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► **To cite this version:**

| Catalin Badea, Michel Crouzeix. Numerical radius and distance to unitaries. 2011. hal-00634780v1

**HAL Id: hal-00634780**

**<https://hal.science/hal-00634780v1>**

Preprint submitted on 23 Oct 2011 (v1), last revised 24 Jan 2012 (v2)

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# Numerical radius and distance to unitaries

Catalin BADEA and Michel CROUZEIX

## Abstract

Denote by  $w(A)$  the numerical radius of a bounded linear operator  $A$  acting on Hilbert space. Suppose that  $A$  is invertible and that  $w(A) \leq 1+\varepsilon$  and  $w(A^{-1}) \leq 1+\varepsilon$  for some  $\varepsilon \geq 0$ . It is shown that  $\inf\{\|A-U\| : U \text{ unitary}\} \leq c\varepsilon^{1/4}$  for some constant  $c > 0$ . This generalizes a result due to J.G. Stampfli, which is obtained for  $\varepsilon = 0$ . An example is given showing that the exponent  $1/4$  is optimal. The more general case of the operator  $\rho$ -radius  $w_\rho(\cdot)$  is discussed for  $1 \leq \rho \leq 2$ .

## 1 Introduction and statement of the results

Let  $H$  be a complex Hilbert space endowed with the inner product  $\langle \cdot, \cdot \rangle$  and the associated norm  $\|\cdot\|$ . We denote by  $\mathcal{B}(H)$  the  $C^*$ -algebra of all bounded linear operators on  $H$  equipped with the operator norm

$$\|A\| = \sup\{\|Ah\| : h \in H, \|h\| = 1\}.$$

It is easy to see that unitary operators can be characterized as invertible contractions with contractive inverses, i.e. as operators  $A$  with  $\|A\| \leq 1$  and  $\|A^{-1}\| \leq 1$ . More generally, if  $A \in \mathcal{B}(H)$  is invertible then

$$\inf\{\|A-U\| : U \text{ unitary}\} = \max\left(\|A\| - 1, 1 - \frac{1}{\|A^{-1}\|}\right).$$

We refer to [6, Theorem 1.3] and [9, Theorem 1] for a proof of this equality using the polar decomposition of bounded operators. It also follows from this proof that if  $A \in \mathcal{B}(H)$  is an invertible operator satisfying  $\|A\| \leq R$  and  $\|A^{-1}\| \leq R$  for some  $R \geq 1$ , then there exists a unitary operator  $U \in \mathcal{B}(H)$  such that  $\|A-U\| \leq R-1$ .

The numerical radius of the operator  $A$  is defined by

$$w(A) = \sup\{|\langle Ah, h \rangle| : h \in H, \|h\| = 1\}.$$

Stampfli has proved in [8] that numerical radius contractivity of  $A$  and of its inverse  $A^{-1}$ , that is  $w(A) \leq 1$  and  $w(A^{-1}) \leq 1$ , imply that  $A$  is unitary. We define a function  $\psi(R)$  for  $R \geq 1$  by

$$\psi(R) = \sup\{\|A\| : A \in \mathcal{B}(H), w(A) \leq R, w(A^{-1}) \leq R\},$$

the supremum being also considered over all Hilbert spaces  $H$ . Then the conditions  $w(A) \leq R$  and  $w(A^{-1}) \leq R$  imply the existence of a unitary operator  $U$  such that  $\|A-U\| \leq \psi(R)-1$ . From the well-known estimate  $w(A) \leq \|A\| \leq 2w(A)$ , we get that  $\psi(R) \leq 2R$ .

Our first aim is to prove the following estimate.

**Theorem 1.1.** *Let  $R \geq 1$ . Then*

$$\psi(R) \leq X(R) + \sqrt{X(R)^2 - 1}, \quad \text{with} \quad X(R) = R + \sqrt{R^2 - 1}. \quad (1)$$

The estimate given in Theorem 1.1 is more accurate than  $\psi(R) \leq 2R$  for  $R$  close to 1, more precisely for  $1 \leq R \leq 1.0290855\dots$ . It also gives  $\psi(1) \leq 1$  (leading to Stampfli's result) and the following asymptotic estimate.

**Corollary 1.2.** *We have*

$$\psi(1+\varepsilon) \leq 1 + \sqrt[4]{8\varepsilon} + O(\varepsilon^{1/2}), \quad \varepsilon \rightarrow 0.$$

Our second aim is to prove that the exponent  $1/4$  in Corollary 1.2 is optimal. This is a consequence of the following result.

**Theorem 1.3.** *Let  $n$  be a positive integer of the form  $n = 8k + 4$ . There exists a  $n \times n$  invertible matrix  $A_n$  with complex entries such that*

$$w(A_n) \leq 1/\cos(\pi/n), \quad w(A_n^{-1}) \leq 1/\cos(\pi/n), \quad \|A_n\| = 1 + \frac{1}{8\sqrt{n}}.$$

Indeed, Theorem 1.3 implies that

$$\psi(1/\cos(\pi/n)) \geq \|A_n\| = 1 + \frac{1}{8\sqrt{n}}.$$

Taking  $1+\varepsilon = 1/\cos(\pi/n)$ , we see that the exponent  $1/4$  cannot be improved.

More generally, we can consider for  $\rho \geq 1$  the  $\rho$ -radius  $w_\rho(A)$  introduced by Sz.-Nagy and Foias (see [5, Chapter 1] and the references therein). Consider the class  $\mathcal{C}_\rho$  of operators  $T \in \mathcal{B}(H)$  which admit unitary  $\rho$ -dilations, i.e. there exist a super-space  $\mathcal{H} \supset H$  and a unitary operator  $U \in \mathcal{B}(\mathcal{H})$  such that

$$T^n = \rho P U^n P^*, \quad \text{for } n = 1, 2, \dots$$

Here  $P$  denotes the orthogonal projection from  $\mathcal{H}$  onto  $H$ . Then the operator  $\rho$ -radius is defined by

$$w_\rho(A) = \inf\{\lambda > 0; \lambda^{-1}A \in \mathcal{C}_\rho\}.$$

From this definition it is easily seen that  $r(A) \leq w_\rho(A) \leq \rho\|A\|$ , where  $r(A)$  denotes the spectral radius of  $A$ . Also,  $w_\rho(A)$  is a non-increasing function of  $\rho$ . Another equivalent definition follows from [5, Theorem 11.1]:

$$w_\rho(A) = \sup_{h \in \mathcal{E}_\rho} (1 - \frac{1}{\rho}) |\langle Ah, h \rangle| + \sqrt{(1 - \frac{1}{\rho})^2 |\langle Ah, h \rangle|^2 + (\frac{2}{\rho} - 1) \|Ah\|^2}, \quad \text{with}$$

$$\mathcal{E}_\rho = \{h \in H; \|h\| = 1 \text{ and } (1 - \frac{1}{\rho})^2 |\langle Ah, h \rangle|^2 - (1 - \frac{2}{\rho}) \|Ah\|^2 \geq 0\}.$$

Notice that  $\mathcal{E}_\rho = \{h \in H; \|h\| = 1\}$  whenever  $1 \leq \rho \leq 2$ . This shows that  $w_1(A) = \|A\|$ ,  $w_2(A) = w(A)$  and  $w_\rho(A)$  is a convex function of  $A$  if  $1 \leq \rho \leq 2$ .

We now define a function  $\psi_\rho(R)$  for  $R \geq 1$  by

$$\psi_\rho(R) = \sup\{\|A\|; A \in \mathcal{B}(H), w_\rho(A) \leq R, w_\rho(A^{-1}) \leq R\}.$$

As before, the conditions  $w_\rho(A) \leq R$  and  $w_\rho(A^{-1}) \leq R$  imply the existence of a unitary operator  $U$  such that  $\|A-U\| \leq \psi_\rho(R)-1$ , and we have  $\psi_\rho(R) \leq \rho R$ . We will generalize the estimate (1) from Theorem 1.1 by proving, for  $1 \leq \rho \leq 2$ , the following result.

**Theorem 1.4.** *For  $1 \leq \rho \leq 2$  we have*

$$\psi_\rho(R) \leq X_\rho(R) + \sqrt{X_\rho(R)^2 - 1}, \tag{2}$$

$$\text{with } X_\rho(R) = \frac{2 + \rho R^2 - \rho + \sqrt{(2 + \rho R^2 - \rho)^2 - 4R^2}}{2R}.$$

**Corollary 1.5.** *For  $1 \leq \rho \leq 2$  we have*

$$\psi_\rho(1+\varepsilon) \leq 1 + \sqrt[4]{8(\rho-1)\varepsilon} + O(\varepsilon^{1/2}), \quad \varepsilon \rightarrow 0.$$

We recover in this way for  $1 \leq \rho \leq 2$  the recent result of Ando and Li [2, Theorem 2.3], namely that  $w_\rho(A) \leq 1$  and  $w_\rho(A^{-1}) \leq 1$  imply  $A$  is unitary. The range  $1 \leq \rho \leq 2$  coincides with the range of those  $\rho \geq 1$  for which  $w_\rho(\cdot)$  is a norm. Contrarily to [2], we have not been able to treat the case  $\rho > 2$ .

The organization of the paper is as follows. In Section 2 we prove Theorem 1.4, which reduces to Theorem 1.1 in the case  $\rho = 2$ . The proof of Theorem 1.3 which shows the optimality of the exponent  $1/4$  in Corollary 1.2 is given in Section 3.

As a concluding remark, we would like to mention that the present developments have been influenced by the recent work of Sano/Uchiyama [7] and Ando/Li [2]. In [3], inspired by the paper of Stampfli [8], we have developed another (more complicated) approach in the case  $\rho = 2$ .

## 2 Proof of Theorem 1.4 about $\psi_\rho$

Let us consider  $M = \frac{1}{2}(A + (A^*)^{-1})$ ; then

$$M^*M - 1 = \frac{1}{4}(A^*A + (A^*A)^{-1} - 2) \geq 0.$$

This implies  $\|M^{-1}\| \leq 1$ . In what follows  $C^{1/2}$  will denote the positive square root of the self-adjoint positive operator  $C$ . The relation  $(A^*A - 2M^*M + 1)^2 = 4M^*M(M^*M - 1)$  yields

$$\begin{aligned} A^*A - 2M^*M + 1 &\leq 2(M^*M)^{1/2}(M^*M - 1)^{1/2}, \\ \text{whence } A^*A &\leq (M^*M)^{1/2} + (M^*M - 1)^{1/2}. \end{aligned}$$

Therefore  $\|A\| \leq \|M\| + \sqrt{\|M\|^2 - 1}$ .

We now assume  $1 \leq \rho \leq 2$ . Then  $w_\rho(\cdot)$  is a norm and the two conditions  $w_\rho(A) \leq R$  and  $w_\rho(A^{-1}) \leq R$  imply  $w_\rho(M) \leq R$ . The desired estimate of  $\psi_\rho(R)$  will follow from the following lemma.

**Lemma 2.1.** *Assume  $\rho \geq 1$ . Then the assumptions  $w_\rho(M) \leq R$  et  $\|M^{-1}\| \leq 1$  imply  $\|M\| \leq X_\rho(R)$ .*

*Proof.* The contractivity of  $M^{-1}$  implies

$$\|u\| \leq \|Mu\|, \quad (\forall u \in H). \tag{3}$$

From a generalization by Durszt [4] of a decomposition due to Ando [1], we can write

$$M = \rho R B^{1/2} U C^{1/2},$$

with  $U$  unitary,  $C$  selfadjoint satisfying  $0 < C < 1$ , and  $B = f(C)$  with  $f(x) = (1-x)/(1-\rho(2-\rho)x)^{-1}$ . Notice that  $f$  is a decreasing function on the segment  $[0, 1]$  and an involution:  $f(f(x)) = x$ . Let  $[\alpha, \beta]$  be the smallest segment containing the spectrum of  $C$ . Then  $[\sqrt{\alpha}, \sqrt{\beta}]$  is the smallest segment containing the spectrum of  $C^{1/2}$  and  $[\sqrt{f(\beta)}, \sqrt{f(\alpha)}]$  is the smallest segment containing the spectrum of  $B^{1/2}$ . We have

$$\|u\| \leq \|Mu\| \leq \rho R \sqrt{f(\alpha)} \|C^{1/2}u\|, \quad (\forall u \in H).$$

Choosing a sequence  $u_n$  of norm-one vectors ( $\|u_n\| = 1$ ) such that  $\|C^{1/2}u_n\|$  tends to  $\sqrt{\alpha}$ , we first get  $1 \leq \rho R \sqrt{\alpha f(\alpha)}$ , i.e.  $1 - (2+\rho R^2-\rho)\rho\alpha + \rho^2 R^2 \alpha^2 \leq 0$ . Consequently we have

$$\frac{2+\rho R^2-\rho - \sqrt{(2+\rho R^2-\rho)^2-4R^2}}{2\rho R^2} \leq \alpha \leq \frac{2+\rho R^2-\rho + \sqrt{(2+\rho R^2-\rho)^2-4R^2}}{2\rho R^2},$$

and

$$\frac{2+\rho R^2-\rho-\sqrt{(2+\rho R^2-\rho)^2-4R^2}}{2\rho R^2} \leq f(\alpha) \leq \frac{2+\rho R^2-\rho+\sqrt{(2+\rho R^2-\rho)^2-4R^2}}{2\rho R^2}.$$

Similarly, noticing that  $\|(M^*)^{-1}\| \leq 1$ ,  $M^* = \rho R C^{1/2} U^* B^{1/2}$  and  $C = f(B)$ , we obtain

$$\frac{2+\rho R^2-\rho-\sqrt{(2+\rho R^2-\rho)^2-4R^2}}{2\rho R^2} \leq \beta \leq \frac{2+\rho R^2-\rho+\sqrt{(2+\rho R^2-\rho)^2-4R^2}}{2\rho R^2}.$$

Therefore

$$\|M\| \leq \rho R \|B^{1/2}\| \|C^{1/2}\| = \rho R \sqrt{f(\alpha)\beta} \leq \frac{2+\rho R^2-\rho+\sqrt{(2+\rho R^2-\rho)^2-4R^2}}{2R}.$$

This shows that  $\|M\| \leq X_\rho(R)$ . □

### 3 The exponent $1/4$ is optimal (Proof of Theorem 1.3)

Consider the family of  $n \times n$  matrices  $A = DBD$ , defined for  $n = 8k + 4$ , by

$$B = I + \frac{1}{2n^{3/2}}E, \quad \text{with} \quad \begin{array}{l} e_{ij} = 1 \quad \text{if } 3k+2 \leq |i-j| \leq 5k+2, \\ e_{ij} = 0 \quad \text{otherwise,} \end{array}$$

$$D = \text{diag}(e^{i\pi/2n}, \dots, e^{(2\ell-1)i\pi/2n}, \dots, e^{(2n-1)i\pi/2n}).$$

We first remark that  $\|A\| = \|B\| = 1 + \frac{1}{8\sqrt{n}}$ . Indeed,  $B$  is a symmetric matrix with non negative entries,  $Be = (1 + \frac{1}{8\sqrt{n}})e$  with  $e^T = (1, 1, 1, \dots, 1)$ . Thus  $\|B\| = r(B) = 1 + \frac{1}{8\sqrt{n}}$ .

Consider now the permutation matrix  $P$  defined by  $p_{ij} = 1$  if  $i = j+1$  modulo  $n$  and  $p_{ij} = 0$  otherwise. Then  $P^{-1}DP = e^{i\pi/n}D$  and  $P^{-1}EP = E$ , whence  $P^{-1}AP = e^{2i\pi/n}A$ . Since  $P$  is a unitary matrix, the numerical range  $W(A) = \{\langle Au, u \rangle; \|u\| = 1\}$  of  $A$  satisfies  $W(A) = W(P^{-1}AP) = e^{2i\pi/n}W(A)$ . Furthermore, from  $A^* = \bar{A}$ , it follows that  $W(A)$  is symmetric with respect to the real axis. Putting all these things together we obtain that the numerical range of  $A$  is invariant by the group of symmetries of a regular  $n$ -sided polygon, and the same property also holds for the numerical range of  $A^{-1}$ .

We postpone the proof of the estimates  $\|\frac{1}{2}(A+A^*)\| \leq 1$  and  $\|\frac{1}{2}(A^{-1}+(A^{-1})^*)\| \leq 1$  to later sections. Using these estimates, we obtain that the numerical range  $W(A)$  is contained in the half-plane  $\{z; \text{Re } z \leq 1\}$ , whence in the regular  $n$ -sided polygon given by the intersection of the half-planes  $\{z; \text{Re}(e^{2i\pi k/n}z) \leq 1\}$ ,  $k = 1, \dots, n$ . Consequently  $w(A) \leq 1/\cos(\pi/n)$ . The proof of  $w(A^{-1}) \leq 1/\cos(\pi/n)$  is similar.

#### 3.1 Proof of $\|\frac{1}{2}(A+A^*)\| \leq 1$ .

It suffices to show that, for every  $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ , we have  $\|u\|^2 - \text{Re}\langle Au, u \rangle \geq 0$ . Let  $\mathcal{E} = \{(i, j); 1 \leq i, j \leq n, 3k+2 \leq |i-j| \leq 5k+2\}$ . The inequality which has to be proved is equivalent to

$$\sum_{i=1}^n 2 \sin^2((i-\frac{1}{2})\frac{\pi}{n}) u_i^2 - \frac{1}{2n^{3/2}} \sum_{i,j \in \mathcal{E}} \cos((i+j-1)\frac{\pi}{n}) u_i u_j \geq 0.$$

Setting  $v_j = u_j \sin((j-\frac{1}{2})\frac{\pi}{n})$ , this may be also written as follows

$$2\|v\|^2 - \langle Mv, v \rangle - \frac{1}{2n^{3/2}} \langle Ev, v \rangle \geq 0, \quad (v \in \mathbb{R}^n). \quad (4)$$

Here  $M$  is the matrix whose entries are defined by

$$m_{ij} = \frac{1}{2n^{3/2}} \cot\left(\left(i - \frac{1}{2}\right)\frac{\pi}{n}\right) \cot\left(\left(j - \frac{1}{2}\right)\frac{\pi}{n}\right), \quad \text{if } (i, j) \in \mathcal{E}, \quad m_{ij} = 0 \quad \text{otherwise.}$$

We will see that the Frobenius (or Hilbert-Schmidt) norm of  $M$  satisfies  $\|M\|_F \leq \frac{9}{32}$ . A fortiori, the operator norm of  $M$  satisfies  $\|M\| \leq \frac{9}{32}$ . Together with  $\|E\| = n/4$ , this shows that  $\|M\| + \frac{1}{2n^{3/2}}\|E\| \leq \frac{13}{32}$ . Property (4) is now verified.

It remains to show that  $\|M\|_F \leq \frac{9}{32}$ . First we notice that  $m_{ij} = m_{ji} = m_{n+1-i, n+1-j}$ , and  $m_{ii} = 0$ . Hence, with  $\mathcal{E}' = \{(i, j) \in \mathcal{E}; i < j, \text{ et } i + j \leq n + 1\}$ ,

$$\|M\|_F^2 = 2 \sum_{i < j} |m_{ij}|^2 \leq 4 \sum_{(i, j) \in \mathcal{E}'} |m_{ij}|^2.$$

We have, for  $(i, j) \in \mathcal{E}'$ ,

$$\begin{aligned} 2j \leq i + j + 5k + 2 \leq n + 5k + 3 = 13k + 7, \quad \text{thus } 3k + 3 \leq j \leq \frac{13k+7}{2}, \\ 2i \leq i + j - 3k - 1 \leq n - 3k - 1 = 5k + 3, \quad \text{thus } 1 \leq i \leq \frac{5k+3}{2}. \end{aligned}$$

This shows that

$$\frac{3\pi}{16} \leq \frac{3k+2}{16k+8}\pi \leq \left(j - \frac{1}{2}\right)\frac{\pi}{n} \leq \frac{13k+6}{16k+8}\pi \leq \pi - \frac{3\pi}{16}, \quad \text{hence } |\cot\left(\left(j - \frac{1}{2}\right)\frac{\pi}{n}\right)| \leq \cot \frac{3\pi}{16} \leq \frac{3}{2}.$$

We also use the estimate  $\cot\left(\left(i - \frac{1}{2}\right)\frac{\pi}{n}\right) \leq n/(\pi(i - \frac{1}{2}))$  to obtain

$$\|M\|_F^2 \leq 4 \sum_{(i, j) \in \mathcal{E}'} |m_{ij}|^2 \leq \frac{4}{4n^3} \frac{n^2}{\pi^2} \sum_{i \geq 1} \frac{1}{(i - 1/2)^2} (2k+1) \frac{9}{4} = \frac{9}{32}.$$

### 3.2 Proof of $\left\|\frac{1}{2}(A^{-1} + (A^{-1})^*)\right\| \leq 1$ .

We start from

$$\begin{aligned} (A^{-1})^* &= D\left(1 + \frac{1}{2n^{3/2}}E\right)^{-1}D \\ &= D^2 - \frac{1}{2n^{3/2}}DED + \frac{1}{4n^3}DE^2\left(1 + \frac{1}{2n^{3/2}}E\right)^{-1}D, \end{aligned}$$

and we want to show that  $\|u\|^2 - \operatorname{Re}\langle A^{-1}u, u \rangle \geq 0$ . As previously, we set  $v_j = u_j \sin\left(\left(j - \frac{1}{2}\right)\frac{\pi}{n}\right)$ . The inequality  $\left\|\frac{1}{2}(A^{-1} + (A^{-1})^*)\right\| \leq 1$  is equivalent to

$$2\|v\|^2 - \langle (M_1 + M_2 + M_3 + M_4)v, v \rangle \geq 0, \quad (v \in \mathbb{R}^n).$$

Here the entries of the matrices  $M_p$ ,  $1 \leq p \leq 4$ , are given by

$$\begin{aligned} (m_1)_{ij} &= -\frac{1}{2n^{3/2}} \cot\left(\left(i - \frac{1}{2}\right)\frac{\pi}{n}\right) \cot\left(\left(j - \frac{1}{2}\right)\frac{\pi}{n}\right)e_{ij}, \\ (m_2)_{ij} &= -\frac{1}{2n^{3/2}} e_{ij}, \\ (m_3)_{ij} &= \frac{1}{4n^3} \cot\left(\left(i - \frac{1}{2}\right)\frac{\pi}{n}\right) \cot\left(\left(j - \frac{1}{2}\right)\frac{\pi}{n}\right)f_{ij}, \\ (m_4)_{ij} &= \frac{1}{4n^3} f_{ij}, \end{aligned}$$

$e_{ij}$  and  $f_{ij}$  respectively denoting the entries of the matrices  $E$  and  $F = E^2(1 + \frac{1}{2n^{3/2}}E)^{-1}$ . Noticing that  $M_1 = -M$ , we have  $\|M_1\| \leq \frac{9}{32}$ ,  $\|M_2\| = \frac{1}{8\sqrt{n}}$ ,  $\|F\| = \frac{n^2/16}{1+1/8\sqrt{n}} \leq \frac{n^2}{16}$  and  $\|M_4\| = \frac{1}{4n^3} \|F\|$ . Now we use

$$\|M_3\|^2 \leq \|M_3\|_F^2 \leq \frac{1}{16n^6} \max_{ij} |f_{ij}|^2 \sum_{i, j} \left| \cot\left(\left(i - \frac{1}{2}\right)\frac{\pi}{n}\right) \right|^2 \left| \cot\left(\left(j - \frac{1}{2}\right)\frac{\pi}{n}\right) \right|^2,$$

together with

$$\sum_{i,j} |\cot((i-\frac{1}{2})\frac{\pi}{n})|^2 |\cot((j-\frac{1}{2})\frac{\pi}{n})|^2 = \left( \sum_i^n |\cot((i-\frac{1}{2})\frac{\pi}{n})|^2 \right)^2 \leq n^4,$$

to obtain

$$\|M_3\|_F \leq \frac{1}{4n} \max_{i,j} |f_{ij}|.$$

Denoting  $e_{ij}^2$  the entries of the matrix  $E^2$ , we easily verify that  $\max_{i,j} |e_{ij}^2| = n/4$ . Therefore  $\max_{i,j} |f_{ij}| \leq \frac{n/4}{1-1/8\sqrt{n}}$ , whence  $\|M_3\|_F \leq \frac{1}{16(1-1/8\sqrt{n})}$ . Finally, we obtain  $\|M_1 + M_2 + M_3 + M_4\| \leq 1$ .

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