbb{N}^d_{n2^n} := \{1, \dots, n2^n\} \times \dots \times \{1, \dots, n2^n\}.$$

Then, for each vector of integers $I = (I^1, \dots, I^d)$ and $K = (K^1, \dots, K^d)$ in $\mathbb{N}^d_{n2^n}$, define the sets A_I and B_K by

$$A_I := \bigcap_{1 \le i \le d} \left\{ \frac{I^i}{2^n} \le X^i < \frac{I^i + 1}{2^n} \right\}, \quad B_K := \bigcap_{1 \le j \le d} \left\{ \frac{I^j}{2^n} \le Y^j < \frac{I^j + 1}{2^n} \right\}.$$

in such a way that the family $\{A_I \cap B_K : (I, K) \in (N^d_{n2^n})^2\}$ is a partition of the set $\{(X, Y) \in [0, n]^d \times [0, n]^d\}$. The rest of the proof is structured in three main steps.

Step 1: construction of X'. We define

$$X^i_n := \sum_{(I,K) \in P_n} \frac{I^i}{2^n} \mathbf{1}_{A_I \bigcap B_K}$$

with $P_n := \left\{ (I, K) \in \left(\mathbb{N}^d_{n2^n}\right)^2 : \mathbb{P}\left(A_I \cap B_K\right) > 0 \right\}$. Defining

$$N_n := \left(\bigcup_{(I,K) \in P_n} (A_I \cap B_K) \right)^C$$

one can see that when $\omega \notin N_n$, we have $\|X^n(\omega) - X(\omega)\| \le 2^{-n}$ for every $n \in \mathbb{N}$. Thus, for every $\omega \in \Omega$, X^n converges and we can define its limit by setting

$$X'(\omega) := \lim_{n \to +\infty} X_n(\omega).$$

By construction, $X' = X$ a.s., since $\mathbb{P}(N_n) \to 0$. Indeed, one has $\mathbb{P}(N_n) = \mathbb{P}(X \ge n\mathbf{1}, Y \ge n\mathbf{1})$ which goes to 0 as $n \to +\infty$, being both X, Y finite valued.

Step 2: construction of Y'. In the same way, we define

$$Y^j_n = \sum_{(I,K) \in P_n} \frac{K^j}{2^n} \mathbf{1}_{A_I \bigcap B_K}.$$

If $\omega \notin N_n$, we have $\|Y^n(\omega) - Y(\omega)\| \le 2^{-n}$ for every $n \in \mathbb{N}$. Therefore, for every ω, Y^n converges and we can define its limit setting

$$Y'(\omega) := \lim_{n \to +\infty} Y_n(\omega)$$

yielding that $Y' = Y$ almost surely.

Step 3: conclusion. The two sequences Y_n and X_n have been constructed in order to have cyclically anticomonotonic random vectors in the limit. Indeed, let $\Omega_n = \Omega \setminus N_n$ and $p \in \mathbb{N}^*$. For $(\omega_1, \omega_2) \in \Omega_n^2$, we

have

$$\begin{aligned}\langle Y^n(\omega_1), X^n(\omega_1) - X^n(\omega_2)\rangle - \langle Y(\omega_1), X(\omega_1) - X(\omega_2)\rangle &= \langle Y^n(\omega_1) - Y(\omega_1), X^n(\omega_1) - X^n(\omega_2)\rangle \\ &\quad + \langle Y(\omega_1), X^n(\omega_1) - X(\omega_1)\rangle \\ &\quad - \langle Y(\omega_1), X^n(\omega_2) - X(\omega_2)\rangle \\ &\leq \frac{n}{2^n} + \frac{n}{2^n} + \frac{n}{2^n}.\end{aligned}$$

Thus, for every $(\omega_1, \ldots, \omega_{p+1}) \in \Omega_n^{p+1}$ with $\omega_{p+1} = \omega_1$, we have

$$\sum_{i=1}^{p} \langle Y^n(\omega_i), X^n(\omega_i) - X^n(\omega_{i+1})\rangle - \sum_{i=1}^{p} \langle Y(\omega_i), X(\omega_i) - X(\omega_{i+1})\rangle \leq 3p\frac{n}{2^n}. \tag{A.1}$$

Consider now the function

$$g_n(\omega_1, \ldots, \omega_p) := \langle Y_n(\omega_1), X_n(\omega_1) - X_n(\omega_2)\rangle + \cdots + \langle Y_n(\omega_p), X_n(\omega_p) - X_n(\omega_1)\rangle.$$

It is constant on the set $\times_{i=1}^{p} A_{I^i} \cap B_{K^i}$, for $(I^i, K^i) \in P_n$, $i = 1, \ldots, p$. Thus, we necessarily have

$$g_n(\omega_1, \ldots, \omega_p) \leq 3p\frac{n}{2^n}.$$

Indeed, if it was not the case, equation (A.1) would yield that for some $(I^i, K^i) \in P_n$, $i = 1, \ldots, p$, and for all $(\omega_1, \ldots, \omega_p) \in \times_{i=1}^{p} A_{I^i} \cap B_{K^i}$ – a strictly positive probability set, by construction – the following inequality holds:

$$\langle Y(\omega_1), X(\omega_1) - X(\omega_2)\rangle + \cdots + \langle Y(\omega_p), X(\omega_p) - X(\omega_1)\rangle > 0$$

which contradicts equation (3.3). We conclude that for every $(\omega_1, \ldots, \omega_p) \in A^p$, where $A := \Omega \setminus \cap_{n\geq 0} N_n$, we have

$$\langle Y'(\omega_1), X'(\omega_1) - X'(\omega_2)\rangle + \cdots + \langle Y'(\omega_p), X'(\omega_p) - X'(\omega_1)\rangle \leq 0$$

with $\mathbb{P}(A) = 1 - \lim_n \mathbb{P}(N_n) = 1$. ■

Proof of Proposition 3.2. First, distinguish between two cases: If for all $X \in \mathcal{L}(X_0)$ (implying in particular that $X \in L^0(\mathbb{R}_+^d)$) we have $\mathbb{E}[XY] = +\infty$, there is nothing to prove. Let us turn to the second case when there exists at least one $X \in \mathcal{L}(X_0)$ such that $\mathbb{E}[XY] < \infty$, so that the infimum in (3.5) is finite.

In the following, we denote C_{X_0} a copula of X_0, and C_Y a copula of Y (we refer to Nelson's book [23] for details on copulas). Consider now the set $\mathcal{C}(X_0, Y)$ of all copulas on $\mathbb{R}^{2d}$, such that for every $C \in \mathcal{C}(X_0, Y)$, the marginal copula of the d first variables is C_{X_0}, and the marginal copula of the d last variables is C_Y, i.e.

$$\begin{aligned}C(u_1, u_2, \ldots, u_d, 1, \ldots, 1) &= C_X(u_1, u_2, \ldots, u_d), \\ C(1, \ldots, 1, v_1, v_2, \ldots, v_d) &= C_Y(u_1, u_2, \ldots, u_d).\end{aligned}$$

It is straightforward to see that the set $\mathcal{C}(X_0, Y)$ is closed with respect to the topology of pointwise convergence on $\mathcal{C}$, the set of all possible copulas on $\mathbb{R}^{2d}$. Furthermore the set $\mathcal{C}$ is compact with respect to this topology (see Deheuvels [7], Theorem 2.3). Thus the set $\mathcal{C}(X_0, Y)$ is itself compact with respect to the topology of pointwise convergence. Let X_n be a sequence in $\mathcal{L}(X_0)$ such that

$$\lim_{n\to+\infty} \mathbb{E}(YX_n) = \inf\{\mathbb{E}(YX) \mid X \in (X_0)\},$$

and let C_n denote the copula of (X_n, Y). Since $\mathcal{C}(X_0, Y)$ is compact, we can assume w.l.o.g. (up to extracting subsequences) that the sequence $C_n \in \mathcal{C}(X_0, Y)$ converges pointwise to a copula $C \in \mathcal{C}(X_0, Y)$. Consider

the random vector $\tilde{X}_0$ such that the copula of $(\tilde{X}_0, Y)$ is C. In particular, $\tilde{X}_0 \in \mathcal{L}(X_0)$. Notice that

$$\begin{aligned}
\mathbb{E}\left[\langle Y, X_n\rangle\right] &= \sum_{i=1}^{d} \mathbb{E}\left[X_n^i Y^i\right] = \sum_{i=1}^{d} \int_0^\infty \int_0^\infty \mathbb{P}(X_n^i > t, Y^i > u) dt du \\
&= \sum_{i=1}^{d} \int_0^\infty \int_0^\infty 1 - \mathbb{P}\left(X_0^i \leq t\right) - \mathbb{P}\left(Y^i \leq u\right) - C_n^i\left(\mathbb{P}\left(X_0^i \leq t\right), \mathbb{P}\left(Y^i \leq u\right)\right) dt du.
\end{aligned}$$

where C_n^i denotes the marginal copula of the vector (X_n^i, Y^i) and where we used the fact that X_n^i has the same law as X_0^i for all i. By the pointwise convergence of C_n to the copula C of $(\tilde{X}_0, Y)$, we deduce that, for every $u \geq 0$, $t \geq 0$ and $i \in \{1, \ldots, d\}$, we have

$$\lim_{n \to +\infty} C_n^i\left(\mathbb{P}\left(X_0^i \leq t\right), \mathbb{P}\left(Y^i \leq u\right)\right) = C^i\left(\mathbb{P}\left(X_0^i \leq t\right), \mathbb{P}\left(Y^i \leq u\right)\right)$$

with C^i is the marginal copula of the vector $(\tilde{X}_0^i, Y^i)$. Therefore by Fatou's lemma and the equation (A.2), we have

$$\begin{aligned}
\mathbb{E}\left[\langle \tilde{X}_0, Y\rangle\right] &= \sum_{i=1}^{d} \int_0^\infty \int_0^\infty 1 - \mathbb{P}\left(X_0^i \leq t\right) - \mathbb{P}\left(Y^i \leq u\right) - C^i\left(\mathbb{P}\left(X_0^i \leq t\right), \mathbb{P}\left(Y^i \leq u\right)\right) dt du \\
&\leq \liminf_{n \to +\infty} \mathbb{E}\left[\langle X_n, Y\rangle\right] \leq \min\{\mathbb{E}\left[XY\right] : X \in \mathcal{L}(X_0)\}
\end{aligned}$$

and thus, $\mathbb{E}[Y\tilde{X}_0] = \min\{\mathbb{E}[YX] \mid X \in \mathcal{L}(X_0)\}$.

Now, let us prove that $\tilde{X}_0$ and Y are cyclically anticomonotonic. Suppose that this is not the case. Thus, by Corollary (3.1) there exist $\varepsilon > 0$, $p \geq 1$ and some non negligible measurable sets $\Omega_1, \ldots, \Omega_p$ such that for all $(\omega_1, \ldots, \omega_p) \in \Omega_1 \times \cdots \times \Omega_p$, we have

$$\langle \tilde{X}_0(\omega_1) - \tilde{X}_0(\omega_2), Y(\omega_1)\rangle + \cdots + \langle \tilde{X}_0(\omega_p) - \tilde{X}_0(\omega_1), Y(\omega_p)\rangle \geq \varepsilon$$

and, being the space $(\Omega, \mathcal{F}, \mathbb{P})$ atomless, we can choose the sets $\Omega_1, \ldots, \Omega_p$ in such a way that $\mathbb{P}(\Omega_1) = \mathbb{P}(\Omega_2) = \cdots = \mathbb{P}(\Omega_p)$. Consider a random vector X', distributed as $\tilde{X}_0$ with

$$\begin{cases} X'_{|\Omega \setminus \cup_{i=1}^p \Omega_i} = (\tilde{X}_0)_{|\Omega \setminus \cup_{i=1}^p \Omega_i} \\ X'_{|\Omega_i} \sim (\tilde{X}_0)_{|\Omega_{i+1}} \text{ for } 1 \leq i \leq p \end{cases}$$

with the convention $\Omega_{p+1} = \Omega_1$.[¶] A consequence of such a construction is that $X'(\Omega_i) = \tilde{X}_0(\Omega_{i+1})$ a.s. for all $i = 1, \ldots, p$. Since X' and $\tilde{X}^0$ coincide on $\Omega \setminus \cup_{i=1}^p \Omega_i$ we have

$$\mathbb{E}\left[\langle X', Y\rangle\right] - \mathbb{E}\left[\langle \tilde{X}_0, Y\rangle\right] = \sum_{i=1}^{p} \mathbb{E}\left[\langle Y, (X' - \tilde{X}_0)_{|\Omega_i}\rangle\right].$$

Moreover, we have by construction that $\sum_{i=1}^p \langle Y, (X' - \tilde{X}_0)_{|\Omega_i}\rangle \leq -\varepsilon$, which implies

$$\mathbb{E}\left[\langle X', Y\rangle\right] - \mathbb{E}\left[\langle \tilde{X}_0, Y\rangle\right] \leq -p\varepsilon.$$

As a consequence, we have $\mathbb{E}\left[X'Y\right] < \mathbb{E}[\tilde{X}_0 Y]$, so that $\tilde{X}_0$ cannot be the minimizer. This is clearly a contradiction and ends the proof. ■

[¶] A construction of such a random vector X' goes as follows: take a random variable U with uniform distribution on $(0, 1)$ and set $X' = \tilde{X}_0$ on Ω_0 and, for $i = 1, \ldots, n$, $X'_{|\Omega_i} = F_{i+1}^{-1}(U_i)$ where U_i is the restriction of U on Ω_i and F_{i+1} is the c.d.f. of the restriction of $\tilde{X}_0$ on Ω_{i+1}. It is easy to verify that X' satisfies the properties listed above. Notice that to perform such a construction we need the assumption that X_0 has a continuous c.d.f..

References

[1] G. Benedetti and L. Campi, Multivariate utility maximization with proportional transaction costs and random endowment. Preprint, 2010. Available at HAL: http://hal.archives-ouvertes.fr/hal-00586377

[2] B. Bouchard, Utility Maximization on the Real Line under Proportional Transaction Costs, *Finance Stochast.*, **6** (2002), 495–516.

[3] L. Campi and M. P. Owen, Multivariate utility maximization with proportional transaction costs. *Finance Stochast.*, **15** (2010), 461–499.

[4] L. Campi and W. Schachermayer, A super-replication theorem in Kabanov's model of transaction costs. *Finance Stochast.*, **10** (2006), 579–596.

[5] G. Carlier, R.-A. Dana and A. Galichon, Pareto efficiency for the concave order and multivariate comonotonicity. *Journal of Economic Theory* (2009), forthcoming.

[6] G. Deelstra, H. Pham and N. Touzi, *dual formulation of the utility maximization problem under transaction costs*, Ann. Appl. Probab. **11** (2001), 1353–1383.

[7] P. Deheuvels, Caractérisation complète des lois extrèmes multivariées et de la convergence des types extrèmes. *Pub. Inst. Stat. Univ. Paris* 23, 3-4 (1978), 1-36.

[8] J. Diestel and J. J. Uhl, *Vector measures*, Mathematical surveys, no. 15, American Mathematical Society, Providence, Rhode Island, 1977.

[9] P.H. Dybvig, Distributional analysis of portfolio choice. *Journal of Business* 63, 3 (1988), 369-393.

[10] P.H. Dybvig, Inefficient dynamic portfolio strategies or how to throw away a million dollars in the stock markets. *Review of Financial Studies* 1, 1 (1988), 67-88.

[11] I. Ekeland, A. Galichon and M. Henry, Comonotonic measures of multivariate risks. *Mathematical Finance* (2009), forthcoming. Available at SSRN: http://ssrn.com/abstract=1115729

[12] I. Ekeland and W. Schachermayer, Law invariant risk measures on $L^\infty(\mathbb{R}^d)$. *Statistics and Decisions* (2011), forthcoming.

[13] P. Guasoni, M. Rásonyi and W. Schachermayer, The fundamental theorem of asset pricing for continuous processes under small transaction costs. *Annals of Finance* 6, 2 (2010), 157-191.

[14] P. Hall and C. C. Heyde. Martingale limit theory and its application. Probability and Mathematical Statistics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.

[15] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*, 2nd Ed., Cambridge University Press, 1952.

[16] E. Jouini and H. Kallal, Martingales and arbitrage in securities markets with transaction costs. *J. Econ. Theory*, **66** (1995), 178–197.

[17] E. Jouini H. and H. Kallal, Efficient trading strategies in the presence of market frictions. *The review of Financial Studies* 14, 2 (2001), 343-369.

[18] E. Jouini and V. Portes, Efficient trading strategies. Preprint, 2005. Available at SSRN: http://ssrn.com/abstract=1000208

[19] Yu. M. Kabanov, Hedging and liquidation under transaction costs in currency markets, *Finance Stochast.* **3** (1999), 237–248.

[20] Yu. M. Kabanov and M. Safarian, *Markets with Transaction Costs: Mathematical Theory*, Springer, Berlin Heidelberg, 2009.

[21] D. Kramkov and W. Schachermayer, The asymptotic elasticity of utility functions and optimal investment in incomplete markets, *Ann. Appl. Probab.* **9** (1999), 904–950.

[22] D. G. Luenberger, *Optimization by vector space methods*, Wiley, New York, 1969.

[23] R.B. Nelson. *An Introduction to Copulas* (Lecture Notes in Statistics No. 139). Springer, New York, 1999.

[24] G. Puccetti and M. Scarsini, Multivariate comonotonicity. *Journal of Multivariate Analysis*, 101 (2010), 291-304.

[25] K. P. S. B. Rao and M. B. Rao, *Theory of charges: a study of finitely additive measures*, Academic Press, London, 1983.

[26] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1972.

[27] L. Ruschendorf and S. Kiesel, On optimal allocation of risk vectors. *Insurance: Mathematics and Economics* **47** (2010), 167-175.

[28] W. Schachermayer, The Fundamental Theorem of Asset Pricing under Proportional Transaction Costs in Finite discrete Time, *Mathematical Finance* **14** (2004), No. 1, 19-48.

[29] C. Zălinescu. *Convex analysis in general vector spaces*, World Scientific, New Jersey, 2002.