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Consistent estimation of a mean pattern in deformable models for high-dimensional shape analysis

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Abstract

We consider the problem of estimating a mean shape from a set of J planar configurations described by a sequence of k landmarks. We study the consistency of a smoothed Procrustean mean when the observations obey a deformable model including some nuisance parameters such as random translations, rotations and scaling. The main contribution of the paper is to analyze the influence of the dimension k of the data and of the number J of observed configurations on the convergence of the smoothed Procrustean estimator to the mean pattern of the model. Some numerical experiments illustrate these results.

Keywords : Shape spaces; Perturbation models; Consistency; Procrustes Means; Nonparametric inference; High-dimensional data; Linear smoothing.

AMS classifications: Primary 62G08; secondary 62H11.

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1 Introduction

1.1 A deformable model for statistical shape analysis

Statistical analysis of planar shapes is the study of random (planar) configurations

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_k \end{pmatrix} \in \mathbb{R}^{k \times 2}$$

described by a set of k landmarks $Y_\ell = (Y_\ell^{(1)}, Y_\ell^{(2)}) \in \mathbb{R}^2$, $\ell = 1, \dots, k$. Since the seminal work of Kendall [Ken84], one considers that the shape of \mathbf{Y} is “what remains when translations, rotations and scaling are filtered out”. More precisely, two configurations $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathbb{R}^{k \times 2}$ are said to have the same shape if there exists a vector $(a, \alpha, b) \in \mathbb{R} \times [-\pi, \pi] \times \mathbb{R}^2$ such that

$$\mathbf{Y}_2 = e^a \mathbf{Y}_1 R_\alpha + \mathbf{1}_k \otimes b, \quad \text{with } R_\alpha = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \quad (1.1)$$

where $\mathbf{1}_k$ denotes the vector of \mathbb{R}^k with all entries equal to one, \otimes denotes the tensor product and $b = (b^{(1)}, b^{(2)})$ is a row vector.

In shape analysis, an important issue is the computation of a sample mean shape from a set of J random planar configurations $\mathbf{Y}_1, \dots, \mathbf{Y}_J$, and the study of its consistency as the number of samples J goes to infinity. A statistical model of shapes must include some nuisance parameters associated to the ambiguity of location, rotation and scaling. In [Goo91], consistent estimation of a mean shape is therefore considered in the following deformable model:

$$\mathbf{Y}_j = e^{a_j^*}(\mathbf{f} + \boldsymbol{\zeta}_j)R_{\alpha_j^*} + \mathbf{1}_k \otimes b_j^*, \quad \text{with } R_{\alpha_j^*} = \begin{pmatrix} \cos(\alpha_j^*) & -\sin(\alpha_j^*) \\ \sin(\alpha_j^*) & \cos(\alpha_j^*) \end{pmatrix}, \quad \text{and } j = 1, \dots, J, \quad (1.2)$$

where the mean pattern $\mathbf{f} \in \mathbb{R}^{k \times 2}$ is an unknown configuration of k landmarks which is also called a population mean in [Goo91] or a perturbation mean in [Huc11]. The error terms $\boldsymbol{\zeta}_j \in \mathbb{R}^{k \times 2}$, $j = 1, \dots, J$ are independent copies of a random perturbation $\boldsymbol{\zeta}$ in $\mathbb{R}^{k \times 2}$ with zero expectation. For $j = 1, \dots, J$, the scaling, rotation and translation parameters $(a_j^*, \alpha_j^*, b_j^*) \in \mathbb{R} \times [-\pi, \pi] \times \mathbb{R}^2$ are independent and identically distributed (i.i.d) random variables independent of the random perturbations $\boldsymbol{\zeta}_j$.

According to Goodall [Goo91] a sample mean pattern $\hat{\mathbf{f}}$ computed from $\mathbf{Y}_1, \dots, \mathbf{Y}_J$ is said to be consistent if, as $J \rightarrow \infty$, it has asymptotically the same shape than the mean pattern \mathbf{f} . Since Goodall's proposal, the deformable model (1.2) has been highly popular in the statistical shape community, and the study of consistent procedures to estimate the shape of the mean pattern \mathbf{f} using this model has been considered by various authors [KM97, Le98, Lel93, Huc11]. In this setting, sample mean patterns obtained by a Procrustes procedure have received a special attention. In particular, it is shown in [KM97, Le98] that, in the very specific case of isotropic perturbations $\boldsymbol{\zeta}_j$, the so-called full and partial Procrustes sample means are consistent estimators of the shape of \mathbf{f} . Nevertheless, these estimators can be inconsistent for non-isotropic perturbations. Therefore, it is generally the belief that consistent statistical inference based on Procrustes analysis is restricted to very limited assumptions on the distribution of the data, see also [DM98, KBCL99, Huc11] for further discussions.

The aim of this paper is to show that a Procrustes sample mean can be considered as a consistent procedure even in the case of non-isotropic perturbations. To this purpose, we propose to exhibit the relation that exists between the dimensionality k of the data and the consistency of smoothed Procrustes sample means in the perturbation model (1.2). Our main result (see Theorem 1.1 below) is that when the dimensionality k is high and the mean pattern \mathbf{f} is the discretization of a sufficiently smooth plane curve, it is possible to build a consistent estimator of the shape of \mathbf{f} in model (1.2) under general assumptions on the perturbation $\boldsymbol{\zeta}_j$. The problem of analyzing high-dimensional 2D configurations (i.e. when the number of landmarks k is high) arises in the statistical study of a set of random points in \mathbb{R}^2 that have been sampled from planar curves. A typical example is the analysis of contours extracted from digital images.

1.2 Main contributions

We consider that the unknown mean pattern $\mathbf{f} \in \mathbb{R}^{k \times 2}$ has been obtained by sampling a planar curve $f : [0, 1] \rightarrow \mathbb{R}^2$ on an equi-spaced design, meaning that

$$\mathbf{f} = \left(f\left(\frac{\ell}{k}\right) \right)_{\ell=1}^k.$$

Under appropriate smoothness assumptions on f , we use a two steps procedure to estimate \mathbf{f} . First, we perform a dimension reduction step by projecting the data into a low-dimensional space of $\mathbb{R}^{k \times 2}$ to eliminate the influence of the random perturbations $\boldsymbol{\zeta}_j$. Then, in a second step, we apply Procrustes analysis in this low-dimensional space to obtain a consistent estimator of \mathbf{f} .

To give a more precise definition of our estimating procedure, introduce the following $k \times k$ matrix

$$\mathbf{A}^\lambda = \left(\frac{1}{k} \sum_{0 \leq |m| \leq \lambda} e^{i2\pi m \frac{\ell - \ell'}{k}} \right)_{\ell, \ell'=1}^k. \quad (1.3)$$

The matrix \mathbf{A}^λ is the smoothing matrix corresponding to a discrete Fourier low pass filter with frequency cutoff $\lambda \in \mathbb{N}$. It is a projection matrix in a sub-space \mathbf{V}^λ of \mathbb{R}^k of dimension $2\lambda + 1$. Then, we project the data on $\mathbf{V}^\lambda \times \mathbf{V}^\lambda \subset \mathbb{R}^{k \times 2}$, and we estimate the scaling, rotation and translation parameters in model (1.2) using M-estimation as follows: denote the scaling parameters by $\mathbf{a} = (a_1, \dots, a_J) \in \mathbb{R}^J$, the rotation parameters by $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_J) \in \mathbb{R}^J$ and the translation parameters by $\mathbf{b} = (b_1, \dots, b_J) \in \mathbb{R}^{2J}$, and introduce the functional,

$$M^\lambda(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) = \frac{1}{Jk} \sum_{j=1}^J \left\| e^{-a_j} \mathbf{A}^\lambda(\mathbf{Y}_j - \mathbf{1}_k \otimes b_j) R_{-\alpha_j} - \frac{1}{J} \sum_{j'=1}^J e^{-a_{j'}} \mathbf{A}^\lambda(\mathbf{Y}_{j'} - \mathbf{1}_k \otimes b_{j'}) R_{-\alpha_{j'}} \right\|_{\mathbb{R}^{k \times 2}}^2, \quad (1.4)$$

where $\|\cdot\|_{\mathbb{R}^{k \times 2}}$ is the standard Euclidean norm in $\mathbb{R}^{k \times 2}$. An M-estimator of

$$(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) = (a_1^*, \dots, a_J^*, \alpha_1^*, \dots, \alpha_J^*, b_1^*, \dots, b_J^*) \in \mathbb{R}^J \times [-\pi, \pi]^J \times \mathbb{R}^{2J}$$

is given by

$$(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda, \hat{\mathbf{b}}^\lambda) \in \underset{(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) \in \Theta_0}{\operatorname{argmin}} M^\lambda(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}), \quad (1.5)$$

where $(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda, \hat{\mathbf{b}}^\lambda) = (\hat{a}_1^\lambda, \dots, \hat{a}_J^\lambda, \hat{\alpha}_1^\lambda, \dots, \hat{\alpha}_J^\lambda, \hat{b}_1^\lambda, \dots, \hat{b}_J^\lambda) \in \mathbb{R}^J \times [-\pi, \pi]^J \times \mathbb{R}^{2J}$ and

$$\Theta_0 = \left\{ (\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) \in [-A, A]^J \times [-\mathcal{A}, \mathcal{A}]^J \times \mathbb{R}^{2J} : \sum_{j=1}^J a_j = 0, \sum_{j=1}^J \alpha_j = 0 \text{ and } \sum_{j=1}^J b_j = 0 \right\}, \quad (1.6)$$

with $A, \mathcal{A} > 0$ being parameters whose values will be discussed below. Finally, the mean pattern \mathbf{f} is estimated by the following smoothed Procrustes mean

$$\hat{\mathbf{f}}^\lambda = \frac{1}{J} \sum_{j=1}^J e^{-\hat{a}_j^\lambda} \left(\mathbf{A}^\lambda \mathbf{Y}_j - \mathbf{1}_k \otimes \hat{b}_j^\lambda \right) R_{-\hat{\alpha}_j^\lambda}. \quad (1.7)$$

To analyze the convergence of the estimator $\hat{\mathbf{f}}^\lambda$ to the mean pattern \mathbf{f} , let us introduce some regularity conditions on the planar curve f and on the covariance structure of the random variable $\boldsymbol{\zeta}$ in $\mathbb{R}^{k \times 2}$. Let $L > 0$, $s > 0$ and define the Sobolev ball of radius L and degree s as

$$H_s(L) = \left\{ f = (f^{(1)}, f^{(2)}) \in L^2([0, 1], \mathbb{R}^2), \sum_{m \in \mathbb{Z}} (1 + |m|^2)^s (|c_m(f^{(1)})|^2 + |c_m(f^{(2)})|^2) < L \right\} \quad (1.8)$$

where $c_m(f) = (c_m(f^{(1)}), c_m(f^{(2)})) = \left(\int_0^1 f^{(1)}(t) e^{-i2\pi mt} dt, \int_0^1 f^{(2)}(t) e^{-i2\pi mt} dt \right) \in \mathbb{C}^2$ is the m -th Fourier coefficient of $f = (f^{(1)}, f^{(2)}) \in L^2([0, 1], \mathbb{R}^2)$, for $m \in \mathbb{Z}$.

Assumption 1. *The function f belongs to $H_s(L)$ for some $L > 0$ and $s \geq \frac{3}{2}$. Moreover, the $k \times 2$ matrix $\mathbf{f} = (f(\frac{\ell}{k}))_{\ell=1}^k$ is of rank two, i.e there is at least two different landmarks in the k -ads composing \mathbf{f} .*

Let $\tilde{\boldsymbol{\zeta}} = (\boldsymbol{\zeta}^{(1)}, \boldsymbol{\zeta}^{(2)}) = (\zeta_1^{(1)}, \dots, \zeta_k^{(1)}, \zeta_1^{(2)}, \dots, \zeta_k^{(2)}) \in \mathbb{R}^{2k}$ be the vectorized version of $\boldsymbol{\zeta} = (\zeta_\ell^{(1)}, \zeta_\ell^{(2)})_{\ell=1}^k \in \mathbb{R}^{k \times 2}$.

Assumption 2. *The random variable $\tilde{\boldsymbol{\zeta}}$ is a centered Gaussian vector in \mathbb{R}^{2k} with covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{2k \times 2k}$. Let $\gamma_{\max}(k)$ be the largest eigenvalue of $\boldsymbol{\Sigma}$. Then,*

$$\lim_{k \rightarrow \infty} \gamma_{\max}(k) k^{-\frac{2s}{2s+1}} = 0,$$

where s is the smoothness parameter defined in Assumption 1.

Note that neither isotropy nor invariance conditions are required on the covariance structure of ζ . The following theorem is the main result of the paper.

Theorem 1.1. *Consider model (1.2) and suppose that the random variables $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*)$ are bounded and belong to $[-\frac{A}{2}, \frac{A}{2}]^J \times [-\frac{A}{2}, \frac{A}{2}]^J \times [-B, B]^{2J}$ for some $0 < A, B$ and $0 < \mathcal{A} < \frac{\pi}{4}$. If Assumptions 1 and 2 hold, and if $\lambda(k) = \lfloor k^{\frac{1}{2s+1}} \rfloor$, then for any $J \geq 2$ there exists $(a_0, \alpha_0, b_0) \in \mathbb{R} \times [-\pi, \pi] \times \mathbb{R}^2$ such that*

$$\frac{1}{k} \|\hat{\mathbf{f}}^\lambda - \mathbf{f}_{\Theta_0}\|_{\mathbb{R}^{k \times 2}}^2 \xrightarrow[k \rightarrow \infty]{} 0, \quad \text{in probability,} \quad (1.9)$$

where $\mathbf{f}_{\Theta_0} = e^{a_0} \mathbf{f} R_{\alpha_0} + \mathbf{1}_k \otimes b_0$. Suppose, in addition, that the random variables $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*)$ have zero expectation in $[-\frac{A}{2}, \frac{A}{2}]^J \times [-\frac{A}{2}, \frac{A}{2}]^J \times [-B, B]^{2J}$ with $A, \mathcal{A} < 0.1$. Then, we have

$$\frac{1}{k} \|\hat{\mathbf{f}}^\lambda - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2 \xrightarrow[k, J \rightarrow \infty]{} 0, \quad \text{in probability.} \quad (1.10)$$

Statement (1.9) means that, under mild assumptions on the covariance structure of the error terms ζ_j , it is possible to consistently estimate the shape of the mean pattern \mathbf{f} when the number of observations J is fixed and the number k of landmarks increases. Note that (a_0, α_0, b_0) depends on J and is given by formula (3.3) in Section 3. To obtain statement (1.10), we assume the condition $A, \mathcal{A} < 0.1$ which means that the random scaling and rotations in model (1.2) are not too large. Also, it is assumed that random scaling, rotations and translations have zero expectation, meaning that the deformations parameters in model (1.2) are centered around the identity. Then, under such assumptions, statement (1.10) shows that one can consistently estimate the true mean pattern \mathbf{f} when both the sample size J and the number of landmarks k go to infinity. These results are consistent with those obtained in [BC11], where we have studied the consistency of Fréchet means in deformable models for signal and image processing.

1.3 Organization of the paper

In Section 2, we recall some properties of the similarity group of the plane, and we describe its action on the mean pattern \mathbf{f} . Then, we discuss General Procrustes Analysis (GPA) and we compare it to our approach. In Section 3 we discuss some identifiability issues in model (1.2). The estimating procedure is described in detail in Section 4. Consistency results are given in Section 5. Some experiments in Section 6 illustrate the numerical performances of our approach. All the proofs are gathered in a technical appendix.

2 Group structure and Generalized Procrustes Analysis

2.1 The similarity group

First let us introduce some notations and definitions that will be useful throughout the paper. The similarity group of the plane is the group (\mathcal{G}, \cdot) generated by isotropic scaling, rotations and translations. The identity element in \mathcal{G} is denoted e and the inverse of $g \in \mathcal{G}$ is denoted by g^{-1} . We parametrize the group \mathcal{G} by a scaling parameter $a \in \mathbb{R}$, an angle $\alpha \in [-\pi, \pi[$ and a translation $b \in \mathbb{R}^2$, and we make no difference between $g \in \mathcal{G}$ and its parametrization $(a, \alpha, b) \in \mathbb{R} \times [-\pi, \pi[\times \mathbb{R}^2$. For all $g_1 = (a_1, \alpha_1, b_1), g_2 = (a_2, \alpha_2, b_2) \in \mathbb{R} \times [-\pi, \pi[\times \mathbb{R}^2$ we have

$$\begin{aligned} g_1 \cdot g_2 &= (a_1, \alpha_1, b_1) \cdot (a_2, \alpha_2, b_2) = (a_1 + a_2, \alpha_1 + \alpha_2, e^{a_1} b_2 R_{\alpha_1} + b_1), \\ g_1^{-1} &= (a_1, \alpha_1, b_1)^{-1} = (-a_1, -\alpha_1, -e^{-a_1} b_1 R_{-\alpha_1}), \\ e &= (0, 0, 0). \end{aligned} \quad (2.1)$$

The action of \mathcal{G} onto $\mathbb{R}^{k \times 2}$ is given by the mapping $(g, \mathbf{x}) \mapsto g \cdot \mathbf{x} := e^a \mathbf{x} R_\alpha + \mathbf{1}_k \otimes b$, for $g = (a, \alpha, b) \in \mathcal{G}$ and $\mathbf{x} \in \mathbb{R}^{k \times 2}$. Note that we use the same symbol “.” for the composition law of \mathcal{G} and its action on $\mathbb{R}^{k \times 2}$. Let

$$\mathbf{1}_{k \times 2} = \mathbf{1}_k \otimes \mathbb{R}^2$$

be the two dimensional linear subspace of $\mathbb{R}^{k \times 2}$ consisting of degenerated configurations, i.e configurations composed of k times the same landmarks. The orthogonal subspace $\mathbf{1}_{k \times 2}^\perp$ is the set of centered configurations. We have the orthogonal decomposition $\mathbb{R}^{k \times 2} = \mathbf{1}_{k \times 2}^\perp \oplus \mathbf{1}_{k \times 2}$, and for any configuration $\mathbf{x} \in \mathbb{R}^{k \times 2}$ we write $\mathbf{x} = \mathbf{x}_0 + \bar{\mathbf{x}} \in \mathbf{1}_{k \times 2}^\perp \oplus \mathbf{1}_{k \times 2}$. We call \mathbf{x}_0 the centered configuration of \mathbf{x} and $\bar{\mathbf{x}} = \mathbf{1}_k \otimes \left(\frac{1}{k} \sum_{\ell=1}^k x_\ell^{(1)}, \frac{1}{k} \sum_{\ell=1}^k x_\ell^{(2)} \right)$ the degenerated configuration associated to \mathbf{x} , see Figure 1 for an illustration.

Definition 2.1. Given a configuration \mathbf{x} in $\mathbb{R}^{k \times 2}$, the *orbit* of \mathbf{x} is defined as the set

$$\mathcal{G} \cdot \mathbf{x} := \{g \cdot \mathbf{x}, g \in \mathcal{G}\} \subset \mathbb{R}^{k \times 2}.$$

This set is also called the shape of \mathbf{x} .

Consider now a degenerated configuration $\mathbf{x} \in \mathbf{1}_{k \times 2}$. Its orbit $\mathcal{G} \cdot \mathbf{x}$ is the entire subspace $\mathbf{1}_{k \times 2}$. Note that the linear subspace $\mathbf{1}_{k \times 2}$ is stable by the action of \mathcal{G} , and that the action of \mathcal{G} on $\mathbf{1}_{k \times 2}$ is not free, meaning that $g_1 \cdot \mathbf{x} = g_2 \cdot \mathbf{x}$ does not imply that $g_1 = g_2$. Now, if $\mathbf{x} \in \mathbb{R}^{k \times 2} \setminus \mathbf{1}_{k \times 2}$ is a non-degenerated configuration of k landmarks, its orbit $\mathcal{G} \cdot \mathbf{x}$ is a sub-manifold of $\mathbb{R}^{k \times 2} \setminus \mathbf{1}_{k \times 2}$ of dimension $\dim(\mathcal{G}) = 4$.

Definition 2.2. Given a configuration $\mathbf{x} \in \mathbb{R}^{k \times 2}$, the stabilizer $I(\mathbf{x})$ is the closed subgroup of \mathcal{G} which leaves \mathbf{x} invariant, namely

$$I(\mathbf{x}) = \{g \in \mathcal{G} : g \cdot \mathbf{x} = \mathbf{x}\}.$$

If $\mathbf{x} \in \mathbf{1}_{k \times 2}$ is a degenerated configuration, it can be written $\mathbf{x} = \mathbf{1}_k \otimes (x^{(1)}, x^{(2)})$ and its stabilizer is non trivial and is equal to $I(\mathbf{x}) = \{(a, \alpha, (x^{(1)}, x^{(2)}) - e^a (x^{(1)}, x^{(2)}) R_\alpha), a \in \mathbb{R}, \alpha \in [-\pi, \pi[\}$. If $\mathbf{x} \in \mathbb{R}^{k \times 2} \setminus \mathbf{1}_{k \times 2}$ is a non-degenerated configuration, its stabilizer $I(\mathbf{x})$ is reduced to the identity $\{e\}$. The action of \mathcal{G} is said free if the stabilizer of any point is reduced to the identity. Hence, the action of \mathcal{G} is free on the set of non-degenerated configurations of k -ads in \mathbb{R}^2 .

Definition 2.3. A section of the orbits of \mathcal{G} is a subset of $\mathbb{R}^{k \times 2}$ containing a unique element of each orbit.

Two well-known examples of sections for the similarity group acting on $\mathbb{R}^{k \times 2}$ are the so-called Bookstein’s and Kendall’s coordinates (see e.g. [DM98] for a precise definition).

2.2 Generalized Procrustes analysis and Kendall’s shape space

Let $\mathbf{x} \in \mathbb{R}^{k \times 2} \setminus \mathbf{1}_{k \times 2}$ be a non-degenerated configuration. Let $H = Id_k - \frac{1}{k} \mathbf{1}_k \mathbf{1}'_k$ be a centering matrix. The effect of translation can be eliminated by centering the configuration \mathbf{x} using the matrix H (see [DM98] for other centering methods), while the effect of isotropic scaling is removed by projecting the centered configuration on a unit sphere, which yields to the so-called pre-shape \mathbf{x}^0 of \mathbf{x} defined as

$$\mathbf{x}^0 = \frac{H \mathbf{x}}{\|H \mathbf{x}\|_{\mathbb{R}^{k \times 2}}} \in \mathbb{R}^{k \times 2}.$$

Consider now the pre-shape sphere defined by $\mathbb{S}_2^k := \{\mathbf{x}^0, \mathbf{x} \in \mathbb{R}^{k \times 2} \setminus \mathbf{1}_{k \times 2}\}$ and see Figure 1 for an illustration. Note that this normalization of the planar configurations amounts to choose a section for the action of the group generated by the translation and scaling in the plane. The Kendall’s shape space is then defined as the quotient of \mathbb{S}_2^k by the group $\mathbf{SO}(2)$ of rotations of the plane, namely

$$\Sigma_2^k := \mathbb{S}_2^k / \mathbf{SO}(2) = \{[\mathbf{x}^0] : \mathbf{x}^0 \in \mathbb{S}_2^k\} \text{ with } [\mathbf{x}^0] = \{\mathbf{x}^0 R_\alpha, \alpha \in [-\pi, \pi[\}.$$

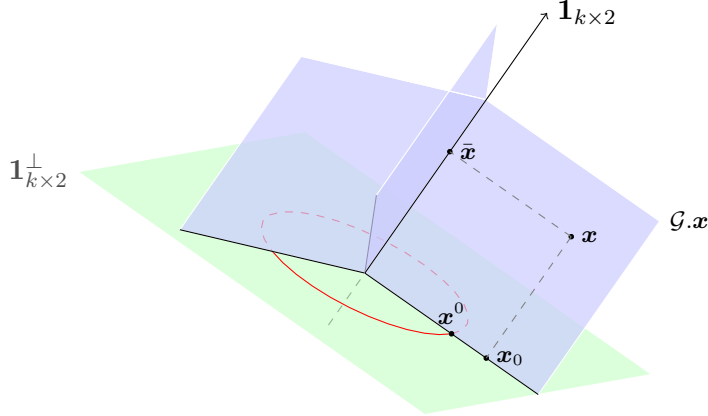


Figure 1: Three orbits of the action of the similarity group \mathcal{G} are represented in blue. The space of centered configurations $\mathbf{1}_{k \times 2}^\perp$ is the green plane. The preshape sphere \mathbb{S}_2^k is the red circle. For a particular $\mathbf{x} \in \mathbb{R}^{k \times 2}$, the centered version is \mathbf{x}_0 and the centered and normalized version is \mathbf{x}^0 . The degenerated configuration associated to \mathbf{x} is $\bar{\mathbf{x}}$.

The space Σ_2^k can be endowed with a Riemannian structure and we refer to [KBCL99] for a detail discussion on its geometric properties.

Let us briefly recall the definition of the so-called partial and full Procrustes distances on \mathbb{S}_2^k . The partial Procrustes distance is defined on the pre-shape sphere \mathbb{S}_2^k as

$$d_P^2(\mathbf{x}^0, \mathbf{y}^0) = \inf_{\alpha \in [-\pi, \pi]} \|\mathbf{x}^0 - \mathbf{y}^0 R_\alpha\|_{\mathbb{R}^{k \times 2}}^2, \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}_2^k.$$

Hence, it is the (Euclidean) distance between the orbits $[\mathbf{x}^0] = \mathbf{SO}(2) \cdot \mathbf{x}^0$ and $[\mathbf{y}^0] = \mathbf{SO}(2) \cdot \mathbf{y}^0$ with $\mathbf{x}^0, \mathbf{y}^0 \in \mathbb{S}_2^k$. Let now \mathcal{H} be the group of transformations of the plane generated by scaling and rotations. The action of $h \in \mathcal{H}$ on the centered configuration \mathbf{x}^0 is defined as $h \cdot \mathbf{x}^0 := e^a \mathbf{x}^0 R_\alpha$ where $h = (a, \alpha) \in \mathbb{R} \times [-\pi, \pi]$. The full Procrustes distance is then defined as

$$d_F^2(\mathbf{x}^0, \mathbf{y}^0) = \inf_{h \in \mathcal{H}} \|\mathbf{x}^0 - h \cdot \mathbf{y}^0\|_{\mathbb{R}^{k \times 2}}^2, \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}_2^k.$$

The full Procrustes sample mean $\hat{\mathbf{Y}}_F$ of $\mathbf{Y}_1, \dots, \mathbf{Y}_J$ (see e.g. [Goo91, DM98]) is defined by $\hat{\mathbf{Y}}_F = \operatorname{argmin}_{\mathbf{x}^0 \in \mathbb{S}_2^k} \sum_{j=1}^J d_F^2(\mathbf{Y}_j^0, \mathbf{x}^0)$. It can equivalently be defined by,

$$\hat{\mathbf{Y}}_F = \frac{1}{J} \sum_{j=1}^J \hat{h}_j \cdot \mathbf{Y}_j^0 \quad (2.2)$$

where $\hat{h}_1, \dots, \hat{h}_J$ are given by the following Procrustes procedure

$$\begin{cases} (\hat{h}_1, \dots, \hat{h}_J) \in \operatorname{argmin}_{(h_1, \dots, h_J) \in \mathcal{H}^J} \left\| \frac{1}{J} \sum_{j=1}^J h_j \cdot \mathbf{Y}_j^0 - \frac{1}{J} \sum_{j'=1}^J h_{j'} \cdot \mathbf{Y}_{j'}^0 \right\|_{\mathbb{R}^{k \times 2}}^2 \\ \text{subject to } \left\| \frac{1}{J} \sum_{j=1}^J h_j \cdot \mathbf{Y}_j^0 \right\|_{\mathbb{R}^{k \times 2}}^2 = 1. \end{cases} \quad (2.3)$$

Obviously, the mean shape \mathbf{f} in model (1.2) does not necessarily belong to \mathbb{S}_2^k , and will generally not have the same orientation than $\hat{\mathbf{Y}}_F$. Therefore, using the Euclidean distance in $\mathbb{R}^{k \times 2}$ to compare $\hat{\mathbf{Y}}_F$ and \mathbf{f} is not meaningful. Moreover, the matching criterion (2.3) is clearly invariant under the action of rotations, meaning that $\hat{\mathbf{Y}}_F R_\alpha$ is a minimizer of (2.3) for any $\alpha \in [-\pi, \pi]$.

To study the consistency of the sample mean $\hat{\mathbf{Y}}_F$, it is classical in the literature on shape analysis to use the Kendall's shape space Σ_2^k . It has then been proved in [KM97, Le98] that, under mild

assumptions on \mathbf{f} , $[\hat{\mathbf{Y}}_F]$ converges almost surely to $[\mathbf{f}^0]$ as $J \rightarrow +\infty$ when the covariance matrix of the random perturbation $\boldsymbol{\zeta}$ in (1.2) is isotropic (see Proposition 1 in [Le98] for a precise definition of isotropy for random variables belonging to $\mathbb{R}^{k \times 2}$). When the random perturbation are non-isotropic, it has been shown in [KM97, Le98] that $[\hat{\mathbf{Y}}_F]$ does not converge to $[\mathbf{f}^0]$ for some specific configurations of mean shape. Note that in [KM97, Le98], the authors also studied the convergence of the so-called partial Procrustes sample mean defined as

$$\hat{\mathbf{Y}}_P \in \operatorname{argmin}_{\mathbf{x}^0 \in \mathbb{S}_2^k} \frac{1}{J} \sum_{j=1}^J d_P^2(\mathbf{x}^0, \mathbf{Y}_j^0) = \operatorname{argmin}_{\mathbf{x}^0 \in \mathbb{S}_2^k} \min_{(\alpha_1, \dots, \alpha_J) \in [-\pi, \pi]^J} \frac{1}{J} \sum_{j=1}^J \|\mathbf{Y}_j^0 R_{\alpha_j} - \mathbf{x}^0\|_{\mathbb{R}^{k \times 2}}^2$$

and they arrived at the same conclusion on the consistency of $\hat{\mathbf{Y}}_P$. Hence, it is commonly the belief that Procrustes sample means can be inconsistent when considering convergence in Σ_2^k and the asymptotic setting $J \rightarrow +\infty$.

Therefore, our approach and GPA share some similarities. They are both based on the estimation of scaling, rotation and translation parameters by a Procrustean procedure which leads to the M-estimators (1.5) and (2.3). To compute a sample mean shape, this M-estimation step is then followed by a standard empirical mean in $\mathbb{R}^{k \times 2}$ of the aligned data using these estimated deformation parameters, see equations (1.7) and (2.2).

However, one of the main differences between the approach developed in this paper and GPA is the choice of the normalization of the data. In GPA, the deformation parameters $\hat{h}_1, \dots, \hat{h}_J$ are computed so that the full Procrustes sample mean $\hat{\mathbf{Y}}_F$ belongs to the pre-shape sphere \mathbb{S}_2^k , see the constraint appearing in (2.3). Therefore, the computation of $\hat{h}_1, \dots, \hat{h}_J$ is somewhat independent of any assumption on the true parameters $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*)$ in model (1.2). In this paper, to ensure the well-posedness of the problem (1.5), we chose to compute the estimator $(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda, \hat{\mathbf{b}}^\lambda)$ by minimizing the matching criterion (1.4) on the constrained set Θ_0 . The choice of the constraints in Θ_0 is motivated by the hypothesis that the true deformation parameters $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*)$ in (1.2) have zero expectation. Another main difference in our approach is the smoothing of the data before applying a Procrustean procedure.

3 Identifiability conditions

Recall that in model (1.2), the random deformations acting on the mean pattern \mathbf{f} are parametrized by a vector $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) = (a_1^*, \dots, a_J^*, \alpha_1^*, \dots, \alpha_J^*, b_1^*, \dots, b_J^*)$ in $\mathbb{R}^J \times [-\pi, \pi]^J \times \mathbb{R}^{2J}$.

Assumption 3. *Let $0 < A, B$ and $0 < \mathcal{A} < \pi$ be three real numbers. The deformation parameters $(\mathbf{a}_j^*, \alpha_j^*, \mathbf{b}_j^*)$, are i.i.d random variables with zero expectation and taking their values in*

$$\Theta^* = \left[-\frac{A}{2}, \frac{A}{2} \right] \times \left[-\frac{\mathcal{A}}{2}, \frac{\mathcal{A}}{2} \right] \times [-B, B]^2.$$

Let $\Theta^* = \left[-\frac{A}{2}, \frac{A}{2} \right]^J \times \left[-\frac{\mathcal{A}}{2}, \frac{\mathcal{A}}{2} \right]^J \times [-B, B]^{2J}$. Under Assumption 3, we have $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) \in \Theta^*$. Note that searching for estimators of the scaling parameters in the compact interval $\left[-\frac{A}{2}, \frac{A}{2} \right]^J$ is an essential condition to ensure the consistency of our procedure. Indeed, the estimation of the deformation parameters $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*)$ and the mean pattern \mathbf{f} is based on the minimization of the criterion (1.4) over the set Θ_0 defined in (1.6). If there were no restriction on the amplitude of the scaling parameter, the degenerate solution $a_j = -\infty$ for all $j = 1, \dots, J$ is always a minimizer of (1.4). Therefore, the minimization has to be performed under additional compact constraints.

3.1 The deterministic criterion D

Let $(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) \in \mathbb{R}^J \times [-\pi, \pi]^J \times \mathbb{R}^{2J}$ and consider the following criterion,

$$D(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) = \frac{1}{Jk} \sum_{j=1}^J \left\| (g_j^{-1} \cdot g_j^*) \cdot \mathbf{f} - \frac{1}{J} \sum_{j'=1}^J (g_{j'}^{-1} \cdot g_{j'}^*) \cdot \mathbf{f} \right\|_{\mathbb{R}^{k \times 2}}^2. \quad (3.1)$$

where $g_j = (a_j, \alpha_j, b_j)$ and $g_j^* = (a_j^*, \alpha_j^*, b_j^*)$ for all $j = 1, \dots, J$. The criterion D is a version without noise of the criterion M^λ defined at (1.4). The estimation procedure described in Section 1 is based on the convergence of the argmins of M^λ toward the argmin of D when k goes to infinity. As a consequence, choosing identifiability conditions amounts to fix a subset Θ_0 of $\mathbb{R}^J \times [-\pi, \pi]^J \times \mathbb{R}^{2J}$ on which D has a unique argmin. In the rest of this section, we determine the zeros of D , and then we fix a convenient constraint set Θ_0 that contains a unique point at which D vanishes.

The criterion D clearly vanishes at $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) \in \mathbb{R}^J \times [-\pi, \pi]^J \times \mathbb{R}^{2J}$. This minimum is not unique since easy algebra implies that

$$D(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) = 0 \quad \iff \quad (g_j^{-1} \cdot g_j^*) \cdot \mathbf{f} = (g_{j'}^{-1} \cdot g_{j'}^*) \cdot \mathbf{f}, \quad \text{for all } j, j' = 1, \dots, J.$$

Suppose now that $\mathbf{f} \notin \mathbf{1}_{2 \times k}$ is a non-degenerated planar configuration. In Section 2.1, we have seen that the action of \mathcal{G} on \mathbf{f} is free, that is, the stabilizer $I(\mathbf{f})$ is reduced to the identity. Thus, we obtain,

$$\begin{aligned} D(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) = 0 & \iff g_j^{-1} \cdot g_j^* = g_{j'}^{-1} \cdot g_{j'}^*, & \text{for all } j, j' = 1, \dots, J \\ & \iff \begin{cases} a_j^* - a_j = a_{j'}^* - a_{j'}, \\ \alpha_j^* - \alpha_j = \alpha_{j'}^* - \alpha_{j'}, \\ e^{-a_j} (b_j^* - b_j) R_{-\alpha_j} = e^{-a_{j'}} (b_{j'}^* - b_{j'}) R_{-\alpha_{j'}} \end{cases} \\ & \text{for all } j, j' = 1, \dots, J \end{aligned}$$

We have proved the following result,

Lemma 3.1. *Let $\mathbf{f} \in \mathbb{R}^{k \times 2}$ be a non-degenerated configuration of k -ads in the plane, i.e. $\mathbf{f} \notin \mathbf{1}_{2 \times k}$. Then, $D(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) = 0$ if and only if $(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b})$ belongs to the set*

$$(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) * \mathcal{G} = \left\{ (\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) * (a_0, \alpha_0, b_0), (a_0, \alpha_0, b_0) \in \mathbb{R} \times [-\pi, \pi] \times \mathbb{R}^2 \right\},$$

where $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) * (a_0, \alpha_0, b_0) = (a_1^* + a_0, \dots, a_J^* + a_0, \alpha_1^* + \alpha_0, \dots, \alpha_J^* + \alpha_0, e^{a_1^*} b_0 R_{\alpha_1^*} + b_1^*, \dots, e^{a_J^*} b_0 R_{\alpha_J^*} + b_J^*) \in \mathbb{R}^J \times [-\pi, \pi]^J \times \mathbb{R}^{2J}$.

Remark 1. Lemma 3.1 is simpler than it appears. By reordering the entries of the vector $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*)$ there is an obvious correspondence between $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) \in \Theta^*$ and $(g_1^*, \dots, g_J^*) \in \mathcal{G}^J$ via the parametrization of the similarity group defined in Section 2.1. Hence, Lemma 3.1 tells us that the criterion D vanishes for all the vectors $(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) \in \mathbb{R}^J \times [-\pi, \pi]^J \times \mathbb{R}^{2J}$ corresponding to the subset of the group \mathcal{G}^J given by

$$(g_1^*, \dots, g_J^*) * \mathcal{G} = \{(g_1^* \cdot g_0, \dots, g_J^* \cdot g_0), g_0 \in \mathcal{G}\} \subset \mathcal{G}^J.$$

The “*” notation is nothing else than the right composition by a same $g_0 \in \mathcal{G}$ of all the entries of a $(g_1, \dots, g_J) \in \mathcal{G}^J$. Hence the subset $(g_1^*, \dots, g_J^*) * \mathcal{G}$ can be interpreted as the orbit of $(g_1^*, \dots, g_J^*) \in \mathcal{G}^J$ under the (right) action of \mathcal{G} . Indeed, \mathcal{G} acts naturally by (right) composition on the all the coordinates of an element of \mathcal{G}^J .

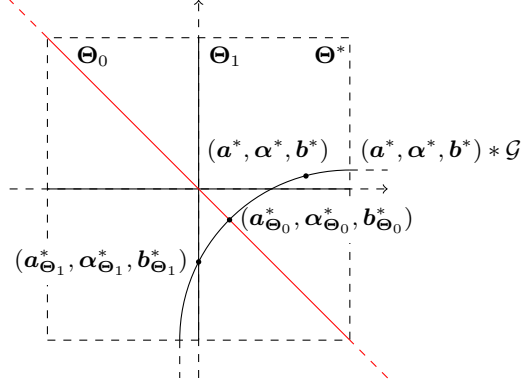


Figure 2: Choice of identifiability conditions when $J = 2$.

3.2 The constraint set Θ_0

By Lemma 3.1, the set Θ_0 must intersect at a unique point, say $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*, \mathbf{b}_{\Theta_0}^*)$, each set $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) * \mathcal{G}$. It is convenient to choose Θ_0 to be of the form $\Theta_0 = \mathbb{R}^J \times [-\pi, \pi]^J \times \mathbb{R}^{2J} \cap \mathcal{L}_0$ where \mathcal{L}_0 is a linear space of \mathbb{R}^{4J} . The linear space \mathcal{L}_0 must be chosen so that for any $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*)$ in Θ^* , there exists a unique point $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*, \mathbf{b}_{\Theta_0}^*)$ in Θ_0 that can be written as $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*, \mathbf{b}_{\Theta_0}^*) = (\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) * (a_0, \alpha_0, b_0)$ for some $(a_0, \alpha_0, b_0) \in \mathbb{R} \times [-\pi, \pi] \times \mathbb{R}^2$.

Remark 2. As we have seen in Remark 1, the set $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) * \mathcal{G}$ can be interpreted as an orbit of the action of \mathcal{G} on \mathcal{G}^J . In this terminology, the set Θ_0 can be viewed as a section of the orbits. Indeed, the section is the set of representatives $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*, \mathbf{b}_{\Theta_0}^*)$ of each orbit. See Figure 2 for an illustration.

Let us consider a choice of Θ_0 motivated by the fact that, under Assumption 3, the random deformation parameters have zero expectation. In this setting, it is natural to impose that the estimated deformation parameters sum up to zero by choosing $\mathcal{L}_0 = \mathbf{1}_{4J}^\perp$, which is the orthogonal of the linear space $\mathbf{1}_{4J} = \mathbf{1}_{4J} \cdot \mathbb{R} \subset \mathbb{R}^{4J}$. Such a choice leads to the set Θ_0 defined equation (1.6) i.e.

$$\Theta_0 = \{(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) \in \Theta^J, (a_1 + \dots + a_J, \alpha_1 + \dots + \alpha_J, b_1 + \dots + b_J) = 0\}.$$

Now, let us show that for any $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) \in \Theta^*$ there exists a unique $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*, \mathbf{b}_{\Theta_0}^*) = (\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) * (a_0, \alpha_0, b_0) \in \Theta_0$ for some $(a_0, \alpha_0, b_0) \in \mathbb{R} \times [-\pi, \pi] \times \mathbb{R}^2$. This amounts to solve the following equations

$$\begin{cases} a_1^* + a_0 + \dots + a_J^* + a_0 = 0, \\ \alpha_1^* + \alpha_0 + \dots + \alpha_J^* + \alpha_0 = 0, \\ e^{a_1^*} b_0 R_{\alpha_1^*} + b_1^* + \dots + e^{a_J^*} b_0 R_{\alpha_J^*} + b_J^* = 0. \end{cases} \quad (3.2)$$

After some computations, we obtain that equations (3.2) are satisfied if and only if

$$(a_0, \alpha_0, b_0) = (-\bar{\mathbf{a}}^*, -\bar{\boldsymbol{\alpha}}^*, -\bar{\mathbf{b}}^* (\overline{e^{\mathbf{a}^*} R_{\boldsymbol{\alpha}^*}})^{-1}), \quad (3.3)$$

where $\bar{\mathbf{a}}^* = \frac{1}{J} \sum_{j=1}^J a_j^* \in \mathbb{R}$, $\bar{\boldsymbol{\alpha}}^* = \frac{1}{J} \sum_{j=1}^J \alpha_j^* \in \mathbb{R}$, $\bar{\mathbf{b}}^* = \frac{1}{J} \sum_{j=1}^J b_j^* \in \mathbb{R}^2$ and $\overline{e^{\mathbf{a}^*} R_{\boldsymbol{\alpha}^*}} = \frac{1}{J} \sum_{j=1}^J e^{a_j^*} R_{\alpha_j^*}$ is a 2×2 invertible matrix. Therefore, $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*, \mathbf{b}_{\Theta_0}^*)$ is uniquely given by

$$([\mathbf{a}_{\Theta_0}^*]_j, [\boldsymbol{\alpha}_{\Theta_0}^*]_j, [\mathbf{b}_{\Theta_0}^*]_j) = (a_j^* - \bar{\mathbf{a}}^*, \alpha_j^* - \bar{\boldsymbol{\alpha}}^*, b_j^* - e^{a_j^*} (\bar{\mathbf{b}}^* (\overline{e^{\mathbf{a}^*} R_{\boldsymbol{\alpha}^*}})^{-1}) R_{\alpha_j^*}),$$

for $j = 1, \dots, J$.

Remark 3. Another possible approach is to fix, say the first observation as a reference, meaning that the criterion D could be optimized on the following subspace of $\mathbb{R}^J \times [-\pi, \pi]^J \times \mathbb{R}^{2J}$

$$\Theta_1 = \{(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) \in [-A, A]^J \times [-\mathcal{A}, \mathcal{A}]^{2J} \times \mathbb{R}^2, (a_1, \alpha_1, b_1) = 0\}.$$

With such a choice, for any $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) \in \Theta^J$, the j -th coordinate of $(\mathbf{a}_{\Theta_1}^*, \boldsymbol{\alpha}_{\Theta_1}^*, \mathbf{b}_{\Theta_1}^*) = (\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) * (a_0, \alpha_0, b_0)$ is given by

$$([\mathbf{a}_{\Theta_1}^*]_j, [\boldsymbol{\alpha}_{\Theta_1}^*]_j, [\mathbf{b}_{\Theta_1}^*]_j) = (a_j^* - a_1^*, \alpha_j^* - \alpha_1^*, b_j^* - e^{a_j^* - a_1^*} b_1^* R_{\alpha_j^* - \alpha_1^*}),$$

where $(a_0, \alpha_0, b_0) = (a_1^*, \alpha_1^*, b_1^*)^{-1} = (-a_1^*, -\alpha_1^*, -e^{-a_1^*} b_1^* R_{-\alpha_1^*})$. A graphical illustration of the choice of identifiability conditions for $J = 2$ is given in Figure 2.

4 The estimating procedure

4.1 A dimension reduction step

We use Fourier filtering to project the data into a low-dimensional space as follows. Assume for convenience that k is odd. For $\mathbf{x} = (\mathbf{x}'_1, \dots, \mathbf{x}'_k)' \in \mathbb{R}^{k \times 2}$ and $m = -\frac{k-1}{2}, \dots, \frac{k-1}{2}$, let

$$c_m(\mathbf{x}) = \sum_{\ell=1}^k \mathbf{x}_\ell e^{-i2\pi m \frac{\ell}{k}} = \left(\sum_{\ell=1}^k x_\ell^{(1)} e^{-i2\pi m \frac{\ell}{k}}, \sum_{\ell=1}^k x_\ell^{(2)} e^{-i2\pi m \frac{\ell}{k}} \right) \in \mathbb{C}^2 \text{ with } \mathbf{x}_\ell = (x_\ell^{(1)}, x_\ell^{(2)}),$$

be the m -th (discrete) Fourier coefficient of \mathbf{x} . Let $\lambda \in \{1, \dots, \frac{k-1}{2}\}$ be a smoothing parameter, and define for each \mathbf{Y}_j the smoothed shapes

$$\hat{\mathbf{f}}_j^\lambda = \left(\frac{1}{k} \sum_{0 \leq |m| \leq \lambda} c_m(\mathbf{Y}_j) e^{i2\pi m \frac{\ell}{k}} \right)_{\ell=1}^k = \mathbf{A}^\lambda \mathbf{Y}_j \in \mathbb{R}^{k \times 2}.$$

In Section 2.1, we have shown that the similarity group is not free on the subset $\mathbf{1}_{k \times 2}$ of degenerated configurations composed of k identical landmarks, see Section 2.1. That is why we are going to treat separately the subspace $\mathbf{1}_{k \times 2}$ and $\mathbf{1}_{k \times 2}^\perp$ by considering the matrices

$$\bar{A} = \frac{1}{k} \mathbf{1}_k \mathbf{1}'_k \quad \text{and} \quad A_0^\lambda = \left(\frac{1}{k} \sum_{0 < |m| \leq \lambda} e^{i2\pi m \frac{\ell - \ell'}{k}} \right)_{\ell, \ell'=1}^k. \quad (4.1)$$

Remark that \bar{A} is a projection matrix on the one dimensional sub-space $\bar{\mathcal{V}} := \mathbf{1}_k \cdot \mathbb{R} = \{c \mathbf{1}_k : c \in \mathbb{R}\}$ of \mathbb{R}^k . The matrix A_0^λ is a projection matrix in a (trigonometric) sub-space \mathcal{V}_0^λ of dimension 2λ . Note that it is included in the linear space $\bar{\mathcal{V}}^\perp = \{x \in \mathbb{R}^k : x' \mathbf{1}_k = 0\}$. Hence, $\mathcal{V}_0^\lambda \times \mathcal{V}_0^\lambda$ is a linear subspace of $\mathbf{1}_{k \times 2}^\perp$ which is the space of the centered configurations. We have,

$$\mathbf{A}^\lambda = A_0^\lambda + \bar{A} \quad \text{and} \quad \mathcal{V}^\lambda = \mathcal{V}_0^\lambda \oplus \bar{\mathcal{V}}. \quad (4.2)$$

Thus, we can write the smoothed shape $\hat{\mathbf{f}}_j^\lambda$ as

$$\hat{\mathbf{f}}_j^\lambda = \mathbf{A}^\lambda \mathbf{Y}_j = A_0^\lambda \mathbf{Y}_j + \bar{A} \mathbf{Y}_j \in (\bar{\mathcal{V}} \times \bar{\mathcal{V}}) \oplus (\mathcal{V}_0^\lambda \times \mathcal{V}_0^\lambda)$$

where $\bar{\mathcal{V}} \times \bar{\mathcal{V}} = \mathbf{1}_{k \times 2}$ and $\mathcal{V}_0^\lambda \times \mathcal{V}_0^\lambda \subset \mathbf{1}_{k \times 2}^\perp$. In other words, $A_0^\lambda \mathbf{Y}_j$ is the smoothed centered configuration associated to \mathbf{Y}_j and $\bar{A} \mathbf{Y}_j$ is the degenerated configuration given by the Euclidean mean of the k landmarks composing \mathbf{Y}_j . Finally, remark that the low pass filter and the action of similarity group commute, that is, we have for all $g \in \mathcal{G}$ and $\mathbf{f} \in \mathbb{R}^{k \times 2}$

$$g(\mathbf{A}^\lambda \mathbf{f}) = e^a \mathbf{A}^\lambda \mathbf{f} R_\alpha + \mathbf{1}_k \otimes b = \mathbf{A}^\lambda (e^a \mathbf{f} R_\alpha + \mathbf{1}_k \otimes b) = \mathbf{A}^\lambda (g \cdot \mathbf{f}).$$

4.2 Estimation of the deformation parameters

Recall that the estimator $(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda, \hat{\mathbf{b}}^\lambda)$ of $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*)$ is defined by the optimization problem (1.5). Nevertheless, thanks to the discussion of Sections 2.1 and 4.1, one can carry out the estimation process in two steps. First, we estimate the rotation and scaling parameters on the space $\mathcal{V}_0^\lambda \times \mathcal{V}_0^\lambda \subset \mathbf{1}_{k \times 2}^\perp$ of the centered configurations. We then use these estimators to estimate the translation parameters which act on $\bar{\mathcal{V}} \times \bar{\mathcal{V}} = \mathbf{1}_{k \times 2}$. Note that this procedure is equivalent to the optimization problem (1.5) as shown by Lemma 4.1 below.

Estimation of rotations and scaling. Define

$$M_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}) = \frac{1}{Jk} \sum_{j=1}^J \left\| e^{-a_j} A_0^\lambda \mathbf{Y}_j R_{-\alpha_j} - \frac{1}{J} \sum_{j'=1}^J e^{-a_{j'}} A_0^\lambda \mathbf{Y}_{j'} R_{-\alpha_{j'}} \right\|_{\mathbb{R}^{k \times 2}}^2.$$

Let $\mathbb{1}_J^\perp = \{(a_1, \dots, a_J) \in \mathbb{R}^J, a_1 + \dots, a_J = 0\}$ and consider the space $\Theta_0^{\mathbf{a}, \boldsymbol{\alpha}} = ([-A, A]^J \cap \mathbb{1}_J^\perp) \times ([-\mathcal{A}, \mathcal{A}]^J \cap \mathbb{1}_J^\perp)$. Then, estimators of the rotation and scaling parameters are given by

$$(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda) \in \underset{(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}}{\operatorname{argmin}} M_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}). \quad (4.3)$$

Estimation of translations. Now that we have computed estimators of the rotation and scaling parameters, let us define the criterion,

$$\bar{M}(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) = \frac{1}{Jk} \sum_{j=1}^J \left\| e^{-a_j} \bar{A}(\mathbf{Y}_j - \mathbb{1}_k \otimes b_j) R_{-\alpha_j} - \frac{1}{J} \sum_{j'=1}^J e^{-a_{j'}} \bar{A}(\mathbf{Y}_{j'} - \mathbb{1}_k \otimes b_{j'}) R_{-\alpha_{j'}} \right\|_{\mathbb{R}^{k \times 2}}^2$$

and the space $\Theta_0^{\mathbf{b}} = \{(b_1^{(1)}, \dots, b_J^{(1)}, b_1^{(2)}, \dots, b_J^{(2)}) \in \mathbb{R}^{2J}, b_1^{(1)} + \dots + b_J^{(1)} = b_1^{(2)} + \dots + b_J^{(2)} = 0\}$. The estimator of the translation parameters is then given by,

$$\hat{\mathbf{b}}^\lambda = \underset{\mathbf{b} \in \Theta_0^{\mathbf{b}}}{\operatorname{argmin}} \bar{M}(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda, \mathbf{b}). \quad (4.4)$$

We emphasize that the estimators of the translation parameters depend on the estimated rotation and scaling parameters. It is shown in the proof of Lemma 4.1 below that we have an explicit expression of $\hat{\mathbf{b}}^\lambda$ given by

$$\hat{\mathbf{b}}^\lambda = (\bar{\mathbf{Y}}_1 - e^{a_1} \bar{\mathbf{Y}} (\overline{e^{\mathbf{a}} R_{\boldsymbol{\alpha}}})^{-1} R_{\alpha_1}, \dots, \bar{\mathbf{Y}}_J - e^{a_J} \bar{\mathbf{Y}} (\overline{e^{\mathbf{a}} R_{\boldsymbol{\alpha}}})^{-1} R_{\alpha_J}) \quad (4.5)$$

where $\overline{e^{\mathbf{a}} R_{\boldsymbol{\alpha}}} = \frac{1}{J} \sum_{j=1}^J e^{a_j} R_{\alpha_j} \in \mathbb{R}^{2 \times 2}$, $\bar{\mathbf{Y}}_j = \frac{1}{k} \sum_{\ell=1}^k [\mathbf{Y}_j]_\ell \in \mathbb{R}^2$, that is $\bar{A} \mathbf{Y}_j = \mathbb{1}_k \otimes \bar{\mathbf{Y}}_j \in \mathbb{R}^{k \times 2}$ is a degenerated configuration, and $\bar{\mathbf{Y}} = \frac{1}{J} \sum_{j=1}^J \bar{\mathbf{Y}}_j \in \mathbb{R}^2$.

This two steps procedure is equivalent to the optimization problem (1.5) as we have the following decompositions $M^\lambda(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) = M_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}) + \bar{M}(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b})$ and $\Theta_0 = \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}} \times \Theta_0^{\mathbf{b}}$, implying following result (see the Appendix for a detailed proof)

$$\text{Lemma 4.1. } (\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda, \hat{\mathbf{b}}^\lambda) \in \underset{(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) \in \Theta_0}{\operatorname{argmin}} M^\lambda(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) \iff \begin{cases} (\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda) \in \underset{(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}}{\operatorname{argmin}} M_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}) \\ \hat{\mathbf{b}}^\lambda = \underset{\mathbf{b} \in \Theta_0^{\mathbf{b}}}{\operatorname{argmin}} \bar{M}(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda, \mathbf{b}). \end{cases}$$

5 Consistency results

In what follows, C, C_0, C_1 denote positive constants whose value may change from line to line. The notation $C(\cdot)$ specifies the dependency of C on some quantities.

5.1 Consistent estimation of the deformation parameters

Rotation and scaling. Recall that the rotation and scaling parameters are estimated separately on the smoothed and centered observations. We have the following result,

Theorem 5.1. *Consider model (1.2) and suppose that Assumptions 1 and 2 hold and that Assumption 3 is verified with $\max\{A, \mathcal{A}\} < 0.1$. Consider $(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda)$ the estimators defined in equation (4.3). If $\lambda = k^{\frac{1}{2s+1}}$ then for all $x > 0$ we have*

$$\mathbb{P} \left(\frac{1}{J} \left\| (\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda) - (\mathbf{a}^*, \boldsymbol{\alpha}^*) \right\|_{\mathbb{R}^{2J}}^2 \geq C_1(L, s, A, \mathcal{A}, \mathbf{f}_0) A_1(k, J, x) + C_2(A, \mathcal{A}) A_2(J, x) \right) \leq 4e^{-x}, \quad (5.1)$$

where $C_1(L, s, A, \mathcal{A}, \mathbf{f}_0), C_2(A, \mathcal{A}) > 0$ are positive constants independent of k and J , $A_1(k, J, x) = F(k^{-\frac{2s}{2s+1}}) + F(V_1(k, J, x))$ with $V_1(k, J, x) = 3\gamma_{\max}(k)k^{-\frac{2s}{2s+1}}(1 + \sqrt{2\frac{x}{Jk}} + 2\frac{x}{Jk})$ and $A_2(J, x) = \left(\frac{x}{3J} + \sqrt{\frac{2x}{J}}\right)^2$, where $F(u) = u + \sqrt{u}$, for all $u \geq 0$.

Remark that a direct consequence of Theorem 5.1 is the consistency of $(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda)$ to $(\mathbf{a}^*, \boldsymbol{\alpha}^*)$ when $k, J \rightarrow \infty$. Indeed, we have $\lim_{u \rightarrow 0} F(u) = 0$ and under Assumption 2 and for any fixed $J \geq 1$ and $x > 0$, the term $V_1(k, J, x)$ tends to zero as k goes to infinity. Hence for any $x > 0$ and $J \geq 1$, we have $\lim_{k \rightarrow \infty} A_1(k, J, x) = 0$. Now, for all $x > 0$, we have $\lim_{J \rightarrow \infty} A_2(J, x) = 0$. Thus a double asymptotic $\min\{k, J\} \rightarrow \infty$ ensures that $\frac{1}{J}\|(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda) - (\mathbf{a}^*, \boldsymbol{\alpha}^*)\|_{\mathbb{R}^{2J}}^2$ tends to 0 in probability.

Note that also that Proposition A.1 (see the Appendix) then ensures the convergence of $(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda)$ to $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)$ as J remains fixed and $k \rightarrow \infty$. It should be also mentioned that the condition $\max\{A, \mathcal{A}\} < 0.1$ is not needed to only ensure the convergence in probability of $(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda)$ to $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)$ is the setting J fixed and $k \rightarrow \infty$. However, this extra condition on A and \mathcal{A} is needed to derive a rate of convergence (depending explicitly on k and J).

Translation parameters. We have the following result,

Theorem 5.2. *Consider the hypothesis and notations of Theorem 5.1 and the estimator $\hat{\mathbf{b}}^\lambda$ given by formula (4.4). Then we have for all $x > 0$,*

$$\mathbb{P}\left(\frac{1}{J}\|\hat{\mathbf{b}}^\lambda - \mathbf{b}^*\|_{\mathbb{R}^{2J}}^2 \geq C_3(L, s, A, \mathcal{A}, B, \mathbf{f})A_3(k, J, x) + C_4(A, \mathcal{A}, B)A_2(J, x)\right) \leq 9e^{-x}, \quad (5.2)$$

where $C_3(L, s, A, \mathcal{A}, B, \mathbf{f}), C_4(A, \mathcal{A}, B) > 0$ are positive constants independent of k and J , $A_3(k, J, x) = F(k^{-\frac{2s}{2s+1}}) + F(V_1(k, J, x)) + V_2(k, J, x)$ with $V_2(k, J, x) = 4\frac{\gamma_{\max}(k)}{k}(1 + \sqrt{2\frac{x}{J}} + \frac{x}{J})$.

Similar comments to those made after Theorem 5.1 are still valid here. For any $J \in \mathbb{N}$ and $x > 0$, we have $\lim_{k \rightarrow \infty} A_3(k, J, x) = 0$, since $V_2(k, J, x)$ tends to 0 as k goes to infinity by Assumption 2. In the double asymptotic setting $k, J \rightarrow \infty$, Theorem 5.2 ensure the consistency of $\hat{\mathbf{b}}^\lambda$ to the true value \mathbf{b}^* of the translation parameters. Remark also that Proposition A.3 (see the Appendix) ensures the convergence in probability of \mathbf{b}^λ to $\mathbf{b}_{\Theta_0}^*$ with only an asymptotic in k (with fixed J).

5.2 Consistent estimation of the mean shape

Theorem 5.3. *Consider model (1.2) and suppose that Assumptions 1 and 2 hold and that Assumption 3 is verified with $\max\{A, \mathcal{A}\} < 0.1$. Consider the estimator $\hat{\mathbf{f}}^\lambda$ defined in (1.7) and let $\lambda = k^{\frac{1}{2s+1}}$. Then, we have for all $x > 0$,*

$$\mathbb{P}\left(\frac{1}{k}\|\hat{\mathbf{f}}^\lambda - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2 \geq C(L, s, A, \mathcal{A}, B, \mathbf{f})\left(A_1(k, J, x) + A_3(k, J, x) + A_2(J, x)\right)\right) \leq 14e^{-x},$$

where $C(L, s, A, \mathcal{A}, B, \mathbf{f}) > 0$ is a constant independent of k and J , $A_1(k, J, x) = F(k^{-\frac{2s}{2s+1}}) + F(V_1(k, J, x))$ with $V_1(k, J, x) = 3\gamma_{\max}(k)k^{-\frac{2s}{2s+1}}(1 + \sqrt{2\frac{x}{Jk}} + 2\frac{x}{Jk})$, $A_3(k, J, x) = V_2(k, J, x) + F(k^{-\frac{2s}{2s+1}}) + F(V_1(k, J, x))$ with $V_2(k, J, x) = 4\frac{\gamma_{\max}(k)}{k}(1 + \sqrt{2\frac{x}{J}} + \frac{x}{J})$ and $A_2(J, x) = \left(\sqrt{\frac{2x}{J}} + \frac{x}{3J}\right)^2$.

The terms $A_1(k, J, x)$, $A_3(k, J, x)$ and $A_2(J, x)$ that appear in the statement of Theorem 5.3 are the same to those appearing in Theorems 5.1 and 5.2. As a consequence, we have $\frac{1}{k}\|\hat{\mathbf{f}}^\lambda - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2 \rightarrow 0$ in probability when $\min\{k, J\} \rightarrow \infty$, see the comments after Theorems 5.1 and 5.2. Let

$$\mathbf{f}_{\Theta_0} = e^{\bar{\mathbf{a}}^*}(\mathbf{f} + \mathbf{1}_k \otimes \bar{\mathbf{b}}^*(\overline{e^{\bar{\mathbf{a}}^*} R_{\bar{\boldsymbol{\alpha}}^*}})^{-1})R_{\bar{\boldsymbol{\alpha}}^*} \quad (5.3)$$

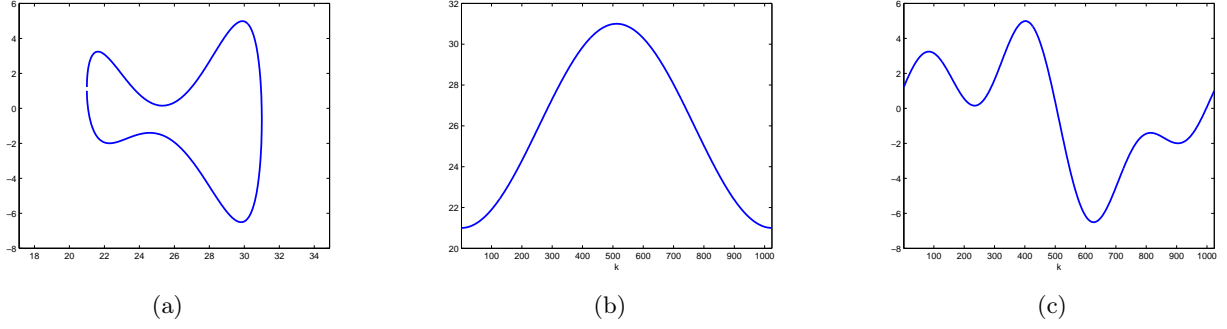


Figure 3: (a) Plot of the mean pattern $\mathbf{f} = (\mathbf{f}^{(1)}, \mathbf{f}^{(2)})$ used in the simulations with $k = 1024$. (b) The first coordinates $\mathbf{f}^{(1)}$. (c) The second coordinates $\mathbf{f}^{(2)}$.

where $\bar{\mathbf{a}}^* = \frac{1}{J} \sum_{j=1}^J a_j^* \in \mathbb{R}$, $\bar{\boldsymbol{\alpha}}^* = \frac{1}{J} \sum_{j=1}^J \alpha_j^* \in \mathbb{R}$, $\bar{\mathbf{b}}^* = \frac{1}{J} \sum_{j=1}^J \mathbf{b}_j^* \in \mathbb{R}^2$ and $\overline{e^{\mathbf{a}^*} R_{\boldsymbol{\alpha}^*}} = \frac{1}{J} \sum_{j=1}^J e^{a_j^*} R_{\alpha_j^*}$ is an invertible 2×2 matrix, see also formula (3.3). A slight modification of the proof of Theorem 5.3 gives the following inequality,

$$\mathbb{P}\left(\frac{1}{k} \|\hat{\mathbf{f}}^\lambda - \mathbf{f}_{\Theta_0}\|^2 \geq C(L, s, A, \mathcal{A}, B, \mathbf{f}) \left(A_1(k, J, x) + A_3(k, J, x) \right)\right) \leq 14e^{-x}.$$

Note that Theorem 5.3 gives rate of convergences of $\hat{\mathbf{f}}^\lambda$ to \mathbf{f} thanks to a concentration inequality. As for the estimation of the deformation parameters, the condition $\max\{A, \mathcal{A}\} < 0.1$ is not needed to only ensure the convergence in probability of $\hat{\mathbf{f}}^\lambda$ to \mathbf{f}_{Θ_0} is the setting J fixed and $k \rightarrow \infty$. However, this extra condition on A and \mathcal{A} is needed to derive a rate of convergence (depending explicitly on k and J) of $\hat{\mathbf{f}}^\lambda$ to \mathbf{f}_{Θ_0} .

6 Numerical experiments

6.1 Description of the data

We make here some numerical simulations to show the effect of the dimension k and the number J of observations on the estimation of the deformation parameters and the mean pattern with data generated by model (1.2). Different types of noise are considered. For all $t \in [0, 1]$, let

$$f(t) = (10 \sin^2(\pi t) + \cos(10\pi) + 20, 2 \sin(6\pi t) - 11 \sin^2(\pi t) + 12 \exp(-25(t - 0.4)^2) + 1).$$

This curve is plotted in Figure 3. The deformation parameters $(a_j^*, \alpha_j^*, b_j^*)$, $j = 1, \dots, J$, are i.i.d uniform random variables taking their values in $\Theta = [-\frac{1}{4}, \frac{1}{4}] \times [-\frac{1}{2}, \frac{1}{2}] \times [-1, 1]^2$. The law of the deformation parameters is supposed to be unknown *a priori* and the minimization is performed on the constraint set

$$\Theta_0 = \left\{ (\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) \in [-1, 1]^J \times \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]^J \times \mathbb{R}^{2J}, \sum_{j=1}^J a_j = 0, \sum_{j=1}^J \alpha_j = 0 \text{ and } \sum_{j=1}^J b_j = 0 \right\}.$$

Recall our notations: the error term is denoted by $\boldsymbol{\zeta} = (\boldsymbol{\zeta}^{(1)}, \boldsymbol{\zeta}^{(2)}) \in \mathbb{R}^{k \times 2}$ and the vectorized version of $\boldsymbol{\zeta}$ is denoted by $\tilde{\boldsymbol{\zeta}} \in \mathbb{R}^{2k}$. The simulations were run with three different kinds of noise.

White noise : the random variable $\tilde{\boldsymbol{\zeta}} = (\zeta_1^{(1)}, \zeta_2^{(1)}, \dots, \zeta_1^{(k)}, \zeta_2^{(k)})' \in \mathbb{R}^{2k}$ is a centered Gaussian vector of variance

$$\boldsymbol{\Sigma}_1 = 4Id_{2k}.$$

We have $\gamma_{\max}(k) = 4$ and this correspond to an isotropic Gaussian noise as in [KM97, Le98], see Figure 4.

Stationary noise : the random variable $\tilde{\zeta}$ is a centered Gaussian vector of variance Σ_2 with

$$\Sigma_2^{\frac{1}{2}} = Id_2 \otimes \left[\frac{1}{2} \exp \left(-\frac{|\ell - \ell'|^2}{100} \right) \right]_{\ell, \ell'=1}^k$$

Hence, Σ_2 is a Toeplitz matrix and it follows from classical matrix theory, see e.g. [HJ90], that $\gamma_{\max}(k)$ is bounded (here $\gamma_{\max} \leq 80$). See Figure 5.

Correlated noise : the random variable $\tilde{\zeta}$ is a centered Gaussian vector of variance

$$\Sigma_3 = Id_2 \otimes P \text{Diag} \left(\frac{\ell^2}{2k}, \ell = 1, \dots, k \right) P'$$

where P is an arbitrary matrix in $\mathbf{SO}(k)$. Hence, in this case $\gamma_{\max}(k) = \frac{k}{4}$ and the level of noise increase with k , see Figure 6.

6.2 Description of the procedure

The estimation procedure follows the guidelines described in Section 4. We are testing the effect of the number J of observations and the number k of landmarks on the estimation of the parameters of interest of model (1.2). All the simulations are performed with $J = 10, 100, 500$ and $k = 20, 50, 100, 1000, 3000$ and for each combination of these two factors the simulations are performed with $M = 30$ repetitions of model (1.2).

Moreover, the estimation are done without and with the pre-smoothing step. In the former case we have $\lambda(k) = \frac{k}{2}$, that is, there is no reduction of the dimension. In the latter case, the smoothing parameter $\lambda(k)$ is fixed manually to ensure a proper reconstruction of the mean pattern f . Note that we need $\lambda > 7$ to get correct results and we took $\lambda_{20} = \lambda_{50} = \lambda_{100} = 7$, $\lambda_{1000} = 11$ and $\lambda_{3000} = 25$.

We use a quasi-Newton trust-region based algorithm to solve the optimization problems (4.3) and (4.4). The formula for the gradient is given in (B.1). All the computations are performed with *Matlab*.

6.3 Results: estimation of the mean pattern

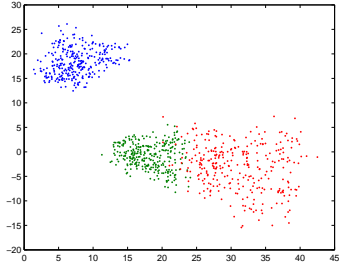
For each of the 30 repetitions of model (1.2) with the possible values of k and J , we compute the quantities $\frac{1}{k} \|\hat{\mathbf{f}}^\lambda - \mathbf{f}_{\Theta_0}^*\|_{\mathbb{R}^{k \times 2}}^2$ where $\hat{\mathbf{f}}^\lambda$ corresponds to the smoothed Procrustes mean of the observations defined in (1.7) and, $\frac{1}{k} \|\hat{\mathbf{f}} - \mathbf{f}_{\Theta_0}^*\|_{\mathbb{R}^{k \times 2}}^2$ where $\hat{\mathbf{f}}$ is the (non smoothed) Procrustes mean of the data. Recall that \mathbf{f}_{Θ_0} is defined by formula (5.3).

Boxplots of the results are given in Figures 7, 8 and 9 for the different kinds of error terms described in Section 6.1. In the figures, the abscissa represents the different values of the number k of landmarks and boxplots in red correspond to $J = 10$ observations, in green to $J = 100$ observations and in blue to $J = 500$ observations.

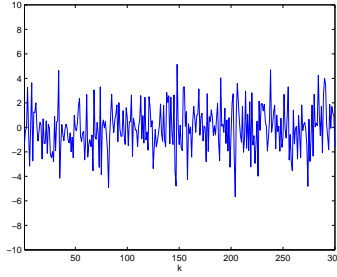
The estimation of the mean pattern with the white noise error term is given by Figure 7. In Figure 7a, for a fixed k , the non-smoothed version $\frac{1}{k} \|\hat{\mathbf{f}} - \mathbf{f}^*\|_{\mathbb{R}^{k \times 2}}^2$ decreases when J increases. Moreover, the values of $\frac{1}{k} \|\hat{\mathbf{f}} - \mathbf{f}^*\|_{\mathbb{R}^{k \times 2}}^2$ remain stable when J remains fixed and k increases. Recall that this framework corresponds to the isotropic Gaussian noise described in [KM97]. The simulations seem to confirm their conclusions and show that in this framework the dimension k is not preponderant. In Figure 7b, the smoothed version $\frac{1}{k} \|\hat{\mathbf{f}}^\lambda - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2$ decreases when J and k increase. The main difference with the non smoothed estimation is the convergence to 0 of $\frac{1}{k} \|\hat{\mathbf{f}}^\lambda - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2$ when J remains fixed and k increases.

In Figure 8, the results of the estimation of the mean pattern are plotted for the stationary noise term. Figure 8a shows us a similar behavior of the non-smoothed Procrustes mean but with non-decreasing values of $\frac{1}{k} \|\hat{\mathbf{f}}^\lambda - \mathbf{f}^*\|_{\mathbb{R}^{k \times 2}}^2$ when k increases and J remains fixed. In Figure 8b, the smoothed Procrustes mean converges as k goes to infinity and the larger J the faster the convergence is.

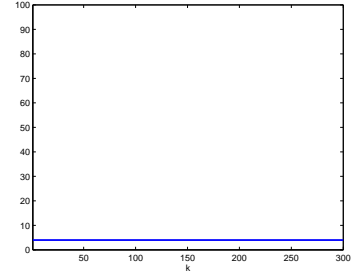
The results of the estimations of the mean pattern with the correlated noise are presented Figure 9. The results that appear Figure 9a are quite different compared to those presented Figures 7a and 8a.



(a) Observations $\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$

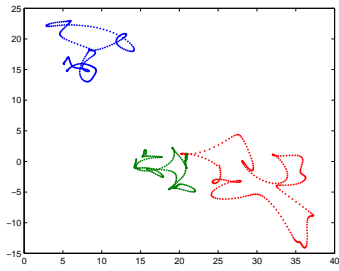


(b) A realization of ζ_1

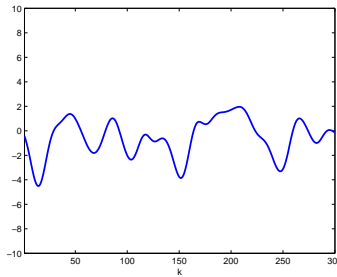


(c) Spectrum of Σ_1

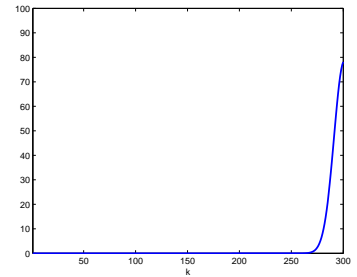
Figure 4: Example of data generated by model (1.2) with white noise in the case $k = 300$.



(a) Observations $\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$

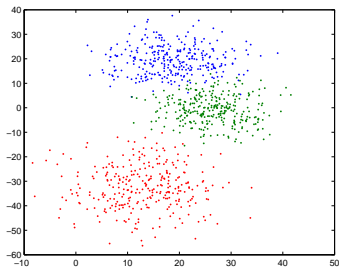


(b) A realization of ζ_1

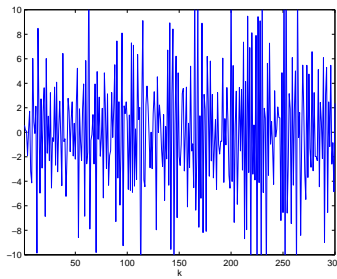


(c) Spectrum of Σ_2

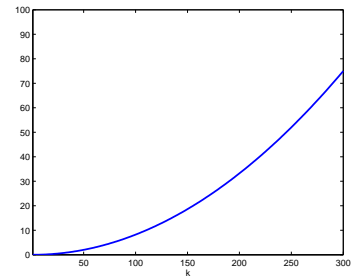
Figure 5: Example of data generated by model (1.2) with stationary noise in the case $k = 300$.



(a) Observations $\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$ ($k = 300$)



(b) A realization of ζ_1 ($k = 300$)



(c) Spectrum of Σ_3

Figure 6: Example of observations generated by model (1.2) with the correlated noise in the case $k = 300$.

The estimation seems to be worst when k increases and J remains fixed. The reason is that the level of noise, measured by $\gamma_{\max}(k)$ is increasing with k . The smoothing step is efficient and the estimations presented Figure 9b have a similar behavior to those of Figures 7b and 8b.

A Proofs

A.1 Proof of Lemma 4.1

Using the decomposition (4.2), we obtain the following identity

$$M^\lambda(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) = M_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}) + \bar{M}(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}). \quad (\text{A.1})$$

Note that we have used the fact that the subspaces \mathcal{V}_0^λ and $\bar{\mathcal{V}}$ are orthogonal and that $\bar{A}\mathbb{1}_k \otimes b_j = \mathbb{1}_k \otimes b_j$, for all $b_j \in \mathbb{R}^2$. Let us also introduce the notation $\bar{\mathbf{Y}}_j = \frac{1}{k} \sum_{\ell=1}^k [\mathbf{Y}_j]_\ell \in \mathbb{R}^2$, that is $\bar{A}\mathbf{Y}_j = \mathbb{1}_k \otimes \bar{\mathbf{Y}}_j \in \mathbb{R}^{k \times 2}$ is a degenerated configuration, and $\bar{\mathbf{Y}} = \frac{1}{J} \sum_{j=1}^J \bar{\mathbf{Y}}_j \in \mathbb{R}^2$. For a fixed $(\mathbf{a}, \boldsymbol{\alpha}) \in \mathbb{R}^J \times [-\pi, \pi]^J$, the functional $\mathbf{b} \mapsto \bar{M}(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b})$ vanishes if and only if there exists a $b_0 \in \mathbb{R}^2$ such that $e^{-a_j}(\bar{A}\mathbf{Y}_j - \mathbb{1}_k \otimes b_j)R_{-\alpha_j} = \mathbb{1}_k \otimes b_0$ for all $j = 1, \dots, J$. Therefore, for this fixed $(\mathbf{a}, \boldsymbol{\alpha})$, there is a unique point $\mathbf{b} = \mathbf{b}(\mathbf{a}, \boldsymbol{\alpha}) := (\bar{\mathbf{Y}}_1 - e^{a_1} \bar{\mathbf{Y}}(\bar{e^a R_\alpha})^{-1} R_{\alpha_1}, \dots, \bar{\mathbf{Y}}_J - e^{a_J} \bar{\mathbf{Y}}(\bar{e^a R_\alpha})^{-1} R_{\alpha_J})$ with $\bar{e^a R_\alpha} = \frac{1}{J} \sum_{j=1}^J e^{a_j} R_{\alpha_j} \in \mathbb{R}^{2 \times 2}$ and which satisfies,

$$\mathbf{b}(\mathbf{a}, \boldsymbol{\alpha}) = \underset{\mathbf{b} \in \Theta_0^{\mathbf{b}}}{\operatorname{argmin}} \bar{M}(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b})$$

Thence, thanks to the decomposition (A.1) and the fact that $\bar{M}(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}(\mathbf{a}, \boldsymbol{\alpha})) = 0$, we have

$$\underset{(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) \in \Theta_0}{\operatorname{argmin}} M^\lambda(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) = \left(\underset{(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}}{\operatorname{argmin}} M_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}), \mathbf{b} \left(\underset{(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}}{\operatorname{argmin}} M_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}) \right) \right),$$

and the claim is proved. \square

A.2 Proof of Theorem 5.1

For all $(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}$ and $(\mathbf{a}^*, \boldsymbol{\alpha}^*) \in [-A, A] \times [-\mathcal{A}, \mathcal{A}]$, we have the following inequality

$$\frac{1}{J} \|(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda) - (\mathbf{a}^*, \boldsymbol{\alpha}^*)\|_{\mathbb{R}^{2J}}^2 \leq \frac{2}{J} \|(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda) - (\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)\|_{\mathbb{R}^{2J}}^2 + \frac{2}{J} \|(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*) - (\mathbf{a}^*, \boldsymbol{\alpha}^*)\|_{\mathbb{R}^{2J}}^2.$$

The proof of Theorem 5.1 is a direct consequence of Proposition A.1 and Lemma A.2 below which control the convergence in probability of the two terms in the right hand side in the preceding inequality.

Proposition A.1. *Consider model (1.2) and suppose that Assumptions 1 and 2 hold and that Assumption 3 is verified with $A, \mathcal{A} < 0.1$. If $\lambda = \lambda(k) = \lfloor k^{\frac{1}{2s+1}} \rfloor$ then there exists a constant $C(L, s, A, \mathcal{A}, \mathbf{f}_0)$ such that for all $x > 0$*

$$\mathbb{P} \left(\frac{1}{J} \|(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda) - (\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)\|_{\mathbb{R}^{2J}}^2 \geq C(L, s, A, \mathcal{A}, \mathbf{f}_0) \left(F(k^{-\frac{2s}{2s+1}}) + F(V_1(k, J, x)) \right) \right) \leq 2e^{-x},$$

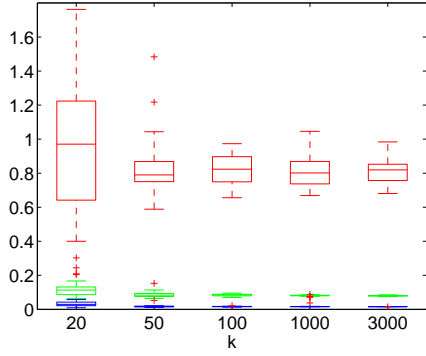
where $V_1(k, J, x) = 3\gamma_{\max}(k)k^{-\frac{2s}{2s+1}} (1 + \sqrt{2\frac{x}{Jk}} + 2\frac{x}{Jk})$ and $F: \mathbb{R}^+ \rightarrow \mathbb{R}$, with $F(u) = u + \sqrt{u}$.

The proof of Proposition A.1 is postponed to Section A.5. The following lemma is a direct consequence of Bernstein's inequality for bounded random variables, see e.g. Proposition 2.9 in [Mas07].

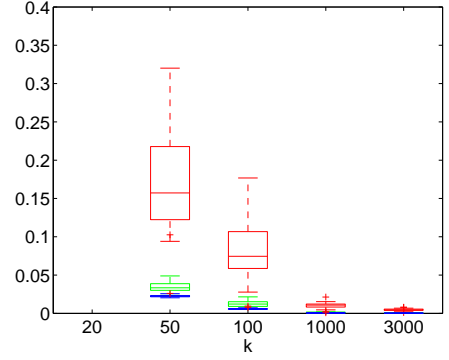
Lemma A.2. *Suppose that Assumption 3 holds and that the random variables (a_j^*, α_j^*) , $j = 1, \dots, J$ have zero expectation in $[-A, A] \times [-\mathcal{A}, \mathcal{A}]$. Then, for any $x > 0$, we have*

$$\mathbb{P} \left(\frac{1}{J} \|(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*) - (\mathbf{a}^*, \boldsymbol{\alpha}^*)\|_{\mathbb{R}^{2J}}^2 \geq C(A, \mathcal{A}) \left(\sqrt{\frac{2x}{J}} + \frac{x}{3J} \right)^2 \right) \leq 4e^{-x},$$

where $C(A, \mathcal{A}) = 4 \max\{A^2, \mathcal{A}^2\}$. \square

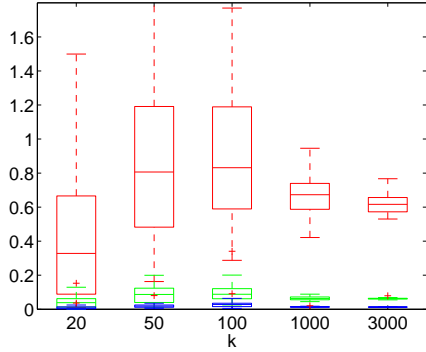


(a) Boxplot of $\frac{1}{k} \|\hat{\mathbf{f}} - \mathbf{f}^*\|_{\mathbb{R}^{k \times 2}}^2$ (estimation without smoothing).

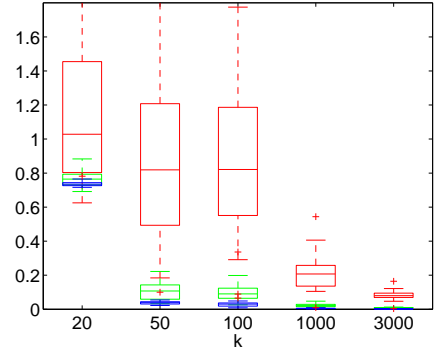


(b) Boxplot of $\frac{1}{k} \|\hat{\mathbf{f}}^\lambda - \mathbf{f}^*\|_{\mathbb{R}^{k \times 2}}^2$ (estimation with smoothing). The missing boxplots for $k = 20$ belong to the range $[0.7, 1.9]$.

Figure 7: Estimation of the mean pattern with the white noise. Boxplot in red correspond to $J = 10$, in green to $J = 100$ and in blue to $J = 500$.

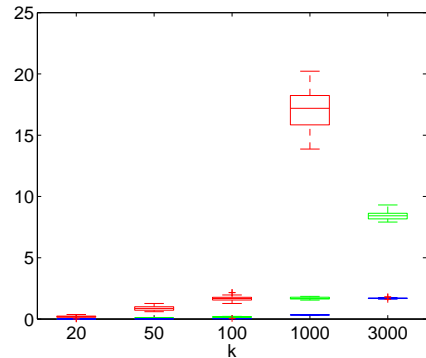


(a) Boxplot of $\frac{1}{k} \|\hat{\mathbf{f}} - \mathbf{f}^*\|_{\mathbb{R}^{k \times 2}}^2$ (estimation without smoothing).

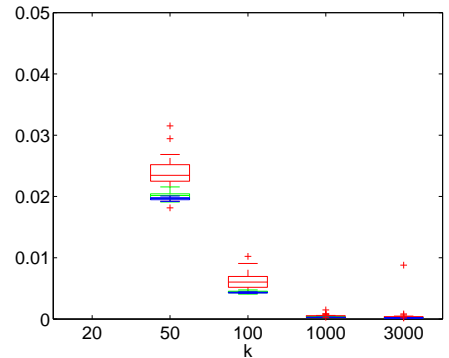


(b) Boxplot of $\frac{1}{k} \|\hat{\mathbf{f}}^\lambda - \mathbf{f}^*\|_{\mathbb{R}^{k \times 2}}^2$ (estimation with smoothing).

Figure 8: Estimation of the mean pattern with the stationary noise. Boxplot in red correspond to $J = 10$, in green to $J = 100$ and in blue to $J = 500$.



(a) Boxplot of $\frac{1}{k} \|\hat{\mathbf{f}} - \mathbf{f}^*\|_{\mathbb{R}^{k \times 2}}^2$ (estimation without smoothing). The missing red boxplot for $k = 20$ and $J = 10$ belongs to the range $[70, 110]$.



(b) Boxplot of $\frac{1}{k} \|\hat{\mathbf{f}}^\lambda - \mathbf{f}^*\|_{\mathbb{R}^{k \times 2}}^2$ (estimation with smoothing). The missing boxplots for $k = 20$ belong to the range $[0.7, 1.9]$.

Figure 9: Estimation of the mean pattern with the correlated noise. Boxplot in red correspond to $J = 10$, in green to $J = 100$ and in blue to $J = 500$.

A.3 Proof of Theorem 5.2

The proof of Theorem 5.2 follows the same guideline as the proof of Theorem 5.1. Consider the inequality

$$\frac{1}{J}\|\hat{\mathbf{b}}^\lambda - \mathbf{b}^*\|_{\mathbb{R}^{2J}}^2 \leq \frac{2}{J}\|\hat{\mathbf{b}}^\lambda - \mathbf{b}_{\Theta_0}^*\|_{\mathbb{R}^{2J}}^2 + \frac{2}{J}\|\mathbf{b}_{\Theta_0}^* - \mathbf{b}^*\|_{\mathbb{R}^{2J}}^2.$$

Theorem 5.2 is now a direct consequence of Proposition A.3 and Lemma A.4.

Proposition A.3. *Under the hypothesis of Proposition A.1, there exists a constant $C(L, s, A, \mathcal{A}, B, \mathbf{f}) > 0$ such that for all $x > 0$,*

$$\mathbb{P}\left(\frac{2}{J}\|\hat{\mathbf{b}}^\lambda - \mathbf{b}_{\Theta_0}^*\|_{\mathbb{R}^{2J}}^2 \geq C(L, s, A, \mathcal{A}, B, \mathbf{f})\left(F(k^{-\frac{-2s}{2s+1}}) + F(V_1(k, J, x)) + V_2(k, J, x)\right)\right) \leq 5e^{-x},$$

where $V_1(k, J, x) = 3\gamma_{\max}(k)k^{-\frac{2s}{2s+1}}(1 + \sqrt{2\frac{x}{Jk}} + 2\frac{x}{Jk})$, $V_2(k, J, x) = \frac{4}{k}\gamma_{\max}(k)(1 + \sqrt{2\frac{x}{J}} + \frac{x}{J})$ and $F(u) = u + \sqrt{u}$, $u \geq 0$.

The proof of Proposition A.3 is postponed to Section A.6.

Lemma A.4. *Suppose that Assumption 3 holds with $\mathcal{A} < \frac{\pi}{4}$ and that the random variables $(a_j^*, \alpha_j^*, b_j^*)$ $j = 1, \dots, J$ have zero expectation in $\Theta \subset \mathbb{R}^{4J}$. For any $x > 0$, we have*

$$\mathbb{P}\left(\frac{2}{J}\|\mathbf{b}_{\Theta_0}^* - \mathbf{b}^*\|_{\mathbb{R}^{2J}}^2 \geq C(A, \mathcal{A}, B)\left(\sqrt{\frac{2x}{J}} + \frac{x}{3J}\right)^2\right) \leq 4e^{-x},$$

where $C(A, \mathcal{A}, B) = 8B^2e^{4A}(\cos \mathcal{A} - \sin \mathcal{A})^{-2}$.

Proof. This result is a consequence of the Bernstein's inequality for bounded random variable. We have

$$\|\mathbf{b}_{\Theta_0}^* - \mathbf{b}^*\|_{\mathbb{R}^{2J}} = \sqrt{\sum_{j=1}^J \|e^{a_j^*} \bar{\mathbf{b}}^* (\overline{e^{a_j^*} R_{\alpha_j^*}})^{-1} R_{\alpha_j^*}\|_{\mathbb{R}^2}^2} \leq e^A \sqrt{J} \|\bar{\mathbf{b}}^* (\overline{e^{a_j^*} R_{\alpha_j^*}})^{-1}\|_{\mathbb{R}^2},$$

where $\overline{e^{a_j^*} R_{\alpha_j^*}} = \frac{1}{J} \sum_{j=1}^J e^{a_j^*} R_{\alpha_j^*}$ is an invertible 2×2 matrix whose smallest eigenvalue is greater than $e^{-A}(\cos \mathcal{A} - \sin \mathcal{A}) > 0$ as $\mathcal{A} < \frac{\pi}{4}$. To see this, remark that the eigenvalues of $\overline{e^{a_j^*} R_{\alpha_j^*}}$ are $\frac{1}{J} \sum_{j=1}^J e^{a_j^*} (\cos \alpha_j^* \pm i \sin \alpha_j^*)$ and we have $|\frac{1}{J} \sum_{j=1}^J e^{a_j^*} (\cos \alpha_j^* \pm i \sin \alpha_j^*)| \geq e^{-A}(\cos \mathcal{A} - \sin \mathcal{A}) > 0$. We now have

$$\frac{1}{\sqrt{J}}\|\mathbf{b}_{\Theta_0}^* - \mathbf{b}^*\|_{\mathbb{R}^{2J}} \leq C(A, \mathcal{A})\|\bar{\mathbf{b}}^*\|_{\mathbb{R}^2},$$

where $C(A, \mathcal{A}) = e^{2A}(\cos \mathcal{A} - \sin \mathcal{A})^{-1}$. Finally, for all $u > 0$ we have $\mathbb{P}(\frac{1}{\sqrt{J}}\|\mathbf{b}_{\Theta_0}^* - \mathbf{b}^*\|_{\mathbb{R}^{2J}} \geq u) \leq \mathbb{P}(C(A, \mathcal{A})\|\bar{\mathbf{b}}^*\|_{\mathbb{R}^2} \geq u)$ and a Bernstein type inequality (see e.g. Proposition 2.9 in [Mas07]) gives us $\mathbb{P}(\|\bar{\mathbf{b}}^*\|_{\mathbb{R}^2} \geq 2B(\sqrt{\frac{2x}{J}} + \frac{x}{3J})) \leq 4e^{-x}$ which yields

$$\mathbb{P}\left(\frac{1}{J}\|\mathbf{b}_{\Theta_0}^* - \mathbf{b}^*\|_{\mathbb{R}^{2J}}^2 \geq C(A, \mathcal{A}, B)\left(\sqrt{\frac{2x}{J}} + \frac{x}{3J}\right)^2\right) \leq 4e^{-x},$$

where $C(A, \mathcal{A}, B) = 4B^2e^{4A}(\cos \mathcal{A} - \sin \mathcal{A})^{-2}$. □

A.4 Proof of Theorem 5.3

Recall the notations introduced Section 2.1: for $\hat{g}_j^\lambda = (\hat{a}_j^\lambda, \hat{\alpha}_j^\lambda, \hat{b}_j^\lambda)$ we have $\hat{g}_j^\lambda \cdot \mathbf{f} = e^{\hat{a}_j^\lambda} \mathbf{f} R_{\hat{\alpha}_j^\lambda} + \hat{b}_j^\lambda$. Then, we have

$$\begin{aligned} \frac{1}{k} \|\hat{\mathbf{f}}^\lambda - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2 &= \frac{1}{k} \left\| \frac{1}{J} \sum_{j=1}^J (\hat{g}_j^\lambda)^{-1} \cdot (\mathbf{A}^\lambda \mathbf{Y}_j) - \mathbf{f} \right\|_{\mathbb{R}^{k \times 2}}^2 \\ &\leq \frac{2}{kJ} \sum_{j=1}^J \left\| \mathbf{A}^\lambda \left((\hat{g}_j^\lambda)^{-1} \cdot \mathbf{Y}_j - \mathbf{f} \right) \right\|_{\mathbb{R}^{k \times 2}}^2 + \frac{2}{k} \left\| \mathbf{A}^\lambda \mathbf{f} - \mathbf{f} \right\|_{\mathbb{R}^{k \times 2}}^2 \\ &\leq \underbrace{\frac{4}{kJ} \sum_{j=1}^J \left\| \mathbf{A}^\lambda \left((g_j^\lambda)^{-1} \cdot g_j^* \cdot \mathbf{f} - \mathbf{f} \right) \right\|_{\mathbb{R}^{k \times 2}}^2}_{\mathbf{V}} + \underbrace{\left\| e^{a_j^* - \hat{a}_j^\lambda} \mathbf{A}^\lambda \zeta_j R_{\alpha_j^* - \hat{\alpha}_j^\lambda} \right\|_{\mathbb{R}^{k \times 2}}^2}_{\mathbf{B}} + \frac{2}{k} \left\| \mathbf{A}^\lambda \mathbf{f} - \mathbf{f} \right\|_{\mathbb{R}^{k \times 2}}^2. \end{aligned}$$

The rest of the proof is devoted to control the terms \mathbf{B} and \mathbf{V} . The term \mathbf{B} is controlled by the bias of the low pass filter. According to Lemma B.1, if Assumption 1 holds and by choosing the optimal frequency cutoff $\lambda = \lambda(k) = \lfloor k^{\frac{1}{2s+1}} \rfloor$, there exists a constant $C(L, s) > 0$ such that

$$\mathbf{B} = \frac{2}{k} \left\| \mathbf{A}^\lambda \mathbf{f} - \mathbf{f} \right\|_{\mathbb{R}^{k \times 2}}^2 \leq C(L, s) k^{-\frac{2s}{2s+1}}. \quad (\text{A.2})$$

The term \mathbf{V} contains two expressions. To bound the first one we use Bessel's inequality and Lemma B.4. More precisely, we have for all $j = 1, \dots, J$

$$\begin{aligned} \frac{1}{k} \left\| \mathbf{A}^\lambda \left((g_j^\lambda)^{-1} \cdot g_j^* \cdot \mathbf{f} - \mathbf{f} \right) \right\|_{\mathbb{R}^{k \times 2}}^2 &\leq \frac{1}{k} \left\| (g_j^\lambda)^{-1} \cdot g_j^* \cdot \mathbf{f} - \mathbf{f} \right\|_{\mathbb{R}^{k \times 2}}^2 \\ &\leq C(A, \mathbf{f}) \left\| (a_j^* - \hat{a}_j^\lambda, \alpha_j^* - \hat{\alpha}_j^\lambda, e^{\hat{a}_j^\lambda} (b_j^* - \hat{b}_j^\lambda) R_{-\hat{\alpha}_j^\lambda}) \right\|_{\mathbb{R}^4}^2 \\ &\leq C(A, \mathbf{f}) \left\| (a_j^* - \hat{a}_j^\lambda, \alpha_j^* - \hat{\alpha}_j^\lambda) \right\|_{\mathbb{R}^2}^2 + C(A, \mathbf{f}) \|b_j^* - \hat{b}_j^\lambda\|_{\mathbb{R}^2}^2 \end{aligned}$$

We can now use Theorem 5.1 and 5.2 to derive the following concentration inequality,

$$\begin{aligned} \mathbb{P} \left(\frac{4}{kJ} \sum_{j=1}^J \left\| \mathbf{A}^\lambda \left((g_j^\lambda)^{-1} \cdot g_j^* \cdot \mathbf{f} - \mathbf{f} \right) \right\|_{\mathbb{R}^{k \times 2}}^2 \right. \\ \left. \geq C(L, s, A, \mathcal{A}, B, \mathbf{f}) \left(A_1(k, J, x) + A_3(k, J, x) + A_2(J, x) \right) \right) \leq 13e^{-x} \end{aligned} \quad (\text{A.3})$$

where $C(L, s, A, \mathcal{A}, B, \mathbf{f}) > 0$ is a constant independent of k and J and A_1, A_2, A_3 are defined in the statement of Theorem 5.1 and 5.2. The second term contained in \mathbf{V} is treated by equation (A.10) below. Hence, formulas (A.3) and (A.10) yield

$$\mathbb{P} \left(\mathbf{V} \geq C(L, s, A, \mathcal{A}, B, \mathbf{f}) \left(A_1(k, J, x) + A_3(k, J, x) + A_2(J, x) \right) \right) \leq 14e^{-x}, \quad (\text{A.4})$$

for some constant $C(L, s, A, \mathcal{A}, B, \mathbf{f}) > 0$. Putting together equations (A.2) and (A.4) gives

$$\mathbb{P} \left(\frac{1}{k} \|\hat{\mathbf{f}}^\lambda - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2 \geq C(L, s, A, \mathcal{A}, B, \mathbf{f}) \left(A_1(k, J, x) + A_3(k, J, x) + A_2(J, x) \right) \right) \leq 14e^{-x}.$$

for some constant $C(L, s, A, \mathcal{A}, B, \mathbf{f}) > 0$. The proof of Theorem 5.3 is completed. \square

A.5 Proof of Proposition A.1

The mean pattern \mathbf{f} can be decomposed as $\mathbf{f} = \bar{\mathbf{f}} + \mathbf{f}_0 \in \mathbf{1}_{k \times 2} \oplus \mathbf{1}_{k \times 2}^\perp$. Then, \mathbf{f}_0 is the centered version of \mathbf{f} and we can consider the criterion,

$$D_0(\mathbf{a}, \boldsymbol{\alpha}) = \frac{1}{Jk} \sum_{j=1}^J \left\| e^{a_j^* - a_j} \mathbf{f}_0 R_{\alpha_j^* - \alpha_j} - \frac{1}{J} \sum_{j'=1}^J e^{a_{j'}^* - a_{j'}} \mathbf{f}_0 R_{\alpha_{j'}^* - \alpha_{j'}} \right\|_{\mathbb{R}^{k \times 2}}^2.$$

We now have,

$$(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda) = \underset{(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}}{\operatorname{argmin}} M_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}) \quad \text{and} \quad (\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*) = \underset{(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}}{\operatorname{argmin}} D_0(\mathbf{a}, \boldsymbol{\alpha}).$$

Then, the convergence of $(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda)$ to $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)$ is guaranteed if $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)$ is uniquely defined and if there is a uniform convergence in probability of M_0^λ to D_0 , see e.g. [vdV98]. This is the aim of Lemmas A.5 and A.6 below.

Lemma A.5. *Let \mathbf{f} be a non-degenerated configuration in $\mathbb{R}^{k \times 2}$, i.e. $\mathbf{f} \notin \mathbf{1}_{k \times 2}$. Then, the argmin of D_0 on $\Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}$ is unique and denoted by $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*) = (a_j - \bar{\mathbf{a}}^*, \alpha_j - \bar{\boldsymbol{\alpha}}^*)$, where $\bar{\mathbf{a}}^* = \frac{1}{J} \sum_{j=1}^J a_j^*$, $\bar{\boldsymbol{\alpha}}^* = \frac{1}{J} \sum_{j=1}^J \alpha_j^*$.*

Proof. As \mathbf{f} is a non-degenerated configuration, we have $\mathbf{f}_0 \neq 0$. Thus, the stabilizer of \mathbf{f} is reduced to the identity, see Section 2.1. Then $D_0(\mathbf{a}, \boldsymbol{\alpha}) = 0$ if and only if there exists $(a_0, \alpha_0) \in \mathbb{R}^2$ such that $(\mathbf{a}, \boldsymbol{\alpha}) = (\mathbf{a}^*, \boldsymbol{\alpha}^*) * (a_0, \alpha_0) = (a_1^* + a_0, \alpha_1^* + \alpha_0, \dots, a_J + a_0, \alpha_J^* + \alpha_0)$. By choosing $(a_0, \alpha_0) = (-\bar{\mathbf{a}}^*, -\bar{\boldsymbol{\alpha}}^*)$ we have $\sum_{j=1}^J (a_j^*, \alpha_j^*) * (a_0, \alpha_0) = 0$. That is $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*) = (\mathbf{a}^*, \boldsymbol{\alpha}^*) * (a_0, \alpha_0) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}$. \square

We now show the uniform convergence in probability,

Lemma A.6. *Suppose that Assumptions 1, 2 and 3 hold and let $F : \mathbb{R} \rightarrow \mathbb{R}$, with $F(u) = u + \sqrt{|u|}$. For any $x > 0$ we have*

$$\mathbb{P} \left(\sup_{(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}} |M_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}) - D_0(\mathbf{a}, \boldsymbol{\alpha})| \geq C(L, s, A, \mathbf{f}_0) \left(F(k^{-\frac{2s}{2s+1}}) + F(V(k, J, x)) \right) \right) \leq 2e^{-x}$$

where $C(L, s, A, \mathbf{f}_0) = e^{2A} \max \left\{ \frac{2}{\sqrt{k}} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}, \frac{\sqrt{2}e^A}{\sqrt{k}} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}, 2 \right\}$ and $V(k, J, x) = 3\gamma_{\max}(k)k^{-\frac{2s}{2s+1}}(1 + F(\frac{2x}{Jk}))$.

Proof. Let us write the following decomposition,

$$M_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}) = \frac{1}{Jk} \sum_{j=1}^J \left\| e^{a_j^* - a_j} \mathbf{f}_0 R_{\alpha_j^* - \alpha_j} - \frac{1}{J} \sum_{j'=1}^J e^{a_{j'}^* - a_{j'}} \mathbf{f}_0 R_{\alpha_{j'}^* - \alpha_{j'}} \right\|_{\mathbb{R}^{k \times 2}}^2 \quad (\text{A.5})$$

$$+ \frac{1}{Jk} \sum_{j=1}^J \left\| e^{a_j^* - a_j} (A_0^\lambda \mathbf{f} - \mathbf{f}_0) R_{\alpha_j^* - \alpha_j} - \frac{1}{J} \sum_{j'=1}^J e^{a_{j'}^* - a_{j'}} (A_0^\lambda \mathbf{f} - \mathbf{f}_0) R_{\alpha_{j'}^* - \alpha_{j'}} \right\|_{\mathbb{R}^{k \times 2}}^2 \quad (\text{A.6})$$

$$+ \frac{2}{Jk} \sum_{j=1}^J \left\langle e^{a_j^* - a_j} \mathbf{f}_0 R_{\alpha_j^* - \alpha_j} - \frac{1}{J} \sum_{j'=1}^J e^{a_{j'}^* - a_{j'}} \mathbf{f}_0 R_{\alpha_{j'}^* - \alpha_{j'}}, \right. \quad (\text{A.7})$$

$$\left. e^{a_j^* - a_j} (A_0^\lambda \mathbf{f} - \mathbf{f}_0) R_{\alpha_j^* - \alpha_j} - \frac{1}{J} \sum_{j'=1}^J e^{a_{j'}^* - a_{j'}} (A_0^\lambda \mathbf{f} - \mathbf{f}_0) R_{\alpha_{j'}^* - \alpha_{j'}} \right\rangle_{\mathbb{R}^{k \times 2}}$$

$$+ \frac{1}{Jk} \sum_{j=1}^J \left\| e^{a_j^* - a_j} A_0^\lambda \boldsymbol{\zeta}_j R_{\alpha_j^* - \alpha_j} - \frac{1}{J} \sum_{j'=1}^J e^{a_{j'}^* - a_{j'}} A_0^\lambda \boldsymbol{\zeta}_{j'} R_{\alpha_{j'}^* - \alpha_{j'}} \right\|_{\mathbb{R}^{k \times 2}}^2 \quad (\text{A.8})$$

$$+ \frac{2}{Jk} \sum_{j=1}^J \left\langle e^{a_j^* - a_j} A_0^\lambda \mathbf{f} R_{\alpha_j^* - \alpha_j} - \frac{1}{J} \sum_{j'=1}^J e^{a_{j'}^* - a_{j'}} A_0^\lambda \mathbf{f} R_{\alpha_{j'}^* - \alpha_{j'}}, \right. \quad (\text{A.9})$$

$$\left. e^{a_j^* - a_j} A_0^\lambda \boldsymbol{\zeta}_j R_{\alpha_j^* - \alpha_j} - \frac{1}{J} \sum_{j'=1}^J e^{a_{j'}^* - a_{j'}} A_0^\lambda \boldsymbol{\zeta}_{j'} R_{\alpha_{j'}^* - \alpha_{j'}} \right\rangle_{\mathbb{R}^{k \times 2}}$$

Then, criterion M_0^λ is viewed as a perturbed version of the criterion $D_0(\mathbf{a}, \boldsymbol{\alpha}) = (\text{A.5})$,

$$M_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}) = D_0(\mathbf{a}, \boldsymbol{\alpha}) + \mathbf{B}_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}) + \mathbf{V}_0^\lambda(\mathbf{a}, \boldsymbol{\alpha})$$

where the bias term is $\mathbf{B}_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}) = (\text{A.6}) + (\text{A.7})$ and the variance term is $\mathbf{V}_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}) = (\text{A.8}) + (\text{A.9})$.

The bias term. We have for any $(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}$,

$$(A.6) = \frac{1}{Jk} \sum_{j=1}^J \left\| e^{a_j^* - a_j} (A_0^\lambda \mathbf{f} - \mathbf{f}_0) R_{\alpha_j^* - \alpha_j} \right\|_{\mathbb{R}^{k \times 2}}^2 - \left\| \frac{1}{J} \sum_{j'=1}^J e^{a_{j'}^* - a_{j'}} (A_0^\lambda \mathbf{f} - \mathbf{f}_0) R_{\alpha_{j'}^* - \alpha_{j'}} \right\|_{\mathbb{R}^{k \times 2}}^2$$

$$\leq e^{2A} \frac{1}{k} \left\| A_0^\lambda \mathbf{f} - \mathbf{f}_0 \right\|_{\mathbb{R}^{k \times 2}}^2.$$

A double application of Cauchy-Schwarz inequality implies that,

$$(A.7) \leq \frac{2}{Jk} \sum_{j=1}^J \left\| e^{a_j^* - a_j} \mathbf{f}_0 R_{\alpha_j^* - \alpha_j} - \frac{1}{J} \sum_{j'=1}^J e^{a_{j'}^* - a_{j'}} \mathbf{f}_0 R_{\alpha_{j'}^* - \alpha_{j'}} \right\|_{\mathbb{R}^{k \times 2}}$$

$$\left\| e^{a_j^* - a_j} (A_0^\lambda \mathbf{f} - \mathbf{f}_0) R_{\alpha_j^* - \alpha_j} - \frac{1}{J} \sum_{j'=1}^J e^{a_{j'}^* - a_{j'}} (A_0^\lambda \mathbf{f} - \mathbf{f}_0) R_{\alpha_{j'}^* - \alpha_{j'}} \right\|_{\mathbb{R}^{k \times 2}}$$

$$\leq \frac{2}{k} \left(\frac{1}{J} \sum_{j=1}^J e^{2A} \|\mathbf{f}_0 R_{\alpha_j^* - \alpha_j}\|_{\mathbb{R}^{k \times 2}}^2 \right)^{\frac{1}{2}} \left(\frac{1}{J} \sum_{j=1}^J e^{2A} \|(A_0^\lambda \mathbf{f} - \mathbf{f}_0) R_{\alpha_j^* - \alpha_j}\|_{\mathbb{R}^{k \times 2}}^2 \right)^{\frac{1}{2}}$$

$$= 2e^{2A} \frac{1}{k} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}} \|A_0^\lambda \mathbf{f} - \mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}.$$

Finally, by using Lemma B.1 we have,

$$\sup_{(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta^{\mathbf{a}, \boldsymbol{\alpha}}} |\mathbf{B}_0^\lambda(\mathbf{a}, \boldsymbol{\alpha})| \leq C_1(L, s, A, \mathbf{f}) \left(k^{-\frac{2s}{2s+1}} + k^{-\frac{s}{2s+1}} \right).$$

where $C_1(L, s, A, \mathbf{f}) = C(L, s) e^{2A} \max \left\{ 2 \frac{1}{\sqrt{k}} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}, 1 \right\}$.

The variance term. First, the term (A.9) is by the Cauchy-Schwarz inequality controlled by $2e^{2A} \frac{1}{\sqrt{k}} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}} \sqrt{(A.8)}$. The term (A.8) is bounded from above by $\frac{1}{Jk} \sum_{j=1}^J e^{2A} \|A_0^\lambda \boldsymbol{\zeta}_j\|_{\mathbb{R}^{k \times 2}}^2$. To derive an upper bound in probability, note that we have the following equality in law,

$$\sum_{j=1}^J \|A_0^\lambda \boldsymbol{\zeta}_j\|_{\mathbb{R}^{k \times 2}}^2 = \boldsymbol{\xi}' \mathbf{B} \boldsymbol{\xi},$$

with $\mathbf{B} = [Id_J \otimes \boldsymbol{\Sigma}^{\frac{1}{2}}] [Id_{2J} \otimes (A_0^\lambda)' A_0^\lambda] [Id_J \otimes \boldsymbol{\Sigma}^{\frac{1}{2}}] \in \mathbb{R}^{2Jk \times 2Jk}$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{2Jk})'$ is a centered Gaussian vector of variance Id_{2Jk} . We have $\text{tr}(Id_J \otimes \boldsymbol{\Sigma}) \leq 2Jk \gamma_{\max}(k)$ and $\text{tr}((A_0^\lambda)' A_0^\lambda) = \frac{2\lambda+1}{k}$. Using a classical concentration inequality for quadratic form of multivariate Gaussian random variables, see e.g. [LM00] Lemma 1, we have for all $x > 0$, $\mathbb{P} \left(\boldsymbol{\xi}' \mathbf{B} \boldsymbol{\xi} \geq 2Jk \gamma_{\max}(k) \frac{2\lambda+1}{k} + 2\gamma_{\max}(k) \frac{2\lambda+1}{k} \sqrt{x 2Jk} + 4x \gamma_{\max}(k) \frac{2\lambda+1}{k} \right) \leq e^{-x}$, which yields together with formula (A.2),

$$\mathbb{P} \left(\frac{1}{Jk} \sum_{j=1}^J e^{2A} \|A_0^\lambda \boldsymbol{\zeta}_j\|_{\mathbb{R}^{k \times 2}}^2 \geq 6e^{2A} \gamma_{\max}(k) k^{-\frac{2s}{2s+1}} \left(1 + \sqrt{2 \frac{x}{Jk}} + 2 \frac{x}{Jk} \right) \right) \leq e^{-x}. \quad (A.10)$$

Hence we have,

$$\mathbb{P} \left(\sup_{(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta^{\mathbf{a}, \boldsymbol{\alpha}}} |\mathbf{V}_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}) + \mathbf{B}_0^\lambda(\mathbf{a}, \boldsymbol{\alpha})| \geq C(L, s, A, \mathbf{f}_0) \left(k^{-\frac{2s}{2s+1}} + k^{-\frac{s}{2s+1}} + V_1(k, J, x) + \sqrt{V_1(k, J, x)} \right) \right) \leq 2e^{-x},$$

where $C(L, s, A, \mathbf{f}_0) = e^{2A} \max \left\{ \frac{2}{\sqrt{k}} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}, \frac{1}{\sqrt{k}} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}} \sqrt{2} e^A, 2 \right\}$ and $V_1(k, J, x) = 3 \frac{\gamma_{\max}(k)}{k^{2s+1}} (1 + \sqrt{2 \frac{x}{Jk}} + 2 \frac{x}{Jk})$. \square

For a fixed $J \in \mathbb{N}$, the convergence of the M-estimator $(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda)$ to $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)$ when $k \rightarrow \infty$ is guaranteed by Lemma A.5 and A.6, see e.g. [vdV98]. Nevertheless, we are able to give a rate of convergence and non-asymptotic bounds in k and J by using the classical inequality,

$$|D_0(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda) - D_0(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)| \leq 2 \sup_{(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}} |D_0(\mathbf{a}, \boldsymbol{\alpha}) - M_0^\lambda(\mathbf{a}, \boldsymbol{\alpha})|.$$

This, together with Lemma A.7 below will prove Proposition A.1.

Lemma A.7. *Assume that $A, \mathcal{A} < 0.1$. There exists a constant $C(A, \mathcal{A}) > 0$ independent of J such that for all $(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}$ we have*

$$|D_0(\mathbf{a}, \boldsymbol{\alpha}) - D_0(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)| \geq C(A, \mathcal{A}, \mathbf{f}_0) \frac{1}{J} \left\| (\mathbf{a} - \mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha} - \boldsymbol{\alpha}_{\Theta_0}^*) \right\|_{\mathbb{R}^{2J}}^2,$$

where $C(A, \mathcal{A}, \mathbf{f}_0) = C(A, \mathcal{A}) \frac{1}{k} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2$.

Proof. By definition, given a $(\mathbf{a}^*, \boldsymbol{\alpha}^*) \in [-A, A]^J \times [-\mathcal{A}, \mathcal{A}]^J$, the point $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)$ is the unique minimum of D_0 on $\Theta_0^{\mathbf{a}, \boldsymbol{\alpha}} = [-A, A]^J \cap \mathbb{1}_{\frac{1}{J}} \times [-\mathcal{A}, \mathcal{A}]^J \cap \mathbb{1}_{\frac{1}{J}}$. Then, for all $(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}$, there exists a $\mathbf{c} = \mathbf{c}(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}$ such that the Taylor expansion of D_0 at $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)$ can be written,

$$\begin{aligned} D_0(\mathbf{a}, \boldsymbol{\alpha}) - D_0(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*) &= \frac{1}{2} (\mathbf{a} - \mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha} - \boldsymbol{\alpha}_{\Theta_0}^*)' [\nabla^2 D_0(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)] (\mathbf{a} - \mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha} - \boldsymbol{\alpha}_{\Theta_0}^*) \\ &\quad + \frac{1}{6} [\nabla^3 D_0(\mathbf{c})] (\mathbf{a} - \mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha} - \boldsymbol{\alpha}_{\Theta_0}^*). \end{aligned}$$

Let $\delta = \max\{A, \mathcal{A}\}$. This, together with Lemma B.2 and B.3 imply that

$$\begin{aligned} D_0(\mathbf{a}, \boldsymbol{\alpha}) - D_0(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*) &\geq \frac{1}{2} (\mathbf{a} - \mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha} - \boldsymbol{\alpha}_{\Theta_0}^*)' [\nabla^2 D_0(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)] (\mathbf{a} - \mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha} - \boldsymbol{\alpha}_{\Theta_0}^*) \\ &\quad - \delta \frac{40}{6} e^{2A} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2 \frac{1}{Jk} \left\| (\mathbf{a} - \mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha} - \boldsymbol{\alpha}_{\Theta_0}^*) \right\|_{\mathbb{R}^{2J}}^2 \\ &\geq \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2 \frac{1}{Jk} \left\| (\mathbf{a} - \mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha} - \boldsymbol{\alpha}_{\Theta_0}^*) \right\|_{\mathbb{R}^{2J}}^2 \left(e^{-2A} - \delta \frac{40}{6} e^{2A} \right). \end{aligned}$$

Hence, one can choose $\delta > 0$ sufficiently small such that $(e^{-2A} - \delta \frac{40}{6} e^{2A})$ is strictly positive for all J and k . For example, we have $(e^{-2\delta} - \delta \frac{40}{6} e^{2\delta}) > 0$, if $\delta < 0.1$. Then, using such a δ it follows that for all $(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}$,

$$|D_0(\mathbf{a}, \boldsymbol{\alpha}) - D_0(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)| \geq C(A, \mathcal{A}) \frac{1}{k} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2 \frac{1}{J} \left\| (\mathbf{a} - \mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha} - \boldsymbol{\alpha}_{\Theta_0}^*) \right\|_{\mathbb{R}^{2J}}^2. \quad \square$$

The proof of Proposition A.1 is almost done. Remark that Lemma A.7 ensures that for all $u \geq 0$ we have $\mathbb{P}\left(\frac{1}{J} \left\| (\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda) - (\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*) \right\|_{\mathbb{R}^{2J}}^2 \geq u\right) \leq \mathbb{P}\left(\frac{2}{C(A, \mathcal{A}, \mathbf{f}_0)} \sup_{(\mathbf{a}, \boldsymbol{\alpha}) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}} |M_0^\lambda(\mathbf{a}, \boldsymbol{\alpha}) - D_0(\mathbf{a}, \boldsymbol{\alpha})| \geq u\right)$. Lemma A.6 ensures that there is a constant $C(L, s, A, \mathcal{A}, \mathbf{f}_0)$ such that for all $x > 0$,

$$\mathbb{P}\left(\frac{1}{J} \left\| (\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda) - (\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*) \right\|_{\mathbb{R}^{2J}}^2 \geq C(L, s, A, \mathcal{A}, \mathbf{f}_0) (F(k^{-\frac{2s}{2s+1}}) + F(V_1(k, J, x)))\right) \leq 2e^{-x}.$$

where $V_1(k, J, x) = \gamma_{\max}(k) k^{-\frac{2s}{2s+1}} (1 + \sqrt{2\frac{x}{Jk}} + 2\frac{x}{Jk})$ and $F: \mathbb{R} \rightarrow \mathbb{R}$, with $F(u) = u + \sqrt{u}$. \square

A.6 Proof of Proposition A.3

First of all remark that, thanks to formulas (3.3) and (4.5), we have explicit expressions of $\mathbf{b}_{\Theta_0}^* = \operatorname{argmin}_{\mathbf{b} \in \Theta_0^{\mathbf{b}}} \bar{D}(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b})$ and of $\hat{\mathbf{b}}^\lambda = \operatorname{argmin}_{\mathbf{b} \in \Theta_0^{\mathbf{b}}} \bar{M}(\hat{\mathbf{a}}^\lambda, \hat{\boldsymbol{\alpha}}^\lambda, \mathbf{b})$. Indeed, we have,

$$\mathbf{b}_{\Theta_0}^* = (b_1^* - e^{\alpha_1^*} \bar{\mathbf{b}}^* (\overline{e^{\alpha^*} R_{\alpha^*}})^{-1} R_{\alpha_1^*}, \dots, b_J^* - e^{\alpha_J^*} \bar{\mathbf{b}}^* (\overline{e^{\alpha^*} R_{\alpha^*}})^{-1} R_{\alpha_J^*}),$$

where $\overline{e^{a^*} R_{\alpha^*}} = \frac{1}{J} \sum_{j=1}^J e^{a_j^*} R_{\alpha_j^*} \in \mathbb{R}^{2 \times 2}$, $\bar{\mathbf{b}}^* = \frac{1}{J} \sum_{j=1}^J b_j^* \in \mathbb{R}^2$ and

$$\hat{\mathbf{b}}^\lambda = (\bar{\mathbf{Y}}_1 - e^{\hat{a}_1^\lambda} \bar{\mathbf{Y}} (\overline{e^{\hat{a}_1^\lambda} R_{\hat{\alpha}_1^\lambda}})^{-1} R_{\hat{\alpha}_1^\lambda}, \dots, \bar{\mathbf{Y}}_J - e^{\hat{a}_J^\lambda} \bar{\mathbf{Y}} (\overline{e^{\hat{a}_J^\lambda} R_{\hat{\alpha}_J^\lambda}})^{-1} R_{\hat{\alpha}_J^\lambda})$$

where $\overline{e^{\hat{a}^\lambda} R_{\hat{\alpha}^\lambda}} = \frac{1}{J} \sum_{j=1}^J e^{\hat{a}_j^\lambda} R_{\hat{\alpha}_j^\lambda} \in \mathbb{R}^{2 \times 2}$, $\bar{\mathbf{Y}}_j = \frac{1}{k} \sum_{\ell=1}^k [\mathbf{Y}_j]_\ell = e^{a_j^*} \bar{\mathbf{f}} R_{\alpha_j^*} + b_j^* + e^{a_j^*} \bar{\boldsymbol{\zeta}}_j R_{\alpha_j^*} \in \mathbb{R}^2$ where $\bar{\mathbf{f}} = \frac{1}{k} \sum_{\ell=1}^k \mathbf{f}_\ell \in \mathbb{R}^2$, $\bar{\boldsymbol{\zeta}}_j = \frac{1}{k} \sum_{\ell=1}^k [\boldsymbol{\zeta}_j]_\ell \in \mathbb{R}^2$ and $\bar{\mathbf{Y}} = \frac{1}{J} \sum_{j=1}^J \bar{\mathbf{Y}}_j = \bar{\mathbf{f}} (\overline{e^{a^*} R_{\alpha^*}}) + \bar{\mathbf{b}}^* + \frac{1}{J} \sum_{j=1}^J e^{a_j^*} \bar{\boldsymbol{\zeta}}_j R_{\alpha_j^*} \in \mathbb{R}^2$. Thence, we have

$$\frac{1}{J} \|\mathbf{b}_{\Theta_0}^* - \hat{\mathbf{b}}^\lambda\|_{\mathbb{R}^{2J}}^2 \leq \frac{1}{J} \sum_{j=1}^J \|\bar{\mathbf{f}} (e^{a_j^*} R_{\alpha_j^*} - e^{\hat{a}_j^\lambda} (\overline{e^{a^*} R_{\alpha^*}}) (\overline{e^{\hat{a}^\lambda} R_{\hat{\alpha}^\lambda}})^{-1} R_{\hat{\alpha}_j^\lambda})\|_{\mathbb{R}^2}^2 \quad (\text{A.11})$$

$$+ \|\bar{\mathbf{b}}^* (e^{a_j^*} (\overline{e^{a^*} R_{\alpha^*}})^{-1} R_{\alpha_j^*} - e^{\hat{a}_j^\lambda} (\overline{e^{\hat{a}^\lambda} R_{\hat{\alpha}^\lambda}})^{-1} R_{\hat{\alpha}_j^\lambda})\|_{\mathbb{R}^2}^2 \quad (\text{A.12})$$

$$+ \|e^{a_j^*} \bar{\boldsymbol{\zeta}}_j R_{\alpha_j^*}\|_{\mathbb{R}^2}^2 + \|e^{\hat{a}_j^\lambda} (\overline{e^{a^*} R_{\alpha^*}}) (\overline{e^{\hat{a}^\lambda} R_{\hat{\alpha}^\lambda}})^{-1} R_{\hat{\alpha}_j^\lambda}\|_{\mathbb{R}^2}^2 \quad (\text{A.13})$$

where $(\overline{e^{a^*} R_{\alpha^*}}) = \frac{1}{J} \sum_{j=1}^J e^{a_j^*} \bar{\boldsymbol{\zeta}}_j R_{\alpha_j^*} \in \mathbb{R}^2$. The rest of the proof is devoted to the control of the terms (A.11), (A.12) and (A.13).

In this section, we denote by $\|\cdot\|_{op}$ the operator norm of a 2×2 matrix, *i.e.* $\|A\|_{op} = |\gamma_{\max}(A)|$ where $\gamma_{\max}(A)$ denotes the largest eigenvalue of a matrix $A \in \mathbb{R}^{2 \times 2}$. Note by the way that the eigenvalues of the matrix $\frac{1}{J} \sum_{j=1}^J e^{a_j} R_{\alpha_j} \in \mathbb{R}^{2 \times 2}$ are $\frac{1}{J} \sum_{j=1}^J e^{a_j} (\cos(\alpha_j) \pm i \sin(\alpha_j))$ for any J and $(a_1, \dots, a_J, \alpha_1, \dots, \alpha_J) \in [-A, A]^J \times [-\mathcal{A}, \mathcal{A}]^J$. It yields $\|e^{-a_j} R_{-\alpha_j}\|_{op} \leq e^A$ and $\|(\overline{e^{\mathcal{A}} R_{\mathcal{A}}})^{-1}\|_{op} \leq e^A (\cos(\mathcal{A}) - \sin(\mathcal{A}))^{-1}$ which is a positive real number since by hypothesis we have $\mathcal{A} < \frac{\pi}{4}$. We are now able to derive an upper bound for (A.11),

$$\begin{aligned} (\text{A.11}) &\leq \|\bar{\mathbf{f}}\|_{\mathbb{R}^2}^2 \frac{1}{J} \sum_{j=1}^J \|e^{a_j^*} R_{\alpha_j^*} - e^{\hat{a}_j^\lambda} (\overline{e^{a^*} R_{\alpha^*}}) (\overline{e^{\hat{a}^\lambda} R_{\hat{\alpha}^\lambda}})^{-1} R_{\hat{\alpha}_j^\lambda}\|_{op}^2 \\ &\leq e^{2A} \|\bar{\mathbf{f}}\|_{\mathbb{R}^2}^2 \frac{1}{J} \sum_{j=1}^J \|e^{a_j^*} R_{\alpha_j^*} (e^{-\hat{a}_j^\lambda} R_{-\hat{\alpha}_j^\lambda}) - (\overline{e^{a^*} R_{\alpha^*}}) (\overline{e^{\hat{a}^\lambda} R_{\hat{\alpha}^\lambda}})^{-1}\|_{op}^2 \\ &\leq 2e^{2A} \|\bar{\mathbf{f}}\|_{\mathbb{R}^2}^2 \frac{1}{J} \sum_{j=1}^J \|e^{-\hat{a}_j^\lambda} R_{-\hat{\alpha}_j^\lambda}\|_{op}^2 \|e^{a_j^*} R_{\alpha_j^*} - e^{\hat{a}_j^\lambda} R_{\hat{\alpha}_j^\lambda}\|_{op}^2 \\ &\quad + 2e^{2A} \|\bar{\mathbf{f}}\|_{\mathbb{R}^2}^2 \|(\overline{e^{\hat{a}^\lambda} R_{\hat{\alpha}^\lambda}})^{-1}\|_{op}^2 \|(\overline{e^{a^*} R_{\alpha^*}}) - (\overline{e^{\hat{a}^\lambda} R_{\hat{\alpha}^\lambda}})\|_{op}^2 \end{aligned}$$

It is now easy to show that there exists a constant $C(A, \mathcal{A}, \mathbf{f}) > 0$ independent of k and J such that

$$(\text{A.11}) \leq C(A, \mathcal{A}, \mathbf{f}) \frac{1}{J} \sum_{j=1}^J \|(a_j^* - \hat{a}_j^\lambda, \alpha_j^* - \hat{\alpha}_j^\lambda)\|_{\mathbb{R}^2}^2. \quad (\text{A.14})$$

The term (A.12) is very similar to the term (A.11) and we have

$$(\text{A.12}) \leq C(A, \mathcal{A}, B) \frac{1}{J} \sum_{j=1}^J \|(a_j^* - \hat{a}_j^\lambda, \alpha_j^* - \hat{\alpha}_j^\lambda)\|_{\mathbb{R}^2}^2, \quad (\text{A.15})$$

for some constant $C(A, \mathcal{A}, B) > 0$ independent of k and J . By using formula (A.14), (A.15) and Proposition A.1 together, there exists a constant $C(L, s, A, \mathcal{A}, B, \mathbf{f})$ independent of k and J such that for all $x > 0$,

$$\mathbb{P}\left((\text{A.11}) + (\text{A.12}) \geq C(L, s, A, \mathcal{A}, B, \mathbf{f}) (F(k^{-\frac{2s}{2s+1}}) + F(V_1(k, J, x)))\right) \leq 4e^{-x}, \quad (\text{A.16})$$

where $V_1(k, J, x)$ is defined in statement of proposition A.1 and $F(u) = u + \sqrt{u}$ for $u \geq 0$.

The term (A.13) can be bounded in probability as follows. First of all, remark that there exists a constant $C(A, \mathcal{A}) > 0$ such that

$$(A.13) \leq C(A, \mathcal{A}) \frac{1}{J} \sum_{j=1}^J \|\bar{\zeta}_j\|_{\mathbb{R}^2}^2.$$

Then, for all $j = 1, \dots, J$, the random variable $\bar{\zeta}_j = \frac{1}{k} \sum_{\ell=1}^k ([\zeta_j^{(1)}]_{\ell}, [\zeta_j^{(2)}]_{\ell}) \in \mathbb{R}^2$ can be written

$$(\bar{\zeta}_j^1, \bar{\zeta}_j^2)' = \frac{1}{k} \begin{pmatrix} \mathbf{1}'_k & \mathbf{0}'_k \\ \mathbf{0}'_k & \mathbf{1}'_k \end{pmatrix} ([\zeta_j^{(1)}]_1, \dots, [\zeta_j^{(1)}]_k, [\zeta_j^{(2)}]_1, \dots, [\zeta_j^{(2)}]_k)'$$

where $\mathbf{0}_k$ is a column vector of k zeros. The random vector $(\bar{\zeta}_j^1, \bar{\zeta}_j^2)'$ is a two dimensional centered Gaussian vector of variance $\mathbf{V} = \frac{1}{k^2} \begin{pmatrix} \mathbf{1}'_k & \mathbf{0}'_k \\ \mathbf{0}'_k & \mathbf{1}'_k \end{pmatrix} \Sigma \begin{pmatrix} \mathbf{1}_k & \mathbf{0}_k \\ \mathbf{0}_k & \mathbf{1}_k \end{pmatrix}$. The 2×2 matrix \mathbf{V} is of trace less or equal to $\frac{2}{k} \gamma_{\max}(k)$. Hence, the random variable $\frac{1}{J} \sum_{j=1}^J \|\bar{\zeta}_j\|_{\mathbb{R}^2}^2 = \frac{1}{J} \sum_{j=1}^J (\bar{\zeta}_j^1, \bar{\zeta}_j^2) (\bar{\zeta}_j^1, \bar{\zeta}_j^2)'$ has the same probability distribution as $\frac{1}{J} \xi' [Id_J \otimes \mathbf{V}] \xi$ where ξ is a centered Gaussian vector of variance Id_{2J} . A standard concentration inequality of χ^2 distribution (see e.g. [LM00] Lemma 1) is $\mathbb{P} \left(\xi' [Id_J \otimes \mathbf{V}] \xi \geq J \frac{4}{k} \gamma_{\max}(k) + \frac{4}{k} \gamma_{\max}(k) \sqrt{2Jx} + x \frac{4}{k} \gamma_{\max}(k) \right) \leq e^{-x}$ for any $x > 0$. It yields that there exists a constant $C(A, \mathcal{A}) > 0$ such that for all $x > 0$

$$\mathbb{P} \left((A.13) \geq C(A, \mathcal{A}) \frac{\gamma_{\max}(k)}{k} \left(1 + \sqrt{2 \frac{x}{J}} + \frac{x}{J} \right) \right) \leq e^{-x}. \quad (A.17)$$

To end the proof remark that for all $u \geq 0$ we have $\mathbb{P}(\frac{1}{J} \|\mathbf{b}_{\Theta_0}^* - \hat{\mathbf{b}}^\lambda\|_{\mathbb{R}^{2J}}^2 \geq u) \leq \mathbb{P}((A.11) + (A.12) + (A.13) \geq u)$. This together with (A.16) and (A.17) yield that there exists a constant $C(L, s, A, \mathcal{A}, B, \mathbf{f}) > 0$ independent of k and J such that for all $x > 0$ we have

$$\mathbb{P} \left(\frac{1}{J} \|\mathbf{b}_{\Theta_0}^* - \hat{\mathbf{b}}^\lambda\|_{\mathbb{R}^{2J}}^2 \geq C(L, s, A, \mathcal{A}, B, \mathbf{f}) (F(k^{-\frac{2s}{2s+1}}) + F(V_1(k, J, x)) + V_2(k, J, x)) \right) \leq 5e^{-x},$$

where $V_2(k, J, x) = \frac{\gamma_{\max}(k)}{k} (1 + \sqrt{2 \frac{x}{J}} + \frac{x}{J})$ and the proof of Proposition A.3 is completed. \square

B Technical Lemma

Lemma B.1. *Assume that Assumption 1 holds, i.e. $f \in H_s(L)$ with $s > \frac{3}{2}$ (see (1.8)) and $\mathbf{f} = (f(\frac{\ell}{k}))_{\ell=1}^k \in \mathbb{R}^{k \times 2}$. If $\lambda(k) = \lfloor k^{\frac{1}{2s+1}} \rfloor$ then there exists a constant $C(L, s)$ such that for all $f \in H_s(L)$ we have*

$$\frac{1}{k} \|\mathbf{A}^\lambda \mathbf{f} - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2 \leq C(L, s) k^{-\frac{2s}{2s+1}},$$

where \mathbf{A}^λ is the projection matrix defined in (4.2).

Proof. Recall the notations introduced Sections 1.1 and 4.1 : $c_m(f) = (c_m(f^{(1)}), c_m(f^{(2)})) \in \mathbb{C}^2$ is the m -th Fourier coefficient of $f \in L^2([0, 1], \mathbb{R}^2)$ and $c_m(\mathbf{f}) = (c_m^{(1)}(\mathbf{f}), c_m^{(2)}(\mathbf{f})) \in \mathbb{C}^2$ is the m -th discrete Fourier coefficient of $\mathbf{f} \in \mathbb{R}^{k \times 2}$. Thus, we have by Parseval's equality

$$\frac{1}{k} \|\mathbf{A}^\lambda \mathbf{f} - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2 = \frac{1}{k} \left\| \left(\frac{1}{k} \sum_{|m| > \lambda} c_m(\mathbf{f}) e^{i2\pi m \frac{\ell}{k}} \right)_{\ell=1}^k \right\|_{\mathbb{R}^{k \times 2}}^2 = \frac{1}{k^2} \sum_{|m| > \lambda} \|c_m(\mathbf{f})\|_{\mathbb{C}^2}^2.$$

where for all $c = (c^{(1)}, c^{(2)}) \in \mathbb{C}^2$, $\|c\|_{\mathbb{C}^2}^2 = |c^{(1)}|^2 + |c^{(2)}|^2$. It yields

$$\frac{1}{k} \|\mathbf{A}^\lambda \mathbf{f} - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2 \leq 2 \sum_{|m| > \lambda} \left\| \frac{1}{k} c_m(\mathbf{f}) - c_m(f) \right\|_{\mathbb{C}^2}^2 + 2 \sum_{|m| > \lambda} \|c_m(f)\|_{\mathbb{C}^2}^2.$$

Firstly, Lemma 1.10 in [Tsy09] ensures that if $s > \frac{1}{2}$ then $|\frac{1}{k}c_m(\mathbf{f}^{(i)}) - c_m(f^{(i)})| \leq C(L, s)k^{-s+\frac{1}{2}}$ for any $m \in \mathbb{N}$ and $i = 1, 2$. Secondly, equation (1.87) in [Tsy09] ensure that if $\lambda(k) = \lfloor k^{\frac{1}{2s+1}} \rfloor$ then $\sum_{|m|>\lambda} |c_m(f^{(i)})|^2 \leq C(L, s)k^{-\frac{2s}{2s+1}}$. Thence, there is a constant $C(L, s)$ independent of k such that

$$\frac{1}{k} \|\mathbf{A}^\lambda \mathbf{f} - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2 \leq C(L, s)(k^{2-2s} + k^{-\frac{2s}{2s+1}}).$$

If $s \geq \frac{3}{2}$ then we have

$$\frac{1}{k} \|\mathbf{A}^\lambda \mathbf{f} - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2 \leq C(L, s)k^{-\frac{2s}{2s+1}}. \quad \square$$

We now give a general expression of the gradient and of the Hessian of the criterion D . Recall that we have

$$D(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) = \frac{1}{J} \sum_{j=1}^J \left\| e^{-a_j} (e^{a_j^*} \mathbf{f} R_{\alpha_j^*} + \mathbf{1}_k \otimes (b_j^* - b_j)) R_{-\alpha_j} + \frac{1}{J} \sum_{j'=1}^J e^{-a_{j'}} (e^{a_{j'}^*} \mathbf{f} R_{\alpha_{j'}^*} + \mathbf{1}_k \otimes (b_{j'}^* - b_{j'})) R_{-\alpha_{j'}} \right\|_{\mathbb{R}^{k \times 2}}^2$$

where $(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) = (a_1, \dots, a_J, \alpha_1, \dots, \alpha_J, b_1, \dots, b_J) \in \mathbb{R}^J \times [-\pi, \pi]^J \times \mathbb{R}^{2J}$. To shorten the formulas below we note $g_j = (g_j^{(1)}, g_j^{(2)}, g_j^{(3)}, g_j^{(4)}) = (a_j, \alpha_j, b_j)$, that is $g_j^{(1)} = a_j$, $g_j^{(2)} = \alpha_j$, $g_j^{(3)} = b_j^{(1)}$ and $g_j^{(4)} = b_j^{(2)}$. Let $\mathbf{f}_{g_j} = g_j^{-1} \cdot g_j^* \cdot \mathbf{f} = e^{-a_j} (e^{a_j^*} \mathbf{f} R_{\alpha_j^*} + \mathbf{1}_k \otimes (b_j^* - b_j)) R_{-\alpha_j}$, and for all $j_1 = 1, \dots, J$ and $p_1 = 1, \dots, 4$,

$$\partial_{g_{j_1}^{(p_1)}} D(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) = \frac{2}{Jk} \left\langle \partial_{g_{j_1}^{(p_1)}} \mathbf{f}_{g_{j_1}}, \mathbf{f}_{g_{j_1}} - \frac{1}{J} \sum_{j'=1}^J \mathbf{f}_{g_{j'}} \right\rangle_{\mathbb{R}^{k \times 2}}. \quad (\text{B.1})$$

The second order derivatives are

$$\partial_{g_{j_2}^{(p_2)}} \partial_{g_{j_1}^{(p_1)}} D(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) = -\frac{2}{J^2 k} \left\langle \partial_{g_{j_1}^{(p_1)}} \mathbf{f}_{g_{j_1}}, \partial_{g_{j_2}^{(p_2)}} \mathbf{f}_{g_{j_2}} \right\rangle_{\mathbb{R}^{k \times 2}}, \quad \text{if } j_1 \neq j_2, \quad (\text{B.2})$$

$$\begin{aligned} \partial_{g_{j_1}^{(p_2)}} \partial_{g_{j_1}^{(p_1)}} D(\mathbf{a}, \boldsymbol{\alpha}, \mathbf{b}) &= \frac{2}{Jk} \left\langle \partial_{g_{j_1}^{(p_2)}} \partial_{g_{j_1}^{(p_1)}} \mathbf{f}_{g_{j_1}}, \left(\mathbf{f}_{g_{j_1}} - \frac{1}{J} \sum_{j'=1}^J \mathbf{f}_{g_{j'}} \right) \right\rangle_{\mathbb{R}^{k \times 2}} \\ &+ \left(\frac{2}{Jk} - \frac{2}{J^2 k} \right) \left\langle \partial_{g_{j_1}^{(p_1)}} \mathbf{f}_{g_{j_1}}, \partial_{g_{j_1}^{(p_2)}} \mathbf{f}_{g_{j_1}} \right\rangle_{\mathbb{R}^{k \times 2}}. \end{aligned} \quad (\text{B.3})$$

The expressions of the gradient and the Hessian of D simplify on the set $(\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) * \mathcal{G}$, see Lemma 3.1. For any $g_0 = (a_0, \alpha_0, b_0) \in \mathbb{R} \times [-\pi, \pi] \times \mathbb{R}^2$, we have $\mathbf{f}_{g_{j_1}^* \cdot g_0} - \frac{1}{J} \sum_{j'=1}^J \mathbf{f}_{g_{j'}^* \cdot g_0} = e^{-a_0} (\mathbf{f} - \mathbf{1}_k \otimes b_0) R_{-\alpha_0} - \frac{1}{J} \sum_{j'=1}^J e^{-a_0} (\mathbf{f} - \mathbf{1}_k \otimes b_0) R_{-\alpha_0} = 0$. It yields that for all $(a_0, \alpha_0, b_0) \in \mathbb{R} \times [-\pi, \pi] \times \mathbb{R}^2$ we have $\nabla D((\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) * (a_0, \alpha_0, b_0)) = 0$, and

$$\partial_{g_{j_2}^{(p_2)}} \partial_{g_{j_1}^{(p_1)}} D((\mathbf{a}^*, \boldsymbol{\alpha}^*, \mathbf{b}^*) * (a_0, \alpha_0, b_0)) = \begin{cases} -\frac{2}{J^2 k} \left\langle \partial_{g_{j_1}^{(p_1)}} \mathbf{f}_{g_{j_1}^* \cdot g_0}, \partial_{g_{j_2}^{(p_2)}} \mathbf{f}_{g_{j_2}^* \cdot g_0} \right\rangle_{\mathbb{R}^{k \times 2}}, & \text{if } j_1 \neq j_2, \\ \left(\frac{2}{Jk} - \frac{2}{J^2 k} \right) \left\langle \partial_{g_{j_1}^{(p_1)}} \mathbf{f}_{g_{j_1}^* \cdot g_0}, \partial_{g_{j_2}^{(p_2)}} \mathbf{f}_{g_{j_1}^* \cdot g_0} \right\rangle_{\mathbb{R}^{k \times 2}}. \end{cases} \quad (\text{B.4})$$

Lemma B.2. *The smallest eigenvalue of $\nabla^2 D_0(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*)$ restricted to the subset Θ_0 is greater than $e^{-2A} \frac{2}{Jk} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2$.*

Proof. In this proof, $\mathbf{h} = (\mathbf{a}, \boldsymbol{\alpha}) \in \mathbb{R}^J \times [-\pi, \pi]^J$ and $\mathbf{f}_{h_j}^0 = e^{a_j^* - a_j} \mathbf{f}_0 R_{\alpha_j^* - \alpha_j}$. We have $\partial_{a_{j_1}} \mathbf{f}_{h_{j_1}}^0 = -\mathbf{f}_{h_{j_1}}^0$ and $\partial_{\alpha_{j_1}} \mathbf{f}_{h_{j_1}}^0 = \mathbf{f}_{h_{j_1}}^0 R_{-\frac{\pi}{2}}$ for all $j_1 = 1, \dots, J$. By using Formulas (B.4), the Hessian of D_0 at the point $(\mathbf{a}_{\Theta_0}^*, \boldsymbol{\alpha}_{\Theta_0}^*) = (\mathbf{a}^* - \bar{\mathbf{a}}^*, \boldsymbol{\alpha}^* - \bar{\boldsymbol{\alpha}}^*) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}$ is given by,

$$\partial_{\alpha_{j_1}} \partial_{\alpha_{j_2}} D_0(\mathbf{a}^* - \bar{\mathbf{a}}^*, \boldsymbol{\alpha}^* - \bar{\boldsymbol{\alpha}}^*) = \partial_{a_{j_1}} \partial_{a_{j_2}} D_0(\mathbf{a}^* - \bar{\mathbf{a}}^*, \boldsymbol{\alpha}^* - \bar{\boldsymbol{\alpha}}^*) = \begin{cases} -\frac{2}{J^2 k} \|e^{\bar{\mathbf{a}}^*} \mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2, & \text{if } j_1 \neq j_2, \\ \left(\frac{2}{Jk} - \frac{2}{J^2 k} \right) \|e^{\bar{\mathbf{a}}^*} \mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2 \end{cases}$$

and the second order cross derivatives are 0. The Hessian of D_0 at $(\mathbf{a}^* - \bar{\mathbf{a}}^*, \boldsymbol{\alpha}^* - \bar{\boldsymbol{\alpha}}^*)$ can be written,

$$\nabla^2 D_0(\mathbf{a}^* - \bar{\mathbf{a}}^*, \boldsymbol{\alpha}^* - \bar{\boldsymbol{\alpha}}^*) = \frac{2}{J^2 k} \|e^{\bar{\mathbf{a}}^*} \mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2 \begin{pmatrix} JId_J - \mathbb{1}_{J \times J} & 0 \\ 0 & JId_J - \mathbb{1}_{J \times J} \end{pmatrix}, \quad (\text{B.5})$$

where Id_J is the identity in \mathbb{R}^J and $\mathbb{1}_J$ is the $J \times J$ matrix of ones. The eigenvalues of $JId_J - \mathbb{1}_{J \times J}$ are 0 with eigenvector $\mathbb{1}_J$ and J with eigenspace $\mathbb{1}_J^\perp$. It yields that on $\Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}$ the smallest eigenvalue of $\nabla^2 D_0(\mathbf{a}^* - \bar{\mathbf{a}}^*, \boldsymbol{\alpha}^* - \bar{\boldsymbol{\alpha}}^*)$ is $e^{2\bar{\mathbf{a}}^*} \frac{2}{Jk} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2$. To finish the proof, remark that $\bar{\mathbf{a}}^* \leq A$. \square

Lemma B.3. *Let $\delta = \max\{A, \mathcal{A}\}$. For all $\mathbf{c} = (a_1, \dots, a_J, \alpha_1, \dots, \alpha_J) \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}$ and $(\mathbf{a}, \boldsymbol{\alpha}) \in \mathbb{R}^{2J}$ we have*

$$|[\nabla^3 D_0(\mathbf{c})](\mathbf{a}, \boldsymbol{\alpha})| \leq 40\delta e^{2A} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2 \frac{1}{Jk} \|(\mathbf{a}, \boldsymbol{\alpha})\|_{\mathbb{R}^{2J}}^2.$$

Proof. In this proof, $\mathbf{h} = (\mathbf{a}, \boldsymbol{\alpha}) \in \mathbb{R}^J \times [-\pi, \pi]^J$, that is $h_j^{(1)} = a_j$ is the parameter of scaling and $h_j^{(2)} = \alpha_j$ is the parameter of rotation. We have $\mathbf{f}_{h_j}^0 = e^{a_j^* - a_j} \mathbf{f}_0 R_{\alpha_j^* - \alpha_j}$. Then, from equations (B.2) and (B.3), it follows that for all $j_1, j_2, j_3 = 1, \dots, J$ and $p_1, p_2, p_3 = 1, \dots, 2$,

$$\begin{aligned} \partial_{h_{j_3}^{(p_3)}} \partial_{h_{j_2}^{(p_2)}} \partial_{h_{j_1}^{(p_1)}} D_0(\mathbf{a}, \boldsymbol{\alpha}) &= 0, & \text{if } j_1 \neq j_2 \text{ and } j_2 \neq j_3 \text{ and } j_1 \neq j_3, \\ \partial_{h_{j_2}^{(p_2)}} \partial_{h_{j_1}^{(p_1)}} \partial_{h_{j_1}^{(p_1)}} D_0(\mathbf{a}, \boldsymbol{\alpha}) &= -\frac{2}{J^2 k} \left\langle \partial_{h_{j_1}^{(p_2)}} \partial_{h_{j_1}^{(p_1)}} \mathbf{f}_{h_{j_1}}^0, \partial_{h_{j_2}^{(p_2)}} \mathbf{f}_{h_{j_2}}^0 \right\rangle_{\mathbb{R}^{k \times 2}}, & \text{if } j_1 \neq j_2, \\ \partial_{h_{j_1}^{(p_3)}} \partial_{h_{j_1}^{(p_2)}} \partial_{h_{j_1}^{(p_1)}} D_0(\mathbf{a}, \boldsymbol{\alpha}) &= \frac{2}{Jk} \left\langle \partial_{h_{j_1}^{(p_3)}} \partial_{h_{j_1}^{(p_2)}} \partial_{h_{j_1}^{(p_1)}} \mathbf{f}_{h_{j_1}}^0, \left(\mathbf{f}_{h_{j_1}}^0 - \frac{1}{J} \sum_{j'=1}^J \mathbf{f}_{h_{j'}}^0 \right) \right\rangle_{\mathbb{R}^{k \times 2}} \\ &\quad + \left(\frac{2}{Jk} - \frac{2}{J^2 k} \right) \left(\left\langle \partial_{h_{j_1}^{(p_2)}} \partial_{h_{j_1}^{(p_1)}} \mathbf{f}_{h_{j_1}}^0, \partial_{h_{j_1}^{(p_3)}} \mathbf{f}_{h_{j_1}}^0 \right\rangle_{\mathbb{R}^{k \times 2}} + \left\langle \partial_{h_{j_1}^{(p_3)}} \partial_{h_{j_1}^{(p_1)}} \mathbf{f}_{h_{j_1}}^0, \right. \right. \\ &\quad \left. \left. \partial_{h_{j_1}^{(p_2)}} \mathbf{f}_{h_{j_1}}^0 \right\rangle_{\mathbb{R}^{k \times 2}} + \left\langle \partial_{h_{j_1}^{(p_3)}} \partial_{h_{j_1}^{(p_2)}} \mathbf{f}_{h_{j_1}}^0, \partial_{h_{j_1}^{(p_1)}} \mathbf{f}_{h_{j_1}}^0 \right\rangle_{\mathbb{R}^{k \times 2}} \right) \end{aligned}$$

By Cauchy-Schwarz inequality we have,

$$\left| \left\langle \partial_{h_{j_1}^{(p_2)}} \partial_{h_{j_1}^{(p_1)}} \mathbf{f}_{h_{j_1}}^0, \partial_{h_{j_2}^{(p_2)}} \mathbf{f}_{h_{j_2}}^0 \right\rangle_{\mathbb{R}^{k \times 2}} \right| \leq \|\partial_{h_{j_1}^{(p_2)}} \partial_{h_{j_1}^{(p_1)}} \mathbf{f}_{h_{j_1}}^0\|_{\mathbb{R}^{k \times 2}} \|\partial_{h_{j_2}^{(p_2)}} \mathbf{f}_{h_{j_2}}^0\|_{\mathbb{R}^{k \times 2}} \leq e^{2A} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2. \quad (\text{B.6})$$

and

$$\begin{aligned} &\left| \left\langle \partial_{h_{j_1}^{(p_3)}} \partial_{h_{j_1}^{(p_2)}} \partial_{h_{j_1}^{(p_1)}} \mathbf{f}_{h_{j_1}}^0, \left(\mathbf{f}_{h_{j_1}}^0 - \frac{1}{J} \sum_{j'=1}^J \mathbf{f}_{h_{j'}}^0 \right) \right\rangle_{\mathbb{R}^{k \times 2}} \right| \\ &\leq \left\| \partial_{h_{j_1}^{(p_3)}} \partial_{h_{j_1}^{(p_2)}} \partial_{h_{j_1}^{(p_1)}} \mathbf{f}_{h_{j_1}}^0 \right\|_{\mathbb{R}^{k \times 2}} \left\| \mathbf{f}_{h_{j_1}}^0 - \frac{1}{J} \sum_{j'=1}^J \mathbf{f}_{h_{j'}}^0 \right\|_{\mathbb{R}^{k \times 2}} \leq 2e^{2A} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2 \quad (\text{B.7}) \end{aligned}$$

For $\kappa = (\kappa_1, \dots, \kappa_{2J}) \in \mathbb{N}^{2J}$, denote by $|\kappa| = \kappa_1 + \dots + \kappa_{2J}$ and

$$(\partial_{\mathbf{h}})^\kappa = (\partial_{a_1})^{\kappa_1} (\partial_{\alpha_1})^{\kappa_2} \dots (\partial_{a_J})^{\kappa_{2J-1}} (\partial_{\alpha_J})^{\kappa_{2J}}.$$

Then, the differential of order 3 of D_0 at $\mathbf{c} \in \Theta_0^{\mathbf{a}, \boldsymbol{\alpha}}$ applied at $(\mathbf{a}, \boldsymbol{\alpha}) \in \mathbb{R}^{2J}$ writes as

$$[\nabla^3 D_0(\mathbf{c})](\mathbf{a}, \boldsymbol{\alpha}) = \sum_{|\kappa|=3} (\partial_{\mathbf{h}})^\kappa D_0(\mathbf{c}) \mathbf{h}^\kappa$$

where $\mathbf{h}^\kappa = a_1^{\kappa_1} \alpha_1^{\kappa_2} \dots a_J^{\kappa_{2J-1}} \alpha_J^{\kappa_{2J}}$. This formula together with equations (B.6) and (B.7) give,

$$\begin{aligned}
\left| [\nabla^3 D_0(\mathbf{c})](\mathbf{a}, \boldsymbol{\alpha}) \right| &= \left| \sum_{p_1, p_2, p_3=1}^2 \sum_{j_1=1}^J \partial_{h_{j_1}^{(p_3)}} \partial_{h_{j_1}^{(p_2)}} \partial_{h_{j_1}^{(p_1)}} D_0(\mathbf{c}) h_{j_1}^{(p_1)} h_{j_1}^{(p_2)} h_{j_1}^{(p_3)} \right. \\
&\quad \left. + 3 \sum_{j_1 \neq j_2=1}^J \partial_{h_{j_2}^{(p_3)}} \partial_{h_{j_1}^{(p_2)}} \partial_{h_{j_1}^{(p_1)}} D_0(\mathbf{c}) h_{j_1}^{(p_1)} h_{j_1}^{(p_2)} h_{j_2}^{(p_3)} \right| \\
&\leq 2e^{2A} \frac{1}{k} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2 \sum_{p_1, p_2, p_3=1}^2 \left(\frac{4}{J} \sum_{j_1=1}^J |h_{j_1}^{(p_1)} h_{j_1}^{(p_2)} h_{j_1}^{(p_3)}| + \frac{6}{J^2} \sum_{j_1 \neq j_2=1}^J |h_{j_1}^{(p_1)} h_{j_1}^{(p_2)} h_{j_2}^{(p_3)}| \right) \\
&\leq 2\delta e^{2A} \frac{1}{k} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2 \sum_{p_1, p_2=1}^2 \left(\frac{4}{J} \sum_{j_1=1}^J |h_{j_1}^{(p_1)} h_{j_1}^{(p_2)}| + \frac{6(J-1)}{J^2} \sum_{j_1=1}^J |h_{j_1}^{(p_1)} h_{j_1}^{(p_2)}| \right) \\
&\leq 20\delta e^{2A} \frac{1}{k} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2 \frac{1}{J} \sum_{j=1}^J \sum_{p_1, p_2=1}^2 |h_j^{(p_1)} h_j^{(p_2)}| \\
&= 20\delta e^{2A} \frac{1}{k} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2 \frac{1}{J} \sum_{j=1}^J \left(\sum_{p_1=1}^2 |h_j^{(p_1)}| \right)^2 \leq 40\delta e^{2A} \frac{1}{k} \|\mathbf{f}_0\|_{\mathbb{R}^{k \times 2}}^2 \frac{1}{J} \sum_{j=1}^J \sum_{p_1=1}^2 |h_j^{(p_1)}|^2.
\end{aligned}$$

And the claim is now proved. \square

Lemma B.4. For all $\mathbf{f} \in \mathbb{R}^{k \times 2}$ and $(a_1, \alpha_1, b_1), (a_2, \alpha_2, b_2) \in [-A, A] \times [-\mathcal{A}, \mathcal{A}] \times \mathbb{R}^2$, let $g_i \cdot \mathbf{f} = e^{a_i} \mathbf{f} R_{\alpha_i} + \mathbf{1}_k \otimes b_i$, $i = 1, 2$. Then, we have

$$\frac{1}{k} \|g_1 \cdot \mathbf{f} - g_2 \cdot \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2 \leq C(A, \mathbf{f}) \|(a_1, \alpha_1, b_1) - (a_2, \alpha_2, b_2)\|_{\mathbb{R}^4}^2,$$

where $C(A, \mathbf{f}) = 2 \max\{4e^{4A} \frac{1}{k} \|\mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2, 1\}$.

Proof. We have

$$\frac{1}{k} \|g_1 \cdot \mathbf{f} - g_2 \cdot \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2 \leq 2e^{2A} \frac{1}{k} \|e^{a_1 - a_2} \mathbf{f} R_{\alpha_1 - \alpha_2} - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2 + 2 \frac{1}{k} \|\mathbf{1}_k \otimes (b_1 - b_2)\|_{\mathbb{R}^{k \times 2}}^2 \quad (\text{B.8})$$

Let now $F(a, \alpha) = \frac{1}{\sqrt{k}} \|e^a \mathbf{f} R_\alpha - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}$. We have $|\partial_a F(a, \alpha)| = \frac{1}{\sqrt{k} \|e^a \mathbf{f} R_\alpha - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}} |\langle e^a \mathbf{f} R_\alpha, e^a \mathbf{f} R_\alpha - \mathbf{f} \rangle_{\mathbb{R}^{k \times 2}}| \leq \frac{e^A}{\sqrt{k}} \|\mathbf{f}\|_{\mathbb{R}^{k \times 2}}$ and $|\partial_\alpha F(a, \alpha)| = \frac{1}{\sqrt{k} \|e^a \mathbf{f} R_\alpha - \mathbf{f}\|_{\mathbb{R}^{k \times 2}}} |\langle e^a \mathbf{f} R_{\alpha + \frac{\pi}{2}}, e^a \mathbf{f} R_\alpha - \mathbf{f} \rangle_{\mathbb{R}^{k \times 2}}| \leq \frac{e^A}{\sqrt{k}} \|\mathbf{f}\|_{\mathbb{R}^{k \times 2}}$. The Euclidean norm of the gradient of F satisfies

$$\|\nabla F(a, \alpha)\|_{\mathbb{R}^2} = \sqrt{|\partial_a F(a, \alpha)|^2 + |\partial_\alpha F(a, \alpha)|^2} \leq \sqrt{2} e^A \frac{1}{\sqrt{k}} \|\mathbf{f}\|_{\mathbb{R}^{k \times 2}}.$$

Since we have $|F(a, \alpha)| = |F(a, \alpha) - F(0, 0)| \leq \sqrt{2} e^A \frac{1}{\sqrt{k}} \|\mathbf{f}\|_{\mathbb{R}^{k \times 2}} \|(a, \alpha)\|_{\mathbb{R}^2}$, equation (B.8) yields

$$\frac{1}{k} \|g_1 \cdot \mathbf{f} - g_2 \cdot \mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2 \leq 2 \max\{4e^{4A} \frac{1}{k} \|\mathbf{f}\|_{\mathbb{R}^{k \times 2}}^2, 1\} (|a_1 - a_2|^2 + |\alpha_1 - \alpha_2|^2 + |b_1^{(1)} - b_2^{(1)}|^2 + |b_1^{(2)} - b_2^{(2)}|^2),$$

which concludes the proof. \square

References

- [BC11] J. Bigot and B. Charlier. On the consistency of fréchet means in deformable models for curve and image analysis. *Electronic Journal of Statistics*, 5:1054–1089, 2011.
- [DM98] I. L. Dryden and K. V. Mardia. *Statistical shape analysis*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Ltd., Chichester, 1998.

- [Goo91] C. Goodall. Procrustes methods in the statistical analysis of shape. *J. Roy. Statist. Soc. Ser. B*, 53(2):285–339, 1991.
- [HJ90] R.A. Horn and C.R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1990.
- [Huc11] S. Huckemann. Inference on 3d procrustes means: Tree bole growth, rank deficient diffusion tensors and perturbation models. *Scand. J. Statist.*, To appear, 2011.
- [KBCL99] D. G. Kendall, D. Barden, T. K. Carne, and H. Le. *Shape and shape theory*. Wiley Series in Probability and Statistics. John Wiley & Sons Ltd., Chichester, 1999.
- [Ken84] D. G. Kendall. Shape manifolds, Procrustean metrics, and complex projective spaces. *Bull. London Math. Soc.*, 16(2):81–121, 1984.
- [KM97] J. T. Kent and K. V. Mardia. Consistency of Procrustes estimators. *J. Roy. Statist. Soc. Ser. B*, 59(1):281–290, 1997.
- [Le98] H. Le. On the consistency of procrustean mean shapes. *Adv. in Appl. Probab.*, 30(1):53–63, 1998.
- [Le1993] S. Lele. Euclidean distance matrix analysis (EDMA): estimation of mean form and mean form difference. *Math. Geol.*, 25(5):573–602, 1993.
- [LM00] B. Laurent and P. Massart. Adaptive estimation of a quadratic functional by model selection. *Ann. Stat.*, 28(5):1302–1338, 2000.
- [Mas07] P. Massart. *Concentration inequalities and model selection*, volume 1896 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.
- [Tsy09] A. Tsybakov. *Introduction to nonparametric estimation*. Springer Series in Statistics. New York, NY: Springer. xii, 214 p., 2009.
- [vdV98] A. W. van der Vaart. *Asymptotic statistics*, volume 3 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 1998.