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# PARABOLIC DELIGNE-LUSZTIG VARIETIES.

FRANÇOIS DIGNE AND JEAN MICHEL

ABSTRACT. Motivated by the Broué conjecture on blocks with abelian defect groups for finite reductive groups, we study “parabolic” Deligne-Lusztig varieties and construct on those which occur in the Broué conjecture an action of a braid monoid, whose action on their  $\ell$ -adic cohomology will conjecturally factor through a cyclotomic Hecke algebra. In order to construct this action, we need to enlarge the set of varieties we consider to varieties attached to a “ribbon category”; this category has a *Garside family*, which plays an important role in our constructions, so we devote the first part of our paper to the necessary background on categories with Garside families.

## 1. INTRODUCTION

In this paper, we study “parabolic” Deligne-Lusztig varieties, one of the main motivations being the Broué conjecture on blocks with abelian defect groups for finite reductive groups.

Let  $\mathbf{G}$  be a connected reductive algebraic group over an algebraic closure  $\overline{\mathbb{F}}_p$  of the prime field  $\mathbb{F}_p$  of characteristic  $p$ . Let  $F$  be an isogeny on  $\mathbf{G}$  such that some power  $F^\delta$  is a Frobenius endomorphism attached to a split structure over the finite field  $\mathbb{F}_{q^\delta}$ ; this defines a real number  $q$  such that  $q^\delta$  is an integral power of  $p$ . When  $\mathbf{G}$  is quasi-simple, any isogeny  $F$  such that the group of fixed points  $\mathbf{G}^F$  is finite is of the above form; such a group  $\mathbf{G}^F$  is called a “finite reductive group” or a “finite group of Lie type”.

Let  $\mathbf{L}$  be an  $F$ -stable Levi subgroup of a (non necessarily  $F$ -stable) parabolic subgroup  $\mathbf{P}$  of  $\mathbf{G}$ . Then, for  $\ell$  a prime number different from  $p$ , Lusztig has constructed a “cohomological induction”  $R_{\mathbf{L}}^{\mathbf{G}}$  which associates to any  $\overline{\mathbb{Q}}_\ell \mathbf{L}^F$ -module a virtual  $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$ -module. We study the particular case  $R_{\mathbf{L}}^{\mathbf{G}}(\text{Id})$ , which is given by the alternating sum of the  $\ell$ -adic cohomology groups of the variety

$$\mathbf{X}_{\mathbf{P}} = \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g\mathbf{P} \cap F(g\mathbf{P}) \neq \emptyset\}$$

on which  $\mathbf{G}^F$  acts on the left. We will construct a monoid of endomorphisms  $M$  of  $\mathbf{X}_{\mathbf{P}}$  related to the braid group, which conjecturally will induce in some cases a cyclotomic Hecke algebra on the cohomology of  $\mathbf{X}_{\mathbf{P}}$ . To construct  $M$  we need to enlarge the set of varieties we consider, to include varieties attached to morphisms in a “ribbon category” — the “parabolic Deligne-Lusztig varieties” of this paper;  $M$  corresponds to the endomorphisms in the “conjugacy category” of this ribbon category of the object attached to  $\mathbf{X}_{\mathbf{P}}$ .

The relationship with Broué’s conjecture for the principal block comes as follows: assume, for some prime number  $\ell \neq p$ , that the  $\ell$ -Sylow  $S$  of  $\mathbf{G}^F$  is abelian. Then Broué’s conjecture predicts in this special case an equivalence of derived categories

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between the principal block of  $\overline{\mathbb{Z}}_\ell \mathbf{G}^F$  and that of  $\overline{\mathbb{Z}}_\ell N_{\mathbf{G}^F}(S)$ . Now  $\mathbf{L} := C_{\mathbf{G}}(S)$  is a Levi subgroup of a (non  $F$ -stable unless  $\ell|q-1$ ) parabolic subgroup  $\mathbf{P}$ ; restricting to unipotent characters and discarding an eventual torsion by changing coefficients from  $\overline{\mathbb{Z}}_\ell$  to  $\overline{\mathbb{Q}}_\ell$ , this translates into conjectures about the cohomology of  $\mathbf{X}_{\mathbf{P}}$ , see 10.1; these conjectures predict in particular that the image in the cohomology of our monoid  $M$  is a cyclotomic Hecke algebra.

The main feature of the ribbon categories we consider is that they have *Garside families*. This concept has appeared in recent work to understand the ordinary and dual monoids attached to the braid groups; in the first part of this paper, we recall its basic properties and go as far as computing the centralizers of “periodic elements”, which is what we need in the applications.

In the second part, we first define the parabolic Deligne-Lusztig varieties which are the aim of our study, and then go on to establish their properties. We extend to this setting in particular all the material in [BM] and [BR2].

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After this paper was written, we received a preprint from Xuhua He and Sian Nie (see [HN]) where, amidst other interesting results, they also prove Theorem 9.1 and Corollary 9.3.

## I. Garside families

This part collects some prerequisites on categories with Garside families. It is mostly self-contained apart from the next section where the proofs are omitted; we refer for them to the book [DDGKM] to appear.

### 2. BASIC RESULTS ON GARSIDE FAMILIES

Given a category  $\mathcal{C}$ , we write  $f \in \mathcal{C}$  to say that  $f$  is a morphism of  $\mathcal{C}$ , and  $\mathcal{C}(x, y)$  for the set of morphisms from  $x \in \text{Obj } \mathcal{C}$  to  $y \in \text{Obj } \mathcal{C}$ . We write  $fg$  for the composition of  $f \in \mathcal{C}(x, y)$  and  $g \in \mathcal{C}(y, z)$ , and  $\mathcal{C}(x)$  for  $\mathcal{C}(x, x)$ . By  $\mathcal{S} \subset \mathcal{C}$  we mean that  $\mathcal{S}$  is a set of morphisms in  $\mathcal{C}$ .

All the categories we consider will be left-cancellative, that is a relation  $hf = hg$  implies  $f = g$ , and right-cancellative, so  $f = g$  is also implied by  $fh = gh$ ; equivalently every morphism is a monomorphism and an epimorphism. We say that  $f$  left divides  $g$ , written  $f \preccurlyeq g$ , if there exists  $h$  such that  $g = fh$ . Similarly we say that  $f$  right divides  $g$  and write  $g \succcurlyeq f$  if there exists  $h$  such that  $g = hf$ .

We denote by  $\mathcal{C}^\times$  the set of invertible morphisms of  $\mathcal{C}$ , and write  $f =^\times g$  if there exists  $h \in \mathcal{C}^\times$  such that  $fh = g$  (or equivalently there exists  $h \in \mathcal{C}^\times$  such that  $f = gh$ ).

**Definition 2.1.** *A Garside family in  $\mathcal{C}$  is a subset  $\mathcal{S} \subset \mathcal{C}$  such that;*

- $\mathcal{S}$  together with  $\mathcal{C}^\times$  generates  $\mathcal{C}$ .
- $\mathcal{C}^\times \mathcal{S} \subset \mathcal{S} \mathcal{C}^\times \cup \mathcal{C}^\times$ .
- For every product  $fg$  with  $f, g \in \mathcal{S} - \mathcal{C}^\times$ , either  $fg \in \mathcal{S} \mathcal{C}^\times$  in which case we say that the 1-term sequence  $(fg)$  is the  $\mathcal{S}$ -normal decomposition of  $fg$ , or we have  $fg = f_1 g_1$ , where  $f_1 \in \mathcal{S}$ ,  $g_1 \in \mathcal{S} \mathcal{C}^\times - \mathcal{C}^\times$  are such that any relation  $h \preccurlyeq kf_1 g_1$  with  $h \in \mathcal{S}$  implies  $h \preccurlyeq kf_1$ ; in this case we say that the 2-term sequence  $(f_1, g_1)$  is an  $\mathcal{S}$ -normal decomposition of  $fg$ .

We extend  $\mathcal{S}$ -normal decompositions to longer lengths by saying that  $(x_1, \dots, x_n)$  is an  $\mathcal{S}$ -normal decomposition of  $x = x_1 \dots x_n$  if for each  $i$  the sequence  $(x_i, x_{i+1})$  is an  $\mathcal{S}$ -normal decomposition. In a category with a Garside family every non-invertible element  $x$  admits an  $\mathcal{S}$ -normal decomposition. We will just say “normal decomposition” if  $\mathcal{S}$  is clear from the context. Normal decompositions are unique up to invertibles, precisely

**Lemma 2.2.** *If  $(x_1, \dots, x_n)$  and  $(x'_1, \dots, x'_{n'})$  are two normal decompositions of  $x$  then  $n = n'$  and for any  $i$  we have  $x_1 \dots x_i =^x x'_1 \dots x'_i$ .*

**Head functions.** We have the following criterion to be Garside:

**Proposition 2.3.** *Let  $\mathcal{S} \subset \mathcal{C}$  together with  $\mathcal{C}^\times$  generate  $\mathcal{C}$ , and let  $H$  be a function  $\mathcal{C} - \mathcal{C}^\times \xrightarrow{H} \mathcal{S}$ . Consider the following properties*

- (i)  $\forall g \in \mathcal{C} - \mathcal{C}^\times, H(g) \preccurlyeq g$ .
- (ii)  $\forall g \in \mathcal{C} - \mathcal{C}^\times, \forall h \in \mathcal{S}, h \preccurlyeq g \Rightarrow h \preccurlyeq H(g)$ .
- (iii)  $\forall f \in \mathcal{C}, \forall g \in \mathcal{C} - \mathcal{C}^\times, H(fg) =^x H(fH(g))$ .
- (iv)  $\mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$  is closed under right-divisor.

Then  $\mathcal{S}$  is Garside if (i), (ii), (iii) hold for some  $H$ , or if (i) and (ii) hold for some  $H$ , and (iv) holds. Conversely if  $\mathcal{S}$  is Garside then (iv) holds and there exists  $H$  satisfying (i) to (iii) above; such a function is called an  $\mathcal{S}$ -head function.

An  $\mathcal{S}$ -head function  $H$  computes the first term of a normal decomposition in the sense that if  $(x_1, \dots, x_n)$  is a normal decomposition of  $x$  then  $H(x) =^x x_1$ .

For  $f \in \mathcal{C}$  we define  $\lg_{\mathcal{S}}(f)$  to be the minimum number  $k$  of morphisms  $s_1, \dots, s_k \in \mathcal{S}$  such that  $s_1 \dots s_k =^x f$ , thus  $\lg_{\mathcal{S}}(f) = 0$  if  $f \in \mathcal{C}^\times$ ; if  $f \notin \mathcal{C}^\times$  then  $\lg_{\mathcal{S}}(f)$  is also the number of terms in a normal decomposition of  $f$ . We have the following property:

**Lemma 2.4.** *Let  $H$  be an  $\mathcal{S}$ -head function, and for  $x \in \mathcal{C} - \mathcal{C}^\times$  let  $x'$  be defined by  $x = H(x)x'$ . Then  $\lg_{\mathcal{S}}(x') < \lg_{\mathcal{S}}(x)$ .*

The following shows that  $\mathcal{S}$  “determines”  $\mathcal{C}$  up to invertibles; we say that a subset  $\mathcal{C}_1$  of  $\mathcal{C}$  is closed under right quotient if an equality  $f = gh$  with  $f, g \in \mathcal{C}_1$  implies  $h \in \mathcal{C}_1$ .

**Lemma 2.5.** *Let  $\mathcal{S}$  be a Garside family in  $\mathcal{C}$ . Let  $\mathcal{C}_1$  be a subcategory of  $\mathcal{C}$  closed under right-quotient which contains  $\mathcal{S}$ . Then  $\mathcal{C} = \mathcal{C}_1\mathcal{C}^\times$  and  $\mathcal{S}$  is a Garside family in  $\mathcal{C}_1$ .*

**Categories with automorphism.** Most categories we want to consider will have no non-trivial invertible element, which simplifies Definition 2.1. The only source of invertible elements will be the following construction.

An automorphism of a category  $\mathcal{C}$  is a functor  $F : \mathcal{C} \rightarrow \mathcal{C}$  which has an inverse. Given an automorphism  $F$  of finite order of the category  $\mathcal{C}$ , we define

**Definition 2.6.** *The semi-direct product category  $\mathcal{C} \rtimes \langle F \rangle$  is the category whose objects are the objects of  $\mathcal{C}$  and whose morphisms with source  $x$  are the pairs  $(g, F^i)$ , which will be denoted by  $gF^i$ , where  $g$  is a morphism of  $\mathcal{C}$  with source  $x$  and  $i$  is an integer. The target of this morphism is  $F^{-i}(\text{target}(g))$ , where  $\text{target}(g)$  is the target of  $g$ . The composition rule is given by  $gF^i \cdot hF^j = gF^i(h)F^{i+j}$  when  $\text{source}(h) = F^{-i}(\text{target}(g))$ .*

The conventions on  $F$  are such that the composition rule is natural. However, they imply that the morphism  $F$  of the semi-direct product category represents the functor  $F^{-1}$ : it is a morphism from the object  $F(A)$  to the object  $A$  and we have the commutative diagram:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow F & & \downarrow F \\ A & \xrightarrow{f} & B \end{array}$$

When  $\mathcal{C}$  has Garside family  $\mathcal{S}$ , we call *Garside automorphism* of  $(\mathcal{C}, \mathcal{S})$ , an automorphism  $F$  which preserves  $\mathcal{S}\mathcal{C}^\times$ .

**Lemma 2.7.** *If  $\mathcal{S}$  is a Garside family in  $\mathcal{C}$ , and  $F$  a Garside automorphism of  $(\mathcal{C}, \mathcal{S})$ , then  $\mathcal{S}$  is also a Garside family in  $\mathcal{C} \rtimes \langle F \rangle$ .*

If  $(f_1, \dots, f_k)$  is an  $\mathcal{S}$ -normal decomposition of  $f \in \mathcal{C}$  then  $(f_1, \dots, f_k F^i)$  is an  $\mathcal{S}$ -normal decomposition of  $f F^i \in \mathcal{C} \rtimes \langle F \rangle$ . Note that if  $\mathcal{C}$  has no non-trivial invertible element, then the only invertibles in  $\mathcal{C} \rtimes \langle F \rangle$  are  $\{F^i\}_{i \in \mathbb{Z}}$ . In general, if  $a, b \in \mathcal{C}$  then  $a F^i \preccurlyeq b F^j$  if and only if  $a \preccurlyeq b$ .

We have the following property

**Proposition 2.8.** *Assume that  $\mathcal{C}$  has a Garside family  $\mathcal{S}$  and has no non-trivial invertible morphisms. Let  $F$  be a Garside automorphism of  $\mathcal{C}$ . Then the subcategory of fixed objects and morphisms  $\mathcal{C}^F$  has a Garside family which consists of the fixed points  $\mathcal{S}^F$ .*

**Gcds and lcms, Noetherianity.** The existence of gcds and lcms are related when  $\mathcal{C}$  is right-Noetherian, which means that there is no infinite sequence  $f_0 \succcurlyeq f_1 \dots \succcurlyeq f_n \succcurlyeq \dots$  where  $f_{i+1}$  is a *proper* right divisor of  $f_i$ , that is we do not have  $f_i =^* f_{i+1}$ . It means equivalently since  $\mathcal{C}$  is left cancellative that there is no infinite sequence  $f_0 \preccurlyeq f_1 \dots \preccurlyeq f_n \preccurlyeq \dots \preccurlyeq f$  where  $f_i$  is a proper left divisor of  $f_{i+1}$ .

We say that  $\mathcal{C}$  admits local right lcms if, whenever  $f$  and  $g$  have a common right multiple, they have a right lcm. We then have:

**Proposition 2.9.** *If  $\mathcal{C}$  is right Noetherian and admits local right lcms, then any family of morphisms of  $\mathcal{C}$  with the same source has a left gcd.*

Here is a more general situation when a Garside family of a subcategory can be determined. If  $\mathcal{C}$  admits local right lcms we say that a subset  $X \subset \mathcal{C}$  is closed under right lcm if whenever two elements of  $X$  have a right lcm in  $\mathcal{C}$  this lcm is in  $X$ .

**Lemma 2.10.** *Let  $\mathcal{S}$  be a Garside family in  $\mathcal{C}$  assumed right-Noetherian and having local right lcms. Let  $\mathcal{S}_1 \subset \mathcal{S}$  be a subfamily such that  $\mathcal{S}_1 \mathcal{C}^\times$  is as a subset of  $\mathcal{S} \mathcal{C}^\times$  closed under right-lcm and right-quotient; then  $\mathcal{S}_1$  is a Garside family in the subcategory  $\mathcal{C}_1$  generated by  $\mathcal{S}_1 \mathcal{C}^\times$ . Moreover  $\mathcal{C}_1$  is a subcategory closed under right-quotient.*

The following lemma about Noetherian categories will also be useful:

**Lemma 2.11.** *Let  $\mathcal{C}$  be a category and  $\mathcal{S}$  be a set of morphisms which generates  $\mathcal{C}$ . Let  $X$  be a set of morphisms of  $\mathcal{C}$  with same source satisfying*

- (i)  $X$  is closed under left divisor and  $X = X \mathcal{C}^\times$ .
- (ii)  $X$  is a bounded and right Noetherian poset.

(iii) If  $f \in X$ ,  $g, h \in \mathcal{S}$  and  $fg, fh \in X$  then  $g$  and  $h$  have a common right-multiple  $m$  such that  $fm \in X$ .

Then  $X$  is the set of left-divisors of some morphism of  $\mathcal{C}$ .

**Garside maps.** An important special case is when  $\mathcal{S}$  is attached to a Garside map. A Garside map is a map  $\text{Obj } \mathcal{C} \xrightarrow{\Delta} \mathcal{C}$  where  $\Delta(x) \in \mathcal{C}(x, -)$  such that  $\mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$  is the set of left divisors of  $\Delta$ . Since by Proposition 2.3(iv) the set  $\mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$  is stable by right divisor, it is also the set of right divisors of  $\Delta$ .

This allows to define a functor  $\Phi$ , first on objects by taking for  $\Phi(x)$  the target of  $\Delta(x)$ , then on morphisms, first on morphisms  $s \in \mathcal{S}$  by, if  $s \in \mathcal{C}(x, -)$  defining  $s'$  by  $ss' = \Delta$  (we omit the source of  $\Delta$  if it is clear from the context) and then  $\Phi(s)$  by  $s'\Phi(s) = \Delta$ . We then extend  $\Delta$  by using normal decompositions; it can be shown that this is well-defined and defines a functor such that for any  $f \in \mathcal{C}$  we have  $f\Delta = \Delta\Phi(f)$ . It can also be shown that the right-cancellativity of  $\mathcal{C}$  implies that  $\Phi$  is an automorphism.

The automorphism  $\Phi$  is a typical example of a Garside automorphism that we will call the *canonical Garside automorphism*.

If  $\mathcal{S}$  is attached to a Garside map, we then have the following properties:

**Proposition 2.12.** (i) If  $f \preccurlyeq g$  then  $\lg_{\mathcal{S}}(f) \leq \lg_{\mathcal{S}}(g)$ .  
(ii) Assume  $f, g, h \in \mathcal{S}$  and  $(f, g)$  is  $\mathcal{S}$ -normal; then  $\lg_{\mathcal{S}}(fgh) \leq 2$  implies  $gh \in \mathcal{S}\mathcal{C}^\times$ .

We will write  $\Delta^p$  for the map which associates to an object  $x$  the morphism  $\Delta(x)\Delta(\Phi(x))\dots\Delta(\Phi^{p-1}(x))$ . For any  $f \in \mathcal{C}(x, -)$  there exists  $p$  such that  $f \preccurlyeq \Delta^p(x)$ .

**Example 2.13.** An example of a category with a Garside family is a *Garside monoid*, which is just the case where  $\mathcal{C}$  has one object. In this case we will say Garside element instead of Garside map. A classical example is given by the *Artin monoid*  $(B^+, \mathbf{S})$  associated to a Coxeter system  $(W, S)$ . Then  $B^+$  is left and right-cancellative, Noetherian, admits local left-lcms and right-lcms and has a Garside family, the canonical lift  $\mathbf{W}$  of  $W$  in  $B^+$ , which consists of the elements whose length with respect to  $\mathbf{S}$  is equal to the length with respect to  $S$  of their image in  $W$ . The Garside family  $\mathbf{W}$  is attached to a Garside element if and only if  $W$  is finite. In this case the Garside element is the lift in  $\mathbf{W}$  of the longest element of  $W$ .

### 3. THE CONJUGACY CATEGORY

The context for this section is a left and right-cancellative category  $\mathcal{C}$ .

**Definition 3.1.** Given a category  $\mathcal{C}$ , we define the conjugacy category  $\text{Ad } \mathcal{C}$  of  $\mathcal{C}$  as the category whose objects are the endomorphisms of  $\mathcal{C}$  and where, for  $w \in \mathcal{C}(A)$  and  $w' \in \mathcal{C}(B)$  we set  $\text{Ad } \mathcal{C}(w, w') = \{x \in \mathcal{C}(A, B) \mid xw' = wx\}$ . We say that  $x$  conjugates  $w$  to  $w'$  and call centralizer of  $w$  the set  $\text{Ad } \mathcal{C}(w)$ . The composition of morphisms in  $\text{Ad } \mathcal{C}$  is given by the composition in  $\mathcal{C}$ , which is compatible with the defining relation for  $\text{Ad } \mathcal{C}$ .

Note that the definition of  $\text{Ad } \mathcal{C}(w, w')$  is what forces the objects of  $\text{Ad } \mathcal{C}$  to be endomorphisms of  $\mathcal{C}$ .

Since  $\mathcal{C}$  is left-cancellative, the data  $x$  and  $w$  determine  $w'$  (resp. since  $\mathcal{C}$  is right-cancellative  $x$  and  $w'$  determine  $w$ ). This allows us to write  $w^x$  for  $w'$  (resp.  ${}^x w'$  for  $w$ ); this illustrates that our category  $\text{Ad } \mathcal{C}$  is a right conjugacy category; we could call left conjugacy category the opposed category.

A proper name for an element of  $\text{Ad } \mathcal{C}(w, w')$  should be a triple  $w \xrightarrow{x} w'$ , since  $x$  by itself does specify neither its source  $w$  nor its target  $w'$ , but we will use just  $x$  when the context makes clear which source  $w$  is meant (or which target is meant). The functor  $I$  which sends  $w \in \text{Obj}(\text{Ad } \mathcal{C})$  to  $\text{source}(w)$  and  $w \xrightarrow{x} w'$  to  $x$  is faithful, though not injective on objects. The faithfulness of  $I$  allows us to identify  $\text{Ad } \mathcal{C}(w, -)$  to the subset  $\{x \in \mathcal{C}(\text{source}(w), -) \mid x \preccurlyeq wx\}$  (resp. identify  $\text{Ad } \mathcal{C}(-, w)$  to the subset  $\{x \in \mathcal{C}(-, \text{source}(w)) \mid xw \succcurlyeq x\}$ ).

It follows that the category  $\text{Ad } \mathcal{C}$  inherits automatically from  $\mathcal{C}$  properties such as cancellativity or Noetherianity. The functor  $I$  maps  $(\text{Ad } \mathcal{C})^\times$  surjectively to  $\mathcal{C}^\times$ , so in particular the subset  $\text{Ad } \mathcal{C}(w, -)$  of  $\mathcal{C}(\text{source}(w), -)$  is closed under multiplication by  $\mathcal{C}^\times$ . In the proofs and statements which follow we identify  $\text{Ad } \mathcal{C}$  to a subset of  $\mathcal{C}$  and  $(\text{Ad } \mathcal{C})^\times$  to  $\mathcal{C}^\times$ ; for the statements obtained about  $\text{Ad } \mathcal{C}$  to make sense, the reader has to check that the sources and target of morphisms viewed as morphisms in  $\text{Ad } \mathcal{C}$  make sense.

**Lemma 3.2.**

- The subset  $\text{Ad } \mathcal{C}$  of  $\mathcal{C}$  is closed under right-quotient, that is if we have an equality  $y = xz$  where  $y \in \text{Ad } \mathcal{C}(w, w')$ ,  $x \in \text{Ad } \mathcal{C}(w, -)$  and  $z \in \mathcal{C}(-, \text{source}(w'))$ , then  $z \in \text{Ad } \mathcal{C}(-, w')$ .
- The subset  $\text{Ad } \mathcal{C}(w, -)$  of  $\mathcal{C}(\text{source}(w), -)$  is closed under right-lcm, in the sense that if  $x, y \in \text{Ad } \mathcal{C}(w, -)$  have a right-lcm in  $\mathcal{C}(\text{source}(w), -)$  then this right-lcm is in  $\text{Ad } \mathcal{C}(w, -)$  and is a right-lcm of  $x$  and  $y$  in  $\text{Ad } \mathcal{C}$ . In particular if  $\mathcal{C}$  admits local right-lcms then so does  $\text{Ad } \mathcal{C}$ .

Similarly  $\text{Ad } \mathcal{C}(-, w)$  is a subset of  $\mathcal{C}(-, \text{source}(w))$  closed under left-lcm and left-quotient.

*Proof.* We show the stability by right-quotient. If  $y, x, z$  are as in the statement, we have  $x \preccurlyeq wx$  and  $yw' = wy$ . By cancellation, let us define  $w''$  by  $xw'' = wx$ . Then from  $xz = y \preccurlyeq wy = wxz = xw''z$  we deduce by cancellation that  $z \preccurlyeq w''z$ , so  $z \in \text{Ad } \mathcal{C}(w, w_1)$  where  $zw_1 = w''z$ . Now since  $y = xz$  the equality  $yw' = wy$  gives  $xzw' = wxz = xw''z = xzw_1$  which shows by cancellation that  $w_1 = w'$ .

We now show stability by right-lcm.  $x, y \in \text{Ad } \mathcal{C}(w, -)$  means that  $x \preccurlyeq wx$  and  $y \preccurlyeq wy$ . Suppose now that  $x$  and  $y$  have a right-lcm  $z$  in  $\mathcal{C}$ . Then  $x \preccurlyeq wz$  and  $y \preccurlyeq wz$  from which it follows that  $z \preccurlyeq wz$ , that is  $z \in \text{Ad } \mathcal{C}(w, -)$ , and  $z$  is necessarily the image of a right-lcm of  $x$  and  $y$  in  $\text{Ad } \mathcal{C}$ .

The proof of the second part is just a mirror symmetry of the above proof.  $\square$

**Proposition 3.3.** *Assume that  $\mathcal{S}$  is a Garside family in  $\mathcal{C}$ ; then  $\text{Ad } \mathcal{C} \cap \mathcal{S}$  is a Garside family in  $\text{Ad } \mathcal{C}$  and  $\mathcal{S}$ -normal decompositions of an element of  $\text{Ad } \mathcal{C}$  are  $\text{Ad } \mathcal{C} \cap \mathcal{S}$ -normal decompositions.*

*Proof.* We will use Proposition 2.3 by showing that  $(\text{Ad } \mathcal{C} \cap \mathcal{S}) \cup \mathcal{C}^\times$  generates  $\text{Ad } \mathcal{C}$  and exhibiting a function  $H : \text{Ad } \mathcal{C} - \mathcal{C}^\times \rightarrow \text{Ad } \mathcal{C} \cap \mathcal{S}$  which satisfies Proposition 2.3(i), (ii) and (iii).

Let  $H$  be a  $\mathcal{S}$ -head function in  $\mathcal{C}$ . We first show that the restriction of  $H$  to  $\text{Ad } \mathcal{C}$  takes its values in  $\text{Ad } \mathcal{C} \cap \mathcal{S}$ . Indeed if  $x \preccurlyeq wx$  then  $H(x) \preccurlyeq H(wx) = {}^x H(wH(x)) \preccurlyeq wH(x)$ .

We now deduce by induction on  $\lg_{\mathcal{S}}$  that  $(\text{Ad } \mathcal{C} \cap \mathcal{S}) \cup \mathcal{C}^\times$  generates  $\text{Ad } \mathcal{C}$ . If  $x \in \text{Ad } \mathcal{C}$  is such that  $\lg_{\mathcal{S}}(x) = 1$  then  $x = s\varepsilon$  with  $s \in \mathcal{S}$  and  $\varepsilon \in \mathcal{C}^\times$ . Since  $\text{Ad } \mathcal{C}$  is closed under multiplication by  $\mathcal{C}^\times$  we have  $s \in \text{Ad } \mathcal{C} \cap \mathcal{S}$ , whence  $x \in (\text{Ad } \mathcal{C} \cap \mathcal{S})\mathcal{C}^\times$ . Assume now that  $x \in \text{Ad } \mathcal{C}$  is such that  $\lg_{\mathcal{S}}(x) = n$  and define  $x'$  by  $x = H(x)x'$ . Since we know that  $H(x) \in \text{Ad } \mathcal{C}$ , we deduce by Lemma 3.2 that  $x' \in \text{Ad } \mathcal{C}$ ; by Lemma 2.4 we have  $\lg_{\mathcal{S}}(x') < n$ , whence the result.

It is obvious that the restriction of  $H$  to  $\text{Ad } \mathcal{C} - \mathcal{C}^\times$  still has properties (i), (ii), (iii) of Proposition 2.3 thus is a head function, which proves that  $\text{Ad } \mathcal{C} \cap \mathcal{S}$  is a Garside family. The assertion about normal decompositions follows.  $\square$

**Simultaneous conjugacy.** A straightforward generalization of conjugacy categories is “simultaneous conjugation categories”, where objects are families of morphisms  $w_1, \dots, w_n$  with same source and target, and morphisms verify  $x \preccurlyeq w_i x$  for all  $i$ . Most statements have a straightforward generalization to this case.

**$F$ -conjugacy.** We want to consider “twisted conjugation” by an automorphism, which will be useful for applications to Deligne-Lusztig varieties, but also for internal applications, with the automorphism being the one induced by a Garside map.

**Definition 3.4.** Let  $F$  be an automorphism of the category  $\mathcal{C}$ . We define the  $F$ -conjugacy category of  $\mathcal{C}$ , denoted by  $F\text{-Ad } \mathcal{C}$ , as the category whose objects are the morphisms in some  $\mathcal{C}(A, F(A))$  and where, for  $w \in \mathcal{C}(A, F(A))$  and  $w' \in \mathcal{C}(B, F(B))$  we set  $F\text{-Ad } \mathcal{C}(w, w') = \{x \in \mathcal{C} \mid xw' = wF(x)\}$ . We say that  $x$   $F$ -conjugates  $w$  to  $w'$  and we call  $F$ -centralizer of a morphism  $w$  of  $\mathcal{C}$  the set  $F\text{-Ad } \mathcal{C}(w)$ .

Note that  $F$ -conjugacy specializes to conjugacy when  $F = \text{Id}$  and that the  $F$ -centralizer of  $x$  is empty unless  $x \in \mathcal{C}(A, F(A))$  for some object  $A$ .

We explore now how these notions relate to conjugation in a semi-direct product category.

- Consider the application which sends  $w \in \mathcal{C}(A, F(A)) \subset \text{Obj}(F\text{-Ad } \mathcal{C})$  to  $wF \in (\mathcal{C} \rtimes \langle F \rangle)(A) \subset \text{Obj}(\text{Ad}(\mathcal{C} \rtimes \langle F \rangle))$ . Since  $x(w'F) = (wF)x$  is equivalent to  $xw' = wF(x)$ , this extends to a functor  $J$  from  $F\text{-Ad } \mathcal{C}$  to  $\text{Ad}(\mathcal{C} \rtimes \langle F \rangle)$ . This functor is clearly an isomorphism onto its image.

The image  $J(\text{Obj}(F\text{-Ad } \mathcal{C}))$  is the subset of  $\mathcal{C} \rtimes \langle F \rangle$  which consists of endomorphisms which lie in  $\mathcal{C}F$ ; and  $J(F\text{-Ad } \mathcal{C})$  identifies via  $I$  to the subset of  $\mathcal{C} \rtimes \langle F \rangle$  whose elements are both in  $\text{Ad}(\mathcal{C} \rtimes \langle F \rangle)$  and in  $\mathcal{C}$ .

As in  $\text{Ad}(\mathcal{C} \rtimes \langle F \rangle)$  there is no morphism between two objects which do not have the same power of  $F$ , the full subcategory that we will denote  $\text{Ad}(\mathcal{C}F)$  of  $\text{Ad}(\mathcal{C} \rtimes \langle F \rangle)$  whose objects are in  $\mathcal{C}F$  is a union of connected components of  $\text{Ad}(\mathcal{C} \rtimes \langle F \rangle)$ ; thus many properties will transfer automatically from  $\text{Ad}(\mathcal{C} \rtimes \langle F \rangle)$  to  $\text{Ad}(\mathcal{C}F)$ .

In particular, if  $\mathcal{C}$  has a Garside family  $\mathcal{S}$  and  $F$  is a Garside automorphism, then  $\mathcal{S}$  is still a Garside family for  $\mathcal{C} \rtimes \langle F \rangle$  by 2.7, and by Proposition 3.3 and the above gives rise to a Garside family  $\mathcal{S} \cap \text{Ad}(\mathcal{C}F)$  of  $\text{Ad}(\mathcal{C}F)$ . The image of  $J$  is the subcategory of  $\text{Ad}(\mathcal{C}F)$  consisting (via  $I$ ) of the morphisms in  $\mathcal{C}$ , thus satisfies the assumptions of Lemma 2.5: it is closed under right quotient, because in a relation  $fg = h$  if  $f$  and  $h$  do not involve  $F$  the same must be true for  $g$ , and contains the Garside family  $\mathcal{S} \cap \text{Ad}(\mathcal{C}F)$  of  $\text{Ad}(\mathcal{C}F)$ .

This will allow to generally translate statements about conjugacy categories to statements about  $F$ -conjugacy categories. For example,  $J^{-1}(\mathcal{S} \cap \text{Ad}(\mathcal{C}F))$  is a Garside family for  $F\text{-Ad } \mathcal{C}$ ; this last family is just  $F\text{-Ad } \mathcal{C} \cap \mathcal{S}$  when identifying  $F\text{-Ad } \mathcal{C}$  to a subset of  $\mathcal{C}$ .

If  $F$  has finite order, since  $(xF)^x = Fx = (xF)^{F^{-1}}$  two morphisms in  $\mathcal{C}F$  are conjugate in  $\mathcal{C} \rtimes \langle F \rangle$  if and only if they are conjugate by a morphism of  $\mathcal{C}$ .

#### 4. THE CYCLIC CONJUGACY CATEGORY

A restricted form of conjugation called “cyclic conjugacy” will be important in applications. In particular, it turns out (a particular case of Proposition 4.5) that two periodic braids are conjugate if and only if they are cyclically conjugate.

**Definition 4.1.** *We define the cyclic conjugacy category  $\text{cyc } \mathcal{C}$  of  $\mathcal{C}$  as the subcategory of  $\text{Ad } \mathcal{C}$  generated by  $\{x \in \text{Ad } \mathcal{C}(w, w') \mid x \preccurlyeq w\}$ .*

That is,  $\text{cyc } \mathcal{C}$  has the same objects as  $\text{Ad } \mathcal{C}$  but contains only the products of elementary conjugations of the form  $w = xy \xrightarrow{x} yx = w'$ . Note that since  $\mathcal{C}$  is left- and right-cancellative, then  $\cup_w \{x \in \text{Ad } \mathcal{C}(w, w') \mid x \preccurlyeq w\} = \cup_w \{x \in \text{Ad } \mathcal{C}(w, w') \mid w' \succcurlyeq x\}$  so cyclic conjugacy “from the left” and “from the right” are the same. To be more precise, the functor which is the identity on objects, and when  $w = xy$  and  $w' = yx$ , sends  $x \in \text{cyc } \mathcal{C}(w, w')$  to  $y \in \text{cyc } \mathcal{C}(w', w)$ , is an isomorphism between  $\text{cyc } \mathcal{C}$  and its opposed category.

**Proposition 4.2.** *Assume  $\mathcal{C}$  is right-Noetherian and admits local right-lcms; if  $\mathcal{S}$  is a Garside family in  $\mathcal{C}$  then the set  $\mathcal{S}_1 = \cup_w \{x \in \text{Ad } \mathcal{C}(w, -) \mid x \preccurlyeq w \text{ and } x \in \mathcal{S}\}$  is a Garside family in  $\text{cyc } \mathcal{C}$ .*

*Proof.* We first observe that  $\mathcal{S}_1 \mathcal{C}^\times$  generates  $\text{cyc } \mathcal{C}$ . Indeed if  $x \preccurlyeq w$  and we choose a decomposition  $x = s_1 \dots s_n$  as a product of morphisms in  $\mathcal{S} \mathcal{C}^\times$  it is clear that each  $s_i$  is in  $\text{cyc } \mathcal{C}$ , so is in  $\mathcal{S}_1$ .

The proposition then results from Lemma 2.10, which applies to  $\text{cyc } \mathcal{C}$  since  $\mathcal{S}_1 \mathcal{C}^\times$  is closed under right-divisor and right-lcm; this is obvious for right-divisor and for right-lcm results from the facts that  $\mathcal{S}$ , being a Garside family, is closed under right-lcm and that a right-lcm of two divisors of  $w$  is a divisor of  $w$ .  $\square$

We also see by Lemma 2.10 that  $\text{cyc } \mathcal{C}$  is closed under right-quotient in  $\text{Ad } \mathcal{C}$ .

We now prove that independently of the choice of a Garside family  $\mathcal{S}$  in  $\mathcal{C}$  the category  $\text{cyc } \mathcal{C}$  has a natural Garside family defined by a Garside map; this Garside family is usually larger than the Garside family  $\mathcal{S}_1$  of Proposition 4.2, since it contains all left divisors of  $w$  even if  $w$  is not in  $\mathcal{S}$ .

**Proposition 4.3.** *Assume  $\mathcal{C}$  is right Noetherian and admits local right-lcms; then the set  $\mathcal{S}' = \cup_w \{x \in \text{Ad } \mathcal{C}(w, -) \mid x \preccurlyeq w\}$  is a Garside family in  $\text{cyc } \mathcal{C}$  attached to the Garside map  $\Delta$  such that  $\Delta(w) = w \in \text{cyc } \mathcal{C}(w)$ ; the corresponding Garside automorphism  $\Phi$  is the identity functor.*

*Proof.* The set  $\mathcal{S}'$  generates  $\text{cyc } \mathcal{C}$  by definition of  $\text{cyc } \mathcal{C}$ . It is closed under right-divisors since  $xy \preccurlyeq w$  implies  $x \preccurlyeq w$  so that  $w^x$  is defined and  $y \preccurlyeq w^x$ ; since  $\mathcal{C}$  is right Noetherian and admits local right-lcms, any two morphisms of  $\mathcal{C}$  with same source have a gcd by Proposition 2.9. We define a function  $H : \text{cyc } \mathcal{C} \rightarrow \mathcal{S}'$  by letting  $H(x)$  be an arbitrarily chosen left-gcd of  $x$  and  $w$  if  $x \in \text{cyc } \mathcal{C}(w, -)$ . Since  $\text{cyc } \mathcal{C}$  is closed under right-divisor, the restriction of  $H$  to non invertible elements

satisfies properties Proposition 2.3 (i), (ii) and (iv), so  $\mathcal{S}'$  is a Garside family for  $\text{cyc } \mathcal{C}$ . The set of morphisms in  $\mathcal{S}'$  with source  $w$  has  $w$  as a lcm. Moreover if  $v$  is a right-divisor of  $\Delta(w) = w$  in  $\text{cyc } \mathcal{C}$ , which defines  $v'$  such that  $w = v'v$ , then  $v' \in \text{cyc } \mathcal{C}(w, vv')$  thus the source of  $v$  is  $vv'$  and  $v$  divides  $vv'$ , so  $v \in \mathcal{S}'$ ; all conditions of Proposition 2.3 are fulfilled, and  $\Delta$  is a Garside map since  $\mathcal{S}'(w, -)$  is the set of left divisors of  $\Delta(w)$ . The equation  $xw^x = wx$  shows that  $\Phi$  is the identity.  $\square$

**Proposition 4.4.** *Assume  $\mathcal{C}$  is right-Noetherian and admits local right-lcms; then the subcategory  $\text{cyc } \mathcal{C}$  of  $\text{Ad } \mathcal{C}$  is closed under left-gcd (that is, a gcd in  $\text{Ad } \mathcal{C}$  of two morphisms in  $\text{cyc } \mathcal{C}$  is in  $\text{cyc } \mathcal{C}$ ).*

*Proof.* Let  $(x_1, \dots, x_n)$  and  $(y_1, \dots, y_m)$  be  $\mathcal{S}'$ -normal decompositions respectively of  $x \in \text{cyc } \mathcal{C}(w, -)$  and  $y \in \text{cyc } \mathcal{C}(w, -)$  where  $\mathcal{S}'$  is as in Proposition 4.3.

We first prove that if  $\gcd(x_1, y_1) =^\times 1$  then  $\gcd(x, y) =^\times 1$  (here we consider left-gcds in  $\text{Ad } \mathcal{C}$ ). We proceed by induction on  $\inf\{m, n\}$ . We write  $\Delta$  for  $\Delta(w)$  when there is no ambiguity on the source  $w$ . We have that  $\gcd(x, y)$  divides

$$\begin{aligned} \gcd(x_1 \dots x_{n-1} \Delta, y_1 \dots y_{m-1} \Delta) &=^\times \gcd(\Delta x_1 \dots x_{n-1}, \Delta y_1 \dots y_{m-1}) \\ &=^\times \Delta \gcd(x_1 \dots x_{n-1}, y_1 \dots y_{m-1}) =^\times \Delta = w, \end{aligned}$$

where the first equality uses that  $\Phi$  is the identity and the one before last results from the induction hypothesis. So we get that  $\gcd(x, y)$  divides  $w$  in  $\text{Ad } \mathcal{C}$ , so  $\gcd(x, y) \in \mathcal{S}'$ ; thus  $\gcd(x, y)$  divides  $x_1$  and  $y_1$ , so is trivial.

We now prove the proposition. If  $\gcd(x_1, y_1) =^\times 1$  then  $\gcd(x, y) =^\times 1$  thus is in  $\text{cyc } \mathcal{C}$  and we are done. Otherwise let  $d_1$  be a gcd of  $x_1$  and  $y_1$  and let  $x^{(1)}, y^{(1)}$  be defined by  $x = d_1 x^{(1)}$ ,  $y = d_1 y^{(1)}$ . Similarly let  $d_2$  be a gcd of the first terms of a normal decomposition of  $x^{(1)}, y^{(1)}$  and let  $x^{(2)}, y^{(2)}$  be the remainders, etc... Since  $\mathcal{C}$  is right-Noetherian the sequence  $d_1, d_1 d_2, \dots$  of increasing divisors of  $x$  must stabilize at some stage  $k$ , which means that the corresponding remainders  $x^{(k)}$  and  $y^{(k)}$  have first terms of their normal decomposition coprime, so by the first part are themselves coprime. Thus  $\gcd(x, y) =^\times d_1 \dots d_k \in \text{cyc } \mathcal{C}$ .  $\square$

We now give a quite general context where cyclic conjugacy is the same as conjugacy.

**Proposition 4.5.** *Let  $\mathcal{C}$  be a right Noetherian category with a Garside map  $\Delta$ , and let  $x$  be an endomorphism of  $\mathcal{C}$  such that for  $n$  large enough we have  $\Delta \preccurlyeq x^n$ . Then for any  $y$  we have  $\text{cyc } \mathcal{C}(x, y) = \text{Ad } \mathcal{C}(x, y)$ .*

*Proof.* We first show that the property  $\Delta \preccurlyeq x^n$  is stable by conjugacy (up to changing  $n$ ). Indeed, if  $u \in \text{Ad } \mathcal{C}(x, -)$  then there exists  $k$  such that  $u \preccurlyeq \Delta^k$ . Then  $(x^u)^{n(k+1)} = (x^{n(k+1)})^u = (u^{-1} x^{n(k+1)}) u$  is divisible by  $\Delta$  since  $\Delta^{k+1} \preccurlyeq x^{n(k+1)}$ .

It follows that it is sufficient to prove that if  $f \in \text{Ad } \mathcal{C}(x, y)$ ,  $f \notin \mathcal{C}^\times$ , then  $\gcd(f, x) \notin \mathcal{C}^\times$ . Indeed write  $f = u f_1$  where  $u = \gcd(f, x)$  then since  $u \in \text{cyc } \mathcal{C}(x, x^u)$  it is sufficient to prove that  $f_1 \in \text{Ad } \mathcal{C}(x^u, y)$  is actually in  $\text{cyc } \mathcal{C}(x^u, y)$ , which we do by induction since  $\mathcal{C}$  is Noetherian and  $x^u$  still satisfies the same condition.

Since as observed any  $u \in \text{Ad } \mathcal{C}(x, -)$  divides some power of  $x$  ( $x^{nk}$  if  $u \preccurlyeq \Delta^k$ ) it is enough to show that if  $u \in \text{Ad } \mathcal{C}(x, -)$ ,  $u \notin \mathcal{C}^\times$  and  $u \preccurlyeq x^n$ , then  $\gcd(u, x) \notin \mathcal{C}^\times$ . We do this by induction on  $n$ . From  $u \in \text{Ad } \mathcal{C}(x, -)$  we have  $u \preccurlyeq xu$ , and from  $u \preccurlyeq x^n$  we deduce  $u \preccurlyeq x \gcd(u, x^{n-1})$ . If  $\gcd(u, x^{n-1}) \in \mathcal{C}^\times$  then  $u \preccurlyeq x$  and we are

done:  $\gcd(x, u) = u$ . Otherwise let  $u_1 = \gcd(u, x^{n-1})$ . We have  $u_1 \preccurlyeq xu_1$ ,  $u_1 \notin \mathcal{C}^\times$  and  $u_1 \preccurlyeq x^{n-1}$  thus we are done by induction.  $\square$

**The  $F$ -cyclic conjugacy.** Let  $F$  be a finite order automorphism of the category  $\mathcal{C}$ . We define  $F$ -cyc  $\mathcal{C}$  as the subcategory of  $F$ -Ad  $\mathcal{C}$  generated by  $\{x \in F\text{-Ad } \mathcal{C}(w, w') \mid x \preccurlyeq w\}$ , or equivalently, since  $\mathcal{C}$  is left- and right-cancellative, by  $\{x \in \text{Ad } \mathcal{C}(w, w') \mid w' \succcurlyeq F(x)\}$ . By the functor  $J$ , the morphisms in  $F$ -cyc  $\mathcal{C}(w, w')$  identify to the morphisms in  $\text{cyc}(\mathcal{C} \rtimes \langle F \rangle)(wF, w'F)$  which lie in  $\mathcal{C}$ . To simplify notation, we will denote by  $\text{cyc } \mathcal{C}(wF, w'F)$  this last set of morphisms. If  $\mathcal{C}$  is right-Noetherian and admits local right-lcms, then  $\mathcal{C} \rtimes \langle F \rangle$  also. If  $\mathcal{S}$  is a Garside family in  $\mathcal{C}$  and  $F$  is a Garside automorphism, and we translate Proposition 4.2 to the image of  $J$  and then to  $F$ -cyc  $\mathcal{C}$ , we get that  $\cup_w \{x \in F\text{-Ad } \mathcal{C}(w, -) \mid x \preccurlyeq w \text{ and } x \in \mathcal{S}\}$  is a Garside family in  $F$ -cyc  $\mathcal{C}$ .

Similarly Proposition 4.3 says that the set  $\cup_w \{x \in F\text{-Ad } \mathcal{C}(w, -) \mid x \preccurlyeq w\}$  is a Garside family in  $F$ -cyc  $\mathcal{C}$  attached to the Garside map  $\Delta$  which sends the object  $w$  to the morphism  $w \in F$ -cyc  $\mathcal{C}(w, F(w))$ ; the associated Garside automorphism is the functor  $F$ .

Finally Proposition 4.4 says that under the assumptions of Proposition 4.3 the subcategory  $F$ -cyc  $\mathcal{C}$  of  $F$ -Ad  $\mathcal{C}$  is closed under left-gcd.

## 5. AN EXAMPLE: RIBBON CATEGORIES

In the context of an Artin monoid  $(B^+, \mathbf{S})$  (see Example 2.13) we want to study the conjugates and the normalizer of a parabolic submonoid (the submonoid generated by a subset of the atoms  $\mathbf{S}$ ). The “ribbon” category that we consider in this section occurs in the work of Paris [Pa] and Godelle [G] on this topic. In Section 8 we will attach parabolic Deligne-Lusztig varieties to the morphisms of the ribbon category and endomorphisms of these varieties to morphisms in the conjugacy category of this ribbon category.

Since most results work in the more general situation of a Garside monoid and a parabolic submonoid we will place ourselves in this context.

**Definition 5.1.** Let  $M$  be a (cancellative) right-Noetherian monoid which admits local right lcm’s. We say that a submonoid  $M'$  is parabolic if it is closed by left-divisor and right-lcm.

**Lemma 5.2.** The above assumption is satisfied when we take for  $M$  an Artin monoid  $B^+$  and for  $M'$  the “parabolic” submonoid  $B_{\mathbf{I}}^+$  generated by  $\mathbf{I} \subset \mathbf{S}$ .

*Proof.* We first show that  $B_{\mathbf{I}}^+$  is closed by left-divisors. Since both sides of each defining relation for  $B^+$  involve the same generators, two equivalent words involve the same generators. Hence if  $xy = z$  with  $z \in B_{\mathbf{I}}^+$  then  $x$  has an expression involving only elements in  $\mathbf{I}$  so is in  $B_{\mathbf{I}}^+$ . This implies also that if two elements have a right-lcm  $\delta$  in  $B_{\mathbf{I}}^+$ , then  $\delta$  is divisible by their right-lcm in  $B^+$ , so has to be equal to that right lcm. It remains to show that two elements which have a common multiple in  $B^+$  have a common multiple (hence a right-lcm) in  $B_{\mathbf{I}}^+$ . Taking heads we see that it is sufficient to prove that two elements of  $\mathbf{W}_{\mathbf{I}}$  which have a common right-multiple in  $\mathbf{W}$  have a common multiple in  $\mathbf{W}_{\mathbf{I}}$ . This is true since any element of  $\mathbf{W}$  can be written uniquely as  $vw$  with  $v \in \mathbf{W}_{\mathbf{I}}$  and  $w$  not divisible by any element of  $\mathbf{I}$ .  $\square$

In the rest of this section we fix a cancellative right-Noetherian monoid  $M$  which admits local right lcms and a Garside family  $\mathcal{S}$  in  $M$ .

**Lemma 5.3.** *Let  $M'$  be a parabolic submonoid of  $M$ . Then any  $u \in M$  has a maximal left-divisor  $\alpha_{M'}(u)$  in  $M'$ .*

*Proof.* The set  $X = \{x \in M' \mid x \preccurlyeq u\}$  is a subset of  $M'$  which satisfies the assumptions of Lemma 2.11: it is closed under left-divisor, it is right-Noetherian and if  $xg$  and  $xh$  are in  $X$  with  $g, h \in M'$ , then  $\text{lcm}(g, h)$  exists, since  $g$  and  $h$  left-divide  $x^{-1}u$ , hence  $x \text{lcm}(g, h)$  is in  $X$  since it divides  $u$  and  $\text{lcm}(g, h) \in M'$ . Thus  $X$  is the set of divisors of some morphism  $\alpha_{M'}(u)$ .  $\square$

**Lemma 5.4.** *Let  $M'$  be a parabolic submonoid of  $M$  and  $\mathcal{S}$  be a Garside family in  $M$ ; assume that  $\mathcal{S}' = \mathcal{S} \cap M'$  together with  $M'^\times$  generates  $M'$ , then  $\mathcal{S}'$  is a Garside family in  $M'$ .*

*Proof.* Let  $H$  be an  $\mathcal{S}$ -head function in  $M$ . Since  $M'$  is closed under left-divisor, for  $g \in M' - \{1\}$  we have  $H(g) \in \mathcal{S}'$ . It is then straightforward that the restriction of  $H$  to  $M' - \{1\}$  satisfies properties (i), (ii) and (iii) of 2.3, whence the result.  $\square$

**The simultaneous conjugacy category.** We now consider a submonoid of  $M$  generated by a subset of the atoms. Let  $\mathbf{S}$  be the set of atoms of  $M$ ; for  $\mathbf{I} \subset \mathbf{S}$  we denote by  $M_{\mathbf{I}}$  the submonoid generated by  $\mathbf{I}$ .

**Assumption 5.5.** *We assume that for  $\mathbf{s} \in \mathbf{S}$  any conjugate  $\mathbf{t}$  in  $M$  of  $\mathbf{s}$  is in  $\mathbf{S}$  (that is, if  $\mathbf{s}f = f\mathbf{t}$  with  $f$  and  $\mathbf{t}$  in  $M$  then  $\mathbf{t} \in \mathbf{S}$ ).*

The above assumption is automatic if  $M$  has homogeneous relations, or equivalently has an additive length function with atoms of length 1. This is clearly the case for Artin monoids.

Under this assumption a conjugate of a subset of  $\mathcal{S}$  is a subset of  $\mathcal{S}$ . In the following we fix an orbit  $\mathcal{I}$  under conjugacy of subsets of  $\mathcal{S}$  and we make the following assumption:

**Assumption 5.6.** *For any  $\mathbf{I} \in \mathcal{I}$  the monoid  $M_{\mathbf{I}}$  is parabolic.*

Let  $\text{Ad}(M, \mathcal{I})$  be the connected component of the simultaneous conjugacy category of  $M$  whose objects are the elements of  $\mathcal{I}$ . A morphism in  $\text{Ad}(M, \mathcal{I})$  with source  $\mathbf{I} \in \mathcal{I}$  is a  $\mathbf{b} \in M$  such that for each  $\mathbf{s} \in \mathbf{I}$  we have  $\mathbf{s}^{\mathbf{b}} \in M$ , which by Assumption 5.5 implies  $\mathbf{s}^{\mathbf{b}} \in \mathbf{S}$ . We denote such a morphism in  $\text{Ad}(M, \mathcal{I})(\mathbf{I}, \mathbf{J})$  by  $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$  where  $\mathbf{J} = \{\mathbf{s}^{\mathbf{b}} \mid \mathbf{b} \in \mathbf{I}\}$ , and in this situation we write  $\mathbf{J} = \mathbf{I}^{\mathbf{b}}$ .

By Proposition 3.3 the set  $\{\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{I}^{\mathbf{b}} \mid \mathbf{b} \in \mathcal{S}\}$  is a Garside family in  $\text{Ad}(M, \mathcal{I})$ .

**The ribbon category.** In our context we will just write  $\alpha_{\mathbf{I}}$  for  $\alpha_{M_{\mathbf{I}}}$  and denote by  $\omega_{\mathbf{I}}(\mathbf{b})$  the element defined by  $\mathbf{b} = \alpha_{\mathbf{I}}(\mathbf{b})\omega_{\mathbf{I}}(\mathbf{b})$ . We say that  $\mathbf{b} \in M$  is  $\mathbf{I}$ -reduced if it is left-divisible by no element of  $\mathbf{I}$ , or equivalently if  $\alpha_{\mathbf{I}}(\mathbf{b}) = 1$ .

**Definition 5.7.** *We define the ribbon category  $M(\mathcal{I})$  as the subcategory of  $\text{Ad}(M, \mathcal{I})$  obtained by restricting the morphisms to the  $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$  such that  $\mathbf{b}$  is  $\mathbf{I}$ -reduced.*

That the above class of morphisms is stable by composition is the object of (ii) in the next proposition; and (i) is a motivation for restricting to the  $\mathbf{I}$ -reduced morphisms by showing that we “lose nothing” in doing so.

**Proposition 5.8.** (i)  $(\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}) \in \text{Ad}(M, \mathcal{I})$  if and only if  $(\mathbf{I} \xrightarrow{\alpha_{\mathbf{I}}(\mathbf{b})} \mathbf{I}) \in \text{Ad}(M, \mathcal{I})$  and  $(\mathbf{I} \xrightarrow{\omega_{\mathbf{I}}(\mathbf{b})} \mathbf{J}) \in M(\mathcal{I})$ .

(ii) If  $(\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}) \in \text{Ad}(M, \mathcal{I})$  then for any  $\mathbf{b}' \in M$  we have  $\alpha_{\mathbf{J}}(\mathbf{b}') = \alpha_{\mathbf{I}}(\mathbf{b}\mathbf{b}')^{\mathbf{b}}$ . In particular if  $(\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}) \in M(\mathcal{I})$  and  $(\mathbf{J} \xrightarrow{\mathbf{b}'} \mathbf{K}) \in \text{Ad}(M, \mathcal{I})$  then  $(\mathbf{I} \xrightarrow{\mathbf{b}\mathbf{b}'} \mathbf{K}) \in M(\mathcal{I})$  if and only if  $(\mathbf{J} \xrightarrow{\mathbf{b}'} \mathbf{K}) \in M(\mathcal{I})$ .

(iii) Let  $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$  and  $\mathbf{I} \xrightarrow{\mathbf{b}'} \mathbf{J}'$  be two morphisms of  $\text{Ad}(M, \mathcal{I})$  and let  $\mathbf{I} \xrightarrow{\mathbf{c}} \mathbf{I}^{\mathbf{c}}$  be their right lcm which by Lemma 3.2 exists and is obtained for  $\mathbf{c}$  the right-lcm in  $M$  of  $\mathbf{b}$  and  $\mathbf{b}'$ ; then if  $\mathbf{b}$  and  $\mathbf{b}'$  are  $\mathbf{I}$ -reduced, then  $\mathbf{c}$  is also.

*Proof.* Let us prove (i). We prove by induction on the length of  $\mathbf{b}$  that if  $\mathbf{s} \in \mathbf{I}$  and  $\mathbf{s}^{\mathbf{b}} \in M$  then  $\mathbf{s}^{\alpha_{\mathbf{I}}(\mathbf{b})} \in \mathbf{I}$ . This will prove (i) in one direction. The converse is obvious.

By Assumption 5.5 we have  $\mathbf{s}\mathbf{b} = \mathbf{b}\mathbf{t}$  for some  $\mathbf{t} \in \mathbf{S}$ . If  $\mathbf{s} \preccurlyeq \mathbf{b}$  we write  $\mathbf{b} = \mathbf{s}\mathbf{b}'$  so that  $\mathbf{s}\mathbf{b}' = \mathbf{b}'\mathbf{t}$ . We have  $\alpha_{\mathbf{I}}(\mathbf{b}) = \mathbf{s}\alpha_{\mathbf{I}}(\mathbf{b}')$  and we are done by induction. If  $\mathbf{s}$  does not divide  $\mathbf{b}$  then the lcm of  $\mathbf{s}$  and  $\alpha_{\mathbf{I}}(\mathbf{b})$  divide  $\mathbf{s}\mathbf{b} = \mathbf{b}\mathbf{t}$  and this lcm can be written  $\mathbf{s}\mathbf{v} = \alpha_{\mathbf{I}}(\mathbf{b})\mathbf{u}$ , with  $\mathbf{v}$  and  $\mathbf{u}$  in  $M_{\mathbf{I}}$  since  $M_{\mathbf{I}}$  is closed by right-lcm. We get then that  $\mathbf{v}$  divides  $\mathbf{b}$ , so divides  $\alpha_{\mathbf{I}}(\mathbf{b})$ ; thus  $\alpha_{\mathbf{I}}(\mathbf{b})\mathbf{u} = \mathbf{v}\mathbf{a}\mathbf{u}$  for some  $\mathbf{a} \in M$ . By Assumption 5.5 we have that  $\mathbf{a}\mathbf{u} \in \mathbf{S}$ , thus  $\mathbf{a} = 1$  and  $\mathbf{u} \in \mathbf{S}$ , hence  $\mathbf{u} \in \mathbf{I}$  which is the result.

Let us prove (ii). For  $\mathbf{s} \in \mathbf{I}$  let  $\mathbf{s}' = \mathbf{s}^{\mathbf{b}} \in \mathbf{J}$ . Assume first that  $\mathbf{s} \not\preccurlyeq \mathbf{b}$ . Then  $\mathbf{b}\mathbf{s}' = \mathbf{s}\mathbf{b}$  is a common multiple of  $\mathbf{s}$  and  $\mathbf{b}$  which has to be their lcm since  $\mathbf{s}'$  is an atom. So for  $\mathbf{s} \in \mathbf{I}$  we have  $\mathbf{s} \preccurlyeq \mathbf{b}\mathbf{s}'$  if and only if  $\mathbf{b}\mathbf{s}' \preccurlyeq \mathbf{b}\mathbf{s}'$ , that is,  $\mathbf{s}^{\mathbf{b}} \preccurlyeq \mathbf{b}'$  whence the result. Now if  $\mathbf{s} \preccurlyeq \mathbf{b}$  we write  $\mathbf{b} = \mathbf{s}^k\mathbf{b}_1$  with  $\mathbf{s} \not\preccurlyeq \mathbf{b}_1$ ; we have  $\mathbf{s}' = \mathbf{s}^{\mathbf{b}_1}$  and the above proof, with  $\mathbf{b}_1$  instead of  $\mathbf{b}$ , applies.

To prove (iii) we will actually show the stronger statement that if for  $\mathbf{b}, \mathbf{c} \in M$  we have  $\mathbf{b} \preccurlyeq \mathbf{c}$ ,  $\mathbf{I}^{\mathbf{b}} \subset \mathbf{S}$  then  $\alpha_{\mathbf{I}}(\mathbf{b}) \preccurlyeq \alpha_{\mathbf{I}}(\mathbf{c})$  (which is obvious) and  $\omega_{\mathbf{I}}(\mathbf{b}) \preccurlyeq \omega_{\mathbf{I}}(\mathbf{c})$  (then in the situation of (iii) we get that  $\omega_{\mathbf{I}}(\mathbf{c})$  is a common multiple of  $\mathbf{b}$  and  $\mathbf{b}'$ , thus  $\mathbf{c} \preccurlyeq \omega_{\mathbf{I}}(\mathbf{c})$ , which is impossible unless  $\alpha_{\mathbf{I}}(\mathbf{c}) = 1$ ). By dividing  $\mathbf{b}$  and  $\mathbf{c}$  by  $\alpha_{\mathbf{I}}(\mathbf{b})$  we may as well assume that  $\alpha_{\mathbf{I}}(\mathbf{b}) = 1$  since  $\mathbf{I}^{\omega_{\mathbf{I}}(\mathbf{b})} \subset \mathbf{S}$  by (i). We write  $\mathbf{c} = \mathbf{b}\mathbf{b}_1$  and  $\mathbf{J} = \mathbf{I}^{\mathbf{b}}$ . By (ii) we have  $\alpha_{\mathbf{I}}(\mathbf{c})^{\mathbf{b}} = \alpha_{\mathbf{J}}(\mathbf{b}_1)$ , whence  $\alpha_{\mathbf{I}}(\mathbf{c})\mathbf{b} = \mathbf{b}\alpha_{\mathbf{J}}(\mathbf{b}_1) \preccurlyeq \mathbf{b}\mathbf{b}_1 = \mathbf{c} = \alpha_{\mathbf{I}}(\mathbf{c})\omega_{\mathbf{I}}(\mathbf{c})$ . Left-canceling  $\alpha_{\mathbf{I}}(\mathbf{c})$  we get  $\mathbf{b} \preccurlyeq \omega_{\mathbf{I}}(\mathbf{c})$  which is what we want since  $\mathbf{b} = \omega_{\mathbf{I}}(\mathbf{b})$ .  $\square$

Note that by Proposition 5.8(i) a morphism in  $M(\mathcal{I})$  with source  $\mathbf{I}$  is the same as an element  $\mathbf{b} \in M$  such that  $\alpha_{\mathbf{I}}(\mathbf{b}) = 1$  and for each  $\mathbf{s} \in \mathbf{I}$  we have  $\mathbf{s}^{\mathbf{b}} \in M$ . We will thus sometimes just denote by  $\mathbf{b}$  such a morphism in  $M(\mathcal{I})$  when the context makes its source clear.

Next proposition shows that  $\mathcal{S} \cap M(\mathcal{I})$  generates  $M(\mathcal{I})$ .

**Proposition 5.9.** All the terms of the normal decomposition in  $\text{Ad}(M, \mathcal{I})$  of a morphism of  $M(\mathcal{I})$  are in  $M(\mathcal{I})$ .

*Proof.* Let  $(\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}) \in M(\mathcal{I})$  and let  $\mathbf{b} = \mathbf{w}_1 \dots \mathbf{w}_k$  be its normal decomposition in  $\text{Ad}(M, \mathcal{I})$  (it is also the normal decomposition in  $M$  by Proposition 3.3). As  $\mathbf{w}_i \in \text{Ad}(M, \mathcal{I})$ , the source of  $\mathbf{w}_i$  is  $\mathbf{I}_i = \mathbf{I}^{\mathbf{w}_1 \dots \mathbf{w}_{i-1}} \subset \mathbf{S}$ . Now,  $\mathbf{w}_1 \dots \mathbf{w}_{i-1} \alpha_{\mathbf{I}_i}(\mathbf{w}_i) \in M_{\mathbf{I}}$  and

$$\mathbf{w}_1 \dots \mathbf{w}_{i-1} \alpha_{\mathbf{I}_i}(\mathbf{w}_i) \preccurlyeq \mathbf{w}_1 \dots \mathbf{w}_{i-1} \alpha_{\mathbf{I}_i}(\mathbf{w}_i) \preccurlyeq \mathbf{w}_1 \dots \mathbf{w}_{i-1} \mathbf{w}_i \preccurlyeq \mathbf{b}$$

so divides  $\alpha_{\mathbf{I}}(\mathbf{b})$ , thus this element has to be 1, whence the result.  $\square$

By Proposition 5.8 items 5.8(ii) and 5.8(iii) the subcategory  $M(\mathcal{I})$  of  $\text{Ad}(M, \mathcal{I})$  is closed under right-quotient and right-lcm. By Lemma 2.10 Proposition 5.8 together with 5.9 implies

**Corollary 5.10.** *The set  $\mathcal{S} \cap M(\mathcal{I}) = \{(\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J}) \in \text{Ad}(M, \mathcal{I}) \mid \mathbf{w} \in \mathcal{S} \text{ and } \alpha_{\mathbf{I}}(\mathbf{w}) = 1\}$  is a Garside family in  $M(\mathcal{I})$ .*

We can describe the atoms of  $M(\mathcal{I})$  when  $M$  is any Garside monoid which has a Garside element and satisfies some additional assumptions. In that case (which includes the particular case of spherical Artin groups) we will give also a convenient criterion to decide whether  $\mathbf{b} \in M$  is in  $M(\mathcal{I})$ . Unless stated otherwise, we assume until the end of this section that  $M$  has a Garside element  $\Delta$ .

**Lemma 5.11.** *Let  $M_{\mathbf{I}}$  be a parabolic submonoid of  $M$  generated by a subset  $\mathbf{I}$  of atoms of  $M$ . Then  $\Delta_{\mathbf{I}} = \alpha_{\mathbf{I}}(\Delta)$  is a Garside element in  $M_{\mathbf{I}}$ .*

*Proof.* Let  $\mathcal{S}$  be the set of divisors of  $\Delta$ ; then  $\mathcal{S} \cap M_{\mathbf{I}}$  generates  $M_{\mathbf{I}}$  so that we can apply Lemma 5.4 which gives that  $\mathcal{S} \cap M_{\mathbf{I}}$  is a Garside family in  $M_{\mathbf{I}}$ . Now the divisors of  $\Delta$  which are in  $M_{\mathbf{I}}$  are by definition of  $\alpha_{\mathbf{I}}$  the divisors of  $\Delta_{\mathbf{I}}$ , so that  $\Delta_{\mathbf{I}}$  is a Garside element in  $M_{\mathbf{I}}$ .  $\square$

We denote by  $\Phi_{\mathbf{I}}$  the associated Garside automorphism. Since  $M_{\mathbf{I}}$  is parabolic,  $\mathbf{I}$  is the whole set of atoms of  $M_{\mathbf{I}}$ , thus  $\Phi_{\mathbf{I}}(\mathbf{I}) = \mathbf{I}$ .

**Proposition 5.12.**  *$M(\mathcal{I})$  has a Garside map defined by the collection of morphisms  $\mathbf{I} \xrightarrow{\Delta_{\mathbf{I}}^{-1}\Delta} \Phi(\mathbf{I})$  for  $\mathbf{I} \in \mathcal{I}$ .*

*Proof.* By definition of  $\Delta_{\mathbf{I}}$ , we have  $\alpha_{\mathbf{I}}(\Delta_{\mathbf{I}}^{-1}\Delta) = 1$ , so that  $\Delta_{\mathbf{I}}^{-1}\Delta$  is an element of  $\mathcal{S} \cap M(\mathcal{I})$ . We have to show that any  $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$  in  $\mathcal{S} \cap M(\mathcal{I})$  divides  $\mathbf{I} \xrightarrow{\Delta_{\mathbf{I}}^{-1}\Delta} \Phi(\mathbf{I})$ , which is equivalent to  $\Delta_{\mathbf{I}}\mathbf{b}$  dividing  $\Delta$ . Since  $\Delta_{\mathbf{I}}$  and  $\mathbf{b}$  divide  $\Delta$ , their right lcm  $\delta$  divides  $\Delta$ . We claim that  $\delta = \Delta_{\mathbf{I}}\mathbf{b}$ . Let us write  $\delta = \mathbf{b}\mathbf{x}$ . We have  $\delta \preccurlyeq \Delta_{\mathbf{I}}\mathbf{b} = \mathbf{b}\Delta_{\mathbf{J}}$ , so that  $\mathbf{x} \preccurlyeq \Delta_{\mathbf{J}}$ . Thus  $\delta = \mathbf{b}\mathbf{x} = \mathbf{y}\mathbf{b}$  with  $\mathbf{y} \preccurlyeq \Delta_{\mathbf{I}}$ . By definition of  $\delta$  we have  $\Delta_{\mathbf{I}} \preccurlyeq \delta = \mathbf{y}\mathbf{b}$ , so that  $\mathbf{y}^{-1}\Delta_{\mathbf{I}} \preccurlyeq \mathbf{b}$  which implies  $\mathbf{y} = \Delta_{\mathbf{I}}$  since  $\alpha_{\mathbf{I}}(\mathbf{b}) = 1$ . Hence  $\delta = \Delta_{\mathbf{I}}\mathbf{b}$  and we are done.  $\square$

**Proposition 5.13.** *Let  $\mathbf{I} \in \mathcal{I}$  and let  $\mathbf{J} \supsetneq \mathbf{I}$  be such that  $M_{\mathbf{J}}$  is parabolic. Then  $\mathbf{I} \xrightarrow{v(\mathbf{J}, \mathbf{I})} \Phi_{\mathbf{J}}(\mathbf{I})$  defined by  $(\mathbf{I} \xrightarrow{\Delta_{\mathbf{J}}} \Phi_{\mathbf{J}}(\mathbf{I})) = (\mathbf{I} \xrightarrow{\Delta_{\mathbf{I}}} \mathbf{I} \xrightarrow{v(\mathbf{J}, \mathbf{I})} \Phi_{\mathbf{J}}(\mathbf{I}))$  is a morphism in  $M(\mathcal{I})$ .*

*Proof.* As noted after Proposition 5.8 we have to show that  $\alpha_{\mathbf{I}}(v(\mathbf{J}, \mathbf{I})) = 1$  and that any  $\mathbf{t} \in \mathbf{I}$  is conjugate by  $v(\mathbf{J}, \mathbf{I})$  to an element of  $M$ . Since  $\Delta_{\mathbf{I}}^{-1}\Delta_{\mathbf{J}}$  divides  $\Delta_{\mathbf{I}}^{-1}\Delta$ , and  $\alpha_{\mathbf{I}}(\Delta_{\mathbf{I}}^{-1}\Delta) = 1$ , by definition of  $\Delta_{\mathbf{I}}$ , we get the first property. The second is clear since by definition  $v(\mathbf{J}, \mathbf{I})$  conjugates  $\mathbf{t}$  to  $\Phi_{\mathbf{J}}(\Phi_{\mathbf{I}}^{-1}(\mathbf{t}))$ .  $\square$

To describe the atoms we now need the following assumption:

**Assumption 5.14.** *Let  $\mathbf{I} \in \mathcal{I}$  and let  $\mathbf{J}$  be the set of atoms of a parabolic submonoid  $M_{\mathbf{J}}$  of  $M$ , strictly containing  $M_{\mathbf{I}}$ , and minimal for this property. Then for any atom  $s \in \mathbf{J} - \mathbf{I}$ , the right-lcm of  $s$  and  $\Delta_{\mathbf{I}}$  is  $\Delta_{\mathbf{J}}$ .*

Note that this assumption holds for Artin monoids since for them a  $\mathbf{J}$  as above is of the form  $\mathbf{I} \cup \{\mathbf{s}\}$  for some atom  $\mathbf{s}$ . We have

**Proposition 5.15.** *Under the Assumptions 5.5, 5.6, and 5.14,*

- (i) Let  $\mathbf{I} \in \mathcal{I}$  and  $g \in M$  such that  $\alpha_{\mathbf{I}}(g) = 1$  and such that there exists  $p > 0$  such that  $(\Delta_{\mathbf{I}}^p)^g \in M$ . Then  $g \in M(\mathcal{I})$ .
- (ii) The atoms of  $M(\mathcal{I})$  are the  $v(\mathbf{J}, \mathbf{I})$  not strictly divisible by another  $v(\mathbf{J}', \mathbf{I})$  for  $\mathbf{I} \in \mathcal{I}$ .

*Proof.* (i) is a generalization of result of Luis Paris [Pa, 5.6]. Since  $M$  is Noetherian, for (i) it suffices to prove that under the assumption  $g$  is either invertible or left divisible by some non-invertible  $v \in M(\mathcal{I})$ ; indeed if  $g = vg'$  where  $\mathbf{I} \xrightarrow{v} \mathbf{I}' \in M(\mathcal{I})$  then by 5.8(ii) we have  $\alpha_{\mathbf{I}'}(g') = 1$  and since  $\mathbf{I}^v = \mathbf{I}'$  we have  $(\Delta_{\mathbf{I}'}^p)^{g'} \in M$ , so (i) is equivalent to the same property for  $g'$  and by Noetherianity the sequence  $g, g', g'', \dots$  thus constructed terminates with an invertible element. Let  $\mathbf{s}$  be an atom such that  $\mathbf{s} \preccurlyeq g$ ; by assumption  $\mathbf{s} \notin M_{\mathbf{I}}$  thus there exists a minimal parabolic submonoid  $M_{\mathbf{J}}$  containing  $\mathbf{s}$  and  $M_{\mathbf{I}}$  since the intersection of parabolic submonoids is parabolic. We will prove that  $v(\mathbf{J}, \mathbf{I}) \preccurlyeq g$  which will thus imply (i). We proceed by decreasing induction on  $p$ . We show that if for  $i > 0$  we have  $\mathbf{t} \preccurlyeq \Delta_{\mathbf{I}}^i g$  for some  $\mathbf{t} \in \mathbf{J} - \mathbf{I}$  (note this holds for  $i = p$  since  $\mathbf{s} \preccurlyeq g \preccurlyeq \Delta_{\mathbf{I}}^p g$ ) then  $v(\mathbf{J}, \mathbf{I}) \preccurlyeq \Delta_{\mathbf{I}}^{i-1} g$ . The right lcm of  $\mathbf{t}$  and  $\Delta_{\mathbf{I}}$  is  $\Delta_{\mathbf{J}}$  by Assumption 5.14 thus from  $\mathbf{t} \preccurlyeq \Delta_{\mathbf{I}}^i g$  and  $\Delta_{\mathbf{I}} \preccurlyeq \Delta_{\mathbf{I}}^i g$  we deduce  $\Delta_{\mathbf{J}} \preccurlyeq \Delta_{\mathbf{I}}^i g$ . Since  $\Delta_{\mathbf{J}} = \Delta_{\mathbf{I}} v(\mathbf{J}, \mathbf{I})$  we get as claimed  $v(\mathbf{J}, \mathbf{I}) \preccurlyeq \Delta_{\mathbf{I}}^{i-1} g$ . Since any atom  $\mathbf{t}'$  such that  $\mathbf{t}' \preccurlyeq v(\mathbf{J}, \mathbf{I})$  is in  $\mathbf{J} - \mathbf{I}$  the induction can go on while  $i - 1 > 0$ .

We get (ii) from the proof of (i): any element  $g \in M(\mathcal{I})$  satisfies the assumption of (i) for  $p = \lg_S(g)$  and  $\mathbf{I}$  equal to the source of  $g$ ; whence the result since in the proof of (i) we have seen that  $g$  is a product of some  $v(\mathbf{J}, \mathbf{K})$ .  $\square$

Though in the current paper we need only finite Coxeter groups, we note that the above description of the atoms also extends to the case of Artin monoids which are associated to infinite Coxeter groups (and thus do not have a Garside element). Proposition 5.16 below can be extracted from the proof of Theorem 0.5 in [G].

In the case of an Artin monoid  $(B^+, \mathbf{S})$  the Garside family of 5.10 in  $B^+(\mathcal{I})$  is  $\mathbf{W} \cap B^+(\mathcal{I}) = \{\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J} \in \text{Ad } B^+(\mathcal{I}) \mid \mathbf{w} \in \mathbf{W} \text{ and } \alpha_{\mathbf{I}}(\mathbf{w}) = 1\}$ . For  $\mathbf{I} \subset \mathbf{S}$  and  $\mathbf{s} \in \mathbf{S}$  we denote by  $\mathbf{I}(\mathbf{s})$  the connected component of  $\mathbf{s}$  in the Coxeter diagram of  $\mathbf{I} \cup \{\mathbf{s}\}$ , that is the vertices of the connected component of  $\mathbf{s}$  in the graph with vertices  $\mathbf{I} \cup \{\mathbf{s}\}$  and an edge between  $\mathbf{s}'$  and  $\mathbf{s}''$  whenever  $\mathbf{s}'$  and  $\mathbf{s}''$  do not commute.

It may be that the subgroup  $W_I$  generated by  $I$  is finite even though  $W$  is not (we say then that  $\mathbf{I}$  is *spherical*), in which case we denote by  $\mathbf{w}_{\mathbf{I}}$  the image in  $\mathbf{W}$  of the longest element of  $W_I$ . With these notations, we have

**Proposition 5.16.** *The atoms of  $B^+(\mathcal{I})$  are the morphisms  $\mathbf{I} \xrightarrow{v(\mathbf{s}, \mathbf{I})} v(\mathbf{s}, \mathbf{I}) \mathbf{I}$  where  $\mathbf{I}$  is in  $\mathcal{I}$  and  $\mathbf{s} \in \mathbf{S} - \mathbf{I}$  is such that  $\mathbf{I}(\mathbf{s})$  is spherical, and where  $v(\mathbf{s}, \mathbf{I}) = \mathbf{w}_{\mathbf{I}(\mathbf{s})} \mathbf{w}_{\mathbf{I}(\mathbf{s}) - \{\mathbf{s}\}}$ .*

## 6. PERIODIC ELEMENTS

**Definition 6.1.** *Let  $\mathcal{C}$  be a category with a Garside map  $\Delta$ ; then an endomorphism  $f$  of  $\mathcal{C}$  is said to be  $(d, p)$ -periodic if  $f^d \in \Delta^p \mathcal{C}^\times$  for some non-zero integers  $d, p$ .*

In the above, we have written  $\Delta^p$  for  $\Delta^p(\text{source}(f))$ .

Note that if  $f$  is  $(d, p)$ -periodic it is also  $(nd, np)$ -periodic for any non-zero integer  $n$ . We call  $d/p$  the *period* of  $f$ . If  $\Phi$  is of finite order, then a conjugate of a periodic element is periodic of the same period (though the minimal pair  $(d, p)$  may change). It can be shown that, up to cyclic conjugacy, the notion of being  $(d, p)$ -periodic depends only on the fraction  $d/p$ ; it results from Proposition 4.5 that two

periodic morphisms are conjugate if and only if they are cyclically conjugate; our interest in periodic elements comes mainly from the fact that one can describe their centralizers.

We deal in this paper with the case  $p = 2$ . We show by elementary computations that a  $(d, 2)$ -periodic element of  $\mathcal{C}$  is the same up to cyclic conjugacy as a  $(d/2, 1)$ -periodic element when  $d$  is even, and get a related characterization when  $d$  is odd.

We denote by  $\mathcal{S}$  a Garside family attached to  $\Delta$  (that is such that  $\mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$  is the set of divisors of  $\Delta$ ).

**Lemma 6.2.** *Let  $f$  be an endomorphism in  $\mathcal{C}$  such that  $f^d \in \Delta^2\mathcal{C}^\times$ , and let  $e = \lfloor \frac{d}{2} \rfloor$ . Then there exists  $g \in \text{Obj}(\text{cyc } \mathcal{C})$  such that  $\text{cyc } \mathcal{C}(f, g) \neq \emptyset$  and  $g^e \in \mathcal{S}\mathcal{C}^\times$  and  $g^d \in \Delta^2\mathcal{C}^\times$ .*

*Further, if  $g$  is as in the conclusion above, that is  $g^d \in \Delta^2\mathcal{C}^\times$  and  $g^e \in \mathcal{S}\mathcal{C}^\times$ , then if  $d$  is even we have  $g^e \in \Delta\mathcal{C}^\times$ , and if  $d$  is odd there exists  $h \in \mathcal{S}\mathcal{C}^\times$  such that  $g = h\Phi(h)\varepsilon$  and  $g^e h = \Delta$ , where  $\varepsilon \in \mathcal{C}^\times$  is defined by  $g^d = \Delta^2\varepsilon$ .*

*Proof.* We will prove by increasing induction on  $i$  that for  $i \leq d/2$  there exists  $v \in \text{cyc } \mathcal{C}$  such that  $(f^v)^i \in \mathcal{S}\mathcal{C}^\times$  and  $(f^v)^d \in \Delta^2\mathcal{C}^\times$ . We start the induction with  $i = 0$  where the result holds trivially with  $v = 1$ .

We consider now the general step: assuming the result for  $i$  such that  $i + 1 \leq d/2$ , we will prove it for  $i + 1$ . We thus have a  $v$  for step  $i$ , thus replacing if needed  $f$  by  $f^v$  we may assume that  $f^i \in \mathcal{S}\mathcal{C}^\times$  and  $f^d \in \Delta^2\mathcal{C}^\times$ ; we will conclude by finding  $v \in \mathcal{S}$  such that  $v \preccurlyeq f$  and  $(f^v)^{i+1} \in \mathcal{S}\mathcal{C}^\times$  and  $(f^v)^d \in \Delta^2\mathcal{C}^\times$ . If  $f^{i+1} \preccurlyeq \Delta$  we have the desired result with  $v = 1$ . We may thus assume that  $\text{lg}_{\mathcal{S}}(f^{i+1}) \geq 2$ . Since  $f^{i+1} \preccurlyeq \Delta^2$  we have actually  $\text{lg}_{\mathcal{S}}(f^{i+1}) = 2$  (see Proposition 2.12(i)); let  $(f^i v, w)$  be a normal decomposition of  $f^{i+1}$  where  $f^i v \in \mathcal{S}$  and  $w \in \mathcal{S}\mathcal{C}^\times$ . As  $f^i v w (f^i v) \preccurlyeq f^i v w (f^i v w) = f^{2(i+1)} \preccurlyeq f^d = \Delta^2$ , we still have  $2 = \text{lg}_{\mathcal{S}}((f^i v) w (f^i v)) = \text{lg}_{\mathcal{S}}((f^i v) w)$ . By Proposition 2.12(ii) we thus have  $w(f^i v) \in \mathcal{S}\mathcal{C}^\times$ . Then  $\mathcal{S}\mathcal{C}^\times \ni w(f^i v) = w((v w)^i) v = (f^v)^{i+1}$  and  $v \preccurlyeq f$ .

So  $v$  will do if we can show  $(f^v)^d \in \Delta^2\mathcal{C}^\times$ . Since  $f^d = \Delta^2\varepsilon$  with  $\varepsilon \in \mathcal{C}^\times$ , we have that  $f$  commutes with  $\Delta^2\varepsilon$ , thus  $f^{i+1}$  also, that is  $\Phi^2(f^{i+1})\varepsilon = \varepsilon f^{i+1}$  or equivalently  $\Phi^2(f^i v)\Phi^2(w)\varepsilon = \varepsilon f^i v w$ . Now  $(\Phi^2(f^i v), \Phi^2(w)\varepsilon)$  is an  $\mathcal{S}\mathcal{C}^\times$ -normal decomposition and since  $(f^i v, w)$  is a normal decomposition, by Lemma 2.2 there exists  $\varepsilon' \in \mathcal{C}^\times$  such that  $\Phi^2(f^i v)\varepsilon' = \varepsilon f^i v$ . We have  $f^i \Delta^2 \Phi^2(v)\varepsilon' = \Delta^2 \Phi^2(f^i v)\varepsilon' = \Delta^2 \varepsilon f^i v = f^i \Delta^2 \varepsilon v$ , the last equality since  $f^i$  commutes with  $\Delta^2\varepsilon$ . Canceling  $f^i \Delta^2$  we get  $\Phi^2(v)\varepsilon' = \varepsilon v$ . We have then  $v(f^v)^d = f^d v = \Delta^2 \varepsilon v = \Delta^2 \Phi^2(v)\varepsilon' = v \Delta^2 \varepsilon'$  whence the result by canceling  $v$  on the left.

We prove now the second part. From  $g^e \in \mathcal{S}\mathcal{C}^\times$  we get that there exists  $h \in \mathcal{S}\mathcal{C}^\times$  such that  $g^e h = \Delta$ . If  $g^d = \Delta^2\varepsilon$  with  $\varepsilon \in \mathcal{C}^\times$  we get  $g^e h \Delta \varepsilon = \Delta^2 \varepsilon = g^d$ , whence by cancellation  $h \Delta \varepsilon = g^e g^a$  with  $a = 1$  if  $d$  is odd and  $a = 0$  if  $d$  is even. We deduce  $g^e g^a = h \Delta \varepsilon = \Delta \Phi(h)\varepsilon = g^e h \Phi(h)\varepsilon$ , thus  $h \Phi(h)\varepsilon = g^a$ .

If  $d$  is odd we get the statement of the lemma, and if  $d$  is even we get  $h \Phi(h) \in \mathcal{C}^\times$ , so  $h \in \mathcal{C}^\times$ , so  $g^e \in \Delta\mathcal{C}^\times$ .  $\square$

**$F$ -periodic elements.** Let us apply Lemma 6.2 to the case of a semi-direct product category  $\mathcal{C} \rtimes \langle F \rangle$  with  $F$  a Garside automorphism of finite order, where  $\mathcal{C}$  has no non-trivial invertible element and the Garside family  $\mathcal{S}$  of  $\mathcal{C} \rtimes \langle F \rangle$  is in  $\mathcal{C}$ . Then a morphism  $yF \in \mathcal{C}F$  is  $(d, p)$  periodic if and only if  $\text{target}(y) = F(\text{source}(y))$  and  $(yF)^d = \Delta^p F^d$ .

From the lemma we can deduce the following.

**Corollary 6.3.** *Assume  $\Phi^2 = \text{Id}$  and that  $yF \in \mathcal{C}F$  satisfies  $(yF)^d = \Delta^2 F^d$ . Then*

- (i) *If  $d = 2e$  is even, there exists  $x$  such that  $\text{cyc } \mathcal{C}(yF, xF) \neq \emptyset$  and  $(xF)^e = \Delta F^e$ . The centralizer of  $xF$  in  $\mathcal{C}$  identifies to  $\text{cyc } \mathcal{C}(xF)$ . Further, we may compute these endomorphisms in the category of fixed points  $(\text{cyc } \mathcal{C})^{\Phi F^e}$  since the morphisms in  $\text{cyc } \mathcal{C}(xF)$  are  $\Phi F^e$ -stable.*
- (ii) *If  $d = 2e+1$  is odd, there exists  $x$  such that  $\text{cyc } \mathcal{C}(yF, xF) \neq \emptyset$  and  $(xF)^d = \Delta^2 F^d$  and  $(xF)^e F^{-e} \preceq \Delta$ . The element  $s$  defined by  $(xF)^e s F^{-e} = \Delta$  is such that, in the category  $\mathcal{C} \rtimes \langle \Lambda \rangle$  with  $\Lambda = \Phi^{-1} F^{-e}$ , we have  $x\Lambda^2 = (s\Lambda)^2$  and  $(s\Lambda)^d = \Delta\Lambda^d$ . The centralizer of  $xF$  in  $\mathcal{C}$  identifies to  $\text{cyc } \mathcal{C}(s\Lambda)$ . Further, we may compute these endomorphisms in the category of fixed points  $(\text{cyc } \mathcal{C})^{F^d}$  since  $\text{cyc } \mathcal{C}(s\Lambda)$  is stable by  $F^d$ .*

Note that 2.8 describes Garside families for the fixed point categories mentioned above.

*Proof.* Lemma 6.2 shows that  $y$  is cyclically  $F$ -conjugate to an  $x$  such that  $(xF)^e \in \mathcal{S}F^e$  and  $(xF)^d = \Delta^2 F^d$  and that if  $d$  is even then  $(xF)^e = \Delta F^e$ . If  $d$  is odd Lemma 6.2 gives the existence of  $h \in \mathcal{S}\mathcal{C}^\times$  such that  $xF = h\Phi(h)F^d$  and that  $(xF)^e h = \Delta$ . Hence we have  $h = sF^{-e}$  with  $s \in \mathcal{S}$ , and  $x = sF^{-e}\Phi(sF^{-e})F^{d-1} = s\Lambda(s)$ . This can be rewritten  $x\Lambda^2 = (s\Lambda)^2$ . Since the elements of  $\text{Ad } \mathcal{C}(xF)$  commute to  $F^d$  and  $xF = x\Lambda^2 F^d$ , we have  $\text{Ad } \mathcal{C}(xF) = \text{Ad } \mathcal{C}(x\Lambda^2)$ ; hence from  $(xF)^e s = \Delta F^e$  we get  $\text{Ad } \mathcal{C}(xF) \subset \text{Ad } \mathcal{C}(s\Lambda)$ . Using  $x\Lambda^2 = (s\Lambda)^2$  we get the reverse inclusion, whence  $\text{Ad } \mathcal{C}(xF) = \text{Ad } \mathcal{C}(s\Lambda)$ .

We get the corollary if we know that the centralizer of  $xF$ , for  $d$  even (resp.  $s\Lambda$ , for  $d$  odd) is the same as  $\text{cyc } \mathcal{C}(xF)$  (resp.  $\text{cyc } \mathcal{C}(s\Lambda)$ ). But this is an immediate consequence of Proposition 4.5.  $\square$

### Conjugacy of periodic elements.

**Theorem 6.4.** *Let  $B^+$  be the Artin monoid (see 2.13) attached to a finite Coxeter group  $(W, S)$ . Then two periodic elements of  $B^+$  of same period are cyclically conjugate.*

*Proof.* This results from the work of David Bessis on the dual braid monoid. Two periodic elements of same period in the classical Artin monoid are also periodic and have equal periods in the dual monoid. By [B1, 11.21], such elements are conjugate in the dual monoid, so are conjugate in the Artin group, hence are conjugate in the classical monoid. By Proposition 4.5 they are cyclically conjugate in the classical monoid.  $\square$

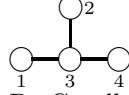
We conjecture that the same results extend to the case of  $F$ -conjugacy, where  $F$  is an automorphism of  $(W, S)$ , which thus induces a Garside automorphism of  $B^+$  via its action of  $\mathbf{W}$ .

We conjecture further that for any conjugacy class  $\mathcal{I}$  of subsets of  $\mathbf{S}$ , all periodic elements in  $\mathcal{C}(\mathcal{I})$  of a given period are conjugate (thus cyclically conjugate); and that this extends also to the case of  $F$ -conjugacy.

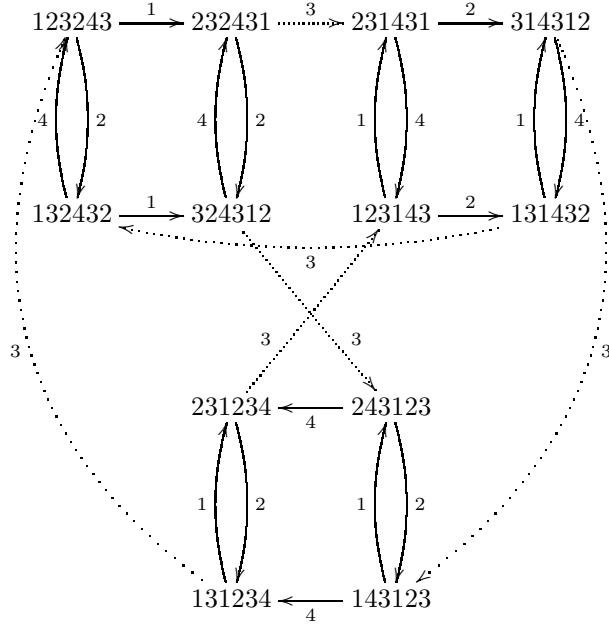
**Two examples.** In two cases we show a picture of the category associated to the centralizer of a periodic element.

We first look at  $\mathcal{C} = B^+(W(D_4))$  and  $\mathbf{w} \in \mathcal{C}$  such that  $\mathbf{w}^2 = \Delta$ ; following Corollary 6.3(i) we describe the component of  $\mathbf{w}$  in the category  $\text{cyc } \mathcal{C}^\Phi$ . As in

Theorem 11.12, we choose  $\mathbf{w}$  given by the word in the generators 123423 where the labeling of the Coxeter diagram is

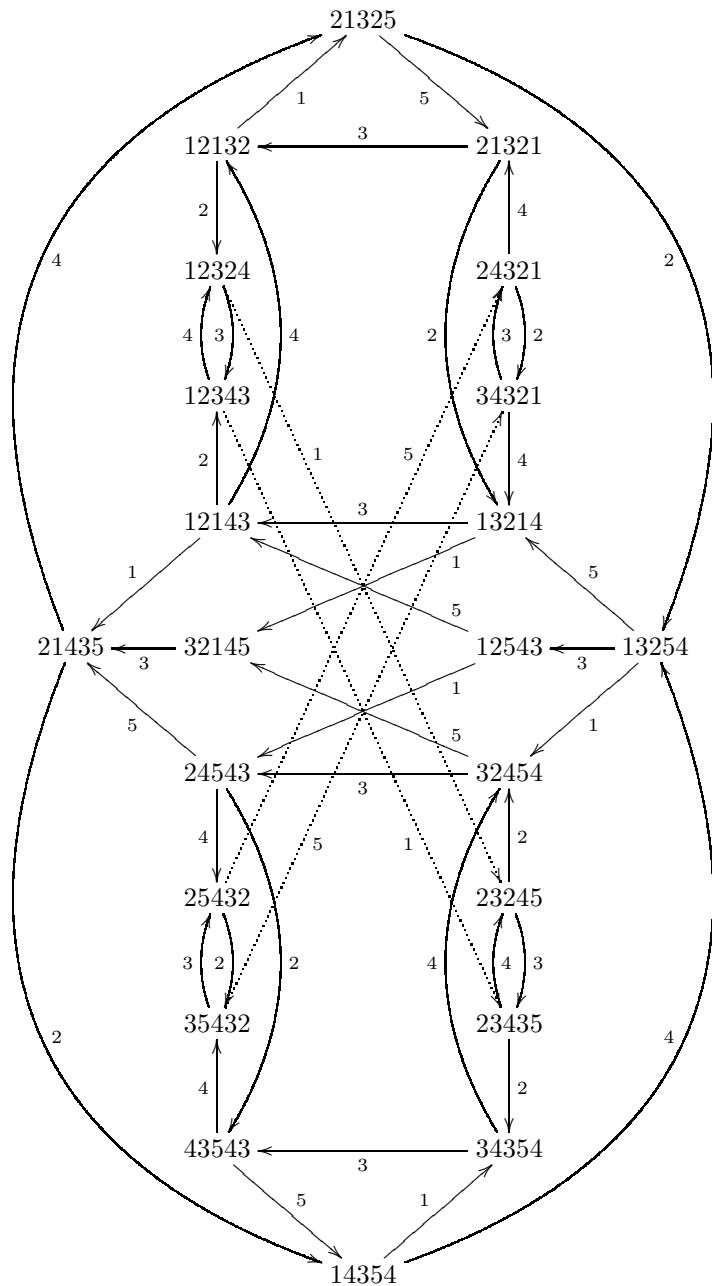


By Corollary 6.3(i) the monoid of endomorphisms  $\text{cyc } \mathcal{C}(\mathbf{w})$  generates  $C_B(\mathbf{w})$ ; by [B1, 12.5(ii)],  $C_B(\mathbf{w})$  is the braid group of  $C_W(w) \simeq G(4, 2, 2)$ . This braid group has presentation  $\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \mid \mathbf{xyz} = \mathbf{yzx} = \mathbf{zxy} \rangle$ . The automorphism  $\mathbf{x} \mapsto \mathbf{y} \mapsto \mathbf{z}$  corresponds to the triality in  $D_4$ . One of the generators  $\mathbf{x}$  corresponds to the morphism 24 in the diagram below. The other generators are the conjugates of the similar morphisms 41 and 21 in the other squares.



We now look at the case of a  $\mathbf{w}$  in the braid monoid  $\mathcal{C} = B^+(W(A_5))$  such that  $\mathbf{w}^3 = \Delta^2$ , and following Corollary 6.3(ii) we describe the component of  $\mathbf{s}\Phi^{-1}$  in the category  $\text{cyc } \mathcal{C} \rtimes \langle \Phi^{-1} \rangle$  where  $\mathbf{s}$  is such that  $\mathbf{w} = \mathbf{s}\Phi(\mathbf{s})$ . By Corollary 6.3(ii) the monoid of endomorphisms  $\text{cyc } \mathcal{C}(\mathbf{s}\Phi^{-1})$  generates  $C_B(\mathbf{w})$  and again by the results of Bessis  $C_B(\mathbf{w})$  is the braid group of  $C_W(w) \simeq G(3, 1, 2)$  (see Theorem 11.5). We choose  $\mathbf{w}$  such that  $\mathbf{s}$  is given by the word 21325 in the generators. The generator of  $C_B(\mathbf{w})$  lifting the generator of order 3 of  $G(3, 1, 2)$  is given by the word 531. The

other one is the conjugate of any of the length 2 cycles 23 in the diagram.



## 7. REPRESENTATIONS INTO BICATEGORIES

We give here a theorem on categories with Garside families which generalizes a result of Deligne [D, 1.11] about representations of spherical braid monoids into a category; just as this theorem of Deligne was used to attach a Deligne-Lusztig variety to an element of the braid group, our theorem will be used to attach a Deligne-Lusztig variety to a morphism of a ribbon category. Note that our theorem covers in particular the case of non-spherical Artin monoids.

We follow the terminology of [McL, XII.6] for bicategories. By “representation of category  $\mathcal{C}$  into bicategory  $X$ ” we mean a morphism of bicategories between  $\mathcal{C}$  viewed as a trivial bicategory into the given bicategory  $X$ . This amounts to give a map  $T$  from  $\text{Obj}(\mathcal{C})$  to the 0-cells of  $X$ , and for  $f \in \mathcal{C}$  of source  $x$  and target  $y$ , an element  $T(f) \in V(T(x), T(y))$  where  $V(T(x), T(y))$  is the category whose objects (resp. morphisms) are the 1-cells of  $X$  with domain  $T(x)$  and codomain  $T(y)$  (resp. the 2-cells between them), together with for each composable pair  $(f, g)$  an isomorphism  $T(f)T(g) \xrightarrow{\sim} T(fg)$  such that the resulting square

$$(7.1) \quad \begin{array}{ccc} T(f)T(f')T(f'') & \xrightarrow{\sim} & T(f f')T(f'') \\ \downarrow \sim & & \downarrow \sim \\ T(f)T(f' f'') & \xrightarrow{\sim} & T(f f' f'') \end{array}$$

commutes.

We define a representation of the Garside family  $\mathcal{S}$  as the same, except that the above square is restricted to the case where  $f, ff'$  and  $ff'f''$  are in  $\mathcal{S}$ , (which implies  $f', f'', f'f'' \in \mathcal{S}$  since  $\mathcal{S}$  is closed under right divisors). We then have

**Theorem 7.2.** *Let  $\mathcal{C}$  be a right Noetherian category which admits local right lcms and has a Garside family  $\mathcal{S}$ . Then any representation of  $\mathcal{S}$  into a bicategory extends uniquely to a representation of  $\mathcal{C}$  into the same bicategory.*

*Proof.* The proof goes exactly as in [D], in that what must be proven is a simple connectedness property for the set of decompositions as a product of elements of  $\mathcal{S}$  of an arbitrary morphism in  $\mathcal{C}$ —this generalizes [D, 1.7] and is used in the same way. In his context, Deligne shows more, the contractibility of the set of decompositions; on the other hand our proof, which follows a suggestion by Serge Bouc to use a version of [Bouc, lemma 6], is simpler and holds in our more general context.

Fix  $g \in \mathcal{C}$  with  $g \notin \mathcal{C}^\times$ . We denote by  $E(g)$  the set of decompositions of  $g$  into a product of elements of  $\mathcal{S} - \mathcal{C}^\times$ .

Then  $E(g)$  is a poset, the order being defined by

$$(g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_n) > (g_1, \dots, g_{i-1}, a, b, g_{i+1}, \dots, g_n)$$

if  $ab = g_i \in \mathcal{S}$ .

We recall the definition of homotopy in a poset  $E$  (a translation of the corresponding notion in a simplicial complex isomorphic as a poset to  $E$ ). A path from  $x_1$  to  $x_k$  in  $E$  is a sequence  $x_1 \dots x_k$  where each  $x_i$  is comparable to  $x_{i+1}$ . The composition of paths is defined by concatenation. Homotopy, denoted by  $\sim$ , is the finest equivalence relation on paths compatible with concatenation and generated by the two following elementary relations:  $xyz \sim xz$  if  $x \leq y \leq z$  and both  $xyx \sim x$  and  $yxy \sim y$  when  $x \leq y$ . Homotopy classes form a groupoid, as the composition of a path with source  $x$  and of the inverse path is homotopic to the constant path at

$x$ . For  $x \in E$  we denote by  $\Pi_1(E, x)$  the fundamental group of  $E$  with base point  $x$ , which is the group of homotopy classes of loops starting from  $x$ .

A poset  $E$  is said to be *simply connected* if it is connected (there is a path linking any two elements of  $E$ ) and if the fundamental group with some (or any) base point is trivial.

Note that a poset with a smallest or largest element  $x$  is simply connected since any path  $(x, y, z, t, \dots, x)$  is homotopic to  $(x, y, x, z, x, t, x, \dots, x)$  which is homotopic to the trivial loop.

**Proposition 7.3.** *The set  $E(g)$  is simply connected.*

*Proof.* First we prove a version of a lemma from [Bouc] on order preserving maps between posets. For a poset  $E$  we put  $E_{\geq x} = \{x' \in E \mid x' \geq x\}$ , which is a simply connected subposet of  $E$  since it has a smallest element. If  $f : X \rightarrow Y$  is an order preserving map it is compatible with homotopy (it corresponds to a continuous map between simplicial complexes), so it induces a homomorphism  $f^* : \Pi_1(X, x) \rightarrow \Pi_1(Y, f(x))$ .

**Lemma 7.4** (Bouc). *Let  $f : X \rightarrow Y$  an order preserving map between two posets. We assume that  $Y$  is connected and that for any  $y \in Y$  the poset  $f^{-1}(Y_{\geq y})$  is connected and non empty. Then  $f^*$  is surjective. If moreover  $f^{-1}(Y_{\geq y})$  is simply connected for all  $y$  then  $f^*$  is an isomorphism.*

*Proof.* Let us first show that  $X$  is connected. Let  $x, x' \in X$ ; we choose a path  $y_0 \dots y_n$  in  $Y$  from  $y_0 = f(x)$  to  $y_n = f(x')$ . For  $i = 0, \dots, n$ , we choose  $x_i \in f^{-1}(Y_{\geq y_i})$  with  $x_0 = x$  and  $x_n = x'$ . Then if  $y_i \geq y_{i+1}$  we have  $f^{-1}(Y_{\geq y_i}) \subset f^{-1}(Y_{\geq y_{i+1}})$  so that there exists a path in  $f^{-1}(Y_{\geq y_{i+1}})$  from  $x_i$  to  $x_{i+1}$ ; otherwise  $y_i < y_{i+1}$ , which implies  $f^{-1}(Y_{\geq y_i}) \supset f^{-1}(Y_{\geq y_{i+1}})$  and there exists a path in  $f^{-1}(Y_{\geq y_i})$  from  $x_i$  to  $x_{i+1}$ . Concatenating these paths gives a path connecting  $x$  and  $x'$ .

We fix now  $x_0 \in X$ . Let  $y_0 = f(x_0)$ . We prove that  $f^* : \Pi_1(X, x_0) \rightarrow \Pi_1(Y, y_0)$  is surjective. Let  $y_0 y_1 \dots y_n$  with  $y_n = y_0$  be a loop in  $Y$ . We lift arbitrarily this loop into a loop  $x_0 \dots x_n$  in  $X$  as above, (where  $x_i - x_{i+1}$  stands for a path from  $x_i$  to  $x_{i+1}$  which is either in  $f^{-1}(Y_{\geq y_i})$  or in  $f^{-1}(Y_{\geq y_{i+1}})$ ). Then the path  $f(x_0 - x_1 - \dots - x_n)$  is homotopic to  $y_0 \dots y_n$ ; this can be seen by induction: let us assume that  $f(x_0 - x_1 - \dots - x_i)$  is homotopic to  $y_0 \dots y_i f(x_i)$ ; then the same property holds for  $i+1$ : indeed  $y_i y_{i+1} \sim y_i f(x_i) y_{i+1}$  as they are two paths in a simply connected set which is either  $Y_{\geq y_i}$  or  $Y_{\geq y_{i+1}}$ ; similarly we have  $f(x_i) y_{i+1} f(x_{i+1}) \sim f(x_i - x_{i+1})$ . Putting things together gives

$$\begin{aligned} y_0 \dots y_i y_{i+1} f(x_{i+1}) &\sim y_0 y_1 \dots y_i f(x_i) y_{i+1} f(x_{i+1}) \\ &\sim f(x_0 - \dots - x_i) y_{i+1} f(x_{i+1}) \\ &\sim f(x_0 - \dots - x_i - x_{i+1}). \end{aligned}$$

We now prove injectivity of  $f^*$  when all  $f^{-1}(Y_{\geq y})$  are simply connected.

We first prove that if  $x_0 \dots x_n$  and  $x'_0 \dots x'_n$  are two loops lifting the same loop  $y_0 \dots y_n$ , then they are homotopic. Indeed, we get by induction on  $i$  that  $x_0 \dots x_i - x'_i$  and  $x'_0 \dots x'_i$  are homotopic paths, using the fact that  $x_{i-1}, x_i, x'_{i-1}$  and  $x'_i$  are all in the same simply connected sub-poset, namely either  $f^{-1}(Y_{\geq y_{i-1}})$  or  $f^{-1}(Y_{\geq y_i})$ .

It remains to prove that we can lift homotopies, which amounts to show that if we lift as above two loops which differ by an elementary homotopy, the liftings

are homotopic. If  $yy'y \sim y$  is an elementary homotopy with  $y < y'$  (resp.  $y > y'$ ), then  $f^{-1}(Y_{\geq y'}) \subset f^{-1}(Y_{\geq y})$  (resp.  $f^{-1}(Y_{\geq y}) \subset f^{-1}(Y_{\geq y'})$ ) and the lifting of  $yy'y$  constructed as above is in  $f^{-1}(Y_{\geq y})$  (resp.  $f^{-1}(Y_{\geq y'})$ ) so is homotopic to the trivial path. If  $y < y' < y''$ , a lifting of  $yy'y''$  constructed as above is in  $f^{-1}(Y_{\geq y})$  so is homotopic to any path in  $f^{-1}(Y_{\geq y})$  with the same endpoints.  $\square$

We now prove Proposition 7.3 by contradiction. If it fails we choose  $g \in \mathcal{C}$  minimal for proper right divisibility such that  $E(g)$  is not simply connected.

Let  $L$  be the set of elements of  $\mathcal{S} - \mathcal{C}^\times$  which are left divisors of  $g$ . For any  $I \subset L$ , since the category admits local right lcms and is right Noetherian, the elements of  $I$  have an lcm. We fix such an lcm  $\Delta_I$ . Let  $E_I(g) = \{(g_1, \dots, g_n) \in E(g) \mid \Delta_I \preccurlyeq g_1\}$ . We claim that  $E_I(g)$  is simply connected for  $I \neq \emptyset$ . This is clear if  $g \in \Delta_I \mathcal{C}^\times$ , in which case  $E_I(g) = \{(g)\}$ . Let us assume this is not the case. In the following, if  $\Delta_I \preccurlyeq a$ , we denote by  $a^I$  the element such that  $a = \Delta_I a^I$ . The set  $E(g^I)$  is defined since  $g \notin \Delta_I \mathcal{C}^\times$ . We apply Lemma 7.4 to the map  $f : E_I(g) \rightarrow E(g^I)$  defined by

$$(g_1, \dots, g_n) \mapsto \begin{cases} (g_2, \dots, g_n) & \text{if } g_1 = \Delta_I \\ (g_1^I, g_2, \dots, g_n) & \text{otherwise} \end{cases}.$$

This map preserves the order and any set  $f^{-1}(Y_{\geq (g_1, \dots, g_n)})$  has a least element, namely  $(\Delta_I, g_1, \dots, g_n)$ , so is simply connected. As by minimality of  $g$  the set  $E(g^I)$  is simply connected Lemma 7.4 implies that  $E_I(g)$  is simply connected.

Let  $Y$  be the set of non-empty subsets of  $L$ . We now apply Lemma 7.4 to the map  $f : E(g) \rightarrow Y$  defined by  $(g_1, \dots, g_n) \mapsto \{s \in L \mid s \preccurlyeq g_1\}$ , where  $Y$  is ordered by inclusion. This map is order preserving since  $(g_1, \dots, g_n) < (g'_1, \dots, g'_n)$  implies  $g_1 \preccurlyeq g'_1$ . We have  $f^{-1}(Y_{\geq I}) = E_I(g)$ , so this set is simply connected. Since  $Y$ , having a greatest element, is simply connected, 7.4 gives that  $E(g)$  is simply connected, whence the proposition.  $\square$

$\square$

## II. Deligne-Lusztig varieties and eigenspaces

In this part, we study the Deligne-Lusztig varieties which give rise to a Lusztig induction functor  $R_L^G(\text{Id})$ ; in Section 8 we generalize these varieties to varieties attached to elements of a ribbon category.

In Section 9 we consider the particular ribbons associated to varieties which play a role in the Broué conjectures, because they are associated to maximal eigenspaces of elements of the Weyl group.

Finally in Section 10 we spell out the geometric form of the Broué conjectures, involving the factorization of the endomorphisms of our varieties in the conjugacy category of the ribbon category through the action of a cyclotomic Hecke algebra on their cohomology.

### 8. PARABOLIC DELIGNE-LUSZTIG VARIETIES

Let  $\mathbf{G}$  be a connected reductive algebraic group over  $\overline{\mathbb{F}}_p$ , and let  $F$  be an isogeny on  $\mathbf{G}$  such that some power  $F^\delta$  is a Frobenius for a split  $\mathbb{F}_{q^\delta}$ -structure (this defines a positive real number  $q$  such that  $q^\delta$  is an integral power of  $p$ ).

Let  $\mathbf{L}$  be an  $F$ -stable Levi subgroup of a (non-necessarily  $F$ -stable) parabolic subgroup  $\mathbf{P}$  of  $\mathbf{G}$  and let  $\mathbf{P} = \mathbf{L}\mathbf{V}$  be the corresponding Levi decomposition of  $\mathbf{P}$ . Let

$$\begin{aligned}\mathbf{X}_{\mathbf{V}} &= \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g\mathbf{V} \cap F(g\mathbf{V}) \neq \emptyset\} = \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g^{-1}Fg \in \mathbf{V}^F\mathbf{V}\} \\ &\simeq \{g \in \mathbf{G} \mid g^{-1}Fg \in \mathbf{V}^F\}/(\mathbf{V} \cap \mathbf{V}^F).\end{aligned}$$

On this variety  $\mathbf{G}^F$  acts by left multiplication and  $\mathbf{L}^F$  acts by right multiplication.

We choose a prime number  $\ell \neq p$ . Then the virtual  $\mathbf{G}^F$ -module  $\mathbf{L}^F$  given by  $M = \sum_i (-1)^i H_c^i(\mathbf{X}_{\mathbf{V}}, \overline{\mathbb{Q}}_{\ell})$  defines the Lusztig induction  $R_{\mathbf{L}}^{\mathbf{G}}$  which by definition maps an  $\mathbf{L}^F$ -module  $\lambda$  to  $M \otimes_{\overline{\mathbb{Q}}_{\ell}\mathbf{L}^F} \lambda$ .

The map  $g\mathbf{V} \mapsto g\mathbf{P}$  makes  $\mathbf{X}_{\mathbf{V}}$  an  $\mathbf{L}^F$ -torsor over

$$\begin{aligned}\mathbf{X}_{\mathbf{P}} &= \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g\mathbf{P} \cap F(g\mathbf{P}) \neq \emptyset\} = \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}Fg \in \mathbf{P}^F\mathbf{P}\} \\ &\simeq \{g \in \mathbf{G} \mid g^{-1}Fg \in \mathbf{P}^F\}/(\mathbf{P} \cap \mathbf{P}^F),\end{aligned}$$

a  $\mathbf{G}^F$ -variety such that  $R_{\mathbf{L}}^{\mathbf{G}}(\text{Id}) = \sum_i (-1)^i H_c^i(\mathbf{X}_{\mathbf{P}}, \overline{\mathbb{Q}}_{\ell})$ . The variety  $\mathbf{X}_{\mathbf{P}}$  is the prototype of the varieties we want to study.

Let  $\mathbf{T} \subset \mathbf{B}$  be a pair of an  $F$ -stable maximal torus and an  $F$ -stable Borel subgroup of  $\mathbf{G}$ . To this choice is associated a basis  $\Pi$  of the root system  $\Phi$  of  $\mathbf{G}$  with respect to  $\mathbf{T}$ , and a Coxeter system  $(W, S)$  for the Weyl group  $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ . Let  $X_{\mathbb{R}} = X(\mathbf{T}) \otimes \mathbb{R}$ ; on the vector space  $X_{\mathbb{R}}$ , the isogeny  $F$  acts as  $q\phi$  where  $\phi$  is of order  $\delta$  and stabilizes the positive cone  $\mathbb{R}^+\Pi$ ; we will still denote by  $\phi$  the induced automorphism of  $(W, S)$ .

To a subset  $I \subset \Pi$  corresponds a subgroup  $W_I \subset W$ , a parabolic subgroup  $\mathbf{P}_I = \coprod_{w \in W_I} \mathbf{B}w\mathbf{B}$ , and the Levi subgroup  $\mathbf{L}_I$  of  $\mathbf{P}_I$  which contains  $\mathbf{T}$ .

Given any  $\mathbf{P} = \mathbf{L}\mathbf{V}$  as above where  $\mathbf{L}$  is  $F$ -stable, there exists  $I \subset \Pi$  such that  $(\mathbf{L}, \mathbf{P})$  is  $\mathbf{G}$ -conjugate to  $(\mathbf{L}_I, \mathbf{P}_I)$ ; if we choose the conjugating element such that it conjugates a maximally split torus of  $\mathbf{L}$  to  $\mathbf{T}$  and a rational Borel subgroup of  $\mathbf{L}$  containing this torus to  $\mathbf{B} \cap \mathbf{L}_I$ , then this element conjugates  $(\mathbf{L}, \mathbf{P}, F)$  to  $(\mathbf{L}_I, \mathbf{P}_I, {}^wF)$  where  ${}^w \in N_{\mathbf{G}}(\mathbf{T})$  is such that  ${}^w\phi I = I$ , where  $w$  is the image of  $w$  in  $W$ .

It will be convenient to consider  $I$  as a subset of  $S$  instead of a subset of  $\Pi$ ; the condition on  $w$  must then be stated as “ $I^w = {}^{\phi}I$  and  $w$  is  $I$ -reduced”. Via the above conjugation, the variety  $\mathbf{X}_{\mathbf{P}}$  is isomorphic to the variety

$$\mathbf{X}(I, w\phi) = \{g\mathbf{P}_I \in \mathbf{G}/\mathbf{P}_I \mid g^{-1}Fg \in \mathbf{P}_I w^F \mathbf{P}_I\}.$$

We will denote by  $\mathbf{X}_{\mathbf{G}}(I, w\phi)$  this variety when there is a possible ambiguity on the group. If we denote by  $\mathbf{U}_I$  the unipotent radical of  $\mathbf{P}_I$ , we have  $\dim \mathbf{X}(I, w\phi) = \dim \mathbf{U}_I - \dim(\mathbf{U}_I \cap {}^wF\mathbf{U}_I) = l(w)$ . The  $\ell$ -adic cohomology of the variety  $\mathbf{X}(I, w\phi)$  gives rise to the Lusztig induction from  $\mathbf{L}_I^{{}^wF}$  to  $\mathbf{G}^F$  of the trivial representation; to avoid ambiguity on the isogenies involved, we will sometimes denote this Lusztig induction by  $R_{\mathbf{L}_I, {}^wF}^{\mathbf{G}, F}(\text{Id})$ .

**Definition 8.1.** *We say that a pair  $(\mathbf{P}, \mathbf{Q})$  of parabolic subgroups is in relative position  $(I, w, J)$ , where  $I, J \subset S$  and  $w \in W$ , if  $(\mathbf{P}, \mathbf{Q})$  is  $\mathbf{G}$ -conjugate to  $(\mathbf{P}_I, {}^w\mathbf{P}_J)$ . We denote this as  $\mathbf{P} \xrightarrow{I, w, J} \mathbf{Q}$ .*

Since any pair  $(\mathbf{P}, \mathbf{Q})$  of parabolic subgroups share a common maximal torus, it has a relative position  $(I, w, J)$  where  $I, J$  is uniquely determined as well as the double coset  $W_I w W_J$ .

Let  $\mathcal{P}_I$  be the variety of parabolic subgroups conjugate to  $\mathbf{P}_I$ ; this variety is isomorphic to  $\mathbf{G}/\mathbf{P}_I$ . Via the map  $g\mathbf{P}_I \mapsto {}^g\mathbf{P}_I$  we have an isomorphism

$$\mathbf{X}(I, w\phi) \simeq \{\mathbf{P} \in \mathcal{P}_I \mid \mathbf{P} \xrightarrow{I, w, \phi_I} {}^F\mathbf{P}\};$$

it is a variety over  $\mathcal{P}_I \times \mathcal{P}_{\phi I}$  by the first and second projection.

**The parabolic braid category  $B^+(\mathcal{I})$ .** In order to have a rich enough monoid of endomorphisms (see Definition 8.23), we need to generalize the pairs  $(I, w\phi)$  which label our varieties to the larger set of morphisms of a “ribbon category” that we proceed to define.

Let  $B^+$  (resp.  $B$ ) denote the Artin-Tits monoid (resp. Artin-Tits group) of  $W$ , and let  $\mathbf{S}$  be its generating set, which is in canonical bijection with  $S$ . To  $I \subset S$  corresponds  $\mathbf{I} \subset \mathbf{S}$  and the submonoid  $B_{\mathbf{I}}^+$  generated by  $\mathbf{I}$ . By Lemma 5.3 every element of  $\mathbf{b} \in B^+$  has a unique longest divisor  $\alpha_{\mathbf{I}}(\mathbf{b})$  in  $B_{\mathbf{I}}^+$ . As in Definition 5.7 we define:

**Definition 8.2.** Let  $\mathcal{I}$  be the set of conjugates of some subset of  $\mathbf{S}$ . Then  $B^+(\mathcal{I})$  is the category whose objects are the elements of  $\mathcal{I}$  and the morphisms from  $\mathbf{I}$  to  $\mathbf{J}$  are the  $\mathbf{b} \in B^+$  such that  $\mathbf{I}^{\mathbf{b}} = \mathbf{J}$  and  $\alpha_{\mathbf{I}}(\mathbf{b}) = 1$ .

If  $\mathbf{b} \in B^+$  determines an element of  $B^+(\mathcal{I})(\mathbf{I}, \mathbf{J})$  for some objects  $\mathbf{I}, \mathbf{J}$  of  $\mathcal{I}$ , we will denote by  $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$  this morphism to lift ambiguity on its source and target.

We have shown in Proposition 5.8 that the above definition makes sense, that is if we have a composition  $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \xrightarrow{\mathbf{c}} \mathbf{K}$  in  $B^+(\mathcal{I})$ , then  $\alpha_{\mathbf{I}}(\mathbf{bc}) = 1$ . When  $\mathcal{I} = \{\emptyset\}$ ,  $B^+(\mathcal{I})$  reduces to the Artin-Tits monoid  $B^+$ .

The canonical lift  $W \xrightarrow{\sim} \mathbf{W}$  of  $W$  in  $B^+$  is denoted by  $w \mapsto \mathbf{w}$ ; it is a Garside family in  $B^+$ . For  $\mathbf{w} \in \mathbf{W}$  we denote by  $w$  its image in  $W$ . By Corollary 5.10 and Proposition 5.12  $B^+(\mathcal{I})$  has a Garside family consisting of the morphisms  $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J}$  where  $\mathbf{w} \in \mathbf{W}$  and a Garside map  $\Delta_{\mathcal{I}}$  given on the object  $\mathbf{I}$  by the morphism  $\mathbf{I} \xrightarrow{\mathbf{w}_I^{-1}\mathbf{w}_0} \mathbf{I}^{\mathbf{w}_0}$  where we denote by  $\mathbf{w}_I$  the lift to  $\mathbf{W}$  of the longest element of  $W_I$ , and write  $\mathbf{w}_0$  for  $\mathbf{w}_S$ . This includes the following:

**Lemma 8.3.** (i)  $\mathcal{S} = \{\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J} \mid \mathbf{w} \in \mathbf{W}\}$  generates  $B^+(\mathcal{I})$ ; specifically, if  $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \in B^+(\mathcal{I})$  and  $(\mathbf{w}_1, \dots, \mathbf{w}_k)$  is the  $\mathbf{W}$ -normal decomposition of  $\mathbf{b}$ , there exist subsets  $\mathbf{I}_i$  with  $\mathbf{I}_1 = \mathbf{I}$ ,  $\mathbf{I}_{k+1} = \mathbf{J}$  such that for all  $i$  we have  $\mathbf{I}_{i+1} = \mathbf{I}_i^{\mathbf{w}_i}$ ; thus  $\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \rightarrow \dots \rightarrow \mathbf{I}_k \xrightarrow{\mathbf{w}_k} \mathbf{J}$  is a decomposition of  $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$  in  $B^+(\mathcal{I})$  as a product of elements of  $\mathcal{S}$ .  
(ii) The relations  $(\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{J} \xrightarrow{\mathbf{w}_2} \mathbf{K}) = (\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{K})$  when  $\mathbf{w} = \mathbf{w}_1\mathbf{w}_2 \in \mathbf{W}$  form a presentation of  $B^+(\mathcal{I})$ .

We set  $\alpha(\mathbf{b})$  to be the left gcd of  $\mathbf{b}$  and  $\mathbf{w}_0$ ; its restriction to  $B^+ - \{1\}$  is an  $\mathcal{S}$ -head function. Lemma 8.3 implies:

**Lemma 8.4.** For  $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{I}' \in B^+(\mathcal{I})$  and  $\mathbf{v} \in B_{\mathbf{I}}^+$  we have  $\alpha(\mathbf{v}\mathbf{w}) = \alpha(\mathbf{v})\alpha(\mathbf{w})$ .

*Proof.* We have  $\alpha(\mathbf{v}\mathbf{w}) = \alpha(\mathbf{v}\alpha(\mathbf{w})) = \alpha(\alpha(\mathbf{w})\mathbf{v}^{\alpha(\mathbf{w})}) = \alpha(\alpha(\mathbf{w})\alpha(\mathbf{v}^{\alpha(\mathbf{w})}))$ , the first and last equalities from Proposition 2.3 (iii). Since by Lemma 8.3(i)  $\mathbf{I}^{\alpha(\mathbf{w})} \subset \mathbf{S}$ , by Lemma 5.11 we have  $\alpha(\mathbf{v}^{\alpha(\mathbf{w})}) = \alpha(\mathbf{v})^{\alpha(\mathbf{w})}$ , so that  $\alpha(\mathbf{v}\mathbf{w}) = \alpha(\alpha(\mathbf{w})\alpha(\mathbf{v})^{\alpha(\mathbf{w})}) = \alpha(\alpha(\mathbf{v})\alpha(\mathbf{w}))$ . Since  $\alpha(\mathbf{w})$  is  $\mathbf{I}$ -reduced we have  $\alpha(\mathbf{v})\alpha(\mathbf{w}) \in \mathbf{W}$ , hence  $\alpha(\alpha(\mathbf{v})\alpha(\mathbf{w})) = \alpha(\mathbf{v})\alpha(\mathbf{w})$ .  $\square$

We now look at the compatibility of morphisms in  $B^+(\mathcal{I})$  with a “parabolic” situation. In our case, the only invertible in  $B^+$  is 1 and we extend the normal decomposition to all of  $B^+$  by deciding that the normal decomposition of 1 is the empty sequence.

**Proposition 8.5.** *Fix  $\mathbf{I} \in \mathcal{I}$ , and for  $\mathbf{J} \subset \mathbf{I}$ , let  $\mathcal{J}$  be the set of  $B_{\mathbf{I}}^+$ -conjugates of  $\mathbf{J}$ . Let  $(\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{I}') \in B^+(\mathcal{I})$  and let  $(\mathbf{J} \xrightarrow{\mathbf{v}} \mathbf{J}') \in B_{\mathbf{I}}^+(\mathcal{J})$ . Let  $(\mathbf{u}_1, \dots, \mathbf{u}_k)$  be the normal decomposition of  $\mathbf{v}\mathbf{w}$  and let  $(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k)$  be the normal decomposition of  $\mathbf{w}$ , with perhaps some 1's added at the end so they have same length; if for each  $i$  we define  $\mathbf{v}_i$  by  $\mathbf{u}_i = \mathbf{v}_i\mathbf{w}_i$  then  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots)$  is the normal decomposition of  $\mathbf{v}$  with perhaps some added 1's at the end.*

*Proof.* We proceed by induction on  $k$ . By Lemma 8.4, we have  $\mathbf{u}_1 = \alpha(\mathbf{v})\alpha(\mathbf{w}) = \mathbf{v}_1\mathbf{w}_1$ , so that  $\mathbf{u}_2 \dots \mathbf{u}_k = \omega(\mathbf{v})^{\alpha(\mathbf{w})}\omega(\mathbf{w})$ . The induction hypothesis applied to  $\omega(\mathbf{v})^{\alpha(\mathbf{w})}$ , which represents both a map in  $B^+(\mathcal{J})$  and an element of  $B_{\mathbf{I}^{\alpha(\mathbf{w})}}^+$ , and to  $\omega(\mathbf{w}) \in B^+(\mathcal{I})$  gives the result.  $\square$

**The varieties  $\mathcal{O}$  attached to  $B^+(\mathcal{I})$ .** In this subsection, we shall define a representation of  $B^+(\mathcal{I})$  into the bicategory  $\mathbf{X}$  of varieties over  $\mathcal{P}_I \times \mathcal{P}_J$ , where  $I, J$  vary over  $\mathcal{I}$ . The bicategory  $\mathbf{X}$  has 0-cells which are the elements of  $\mathcal{I}$ , has 1-cells with domain  $\mathbf{I}$  and codomain  $\mathbf{J}$  which are the  $\mathcal{P}_I \times \mathcal{P}_J$ -varieties and has 2-cells which are isomorphisms of  $\mathcal{P}_I \times \mathcal{P}_J$ -varieties. We denote by  $V(\mathbf{I}, \mathbf{J})$  the category whose objects (resp. morphisms) are the 1-cells with domain  $\mathbf{I}$  and codomain  $\mathbf{J}$  (resp. the 2-cells between them); in other words,  $V(\mathbf{I}, \mathbf{J})$  is the category of  $\mathcal{P}_I \times \mathcal{P}_J$ -varieties endowed with the isomorphisms of  $\mathcal{P}_I \times \mathcal{P}_J$ -varieties. The horizontal composition bifunctor  $V(\mathbf{I}, \mathbf{J}) \times V(\mathbf{J}, \mathbf{K}) \rightarrow V(\mathbf{I}, \mathbf{K})$  is given by the fibered product over  $\mathcal{P}_J$ . The vertical composition is given by the composition of isomorphisms.

The representation of  $B^+(\mathcal{I})$  in  $\mathbf{X}$  we construct will be denoted by  $T$ , following the notations of Section 7. We will also write  $\mathcal{O}(\mathbf{I}, \mathbf{b})$  for  $T(\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J})$ , to lighten the notation. We first define  $T$  on the Garside family  $\mathcal{S}$ .

**Definition 8.6.** *For  $(\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J}) \in \mathcal{S}$ , if  $I, w, J$  are the images in  $W$  of  $\mathbf{I}, \mathbf{w}, \mathbf{J}$  respectively, we define  $\mathcal{O}(\mathbf{I}, \mathbf{w})$  to be the variety  $\{(\mathbf{P}, \mathbf{P}') \in \mathcal{P}_I \times \mathcal{P}_J \mid \mathbf{P} \xrightarrow{I, w, J} \mathbf{P}'\}$ .*

The following lemma constructs the isomorphism  $T(f)T(g) \xrightarrow{\sim} T(fg)$  when  $f, g, fg \in \mathcal{S}$ :

**Lemma 8.7.** *Let  $(\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \xrightarrow{\mathbf{w}_2} \mathbf{J}) = (\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J})$  where  $\mathbf{w} = \mathbf{w}_1\mathbf{w}_2 \in \mathbf{W}$  be a defining relation of  $B^+(\mathcal{I})$ . Then  $(p', p'') : \mathcal{O}(I, \mathbf{w}_1) \times_{\mathcal{P}_{I_2}} \mathcal{O}(I_2, \mathbf{w}_2) \xrightarrow{\sim} \mathcal{O}(I, \mathbf{w}_1\mathbf{w}_2)$  is an isomorphism, where  $p'$  and  $p''$  are respectively the first and last projections..*

*Proof.* First notice that for two parabolic subgroups  $(\mathbf{P}', \mathbf{P}'') \in \mathcal{P}_I \times \mathcal{P}_J$  we have  $\mathbf{P}' \xrightarrow{I, w, J} \mathbf{P}''$  if and only if the pair  $(\mathbf{P}', \mathbf{P}'')$  is conjugate to a pair containing termwise the pair  $(\mathbf{B}, {}^w\mathbf{B})$ . This shows that if  $\mathbf{P}' \xrightarrow{I, w_1, I_2} \mathbf{P}_1$  and  $\mathbf{P}_1 \xrightarrow{I_2, w_2, J} \mathbf{P}''$  then  $\mathbf{P}' \xrightarrow{I, w_1 w_2, J} \mathbf{P}''$ , so  $(p', p'')$  goes to the claimed variety.

Conversely, we have to show that given  $\mathbf{P}' \xrightarrow{I, w, J} \mathbf{P}''$  there is a unique  $\mathbf{P}_1$  such that  $\mathbf{P}' \xrightarrow{I, w_1, I_2} \mathbf{P}_1 \xrightarrow{I_2, w_2, J} \mathbf{P}''$ . The image of  $(\mathbf{B}, {}^w\mathbf{B})$  by the conjugation which sends  $(\mathbf{P}_I, {}^w\mathbf{P}_J)$  to  $(\mathbf{P}', \mathbf{P}'')$  is a pair of Borel subgroups  $(\mathbf{B}' \subset \mathbf{P}', \mathbf{B}'' \subset \mathbf{P}'')$  in position  $w$ . Since  $l(w_1) + l(w_2) = l(w)$ , there is a unique Borel subgroup  $\mathbf{B}_1$  such that  $\mathbf{B}' \xrightarrow{w_1} \mathbf{B}_1 \xrightarrow{w_2} \mathbf{B}''$ . The unique parabolic subgroup of type  $I_2$  containing  $\mathbf{B}_1$

has the desired relative positions, so  $\mathbf{P}_1$  exists. And any other parabolic subgroup  $\mathbf{P}'_1$  which has the desired relative positions contains a Borel subgroup  $\mathbf{B}'_1$  such that  $\mathbf{B}' \xrightarrow{w_1} \mathbf{B}'_1 \xrightarrow{w_2} \mathbf{B}''$  (take for  $\mathbf{B}'_1$  the image of  ${}^{w_1}\mathbf{B}$  by the conjugation which maps  $(\mathbf{P}_I, {}^{w_1}\mathbf{P}_{I_2})$  to  $(\mathbf{P}', \mathbf{P}'_1)$ ), which implies that  $\mathbf{B}'_1 = \mathbf{B}_1$  and thus  $\mathbf{P}'_1 = \mathbf{P}_1$ . Thus our map is bijective on points. To show it is an isomorphism, it is sufficient to check that its target is a normal variety, which is given by

**Lemma 8.8.** *For  $(\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J}) \in \mathcal{S}$  the variety  $\mathcal{O}(\mathbf{I}, \mathbf{w})$  is smooth.*

*Proof.* Consider the locally trivial fibrations with smooth fibers given by  $\mathbf{G} \times \mathbf{G} \xrightarrow{p} \mathcal{P}_I \times \mathcal{P}_J : (g_1, g_2) \mapsto ({}^{g_1}\mathbf{P}_I, {}^{g_2 w}\mathbf{P}_J)$  and  $\mathbf{G} \times \mathbf{G} \xrightarrow{q} \mathbf{G} : (g_1, g_2) \mapsto g_1^{-1}g_2$ . It is easy to check that  $\mathcal{O}(\mathbf{I}, \mathbf{w}) = p(q^{-1}({}^w\mathbf{P}_J))$  thus by for example [DMR, 2.2.3] it is smooth.  $\square$

$\square$

From the above lemma we see also that the square 7.1 commutes for elements of  $\mathcal{S}$ , since the isomorphism “forgetting the middle parabolic” has clearly the corresponding property. We have thus defined a representation  $T$  of  $\mathcal{S}$  in  $\mathbf{X}$ .

The extension of  $T$  to the whole of  $B^+(\mathcal{I})$  associates to a composition  $\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \rightarrow \dots \rightarrow \mathbf{I}_k \xrightarrow{\mathbf{w}_k} \mathbf{J}$  with  $\mathbf{w}_i \in \mathbf{W}$  the variety

$$\mathcal{O}(\mathbf{I}, \mathbf{w}_1) \times_{\mathcal{P}_{I_2}} \dots \times_{\mathcal{P}_{I_k}} \mathcal{O}(\mathbf{I}_k, \mathbf{w}_k) = \{(\mathbf{P}_1, \dots, \mathbf{P}_{k+1}) \mid \mathbf{P}_i \xrightarrow{I_i, w_i, I_{i+1}} \mathbf{P}_{i+1}\},$$

where  $I_1 = I$  and  $I_{k+1} = J$ . It is a  $\mathcal{P}_I \times \mathcal{P}_J$ -variety via the first and last projections mapping respectively  $(\mathbf{P}_1, \dots, \mathbf{P}_{k+1})$  to  $\mathbf{P}_1$  and  $\mathbf{P}_{k+1}$ , and Lemma 8.7 shows that up to isomorphism it does not depend on the chosen decomposition of  $\mathbf{I} \xrightarrow{\mathbf{w}_1 \dots \mathbf{w}_k} \mathbf{J}$ . Theorem 7.2 shows that there is actually a unique isomorphism between the various models attached to different decompositions, so  $T$  defines a variety for any element of  $B^+(\mathcal{I})$ .

**Definition 8.9.** *For  $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \in B^+(\mathcal{I})$  we denote by  $\mathcal{O}(\mathbf{I}, \mathbf{b})$  the variety defined by Theorem 7.2. For any decomposition  $(\mathbf{I} \xrightarrow{\mathbf{b}} \phi\mathbf{I}) = (\mathbf{I}_1 \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \rightarrow \dots \xrightarrow{\mathbf{w}_k} \phi\mathbf{I})$  in elements of  $\mathcal{S}$  it has the model  $\{(\mathbf{P}_1, \dots, \mathbf{P}_{k+1}) \mid \mathbf{P}_i \xrightarrow{I_i, w_i, I_{i+1}} \mathbf{P}_{i+1}\}$ .*

**The Deligne-Lusztig varieties attached to  $B^+(\mathcal{I})$ .** The automorphism  $\phi$  lifts naturally to an automorphism of  $B^+$  which stabilizes  $\mathbf{S}$ , which we will still denote by  $\phi$ , by abuse of notation. If  $(\mathbf{I} \xrightarrow{\mathbf{w}} \phi\mathbf{I}) \in \mathcal{S}$ , then  $\mathbf{X}(\mathbf{I}, w\phi)$  is the intersection of  $\mathcal{O}(\mathbf{I}, \mathbf{w})$  with the graph of  $F$ , that is, points whose image under  $(p', p'')$  has the form  $(\mathbf{P}, {}^F\mathbf{P})$ . More generally,

**Definition 8.10.** *Let  $\mathbf{I} \xrightarrow{\mathbf{b}} \phi\mathbf{I}$  be any morphism of  $B^+(\mathcal{I})$ ; we define the variety  $\mathbf{X}(\mathbf{I}, \mathbf{b}\phi)$  as the intersection of  $\mathcal{O}(\mathbf{I}, \mathbf{b})$  with the graph of  $F$ . For any decomposition  $(\mathbf{I} \xrightarrow{\mathbf{b}} \phi\mathbf{I}) = (\mathbf{I}_1 \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \rightarrow \dots \xrightarrow{\mathbf{w}_k} \phi\mathbf{I})$  in elements of  $\mathcal{S}$  the variety  $\mathcal{O}(\mathbf{I}, \mathbf{b})$  has the model  $\{(\mathbf{P}_1, \dots, \mathbf{P}_{k+1}) \mid \mathbf{P}_i \xrightarrow{I_i, w_i, I_{i+1}} \mathbf{P}_{i+1} \text{ and } \mathbf{P}_{k+1} = F(\mathbf{P}_1)\}$ .*

The above model may be interpreted as an “ordinary” parabolic Deligne-Lusztig variety in a group which is a descent of scalars:

**Proposition 8.11.** *Let  $\mathbf{I} = \mathbf{I}_1 \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \rightarrow \dots \rightarrow \mathbf{I}_k \xrightarrow{\mathbf{w}_k} \phi\mathbf{I}$  be a decomposition into elements of  $\mathcal{S}$  of  $\mathbf{I} \xrightarrow{\mathbf{b}} \phi\mathbf{I} \in B^+(\mathcal{I})$ , let  $F_1$  be the isogeny of  $\mathbf{G}^k$  defined by*

$F_1(g_1, \dots, g_k) = (g_2, \dots, g_k, F(g_1))$  and let  $\phi_1$  be the corresponding automorphism of  $W^k$ . Then  $\mathbf{X}_{\mathbf{G}}(\mathbf{I}, \mathbf{b}\phi) \simeq \mathbf{X}_{\mathbf{G}^k}(I_1 \times \dots \times I_k, (w_1, \dots, w_k)\phi_1)$ . By this isomorphism the action of  $F^\delta$  corresponds to that of  $F_1^{k\delta}$  and the action of  $\mathbf{G}^F$  corresponds to that of  $(\mathbf{G}^k)^{F_1}$ .

*Proof.* An element  $\mathbf{P}_1 \times \dots \times \mathbf{P}_k \in \mathbf{X}_{\mathbf{G}^k}(I_1 \times \dots \times I_k, (w_1, \dots, w_k)\phi_1)$  by definition satisfies

$$\mathbf{P}_1 \times \dots \times \mathbf{P}_k \xrightarrow{I_1 \times \dots \times I_k, (w_1, \dots, w_k), I_2 \times \dots \times I_k \times {}^\phi I_1} \mathbf{P}_2 \times \dots \times \mathbf{P}_k \times {}^F \mathbf{P}_1$$

thus is equivalently given by a sequence  $(\mathbf{P}_1, \dots, \mathbf{P}_{k+1})$  such that  $\mathbf{P}_i \xrightarrow{I_i, w_i, I_{i+1}} \mathbf{P}_{i+1}$  with  $\mathbf{P}_{k+1} = {}^F \mathbf{P}_1$  and  $I_{k+1} = {}^\phi I_1$ , which is the same as an element

$$(\mathbf{P}_1, \dots, \mathbf{P}_{k+1}) \in \mathcal{O}(\mathbf{I}_1, \mathbf{w}_1) \times_{\mathcal{P}_{I_2}} \mathcal{O}(\mathbf{I}_2, \mathbf{w}_2) \dots \times_{\mathcal{P}_{I_{k-1}}} \mathcal{O}(\mathbf{I}_k, \mathbf{w}_k)$$

such that  $\mathbf{P}_{k+1} = {}^F \mathbf{P}_1$ . But this is a model of  $\mathbf{X}_{\mathbf{G}}(\mathbf{I}, \mathbf{b}\phi)$  as explained above.

One checks easily that this sequence of identifications is compatible with the actions of  $F^\delta$  and  $\mathbf{G}^F$  as described by the proposition.  $\square$

**Proposition 8.12.** *The variety  $\mathbf{X}(\mathbf{I}, \mathbf{b}\phi)$  is irreducible if and only if  $\mathbf{I} \cup c(\mathbf{b})$  meets all the orbits of  $\phi$  on  $\mathbf{S}$ , where  $c(\mathbf{b})$  is the set of elements of  $\mathbf{S}$  which appear in a decomposition of  $\mathbf{b}$ .*

*Proof.* This is, using Proposition 8.11, an immediate translation in our setting of the result [BR, Theorem 2] of Bonnafé-Rouquier.  $\square$

**The varieties  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$ .** The conjugation which transforms  $\mathbf{X}_{\mathbf{P}}$  into  $\mathbf{X}(I, w\phi)$  maps  $\mathbf{X}_{\mathbf{V}}$  to the  $\mathbf{G}^F$ -variety  $\mathbf{L}_I^{\dot{w}F}$  given by

$$\tilde{\mathbf{X}}(I, \dot{w}F) = \{g\mathbf{U}_I \in \mathbf{G}/\mathbf{U}_I \mid g^{-1}Fg \in \mathbf{U}_I \dot{w}^F \mathbf{U}_I\},$$

where  $\dot{w}$  is a representative of  $w$  (any representative can be obtained by choosing an appropriate conjugation). The map  $g\mathbf{U}_I \mapsto g\mathbf{P}_I$  makes  $\tilde{\mathbf{X}}(I, \dot{w}F)$  a  $\mathbf{L}_I^{\dot{w}F}$ -torsor over  $\mathbf{X}(I, w\phi)$ . We will sometimes write  $\tilde{\mathbf{X}}(I, \dot{w}F)$  to separate the Frobenius endomorphism from the representative of the Weyl group element. This will be especially useful when the ambient group is a Levi subgroup with Frobenius endomorphism of the form  $\dot{x}F$ .

In this section, we define a variety  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$  which generalizes  $\tilde{\mathbf{X}}(I, \dot{w}F)$  by replacing  $\dot{w}$  by elements of the braid group. Since  $\dot{w}$  represents a choice of a lift of  $w$  to  $N_{\mathbf{G}}(\mathbf{T})$ , we have to make uniformly such choices for all elements of the braid group, which we do by using a “Tits homomorphism”.

First, we need, when  $\mathbf{w} \in \mathbf{W}$ , to define a variety  $\tilde{\mathcal{O}}(I, \dot{w})$  “above”  $\mathcal{O}(\mathbf{I}, \mathbf{w})$  such that  $\tilde{\mathbf{X}}(I, \dot{w}F)$  is the intersection of  $\tilde{\mathcal{O}}(I, \dot{w})$  with the graph of  $F$ , and then we extend this construction to  $\mathcal{B}^+(\mathcal{I})$ .

**Definition 8.13.** *Let  $(\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J}) \in \mathcal{S}$ , and let  $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$  be a representative of  $w$ . We define  $\tilde{\mathcal{O}}(I, \dot{w}) = \{(g\mathbf{U}_I, g'\mathbf{U}_J) \in \mathbf{G}/\mathbf{U}_I \times \mathbf{G}/\mathbf{U}_J \mid g^{-1}g' \in \mathbf{U}_I \dot{w} \mathbf{U}_J\}$ .*

We can prove an analogue of Lemma 8.7.

**Lemma 8.14.** *Let  $(\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \xrightarrow{\mathbf{w}_2} \mathbf{J}) = (\mathbf{I} \xrightarrow{\mathbf{w}_1 \mathbf{w}_2} \mathbf{J})$  where  $\mathbf{w}_1 \mathbf{w}_2 \in \mathbf{W}$  be a defining relation of  $B^+(\mathcal{I})$ , and let  $\dot{w}_1, \dot{w}_2$  be representatives of the images of  $\mathbf{w}_1$  and  $\mathbf{w}_2$  in  $W$ . Then  $(p', p'') : \tilde{\mathcal{O}}(I, \dot{w}_1) \times_{\mathbf{G}/\mathbf{U}_{I_2}} \tilde{\mathcal{O}}(I_2, \dot{w}_2) \xrightarrow{\sim} \tilde{\mathcal{O}}(I, \dot{w}_1 \dot{w}_2)$  is an isomorphism where  $p'$  and  $p''$  are the first and last projections.*

*Proof.* We first note that if  $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J} \in B^+(\mathcal{I})$  and  $\dot{w}$  is a representative in  $N_{\mathbf{G}}(\mathbf{T})$  of the image of  $\mathbf{w}$  in  $W$ , then  $\mathbf{U}_I \dot{w} \mathbf{U}_J$  is isomorphic by the product morphism to the direct product of varieties  $(\mathbf{U}_I \cap {}^w \mathbf{U}_J^-) \dot{w} \times \mathbf{U}_J$ , where  $\mathbf{U}_J^-$  is the unipotent radical of the parabolic subgroup opposed to  $\mathbf{P}_J$  containing  $\mathbf{T}$ . We now use the lemma:

**Lemma 8.15.** *Under the assumptions of Lemma 8.14, the product gives an isomorphism  $(\mathbf{U}_I \cap {}^{\dot{w}_1} \mathbf{U}_{I_2}^-) \dot{w}_1 \times (\mathbf{U}_{I_2} \cap {}^{\dot{w}_2} \mathbf{U}_J^-) \dot{w}_2 \xrightarrow{\sim} (\mathbf{U}_I \cap {}^{\dot{w}_1 \dot{w}_2} \mathbf{U}_J^-) \dot{w}_1 \dot{w}_2$ .*

*Proof.* As a product of root subgroups, we have  $\mathbf{U}_I \cap {}^w \mathbf{U}_J^- = \prod_{-\alpha \in {}^w N(w)} \mathbf{U}_\alpha$ , where  $N(w) = \{\alpha \in \Phi^+ \mid {}^w \alpha \in \Phi^-\}$ . The lemma is then a consequence of the equality  $N(w_1)^{w_2} \coprod N(w_2) = N(w_1 w_2)$  when  $l(w_1) + l(w_2) = l(w_1 w_2)$ .  $\square$

The lemma proves in particular that if  $g_1^{-1} g_2 \in \mathbf{U}_I \dot{w}_1 \mathbf{U}_{I_2}$  and  $g_2^{-1} g_3 \in \mathbf{U}_{I_2} \dot{w}_2 \mathbf{U}_J$  then  $g_1^{-1} g_3 \in \mathbf{U}_I \dot{w}_1 \mathbf{U}_{I_2} \dot{w}_2 \mathbf{U}_J = (\mathbf{U}_I \cap {}^{\dot{w}_1} \mathbf{U}_{I_2}^-) \dot{w}_1 (\mathbf{U}_{I_2} \cap {}^{\dot{w}_2} \mathbf{U}_J^-) \dot{w}_2 \mathbf{U}_J = (\mathbf{U}_I \cap {}^{\dot{w}_1 \dot{w}_2} \mathbf{U}_J^-) \dot{w}_1 \dot{w}_2 \mathbf{U}_J = \mathbf{U}_I \dot{w}_1 \dot{w}_2 \mathbf{U}_J$ , so the image of the morphism  $(p', p'')$  in Lemma 8.14 is indeed in the variety  $\tilde{\mathcal{O}}(I, \dot{w}_1 \dot{w}_2)$ .

Conversely, we have to show that given  $(g_1 \mathbf{U}_I, g_3 \mathbf{U}_J) \in \tilde{\mathcal{O}}(I, \dot{w}_1 \dot{w}_2)$ , there exists a unique  $g_2 \mathbf{U}_{I_2}$  such that  $(g_1 \mathbf{U}_I, g_2 \mathbf{U}_{I_2}) \in \tilde{\mathcal{O}}(I, \dot{w}_1)$  and  $(g_2 \mathbf{U}_{I_2}, g_3 \mathbf{U}_J) \in \tilde{\mathcal{O}}(I_2, \dot{w}_2)$ . The varieties involved being invariant by left translation by  $\mathbf{G}$ , it is enough to solve the problem when  $g_1 = 1$ . Then we have  $g_3 \in \mathbf{U}_I \dot{w}_1 \dot{w}_2 \mathbf{U}_J$ , and the conditions for  $g_2 \mathbf{U}_{I_2}$  is that  $g_2 \mathbf{U}_{I_2} \subset \mathbf{U}_I \dot{w}_1 \mathbf{U}_{I_2}$ . Any such coset has then a unique representative in  $(\mathbf{U}_I \cap {}^{\dot{w}_1} \mathbf{U}_{I_2}^-) \dot{w}_1$  and we will look for such a representative  $g_2$ . But we must have  $g_2^{-1} g_3 \in \mathbf{U}_{I_2} \dot{w}_2 \mathbf{U}_J = (\mathbf{U}_{I_2} \cap {}^{\dot{w}_2} \mathbf{U}_J^-) \dot{w}_2 \mathbf{U}_J$  and since by the lemma the product gives an isomorphism between  $(\mathbf{U}_I \cap {}^{\dot{w}_1} \mathbf{U}_{I_2}^-) \dot{w}_1 \times (\mathbf{U}_{I_2} \cap {}^{\dot{w}_2} \mathbf{U}_J^-) \dot{w}_2 \mathbf{U}_J$  and  $\mathbf{U}_I \dot{w}_1 \dot{w}_2 \mathbf{U}_J$ , the element  $g_3$  can be decomposed in one and only one way in a product  $g_2(g_2^{-1} g_3)$  satisfying the conditions. To conclude as in 8.7 we show that the variety  $\tilde{\mathcal{O}}(I, \dot{w}_1 \dot{w}_2)$  is smooth. An argument similar to the proof of 8.8, replacing  $\mathcal{P}_I$  and  $\mathcal{P}_J$  by  $\mathbf{G}/\mathbf{U}_I$  and  $\mathbf{G}/\mathbf{U}_J$  respectively gives the result.  $\square$

We will now use a Tits homomorphism, which is a homomorphism  $B \xrightarrow{t} N_{\mathbf{G}}(\mathbf{T})$  which factors the projection  $B \rightarrow W$  (their existence is proved in [T]). Theorem 7.2 implies that, setting  $T(\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J}) = \tilde{\mathcal{O}}(I, t(\mathbf{w}))$  for  $(\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J}) \in \mathcal{S}$  and replacing Lemma 8.7 by Lemma 8.14, we can define a representation of  $B^+(\mathcal{I})$  in the bicategory  $\tilde{\mathbf{X}}$  of varieties above  $\mathbf{G}/\mathbf{U}_I \times \mathbf{G}/\mathbf{U}_J$  for  $I, J \in \mathcal{I}$ .

**Definition 8.16.** *The above representation defines for any  $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \in B^+(\mathcal{I})$  a variety  $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$  which for any decomposition  $(\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}) = (\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \rightarrow \dots \rightarrow \mathbf{I}_k \xrightarrow{\mathbf{w}_k} \mathbf{J})$  into elements of  $\mathcal{S}$  has the model  $\tilde{\mathcal{O}}(I, t(\mathbf{w}_1)) \times_{\mathbf{G}/\mathbf{U}_{I_2}} \dots \times_{\mathbf{G}/\mathbf{U}_{I_k}} \tilde{\mathcal{O}}(I_k, t(\mathbf{w}_k))$ .*

**Proposition 8.17.** *There exists a Tits homomorphism  $t$  which is  $F$ -equivariant, that is such that  $t(\phi(\mathbf{b})) = F(t(\mathbf{b}))$ .*

*Proof.* To any simple reflection  $s \in S$  is associated a quasi-simple subgroup  $\mathbf{G}_s$  of rank 1 of  $\mathbf{G}$ , generated by the root subgroups  $\mathbf{U}_{\alpha_s}$  and  $\mathbf{U}_{-\alpha_s}$ ; the 1-parameter subgroup of  $\mathbf{T}$  given by  $\mathbf{T} \cap \mathbf{G}_s$  is a maximal torus of  $\mathbf{G}_s$ . By [T, Theorem 4.4] if for any  $s \in S$  we choose a representative  $\dot{s}$  of  $s$  in  $\mathbf{G}_s$ , then these representatives satisfy the braid relations, which implies that  $\mathbf{s} \mapsto \dot{s}$  induces a well defined Tits homomorphism. We claim that if  $s$  is fixed by some power  $\phi^d$  of  $\phi$  then there exists  $\dot{s} \in \mathbf{G}_s$  fixed by  $F^d$ ; we then get an  $F$ -equivariant Tits homomorphism by choosing arbitrarily  $\dot{s}$  for one  $s$  in each orbit of  $\phi$ . If  $s$  is fixed by  $\phi^d$  then  $\mathbf{G}_s$  is stable

by  $F^d$ ; the group  $\mathbf{G}_s$  is isomorphic to either  $SL_2$  or  $PSL_2$  and  $F^d$  is a Frobenius endomorphism of this group. In either case the simple reflection  $s$  of  $\mathbf{G}_s$  has an  $F^d$ -stable representative in  $N_{\mathbf{G}_s}(\mathbf{T} \cap \mathbf{G}_s)$ .  $\square$

**Notation 8.18.** *We assume now that we have chosen, once and for all, an  $F$ -equivariant Tits homomorphism  $t$  which is used to define the varieties  $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$ . For  $w \in W$  we will write  $\dot{w}$  for  $t(\mathbf{w})$  where  $\mathbf{w} \in \mathbf{W}$  is the canonical lift of  $w$ .*

**Definition 8.19.** *For any morphism  $(\mathbf{I} \xrightarrow{\mathbf{b}} \phi\mathbf{I}) \in B^+(\mathcal{I})$  we define  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{b}\phi) = \{x \in \tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b}) \mid p''(x) = F(p'(x))\}$ .*

When  $\mathbf{w} \in \mathbf{W}$  we have  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) = \tilde{\mathbf{X}}(\mathbf{I}, \dot{w}F)$  (the variety defined at the beginning of this section).

**Lemma 8.20.** *For any  $(\mathbf{I} \xrightarrow{\mathbf{w}} \phi\mathbf{I}) \in B^+(\mathcal{I})$ , there is a natural projection  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \xrightarrow{\pi} \mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$  which makes  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$  a  $\mathbf{L}_I^{t(\mathbf{w})F}$ -torsor over  $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$ , where the action of  $\mathbf{L}_I^{t(\mathbf{w})F}$  is compatible with the first projection  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \rightarrow \mathbf{G}/\mathbf{U}_I$ .*

*Proof.* Let  $\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \rightarrow \cdots \rightarrow \mathbf{I}_r \xrightarrow{\mathbf{w}_r} \phi\mathbf{I}$  be a decomposition into elements of  $\mathcal{S}$  of  $\mathbf{I} \xrightarrow{\mathbf{w}} \phi\mathbf{I}$ , so that  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$  identifies to the set of sequences  $(g_1 \mathbf{U}_I, g_2 \mathbf{U}_{I_2}, \dots, g_r \mathbf{U}_{I_r})$  such that  $g_j^{-1} g_{j+1} \in \mathbf{U}_{I_j} t(\mathbf{w}_j) \mathbf{U}_{I_{j+1}}$  for  $j < r$  and  $g_r^{-1} F g_1 \in \mathbf{U}_{I_r} t(\mathbf{w}_r) \mathbf{U}_{\phi I}$ . We define  $\pi$  by  $g_j \mathbf{U}_{I_j} \mapsto {}^{g_j} \mathbf{P}_{I_j}$ . It is easy to check that the morphism  $\pi$  thus defined commutes with an “elementary morphism” in the bicategories of varieties  $\tilde{\mathbf{X}}$  or  $\mathbf{X}$  consisting of passing from the decomposition  $(\mathbf{w}_1, \dots, \mathbf{w}_i, \mathbf{w}_{i+1}, \dots, \mathbf{w}_r)$  to  $(\mathbf{w}_1, \dots, \mathbf{w}_i \mathbf{w}_{i+1}, \dots, \mathbf{w}_r)$  when  $(\mathbf{I}_i \xrightarrow{\mathbf{w}_i \mathbf{w}_{i+1}} \mathbf{I}_{i+2}) \in \mathcal{S}$ . Thus by 7.1 the morphism  $\pi$  is well-defined independently of the decomposition chosen of  $\mathbf{w}$ . We claim that  $\pi$  makes  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$  a  $\mathbf{L}^{t(\mathbf{w})F}$ -torsor over  $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$ . Indeed, the fiber  $\pi^{-1}(({}^{g_1} \mathbf{P}_I, {}^{g_2} \mathbf{P}_{I_2}, \dots, {}^{g_r} \mathbf{P}_{I_r}))$  consists of the  $(g_1 l_1 \mathbf{U}_I, \dots, g_r l_r \mathbf{U}_{I_r}) \in \tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$  with  $l_j \in \mathbf{L}_{I_j}$ , that is such that

$$\begin{aligned} \text{for } j < r \text{ we have } g_j^{-1} g_{j+1} &\in (\mathbf{U}_{I_j} t(\mathbf{w}_j) \mathbf{U}_{I_{j+1}}) \cap l_j (\mathbf{U}_{I_j} t(\mathbf{w}_j) \mathbf{U}_{I_{j+1}}) l_{j+1}^{-1} \\ \text{and } g_r^{-1} F g_1 &\in (\mathbf{U}_{I_r} t(\mathbf{w}_r) \mathbf{U}_{\phi I}) \cap l_r (\mathbf{U}_{I_r} t(\mathbf{w}_r) \mathbf{U}_{\phi I})^F l_1^{-1}. \end{aligned}$$

Now

$$(\mathbf{U}_{I_j} t(\mathbf{w}_j) \mathbf{U}_{I_{j+1}}) \cap l_j (\mathbf{U}_{I_j} t(\mathbf{w}_j) \mathbf{U}_{I_{j+1}}) l_{j+1}^{-1} = (\mathbf{U}_{I_j} t(\mathbf{w}_j) \mathbf{U}_{I_{j+1}}) \cap \mathbf{U}_{I_j} t(\mathbf{w}_j) \mathbf{U}_{I_{j+1}} l_j^{t(\mathbf{w}_j)} l_{j+1}^{-1}$$

and the intersection is non-empty if and only if  $\mathbf{U}_{I_j}^{t(\mathbf{w}_j)} \cap \mathbf{U}_{I_{j+1}} l_j^{t(\mathbf{w}_j)} l_{j+1}^{-1} \neq \emptyset$ , which, since  $\mathbf{P}_{I_j}^{t(\mathbf{w}_j)}$  and  $\mathbf{P}_{I_{j+1}}$  are two parabolic subgroups with the same Levi subgroup, occurs only if  $l_j^{t(\mathbf{w}_j)} = l_{j+1}$ . Similarly we get  $l_r^{t(\mathbf{w}_r)} = {}^F l_1$ , so in the end the fiber is given by the  $l_1$  such that  $l_1 = {}^{t(\mathbf{w})F} l_1$ .  $\square$

We give an analogue of Proposition 8.11 for  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{b}\phi)$ .

**Proposition 8.21.** *Let  $\mathbf{I} = \mathbf{I}_1 \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \rightarrow \cdots \rightarrow \mathbf{I}_k \xrightarrow{\mathbf{w}_k} \phi\mathbf{I}$  be a decomposition into elements of  $\mathcal{S}$  of  $\mathbf{I} \xrightarrow{\mathbf{b}} \phi\mathbf{I} \in B^+(\mathcal{I})$ , let  $F_1$  be the isogeny of  $\mathbf{G}^k$  as in Proposition 8.11.*

*Then  $\tilde{\mathbf{X}}_{\mathbf{G}}(\mathbf{I}, \mathbf{b}\phi) \simeq \tilde{\mathbf{X}}_{\mathbf{G}^k}(I_1 \times \dots \times I_k, (\dot{w}_1, \dots, \dot{w}_k) F_1)$ . By this isomorphism the action of  $F^\delta$  corresponds to that of  $F_1^{k\delta}$ , the action of  $\mathbf{G}^F$  corresponds to that of  $(\mathbf{G}^k)^{F_1}$ , and the action of  $\mathbf{L}_I^{t(\mathbf{b})F}$  corresponds to that of  $(\mathbf{L}_{I_1} \times \dots \times \mathbf{L}_{I_k})^{(\dot{w}_1, \dots, \dot{w}_k) F_1}$ .*

*Proof.* An element  $x_1 \mathbf{U}_{I_1} \times \dots \times x_k \mathbf{U}_{I_k} \in \tilde{\mathbf{X}}_{\mathbf{G}^k}(I_1 \times \dots \times I_k, (\dot{w}_1, \dots, \dot{w}_k)F_1)$  by definition satisfies  $(x_i \mathbf{U}_{I_i}, x_{i+1} \mathbf{U}_{I_{i+1}}) \in \tilde{\mathcal{O}}(I_i, \dot{w}_i)$  for  $i = 1, \dots, k$ , where we have put  $I_{k+1} = {}^F I_1$  and  $x_{k+1} \mathbf{U}_{I_{k+1}} = {}^F(x_1 \mathbf{U}_{I_1})$ . This is the same as an element in the intersection of  $\tilde{\mathcal{O}}(\mathbf{I}_1, \mathbf{w}_1) \times_{\mathbf{G}/\mathbf{U}_{I_2}} \tilde{\mathcal{O}}(\mathbf{I}_2, \mathbf{w}_2) \dots \times_{\mathbf{G}/\mathbf{U}_{I_{k-1}}} \tilde{\mathcal{O}}(\mathbf{I}_k, \mathbf{w}_k)$  with the graph of  $F$ . Since, by definition, we have

$$\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b}) \simeq \tilde{\mathcal{O}}(\mathbf{I}_1, \mathbf{w}_1) \times_{\mathbf{G}/\mathbf{U}_{I_2}} \tilde{\mathcal{O}}(\mathbf{I}_2, \mathbf{w}_2) \dots \times_{\mathbf{G}/\mathbf{U}_{I_{k-1}}} \tilde{\mathcal{O}}(\mathbf{I}_k, \mathbf{w}_k),$$

via this last isomorphism we get an element of  $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$  which is in  $\tilde{\mathbf{X}}_{\mathbf{G}}(\mathbf{I}, \mathbf{b}\phi)$ .

One checks easily that this sequence of identifications is compatible with the actions of  $F^\delta$ , of  $\mathbf{G}^F$  and of  $\mathbf{L}_I^{t(\mathbf{b})F}$  as described by the proposition.  $\square$

We give an isomorphism which reflects the transitivity of Lusztig's induction.

**Proposition 8.22.** *Let  $\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi \mathbf{I} \in B^+(\mathcal{I})$ , and let  $w$  be the image of  $\mathbf{w}$  in  $W$ ; the automorphism  $w\phi$  lifts to an automorphism that we will still denote by  $w\phi$  of  $B_{\mathbf{I}}^+$ . For  $\mathbf{J} \subset \mathbf{I}$ , let  $\mathcal{J}$  be the set of  $B_{\mathbf{I}}^+$ -conjugates of  $\mathbf{J}$  and let  $\mathbf{J} \xrightarrow{\mathbf{v}} {}^{w\phi} \mathbf{J} \in B_{\mathbf{I}}^+(\mathcal{J})$ . Then*

- (i) *We have an isomorphism  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \times_{\mathbf{L}_I^{t(\mathbf{w})F}} \tilde{\mathbf{X}}_{\mathbf{L}_I}(\mathbf{J}, \mathbf{v}w\phi) \xrightarrow{\sim} \tilde{\mathbf{X}}(\mathbf{J}, \mathbf{v}\mathbf{w}\phi)$  of  $\mathbf{G}^F$ -varieties- $\mathbf{L}_J^{t(\mathbf{v}\mathbf{w})F}$ . This isomorphism is compatible with the action of  $F^n$  for any  $n$  such that  $\mathbf{I}$ ,  $\mathbf{J}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  are  $\phi^n$ -stable.*
- (ii) *Through the quotient by  $\mathbf{L}_J^{t(\mathbf{v}\mathbf{w})F}$  (see Lemma 8.20) we get an isomorphism of  $\mathbf{G}^F$ -varieties*

$$\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \times_{\mathbf{L}_I^{t(\mathbf{w})F}} \mathbf{X}_{\mathbf{L}_I}(\mathbf{J}, \mathbf{v}w\phi) \xrightarrow{\sim} \mathbf{X}(\mathbf{J}, \mathbf{v}\mathbf{w}\phi).$$

*Proof.* We first look at the case  $\mathbf{w}, \mathbf{v} \in \mathbf{W}$  (which implies  $\mathbf{v}\mathbf{w} \in \mathbf{W}$ ), in which case the isomorphism we seek is

$$\tilde{\mathbf{X}}(I, \dot{w}F) \times_{\mathbf{L}_I^{\dot{w}F}} \tilde{\mathbf{X}}_{\mathbf{L}_I}(J, \dot{v}\dot{w}F) \xrightarrow{\sim} \tilde{\mathbf{X}}(J, \dot{v}\dot{w}F)$$

where  $v$  is the image of  $\mathbf{v}$  in  $W$ . This is the content of Lusztig's proof of the transitivity of his induction (see [Lu, lemma 3]), that we recall and detail in our context. We claim that  $(g\mathbf{U}_I, l\mathbf{V}_J) \mapsto g\mathbf{U}_I l\mathbf{V}_J = gl\mathbf{U}_J$  where  $\mathbf{V}_J = \mathbf{L}_I \cap \mathbf{U}_J$  induces the isomorphism we want. We have

$$\mathbf{U}_J \dot{v} \dot{w}^F \mathbf{U}_J = \mathbf{U}_I \mathbf{V}_J \dot{v} \dot{w}^F \mathbf{V}_J^F \mathbf{U}_I = \mathbf{U}_I \mathbf{V}_J \dot{v} \dot{w}^F \mathbf{V}_J \dot{w}^F \mathbf{U}_I.$$

Since  $\mathbf{V}_J \dot{v} \dot{w}^F \mathbf{V}_J$  is in  $\mathbf{L}_I$ , so normalizes  $\mathbf{U}_I$  we get finally

$$\mathbf{U}_J \dot{v} \dot{w}^F \mathbf{U}_J = \mathbf{V}_J \dot{v} \dot{w}^F \mathbf{V}_J \mathbf{U}_I \dot{w}^F \mathbf{U}_I.$$

Hence if  $(g\mathbf{U}_I, l\mathbf{V}_J) \in \tilde{\mathbf{X}}(I, \dot{w}F) \times \tilde{\mathbf{X}}_{\mathbf{L}_I}(\dot{v}\dot{w}\phi)$ , we have

$$\begin{aligned} (gl)^{-1}F(gl) &\in l^{-1} \mathbf{U}_I \dot{w}^F \mathbf{U}_I^F l = l^{-1} \mathbf{U}_I \dot{w}^F l \dot{w}^F \mathbf{U}_I \\ &= l^{-1} \dot{w}^F l \mathbf{U}_I \dot{w}^F \mathbf{U}_I \subset \mathbf{V}_J \dot{v} \dot{w}^F \mathbf{V}_J \mathbf{U}_I \dot{w}^F \mathbf{U}_I = \mathbf{U}_J \dot{v} \dot{w}^F \mathbf{U}_J. \end{aligned}$$

Hence we have defined a morphism  $\tilde{\mathbf{X}}(I, \dot{w}F) \times \tilde{\mathbf{X}}_{\mathbf{L}_I}(\dot{v}\dot{w}\phi) \rightarrow \tilde{\mathbf{X}}(J, \dot{v}\dot{w}F)$  of  $\mathbf{G}^F$ -varieties- $\mathbf{L}_J^{\dot{v}\dot{w}F}$ . We show now that it is surjective. The product  $\mathbf{L}_I \cdot (\mathbf{U}_I \dot{w}^F \mathbf{U}_I)$  is direct: a computation shows that this results from the unicity in the decomposition  $\mathbf{P}_I \cap {}^{\dot{w}F} \mathbf{U}_I = \mathbf{L}_I \cdot (\mathbf{U}_I \cap {}^{\dot{w}F} \mathbf{U}_I)$ . Hence an element  $x^{-1}F x \in \mathbf{U}_J \dot{v} \dot{w}^F \mathbf{U}_J$  defines unique elements  $l \in \mathbf{V}_J \dot{v} \dot{w}^F \mathbf{V}_J$  and  $u \in \mathbf{U}_I \dot{w}^F \mathbf{U}_I$  such that  $x^{-1}F x = lu$ . If, using Lang's theorem, we write  $l = l'^{-1} \dot{w}^F l'$  with  $l' \in \mathbf{L}_I$ , the element  $g = xl'^{-1}$  satisfies  $g^{-1}F g = l'^{-1} x^{-1} F x F l'^{-1} = {}^{\dot{w}F} l' u^F l'^{-1} \in {}^{\dot{w}F} l' \mathbf{U}_I \dot{w}^F \mathbf{U}_I F l'^{-1} = \mathbf{U}_I \dot{w}^F \mathbf{U}_I$ . Hence  $(g\mathbf{U}_I, l'\mathbf{V}_J)$  is a preimage of  $x\mathbf{U}_J$  in  $\tilde{\mathbf{X}}(I, \dot{w}F) \times \tilde{\mathbf{X}}_{\mathbf{L}_I}(J, \dot{v}\dot{w}\phi)$ .

Let us look now at the fibers of the above morphism. If  $g' \mathbf{U}_I l' \mathbf{V}_J = g \mathbf{U}_I l \mathbf{V}_J$  then  $g'^{-1} g \in \mathbf{P}_I$  so up to  $\mathbf{U}_I$  we may assume  $g' = g\lambda$  with  $\lambda \in \mathbf{L}_I$ ; we have then  $\lambda l' \mathbf{U}_J = l \mathbf{U}_J$ , so that  $l^{-1} \lambda l' \in \mathbf{U}_J \cap \mathbf{L}_I = \mathbf{V}_J$ ; moreover if  $g\lambda \mathbf{U}_I \in \tilde{\mathbf{X}}(I, \dot{w}F)$  with  $\lambda \in \mathbf{L}_I$ , then  $\lambda^{-1} \mathbf{U}_I \dot{w}^F \mathbf{U}_I^F \lambda = \mathbf{U}_I \dot{w}^F \mathbf{U}_I$  which implies  $\lambda \in \mathbf{L}_I^{\dot{w}F}$ . Conversely, the action of  $\lambda \in \mathbf{L}_I^{\dot{w}F}$  given by  $(g \mathbf{U}_I, l \mathbf{V}_J) \mapsto (g\lambda \mathbf{U}_I, \lambda^{-1} l \mathbf{V}_J)$  preserves the subvariety  $\tilde{\mathbf{X}}(I, \dot{w}F) \times_{\mathbf{L}_I} (\dot{v}w\phi)$ , of  $\mathbf{G}/\mathbf{U}_I \times \mathbf{L}_I/\mathbf{V}_J$ . Hence the fibers are the orbits under this action of  $\mathbf{L}_I^{\dot{w}F}$ .

Now the morphism  $j : (g \mathbf{U}_I, l \mathbf{V}_J) \mapsto g l \mathbf{U}_J$  is an isomorphism  $\mathbf{G}/\mathbf{U}_I \times_{\mathbf{L}_I} \mathbf{L}_I/\mathbf{V}_J \simeq \mathbf{G}/\mathbf{U}_J$  since  $g \mathbf{U}_J \mapsto (g \mathbf{U}_I, \mathbf{V}_J)$  is its inverse. By what we have seen above the restriction of  $j$  to the closed subvariety  $\tilde{\mathbf{X}}(I, \dot{w}F) \times_{\mathbf{L}_I^{\dot{w}F}} \tilde{\mathbf{X}}_{\mathbf{L}_I}(J, \dot{v}\dot{w}\phi)$  maps this variety surjectively on the closed subvariety  $\tilde{\mathbf{X}}(J, \dot{v}\dot{w}F)$  of  $\mathbf{G}/\mathbf{U}_J$ , hence we get the isomorphism we want.

We now consider the case of generalized varieties. Let  $k$  be the number of terms of the normal decomposition of  $\mathbf{vw}$  and let  $\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \xrightarrow{\mathbf{w}_2} \mathbf{I}_3 \rightarrow \cdots \rightarrow \mathbf{I}_r \xrightarrow{\mathbf{w}_r} {}^\phi \mathbf{I}$  be the normal decomposition of  $\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi \mathbf{I}$ , perhaps extended by some identity morphisms. We have  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \simeq \tilde{\mathbf{X}}(I_1 \times I_2 \times \cdots \times I_k, (t(\mathbf{w}_1), \dots, t(\mathbf{w}_k))F_1)$ , where  $F_1$  is as in Proposition 8.11. Let us write  $(\mathbf{v}_1 \mathbf{w}_1, \dots, \mathbf{v}_k \mathbf{w}_k)$  for the normal decomposition of  $\mathbf{vw}$ , with same notation as in Proposition 8.5. Let  $J_1 = J$  and  $J_{j+1} = J_j^{v_j w_j} \subset I_{j+1}$  for  $j = 1, \dots, k-1$ . We apply the first part of the proof to the group  $\mathbf{G}^k$  with isogeny  $F_1$  with  $I, J, w$ , and  $v$  replaced respectively by  $I_1 \times \cdots \times I_k, J_1 \times \cdots \times J_k, (w_1, \dots, w_k)$  and  $(v_1, \dots, v_k)$ . Using the isomorphisms from Proposition 8.21;

$$\tilde{\mathbf{X}}_{\mathbf{G}^k}(J_1 \times \cdots \times J_k, (\dot{v}_1 \dot{w}_1, \dots, \dot{v}_k \dot{w}_k)\phi_1) \simeq \tilde{\mathbf{X}}(\mathbf{J}, \mathbf{vw}\phi)$$

and

$$\tilde{\mathbf{X}}_{\mathbf{L}_{I_1} \times \cdots \times I_k}(J_1 \times \cdots \times J_k, (v_1, \dots, v_k).(t(\mathbf{w}_1), \dots, t(\mathbf{w}_k))F_1) \simeq \tilde{\mathbf{X}}_{\mathbf{L}_J}(\mathbf{J}, \mathbf{vw}\phi),$$

we get (i). Now (ii) is immediate from (i) taking the quotient on both sides by  $\mathbf{L}_J^{t(\mathbf{vw})F}$ .  $\square$

**Endomorphisms of parabolic Deligne-Lusztig varieties — the conjugacy category  $\mathcal{D}^+(\mathcal{I})$ .**

**Definition 8.23.** *Given any morphism  $\mathbf{I} \xrightarrow{\mathbf{v}} \mathbf{J} \in B^+(\mathcal{I})$  which is a left divisor of  $\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi \mathbf{I}$  we define morphisms of varieties:*

(i)  $D_{\mathbf{v}} : \mathbf{X}(\mathbf{I}, \mathbf{w}\phi) \rightarrow \mathbf{X}(\mathbf{J}, \mathbf{v}^{-1} \mathbf{w}\phi \mathbf{v})$  as the restriction of the morphism

$$(a, b) \mapsto (b, {}^F a) : \mathcal{O}(\mathbf{I}, \mathbf{w}) = \mathcal{O}(\mathbf{I}, \mathbf{v}) \times_{\mathbf{P}_J} \mathcal{O}(\mathbf{J}, \mathbf{v}^{-1} \mathbf{w}) \rightarrow \mathcal{O}(\mathbf{J}, \mathbf{v}^{-1} \mathbf{w}) \times_{\mathbf{P}_{\phi_I}} \mathcal{O}({}^\phi \mathbf{I}, {}^\phi \mathbf{v}) = \mathcal{O}(\mathbf{J}, \mathbf{v}^{-1} \mathbf{w}^\phi \mathbf{v}).$$

(ii)  $\tilde{D}_{\mathbf{v}} : \tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \rightarrow \tilde{\mathbf{X}}(\mathbf{J}, \mathbf{v}^{-1} \mathbf{w}\phi \mathbf{v})$  as the restriction of the morphism

$$(a, b) \mapsto (b, {}^F a) : \tilde{\mathcal{O}}(\mathbf{I}, \mathbf{w}) = \tilde{\mathcal{O}}(\mathbf{I}, \mathbf{v}) \times_{\mathbf{G}/\mathbf{U}_J} \tilde{\mathcal{O}}(\mathbf{J}, \mathbf{v}^{-1} \mathbf{w}) \rightarrow \tilde{\mathcal{O}}(\mathbf{J}, \mathbf{v}^{-1} \mathbf{w}) \times_{\mathbf{G}/\mathbf{U}_{\phi_I}} \tilde{\mathcal{O}}({}^\phi \mathbf{I}, {}^\phi \mathbf{v}) = \tilde{\mathcal{O}}(\mathbf{J}, \mathbf{v}^{-1} \mathbf{w}^\phi \mathbf{v}).$$

Note that the existence of well-defined decompositions as above of  $\mathcal{O}(\mathbf{I}, \mathbf{w})$  and of  $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{w})$  are consequences of Theorem 7.2. We have written  $\mathbf{v}^{-1} \mathbf{w}\phi \mathbf{v}$  for  $\mathbf{v}^{-1} \mathbf{w}^\phi \mathbf{v}\phi$ .

Note that when  $\mathbf{v}$ ,  $\mathbf{w}$  and  $\mathbf{v}^{-1} \mathbf{w}^\phi \mathbf{v}$  are in  $\mathbf{W}$  the endomorphism  $D_{\mathbf{v}}$  maps  $g \mathbf{P}_I \in \mathbf{X}(I, \mathbf{w}\phi)$  to  $g' \mathbf{P}_J \in \mathbf{X}(J, \mathbf{v}^{-1} \mathbf{w}\phi \mathbf{v})$  such that  $g^{-1} g' \in \mathbf{P}_I v \mathbf{P}_J$  and  $g'^{-1} {}^F g \in \mathbf{P}_J v^{-1} w^F \mathbf{P}_I$  and similarly for  $\tilde{D}_{\mathbf{v}}$ .

Note also that  $D_{\mathbf{v}}$  and  $\tilde{D}_{\mathbf{v}}$  are equivalences of étale sites; indeed, the proof of [DMR, 3.1.6] applies without change in our case.

The definition of  $\tilde{D}_{\mathbf{v}}$  and  $D_{\mathbf{v}}$  shows the following property:

**Lemma 8.24.** *The following diagram is commutative:*

$$\begin{array}{ccc} \tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) & \xrightarrow{\tilde{D}_{\mathbf{v}}} & \tilde{\mathbf{X}}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v}) \\ \downarrow & & \downarrow \\ \mathbf{X}(\mathbf{I}, \mathbf{w}\phi) & \xrightarrow{D_{\mathbf{v}}} & \mathbf{X}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v}) \end{array}$$

where the vertical arrows are the respective quotients by  $\mathbf{L}_I^{t(\mathbf{w})F}$  and  $\mathbf{L}_J^{t(\mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v})F}$  (see Lemma 8.20); for  $l \in \mathbf{L}_I^{t(\mathbf{w})F}$  we have  $\tilde{D}_{\mathbf{v}} \circ l = l^{t(\mathbf{v})} \circ \tilde{D}_{\mathbf{v}}$ .

**Definition 8.25.** We denote by  $\mathcal{D}^+(\mathcal{I})$  the category  $\phi$ -cyc  $B^+(\mathcal{I})$ ; that is the objects of  $\mathcal{D}^+(\mathcal{I})$  are the morphisms in  $B^+(\mathcal{I})$  of the form  $\mathbf{I} \xrightarrow{\mathbf{w}} \phi\mathbf{I}$  and the morphisms are generated by the “simple” morphisms that we will denote by  $\text{ad } \mathbf{v}$ , for  $\mathbf{v} \preccurlyeq \mathbf{w}$ ; such a morphism, more formally denoted by  $\mathbf{I} \xrightarrow{\text{ad } \mathbf{v}} \mathbf{J}$ , where  $\mathbf{J} = \mathbf{I}^{\mathbf{v}}$ , goes from  $\mathbf{I} \xrightarrow{\mathbf{w}} \phi\mathbf{I}$  to  $\mathbf{J} \xrightarrow{\mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v}} \phi\mathbf{J}$ . The relations are given by the equalities  $\text{ad } \mathbf{v}_1 \dots \text{ad } \mathbf{v}_k = \text{ad } \mathbf{v}'_1 \dots \text{ad } \mathbf{v}'_{k'}$  whenever  $\text{ad } \mathbf{v}_i$  are simple and  $\mathbf{v}_1 \dots \mathbf{v}_k = \mathbf{v}'_1 \dots \mathbf{v}'_{k'}$  in  $B^+$ .

If  $\mathbf{v} = \mathbf{v}_1 \dots \mathbf{v}_k \in B^+$  with the  $\text{ad } \mathbf{v}_i$  simple morphisms of  $\mathcal{D}^+(\mathcal{I})$ , we will still denote by  $\mathbf{I} \xrightarrow{\text{ad } \mathbf{v}} \mathbf{J}$  the composed morphism of  $\mathcal{D}^+(\mathcal{I})$ .

As a further consequence of Theorem 7.2, the map which sends a simple morphism  $\text{ad } \mathbf{v}$  to  $D_{\mathbf{v}}$  extends to a natural morphism of monoids  $\mathcal{D}^+(\mathcal{I})(\mathbf{I} \xrightarrow{\mathbf{w}} \phi\mathbf{I}) \rightarrow \text{End}_{\mathbf{G}^F}(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$ , whose image consists of equivalences of étale sites. We still denote by  $D_{\mathbf{v}}$  the image of  $\mathbf{v}$  by this morphism.

By Proposition 4.2 the category  $\mathcal{D}^+(\mathcal{I})$  has a Garside family consisting of the simple morphisms. Those of source  $\mathbf{I} \xrightarrow{\mathbf{w}} \phi\mathbf{I}$  correspond to the set of  $\mathbf{v} \preccurlyeq \mathbf{w}$  such that  $\mathbf{I}^{\mathbf{v}} \subset \mathbf{S}$ . For  $\mathbf{J} \subset \mathbf{I}$  we will denote by  $\mathcal{J}$  the set of  $B_{\mathbf{I}}^+$ -conjugates of  $\mathbf{J}$  and by  $\mathcal{D}_{\mathbf{I}}^+(\mathcal{J})$  the analogous category where  $B^+$  is replaced by  $B_{\mathbf{I}}^+$  and  $\mathcal{I}$  by  $\mathcal{J}$ .

**Proposition 8.26.** *With same assumptions and notation as in Proposition 8.22, let  $\mathbf{J} \xrightarrow{\mathbf{x}} \mathbf{J}^{\mathbf{x}} \in B_{\mathbf{I}}^+(\mathcal{J})$  be a left divisor of  $\mathbf{J} \xrightarrow{\mathbf{v}} \mathbf{v}\phi\mathbf{J}$ . The following diagram is commutative:*

$$\begin{array}{ccc} \tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \times_{\mathbf{L}_I^{\dot{\mathbf{w}}F}} \tilde{\mathbf{X}}_{\mathbf{L}_I}(\mathbf{J}, \mathbf{v} \cdot w\phi) & \xrightarrow{\sim} & \tilde{\mathbf{X}}(\mathbf{J}, \mathbf{v}\mathbf{w}\phi) \\ \text{Id} \times \tilde{D}_{\mathbf{x}} \downarrow & & \downarrow \tilde{D}_{\mathbf{x}} \\ \tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \times_{\mathbf{L}_I^{\dot{\mathbf{w}}F}} \tilde{\mathbf{X}}_{\mathbf{L}_I}(\mathbf{J}^{\mathbf{x}}, \mathbf{x}^{-1}(\mathbf{v} \cdot w\phi)\mathbf{x}) & \xrightarrow{\sim} & \tilde{\mathbf{X}}(\mathbf{J}^{\mathbf{x}}, \mathbf{x}^{-1}\mathbf{v}\mathbf{w}\phi\mathbf{x}) \end{array}$$

*Proof.* Decomposing  $\mathbf{x}$  into a product of simples in  $\mathcal{D}_{\mathbf{I}}^+(\mathcal{J})$  the definitions show that it is sufficient to prove the result for  $\mathbf{x} \in \mathbf{W}$ . We use then Proposition 8.21 to reduce the proof to the case where  $\mathbf{v}\mathbf{w}$  and  $\mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v}$  are in  $\mathbf{W}$  (in which case  $\mathbf{w}$  and  $\mathbf{v}^{-1}\mathbf{w}\phi\mathbf{w}$  are in  $\mathbf{W}$  too). We can make this reduction if we know that the isomorphism of Proposition 8.21 is compatible with the action of  $D_{\mathbf{x}}$  for  $\mathbf{x} \in \mathbf{W}$  (we will then use this fact in  $\mathbf{G}$  and in  $\mathbf{L}_I$ ). Take  $(\mathbf{I}, \mathbf{y}, \phi\mathbf{I}) \in B^+(\mathcal{I})$  and  $\mathbf{x} \in \mathbf{W}$  such that  $\mathbf{I} \xrightarrow{\mathbf{x}} \mathbf{I}^{\mathbf{x}}$  is a left divisor of  $\mathbf{I} \xrightarrow{\mathbf{y}} \phi\mathbf{I}$ . Let  $\mathbf{y} = \mathbf{y}_1 \dots \mathbf{y}_k$  be a decomposition of  $\mathbf{y}$  as a product of elements of  $\mathbf{W}$  such that  $\mathbf{x} = \mathbf{y}_1$ . The endomorphism  $D_{\mathbf{x}}$  maps the sequence

$(g_1 \mathbf{U}_1, \dots, g_k \mathbf{U}_k)$  such that  $g_i^{-1} g_{i+1} \in \mathbf{U}_i \dot{y}_i \mathbf{U}_{i+1}$  and  $g_k^{-1} {}^F g_1 \in \mathbf{U}_k \dot{y}_k {}^F \mathbf{U}_1$  to the sequence  $(g_2 \mathbf{U}_2, \dots, g_k \mathbf{U}_k, {}^F g_1 {}^F \mathbf{U}_1)$ . On the other hand, via the isomorphism of Proposition 8.21, using the decomposition  $(\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k, 1)$  of  $\mathbf{y}$ , the sequence  $(g_1 \mathbf{U}_1, \dots, g_k \mathbf{U}_k)$  corresponds to  $((g_1, \dots, g_k, {}^F g_1)(\mathbf{U}_1, \dots, \mathbf{U}_k, {}^F \mathbf{U}_1) \in \tilde{X}_{\mathbf{G}^{k+1}}(I_1 \times \dots \times I_k \times {}^F I_1, (\dot{y}_1, \dots, \dot{y}_k, 1) F_1)$ . This element is mapped by  $D_{(\dot{y}_1, \dots, \dot{y}_k, 1)}$  to the element  $(g_2, g_2, \dots, g_k, {}^F g_1)(\mathbf{U}_2, \mathbf{U}_2, \dots, \mathbf{U}_k, {}^F \mathbf{U}_1)$  which is in  $\tilde{X}_{\mathbf{G}^{k+1}}(I_2 \times I_2 \times I_3 \times \dots \times I_k \times {}^F I_1, (1, \dot{y}_2, \dots, \dot{y}_k, {}^F y_1) F_1)$ . Since this last element corresponds by the isomorphism of Proposition 8.21 to  $(g_2 \mathbf{U}_2, \dots, g_k \mathbf{U}_k, {}^F g_1 {}^F \mathbf{U}_1)$ , we have proved the compatibility we want.

Assume now  $\mathbf{v} \mathbf{w}$  and  $\mathbf{v}^{-1} \mathbf{w}^\phi \mathbf{v}$  in  $\mathbf{W}$ . We start with  $(g \mathbf{U}_I, l \mathbf{V}_J) \in \tilde{\mathbf{X}}(I, \dot{w} F) \times \tilde{\mathbf{X}}_{\mathbf{L}_I}(J, v w \phi)$ . This element is mapped by the top isomorphism of the diagram to  $g l \mathbf{U}_J$ . As we have seen above Lemma 8.24 it is mapped by  $\text{Id} \times D_{\mathbf{x}}$  to  $(g \mathbf{U}_I, l' \mathbf{V}_{J^x})$  where  $l'^{-1} l' \in \mathbf{V}_{J^x} \mathbf{V}_{J^x}$  and  $l'^{-1} \dot{w} F l' \in \mathbf{V}_{J^x} x^{-1} v w {}^F \mathbf{V}_J$ . This element is mapped to  $g l' \mathbf{U}_{J^x}$  by the bottom isomorphism of the diagram. We have to check that  $g l' \mathbf{U}_{J^x} = D_{\mathbf{x}}(g l \mathbf{U}_J)$ . But  $(g l)^{-1} g l' = l'^{-1} l'$  is in  $\mathbf{V}_{J^x} \mathbf{V}_{J^x} \subset \mathbf{U}_{J^x} \mathbf{U}_{J^x}$  and

$$(g l)^{-1} F(g l) = l'^{-1} g^{-1} F g l \in l'^{-1} \mathbf{U}_I \dot{w} {}^F \mathbf{U}_I F l = \mathbf{U}_I l'^{-1} \dot{w} F l \dot{w} {}^F \mathbf{U}_I \\ \subset \mathbf{U}_I \mathbf{V}_{J^x} x^{-1} v w {}^F \mathbf{V}_J {}^F \mathbf{U}_I = \mathbf{U}_{J^x} x^{-1} v w {}^F \mathbf{U}_J,$$

so that  $(g l' \mathbf{U}_{J^x}) = D_{\mathbf{x}}(g l \mathbf{U}_J)$ .  $\square$

Using Proposition 8.22(ii) and Lemma 8.24 we get

**Corollary 8.27.** *The following diagram is commutative:*

$$\begin{array}{ccc} \tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w} \phi) \times_{\mathbf{L}_I^{\dot{w} F}} \mathbf{X}_{\mathbf{L}_I}(\mathbf{J}, \mathbf{v} \cdot w \phi) & \xrightarrow{\sim} & \mathbf{X}(\mathbf{J}, \mathbf{v} \mathbf{w} \phi) \\ \text{Id} \times D_{\mathbf{x}} \downarrow & & \downarrow D_{\mathbf{x}} \\ \tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w} \phi) \times_{\mathbf{L}_I^{\dot{w} F}} \mathbf{X}_{\mathbf{L}_I}(\mathbf{J}^x, \mathbf{x}^{-1}(\mathbf{v} \cdot w \phi) \mathbf{x}) & \xrightarrow{\sim} & \mathbf{X}(\mathbf{J}^x, \mathbf{x}^{-1} \mathbf{v} \mathbf{w} \phi \mathbf{x}) \end{array}$$

We now give a general case where we can describe  $\mathcal{D}^+(\mathcal{I})(\mathbf{I} \xrightarrow{\mathbf{w}} \phi \mathbf{I})$ .

**Theorem 8.28.** *Assume that some power of  $\mathbf{w} \phi$  is divisible on the left by  $\mathbf{w}_{\mathbf{I}}^{-1} \mathbf{w}_0$ . Then  $\mathcal{D}^+(\mathcal{I})(\mathbf{I} \xrightarrow{\mathbf{w}} \phi \mathbf{I})$  consists of the morphisms  $\mathbf{I} \xrightarrow{\text{ad } \mathbf{b}} \mathbf{I}$  where  $\mathbf{b}$  runs over the submonoid  $B_{\mathbf{w}}^+ = \{\mathbf{b} \in C_{B^+}(\mathbf{w} \phi) \mid \mathbf{I}^{\mathbf{b}} = \mathbf{I} \text{ and } \alpha_{\mathbf{I}}(\mathbf{b}) = 1\}$ .*

*Proof.* This is an immediate translation of Proposition 4.5, since the Garside map of  $B^+(\mathcal{I})$  is  $\mathbf{I} \xrightarrow{\mathbf{w}_{\mathbf{I}}^{-1} \mathbf{w}_0} \mathbf{I}^{\mathbf{w}_0}$ ; the submonoid  $B_{\mathbf{w}}^+$  is the centralizer of the morphism  $\mathbf{I} \xrightarrow{\mathbf{w}} \phi \mathbf{I}$  of  $B^+(\mathcal{I})$ .  $\square$

Note that if  $k$  is the smallest power such that  $\phi^k \mathbf{I} = \mathbf{I}$  and  $\phi^k \mathbf{w} = \mathbf{w}$ , then  $\mathbf{w}^{(k)} := \mathbf{w} \phi \mathbf{w} \dots \phi^{k-1} \mathbf{w}$  is in  $B_{\mathbf{w}}^+$ . Since  $\mathbf{I} \xrightarrow{\text{ad } \mathbf{w}} \phi \mathbf{I}$  is the Garside map of  $\mathcal{D}^+(\mathcal{I})$  described in Proposition 4.3, it follows that under the assumptions of Theorem 8.28 every element of  $B_{\mathbf{w}}^+$  divides a power of  $\mathbf{w}^{(k)}$ . In particular, in the case  $\mathbf{I} = \emptyset$ , the group  $C_B(\mathbf{w} \phi)$  is generated as a monoid, with the notations of [DM2, 2.1], by  $\text{End}_{\mathcal{D}^+}(\mathbf{w})$  and  $(\mathbf{w}^{(k)})^{-1}$ . Thus Theorem 8.28 in this particular case gives a positive answer to conjecture [DM2, 2.1].

**Definition 8.29.** *We define  $\pi = \mathbf{w}_0^2$  (it is a generator of the center of the pure braid group) and similarly for  $I \subset S$  we define  $\pi_{\mathbf{I}} = \mathbf{w}_{\mathbf{I}}^2$ .*

As an example of Theorem 8.28 we get  $\mathcal{D}^+(\mathcal{I})(\mathbf{I} \xrightarrow{\pi/\pi_{\mathbf{I}}} {}^\phi \mathbf{I}) = B^+(\mathcal{I})(\mathbf{I})^\phi$ .

**Affineness.** Until the end of the text, we will consider varieties which satisfy the assumption of Theorem 8.28. They have many nice properties. We show in this subsection that they are affine, by adapting the proof of Bonnafé and Rouquier [BR2] to our case; we use the existence of the varieties  $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$  and  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{b}\phi)$  to replace doing a quotient by  $\mathbf{L}_I$  by doing a quotient by  $\mathbf{L}_I^F$ .

**Proposition 8.30.** *Assume the morphism  $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \in B^+(\mathcal{I})$  is left-divisible by  $\Delta_{\mathcal{I}}$ . Then the variety  $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$  is affine.*

*Proof.* By assumption there exists a decomposition into elements of  $\mathcal{S}$  of  $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$  of the form  $\mathbf{I} \xrightarrow{\mathbf{w}_I^{-1}\mathbf{w}_0} \mathbf{I}_1 \xrightarrow{\mathbf{v}_1} \mathbf{I}_2 \xrightarrow{\mathbf{v}_2} \mathbf{I}_3 \rightarrow \dots \rightarrow \mathbf{I}_r \xrightarrow{\mathbf{v}_r} \mathbf{J}$ . We show that the map  $\varphi$  defined by:

$$\begin{aligned} \mathbf{G} \times \prod_{i=1}^{i=r} (\mathbf{U}_{I_i} \cap {}^{v_i} \mathbf{U}_{I_{i+1}}^-) \dot{v}_i &\rightarrow \\ \tilde{\mathcal{O}}(I, \dot{w}_I^{-1} \dot{w}_0) \times_{\mathbf{G}/\mathbf{U}_{I_1}} \tilde{\mathcal{O}}(I_1, \dot{v}_1) \dots \times_{\mathbf{G}/\mathbf{U}_{I_r}} \tilde{\mathcal{O}}(I_r, \dot{v}_r) \\ (g, h_1, \dots, h_r) &\mapsto \\ (g\mathbf{U}_I, g\dot{w}_I^{-1} \dot{w}_0 \mathbf{U}_{I_1}, g\dot{w}_I^{-1} \dot{w}_0 h_1 \mathbf{U}_{I_2}, \dots, g\dot{w}_I^{-1} \dot{w}_0 h_1 \dots h_r \mathbf{U}_J) \end{aligned}$$

is an isomorphism; since the first variety is a product of affine varieties this will prove our claim.

Since  $\mathbf{U}_{I_i} \dot{v}_i \mathbf{U}_{I_{i+1}}$  is isomorphic to  $(\mathbf{U}_{I_i} \cap {}^{v_i} \mathbf{U}_{I_{i+1}}^-) \dot{v}_i \times \mathbf{U}_{I_{i+1}}$ , by composition with the first projection we get a morphism  $\eta_i : \mathbf{U}_{I_i} \dot{v}_i \mathbf{U}_{I_{i+1}} \rightarrow (\mathbf{U}_{I_i} \cap {}^{v_i} \mathbf{U}_{I_{i+1}}^-) \dot{v}_i$  for  $i = 1, \dots, r$ , where  $I_{r+1} = J$ . For  $x = (g\mathbf{U}_I, g_1 \mathbf{U}_{I_1}, g_2 \mathbf{U}_{I_2}, \dots, g_r \mathbf{U}_{I_r}, g_{r+1} \mathbf{U}_J)$  in  $\tilde{\mathcal{O}}(I, \dot{w}_I^{-1} \dot{w}_0) \times_{\mathbf{G}/\mathbf{U}_{I_1}} \tilde{\mathcal{O}}(I_1, \dot{v}_1) \dots \times_{\mathbf{G}/\mathbf{U}_{I_r}} \tilde{\mathcal{O}}(I_r, \dot{v}_r)$  we put  $\psi(x) = g\eta(g^{-1}g_1)$ ,  $\psi_1(x) = \psi(x)\dot{w}_0$ ,  $\psi_i(x) = \eta_i((\psi(x)\psi_1(x) \dots \psi_{i-1}(x))^{-1}g_i)$ . We claim that the maps  $\psi$  (resp.  $\psi_i$ ) are well defined, that is do not depend on the representative  $g$  (resp.  $g_i$ ) chosen; the morphism  $x \mapsto (\psi(x), \psi_1(x), \dots, \psi_r(x))$  is then clearly inverse to  $\varphi$ . Since  $\eta_i(hu) = \eta_i(h)$  for all  $h \in \mathbf{U}_{I_i} \dot{v}_i \mathbf{U}_{I_{i+1}}$  and all  $u \in \mathbf{U}_{I_{i+1}}$ , we get that all  $\psi_i$  are well-defined. Since moreover  $\eta(uh) = u\eta(h)$  for all  $h \in \mathbf{U}_I \dot{w}_I^{-1} \dot{w}_0 \mathbf{U}_{I_1}$  and all  $u \in \mathbf{U}_I$ , we get that  $\psi$  also is well-defined, whence our claim.  $\square$

**Proposition 8.31.** *Assume that we are under the assumptions of Theorem 8.28, that is  $(\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi \mathbf{I}) \in B^+(\mathcal{I})$  has some power divisible by  $\Delta_{\mathcal{I}}$ , or equivalently some power of  $\mathbf{w}\phi$  is divisible on the left by  $\mathbf{w}_I^{-1}\mathbf{w}_0$ . Assume further that the Tits homomorphism  $t$  has been chosen  $F$ -equivariant. Then  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$  is affine.*

*Proof.* Let us define  $k$  as the smallest integer such that  ${}^{\phi^k} \mathbf{I} = \mathbf{I}$ ,  ${}^{\phi^k} \mathbf{w} = \mathbf{w}$  and  $\mathbf{w}_I^{-1}\mathbf{w}_0 \preceq \mathbf{w}^{(k)}$ , where  $\mathbf{w}^{(k)} := \mathbf{w}^\phi \mathbf{w} \dots {}^{\phi^{k-1}} \mathbf{w}$ .

We will embed  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$  as a closed subvariety in  $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{w}^{(k)})$ , which will prove it to be affine.

Let  $\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \xrightarrow{\mathbf{w}_2} \mathbf{I}_3 \rightarrow \cdots \rightarrow \mathbf{I}_r \xrightarrow{\mathbf{w}_r} {}^\phi \mathbf{I}$  be a decomposition of  $\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi \mathbf{I}$  into elements of  $\mathcal{S}$ , so that  $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{w}^{(k)})$  identifies to the set of sequences

$$\begin{aligned} & (g_{1,1} \mathbf{U}_I, g_{1,2} \mathbf{U}_{I_2}, \dots, g_{1,r} \mathbf{U}_{I_r}, \\ & g_{2,1} \mathbf{U}_{\phi I}, g_{2,2} \mathbf{U}_{\phi I_2}, \dots, g_{2,r} \mathbf{U}_{\phi I_r}, \\ & \dots, \\ & g_{k,1} \mathbf{U}_{\phi^{k-1} I}, g_{k,2} \mathbf{U}_{\phi^{k-1} I_2}, \dots, g_{k,r} \mathbf{U}_{\phi^{k-1} I_r}, \\ & g_{k+1,1} \mathbf{U}_I) \end{aligned}$$

such that for  $j < r$  we have  $g_{i,j}^{-1} g_{i,j+1} \in \mathbf{U}_{\phi^{i-1} I_j} {}^{F^{i-1}} \dot{w}_j \mathbf{U}_{\phi^{i-1} I_{j+1}}$  and  $g_{i,r}^{-1} g_{i+1,1} \in \mathbf{U}_{\phi^{i-1} I_r} {}^{F^{i-1}} \dot{w}_r \mathbf{U}_{\phi^i I}$ ; note that we have used the  $F$ -equivariance of  $t$  to write  ${}^{F^i} \dot{w}_j$  for  $t({}^{\phi^i} \mathbf{w}_j)$ .

Similarly  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$  identifies to the set of sequences  $(g_1 \mathbf{U}_I, g_2 \mathbf{U}_{I_2}, \dots, g_r \mathbf{U}_{I_r})$  such that  $g_j^{-1} g_{j+1} \in \mathbf{U}_{I_j} \dot{w}_j \mathbf{U}_{I_{j+1}}$  for  $j < r$  and  $g_r^{-1} {}^F g_1 \in \mathbf{U}_{I_r} \dot{w}_r \mathbf{U}_{\phi I}$ . It is thus clear that the map

$$\begin{aligned} (g_1 \mathbf{U}_I, g_2 \mathbf{U}_{I_2}, \dots, g_r \mathbf{U}_{I_r}) \mapsto & (g_1 \mathbf{U}_I, g_2 \mathbf{U}_{I_2}, \dots, g_r \mathbf{U}_{I_r}, \\ & {}^F g_1 \mathbf{U}_{\phi I}, {}^F g_2 \mathbf{U}_{\phi I_2}, \dots, {}^F g_r \mathbf{U}_{\phi I_r}, \\ & \dots, \\ & {}^{F^{k-1}} g_1 \mathbf{U}_{\phi^{k-1} I}, \dots, {}^{F^{k-1}} g_r \mathbf{U}_{\phi^{k-1} I_r}, {}^F g_1 \mathbf{U}_I) \end{aligned}$$

identifies  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$  to the closed subvariety of  $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{w}^{(k)})$  defined by  $g_{i+1,j} \mathbf{U}_{\phi^i I_j} = {}^F(g_{i,j} \mathbf{U}_{\phi^{i-1} I_j})$  for all  $i, j$ .  $\square$

**Corollary 8.32.** *Under the assumptions of Theorem 8.28, that is  $(\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi \mathbf{I}) \in B^+(\mathcal{I})$  has some power divisible by  $\Delta_{\mathcal{I}}$ , or equivalently some power of  $\mathbf{w}\phi$  is divisible on the left by  $\mathbf{w}_{\mathbf{I}}^{-1} \mathbf{w}_0$ , the variety  $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$  is affine.*

*Proof.* Indeed, by Proposition 8.31 and Lemma 8.20, it is the quotient of an affine variety by a finite group, so is affine.  $\square$

**Shintani descent identity.** In this subsection we give a formula for the Lefschetz number of a variety  $\mathbf{X}(\mathbf{I}, \mathbf{w}F)$  which we deduce from a “Shintani descent identity”.

Let  $m$  be a multiple of  $\delta$  and let  $e_{\mathbf{B}} = |\mathbf{B}^{F^m}|^{-1} \sum_{b \in \mathbf{B}^{F^m}} b$ ; the  $\mathbf{G}^{F^m}$ -module  $\overline{\mathbb{Q}}_{\ell}[(\mathbf{G}/\mathbf{B})^{F^m}]$  identifies with  $\overline{\mathbb{Q}}_{\ell}[\mathbf{G}^{F^m}]e_{\mathbf{B}}$ . Its endomorphism algebra  $\mathcal{H}_{q^m}(W) := \text{End}_{\mathbf{G}^{F^m}}(\overline{\mathbb{Q}}_{\ell}[(\mathbf{G}/\mathbf{B})^{F^m}])$  identifies with  $e_{\mathbf{B}} \overline{\mathbb{Q}}_{\ell}[\mathbf{G}^{F^m}]e_{\mathbf{B}}$  acting by right multiplication. It has a basis consisting of the operators  $T_w = |\mathbf{B}^{F^m} \cap {}^w \mathbf{B}^{F^m}| \sum_{g \in \mathbf{B}^{F^m} w \mathbf{B}^{F^m}} g = e_{\mathbf{B}} w e_{\mathbf{B}}$  for  $w \in W$ , since  $W$  is a set of representatives of  $\mathbf{B}^{F^m} \backslash \mathbf{G}/\mathbf{B}^{F^m}$  (see [Bou] IV, §2 exercice 22). If we identify  $\mathbf{G}/\mathbf{B}$  to the variety  $\mathcal{B}$  of Borel subgroups of  $\mathbf{G}$ , the operator  $T_w$  becomes

$$T_w : \mathbf{B}' \mapsto \sum_{\{\mathbf{B}'' \in \mathcal{B}^{F^m} \mid \mathbf{B}'' \xrightarrow{w} \mathbf{B}'\}} \mathbf{B}''.$$

Similarly the algebra  $\mathcal{H}_{q^m}(W, W_I) := \text{End}_{\mathbf{G}^{F^m}}(\overline{\mathbb{Q}}_{\ell}[(\mathbf{G}/\mathbf{P}_I)^{F^m}])$  has a  $\overline{\mathbb{Q}}_{\ell}$ -basis consisting of the operators  $X_w = |\mathbf{P}_I^{F^m} \cap {}^w \mathbf{P}_I^{F^m}| \sum_{g \in \mathbf{P}_I^{F^m} w \mathbf{P}_I^{F^m}} g = e_{\mathbf{P}_I} w e_{\mathbf{P}_I}$  where  $e_{\mathbf{P}_I} = |\mathbf{P}_I^{F^m}|^{-1} \sum_{p \in \mathbf{P}_I^{F^m}} p$  and  $w$  runs over a set of representatives of the double

cosets  $\mathbf{P}_I^{F^m} \backslash \mathbf{G}^{F^m} / \mathbf{P}_I^{F^m} \simeq W_I \backslash W / W_I$ . Identifying  $\mathbf{G}/\mathbf{P}_I$  to the variety  $\mathcal{P}_I$  of the parabolic subgroups  $\mathbf{G}$ -conjugate to  $\mathbf{P}_I$  we have

$$X_w : \mathbf{P} \mapsto \sum_{\substack{\mathbf{P}' \in \mathcal{P}_I^{F^m} | \mathbf{P}' \xrightarrow{I, w, I} \mathbf{P}}} \mathbf{P}',$$

The multiplication by the idempotent  $X_1 = e_{\mathbf{P}_I} = \sum_{v \in W_I} |\mathbf{B}^{F^m} \cap {}^v \mathbf{B}^{F^m}|^{-1} T_v$  makes  $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]$  into a direct factor of  $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{B})^{F^m}]$  and the equality  $X_w = X_1 T_w X_1$  is compatible with this inclusion. Note that this inclusion maps a parabolic  $\mathbf{P}$  conjugate to  $\mathbf{P}_I$  in  $\mathbf{G}^{F^m}$  to the sum of all  $F^m$ -stable Borel subgroups of  $\mathbf{P}$ .

We may define a  $\overline{\mathbb{Q}}_\ell$ -representation of  $B^+(\mathcal{I})(\mathbf{I})$  on  $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]$  by sending  $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{I}$  to the operator  $X_{\mathbf{w}} \in \mathcal{H}(W, W_I)$  defined by

$$X_{\mathbf{w}}(\mathbf{P}) = \sum_{\{x \in \mathcal{O}(\mathbf{I}, \mathbf{w})^{F^m} | p''(x) = \mathbf{P}\}} p'(x).$$

The operator  $X_{\mathbf{w}}$  identifies to  $X_1 T_{\mathbf{w}} X_1 = X_1 T_{\mathbf{w}}$ , the last equality since  $\mathbf{I}^{\mathbf{w}} = \mathbf{I}$ . When  $\mathbf{w} \in \mathbf{W}$ , with image  $w$  in  $W$ , the operators  $X_{\mathbf{w}}$  and  $X_w$  coincide. In the particular case where  $I = \emptyset$  we get an operator denoted by  $T_{\mathbf{w}}$ , defined for any  $\mathbf{w}$  in  $B^+$ .

Similarly, to  $(\mathbf{I} \xrightarrow{\mathbf{w}} \phi \mathbf{I}) \in B^+(\mathcal{I})$ , we associate an endomorphism  $X_{\mathbf{w}\phi}$  of  $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]$  by the formula

$$X_{\mathbf{w}\phi}(\mathbf{P}) = \sum_{\{x \in \mathcal{O}(\mathbf{I}, \mathbf{w})^{F^m} | p''(x) = F(\mathbf{P})\}} p'(x).$$

When  $\phi(I) = I$  we have  $X_{\mathbf{w}\phi} = X_{\mathbf{w}\phi}$ . In general we have  $X_{\mathbf{w}\phi} = X_1 T_{\mathbf{w}\phi} \phi$  on  $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]$  seen as a subspace of  $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{B})^{F^m}]$ : on the latter representation one can separate the action of  $F$ ; the operator  $F$  sends the submodule  $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]$  to  $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_{\phi(I)})^{F^m}]$  which is sent back to  $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]$  by  $X_1 T_{\mathbf{w}}$ . The endomorphism  $X_{\mathbf{w}\phi}$  commutes with  $\mathbf{G}^{F^m}$  like  $F$ , hence normalizes  $\mathcal{H}_{q^m}(W, W_I)$ ; its action identifies to the conjugation action of  $T_{\mathbf{w}\phi}$  on  $\mathcal{H}_{q^m}(W, W_I)$  inside  $\mathcal{H}_{q^m}(W) \rtimes \langle \phi \rangle$ .

Recall that the Shintani descent  $\text{Sh}_{F^m/F}$  is the “norm” map which maps the  $F$ -class of  $g' = h \cdot {}^F h^{-1} \in \mathbf{G}^{F^m}$  to the class of  $g = h^{-1} \cdot {}^{F^m} h \in \mathbf{G}^F$ .

**Proposition 8.33** (Shintani descent identity). *Let  $\mathbf{I} \xrightarrow{\mathbf{w}} \phi \mathbf{I}$  be a morphism of  $B^+(\mathcal{I})$ , and let  $m$  be a multiple of  $\delta$ . Then*

$$(g \mapsto |\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m}|) = \text{Sh}_{F^m/F}(g' \mapsto \text{Trace}(g' X_{\mathbf{w}\phi} | \overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}])).$$

*Proof.* Let  $g = h^{-1} \cdot {}^{F^m} h$  and  $g' = h \cdot {}^F h^{-1}$ , so that the class of  $g$  is  $\text{Sh}_{F^m/F}$  of the  $F$ -class of  $g'$ ; we have  $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m} = \{x \in \mathcal{O}(\mathbf{I}, \mathbf{w}) \mid {}^{F^m} h x = {}^h x \text{ and } p''({}^h x) = {}^{g'F} p'({}^h x)\}$ . Taking  ${}^h x$  as a variable in the last formula we get  $|\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m}| = |\{x \in \mathcal{O}(\mathbf{I}, \mathbf{w})^{F^m} \mid p''(x) = {}^{g'F} p'(x)\}|$ . Putting  $\mathbf{P} = p'(x)$  this last number becomes  $\sum_{\mathbf{P} \in \mathcal{P}_I^{F^m}} |\{x \in \mathcal{O}(\mathbf{I}, \mathbf{w})^{F^m} \mid p'(x) = \mathbf{P} \text{ and } p''(x) = {}^{g'F} \mathbf{P}\}|$ . On the other hand the trace of  $g' X_{\mathbf{w}\phi}$  is the sum over  $\mathbf{P} \in \mathcal{P}_I^{F^m}$  of the coefficient of  $\mathbf{P}$  in  $\sum_{\{x \in \mathcal{O}(\mathbf{I}, \mathbf{w})^{F^m} \mid p''(x) = F(\mathbf{P})\}} g' p'(x)$ . This coefficient is equal to  $|\{x \in \mathcal{O}(\mathbf{I}, \mathbf{w})^{F^m} \mid g' p'(x) = \mathbf{P} \text{ and } p''(x) = F(\mathbf{P})\}| = |\{x \in \mathcal{O}(\mathbf{I}, \mathbf{w})^{F^m} \mid p'(x) = \mathbf{P} \text{ and } p''(x) = {}^{g'F} \mathbf{P}\}|$ , this last equality by changing  ${}^h x$  into  $x$ .  $\square$

By, for example, [DM1, II, 3.1] the algebras  $\mathcal{H}_{q^m}(W)$  and  $\mathcal{H}_{q^m}(W) \rtimes \langle \phi \rangle$  split over  $\overline{\mathbb{Q}}_\ell[q^{m/2}]$ ; corresponding to the specialization  $q^{m/2} \mapsto 1 : \mathcal{H}_{q^m}(W) \rightarrow \overline{\mathbb{Q}}_\ell W$ ,

there is a bijection  $\chi \mapsto \chi_{q^m} : \text{Irr}(W) \rightarrow \text{Irr}(\mathcal{H}_{q^m}(W))$ . Choosing an extension  $\tilde{\chi}$  to  $W \rtimes \langle \phi \rangle$  of each character in  $\text{Irr}(W)^\phi$ , we get a corresponding extension  $\tilde{\chi}_{q^m} \in \text{Irr}(\mathcal{H}_{q^m}(W) \rtimes \langle \phi \rangle)$  which takes its values in  $\overline{\mathbb{Q}}_\ell[q^{m/2}]$ . If  $U_\chi \in \text{Irr}(\mathbf{G}^{F^m})$  is the corresponding character of  $\mathbf{G}^{F^m}$ , we get a corresponding extension  $U_{\tilde{\chi}}$  of  $U_\chi$  to  $\mathbf{G}^{F^m} \rtimes \langle F \rangle$  (see [DM1, III théorème 1.3]). With these notations, the Shintani descent identity becomes

**Proposition 8.34.**

$$(g \mapsto |\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m}|) = \sum_{\chi \in \text{Irr}(W)^\phi} \tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}} \phi) \text{Sh}_{F^m/F} U_{\tilde{\chi}}$$

and the only characters  $\chi$  in that sum which give a non-zero contribution are those which are a component of  $\text{Ind}_{W_I}^W \text{Id}$ .

*Proof.* We have  $\text{Trace}(g' X_{\mathbf{w}\phi} \mid \overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]) = \text{Trace}(g' X_1 T_{\mathbf{w}} \phi \mid \overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{B})^{F^m}])$  since  $X_1$  is the projector onto  $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]$ . Hence  $(g \mapsto |\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m}|) = \sum_{\chi \in \text{Irr}(W)^\phi} \tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}} \phi) \text{Sh}_{F^m/F} U_{\tilde{\chi}}$ . Since  $X_1$  acts by 0 on the representation of character  $\chi$  if  $\chi$  is not a component of  $\text{Ind}_{W_I}^W \text{Id}$ , we get the second assertion.  $\square$

Finally, if  $\lambda_\rho$  is the root of unity attached to  $\rho \in \mathcal{E}(\mathbf{G}^F, 1)$  as in [DMR, 3.3.4], the above formula translates, using [DM1, III, 2.3(ii)] as

**Corollary 8.35.**

$$|\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m}| = \sum_{\rho \in \mathcal{E}(\mathbf{G}^F, 1)} \lambda_\rho^{m/\delta} \rho(g) \sum_{\chi \in \text{Irr}(W)^\phi} \tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}} \phi) \langle \rho, R_{\tilde{\chi}} \rangle_{\mathbf{G}^F},$$

where  $R_{\tilde{\chi}} = |W|^{-1} \sum_{w \in W} \tilde{\chi}(w\phi) R_{\mathbf{F}_w}^{\mathbf{G}}(\text{Id})$ . The only characters  $\chi$  in the above sum which give a non-zero contribution are those which are a component of  $\text{Ind}_{W_I}^W \text{Id}$ .

Using the Lefschetz formula and taking the “limit for  $m \rightarrow 0$ ” (see for example [DMR, 3.3.8]) we get the equality of virtual characters

**Corollary 8.36.**

$$\sum_i (-1)^i H_c^i(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell) = \sum_{\{\chi \in \text{Irr}(W)^\phi \mid \langle \text{Res}_{W_I}^W \chi, \text{Id} \rangle_{W_I} \neq 0\}} \tilde{\chi}(x_1 w\phi) R_{\tilde{\chi}},$$

where  $w$  is the image of  $\mathbf{w}$  in  $W$  and  $x_1 = |W_I|^{-1} \sum_{v \in W_I} v$ .

**Cohomology.** If  $\pi$  is the projection of Lemma 8.20, the sheaf  $\pi_! \overline{\mathbb{Q}}_\ell$  decomposes into a direct sum of sheaves indexed by the irreducible characters of  $\mathbf{L}_{\mathbf{I}}^{t(\mathbf{w})F}$ . We will denote by  $\mathbf{St}$  the subsheaf indexed by the Steinberg character of  $\mathbf{L}_{\mathbf{I}}^{t(\mathbf{w})F}$ .

In the particular case where  $\mathbf{I} = \emptyset$  we write  $\mathbf{X}(\mathbf{w}\phi)$  for  $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$ . Quite a few theorems are known about the  $\ell$ -adic cohomology of these varieties (see [DMR]). The following corollary of Proposition 8.22 relates the cohomology of a general variety to this particular case; its part (ii) is a refinement of Corollary 8.36.

**Corollary 8.37.** Let  $\mathbf{I} \xrightarrow{\mathbf{w}} \phi \mathbf{I} \in B^+(\mathcal{I})$ .

(i) For all  $\mathbf{v} \in B_{\mathbf{I}}^+$  and all  $i$  we have the following inclusions of  $\mathbf{G}^F \rtimes \langle F^\delta \rangle$ -modules:

$$H_c^i(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell) \subset H_c^{i+2l(\mathbf{v})}(\mathbf{X}(\mathbf{v}\mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)(-l(\mathbf{v}))$$

and

$$H_c^i(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \mathbf{St}) \subset H_c^{i+l(\mathbf{v})}(\mathbf{X}(\mathbf{v}\mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)$$

(ii) For all  $i$  we have the following equality of  $\mathbf{G}^F \rtimes \langle F^\delta \rangle$ -modules:

$$H_c^i(\mathbf{X}(\underline{\mathbf{w}}_I \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell) = \sum_{j+2k=i} H_c^j(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell) \otimes \overline{\mathbb{Q}}_\ell^{n_{I,k}}(k)$$

where  $n_{I,k} = |\{v \in W_I \mid l(v) = k\}|$ , where  $w_I$  is the longest element of  $W_I$  and the variety  $\mathbf{X}(\underline{\mathbf{w}}_I \mathbf{w}\phi)$  is the union  $\bigcup_{\mathbf{v} \in \mathbf{W}_I} \mathbf{X}(\mathbf{v}\mathbf{w}\phi)$  as defined in [DMR, 2.3.2].

*Proof.* For getting (i), we apply the Künneth formula to the isomorphism of Proposition 8.22 when  $\mathbf{J} = \emptyset$ . If we decompose the equality given by the Künneth formula according to the characters of  $\mathbf{L}_I^{t(\mathbf{w})F}$ , we get

$$\bigoplus_{\chi \in \text{Irr}(\mathbf{L}_I^{t(\mathbf{w})F})} \bigoplus_j H_c^{i-j}(\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)_\chi \otimes_{\mathbf{L}_I^{t(\mathbf{w})F}} H_c^j(\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}\mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)_{\overline{\chi}} \simeq H_c^i(\mathbf{X}(\mathbf{v}\mathbf{w}\phi), \overline{\mathbb{Q}}_\ell).$$

We now use that  $H_c^i(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell) = H_c^i(\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)_{\text{Id}}$ , and  $H_c^i(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \mathbf{St}) = H_c^i(\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)_{\text{St}}$  where  $\text{Id}$  and  $\text{St}$  denote the identity and Steinberg characters of  $\mathbf{L}_I^{t(\mathbf{w})F}$ , and the facts that

- the only  $j$  such that  $H_c^j(\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}\mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)_{\text{Id}}$  is non-trivial is  $j = 2l(\mathbf{v})$  and in that case the cohomology group has dimension 1 and  $t(\mathbf{w}F)$  acts by  $q^{l(\mathbf{v})}$  (see [DMR, 3.3.14]).
- the only  $j$  such that  $H_c^j(\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}\mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)_{\text{St}}$  is non-trivial is  $j = l(\mathbf{v})$  and that isotypic component is of multiplicity one, with trivial action of  $t(\mathbf{w}F)$  (see [DMR, 3.3.15]).

Hence we have

$$\bigoplus_j H_c^{i-j}(\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)_{\text{Id}} \otimes H_c^j(\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}\mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)_{\text{Id}} = H_c^{i-2l(\mathbf{v})}(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)(l(\mathbf{v})),$$

and similarly

$$\bigoplus_j H_c^{i-j}(\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)_{\text{St}} \otimes H_c^j(\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}\mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)_{\text{St}} = H_c^{i-l(\mathbf{v})}(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell).$$

We now prove (ii). Let  $\mathcal{B}_I$  be the variety of Borel subgroups of  $\mathbf{L}_I$ , identified to  $\mathbf{L}_I/\mathbf{B}_I$ . We first prove that we have an isomorphism  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \times_{\mathbf{L}_I^{w\phi}} \mathcal{B}_I \xrightarrow{\sim} \mathbf{X}(\underline{\mathbf{w}}_I \mathbf{w}\phi)$ . The variety  $\mathbf{X}(\underline{\mathbf{w}}_I \mathbf{w}\phi)$  is the union  $\bigcup_{\mathbf{v} \in \mathbf{W}_I} \mathbf{X}(\mathbf{v}\mathbf{w}\phi)$ . The variety  $\mathcal{B}_I$  is the union of the varieties  $\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}\mathbf{w}\phi)$  when  $\mathbf{v}$  runs over  $\mathbf{W}_I$ . The isomorphisms given by Proposition 8.22 when  $\mathbf{J} = \emptyset$  and  $\mathbf{v}$  running over  $\mathbf{W}_I$  can be glued together since they are defined by a formula independent of  $\mathbf{v}$ . We thus get a bijective morphism  $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \times_{\mathbf{L}_I^{w\phi}} \mathcal{B}_I \rightarrow \mathbf{X}(\underline{\mathbf{w}}_I \mathbf{w}\phi)$  which is an isomorphism since  $\mathbf{X}(\underline{\mathbf{w}}_I \mathbf{w}\phi)$  is normal (see [DMR, 2.3.5]). We now get (ii) from the fact that  $H_c^k(\mathcal{B}_I, \overline{\mathbb{Q}}_\ell)$  is 0 if  $k$  is odd and if  $k = 2k'$  is a trivial  $\mathbf{L}_I^{t(\mathbf{w})F}$ -module of dimension  $n_{I,k'}$ , where  $F$  acts by the scalar  $q^{k'}$ ; this results for example from the cellular decomposition into affine spaces given by the Bruhat decomposition and the fact that the action of  $\mathbf{L}_I^{t(\mathbf{w})F}$  extends to the connected group  $\mathbf{L}_I$ .  $\square$

**Corollary 8.38.** (i) The  $\mathbf{G}^F$ -module  $H_c^i(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)$  is unipotent. The eigenvalues of  $F^\delta$  on an irreducible  $\mathbf{G}^F$ -submodule  $\rho$  of  $H_c^i(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)$  are in  $q^{\delta\mathbb{N}} \lambda_\rho \omega_\rho$ , where  $\lambda_\rho$  is as in 8.35 and  $\omega_\rho$  is the element of  $\{1, q^{\delta/2}\}$  attached to  $\rho$  as in [DMR, 3.3.4]; they are both independent of  $i$  and  $\mathbf{w}$ .

(ii) We have  $H_c^i(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell) = 0$  unless  $l(\mathbf{w}) \leq i \leq 2l(\mathbf{w})$ .

- (iii) The eigenvalues of  $F^\delta$  on  $H_c^i(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)$  are of absolute value less than  $q^{\delta i/2}$ .
- (iv) The Steinberg representation does not occur in any cohomology group of  $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$  unless  $\mathbf{I} = \emptyset$  in which case it occurs with multiplicity 1 in  $H_c^{l(\mathbf{w})}(\mathbf{X}(\mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)$ , associated to the eigenvalue 1 of  $F^\delta$ .
- (v) The trivial representation occurs with multiplicity 1 in  $H_c^{2l(\mathbf{w})}(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)$ , associated to the eigenvalue  $q^{\delta l(\mathbf{w})}$  of  $F^\delta$ , and does not occur in any other cohomology group of  $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$ .

*Proof.* (i) is a straightforward consequence of Corollary 8.37(i) since the result is known for  $H_c^j(\mathbf{X}(\mathbf{v}\mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)$  (see [DMR, 3.3.4] and [DMR, 3.3.10 (i)]).

(ii) and (iii) are similarly a straightforward consequence of Corollary 8.37(i) applied with  $\mathbf{v} = 1$  and of [DMR, 3.3.22] and [DMR, 3.3.10(i)].

For (iv), we first note that by Corollary 8.37(i) applied with  $\mathbf{v} = 1$  and [DMR, 3.3.15] the Steinberg representation has multiplicity at most 1 in  $H_c^{l(\mathbf{w})}(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell)$ , associated to the eigenvalue 1 of  $F^\delta$ , and does not occur in any other cohomology group of  $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$ . To see when it does occur, it is enough then to use Proposition 8.34 and the Lefschetz formula. The only  $U_{\tilde{\chi}}$  such that the Steinberg representation has a non-zero scalar product with  $\mathrm{Sh}_{F^m/F} U_{\tilde{\chi}}$  is the Steinberg representation, and for the corresponding  $\tilde{\chi}$  we have

$$\tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}} \phi) = \begin{cases} (-1)^{l(\mathbf{w})} & \text{if } \mathbf{I} = \emptyset \\ 0 & \text{otherwise} \end{cases}.$$

(v) is similarly a consequence of Corollary 8.37(i), [DMR, 3.3.14], 8.34, the Lefschetz formula, and that if  $\tilde{\chi}_{q^m}$  corresponds to the trivial representation we have  $\tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}} \phi) = q^{ml(\mathbf{w})}$ .  $\square$

## 9. EIGENSPACES AND ROOTS OF $\pi/\pi_{\mathbf{I}}$

Let  $\ell \neq p$  be a prime such that the  $\ell$ -Sylow  $S$  of  $\mathbf{G}^F$  is abelian.

Then “generic block theory” (see [BMM]) associates to  $\ell$  a root of unity  $\zeta$  and some  $w\phi \in W\phi$  such that its  $\zeta$ -eigenspace in  $V$  in  $X := X_{\mathbb{R}} \otimes \mathbb{C}$  is non-zero and maximal among  $\zeta$ -eigenspaces of elements of  $W\phi$ ; for any such  $\zeta$ , there exists a unique minimal subtorus  $\mathbf{S}$  of  $\mathbf{T}$  such that  $V \subset X(\mathbf{S}) \otimes \mathbb{C}$ . If the coset  $W\phi$  is rational  $X(\mathbf{S}) \otimes \mathbb{C}$  is the kernel of  $\Phi(w\phi)$ , where  $\Phi$  is the  $d$ -th cyclotomic polynomial, if  $d$  is the order of  $\zeta$ . Otherwise, in the “very twisted” cases  ${}^2B_2$ ,  ${}^2F_4$  (resp.  ${}^2G_2$ ) we have to take for  $\Phi$  the irreducible cyclotomic polynomial over  $\mathbb{Q}(\sqrt{2})$  (resp.  $\mathbb{Q}(\sqrt{3})$ ) of which  $\zeta$  is a root. The torus  $\mathbf{S}$  is then called a  $\Phi$ -Sylow; we have  $|\mathbf{S}^F| = \Phi(q)^{\dim V}$ .

The relationship with  $\ell$  is that  $S$  is a subgroup of  $\mathbf{S}^F$ , and thus that  $|\mathbf{G}^F|/|\mathbf{S}^F|$  is prime to  $\ell$ ; we have  $N_{\mathbf{G}^F}(S) = N_{\mathbf{G}^F}(\mathbf{S}) = N_{\mathbf{G}^F}(\mathbf{L})$  where  $\mathbf{L} := C_{\mathbf{G}}(\mathbf{S})$  is a Levi subgroup of  $\mathbf{G}$  whose Weyl group is  $C_W(V)$ . Conversely, any maximal  $\zeta$ -eigenspace for any  $\zeta$  determines some primes  $\ell$  with abelian Sylow, those which divide  $\Phi(q)^{\dim V}$  and no other cyclotomic factor of  $|\mathbf{G}^F|$ .

The classes  $C_W(V)w\phi$ , where  $V = \mathrm{Ker}(w\phi - \zeta)$  is maximal, form a single orbit under  $W$ -conjugacy [see eg. [Br, 5.6(i)]]; the maximality implies that all elements of  $C_W(V)w\phi$  have same  $\zeta$ -eigenspace.

We will see in Theorem 9.1(i) that up to conjugacy we may assume that  $C_W(V)$  is a standard parabolic group  $W_I$ ; then the Broué conjectures predict that for an

appropriate choice of coset  $C_W(V)w\phi$  in its  $N_W(W_I)$ -conjugacy class the cohomology complex of the variety  $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$  should be a tilting complex realizing a derived equivalence between the unipotent parts of the  $\ell$ -principal blocks of  $\mathbf{G}^F$  and of  $N_{\mathbf{G}^F}(\mathbf{S})$ . We want to describe explicitly what should be a “good” choice of  $w$  (see Definition 9.12).

Since it is no more effort to have a result in the context of any finite real reflection group than for a context which includes the Ree and Suzuki groups, we give a more general statement.

In what follows we look at real reflection cosets  $W\phi$  of finite order, that is  $W$  is a finite reflection group acting on the real vector space  $X_{\mathbb{R}}$  and  $\phi$  is an element of  $N_{\mathrm{GL}(X_{\mathbb{R}})}(W)$ , such that  $W\phi$  is of finite order  $\delta$ , that is  $\delta$  is the smallest integer such that  $(W\phi)^{\delta} = W$  (equivalently  $\phi$  is of finite order). Since  $W$  is transitive on the chambers of the real hyperplane arrangement it determines, one can always choose  $\phi$  in its coset so that it preserves a chamber of this arrangement. Such elements  $\phi$  are the 1-regular elements of the coset (they have a fixed point outside the reflecting hyperplanes), thus are of order  $\delta$ .

**Theorem 9.1.** *Let  $W\phi \subset \mathrm{GL}(X_{\mathbb{R}})$  be a finite order real reflection coset, such that  $\phi$  preserves a chamber of the hyperplane arrangement on  $X_{\mathbb{R}}$  determined by  $W$ , thus induces an automorphism of the Coxeter system  $(W, S)$  determined by this chamber. We call again  $\phi$  the induced automorphism of the braid group  $B$  of  $W$ , and denote by  $\mathbf{S}, \mathbf{W}$  the lifts of  $S, W$  to  $B$  (see around Definition 8.2).*

Let  $\zeta_d = e^{2i\pi/d}$  and let  $V$  be a subspace of  $X := X_{\mathbb{R}} \otimes \mathbb{C}$  on which some element of  $W\phi$  acts by  $\zeta_d$ . Then we may choose  $V$  in its  $W$ -orbit such that:

- (i)  $C_W(V) = W_I$  for some  $I \subset S$ .
- (ii) If  $W_I w\phi$  is the  $W_I$ -coset of elements which act by  $\zeta_d$  on  $V$ , where  $w$  is  $I$ -reduced, then when  $d \neq 1$  we have  $l(w) = (2/d)l(w_0 w_I^{-1})$  and  $l((w\phi)^i \phi^{-i}) = il(w)$  if  $2i \leq d$ .

Further, when  $d \neq 1$  the lift  $\mathbf{w} \in \mathbf{W}$  of a  $w$  as in (ii) satisfies  ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$  and  $(\mathbf{w}\phi)^d = \phi^d \mathbf{\pi}/\mathbf{\pi_I}$ , where  $\mathbf{I} \subset \mathbf{S}$  is the lift of  $I$ .

Finally note that if  $d = 1$  then  $w = 1$  in (ii) and we may lift it to  $\mathbf{w} := \mathbf{\pi}/\mathbf{\pi_I}$  and we still have  ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$  and  $(\mathbf{w}\phi)^d = \mathbf{\pi}/\mathbf{\pi_I}\phi^d$ .

Note that in particular, for the  $w$  in (ii) we have  $(w\phi)^d = \phi^d$ .

*Proof.* Since  $W\langle\phi\rangle$  is finite, we may find a scalar product on  $X_{\mathbb{R}}$  (extending to an Hermitian product on  $X$ ) invariant by  $W$  and  $\phi$ . The subspace  $X'_{\mathbb{R}}$  of  $X_{\mathbb{R}}$  on which  $W$  acts non-trivially (the subspace spanned by the root lines of  $W$ ) identifies to the reflection representation of the Coxeter system  $(W, S)$  (see for example [Bou, chap. 5, §3]). We will use the root system  $\Phi$  on  $X'_{\mathbb{R}}$  consisting of the vectors of length 1 for this scalar product along the root lines of  $W$ , which is thus preserved by  $W\langle\phi\rangle$ . The strategy for the proof of (i) will be, rather than change  $V$ , to choose an order on  $\Phi$  such that the corresponding basis makes  $C_W(V)$  a standard parabolic subgroup of  $W$ .

Let  $v$  be a regular vector in  $V$ , that is  $v \in V$  such that  $C_W(v) = C_W(V)$ . Multiplying  $v$  if needed by a complex number of absolute value 1, we may assume that for any  $\alpha \in \Phi$  we have  $\Re\langle v, \alpha \rangle = 0$  if and only if  $\langle v, \alpha \rangle = 0$ . Then there exists an order on  $\Phi$  such that  $\Phi^+ \subset \{\alpha \in \Phi \mid \Re(\langle v, \alpha \rangle) \geq 0\}$ . Let  $\Pi$  be the corresponding basis and let  $I = \{\alpha \in \Pi \mid \Re(\langle v, \alpha \rangle) = 0\}$ . Then for  $\alpha \in \Phi$  we have  $\alpha \in \Phi_I$  if and only if  $\langle v, \alpha \rangle = 0$ , thus  $C_W(V) = C_W(v) = W_I$ . This proves (i).

We prove now (ii). The element  $w\phi$  sends  $v$  to  $\zeta_d v$ , thus preserves  $\Phi_I$ , and since we chose  $w$  to be  $I$ -reduced we have  ${}^{w\phi}I = I$ .

Note that  $(w\phi)^d = \phi^d$ . Indeed  $(w\phi)^d$  fixes  $v$ , thus preserves the sign of any root not in  $\Phi_I$ ; as  ${}^{w\phi}I = I$ , it also preserves the sign of roots in  $\Phi_I$ . It is thus equal to the only element  $\phi^d$  of  $W\phi^d$  which preserves the signs of all roots. We get also that  ${}^{\phi^d}I = I$ .

Since  $\langle v, {}^{(w\phi)^m}\alpha \rangle = \langle {}^{(w\phi)^{-m}}v, \alpha \rangle = \zeta_d^{-m} \langle v, \alpha \rangle$ , we get that all orbits of  $w\phi$  on  $\Phi - \Phi_I$  have cardinality a multiple of  $d$ ; it is thus possible by partitioning suitably those orbits, to get a partition of  $\Phi - \Phi_I$  in subsets  $\mathcal{O}$  of the form  $\{\alpha, {}^{w\phi}\alpha, \dots, {}^{(w\phi)^{d-1}}\alpha\}$ ; and the numbers  $\{\langle v, \beta \rangle \mid \beta \in \mathcal{O}\}$  for a given  $\mathcal{O}$  form the vertices of a regular  $d$ -gon centered at  $0 \in \mathbb{C}$ ; the action of  $w\phi$  is the rotation by  $-2\pi/d$  of this  $d$ -gon. Looking at the real parts of the vertices of this  $d$ -gon, we see that for  $m \leq d/2$ , exactly  $m$  positive roots in  $\mathcal{O}$  are sent to negative roots by  $(w\phi)^m$ . Since this holds for all  $\mathcal{O}$ , we get that for  $m \leq d/2$  we have  $l(\phi^{-m}(w\phi)^m) = \frac{m|\Phi - \Phi_I|}{d}$ ; thus if  $\mathbf{w}$  is the lift of  $w$  to  $\mathbf{W}$  we have  $(\mathbf{w}\phi)^i \in \mathbf{W}\phi^i$  if  $2i \leq d$ .

If  $d = 1$  since  $w\phi = \phi$  we have  $w = 1$  so we may lift it to  $\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$  as stated. Otherwise we finish with the following

**Lemma 9.2.** *Assume that  ${}^{w\phi}W_I = W_I$ , that  $w$  is  $I$ -reduced, that  $(w\phi)^d = \phi^d$  and that  $l((w\phi)^i\phi^{-i}) = (2i/d)l(w_0w_I^{-1})$  if  $2i \leq d$ . Then if  $\mathbf{w}$  is the lift of  $w$  to  $\mathbf{W}$  we have  ${}^{w\phi}\mathbf{I} = \mathbf{I}$  and if  $d \neq 1$  we have  $(\mathbf{w}\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$ .*

*Proof.* Since  $w$  is  $I$ -reduced and  $w\phi$  normalizes  $W_I$  we get that  $w\phi$  stabilizes  $I$ , which lifts to the braid group as  ${}^{w\phi}\mathbf{I} = \mathbf{I}$ .

Assume first  $d$  even and let  $d = 2d'$  and  $x = \phi^{-d'}(w\phi)^{d'}$ . Then  $l(x) = (1/2)l(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}) = l(w_0) - l(w_I)$  and since  $x$  is reduced- $I$  it is equal to the only reduced- $I$  element of that length which is  $w_0w_I^{-1}$ . Since the lengths add we can lift the equality  $(w\phi)^{d'} = \phi^{d'}w_0w_I^{-1}$  to the braid monoid as  $(\mathbf{w}\phi)^{d'} = \phi^{d'}\mathbf{w}_0\mathbf{w}_I^{-1}$ . By a similar reasoning using that  $(w\phi)^{d'}\phi^{-d'}$  is the unique  $I$ -reduced element of its length, we get also  $(\mathbf{w}\phi)^{d'} = \mathbf{w}_I^{-1}\mathbf{w}_0\phi^{d'}$ . Thus  $(\mathbf{w}\phi)^d = \mathbf{w}_I^{-1}\mathbf{w}_0\phi^{d'}\phi^{d'}\mathbf{w}_0\mathbf{w}_I^{-1} = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$ , where the last equality uses that  $\phi^d = (w\phi)^d$  preserves  $\mathbf{I}$ , whence the lemma in this case.

Assume now that  $d = 2d' + 1$ ; then  $(w\phi)^{d'}\phi^{-d'}$  is  $I$ -reduced and  $\phi^{-d'}(w\phi)^{d'}$  is reduced- $I$ . Using that any reduced- $\mathbf{I}$  element of  $\mathbf{W}$  is a right divisor of  $\mathbf{w}_0\mathbf{w}_I^{-1}$  (resp. any  $\mathbf{I}$ -reduced element of  $\mathbf{W}$  is a left divisor of  $\mathbf{w}_I^{-1}\mathbf{w}_0$ ), we get that there exists  $\mathbf{t}, \mathbf{u} \in \mathbf{W}$  such that  $\phi^{d'}\mathbf{w}_I^{-1}\mathbf{w}_0 = \mathbf{t}(\mathbf{w}\phi)^{d'}$  and  $\mathbf{w}_0\mathbf{w}_I^{-1}\phi^{d'} = (\mathbf{w}\phi)^{d'}\mathbf{u}$ . Thus  $\phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}} = \mathbf{w}_0\mathbf{w}_I^{-1}\phi^{d'}\mathbf{w}_I^{-1}\mathbf{w}_0 = (\mathbf{w}\phi)^{d'}\mathbf{u}\phi^{d'}(\mathbf{w}\phi)^{d'}$ , the first equality since  ${}^{\phi^d}I = I$ . The image in  $W\phi^d$  of the left-hand side is  $\phi^d$ , and  $(w\phi)^d = \phi^d$ . We deduce that the image in  $W\phi$  of  $\mathbf{u}\phi^{d'}$  is  $w\phi$ . If  $d \neq 1$  then  $d' \neq 0$  and we have  $l(\mathbf{u}) = l(\mathbf{t}) = l(\mathbf{w})/2$ ; thus  $\mathbf{u}\phi^{d'} = \mathbf{w}\phi$  and  $(\mathbf{w}\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$ .  $\square$

$\square$

Note that Theorem 9.1 only handles the case of eigenspaces for the eigenvalue  $\zeta_d$ , and not for another primitive  $d$ -th root of unity  $\zeta_d^k$ . However, note that if the coset  $W\phi$  preserves a  $\mathbb{Q}$ -structure on  $X_{\mathbb{R}}$  (which is the case for cosets associated to finite reductive groups, except for the “very twisted” cases  ${}^2B_2$ ,  ${}^2G_2$  and  ${}^2F_4$ ), then if  $\zeta_d^k$  is an eigenvalue of  $w\phi$ , the Galois conjugate  $\zeta_d$  is also an eigenvalue, for a Galois conjugate eigenspace. In general, since we assume  $W\phi$  real, we may assume  $2k \leq d$  since if  $\zeta_d^k$  is an eigenvalue of  $w\phi$  the complex conjugate  $\zeta_d^{d-k}$  is also an

eigenvalue, for the complex conjugate eigenspace. In this last case we may say the following (here we assume  $d \neq 1$ ):

**Corollary 9.3.** *In the situation of Theorem 9.1, let  $\zeta = \zeta_d^k$  with  $k$  prime to  $d$  and  $2k \leq d$ , and let  $V$  be a subspace of  $X$  on which some element of  $W\phi$  acts by  $\zeta$ . Then we may choose  $V$  in its  $W$ -orbit such that:*

- (i)  $C_W(V) = W_I$  for some  $I \subset S$ .
- (ii) If  $W_Iw\phi$  is the  $W_I$ -coset of elements which act by  $\zeta$  on  $V$ , and  $w$  is the unique  $I$ -reduced element of that coset, then  $l(w) = (2k/d)(l(w_0w_I^{-1}))$  and  $l((w\phi)^i\phi^{-i}) = il(w)$  if  $2ik \leq d$ .

Further, if  $\mathbf{w}$  is the lift of  $w$  as in (ii) to  $\mathbf{W}$  and  $\mathbf{I} \subset \mathbf{S}$  is the lift of  $I$ , then  ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$  and  $(\mathbf{w}\phi)^d = \phi^d(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^k$ .

*Proof.* The proof of (i) in Theorem 9.1 does not use that the eigenvalue is  $\zeta_d$ , so still applies. The beginning of the proof of (ii) also applies and proves that in the  $W$ -orbit we may choose  $w$  such that  $C_W(V) = W_I$ ,  $(w\phi)^d = \phi^d$  and  ${}^w\phi I = I$ .

Let  $d', k'$  be positive integers such that  $kk' = 1 + dd'$ , and let  $w_1\phi_1 = (w\phi)^{k'}$ , where  $\phi_1 = \phi^{k'}$ . Then  $w_1\phi_1$  acts on  $V$  by  $\zeta_d$ , so we may apply Theorem 9.1 to it. We have  $(w_1\phi_1)^k = (w\phi)^{kk'} = (w\phi)^{1+dd'} = (w\phi)(w\phi)^{dd'} = (w\phi)\phi^{dd'}$ , thus  $W_Iw\phi = (W_Iw_1\phi_1)^k\phi^{1-kk'}$ , thus  $(W_Iw\phi)^i\phi^{-i} = (W_Iw_1\phi_1)^{ki}\phi_1^{-ki}$ , whence (ii).

Finally, by Theorem 9.1 the lift  $\mathbf{w}_1$  of  $w_1$  to  $B$  satisfies  ${}^{\mathbf{w}_1\phi_1}\mathbf{I} = \mathbf{I}$  and  $(\mathbf{w}_1\phi_1)^d = \phi_1^d\boldsymbol{\pi}/\boldsymbol{\pi}_I$ , thus if we define  $\mathbf{w}$  by  $(\mathbf{w}_1\phi_1)^k = \mathbf{w}\phi^{1+dd'}$ , then  $\mathbf{w}$  is the lift of  $w$  and satisfies the last part of the corollary, using  ${}^{\phi^d}I = I$ .  $\square$

We give now a converse.

**Theorem 9.4.** *Let  $(W, S)$ ,  $\phi$ ,  $X_{\mathbb{R}}$ ,  $X$ ,  $\mathbf{S}$ ,  $B$ ,  $B^+$  be as in Theorem 9.1. For  $d \in \mathbb{N}$ , let  $\mathbf{w} \in B^+$  be such that  $(\mathbf{w}\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$  for some  $\phi^d$ -stable  $\mathbf{I} \subset \mathbf{S}$ . Then*

- (i)  ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$ .

*Denote by  $w$  and  $I$  the images in  $W$  of  $\mathbf{w}$  and  $\mathbf{I}$ , let  $\zeta_d = e^{2i\pi/d}$ , let  $V \subset X$  be the  $\zeta_d$ -eigenspace of  $w\phi$ , and let  $X^{W_I}$  be the fixed point space of  $W_I$ ; then*

- (ii)  $W_I = C_W(X^{W_I} \cap V)$ , in particular  $C_W(V) \subset W_I$ .

*Further, the following two assertions are equivalent:*

- (iii)  $\mathbf{w}$  is maximal, that is, there do not exist a  $\phi^d$ -stable  $\mathbf{J} \subsetneq \mathbf{I}$  and  $\mathbf{v} \in B_{\mathbf{I}}^+$  such that  $(\mathbf{v}\mathbf{w}\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}$ .
- (iv) No element of the coset  $W_Iw\phi$  has a non-zero  $\zeta_d$ -eigenvector on the subspace spanned by the root lines of  $W_I$ .

*Proof.* Notice that, since  $(\mathbf{w}\phi)^d = (\boldsymbol{\pi}_{\mathbf{I}})^{-1}\boldsymbol{\pi}\phi^d$  implies  $\alpha_{\mathbf{I}}(\mathbf{w}) = 1$ , condition (i) is equivalent to require that  $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi^d}\mathbf{I}$  is a morphism in the category  $B^+(\mathcal{I})$  (this morphism is then by assumption a  $d$ -th root of  $\Delta_{\mathcal{I}}^2$ ).

To prove (i) notice that by assumption  $\mathbf{w}\phi$  commutes to  $\phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$ , thus, since  $\boldsymbol{\pi}$  is central and  $\phi$ -stable, it commutes to  $\boldsymbol{\pi}_{\mathbf{I}}\phi^{-d}$ . Thus, if  $\delta$  is the order of  $\phi$ , since  $\boldsymbol{\pi}_{\mathbf{I}}$  is  $\phi^d$ -stable,  $\mathbf{w}\phi$  commutes to  $\boldsymbol{\pi}_{\mathbf{I}}^{\delta}$ , hence  $(\boldsymbol{\pi}_{\mathbf{I}}^{\delta})^{\mathbf{w}} = \boldsymbol{\pi}_{\phi^d\mathbf{I}}^{\delta}$ . By Proposition 5.15(i) we deduce (i).

In our setting Lemma 6.2 thus reduces to the following generalization of [BM, lemme 6.9]

**Lemma 9.5.** *Let  $\mathbf{w} \in B^+$  and  $\mathbf{I} \subset \mathbf{S}$  be a  $\phi^d$ -stable subset such that  $(\mathbf{w}\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$ . Then there exists  $\mathbf{v} \in (B^+)^{\phi^d}$  such that  $(\mathbf{w}\phi)^{\mathbf{v}} \in B^+\phi$ ,  $\mathbf{I}^{\mathbf{v}} \subset \mathbf{S}$  and*

$((\mathbf{w}\phi)^{\mathbf{v}})^{\lfloor \frac{d}{2} \rfloor} \in \mathbf{W}\phi^{\lfloor \frac{d}{2} \rfloor}$ . Further,  $\text{ad } \mathbf{v}$  defines a morphism in  $\mathcal{D}^+(\mathcal{I})^{\phi^d}$  (that is, the conjugation is by “ $\phi^d$ -stable cyclic permutations”).

Thus if we define  $\mathbf{w}'$  and  $\mathbf{J}$  by  $(\mathbf{w}\phi)^{\mathbf{v}} = \mathbf{w}'\phi$  and  $\mathbf{I}^{\mathbf{v}} = \mathbf{J}$ , we have  $(\mathbf{w}'\phi)^d = \phi^d \mathbf{w}' \mathbf{J} / \mathbf{w}' \mathbf{J}$  and  $\mathbf{w}'\phi \mathbf{J} = \mathbf{J}$ .

As (ii) and the equivalence of (iii) and (iv) are invariant by a conjugacy in  $B$  which sends  $\mathbf{w}\phi$  to  $B^+\phi$  and  $\mathbf{I}$  to another subset of  $\mathbf{S}$ , we may replace  $(\mathbf{w}\phi, \mathbf{I})$  by a conjugate as in Lemma 9.5, thus assume that  $w$  and  $I$  satisfy the assumptions of the next lemma.

To state the next lemma we extend the length function from  $W$  to  $W \rtimes \langle \phi \rangle$  by setting  $l(w\phi^i) = l(w)$ .

**Lemma 9.6.** *Let  $w \in W, I \subset S$  be such that  $(w\phi)^d = \phi^d$ ,  $w\phi I = I$  and such that  $l((w\phi)^i) = \frac{2i}{d} l(w_I^{-1} w_0)$  for any  $i \leq d/2$ . We have*

- (i) *If  $\Phi$  be a  $\phi$ -stable root system for  $W$  (as in the proof of Theorem 9.1), then  $\Phi - \Phi_I$  is the disjoint union of sets of the form  $\{\alpha, {}^{w\phi}\alpha, \dots, {}^{(w\phi)^{d-1}}\alpha\}$  with  $\alpha, {}^{w\phi}\alpha, \dots, {}^{(w\phi)^{\lfloor d/2 \rfloor-1}}\alpha$  of same sign and  ${}^{(w\phi)^{\lfloor d/2 \rfloor}}\alpha, \dots, {}^{(w\phi)^{d-1}}\alpha$  of the opposite sign.*
- (ii) *The order of  $w\phi$  is  $\text{lcm}(d, \delta)$ .*
- (iii) *If  $d > 1$ , then  $W_I = C_W(X^{W_I} \cap \ker(w\phi - \zeta_d))$ .*

*Proof.* The statement is empty for  $d = 1$  so in the following proof we assume  $d > 1$ .

For  $x \in W \rtimes \langle \phi \rangle$  let  $N(x) = \{\alpha \in \Phi^+ \mid {}^x\alpha \in \Phi^-\}$ ; it is well known that for  $x \in W$  we have  $l(x) = |N(x)|$ . This still holds for  $x = w\phi^i \in W \rtimes \langle \phi \rangle$  since  $N(w\phi^i) = \phi^{-i} N(w)$ . It follows that for  $x, y \in W \rtimes \langle \phi \rangle$  we have  $l(xy) = l(x) + l(y)$  if and only if  $N(xy) = N(y) \coprod {}^{y^{-1}}N(x)$ . In particular  $l((w\phi)^i) = il(w\phi)$  for  $i \leq d/2$  implies  ${}^{(w\phi)^{-i}}N(w\phi) \subset \Phi^+$  for  $i \leq d/2 - 1$ .

Let us partition each  $w\phi$ -orbit in  $\Phi - \Phi_I$  into “pseudo-orbits” of the form  $\{\alpha, {}^{w\phi}\alpha, \dots, {}^{(w\phi)^{k-1}}\alpha\}$ , where  $k$  is minimal such that  ${}^{(w\phi)^k}\alpha = \phi^k\alpha$  (then  $k$  divides  $d$ ); a pseudo-orbit is an orbit if  $\phi = 1$ . The action of  $w\phi$  defines a cyclic order on each pseudo-orbit. The previous paragraph shows that when there is a sign change in a pseudo-orbit, at least the next  $\lfloor d/2 \rfloor$  roots for the cyclic order have the same sign. On the other hand, as  $\phi^k$  preserves  $\Phi^+$ , each pseudo-orbit contains an even number of sign changes. Thus if there is at least one sign change we have  $k \geq 2\lfloor d/2 \rfloor$ . Since  $k$  divides  $d$ , we must have  $k = d$  for pseudo-orbits which have a sign change, and then they have exactly two sign changes. As the total number of sign changes is  $2l(w) = 2|\Phi - \Phi_I|/d$ , there are  $|\Phi - \Phi_I|/d$  pseudo-orbits with sign changes; their total cardinality is  $|\Phi - \Phi_I|$ , thus there are no other pseudo-orbits and up to a cyclic permutation we may assume that each pseudo-orbit consists of  $\lfloor d/2 \rfloor$  roots of the same sign followed by  $d - \lfloor d/2 \rfloor$  of the opposite sign. We have proved (i).

Let  $d' = \text{lcm}(d, \delta)$ . The proof of (i) shows that the order of  $w\phi$  is a multiple of  $d$ . Since the order of  $(w\phi)^d = \phi^d$  is  $d'/d$ , we get (ii).

We now prove (iii). Let  $V = \ker(w\phi - \zeta_d)$ . Since  $W\langle \phi \rangle$  is finite, we may find a scalar product on  $X$  invariant by  $W$  and  $\phi$ . We have then  $X^{W_I} = \Phi_I^\perp$ . The map  $p = \frac{1}{d'} \sum_{i=0}^{d'-1} \zeta_d^{-i} (w\phi)^i$  is the (unique up to scalar)  $w\phi$ -invariant projector on  $V$ , thus is the orthogonal projector on  $V$ .

We claim that  $p(\alpha) \notin \langle \Phi_I \rangle$  for any  $\alpha \in \Phi - \Phi_I$ . As  $p((w\phi)^i \alpha) = \zeta_d^i p(\alpha)$  it is enough to assume that  $\alpha$  is the first element of a pseudo-orbit; replacing if

needed  $\alpha$  by  $-\alpha$  we may even assume  $\alpha \in \Phi^+$ . Looking at imaginary parts, we have  $\Im(\zeta_d^i) \geq 0$  for  $0 \leq i < \lfloor d/2 \rfloor$ , and  $\Im(\zeta_d^i) < 0$  for  $\lfloor d/2 \rfloor \leq i < d$ . Let  $\lambda$  be a linear form such that  $\lambda$  is 0 on  $\Phi_I$  and is real strictly positive on  $\Phi^+ - \Phi_I$ ; we have  $\lambda((w\phi)^i \alpha) > 0$  for  $0 \leq i < \lfloor d/2 \rfloor$ , and  $\lambda((w\phi)^i \alpha) < 0$  for  $\lfloor d/2 \rfloor \leq i < d$ ; it follows that  $\Im(\lambda(\zeta_d^i (w\phi)^i \alpha)) = \Im(\zeta_d^i \lambda((w\phi)^i \alpha)) > 0$  for all elements of the pseudo-orbit. If  $d' = d$  we have thus  $\Im(\lambda(p(\alpha))) > 0$ , in particular  $p(\alpha) \notin \langle \Phi_I \rangle$ . If  $d' > d$ , since  $\phi^d \alpha$  is also a positive root and the first term of the next pseudo-orbit the same computation applies to the other pseudo-orbits and we conclude the same way.

Now  $C_W(X^{W_I} \cap V)$  is generated by the reflections whose root is orthogonal to  $X^{W_I} \cap V$ , that is whose root is in  $\langle \Phi_I \rangle + V^\perp$ . If  $\alpha$  is such a root we have  $p(\alpha) \in \langle \Phi_I \rangle$ , whence  $\alpha \in \Phi_I$  by the above claim. This proves that  $C_W(X^{W_I} \cap V) \subset W_I$ . Since the reverse inclusion is true, we get (iii).  $\square$

We return to the proof of Theorem 9.4. Assertion (iii) of Lemma 9.6 gives the first assertion of the theorem. We now show  $\neg(\text{iii}) \Rightarrow \neg(\text{iv})$ . If  $\mathbf{w}$  is not maximal, there exists a  $\phi^d$ -stable  $\mathbf{J} \subsetneq \mathbf{I}$  and  $\mathbf{v} \in B_{\mathbf{I}}^+$  such that  $(\mathbf{v}\mathbf{w}\phi)^d = \phi^d \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}$ , which implies  $\mathbf{v}\mathbf{w}\phi \mathbf{J} = \mathbf{J}$ . If we denote by  $\psi$  the automorphism of  $B_{\mathbf{I}}$  induced by the automorphism  $\mathbf{w}\phi$  of  $\mathbf{I}$ , we have  $\mathbf{v}\psi \mathbf{J} = \mathbf{J}$  and  $(\mathbf{v}\psi)^d = \psi^d \boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}}$ . Let  $X_I$  be the subspace of  $X$  spanned by  $\Phi_I$ . It follows from the first part of the theorem applied with  $X$ ,  $\phi$ ,  $\mathbf{w}$  and  $w$  respectively replaced with  $X_I$ ,  $\psi$ ,  $\mathbf{v}$  and  $v$  that  $v\psi = vw\phi$  has a non-zero  $\zeta_d$ -eigenspace in  $X_I$ , since if  $V'$  is the  $\zeta_d$ -eigenspace of  $vw\phi$  we get  $C_{W_I}(V') \subset W_J \subsetneq W_I$ ; this contradicts (iv).

We show finally that  $\neg(\text{iv}) \Rightarrow \neg(\text{iii})$ . If some element of  $W_I\psi$  has a non-zero  $\zeta_d$ -eigenvector on  $X_I$ , by Theorem 9.1 applied to  $W_I\psi$  acting on  $X_I$  we get the existence of  $\mathbf{J} \subsetneq \mathbf{I}$  and  $\mathbf{v} \in B_{\mathbf{I}}^+$  satisfying  $\mathbf{v}\psi \mathbf{J} = \mathbf{J}$  and  $(\mathbf{v}\psi)^d = \psi^d \boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}}$ . Using that  $(\mathbf{w}\phi)^d = \phi^d \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$ , it follows that  $(\mathbf{v}\mathbf{w}\phi)^d = (\mathbf{w}\phi)^d \boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}} = \phi^d \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}} \cdot \boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}} = \phi^d \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}$  so  $\mathbf{w}$  is not maximal.  $\square$

The maximality condition (iii) or (iv) of Theorem 9.4 is equivalent to the conjunction of two others, thanks to the following lemma which holds for any complex reflection coset and any  $\zeta$ .

**Lemma 9.7.** *Let  $W$  be finite a (pseudo)-reflection group on the complex vector space  $X$  and let  $\phi$  be an automorphism of  $X$  of finite order which normalizes  $W$ . Let  $V$  be the  $\zeta$ -eigenspace of an element  $w\phi \in W\phi$ . Assume that  $W'$  is a parabolic subgroup of  $W$  which is  $w\phi$ -stable and such that  $C_W(V) \subset W'$ , and let  $X'$  denote the subspace of  $X$  spanned by the root lines of  $W'$ . Then the condition*

- (i)  $V \cap X' = 0$ .

*is equivalent to*

- (ii)  $C_W(V) = W'$ .

*While the stronger condition*

- (iv) *No element of the coset  $W'w\phi$  has a non-zero  $\zeta$ -eigenvector on  $X'$ .*

*is equivalent to the conjunction of (ii) and*

- (iii) *the space  $V$  is maximal among the  $\zeta$ -eigenspaces of elements of  $W\phi$ .*

*Proof.* Since  $W\langle\phi\rangle$  is finite we may endow  $X$  with a  $W\langle\phi\rangle$ -invariant scalar product, which we shall do.

We show (i)  $\Leftrightarrow$  (ii). Assume (i); since  $w\phi$  has no non-zero  $\zeta$ -eigenvector in  $X'$  and  $X'$  is  $w\phi$ -stable, we have  $V \perp X'$ , so that  $W' \subset C_W(V)$ , whence (ii) since the

reverse inclusion is true by assumption. Conversely, (ii) implies that  $V \subset X'^\perp$  thus  $V \cap X' = 0$ .

We show (iv)  $\Rightarrow$  (iii). There exists an element of  $W\phi$  whose  $\zeta$ -eigenspace  $V_1$  is maximal with  $V \subset V_1$ . Then  $C_W(V_1) \subset C_W(V) \subset W'$  and the  $C_W(V_1)$ -coset of elements of  $W\phi$  which act by  $\zeta$  on  $V_1$  is a subset of the coset  $C_W(V)w\phi$  of elements which act by  $\zeta$  on  $V$ . Thus this coset is of the form  $C_W(V_1)vw\phi$  for some  $v \in W'$ . By (i)  $\Rightarrow$  (ii) applied with  $w\phi$  replaced by  $vw\phi$  we get  $C_W(V_1) = W'$ . Since  $v \in W'$  this implies that  $vw\phi$  and  $w\phi$  have same action on  $V_1$  so that  $w\phi$  acts by  $\zeta$  on  $V_1$ , thus  $V_1 \subset V$ .

Conversely, assume that (ii) and (iii) are true. If there exists  $v \in W'$  such that  $vw\phi$  has a non-zero  $\zeta$ -eigenvector in  $X'$ , then since  $v$  acts trivially on  $V$  by (ii), the element  $vw\phi$  acts by  $\zeta$  on  $V$  and on a non-zero vector of  $X'$  so has a  $\zeta$ -eigenspace strictly larger than  $V$ , contradicting (iii).  $\square$

Let us give now examples which illustrate the need for the conditions in Theorem 9.4 and Lemma 9.7.

We first give an example where  $\mathbf{w}\phi$  is a root of  $\pi/\pi_{\mathbf{I}}$  but is not maximal in the sense of Theorem 9.4(iii) and  $\ker(w\phi - \zeta)$  is not maximal: let us take  $W = W(A_3)$ ,  $\phi = 1$ ,  $d = 2$ ,  $\zeta = -1$ ,  $\mathbf{I} = \{\mathbf{s}_2\}$  (where the conventions for the generators of  $W$  are as in the appendix, see Subsection 11.2),  $\mathbf{w} = \mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0$ . We have  $\mathbf{w}^2 = \pi/\pi_{\mathbf{I}}$  but  $\ker(w+1)$  is not maximal: its dimension is 1 and a 2-dimensional  $-1$ -eigenspace is obtained for  $\mathbf{w} = \mathbf{w}_0$ .

In the above example we still have  $C_W(V) = W_I$  but even this need not happen; at the same time we illustrate that the maximality of  $V = \ker(w\phi - \zeta)$  does not imply the maximality of  $\mathbf{w}$  if  $C_W(V) \subsetneq W_I$ ; we take  $W = W(A_3)$ ,  $\phi = 1$ ,  $d = 2$ ,  $\zeta = -1$ , but this time  $I = \{\mathbf{s}_1, \mathbf{s}_3\}$ ,  $\mathbf{w} = \mathbf{w}_I^{-1}\mathbf{w}_0$ . We have  $\mathbf{w}^2 = \pi/\pi_{\mathbf{I}}$  and  $\ker(w+1)$  is maximal ( $w$  is conjugate to  $w_0$ , thus  $-1$ -regular) but  $\mathbf{w}$  is not maximal. In this case  $C_W(V) = \{1\}$ .

The smallest example with a maximal  $\mathbf{w}\phi$  and non-trivial  $\mathbf{I}$  is for  $W = W(A_4)$ ,  $\phi = 1$ ,  $d = 3$ ,  $\mathbf{w} = \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2$  and  $\mathbf{I} = \{\mathbf{s}_3\}$ . Then  $\mathbf{w}^3 = \pi/\pi_{\mathbf{I}}$ ; this corresponds to the smallest example with a non-regular eigenvalue:  $\zeta_3$  is not regular in  $A_4$ .

Finally we give an example which illustrates the necessity of the condition  $\phi^d(\mathbf{I}) = \mathbf{I}$  in 9.4. We take  $W = W(D_4)$  and for  $\phi$  the triality automorphism  $\mathbf{s}_1 \mapsto \mathbf{s}_4 \mapsto \mathbf{s}_2$ . Let  $\mathbf{v} = \mathbf{w}_0\mathbf{s}_1^{-1}\mathbf{s}_2^{-1}\mathbf{s}_4^2$ . Then, for  $\mathbf{I} = \{\mathbf{s}_1\}$  we have  $(\mathbf{w}\phi)^2 = \pi/\pi_{\mathbf{I}}\phi^2$ , but  $\mathbf{I}^{\mathbf{w}\phi} = \{\mathbf{s}_4\}$ . The other statements of 9.4 also fail: if  $V$  is the  $-1$ -eigenspace of  $w\phi$  the group  $C_W(V)$  is the parabolic subgroup generated by  $s_1, s_2$  and  $s_4$ .

**Lemma 9.8.** *Let  $W\phi$  be a complex reflection coset and let  $V$  be the  $\zeta$ -eigenspace of  $w\phi \in W\phi$ ; then*

- (i)  $N_W(V) = N_W(C_W(V)w\phi)$ .
- (ii) *If  $W\phi$  is real, and  $C_W(V) = W_I$  where  $(W, S)$  is a Coxeter system and  $I \subset S$ , and  $w$  is  $I$ -reduced, then the subgroup  $\{v \in C_W(w\phi) \cap N_W(W_I) \mid v \text{ is } I\text{-reduced}\}$  is a section of  $N_W(V)/C_W(V)$  in  $W$ .*

*Proof.* Let  $W_1$  denote the parabolic subgroup  $C_W(V)$ . All elements of  $W_1w\phi$  have the same  $\zeta$ -eigenspace  $V$ , so  $N_W(W_1w\phi)$  normalizes  $V$ ; conversely, an element of  $N_W(V)$  normalizes  $W_1$  and conjugates  $w\phi$  to an element  $w'\phi$  with same  $\zeta$ -eigenspace, thus  $w$  and  $w'$  differ by an element of  $W_1$ , whence (i).

For the second item,  $N_W(W_I w\phi)/W_I$  admits as a section the set of  $I$ -reduced elements, and such an element will conjugate  $w\phi$  to the element of the coset  $W_I w\phi$  which is  $I$ -reduced, so will centralize  $w\phi$ .  $\square$

Recall that given a category  $\mathcal{C}$  with a Garside map  $\Delta$  and a Garside automorphism  $\phi$ , we can consider the semi-direct product of  $\mathcal{C}$  by  $\phi$  (see Definition 2.6). Then a morphism  $\mathbf{w}\phi \in C\phi$  is  $(p, q)$ -periodic if  $\text{target}(\mathbf{w}) = \phi(\text{source}(\mathbf{w}))$  and  $(\mathbf{w}\phi)^p = \Delta^q\phi^p$ . An element satisfying the assumption of Theorem 9.4 is thus a  $(d, 2)$ -periodic element of  $B^+(\mathcal{I})\phi$ , since  $\Delta_{\mathcal{I}}^2$  starting from the object  $\mathbf{I}$  is  $\mathbf{I} \xrightarrow{\pi/\pi_{\mathbf{I}}} \mathbf{I}$ . Lemma 9.5 shows that such an element is cyclically conjugate to an element which satisfies in addition  $(\mathbf{w}\phi)^{d'} \in \mathbf{W}\phi^{d'}$ , where  $d' = \lfloor \frac{d}{2} \rfloor$ . We will call *good* a periodic element which satisfies the above condition.

The following proposition, which rephrases Corollary 6.3 in our setting, shows that it makes sense to write a period of the form  $(d, 2)$  as a fraction  $d/2$ , since it shows that when  $2|d$ , a good  $(d, 2)$ -periodic element such that  $(\mathbf{w}\phi)^d = \Delta_{\mathcal{I}}^2$  satisfies  $(\mathbf{w}\phi)^{d/2} = \Delta_{\mathcal{I}}$ . We will thus call such elements  $d/2$ -periodic. In [DDGKM] the analogous statement is shown for a general  $p/q$ .

**Proposition 9.9.** *Assume the morphism  $\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi\mathbf{I}$  is good  $d/2$ -periodic (which means that  $\mathbf{w} \in B^+$  satisfies  $\mathbf{w}^\phi\mathbf{I} = \mathbf{I}$ ,  $(\mathbf{w}\phi)^d = \phi^d\pi/\pi_{\mathbf{I}}$  and that in addition  $(\mathbf{w}\phi)^{d'} \in \mathbf{W}\phi^{d'}$ , where  $d' = \lfloor \frac{d}{2} \rfloor$ ). Then if  $d$  is even we have  $(\mathbf{w}\phi)^{d'} = \mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0\phi^{d'}$ , and if  $d$  is odd there exists  $\mathbf{u} \in \mathbf{W}^{\Phi^d}$  with  $\mathbf{I}^{\mathbf{u}} \subset \mathbf{S}$  such that  $\mathbf{w}\phi = \mathbf{u}\phi \cdot {}^{\mathbf{w}_0\phi^{d'}}\mathbf{u}$  and  $(\mathbf{w}\phi)^{d'}\mathbf{u} = \mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0\phi^{d'}$ .*

Let us define the  $\zeta$ -rank of a (complex) reflection coset  $W\phi \subset \text{GL}(X)$  as the maximal dimension of a  $\zeta$ -eigenspace of an element of  $W\phi$ , and the  $\zeta$ -rank of an element of  $W\phi$  as the dimension of its  $\zeta$ -eigenspace.

Let us say that a periodic element of  $B^+(\mathcal{I})\phi$  is maximal if it is maximal in the sense of Theorem 9.4(iii). Another way to state the maximality of a periodic element is to require that  $|\mathbf{I}|$  be no more than the rank of the centralizer of a maximal  $\zeta_d$ -eigenspace: indeed if  $\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi\mathbf{I}$  is not maximal there exists  $\mathbf{J}$  and  $\mathbf{v}$  as in Theorem 9.4(iii) and, since Theorem 9.4(iii) implies Lemma 9.7(iii), the element  $v\mathbf{w}\phi$  has maximal  $\zeta_d$ -rank, and the centralizer of its  $\zeta_d$ -eigenspace has rank  $|\mathbf{J}| < |\mathbf{I}|$ .

A particular case of Theorems 9.1 and 9.4 is

**Corollary 9.10.** *Let  $V'$  be the  $\zeta_d$ -eigenspace of an element of  $W\phi$  of maximal  $\zeta_d$ -rank. Then there is a  $W$ -conjugate  $V$  of  $V'$  and  $I \subset S$  such that  $C_W(V) = W_I$  and the  $w\phi$  defined in Theorem 9.1(ii) induces a  $d/2$ -periodic  $\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi\mathbf{I}$  which is maximal. Conversely, for a  $d/2$ -periodic maximal  $\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi\mathbf{I}$  the image  $w\phi$  in  $W\phi$  has maximal  $\zeta_d$ -rank.*

**Lemma 9.11.** *Let  $W\phi \subset \text{GL}(X_{\mathbb{R}})$  be a finite order real reflection coset such that  $\phi$  preserves the chamber of the corresponding hyperplane arrangement determining the Coxeter system  $(W, S)$ .*

*Let  $w \in W$  and  $I \subset S$  and let  $\mathbf{w} \in \mathbf{W}$  and  $\mathbf{I} \subset \mathbf{S}$  be their lifts; let  $\mathcal{I}$  be the conjugacy orbit of  $\mathbf{I}$ , then  $\mathbf{w}$  induces a morphism  $\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi\mathbf{I} \in B^+(\mathcal{I})$  if and only if:*

- (i)  ${}^w\phi I = I$  and  $w$  is  $I$ -reduced.

*For  $d > 1$ , the above morphism  $\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi\mathbf{I}$  is good  $d/2$ -periodic if and only if the following two additional conditions are satisfied.*

- (ii)  $l((w\phi)^i\phi^{-i}) = \frac{2i}{d}l(w_I^{-1}w_0)$  for  $0 < i \leq \lfloor \frac{d}{2} \rfloor$ .
- (iii)  $(w\phi)^d = \phi^d$ .

If, moreover,

- (iv)  $W_I w\phi$  has  $\zeta_d$ -rank 0 on the subspace spanned by the root lines of  $W_I$ , then  $w\phi$  is maximal in the sense of Theorem 9.4(iii).

*Proof.* By definition  $w$  induces a morphism  $\mathbf{I} \xrightarrow{w} \phi\mathbf{I}$  if and only if it satisfies (i). By definition again if this morphism is good  $d/2$ -periodic then (ii) and (iii) are satisfied. Conversely, Lemma 9.2 shows that the morphism induced by the lift of a  $w$  satisfying (i), (ii), (iii) is good  $d/2$ -periodic.

Property (iv) means that no element  $vw\phi$  with  $v \in W_I$  has an eigenvalue  $\zeta_d$  on the subspace spanned by the root lines of  $W_I$  which is exactly the characterization of Theorem 9.4(iv) of a maximal element.  $\square$

Note that  $d$  and  $I$  in the above assumptions (i), (ii), (iii) are uniquely determined by  $w$  since  $d$  is the smallest power of  $w\phi$  which is a power of  $\phi$  and  $I$  is given uniquely by  $(w\phi)^d = \pi/\pi_{\mathbf{I}}\phi^d$ .

**Definition 9.12.** We say that  $w\phi \in W\phi$  is  $\zeta_d$ -good (relative to  $W\phi$  and  $I$ ) if it satisfies (i), (ii), (iii) in Lemma 9.11.

We say  $w\phi$  is  $\zeta_d$ -good maximal if it satisfies in addition (iv) in Lemma 9.11.

In particular,  $\zeta_d$ -good elements lift to good  $d/2$ -periodic elements, and  $\zeta_d$ -good maximal elements lift to good maximal  $d/2$ -periodic elements.

The  $\zeta_d$ -good maximal elements belong to a single conjugacy class of  $W$ . The following lemma applied with  $\zeta = \zeta_d$  gives a characterization of this class.

**Lemma 9.13.** Let  $W\phi$  be a finite order real reflection coset such that  $\phi$  preserves a chamber of the corresponding hyperplane arrangement. The elements of  $W\phi$  which have a  $\zeta$ -eigenspace  $V$  of maximal dimension and among those, have the largest dimension of fixed points, are conjugate.

*Proof.* Let  $w$  and  $V$  be as in the lemma. Since, by [S, Theorem 3.4(iii) and Theorem 6.2(iii)], the maximal  $\zeta$ -eigenspaces are conjugate, we may fix  $V$ . Since  $C_W(V)$  is a parabolic subgroup of the Coxeter group  $W$  normalized by  $w\phi$ , the coset  $C_W(V)w\phi$  is a real reflection coset; in this coset there are 1-regular elements, which are those which preserve a chamber of the corresponding real hyperplane arrangement; the 1-regular elements have maximal 1-rank, that is have the largest dimension of fixed points, and they form a single  $C_W(V)$ -orbit under conjugacy, whence the lemma.  $\square$

**Lemma 9.14.** Let  $w\phi$  be a  $\zeta_d$ -good maximal element, let  $I$  be as in Lemma 9.11 and let  $V_1$  be the fixed point subspace of  $w\phi$  in the space spanned by the root lines of  $W_I$ ; then  $w\phi$  is regular in the coset  $C_W(V_1)w\phi$ .

*Proof.* Let  $W' = C_W(V_1)$ ; we first note that since  $w\phi$  normalizes  $V_1$  it normalizes also  $W'$ , so  $W'w\phi$  is indeed a reflection coset. We have thus only to prove that  $C_{W'}(V)$  is trivial, where  $V$  is the  $\zeta_d$ -eigenspace of  $w\phi$ . This last group is generated by the reflections with respect to roots both orthogonal to  $V$  and to  $V_1$ , which are the roots of  $W_I = C_W(V)$  orthogonal to  $V_1$ . Since  $w\phi$  preserves a chamber of  $W_I$ , the sum  $v$  of the positive roots of  $W_I$  with respect to the order defined by this chamber is in  $V_1$  and is in the chamber: this is well known for a true root system;

here we have taken all the roots to be of length 1 but the usual proof (see [Bou, Chapitre VI §1, Proposition 29]) is still valid. Since no root is orthogonal to a vector  $v$  inside a chamber,  $W_I$  has no root orthogonal to  $V_1$ , hence  $C_{W'}(V) = \{1\}$ .  $\square$

Note that the map  $C_{W'}(w\phi) = N_{W'}(V) \rightarrow N_W(V)/C_W(V)$  in the above proof is injective, but not always surjective: if  $W$  of type  $E_7$ , if  $\phi = \text{Id}$  and  $\zeta = i$ , a fourth root of unity, then  $N_W(V)/C_W(V)$  is the complex reflection group  $G_8$ , while  $W'$  is of type  $D_4$  and  $N_{W'}(V)/C_{W'}(V)$  is the complex reflection group  $G(4, 2, 2)$ . However, we will see in appendix 1 that there are only 4 such cases for irreducible groups  $W$ ; to see in the other cases that  $C_{W'}(w\phi) \simeq N_W(V)/C_W(V)$  it is sufficient to check that they have same reflection degrees, which is a simple arithmetic check on the reflection degrees of  $W$  and  $W'$ .

## 10. CONJECTURES

The following conjectures extend those of [DM2, §2]. They are a geometric form of Broué conjectures.

**Conjecture 10.1.** *Let  $\mathbf{I} \xrightarrow{\mathbf{w}} \phi\mathbf{I} \in B^+(\mathcal{I})$  be a maximal  $d/2$ -periodic morphism. Then*

- (i) *The group  $B_{\mathbf{w}}$  generated by the monoid  $B_{\mathbf{w}}^+$  of Theorem 8.28 is isomorphic to the braid group of the complex reflection group  $W_w := N_W(W_I w\phi)/W_I$ .*
- (ii) *The natural morphism  $\mathcal{D}^+(\mathcal{I})(\mathbf{I} \xrightarrow{\mathbf{w}} \phi\mathbf{I}) \rightarrow \text{End}_{\mathbf{G}^F}(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$  (see below Definition 8.25) gives rise to a morphism  $B_{\mathbf{w}} \rightarrow \text{End}_{\mathbf{G}^F} H_c^*(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$  which factors through a special representation of a  $\zeta_d$ -cyclotomic Hecke algebra  $\mathcal{H}_{\mathbf{w}}$  for  $W_w$ .*
- (iii) *The odd and even  $H_c^i(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$  are disjoint, and the above morphism extends to a surjective morphism  $\overline{\mathbb{Q}}_{\ell}[B_{\mathbf{w}}] \rightarrow \text{End}_{\mathbf{G}^F}(H_c^*(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)))$ .*

**Lemma 10.2.** *Let  $\mathbf{I} \xrightarrow{\mathbf{w}} \phi\mathbf{I} \in B^+(\mathcal{I})$  be a maximal  $d/2$ -periodic morphism and assume Conjecture 10.1; then for any  $i \neq j$  the  $\mathbf{G}^F$ -modules  $H_c^i(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$  and  $H_c^j(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$  are disjoint.*

*Proof.* Since the image of the morphism of Conjecture 10.1(ii) consists of equivalences of étale sites, it follows that the action of  $\mathcal{H}_{\mathbf{w}}$  on  $H_c^*(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$  preserves individual cohomology groups. The surjectivity of the morphism of (iii) implies that for  $\rho \in \text{Irr}(\mathbf{G}^F)$ , the  $\rho$ -isotypic part of  $H_c^*(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$  affords an irreducible  $\mathcal{H}_{\mathbf{w}}$ -module; this would not be possible if this  $\rho$ -isotypic part was spread over several distinct cohomology groups.  $\square$

We will now explore the information given by the Shintani descent identity on the above conjectures

**Lemma 10.3.** *Let  $\mathbf{I} \xrightarrow{\mathbf{w}} \phi\mathbf{I} \in B^+(\mathcal{I})$  be a  $d/2$ -periodic morphism. With the notations of Proposition 8.34, we have  $\tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}} \phi) = q^{m \frac{l(\boldsymbol{\pi}) - l(\boldsymbol{\pi}_1) - a_{\chi} - A_{\chi}}{d}} \tilde{\chi}(e_I w F)$  for  $\chi \in \text{Irr}(W)^{\phi}$ , where  $a_{\chi}$  (resp.  $A_{\chi}$ ) is the valuation (resp. the degree) of the generic degree of  $\chi$  and  $e_I = |W_I|^{-1} \sum_{v \in W_I} v$ .*

*Proof.* We have  $(X_1 T_{\mathbf{w}} \phi)^d = X_1 T_{\boldsymbol{\pi}} / T_{\boldsymbol{\pi}_1} \phi^d = q^{-l(\boldsymbol{\pi}_1)} X_1 T_{\boldsymbol{\pi}} \phi^d$  since  $X_1$  commutes with  $T_{\mathbf{w}} \phi$  and since for any  $v \in W_I$  we have  $X_1 T_v = q^{l(v)} T_v$ . Since  $T_{\boldsymbol{\pi}}$  acts on the representation of character  $\chi_{q^m}$  as the scalar  $q^{m(l(\boldsymbol{\pi}) - a_{\chi} - A_{\chi})}$  (see [BM, Corollary 4.20]), it follows that all the eigenvalues of  $X_1 T_{\mathbf{w}} \phi$  on this representation are equal

to  $q^{m \frac{l(\pi) - l(\pi_{\mathbf{I}}) - a_{\chi} - A_{\chi}}{d}}$  times a root of unity. To compute the sum of these roots of unity, we may use the specialization  $q^{m/2} \mapsto 1$ , through which  $\tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}} \phi)$  specializes to  $\tilde{\chi}(e_I w \phi)$ .  $\square$

**Proposition 10.4.** *Let  $\mathbf{I} \xrightarrow{\mathbf{w}} \phi \mathbf{I} \in B^+(\mathcal{I})$  be a  $d/2$ -periodic morphism. For any  $m$  multiple of  $\delta$ , we have*

$$|\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m}| = \sum_{\rho \in \mathcal{E}(\mathbf{G}^F, 1)} \lambda_{\rho}^{m/\delta} q^{m \frac{l(\pi) - l(\pi_{\mathbf{I}}) - a_{\rho} - A_{\rho}}{d}} \langle \rho, R_{\mathbf{L}_I, \dot{w}F}^{\mathbf{G}, F} \text{Id} \rangle_{\mathbf{G}^F} \rho(g),$$

where  $a_{\rho}$  and  $A_{\rho}$  are respectively the valuation and the degree of the generic degree of  $\rho$ .

*Proof.* We start with Corollary 8.35, whose statement reads, using the value of  $\tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}} \phi)$  given by Lemma 10.3:

$$\begin{aligned} |\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m}| &= \sum_{\rho \in \mathcal{E}(\mathbf{G}^F, 1)} \lambda_{\rho}^{m/\delta} \rho(g) \\ &\quad \sum_{\chi \in \text{Irr}(W)^{\phi}} q^{m \frac{l(\pi) - l(\pi_{\mathbf{I}}) - a_{\chi} - A_{\chi}}{d}} \tilde{\chi}(e_I w \phi) \langle \rho, R_{\tilde{\chi}} \rangle_{\mathbf{G}^F}. \end{aligned}$$

Using that for any  $\rho$  such that  $\langle \rho, R_{\tilde{\chi}} \rangle_{\mathbf{G}^F} \neq 0$  we have  $a_{\rho} = a_{\chi}$  and  $A_{\rho} = A_{\chi}$  (see [BM] around (2.4)) the right-hand side can be rewritten

$$\sum_{\rho \in \mathcal{E}(\mathbf{G}^F, 1)} \lambda_{\rho}^{m/\delta} q^{m \frac{l(\pi) - l(\pi_{\mathbf{I}}) - a_{\rho} - A_{\rho}}{d}} \rho(g) \langle \rho, \sum_{\chi \in \text{Irr}(W)^{\phi}} \tilde{\chi}(e_I w \phi) R_{\tilde{\chi}} \rangle_{\mathbf{G}^F}.$$

The proposition is now just a matter of observing that

$$\begin{aligned} \sum_{\chi \in \text{Irr}(W)^{\phi}} \tilde{\chi}(e_I w \phi) R_{\tilde{\chi}} &= |W_I|^{-1} \sum_{v \in W_I} \sum_{\chi \in \text{Irr}(W)^{\phi}} \tilde{\chi}(vw \phi) R_{\tilde{\chi}} = \\ &= |W_I|^{-1} \sum_{v \in W_I} R_{\mathbf{T}_{vw}}^{\mathbf{G}}(\text{Id}) = R_{\mathbf{L}_I, \dot{w}F}^{\mathbf{G}, F}(\text{Id}). \end{aligned}$$

Where the last equality is obtained by transitivity of  $R_{\mathbf{L}}^{\mathbf{G}}$  and the equality  $\text{Id}_{\mathbf{L}_I^{\dot{w}F}} = |W_I|^{-1} \sum_{v \in W_I} R_{\mathbf{T}_{vw}}^{\mathbf{L}_I, \dot{w}F}(\text{Id})$ , a torus  $\mathbf{T}$  of  $\mathbf{L}_I$  of type  $v$  for the isogeny  $\dot{w}F$  being conjugate to  $\mathbf{T}_{vw}$  in  $\mathbf{G}$ .  $\square$

**Corollary 10.5.** *Let  $\mathbf{I} \xrightarrow{\mathbf{w}} \phi \mathbf{I} \in B^+(\mathcal{I})$  be a maximal  $d/2$ -periodic morphism and assume Conjecture 10.1; then for any  $\rho \in \text{Irr}(\mathbf{G}^F)$  such that  $\langle \rho, R_{\mathbf{L}_I, \dot{w}F}^{\mathbf{G}, F}(\text{Id}) \rangle_{\mathbf{G}^F} \neq 0$  the isogeny  $F^{\delta}$  has a single eigenvalue on the  $\rho$ -isotypic part of  $H_c^*(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$ , equal to  $\lambda_{\rho} q^{\delta \frac{l(\pi/\pi_{\mathbf{I}}) - a_{\rho} - A_{\rho}}{d}}$ .*

*Proof.* This follows immediately, in view of Lemma 10.2, from the comparison between Proposition 10.4 and the Lefschetz formula:

$$|\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m}| = \sum_i (-1)^i \text{Trace}(gF^m | H_c^i(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_{\ell})).$$

$\square$

In view of Corollary 8.38(i) it follows that if  $\langle \rho, R_{\mathbf{L}_I}^{\mathbf{G}}(\text{Id}) \rangle_{\mathbf{G}^F} \neq 0$  then if  $\omega_{\rho} = 1$  then  $\frac{l(\pi/\pi_{\mathbf{I}}) - a_{\rho} - A_{\rho}}{d} \in \mathbb{N}$ , and if  $\omega_{\rho} = \sqrt{q^{\delta}}$  then  $\frac{l(\pi/\pi_{\mathbf{I}}) - a_{\rho} - A_{\rho}}{d} \in \mathbb{N} + 1/2$ .

Assuming Conjectures 10.1, we choose once and for all a specialization  $q^{1/a} \mapsto \zeta^{1/a}$ , where  $a \in \mathbb{N}$  is large enough such that  $\mathcal{H}_w \otimes \overline{\mathbb{Q}}_\ell[q^{1/a}]$  is split. This gives a bijection  $\varphi \mapsto \varphi_q : \text{Irr}(W_w) \rightarrow \text{Irr}(\mathcal{H}_w)$ , and the conjectures give a further bijection  $\varphi \mapsto \rho_\varphi$  between  $\text{Irr}(W_w)$  and the set  $\{\rho \in \text{Irr}(\mathbf{G}^F) \mid \langle \rho, R_{\mathbf{L}_I}^{\mathbf{G}}(\text{Id}) \rangle_{\mathbf{G}^F} \neq 0\}$ , which is such that  $\langle \rho_\varphi, R_{\mathbf{L}_I}^{\mathbf{G}}(\text{Id}) \rangle_{\mathbf{G}^F} = \varphi(1)$ .

**Corollary 10.6.** *Under the assumptions of Corollary 10.5, if  $\omega_\varphi$  is the central character of  $\varphi$ , then*

$$\lambda_{\rho_\varphi} = \omega_\varphi((w\phi)^\delta) \zeta^{-\delta \frac{l(\pi/\pi_1) - a_{\rho_\varphi} - A_{\rho_\varphi}}{d}}.$$

*Proof.* We first note that it makes sense to apply  $\omega_\varphi$  to  $(w\phi)^\delta$ , since  $(w\phi)^\delta$  is a central element of  $W_w$ . Actually  $(w\phi)^\delta$  is a central element of  $B_w$  and maps by the morphism of Conjecture 10.1(iii) to  $F^\delta$ , thus the eigenvalue of  $F^\delta$  on the  $\rho_\varphi$ -isotypic part of  $H_c^*(\mathbf{X}(\mathbf{I}, w\phi))$  is equal to  $\omega_{\varphi_q}((w\phi)^\delta)$ ; thus  $\omega_{\varphi_q}((w\phi)^\delta) = \lambda_{\rho_\varphi} q^{\delta \frac{l(\pi/\pi_1) - a_{\rho_\varphi} - A_{\rho_\varphi}}{d}}$ . The statement follows by applying the specialization  $q^{1/a} \mapsto \zeta^{1/a}$  to this equality.  $\square$

## 11. APPENDIX 1: GOOD $\zeta_d$ -MAXIMAL ELEMENTS IN REDUCTIVE GROUPS

We will describe, in a reductive group  $\mathbf{G}$ , for each  $d$ , a  $\zeta_d$ -good maximal element  $w\phi$  relative to  $W\phi$  and some  $I \subset S$ . Thus the variety  $\mathbf{X}(\mathbf{I}, w\phi)$  will be the one whose cohomology should be a tilting complex for the Broué conjectures for an  $\ell$  dividing  $\Phi(q)$  ( $\Phi$  as in the introduction of Section 9).

Since such an element depends only on the Weyl group, we may assume that  $\mathbf{G}$  is semi-simple and simply connected. Now, a semi-simple and simply connected group is a direct product of restrictions of scalars of simply connected quasi-simple groups. A  $\zeta_d$ -good (resp. maximal) element in a direct product is the product of a  $\zeta_d$ -good (resp. maximal) element in each component. So we reduce immediately to the case of restriction of scalars.

**11.1. Restrictions of scalars.** A restriction of scalars is a group of the form  $\mathbf{G}^n$ , with an isogeny  $F_1$  such that  $F_1(x_0, \dots, x_{n-1}) = (x_1, \dots, x_{n-1}, F(x_0))$ . Thus  $(\mathbf{G}^n)^{F_1} \simeq \mathbf{G}^F$ .

If  $F$  induces  $\phi$  on the Weyl group  $W$  of  $G$  then  $(\mathbf{G}^n, F_1)$  corresponds to the reflection coset  $W^n \cdot \sigma$ , where  $\sigma(x_1, \dots, x_n) = (x_2, \dots, x_n, \phi(x_1))$ .

In the first two propositions of this section, we will study such a “restriction of scalars” for arbitrary complex reflection cosets. Thus we start with a reflection coset  $W\phi$ , with  $W \subset \text{GL}(V)$  a complex reflection group where  $V = \mathbb{C}^r$ , and  $\phi \in N_{\text{GL}(V)}(W)$ . We denote by  $\delta$  the order of  $W\phi$  (the minimal  $i$  such that  $(W\phi)^i = W$ ). We want to study the eigenvalues of elements in the coset  $W^n \cdot \sigma \subset \text{GL}(V^n)$ , where  $\sigma(x_1, \dots, x_n) = (x_2, \dots, x_n, \phi(x_1))$ ; we say that this coset is a *restriction of scalars* of the coset  $W\phi$ .

Recall (see for example [Br]) that, if  $S_W$  is the coinvariant algebra of  $W$  (the quotient of the symmetric algebra of  $V^*$  by the ideal generated by the  $W$ -invariants of positive degree), for any  $W$ -module  $X$  the graded vector space  $(S_W \otimes X^*)^W$  admits a homogeneous basis formed of eigenvectors of  $\phi$ . The degrees of the elements of this basis are called the  $X$ -exponents of  $W$  and the corresponding eigenvalues of  $\phi$  the  $X$ -factors of  $W\phi$ . For  $X = V$ , the  $V$ -exponents  $n_i$  satisfy  $n_i = d_i - 1$  where the  $d_i$ 's are the reflection degrees of  $W$ , and the  $V$ -factors  $\varepsilon_i$  are equal to the

factors of  $W\phi$ . For  $X = V^*$ , the  $n_i - 1$  where  $n_i$  are the  $V^*$ -exponents are called the codegrees  $d_i^*$  of  $W$  and the corresponding  $V^*$ -factors  $\varepsilon_i^*$  are called the cofactors of  $W\phi$ . By Springer [S, 6.4], the  $\zeta$ -rank of  $W\phi$  is equal to  $|\{i \mid \zeta^{d_i} = \varepsilon_i\}|$ . By analogy with the  $\zeta$ -rank, we define the  $\zeta$ -corank of  $W\phi$  as  $|\{i \mid \zeta^{d_i^*} = \varepsilon_i^*\}|$ . By for example [Br, 5.19.2] an eigenvalue is regular if it has same rank and corank.

**Proposition 11.1.** *Let  $W^n \cdot \sigma$  be a restriction of scalars of the complex reflection coset  $W\phi$ . Then the  $\zeta$ -rank (resp. corank) of  $W^n \cdot \sigma$  is equal to the  $\zeta^n$ -rank (resp. corank) of  $W\phi$ .*

*In particular,  $\zeta$  is regular for  $W^n \cdot \sigma$  if and only if  $\zeta^n$  is regular for  $W \cdot \phi$ .*

*Proof.* The pairs of a reflection degree and the corresponding factor of  $\sigma$  for the coset  $W^n \cdot \sigma$  are the pairs  $(d_i, \zeta_n^j \sqrt[n]{\varepsilon_i})$ , where  $i \in \{1, \dots, r\}$ ,  $j \in \{1, \dots, n\}$  and where  $\sqrt[n]{\varepsilon_i}$  represents an  $n$ -th root of  $\varepsilon_i$  (that we choose arbitrarily for each  $i$ ).

Similarly, the pairs of a reflection codegree and the corresponding cofactor are  $(d_i^*, \zeta_n^j \sqrt[n]{\varepsilon_i^*})$ .

In particular the  $\zeta$ -rank of  $W^n \cdot \sigma$  is  $|\{(i, j) \mid \zeta^{d_i} = \zeta_n^j \sqrt[n]{\varepsilon_i}\}|$  and the  $\zeta$ -corank is  $|\{(i, j) \mid \zeta^{d_i^*} = \zeta_n^j \sqrt[n]{\varepsilon_i^*}\}|$ .

Given  $a \in \mathbb{N}$ , there is at most one  $j$  such that the equality  $\zeta^a = \zeta_n^j \sqrt[n]{\varepsilon_i}$  holds, and there is one  $j$  if and only if  $\zeta^{na} = \varepsilon_i$ . Thus we have

$$|\{(i, j) \mid \zeta^{d_i} = \zeta_n^j \sqrt[n]{\varepsilon_i}\}| = |\{i \mid \zeta^{nd_i} = \varepsilon_i\}|$$

and similarly for the corank, whence the two assertions of the statement.  $\square$

We assume now that  $\zeta = \zeta_d$ ; note that  $\zeta_d^n$  is a  $d/k$ -th root of unity, where  $k = \gcd(n, d)$ , but it is not a distinguished root of unity. We have however the following:

**Proposition 11.2.** *Let  $W^n \cdot \sigma$  be a restriction of scalars of the complex reflection coset  $W\phi$  and for  $d \in \mathbb{N}$  let  $k = \gcd(n, d)$ ; then there exists  $m$  such that  $m(n/k) \equiv 1 \pmod{d/k}$  and  $\gcd(m, \delta) = 1$ , and for such an  $m$  the  $\zeta_d$ -rank (resp. corank) of  $W^n \cdot \sigma$  is equal to the  $\zeta_{d/k}$ -rank (resp. corank) of  $W \cdot \phi^m$ .*

*Proof.* We first show that  $m$  exists. Choose an  $m$  such that  $m(n/k) \equiv 1 \pmod{d/k}$ . Since  $m$  is prime to  $d/k$  it is prime to  $\gcd(d/k, \delta)$ . By adding to  $m$  a multiple of  $d/k$  we can add modulo  $\delta$  any multiple of  $\gcd(d/k, \delta)$ , thus we can reach a number prime to  $\delta$ , using the general fact that for any divisor  $\delta'$  of  $\delta$ , the natural projection  $\mathbb{Z}/\delta\mathbb{Z} \xrightarrow{\theta} \mathbb{Z}/\delta'\mathbb{Z}$  is such that  $\theta((\mathbb{Z}/\delta\mathbb{Z})^\times) \supset (\mathbb{Z}/\delta'\mathbb{Z})^\times$ .

By Proposition 11.1, the  $\zeta_d$ -rank (resp. corank) of  $W^n \cdot \sigma$  is equal to the  $\zeta_d^n = \zeta_{d/k}^{n/k}$ -rank (resp. corank) of  $W \cdot \phi$ . Now  $\varepsilon_i^m$  (resp.  $\varepsilon_i^{*m}$ ) are the factors (resp. cofactors) of  $W \cdot \phi^m$  and since  $m$  is prime to  $\delta$  and  $\varepsilon_i^\delta = 1$ , we have  $|\{i \mid \zeta^{d_i} = \varepsilon_i\}| = |\{i \mid (\zeta^m)^{d_i} = \varepsilon_i^m\}|$ , (similarly for  $d_i^*, \varepsilon_i^*$ ); thus the  $\zeta_{d/k}^{n/k}$ -rank (resp. corank) of  $W \cdot \phi$  is equal in turn to the  $\zeta_{d/k}^{m \cdot n/k}$ -rank (resp. corank) of  $W \cdot \phi^m$ . Now, since  $m(n/k) \equiv 1 \pmod{d/k}$ , we have  $\zeta_{d/k}^{m \cdot n/k} = \zeta_{d/k}$ .  $\square$

We now assume, until the end of the subsection, that  $W\phi$  is a real reflection coset of order  $\delta$ , that  $\phi$  preserves a chamber corresponding to the Coxeter system  $(W, S)$ , and that  $\zeta = \zeta_d$  is a distinguished root of unity. We will use the criteria of Lemma 9.11 to check that an element is  $\zeta_d$ -good (resp. maximal).

**Proposition 11.3.** *Under the assumptions of Proposition 11.2, let  $v\phi^m$  be a  $\zeta_{d/k}$ -good element relative to  $W\phi^m$  and  $I$ . Then*

- *If either  $k = 1$  or  $d/k$  is even, define  $w = (w_0, \dots, w_{n-1}) \in W^n$  by  $w_{ik} = \phi^{im}(v)$ , and  $w_j = 1$  if  $j \not\equiv 0 \pmod{k}$*
- *If  $d/k$  is odd and  $k \neq 1$ , by Proposition 9.9 there exists  $v_1, v_2 \in W$  such that  $v\phi^m = v_1\phi^m v_2$  and  $(v\phi^m)^{(\frac{d}{k}-1)/2}v_1 = w_I^{-1}w_0\phi^{m(\frac{d}{k}-1)/2}$ ; define  $w = (w_0, \dots, w_{n-1}) \in W^n$  by*

$$w_j = \begin{cases} \phi^{im}(v_2) & \text{if } j = ik \\ \phi^{(i+1)m}(v_1) & \text{if } j = ik + \lfloor \frac{k}{2} \rfloor \\ 1 & \text{if } j \not\equiv 0, \lfloor \frac{k}{2} \rfloor \pmod{k} \end{cases}$$

*In each case  $w\sigma$  is a  $\zeta_d$ -good element relative to  $W^n\sigma$  and  $\underline{I}$  where  $\underline{I} = (I_0, \dots, I_{n-1}) \subset S^n$  with  $I_j = {}^{w_j w_{j+1} \dots w_{n-1} \phi} I$  and we have  $N_{W^n}(W_{\underline{I}} w\sigma)/W_{\underline{I}} \simeq N_W(W_I v\phi^m)/W_I$ .*

*If moreover  $v\phi^m$  is maximal then  $w\sigma$  is also maximal.*

*Proof.* To lighten the notation, we set  $n' = n/k$  and  $d' = d/k$ .

We recall that  $v\phi^m$  being  $\zeta_{d'}$ -good means  ${}^{v\phi^m} I = I$  and  $v$  is  $I$ -reduced,  $(v\phi^m)^{d'} = \phi^{md'}$ , and  $l((v\phi^m)^i \phi^{-im}) = 2i/d' \cdot l(w_I^{-1}w_0)$  for  $0 \leq i \leq \lfloor \frac{d'}{2} \rfloor$ . We have to show the same conditions for  $w\sigma$ , that is

- (i)  ${}^{w\sigma}(I_0, \dots, I_{n-1}) = (I_0, \dots, I_{n-1})$  and  $w$  is  $(I_0, \dots, I_{n-1})$ -reduced.
- (ii)  $(w\sigma)^d = \sigma^d$ .
- (iii)  $l((w\sigma)^i \sigma^{-i}) = \frac{2in}{d} l(w_I^{-1}w_0)$  for  $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$ .

We first note:

**Lemma 11.4.**  $\phi^{d'}$  stabilizes  $v$  and  $I$  (thus  $\phi^{\gcd(d', \delta)}$  also).

*Proof.* As  $(v\phi^m)^{d'} = \phi^{md'}$ , we find that  $\phi^{md'}$  stabilizes  $v\phi^m$  and  $I$ , thus  $v$  and  $I$ . Since  $m$  is invertible modulo  $\delta$ , we get that  $\phi^{d'}$  stabilizes  $v$  and  $I$ .  $\square$

We first check that  $\underline{I} \subset S^n$ . In the case  $d'$  even, each  $I_j$  is of the form  $\phi^{im}(v)\phi^{(i+1)m}(v)\dots\phi^{(n'-1)m}(v)\phi I$  (where  $ik$  is the smallest multiple of  $k$  greater than  $j$ ).

If  $d'$  is odd  $I_j$  is either as above or of the form  $\phi^{(i-1)m}(v_2)\phi^{im}(v)\phi^{(i+1)m}(v)\dots\phi^{(n'-1)m}(v)\phi I$ . In the first case, since  $1 - mn' \equiv 0 \pmod{d'}$  and  $\phi^{d'}$  stabilizes  $I$ , by Lemma 11.4, we can write

$$I_j = \phi^{im}(v)\phi^{(i+1)m}(v)\dots\phi^{(n'-1)m}(v)\phi^{mn'} I = \phi^{im}(v\phi^m)^{n'-i} I = \phi^{im} I \subset S.$$

In the second case, if we put  $J = I^{v_1} = {}^{v_2\phi^m} I$ , a subset of  $S$  by Proposition 9.9, we get  $I_j = \phi^{(i-1)m}(v_2)\phi^{im} I = \phi^{(i-1)m} J$ .

We now check (i). The verification of  ${}^{w\sigma}(I_0, \dots, I_{n-1}) = (I_0, \dots, I_{n-1})$  reduces to  ${}^{w_0 w_1 \dots w_{n-1} \phi} I = I$ , which itself reduces to  ${}^{v\phi^m(v)\dots\phi^{(n'-1)m}(v)\phi} I = I$ , which is true by the case  $i = 0$  of the above computation. Similarly, checking that  $w\sigma$  is  $(I_0, \dots, I_{n-1})$ -reduced reduces to the check that for each  $j$  the element  $w_j$  is  $I_j$ -reduced, where  $I_j = {}^{w_j w_{j+1} \dots w_{n-1} \phi} I = I^{w_0 \dots w_{j-1}}$ , or equivalently that  $w_0 \dots w_{j-1}$  is  $I$ -reduced. Thus in the  $d'$  even case we have to check that  $v\phi^m(v) \dots \phi^{im}(v)$  is  $I$ -reduced for  $0 \leq i < n'$ . This results from the fact that  $v$  is  $I$ -reduced and that  $v\phi^m$  normalizes  $I$ . In the  $d'$  odd case we have also to check that  $v\phi^m(v) \dots \phi^{(i-1)m}(v)\phi^{im}(v_1)$  is  $I$ -reduced, which follows from the fact that  $v$  is  $I$ -reduced, that  $v\phi^m$  normalizes  $I$  and that  $v_1$  is also  $I$ -reduced, which we know by Proposition 9.9.

For checking (ii) and (iii) we compute  $(w\sigma)^i$ . For any  $(w_0, \dots, w_{n-1}) \in W^n$  we have  $\sigma(w_0, \dots, w_{n-1}) = (w_1, \dots, w_{n-1}, \phi(w_0))\sigma$ , thus we find that if we define for all  $j$  the element  $w_j = \phi^{\frac{j-j_0}{n}}(w_{j_0}) = \phi^{\lfloor \frac{j}{n} \rfloor}(w_{j_0})$  where  $j_0 \equiv j \pmod{n}$  and  $0 \leq j_0 < n$ , we have

$$(w\sigma)^i = (w_0 \dots w_{i-1}, w_1 \dots w_i, \dots, w_{n-1} \dots w_{i+n-2})\sigma^i.$$

Each product  $w_u w_{u+1} \dots w_{u+i-1}$  appearing in the above expression is, up to applying a power of  $\phi$ , of the form  $(v\phi^m)^j \phi^{-mj}$  or in the  $d'$  odd case additionally of one of the forms  $(v_2 \phi^m v_1)^j \phi^{-mj}$ ,  $(v_1 \phi^m v_2)^j v_1 \phi^{-mj}$  or  $v_2 (v_1 \phi^m v_2)^j \phi^{-mj}$ , for some  $j$  which depends on  $u$  and  $i$ . If  $i$  is a multiple of  $k$  the last two forms do not appear and  $j = i/k$ . In particular if  $i = d$  we get either  $(v\phi^m)^{d/k} \phi^{-md/k}$  or  $(v_2 \phi^m v_1)^{d/k} \phi^{-md/k}$ . Since  $(v\phi^m)^{d'} = \phi^{d'm}$  we have also  $(v_2 \phi^m v_1)^{d'} = v_1^{-1} (v\phi^m)^{d'} v_1 = \phi^{md'}$ , since  $v$ , hence  $v_1$ , is  $\phi^{md'}$ -stable, whence (ii).

To check (iii) it is enough check it for  $i = 1$ , which is clear since  $l(w) = n'l(v) = \frac{2in}{d}l(w_I^{-1}w_0)$  and  $l(v) = l(v_1) + l(\phi^m(v_2))$  (by Proposition 9.9) and to check that in a product  $w_u w_{u+1} \dots w_{u+i-1}$  the lengths add for all  $i \leq \lfloor \frac{d}{2} \rfloor$ : the lengths will then add in  $(w\sigma)^i$  for  $i \leq \lfloor \frac{d}{2} \rfloor$  which gives (iii). In the  $d'$  even case this is a result of the lengths adding for  $(v\phi^m)^j \phi^{-mj}$ . In the  $d'$  odd case, we know by Proposition 9.9 that the lengths add in a product of at most  $d'$  terms of the form  $v_1^{\phi^m} v_2^{\phi^m} v_1 \dots$  or of the form  $v_2^{\phi^m} v_1^{\phi^m} v_2 \dots$ . We claim that to get more than  $d'$  non-trivial terms in the product  $w_u w_{u+1} \dots w_{u+i-1}$  we need  $i > \lfloor \frac{d}{2} \rfloor$ . The maximal number of non-trivial terms is obtained when the first or the last term is non trivial. To get  $d' + 1$  non-trivial terms we need  $i \geq \frac{d'+1}{2}k + \lfloor \frac{k}{2} \rfloor$ , since  $d' + 1$  is even. But  $\frac{d'+1}{2}k + \lfloor \frac{k}{2} \rfloor = \lfloor \frac{kd'}{2} \rfloor + k > \frac{d}{2}$ , whence our claim.

Computing now  $N_{W^n}(W_{\underline{I}} w\sigma)$ , we find that  $(g_0, \dots, g_{n-1})$  normalizes  $(W_{\underline{I}} w\sigma)$  if and only if:

$$\begin{aligned} g_0 W_{I_0} &= W_{I_0}^{w_0} g_1 \\ &\dots = \dots \\ g_{n-2} W_{I_{n-2}} &= W_{I_{n-2}}^{w_{n-2}} g_{n-1} \\ g_{n-1} W_{I_{n-1}} &= W_{I_{n-1}}^{w_{n-1}\phi} g_0 \end{aligned}$$

which, using the value  $I_j = {}^{w_j \dots w_{n-1}\phi} I = I^{w_0 \dots w_{j-1}}$  becomes

$$\begin{aligned} g_0 W_I &= W_I^{w_0} g_1 \\ &\dots = \dots \\ {}^{w_0 \dots w_{n-3}} g_{n-2} W_I &= W_I^{w_0 \dots w_{n-2}} g_{n-1} \\ {}^{w_0 \dots w_{n-2}} g_{n-1} W_I &= W_I^{w_0 \dots w_{n-1}\phi} g_0 \end{aligned}$$

We now notice that an equality  $aW_I = W_I b$  is equivalent to:  $a$  normalizes  $W_I$ , and  $aW_I = bW_I$ . Thus our equations are equivalent to:  $g_0$  normalizes  $W_I$ , the cosets  $W_I g_0, \dots, W_I {}^{w_0 \dots w_{n-2}} g_{n-1}$  are equal (thus determined by  $g_0$ ) and  $W_I g_0 = W_I {}^{w_0 \dots w_{n-1}\phi} g_0$ . The last equality means that  $g_0$  normalizes  $W_I w_0 \dots w_{n-1}\phi$ ; we find  $N_{W^n}(W_{\underline{I}} w\sigma)/W_{\underline{I}} \simeq N_W(W_I w_0 \dots w_{n-1}\phi)/W_I = N_W(W_I (v\phi^m)^{n'} \phi^{1-mn'})/W_I$ . Since  $1 - mn' \equiv 0 \pmod{d'}$ , by Lemma 11.4  $\phi^{1-mn'}$  commutes with  $v\phi^m$ , thus  $((v\phi^m)^{n'} \phi^{1-mn'})^m = (v\phi^m)^{n'm} \phi^{m-mn'm}$ . Let us write  $n'm = ad' + 1$ ; using that  $(v\phi^m)^{d'} = \phi^{md'}$  we get  $(v\phi^m)^{n'm} \phi^{m-mn'm} = (v\phi^m)^{ad'+1} \phi^{-amn'} = v\phi^m$ , thus the above coset has same normalizer as  $W_I v\phi^m$ .

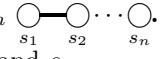
Assume now that  $v\phi^m$  is maximal, that is  $W_I v\phi^m$  has  $\zeta_{d'}$ -rank equal to 0. We prove the same for  $w\sigma$ , that is  $(W_{I_0} \times \dots \times W_{I_{n-1}})w\sigma$  has  $\zeta_d$ -rank 0. Identifying  $I_j$  to  $I$  via  $w_j \dots w_{n-1}\phi$ , the coset  $(W_{I_0} \times \dots \times W_{I_{n-1}})w\sigma$  identifies to  $W_I^n\sigma'$  where

$$\begin{aligned}\sigma'(x_0, \dots, x_{n-1}) &= (x_1, \dots, x_{n-1}, (w_0 \dots w_{n-1}\phi)(x_0)) \\ &= (x_1, \dots, x_{n-1}, ((v\phi^m)^{n'}\phi^{1-mn'})(x_0)),\end{aligned}$$

since in each case we have  $w_0 \dots w_{n-1}\phi = ((v\phi^m)^{n'}\phi^{1-mn'})(x_0)$ . Now by Proposition 11.2 the  $\zeta_d$ -rank of this last coset is equal to the  $\zeta_{d'}$ -rank of the coset  $W((v\phi^m)^{n'}\phi^{1-mn'})^m$ . But we have checked above that  $((v\phi^m)^{n'}n\phi^{1-mn'})^m = v\phi^m$ , thus the sought  $\zeta_{d'}$ -rank is the same as the  $\zeta_{d'}$ -rank of  $W_I v\phi^m$  which is 0 by assumption.  $\square$

**11.2. Case of irreducible Coxeter cosets.** We now look at the case of quasi-simple simply connected reductive groups  $\mathbf{G}$ , or equivalently at the case of irreducible Coxeter cosets  $W\phi$ . We will look at any real Coxeter coset  $W\phi$  since it is not much more effort than to look just at the rational ones.

We use the classification. We are going to give, for each irreducible type and each possible  $d$ , a representative  $w\phi$  of the  $\zeta_d$ -good maximal elements, describing the corresponding  $I$ ; since conjecturally for a given  $d$  all such elements are conjugate in the braid group, this describes all the  $\zeta_d$ -good maximal elements. We also describe the relative complex reflection group  $W(w\phi) := N_W(V)/C_W(V)$ , where  $V$  is the  $\zeta_d$ -eigenspace of  $w\phi$ . In the cases where the injection  $C_{W'}(w\phi) \rightarrow N_W(V)/C_W(V) = W(w\phi)$  of the remark after Lemma 9.14, is surjective, where  $W' = C_W(V_1)$  and  $V_1$  is the fixed point subspace of  $w\phi$  in the space spanned by the root lines of  $W_I$ , we use it to deduce  $W(w\phi)$  from  $W' = C_W(V_1)$  since the centralizers of regular elements are known (see [BM, Annexe 1]).

**Types  $A_n$  and  ${}^2A_n$**    ${}^2A_n$  is defined by the diagram automorphism  $\phi$  which exchanges  $s_i$  and  $s_{n+1-i}$ .

For any integer  $1 < d \leq n+1$ , we define

$$v_d = s_1 s_2 \dots s_{n-\lfloor \frac{d}{2} \rfloor} s_n s_{n-1} \dots s_{\lfloor \frac{d+1}{2} \rfloor} \text{ and } J_d = \{s_i \mid \lfloor \frac{d+1}{2} \rfloor + 1 \leq i \leq n - \lfloor \frac{d}{2} \rfloor\}.$$

If  $d$  is odd we have  $v_d = v'_d\phi v'_d$ , where  $v'_d = s_1 s_2 \dots s_{n-\lfloor \frac{d}{2} \rfloor}$ .

Now, for  $1 < d \leq n+1$ , let  $kd$  be the largest multiple of  $d$  less than or equal to  $n+1$ , so that  $\frac{n+1}{2} < kd \leq n+1$  and  $k = \lfloor \frac{n+1}{d} \rfloor$ . We then define  $w_d = v_{kd}^k$ ,  $I_d = J_{kd}$  and if  $d$  is odd we define  $w'_d$  by

$$w'_d\phi = \begin{cases} (v'_{kd}\phi)^k & \text{if } k \text{ is odd,} \\ v_{kd}^{k/2}\phi & \text{if } k \text{ is even,} \end{cases}$$

**Theorem 11.5.** For  $W = W(A_n)$ ,  $\zeta_d$ -good maximal elements exist for  $1 < d \leq n+1$ ; a representative is  $w_d$ , with  $I = I_d$  and  $W(w_d) = G(d, 1, \lfloor \frac{n+1}{d} \rfloor)$ .

For  $W\phi$ ,  $\zeta_d$ -good maximal elements exist for the following  $d$  with representatives as follows:

- $d \equiv 0 \pmod{4}$ ,  $1 < d \leq n+1$ ; a representative is  $w_d\phi$  with  $I = I_d$  and  $W(w_d\phi) = G(d, 1, \lfloor \frac{n+1}{d} \rfloor)$ .
- $d \equiv 2 \pmod{4}$ ,  $1 < d \leq 2(n+1)$ ; a representative is  $w'_{d/2}\phi$  with  $I = I_{d/2}$  and  $W(w'_{d/2}\phi) = G(d/2, 1, \lfloor \frac{2(n+1)}{d} \rfloor)$ .

- $d$  odd,  $1 < d \leq \frac{n+1}{2}$ . If  $d \neq 1$  a representative is  $w_{2d}^2 \phi$  with  $I = I_{2d}$  and  $W(w_{2d}^2 \phi) = G(2d, 1, \lfloor \frac{n+1}{2d} \rfloor)$ .

*Proof.* We identify the Weyl group of type  $A_n$  as usual with  $\mathfrak{S}_{n+1}$  by  $s_i \mapsto (i, i+1)$ ; the automorphism  $\phi$  maps to the exchange of  $i$  and  $n+2-i$ . An easy computation shows that the element  $v_d$  maps to the  $d$ -cycle  $(1, 2, \dots, \lfloor \frac{d+1}{2} \rfloor, n+1, n, \dots, n+2-\lfloor \frac{d}{2} \rfloor)$  and that for  $d$  odd  $v'_d$  maps to the cycle  $(1, 2, \dots, n-\frac{d-3}{2})$ .

**Lemma 11.6.** *If  $d$  is even  $v_d$  and  $w_d$  are  $\phi$ -stable. If  $d$  is odd we have  $w_d = w'_d \cdot \phi w'_d$ .*

*Proof.* That  $d$  is even implies  $\lfloor \frac{d+1}{2} \rfloor = \lfloor \frac{d}{2} \rfloor$ , thus in the above cycle  $\phi$  exchanges the two sequences  $1, 2, \dots, \lfloor \frac{d+1}{2} \rfloor$  and  $n+1, n, \dots, n+2-\lfloor \frac{d}{2} \rfloor$ , thus  $v_d$  is  $\phi$ -stable. The same follows for  $w_d$ , with  $k = \lfloor \frac{n+1}{d} \rfloor$ , since  $kd$  is even if  $d$  is even.

For  $d$  odd we have

$$w'_d \cdot \phi w'_d = (w'_d \phi)^2 = \begin{cases} (v'_{kd} \phi)^{2k} & \text{if } k \text{ is odd,} \\ v_{kd}^{k/2} \cdot \phi(v_{kd}^{k/2}) & \text{if } k \text{ is even.} \end{cases}$$

If  $k$  is odd we have  $(v'_{kd} \phi)^{2k} = (v'_{kd} \phi v'_{kd})^k = v'_{kd}^k = w_d$ . If  $k$  is even then  $v_{kd}$  is  $\phi$ -stable thus  $v_{kd}^{k/2} \cdot \phi(v_{kd}^{k/2}) = v_{kd}^k = w_d$ .  $\square$

**Lemma 11.7.** *For  $1 < d \leq n+1$ ,*

- the element  $v_d$  is  $J_d$ -reduced and stabilizes  $J_d$ .
- the element  $w_d$  is  $I_d$ -reduced and stabilizes  $I_d$ .
- for  $d$  odd, the element  $v'_d$  is  $J_d$ -reduced and  $v'_d \phi$  stabilizes  $J_d$ .
- for  $d$  odd, the element  $w'_d$  is  $I_d$ -reduced and  $w'_d \phi$  stabilizes  $I_d$ .

*Proof.* The property for  $w_d$  (resp.  $w'_d$ ) follows from that for  $v_d$  (resp.  $v'_d$ ) and the definitions since being  $I_d$ -reduced and stabilizing  $I_d$  are properties stable by taking a power.

It is clear on the expression of  $v_d$  as a cycle that it fixes  $i$  and  $i+1$  if  $s_i \in J_d$  thus it fixes the simple roots corresponding to  $J_d$ , whence the lemma for  $v_d$ .

For  $d$  odd,  $1 < d \leq n+1$ , an easy computation shows that  $v'_d = (1, 2, \dots, n-\frac{d-3}{2})$ , and that  $v'_d \phi$  preserves the simple roots corresponding to  $J_d$ .  $\square$

**Lemma 11.8.** *For  $1 < d \leq n+1$  and for  $0 < i \leq \lfloor \frac{d}{2} \rfloor$ , we have*

- $l(v_d^i) = \frac{2i}{d} l(w_{J_d}^{-1} w_0)$  and  $l(w_d^i) = \frac{2i}{d} l(w_{I_d}^{-1} w_0)$
- (for  $d$  odd)  $l((v'_d \phi)^i \phi^{-i}) = \frac{i}{d} l(w_{J_d}^{-1} w_0)$  and  $l((w'_d \phi)^i \phi^{-i}) = \frac{i}{d} l(w_{I_d}^{-1} w_0)$ .

*Proof.* It is straightforward to see that the result for  $w_d$  (resp.  $w'_d$ ) results from the result for  $v_d$  (resp.  $v'_d$  or  $v_d$ ) and the definitions.

Note that the group  $W_{J_d}$  is of type  $A_{n-d}$ , thus  $l(w_{J_d}^{-1} w_0) = \frac{n(n+1)}{2} - \frac{(n-d)(n-d+1)}{2} = \frac{(2n-d+1)d}{2}$ .

We first prove the result for  $v_d$  and  $v'_d$  when  $i = 1$ . For odd  $d$  we have by definition  $l(v'_d) = n - \frac{d-1}{2} = \frac{2n-d+1}{2}$  which is the formula we want for  $v'_d$ . To find the length of  $v_d$  one can use that  $s_n s_{n-1} \dots s_{\lfloor \frac{d+1}{2} \rfloor}$  is  $\{s_1, s_2, \dots, s_{n-1}\}$ -reduced, thus adds to  $s_1 s_2 \dots s_{n-\lfloor \frac{d}{2} \rfloor}$ , which gives  $l(v_d) = 2n - d + 1$ , the result for  $v_d$ .

We now show by direct computation that when  $d$  is even  $v_d^{d/2} = w_{J_d}^{-1} w_0$ . Raising the cycle  $(1, 2, \dots, \frac{d}{2}, n+1, n, \dots, n+2-\frac{d}{2})$  to the  $d/2$ -th power we get  $(1, n+1)(2, n) \dots (\frac{d}{2}, n+2-\frac{d}{2})$  which gives the result since  $w_{J_d} = (\frac{d}{2}+1, n+$

$1 - \frac{d}{2}) \dots (\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n+1}{2} \rfloor)$ . The lemma follows for  $v_d$  with  $d$  even since its truth for  $i = 1$  and  $i = \frac{d}{2}$  implies its truth for all  $i$  between these values.

We show now similarly that for odd  $d$  we have  $(v'_d \phi)^d = w_{J_d}^{-1} w_0 \phi^d$ . Since  $\phi$  acts on  $W$  by the inner automorphism given by  $w_0$ , this is the same as  $(v'_d w_0)^d = w_{J_d}$ . We find that  $(1, 2, \dots, n - \frac{d-3}{2}) w_0 = (1, n+1, 2, n, 3, n-1, \dots, n - \frac{d-5}{2}, \frac{d+1}{2}) (\frac{d+3}{2}, n - \frac{d-3}{2}) \dots (\lfloor \frac{n+3}{2} \rfloor, \lfloor \frac{n+4}{2} \rfloor)$  as a product of disjoint cycles, which gives the result since  $(1, n+1, 2, n, 3, n-1, \dots, n - \frac{d-5}{2}, \frac{d+1}{2})$  is a  $d$ -cycle and  $(\frac{d+3}{2}, n - \frac{d-3}{2}) \dots (\lfloor \frac{n+3}{2} \rfloor, \lfloor \frac{n+4}{2} \rfloor) = w_{J_d}$ . This proves the lemma for  $w'_d$  by interpolating the other values of  $i$  as above.

It remains the case of  $v_d$  for odd  $d$ . We then have  $v_d = (v'_d \phi)^2$  where the lengths add, and we deduce the result for  $v_d$  from the result for  $v'_d$ .  $\square$

**Lemma 11.9.** *The following elements are  $\zeta_d$ -good*

- For  $1 < d \leq n+1$ , the elements  $v_d$  and  $w_d$ .
- For  $d \equiv 0 \pmod{4}$ ,  $d \leq n+1$  the elements  $v_d \phi$  and  $w_d \phi$ .
- For  $d \equiv 2 \pmod{4}$ ,  $d \leq 2(n+1)$  the elements  $v'_{d/2} \phi$  and  $w'_{d/2} \phi$ .
- For  $d$  odd,  $d \leq \frac{n+1}{2}$  the elements  $v_{2d}^2 \phi$  and  $w_{2d}^2 \phi$ .

*Proof.* In view of the previous lemmas, the only thing left to check is that in each case, the chosen element  $x$  in  $W$  (resp.  $W\phi$ ) satisfies  $x^d = 1$  (resp.  $(x\phi)^d = \phi^d$ ). Once again, it is easy to check that the property for  $w_d$  (resp.  $w'_d$ ) results from that for  $v_d$  (resp.  $v'_d$  or  $v_d$ ) and the definitions.

It is clear that  $v_d^d = 1$  since then it is a  $d$ -cycle, from which it follows that when  $d \equiv 2 \pmod{4}$  we have  $(v'_{d/2} \phi)^d = v'_{d/2}^{d/2} = 1$ . The other cases are obvious.  $\square$

To prove the theorem, it remains to check that:

- The possible  $d$  for which the  $\zeta_d$ -rank of  $W$  (resp.  $W\phi$ ) is non-zero are as described in the theorem. In the untwisted case they are the divisors of one of the degrees, which are  $2, \dots, n+1$ . In the twisted case the pairs of degrees and factors are  $(2, 1), \dots, (i, (-1)^i), \dots, (n+1, (-1)^{n+1})$  and we get the given list by the formula for the  $\zeta_d$ -rank recalled above Proposition 11.1.
- The coset  $W_I w\phi$  has  $\zeta_d$ -rank 0 on the subspace spanned by the root lines of  $W_I$ . For this we first have to describe the type of the coset, which is a consequence of the analysis we did to show that  $w\phi$  stabilizes  $I$ . We may assume  $I$  non-empty.

Let us look first at the untwisted case. We found that  $w_d$  acts trivially on  $I_d$ , so the coset is of untwisted type  $A_{n-kd}$  where  $k = \lfloor \frac{n+1}{d} \rfloor$ . Since  $1 + n - kd < d$  by construction, this coset has  $\zeta_d$ -rank 0.

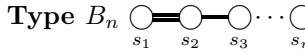
In the twisted case, if  $d \equiv 0 \pmod{4}$ , the coset is  $W_{I_d} w_d \phi$ , which since  $w_d$  acts trivially on  $I_d$  and  $\phi$  acts by the non-trivial diagram automorphism, is of type  ${}^2 A_{n-kd}$  where  $k = \lfloor \frac{n+1}{d} \rfloor$ . Since  $n - kd = n - \lfloor \frac{n+1}{d} \rfloor d < d - 1$ , this coset has  $\zeta_d$ -rank 0.

If  $d$  is odd, the coset is  $W_{I_{2d}} w_{2d}^2 \phi$ , which since  $w_{2d}$  acts trivially on  $I_{2d}$  and  $\phi$  acts by the non-trivial diagram automorphism, is of type  ${}^2 A_{n-2kd}$  where  $k = \lfloor \frac{n+1}{2d} \rfloor$ . Since  $n - 2kd = n - \lfloor \frac{n+1}{2d} \rfloor 2d < 2d$ , this coset has  $\zeta_d$ -rank 0.

Finally, if  $d \equiv 2 \pmod{4}$ , the coset is  $W_{I_{d/2}} w'_{d/2} \phi$ . Let  $k = \lfloor \frac{2(n+1)}{d} \rfloor$ ; then  $W_{I_{d/2}}$  is of type  $A_{n-kd/2}$ . If  $k$  is even then  $w'_{d/2} = w_{kd/2}^{k/2}$  and the coset is of type  ${}^2 A_{n-kd/2}$ . Since  $n - kd/2 < d/2 - 1$ , this coset has  $\zeta_d$ -rank 0. Finally if  $k$  is odd  $w'_{d/2} \phi = (w'_{kd/2} \phi)^k$ . Since  $kd/2$  is odd, we found that  $w'_{kd/2} \phi$  acts trivially on  $I_{d/2}$  so the coset is of type  $A_{n-kd/2}$ , and has also has  $\zeta_d$ -rank 0.

• Determine the group  $W(w\phi)$  (resp.  $W(w)$ ) in each case, We first give  $V_1$  and the coset  $C_W(V_1)w\phi$  or  $C_W(V_1)w$ . In the untwisted case  $w_d$  acts trivially on the roots of  $W_{I_d}$ , hence  $V_1$  is spanned by these roots and  $C_W(V_1)$  is generated by the reflection with respect to the roots orthogonal to those, which gives that  $C_W(V_1)$  is of type  $A_{d[\frac{n+1}{d}]-1}$  if  $d \nmid n$  and  $A_n$  otherwise. In the twisted case if  $d \equiv 0 \pmod{4}$  since  $w_d$  acts trivially on the roots of  $W_{I_d}$  the space  $V_1$  is spanned by the sums of the orbits of the roots under  $\phi$  which is the non-trivial automorphism of that root system. Hence the type of the coset  $C_W(V_1)w_d\phi$  is  ${}^2A_{d[\frac{n+1}{d}]-1}$  if  $n$  is odd and  ${}^2A_{d[\frac{n+1}{d}]}$  if  $n$  is even. If  $d$  is odd a similar computation gives that the type of the coset  $C_W(V_1)w_{2d}^2\phi$  is  ${}^2A_{2d[\frac{n+1}{2d}]-1}$  if  $n$  is odd and  ${}^2A_{2d[\frac{n+1}{2d}]}$  if  $n$  is even. If  $d \equiv 2 \pmod{4}$   $w'_{d/2}\phi$  acts also by the non-trivial automorphism on  $W_{I_{d/2}}$  and we get that the coset  $C_W(V_1)w'_{d/2}\phi$  is of type  ${}^2A_{\frac{d}{2}[\frac{2(n+1)}{d}]-1}$  if  $n$  and  $[\frac{2(n+1)}{d}]$  have the same parity and  ${}^2A_{\frac{d}{2}[\frac{2(n+1)}{d}]-1}$  otherwise.

Knowing the type of the coset in each case, we deduce the group  $W(w\phi)$  (resp.  $W(w)$ ) as in the remark at the beginning of Subsection 11.2.  $\square$

**Type  $B_n$**   For  $d$  even,  $2 \leq d \leq 2n$  we define

$$v_d = s_{n+1-d/2} \dots s_2 s_1 s_2 \dots s_n \text{ and } J_d = \{s_i \mid 1 \leq i \leq n-d/2\}.$$

Note that  $v_{2n}$  is the Coxeter element  $s_1 s_2 \dots s_n$ . Now for  $1 \leq d \leq 2n$ , that we require even if  $d > n$ , we define  $w_d$  as follows: let  $kd$  be the largest even multiple of  $d$  less than or equal to  $2n$  so that  $k = \lfloor \frac{2n}{d} \rfloor$  if  $d$  is even and  $k = 2\lfloor \frac{n}{d} \rfloor$  if  $d$  is odd. We define  $w_d = v_{kd}^k$  and  $I_d = J_{kd}$ .

**Theorem 11.10.** *For  $W = W(B_n)$ ,  $\zeta_d$ -good maximal elements exist for odd  $d$  less than or equal to  $n$  and even  $d$  less than or equal to  $2n$ . A representative is  $w_d$ , with  $I = I_d$ ; we have  $W(w_d) = G(d, 1, \lfloor \frac{2n}{d} \rfloor)$  if  $d$  is even and  $W(w_d) = G(2d, 1, \lfloor \frac{n}{d} \rfloor)$  if  $d$  is odd.*

*Proof.* We identify as usual the Weyl group of type  $B_n$  to the group of signed permutations on  $\{1, \dots, n\}$  by  $s_i \mapsto (i-1, i)$  for  $i \geq 2$  and  $s_1 \mapsto (1, -1)$ . The element  $v_d$  maps to the  $d$ -cycle (or signed  $d/2$ -cycle) given by  $(n+1-d/2, n+2-d/2, \dots, n-1, n, d/2-n-1, d/2-n-2, \dots, -n)$ . This element normalizes  $J_d$  and acts trivially on the corresponding roots, so is  $J_d$ -reduced. The same is thus true for  $w_d$  and  $I_d$ , since these properties carry to powers.

**Lemma 11.11.** *For  $0 < i \leq \lfloor \frac{d}{2} \rfloor$  we have  $l(v_d^i) = \frac{2i}{d}l(w_{J_d}^{-1}w_0)$  and  $l(w_d^i) = \frac{2i}{d}l(w_{I_d}^{-1}w_0)$ .*

*Proof.* As in Lemma 11.8 it is sufficient to prove the lemma for  $v_d$ , which we do now. To find the length of  $v_d$  we note that  $s_1 s_2 \dots s_n$  is  $\{s_2, s_3, \dots, s_n\}$ -reduced so that the lengths of  $s_{n+1-d/2} \dots s_2$  and of  $s_1 s_2 \dots s_n$  add, whence  $l(v_d) = 2n - d/2$ . Since  $l(w_0) = n^2$  and  $l(w_{J_d}) = (n-d/2)^2$  we have  $l(w_{I_d}^{-1}w_0) = nd - d^2/4$ , which gives the result for  $i = 1$ . Written as permutations  $w_0$  is the product of all sign changes and  $w_{I_d}$  is the product of all sign changes on the set  $\{1, \dots, n-d/2\}$ ; a direct computation shows that  $v_d^{d/2}$  is the product of all sign changes on  $\{n+1-d/2, \dots, n\}$ , hence  $v_d^{d/2} = w_{I_d}^{-1}w_0$ . The lemma follows for the other values of  $d$ .  $\square$

Since  $v_d^{d/2} = w_{I_d}^{-1} w_0$  we have  $v_d^d = 1$ , so the same property is true for  $w_d$ , thus the above lemma shows that  $v_d$  and  $w_d$  are  $\zeta_d$ -good elements.

Note also that Theorem 11.10 describes all  $d$  such that  $W$  has non-zero  $\zeta_d$ -rank since the degrees of  $W(B_n)$  are all the even integers from 2 to  $2n$ . We prove now the maximality property 9.11(iv) for  $w_d$ . If  $k$  is as in the definition of  $w_d$ , the group  $W_{I_d}$  is a Weyl group of type  $B_{n-kd/2}$  and  $w_d$  acts trivially on  $I_d$ . Since  $n-kd/2 < d$  the  $\zeta_d$ -rank of  $W_{I_d} w_d$  is zero on the subspace spanned by the roots corresponding to  $I_d$ .

It remains to get the type of  $W(w_d)$ . Since  $w_d$  acts trivially on  $I_d$  the space  $V_1$  of Lemma 9.14 is spanned by the root lines of  $W_{I_d}$  and  $C_W(V_1)$  is spanned by the roots orthogonal to those, so is of type  $B_{kd/2}$ . We then deduce the group  $W(w_d)$  as in the remark at the beginning of Subsection 11.2, as the centralizer of a  $\zeta_d$ -regular element in a group of type  $B_{kd/2}$ .  $\square$

**Types  $D_n$  and  ${}^2D_n$**    ${}^2D_n$  is defined by the diagram automorphism  $\phi$  which exchanges  $s_1$  and  $s_2$  and fixes  $s_i$  for  $i > 2$ .

For  $d$  even,  $2 \leq d \leq 2(n-1)$  we define

$$v_d = s_{n+1-d/2} \dots s_3 s_2 s_1 s_3 \dots s_n \text{ and } J_d = \begin{cases} \emptyset & \text{if } d = 2(n-1) \\ \{s_i \mid 1 \leq i \leq n-d/2\} & \text{otherwise.} \end{cases}$$

Note that  $v_{2(n-1)}$  is a Coxeter element. Then for  $1 \leq d \leq 2(n-1)$ , that we require even if  $d > n$ , we let  $kd$  be the largest even multiple of  $d$  less than  $2n$ , so that  $k = \lfloor \frac{2n-2}{d} \rfloor$  if  $d$  is even and  $k = 2\lfloor \frac{n-1}{d} \rfloor$  if  $d$  is odd, and define  $w_d = v_{kd}^k$  and  $I_d = J_{kd}$ .

Note that  $v_d$ , and thus  $w_d$ , are  $\phi$ -stable.

**Theorem 11.12.**

- For  $W = W(D_n)$  there exist  $\zeta_d$ -good maximal elements for odd  $d$  less than or equal to  $n$  and even  $d$  less than or equal to  $2(n-1)$ .

When  $d$  does not divide  $n$  a representative is  $w_d$ , with  $I = I_d$ ; in this case, if  $d$  is odd  $W(w_d) = G(2d, 1, \lfloor \frac{n-1}{d} \rfloor)$  and if  $d$  is even  $W(w_d) = G(d, 1, \lfloor \frac{2n-2}{d} \rfloor)$ .

If  $d|n$  a representative is  $w_n^{n/d}$  where  $w_n = s_1 s_2 s_3 \dots s_n s_2 s_3 \dots s_{n-1}$ . In this case  $I = \emptyset$  and  $W(w_n^{n/d}) = G(2d, 2, n/d)$ .

- For  $W\phi$  there exist  $\zeta_d$ -good maximal elements for odd  $d$  less than  $n$ , for even  $d$  less than  $2(n-1)$  and for  $d = 2n$ . Except in the case when  $d$  divides  $2n$  and  $2n/d$  is odd a representative is  $w_d\phi$ , with  $I = I_d$  and  $W(w_d\phi) = G(2d, 1, \lfloor \frac{n-1}{d} \rfloor)$  if  $d$  is odd and  $W(w_d\phi) = G(d, 1, \lfloor \frac{2n-2}{d} \rfloor)$  if  $d$  is even. In the excluded case a representative is  $(w_{2n}\phi)^{2n/d}$  where  $w_{2n} = s_1 s_3 s_4 \dots s_n$ . In this case  $I = \emptyset$  and  $W((w_{2n}\phi)^{2n/d}) = G(d, 2, 2n/d)$ .

*Proof.* The cases  $D_n$  with  $d|n$  or  ${}^2D_n$  with  $d|2n$  and  $2n/d$  odd involve regular elements, so are dealt with in [BM]. We thus consider only the other cases.

We identify the Weyl group of type  $D_n$  to the group of signed permutations on  $\{1, \dots, n\}$  with an even number of sign changes, by mapping  $s_i$  to  $(i-1, i)$  for  $i \neq 2$  and  $s_2$  to  $(1, -2)(-1, 2)$ . For  $d$  even  $v_d$  maps to  $(1, -1)(n+1-d/2, n+2-d/2, \dots, n-1, n, d/2-n-1, \dots, 1-n, -n)$ . This element normalizes  $J_d$ : when  $J_d \neq \emptyset$ , it exchanges the simple roots corresponding to  $s_1$  and  $s_2$  and acts trivially

on the other simple roots indexed by  $J_d$ , so it is  $J_d$ -reduced. It follows that  $w_d$  normalizes  $I_d$  and is  $I_d$ -reduced.

**Lemma 11.13.** *For  $0 < i \leq \lfloor \frac{d}{2} \rfloor$  we have  $l(v_d^i) = \frac{2i}{d}l(w_{J_d}^{-1}w_0)$  and  $l(w_d^i) = \frac{2i}{d}l(w_{I_d}^{-1}w_0)$ .*

*Proof.* As in Lemma 11.8 it is sufficient to prove the lemma for  $v_d$ . To find the length of  $v_d$  we note that  $s_2s_1s_3s_4\dots s_n$  is  $\{s_3, \dots, s_n\}$ -reduced so that the lengths of  $s_{n+1-d/2}\dots s_3$  and of  $s_2s_1s_3\dots s_n$  add, whence  $l(v_d) = 2n - 1 - d/2$ . Since  $l(w_0) = n^2 - n$  and  $l(w_{J_d}) = (n - d/2)^2 - (n - d/2)$ , we have  $l(w_{J_d}^{-1}w_0) = d/2(2n - 1 - d/2)$ , which gives the result for  $i = 1$ . Written as permutations  $w_0 = (1, -1)^n(2, -2)\dots(n, -n)$  and  $w_{J_d} = (1, -1)^{n-d/2}(2, -2)\dots(n-d/2, d/2-n)$ ; a direct computation shows that  $v_d^{d/2} = (1, -1)^{d/2}(n+1-d/2, d/2-n-1)\dots(n, -n)$ , hence  $v_d^{d/2} = w_{J_d}^{-1}w_0$ . The lemma follows for smaller  $i$ .  $\square$

Since  $v_d^{d/2} = w_{J_d}^{-1}w_0$  and  $J_d$  is  $w_0$  stable we have  $v_d^d = 1$ , so the same property follows for  $w_d$  which shows that  $v_d$  and  $w_d$  are  $\zeta_d$ -good elements.

We also note that the theorem describes all  $d$  such that the  $\zeta_d$ -rank is not zero, since the degrees of  $W(D_n)$  are all the even integers from 2 to  $2n - 2$  and  $n$ , and in the twisted case the factor associated to the degree  $n$  is -1 and the other factors are equal to 1.

Since  $w_d$  is  $\phi$ -stable the element  $w_d\phi$  is also  $\zeta_d$ -good.

We now check Lemma 9.11(iv), that is that the  $\zeta_d$ -rank of  $W_{I_d}w_d$  in the untwisted case, resp.  $W_{I_d}w_d\phi$  in the twisted case is 0 on the subspace spanned by the roots corresponding to  $I_d$ . This property is clear if  $I_d = \emptyset$ . Otherwise:

- In the untwisted case the type of the coset is  $D_{n-kd/2}$  if  $k$  is even and  ${}^2D_{n-kd/2}$  if  $k$  is odd, where  $k$  is as in the definition of  $w_d$ . In both cases the set of values  $i$  such that the  $\zeta_i$ -rank is not 0 consists of the even  $i$  less than  $2n - kd$ , the odd  $i$  less than  $n - kd/2$  and in the twisted case ( $k$  odd)  $i = 2n - kd$ . Since if  $d$  is even we have  $2n - kd \leq d$  and if  $d$  is odd we have  $n - kd/2 \leq d$ , the only case where  $d$  could be in this set is  $k$  odd and  $d = 2n - kd$ , which means that  $\frac{k+1}{2}d = n$ . But  $d$  is assumed not to divide  $n$ , so this case does not happen.

- In the twisted case the type of the coset is  $D_{n-kd/2}$  if  $k$  is odd and  ${}^2D_{n-kd/2}$  if  $k$  is even. In both cases the set of values  $i$  such that the  $\zeta_i$ -rank is not 0 consists of the even  $i$  less than  $2n - kd$ , the odd  $i$  less than  $n - kd/2$  and in the twisted case ( $k$  even)  $i = 2n - kd$ . Since if  $d$  is even we have  $2n - kd \leq d$  and if  $d$  is odd we have  $n - kd/2 \leq d$ , the only case where  $d$  could be in this set is  $k$  even and  $d = 2n - kd$ , which means that  $(k+1)d = 2n$ . But this is precisely the excluded case.

We now give  $C_W(V_1)$ , where  $V_1$  is as in Lemma 9.14, in each case where  $I$  is not empty. In the untwisted case, if  $d$  is odd the group  $C_W(V_1)$  is of type  $D_{d\lfloor \frac{n-1}{d} \rfloor}$ ; if  $d$  is even the group  $C_W(V_1)$  is of type  $D_{\frac{d}{2}\lfloor \frac{2n-2}{d} \rfloor + 1}$  if  $\lfloor \frac{2n-2}{d} \rfloor$  is odd and  $D_{\frac{d}{2}\lfloor \frac{2n-2}{d} \rfloor}$  if  $\lfloor \frac{2n-2}{d} \rfloor$  is even. In the twisted case, if  $d$  is odd the coset  $C_W(V_1)w\phi$  is of type  ${}^2D_{d\lfloor \frac{n-1}{d} \rfloor + 1}$  and if  $d$  is even the coset is of type  ${}^2D_{\frac{d}{2}\lfloor \frac{2n-2}{d} \rfloor + 1}$  if  $\lfloor \frac{2n-2}{d} \rfloor$  is even and  $D_{\frac{d}{2}\lfloor \frac{2n-2}{d} \rfloor}$  if  $\lfloor \frac{2n-2}{d} \rfloor$  is odd. In all cases except if  $d$  is even and  $\lfloor \frac{2n-2}{d} \rfloor$  is even (resp. odd) in the untwisted case (resp. twisted case) we then deduce the group  $W(w\phi)$  (resp.  $W(w)$ ) as in the remarks at the beginning of Subsection 11.2 and after Lemma 9.14, since in these cases the centralizer of the regular element  $w\phi$  (resp.  $w$ ) in the parabolic subgroup  $W' = C_W(V_1)$  has the (known) reflection degrees of  $W(w\phi)$

(resp.  $W(w)$ ). In the excluded cases the group  $C_{W'}(w\phi)$  or  $C_{W'}(w)$  is isomorphic to  $G(d, 2, \lfloor \frac{2n-2}{d} \rfloor)$  which does not have the reflection degrees of  $W(w\phi)$ , resp.  $W(w)$ . This means that the morphism of the remark after Lemma 9.14 is not surjective. We can prove in this case that  $W(w\phi)$  or  $W(w)$  is  $G(d, 1, \lfloor \frac{2n-2}{d} \rfloor)$  since it is an irreducible complex reflection group by [Br, 5.6.6] and it is the only one which has the right reflection degrees apart from the exceptions in low rank given by  $G_5, G_{10}, G_{15}, G_{18}, G_{26}$ ; we can exclude these since they do not have  $G(d, 2, \lfloor \frac{2n-2}{d} \rfloor)$  as a reflection subgroup.  $\square$

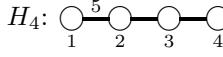
**Types  $I_2(n)$  and  ${}^2I_2(n)$ .** All eigenvalues  $\zeta$  such that the  $\zeta$ -rank is non-zero are regular, so this case can be found in [BM].

**Exceptional types.** Below are tables for exceptional finite Coxeter groups giving information on  $\zeta_d$ -good maximal elements for each  $d$ . They were obtained with the GAP package **Chevie** (see [Chevie]): first, the conjugacy class of good  $\zeta_d$ -maximal elements as described in Lemma 9.13 was determined; then we determined  $I$  for an element of that class, which gave  $l(w_I)$ . The next step was to determine the elements of the right length  $2(l(w_0) - l(w_I))/d$  in that conjugacy class; this required care in large groups like  $E_8$ . The best algorithm is to start from an element of minimal length in the class (known by [GP]) and conjugate by Coxeter generators until all elements of the right length are reached.

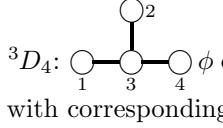
In the following tables, we give for each possible  $d$  and each possible  $I$  for that  $d$  a representative good  $w\phi$ , and give the number of possible  $w\phi$ . We then describe the coset  $W_I w\phi$  by giving, if  $I \neq \emptyset$ , in the column  $I$  the permutation induced by  $w\phi$  of the nodes of the Coxeter diagram indexed by  $I$ . Then we describe the isomorphism type of the complex reflection group  $N_W(W_I w\phi)/W_I = N_W(V)/C_W(V)$ , where  $V$  is the  $\zeta_d$ -eigenspace of  $w\phi$ . Finally, in the cases where  $I \neq \emptyset$ , we give the isomorphism type of  $W' = C_W(V_1)$ , where  $V_1$  is the 1-eigenspace of  $w\phi$  on the subspace spanned by the root lines of  $I$ . We note that there are 4 cases where  $N_{W'}(V)/C_{W'}(V) \not\leq N_W(V)/C_W(V)$ : for  $d = 5$  in  ${}^2E_6$ , for  $d = 4$  or 5 in  $E_7$  and for  $d = 9$  in  $E_8$ .

$H_3$ :  The reflection degrees are 2, 6, 10.

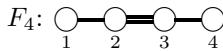
$d$	representative $w$	#good $w$	$C_W(w)$
10	$w_{10} = 123$	4	$Z_{10}$
6	$w_6 = 32121$	6	$Z_6$
5	$w_{10}^2$	4	$Z_{10}$
3	$w_6^2$	6	$Z_6$
2	$w_0$	1	$H_3$
1	.	1	$H_3$

$H_4$ :  The reflection degrees are 2, 12, 20, 30.

$d$	representative $w$	#good $w$	$C_W(w)$
30	$w_{30} = 1234$	8	$Z_{30}$
20	$w_{20} = 432121$	12	$Z_{20}$
15	$w_{30}^2$	8	$Z_{30}$
12	$w_{12} = 2121432123$	22	$Z_{12}$
10	$w_{30}^3$ or $w_{20}^2$	24	$G_{16}$
6	$w_{30}^5$ or $w_{12}^2$	40	$G_{20}$
5	$w_{30}^6$ or $w_{20}^4$	24	$G_{16}$
4	$w_{20}^5$ or $w_{12}^3$	60	$G_{22}$
3	$w_{30}^{10}$ or $w_{12}^4$	40	$G_{20}$
2	$w_0$	1	$H_4$
1	.	1	$H_4$

${}^3D_4$ :   $\phi$  does the permutation  $(1, 2, 4)$ . The reflection degrees are 2, 4, 4, 6 with corresponding factors  $1, \zeta_3, \zeta_3^2, 1$ .

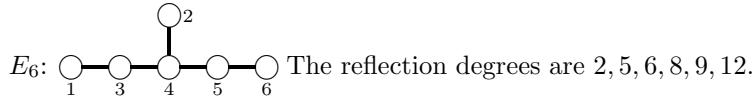
$d$	representative $w\phi$	#good $w\phi$	$C_W(w\phi)$
12	$w_{12}\phi = 13\phi$	6	$Z_4$
6	$w_6\phi = 1243\phi$	8	$G_4$
3	$w_6^2\phi$	8	$G_4$
2	$w_0\phi$	1	$G_2$
1	$\phi$	1	$G_2$

$F_4$ :  The reflection degrees are 2, 6, 8, 12.

$d$	representative $w$	#good $w$	$C_W(w)$
12	$w_{12} = 1234$	8	$Z_{12}$
8	$w_8 = 214323$	14	$Z_8$
6	$w_{12}^2$	16	$G_5$
4	$w_{12}^3$ or $w_8^2$	12	$G_8$
3	$w_{12}^4$	16	$G_5$
2	$w_0$	1	$F_4$
1	.	1	$F_4$

${}^2F_4$ :  $\phi$  does the permutation  $(1, 4)(2, 3)$ . The factors, in increasing order of the degrees, are  $1, -1, 1, -1$ .

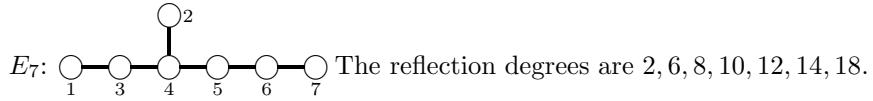
$d$	representative $w\phi$	#good $w\phi$	$C_W(w\phi)$
24	$w_{24}\phi = 12\phi$	6	$Z_{12}$
12	$w_{12}\phi = 3231\phi$	10	$Z_6$
8	$(w_{24}\phi)^3$	12	$G_8$
4	$(w_{12}\phi)^3$	24	$G_{12}$
2	$w_0\phi$	1	$I_2(8)$
1	$\phi$	1	$I_2(8)$



$d$	representative $w$	#good $w$	$I$	$N_W(W_I w)/W_I$	$C_W(V_1)$
12	$w_{12} = 123654$	8		$Z_{12}$	
9	$w_9 = 12342654$	24		$Z_9$	
8	$w_8 = 123436543$	14		$Z_8$	
6	$w_{12}^2$	16		$G_5$	
5	24231454234565 12435423456543 12314235423654	8 8 8	(3) (4) (5)	$Z_5$	$A_5$
4	$w_8^2$ or $w_{12}^3$	12		$G_8$	
3	$w_{12}^4$ or $w_9^3$	80		$G_{25}$	
2	$w_0$	1		$F_4$	
1	.	1		$E_6$	

${}^2E_6$ :  $\phi$  does the permutation  $(1, 6)(3, 5)$ . The factors, in increasing order of the degrees, are  $1, -1, 1, 1, -1, 1$ .

$d$	representative $w\phi$	#good $w\phi$	$I$	$N_W(W_I w\phi)/W_I$	$C_W(V_1)w\phi$
18	$w_{18}\phi = 1234\phi$	24		$Z_9$	
12	$w_{12}\phi = 123654\phi$	8		$Z_{12}$	
10	2431543 $\phi$ 5423145 $\phi$ 3143542 $\phi$	8 8 8	(3) (4) (5)	$Z_5$	${}^2A_5$
8	$w_8\phi = 123436543\phi$	14		$Z_8$	
6	$(w_{18}\phi)^3$	80		$G_{25}$	
4	$(w_{12}\phi)^3$	12		$G_8$	
3	$w_{12}^4\phi$	16		$G_5$	
2	$w_0\phi$	1		$E_6$	
1	$\phi$	1		$F_4$	



$d$	representative $w$	#good $w$	$I$	$N_W(W_I w)/W_I$	$C_W(V_1)$
18	$w_{18} = 1234567$	64		$Z_{18}$	
14	$w_{14} = 123425467$	160		$Z_{14}$	
12	$w_{12} = 1342546576$	8	$(2, 5, 7)$	$Z_{12}$	$E_6$
10	$w_{10a} = 134254234567$	8	$(2, 4)$	$Z_{10}$	$D_6$
	$w_{10b} = 243154234567$	8	$(3, 4)$		
	$w_{10c} = 124354265437$	8	$(4, 5)$		
9	$w_{18}^2$	64		$Z_{18}$	
8	$134234542346576$	14	$(2)(5, 7)$	$Z_8$	$D_5$
7	$w_{14}^2$	160		$Z_{14}$	
6	$w_{18}^3$ or $w_{12}^2$	800		$G_{26}$	
5	$w_{10a}^2$	8	$(2)(4)$	$Z_{10}$	$A_5$
	$w_{10b}^2$	8	$(3)(4)$		
	$w_{10c}^2$	8	$(4)(5)$		
4	$w_8^2$ or $w_{12}^3$	12	$(2)(5)(7)$	$G_8$	$D_4$
3	$w_{18}^6$ or $w_{12}^4$	800		$G_{26}$	
2	$w_0$	1		$E_7$	
1	.	1		$E_7$	



$d$	representative $w$	#good $w$	$I$	$N_W(W_I w)/W_I$	$C_W(V_1)$
30	$w_{30} = 12345678$	128		$Z_{30}$	
24	$w_{24} = 1234254678$	320		$Z_{24}$	
20	$w_{20} = 123425465478$	624		$Z_{20}$	
18	$w_{18a} = 1342542345678$	16	(2, 4)	$Z_{18}$	$E_7$
	$w_{18b} = 2431542345678$	16	(3, 4)		
	$w_{18c} = 1243542654378$	16	(4, 5)		
15	$w_{30}^2$	128		$Z_{30}$	
14	$w_{14a} = 13423454234565768$	128	(2)	$Z_{14}$	$E_7$
	$w_{14b} = 24231454234565768$	88	(3)		
	$w_{14c} = 12435423456543768$	108	(4)		
	$w_{14d} = 12342543654276548$	68	(5)		
12	$w_{24}^2$	2696		$G_{10}$	
10	$w_{30}^3$ or $w_{20}^2$	3370		$G_{16}$	
9	$w_{18a}^2$	16	(2)(4)	$Z_{18}$	$E_6$
	$w_{18b}^2$	16	(3)(4)		
	$w_{18c}^2$	16	(4)(5)		
8	$w_{24}^3$	7748		$G_9$	
7	$w_{14a}^2$	128	(2)	$Z_{14}$	$E_7$
	$w_{14b}^2$	88	(3)		
	$w_{14c}^2$	108	(4)		
	$w_{14d}^2$	68	(5)		
6	$w_{30}^5$ or $w_{24}^4$	4480		$G_{32}$	
5	$w_{30}^6$ or $w_{20}^4$	3370		$G_{16}$	
4	$w_{24}^6$ or $w_{20}^5$	15120		$G_{31}$	
3	$w_{30}^{10}$ or $w_{24}^8$	4480		$G_{32}$	
2	$w_0$	1		$E_8$	
1	.	1		$E_8$	

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