



HAL
open science

Numerical radius and distance from unitary operators

Catalin Badea, Michel Crouzeix

► **To cite this version:**

Catalin Badea, Michel Crouzeix. Numerical radius and distance from unitary operators. *Operators and Matrices*, 2013, 7 (2), pp.285-292. 10.7153/oam-07-16 . hal-00634780v2

HAL Id: hal-00634780

<https://hal.science/hal-00634780v2>

Submitted on 24 Jan 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Numerical radius and distance from unitary operators

Catalin BADEA and Michel CROUZEIX

Abstract

Denote by $w(A)$ the numerical radius of a bounded linear operator A acting on Hilbert space. Suppose that A is invertible and that $w(A) \leq 1+\varepsilon$ and $w(A^{-1}) \leq 1+\varepsilon$ for some $\varepsilon \geq 0$. It is shown that $\inf\{\|A-U\| : U \text{ unitary}\} \leq c\varepsilon^{1/4}$ for some constant $c > 0$. This generalizes a result due to J.G. Stampfli, which is obtained for $\varepsilon = 0$. An example is given showing that the exponent $1/4$ is optimal. The more general case of the operator ρ -radius $w_\rho(\cdot)$ is discussed for $1 \leq \rho \leq 2$.

1 Introduction and statement of the results

Let H be a complex Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$. We denote by $\mathcal{B}(H)$ the C^* -algebra of all bounded linear operators on H equipped with the operator norm

$$\|A\| = \sup\{\|Ah\| : h \in H, \|h\| = 1\}.$$

It is easy to see that unitary operators can be characterized as invertible contractions with contractive inverses, i.e. as operators A with $\|A\| \leq 1$ and $\|A^{-1}\| \leq 1$. More generally, if $A \in \mathcal{B}(H)$ is invertible then

$$\inf\{\|A-U\| : U \text{ unitary}\} = \max\left(\|A\| - 1, 1 - \frac{1}{\|A^{-1}\|}\right).$$

We refer to [6, Theorem 1.3] and [9, Theorem 1] for a proof of this equality using the polar decomposition of bounded operators. It also follows from this proof that if $A \in \mathcal{B}(H)$ is an invertible operator satisfying $\|A\| \leq r$ and $\|A^{-1}\| \leq r$ for some $r \geq 1$, then there exists a unitary operator $U \in \mathcal{B}(H)$ such that $\|A-U\| \leq r-1$.

The numerical radius of the operator A is defined by

$$w(A) = \sup\{|\langle Ah, h \rangle| : h \in H, \|h\| = 1\}.$$

Stampfli has proved in [8] that numerical radius contractivity of A and of its inverse A^{-1} , that is $w(A) \leq 1$ and $w(A^{-1}) \leq 1$, imply that A is unitary. We define a function $\psi(r)$ for $r \geq 1$ by

$$\psi(r) = \sup\{\|A\| : A \in \mathcal{B}(H), w(A) \leq r, w(A^{-1}) \leq r\},$$

the supremum being also considered over all Hilbert spaces H . Then the conditions $w(A) \leq r$ and $w(A^{-1}) \leq r$ imply $\max(\|A\|-1, 1-\|A^{-1}\|^{-1}) \leq \max(\|A\|-1, \|A^{-1}\|-1) \leq \psi(r)-1$, hence the existence of a unitary operator U such that $\|A-U\| \leq \psi(r)-1$. We have the two-sided estimate

$$r + \sqrt{r^2 - 1} \leq \psi(r) \leq 2r.$$

The upper bound follows from the well-known inequalities $w(A) \leq \|A\| \leq 2w(A)$, while the lower bound is obtained by choosing $H = \mathbb{C}^2$ and

$$A = \begin{pmatrix} 1 & 2y \\ 0 & -1 \end{pmatrix} \quad \text{with } y = \sqrt{r^2 - 1},$$

in the definition of ψ . Indeed, we have $A = A^{-1}$, $w(A) = \sqrt{1+y^2} = r$, and $\|A\| = y + \sqrt{1+y^2} = r + \sqrt{r^2-1}$.

Our first aim is to improve the upper estimate.

Theorem 1.1. *Let $r \geq 1$. Then*

$$\psi(r) \leq X(r) + \sqrt{X(r)^2-1}, \quad \text{with } X(r) = r + \sqrt{r^2-1}. \quad (1)$$

The estimate given in Theorem 1.1 is more accurate than $\psi(r) \leq 2r$ for r close to 1, more precisely for $1 \leq r \leq 1.0290855\dots$. It also gives $\psi(1) = 1$ (leading to Stampfli's result) and the following asymptotic estimate.

Corollary 1.2. *We have*

$$\psi(1+\varepsilon) \leq 1 + \sqrt[4]{8\varepsilon} + O(\varepsilon^{1/2}), \quad \varepsilon \rightarrow 0.$$

Our second aim is to prove that the exponent $1/4$ in Corollary 1.2 is optimal. This is a consequence of the following result.

Theorem 1.3. *Let n be a positive integer of the form $n = 8k + 4$. There exists a $n \times n$ invertible matrix A_n with complex entries such that*

$$w(A_n) \leq \frac{1}{\cos \frac{\pi}{n}}, \quad w(A_n^{-1}) \leq \frac{1}{\cos \frac{\pi}{n}}, \quad \|A_n\| = 1 + \frac{1}{8\sqrt{n}}.$$

Indeed, Theorem 1.3 implies that

$$\psi\left(\frac{1}{\cos \frac{\pi}{n}}\right) \geq \|A_n\| = 1 + \frac{1}{8\sqrt{n}}.$$

Taking $1+\varepsilon = 1/\cos \frac{\pi}{n} = 1 + \frac{\pi^2}{2n^2} + O(\frac{1}{n^4})$, we see that the exponent $\frac{1}{4}$ cannot be improved.

More generally, we can consider for $\rho \geq 1$ the ρ -radius $w_\rho(A)$ introduced by Sz.-Nagy and Foias (see [5, Chapter 1] and the references therein). Consider the class \mathcal{C}_ρ of operators $T \in \mathcal{B}(H)$ which admit unitary ρ -dilations, i.e. there exist a super-space $\mathcal{H} \supset H$ and a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that

$$T^n = \rho P U^n P^*, \quad \text{for } n = 1, 2, \dots$$

Here P denotes the orthogonal projection from \mathcal{H} onto H . Then the operator ρ -radius is defined by

$$w_\rho(A) = \inf\{\lambda > 0; \lambda^{-1}A \in \mathcal{C}_\rho\}.$$

From this definition it is easily seen that $r(A) \leq w_\rho(A) \leq \rho\|A\|$, where $r(A)$ denotes the spectral radius of A . Also, $w_\rho(A)$ is a non-increasing function of ρ . Another equivalent definition follows from [5, Theorem 11.1]:

$$w_\rho(A) = \sup_{h \in \mathcal{E}_\rho} \left\{ \left(1 - \frac{1}{\rho}\right) |\langle Ah, h \rangle| + \sqrt{\left(1 - \frac{1}{\rho}\right)^2 |\langle Ah, h \rangle|^2 + \left(\frac{2}{\rho} - 1\right) \|Ah\|^2} \right\}, \quad \text{with}$$

$$\mathcal{E}_\rho = \{h \in H; \|h\| = 1 \text{ and } (1 - \frac{1}{\rho})^2 |\langle Ah, h \rangle|^2 - (1 - \frac{2}{\rho}) \|Ah\|^2 \geq 0\}.$$

Notice that $\mathcal{E}_\rho = \{h \in H; \|h\| = 1\}$ whenever $1 \leq \rho \leq 2$. This shows that $w_1(A) = \|A\|$, $w_2(A) = w(A)$ and $w_\rho(A)$ is a convex function of A if $1 \leq \rho \leq 2$.

We now define a function $\psi_\rho(r)$ for $r \geq 1$ by

$$\psi_\rho(r) = \sup\{\|A\|; A \in \mathcal{B}(H), w_\rho(A) \leq r, w_\rho(A^{-1}) \leq r\}.$$

As before, the conditions $w_\rho(A) \leq r$ and $w_\rho(A^{-1}) \leq r$ imply the existence of a unitary operator U such that $\|A-U\| \leq \psi_\rho(r)-1$, and we have $\psi_\rho(r) \leq \rho r$. We will generalize the estimate (1) from Theorem 1.1 by proving, for $1 \leq \rho \leq 2$, the following result.

Theorem 1.4. For $1 \leq \rho \leq 2$ we have

$$\psi_\rho(r) \leq X_\rho(r) + \sqrt{X_\rho(r)^2 - 1}, \quad (2)$$

with
$$X_\rho(r) = \frac{2 + \rho r^2 - \rho + \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2r}.$$

Corollary 1.5. For $1 \leq \rho \leq 2$ we have

$$\psi_\rho(1+\varepsilon) \leq 1 + \sqrt[4]{8(\rho-1)\varepsilon} + O(\varepsilon^{1/2}), \quad \varepsilon \rightarrow 0.$$

We recover in this way for $1 \leq \rho \leq 2$ the recent result of Ando and Li [2, Theorem 2.3], namely that $w_\rho(A) \leq 1$ and $w_\rho(A^{-1}) \leq 1$ imply A is unitary. The range $1 \leq \rho \leq 2$ coincides with the range of those $\rho \geq 1$ for which $w_\rho(\cdot)$ is a norm. Contrarily to [2], we have not been able to treat the case $\rho > 2$.

The organization of the paper is as follows. In Section 2 we prove Theorem 1.4, which reduces to Theorem 1.1 in the case $\rho = 2$. The proof of Theorem 1.3 which shows the optimality of the exponent $1/4$ in Corollary 1.2 is given in Section 3.

As a concluding remark, we would like to mention that the present developments have been influenced by the recent work of Sano/Uchiyama [7] and Ando/Li [2]. In [3], inspired by the paper of Stampfi [8], we have developed another (more complicated) approach in the case $\rho = 2$.

2 Proof of Theorem 1.4 about ψ_ρ

Let us consider $M = \frac{1}{2}(A + (A^*)^{-1})$; then

$$M^*M - 1 = \frac{1}{4}(A^*A + (A^*A)^{-1} - 2) \geq 0.$$

This implies $\|M^{-1}\| \leq 1$. In what follows $C^{1/2}$ will denote the positive square root of the self-adjoint positive operator C . The relation $(A^*A - 2M^*M + 1)^2 = 4M^*M(M^*M - 1)$ yields

$$A^*A - 2M^*M + 1 \leq 2(M^*M)^{1/2}(M^*M - 1)^{1/2},$$

whence
$$A^*A \leq ((M^*M)^{1/2} + (M^*M - 1)^{1/2})^2.$$

Therefore $\|A\| \leq \|M\| + \sqrt{\|M\|^2 - 1}$.

We now assume $1 \leq \rho \leq 2$. Then $w_\rho(\cdot)$ is a norm and the two conditions $w_\rho(A) \leq r$ and $w_\rho(A^{-1}) \leq r$ imply $w_\rho(M) \leq r$. The desired estimate of $\psi_\rho(r)$ will follow from the following auxiliary result.

Lemma 2.1. Assume $\rho \geq 1$. Then the assumptions $w_\rho(M) \leq r$ and $\|M^{-1}\| \leq 1$ imply $\|M\| \leq X_\rho(r)$.

Proof. The contractivity of M^{-1} implies

$$\|u\| \leq \|Mu\|, \quad (\forall u \in H). \quad (3)$$

As $w_\rho(M) \leq r$, it follows from a generalization by Durszt [4] of a decomposition due to Ando [1], that the operator M can be decomposed as

$$M = \rho r B^{1/2}UC^{1/2},$$

with U unitary, C selfadjoint satisfying $0 < C < 1$, and $B = f(C)$ with $f(x) = (1-x)/(1-\rho(2-\rho)x)^{-1}$. Notice that f is a decreasing function on the segment $[0, 1]$ and an involution: $f(f(x)) = x$. Let $[\alpha, \beta]$ be the smallest segment containing the spectrum of C . Then $[\sqrt{\alpha}, \sqrt{\beta}]$ is the smallest segment

containing the spectrum of $C^{1/2}$ and $[\sqrt{f(\beta)}, \sqrt{f(\alpha)}]$ is the smallest segment containing the spectrum of $B^{1/2}$. We have

$$\|u\| \leq \|Mu\| \leq \rho r \sqrt{f(\alpha)} \|C^{1/2}u\|, \quad (\forall u \in H).$$

Choosing a sequence u_n of norm-one vectors ($\|u_n\| = 1$) such that $\|C^{1/2}u_n\|$ tends to $\sqrt{\alpha}$, we first get $1 \leq \rho r \sqrt{\alpha f(\alpha)}$, i.e. $1 - (2 + \rho r^2 - \rho)\rho\alpha + \rho^2 r^2 \alpha^2 \leq 0$. Consequently we have

$$\frac{2 + \rho r^2 - \rho - \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2\rho r^2} \leq \alpha \leq \frac{2 + \rho r^2 - \rho + \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2\rho r^2},$$

and by $\alpha = f(f(\alpha))$

$$\frac{2 + \rho r^2 - \rho - \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2\rho r^2} \leq f(\alpha) \leq \frac{2 + \rho r^2 - \rho + \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2\rho r^2}.$$

Similarly, noticing that $\|(M^*)^{-1}\| \leq 1$, $M^* = \rho r C^{1/2}U^*B^{1/2}$ and $C = f(B)$, we obtain

$$\frac{2 + \rho r^2 - \rho - \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2\rho r^2} \leq \beta \leq \frac{2 + \rho r^2 - \rho + \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2\rho r^2}.$$

Therefore

$$\|M\| \leq \rho r \|B^{1/2}\| \|C^{1/2}\| = \rho r \sqrt{f(\alpha)\beta} \leq \frac{2 + \rho r^2 - \rho + \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2r}.$$

This shows that $\|M\| \leq X_\rho(r)$. □

3 The exponent $1/4$ is optimal (Proof of Theorem 1.3)

Consider the family of $n \times n$ matrices $A = DBD$, defined for $n = 8k + 4$, by

$$\begin{aligned} D &= \text{diag}(e^{i\pi/2n}, \dots, e^{(2\ell-1)i\pi/2n}, \dots, e^{(2n-1)i\pi/2n}), \\ B &= I + \frac{1}{2n^{3/2}}E, \quad \text{where } E \text{ is a matrix whose entries are defined as} \\ e_{ij} &= 1 \quad \text{if } 3k + 2 \leq |i - j| \leq 5k + 2, \quad e_{ij} = 0 \quad \text{otherwise.} \end{aligned}$$

We first remark that $\|A\| = \|B\| = 1 + \frac{1}{8\sqrt{n}}$. Indeed, B is a symmetric matrix with non negative entries, $Be = (1 + \frac{1}{8\sqrt{n}})e$ with $e^T = (1, 1, 1, \dots, 1)$. Thus $\|B\| = r(B) = 1 + \frac{1}{8\sqrt{n}}$ by the Perron-Frobenius theorem.

Consider now the permutation matrix P defined by $p_{ij} = 1$ if $i = j + 1$ modulo n and $p_{ij} = 0$ otherwise and the diagonal matrix $\Delta = \text{diag}(1, \dots, 1, -1)$. Then $P^{-1}DP = e^{i\pi/n}\Delta D$ and $P^{-1}EP = E$, whence $(P\Delta)^{-1}AP\Delta = e^{2i\pi/n}A$. Since $P\Delta$ is a unitary matrix, the numerical range $W(A) = \{\langle Au, u \rangle; \|u\| = 1\}$ of A satisfies $W(A) = W((P\Delta)^{-1}AP\Delta) = e^{2i\pi/n}W(A)$. This shows that the numerical range of A is invariant by the rotation of angle $2\pi/n$ centered in 0, and the same property also holds for the numerical range of A^{-1} .

We postpone the proof of the estimates $\|\frac{1}{2}(A + A^*)\| \leq 1$ and $\|\frac{1}{2}(A^{-1} + (A^{-1})^*)\| \leq 1$ to later sections. Using these estimates, we obtain that the numerical range $W(A)$ is contained in the half-plane $\{z; \text{Re } z \leq 1\}$, whence in the regular n -sided polygon given by the intersection of the half-planes $\{z; \text{Re}(e^{2i\pi k/n}z) \leq 1\}$, $k = 1, \dots, n$. Consequently $w(A) \leq 1/\cos(\pi/n)$. The proof of $w(A^{-1}) \leq 1/\cos(\pi/n)$ is similar.

3.1 Proof of $\|\frac{1}{2}(A+A^*)\| \leq 1$.

Since the (ℓ, j) -entry of A is $e^{(\ell+j-1)i\frac{\pi}{n}}\left(\delta_{\ell,j} + \frac{e_{\ell,j}}{2n^{3/2}}\right)$, the matrix $\frac{1}{2}(A+A^*)$ is a real symmetric matrix whose (i, j) -entry is $\cos\left(\left(i+j-1\right)\frac{\pi}{n}\right)\left(\delta_{i,j} + \frac{e_{i,j}}{2n^{3/2}}\right)$. It suffices to show that, for every $u = (u_1, \dots, u_n)^T \in \mathbb{R}^n$, we have $\|u\|^2 - \operatorname{Re}\langle Au, u \rangle \geq 0$. Let $\mathcal{E} = \{(i, j); 1 \leq i, j \leq n, 3k+2 \leq |i-j| \leq 5k+2\}$. The inequality which has to be proved is equivalent to

$$\sum_{i=1}^n 2 \sin^2\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right) u_i^2 - \frac{1}{2n^{3/2}} \sum_{i,j \in \mathcal{E}} \cos\left(\left(i+j-1\right)\frac{\pi}{n}\right) u_i u_j \geq 0.$$

Setting $v_j = u_j \sin\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right)$, this may be also written as follows

$$2\|v\|^2 - \langle Mv, v \rangle + \frac{1}{2n^{3/2}} \langle Ev, v \rangle \geq 0, \quad (v \in \mathbb{R}^n). \quad (4)$$

Here M is the matrix whose entries are defined by

$$m_{ij} = \frac{1}{2n^{3/2}} \cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right) \cot\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right), \quad \text{if } (i, j) \in \mathcal{E}, \quad m_{ij} = 0 \quad \text{otherwise.}$$

We will see that the Frobenius (or Hilbert-Schmidt) norm of M satisfies $\|M\|_F \leq \sqrt{9/32} < 3/4$. A fortiori, the operator norm of M satisfies $\|M\| \leq \frac{3}{4}$. Together with $\|E\| = n/4$, this shows that $\|M\| + \frac{1}{2n^{3/2}}\|E\| \leq \frac{7}{8}$. Property (4) is now verified.

It remains to show that $\|M\|_F^2 \leq \frac{9}{32}$. First we notice that $m_{ij} = m_{ji} = m_{n+1-i, n+1-j}$, and $m_{ii} = 0$. Hence, with $\mathcal{E}' = \{(i, j) \in \mathcal{E}; i < j \text{ and } i+j \leq n+1\}$,

$$\|M\|_F^2 = 2 \sum_{i < j} |m_{ij}|^2 \leq 4 \sum_{(i,j) \in \mathcal{E}'} |m_{ij}|^2.$$

We have, for $(i, j) \in \mathcal{E}'$,

$$\begin{aligned} 2j \leq i+j+5k+2 \leq n+5k+3 = 13k+7, \quad \text{thus } 3k+3 \leq j \leq \frac{13k+7}{2}, \\ 2i \leq i+j-3k-2 \leq n-3k-1 = 5k+3, \quad \text{thus } 1 \leq i \leq \frac{5k+3}{2}. \end{aligned}$$

This shows that

$$\frac{3\pi}{16} \leq \frac{3k+2}{16k+8}\pi \leq \left(j-\frac{1}{2}\right)\frac{\pi}{n} \leq \frac{13k+6}{16k+8}\pi \leq \pi - \frac{3\pi}{16}, \quad \text{hence } |\cot\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right)| \leq \cot \frac{3\pi}{16} \leq \frac{3}{2}.$$

We also use the estimate $\cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right) \leq n/(\pi(i-\frac{1}{2}))$ and the classical relation $\sum_{i \geq 1} (i-1/2)^{-2} = \pi^2/2$ to obtain

$$\|M\|_F^2 \leq 4 \sum_{(i,j) \in \mathcal{E}'} |m_{ij}|^2 \leq \frac{4}{4n^3} \frac{n^2}{\pi^2} \sum_{i \geq 1} \frac{1}{(i-1/2)^2} (2k+1) \frac{9}{4} = \frac{9}{32}.$$

3.2 Proof of $\|\frac{1}{2}(A^{-1}+(A^{-1})^*)\| \leq 1$.

We start from

$$\begin{aligned} (A^{-1})^* &= D\left(1 + \frac{1}{2n^{3/2}}E\right)^{-1}D \\ &= D^2 - \frac{1}{2n^{3/2}}DED + \frac{1}{4n^3}DE^2\left(1 + \frac{1}{2n^{3/2}}E\right)^{-1}D, \end{aligned}$$

and we want to show that $\|u\|^2 - \operatorname{Re}\langle A^{-1}u, u \rangle \geq 0$. As previously, we set $v_j = u_j \sin\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right)$. The inequality $\|\frac{1}{2}(A^{-1}+(A^{-1})^*)\| \leq 1$ is equivalent to

$$2\|v\|^2 - \langle (M_1 + M_2 + M_3 + M_4)v, v \rangle \geq 0, \quad (v \in \mathbb{R}^n).$$

Here the entries of the matrices M_p , $1 \leq p \leq 4$, are given by

$$\begin{aligned}(m_1)_{ij} &= -\frac{1}{2n^{3/2}} \cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right) \cot\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right) e_{ij}, \\(m_2)_{ij} &= \frac{1}{2n^{3/2}} e_{ij}, \\(m_3)_{ij} &= \frac{1}{4n^3} \cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right) \cot\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right) f_{ij}, \\(m_4)_{ij} &= -\frac{1}{4n^3} f_{ij},\end{aligned}$$

e_{ij} and f_{ij} respectively denoting the entries of the matrices E and $F = E^2(1 + \frac{1}{2n^{3/2}}E)^{-1}$. Noticing that $M_1 = -M$, we have $\|M_1\| \leq \frac{3}{4}$, $\|M_2\| = \frac{1}{8\sqrt{n}}$, $\|F\| \leq \frac{n^2/16}{1-1/(8\sqrt{n})} \leq \frac{n^2}{14}$ and $\|M_4\| = \frac{1}{4n^3} \|F\|$. Now we use

$$\|M_3\|^2 \leq \|M_3\|_F^2 \leq \frac{1}{16n^6} \max_{i,j} |f_{ij}|^2 \sum_{i,j} |\cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right)|^2 |\cot\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right)|^2,$$

together with

$$\begin{aligned}\sum_{i,j} |\cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right)|^2 |\cot\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right)|^2 &= \left(\sum_{i=1}^n |\cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right)|^2\right)^2 \\ &\leq 4 \left(\sum_{i=1}^{n/2} |\cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right)|^2\right)^2 \leq n^4,\end{aligned}$$

to obtain

$$\|M_3\| \leq \frac{1}{4n} \max_{i,j} |f_{ij}|.$$

Using the notation $\|E\|_\infty := \max\{\|Eu\|_\infty; u \in \mathbb{C}^n, \|u\|_\infty \leq 1\}$ for the operator norm induced by the maximum norm in \mathbb{C}^d , it holds $\|E\|_\infty = n/4$, whence $\|\frac{1}{2n^{3/2}}E\|_\infty \leq 1/8$ and thus $\|(1 + \frac{1}{2n^{3/2}}E)^{-1}\|_\infty \leq \frac{1}{1-1/8} = \frac{8}{7}$. This shows that

$$\max_{i,j} |f_{ij}| \leq \|(1 + \frac{1}{2n^{3/2}}E)^{-1}\|_\infty \max_{i,j} |e_{ij}^2| \leq \frac{2n}{7},$$

by denoting e_{ij}^2 the entries of the matrix E^2 and noticing that $\max_{i,j} |e_{ij}^2| = n/4$. Finally, we obtain $\|M_3\| \leq \frac{1}{14}$ and $\|M_1 + M_2 + M_3 + M_4\| \leq \frac{3}{4} + \frac{1}{8} + \frac{1}{14} + \frac{1}{56} < 1$.

Acknowledgement. The authors thank the referees for a careful reading of the article and for their useful remarks and suggestions. The first-named author was supported in part by ANR Projet Blanc DYNOP.

References

- [1] T. Ando, Structure of operators with numerical radius one, *Acta Sci. Math. (Szeged)* **34** (1973), 11–15
- [2] T. Ando, C.K. Li, Operator radii and unitary operators, *Oper. Matrices* **4** (2010), no. 2, 273–281.
- [3] M. Crouzeix, The annulus as a K-spectral set, to appear in *Comm. Pure Appl. Anal.*
- [4] E. Durszt, Factorization of operators in C_ρ class, *Acta Sci. Math. (Szeged)* **37** (1975), 195–200.

- [5] B.-Sz. Nagy, C. Foiaş, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Masson, Paris, 1967.
- [6] D.D. Rogers, Approximation by unitary and essentially unitary operators, *Acta Sci. Math.* (Szeged) **39** (1977), no. 1-2, 3141–151.
- [7] T. Sano, A. Uchiyama, Numerical radius and unitarity, *Acta Sci. Math.* (Szeged) **76** (2010), 581–584.
- [8] J.G. Stampfli, Minimal range theorems for operators with thin spectra, *Pacific Journal of Math.* **23** (1967), 601–612.
- [9] A. F. M. ter Elst, Approximation by unitary operators, *Acta Sci. Math.* (Szeged) **54** (1990), 145–149.

C. B. : Laboratoire Paul Painlevé, UMR no. 8524, Bât. M2, Université Lille 1, 59655 Villeneuve d'Ascq Cedex, France.

E-mail : `badea@math.univ-lille1.fr`

M. C. : Institut de Recherche Mathématique de Rennes, UMR no. 6625, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France

E-mail : `michel.crouzeix@univ-rennes1.fr`