Parabolic Deligne-Lusztig varieties

François Digne, Jean Michel

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FRANÇOIS DIGNE AND JEAN MICHEL

Abstract. Motivated by the Broué conjecture on blocks with abelian defect groups for finite reductive groups, we study "parabolic" Deligne-Lusztig varieties and construct on those which occur in the Broué conjecture an action of a braid monoid, whose action on their ℓ-adic cohomology will conjecturally factor trough a cyclotomic Hecke algebra. In order to construct this action, we need to enlarge the set of varieties we consider to varieties attached to a "ribbon category"; this category is a *Garside category*, which plays an important role in our proof, so we devote the first part of our paper to the necessary background on Garside categories.

1. Introduction

In this paper, we study "parabolic" Deligne-Lusztig varieties, one of the main motivations being the Broué conjecture on blocks with abelian defect groups for finite reductive groups.

Let $\mathbf{G}$ be a connected reductive algebraic group over an algebraic closure $\overline{\mathbb{F}}_p$ of the prime field $\mathbb{F}_p$ of characteristic p. Let F be an isogeny on $\mathbf{G}$ such that some power F^δ is a Frobenius endomorphism attached to a split structure over the finite field $\mathbb{F}_{q^\delta}$; this defines a real number q such that q^δ is an integral power of p. When $\mathbf{G}$ is quasi-simple, any isogeny F such that the group of fixed points $\mathbf{G}^F$ is finite is of the above form; such a group $\mathbf{G}^F$ is called a "finite reductive group" or a "finite group of Lie type".

Let $\mathbf{L}$ be an F-stable Levi subgroup of a (non necessarily F-stable) parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Then, for ℓ a prime number different from p, Lusztig has constructed a "cohomological induction" $R_{\mathbf{L}}^{\mathbf{G}}$ which associates to any $\overline{\mathbb{Q}}_\ell\mathbf{L}^F$-module a virtual $\overline{\mathbb{Q}}_\ell\mathbf{G}^F$-module. We study the particular case $R_{\mathbf{L}}^{\mathbf{G}}(\mathrm{Id})$, which is given by the alternating sum of the ℓ-adic cohomology groups of the variety

$$\mathbf{X}_{\mathbf{P}} = \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g\mathbf{P} \cap F(g\mathbf{P}) \neq \emptyset\}$$

on which $\mathbf{G}^F$ acts on the left. We will construct a monoid of endomorphisms M of $\mathbf{X}_{\mathbf{P}}$ related to the braid group, which conjecturally will induce in some cases a cyclotomic Hecke algebra on the cohomology of $\mathbf{X}_{\mathbf{P}}$. To construct M we need to enlarge the set of varieties we consider, to include varieties attached to morphisms in a "ribbon category" — the most general "parabolic Deligne-Lusztig varieties" that we consider; M corresponds to the endomorphisms in the "conjugacy category" of this ribbon category of the object attached to $\mathbf{X}_{\mathbf{P}}$.

The relationship with Broué's conjecture for the principal block comes as follows: assume, for some prime number $\ell \neq p$, that the ℓ-Sylow S of $\mathbf{G}^F$ is abelian. Then

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Broué's conjecture predicts in this special case an equivalence of derived categories between the principal block of $\overline{\mathbb{Z}}_\ell \mathbf{G}^F$ and that of $\overline{\mathbb{Z}}_\ell N_{\mathbf{G}^F}(S)$. Now $\mathbf{L} := C_{\mathbf{G}}(S)$ is a Levi subgroup of a (non F-stable unless $\ell | q-1$) parabolic subgroup $\mathbf{P}$; restricting to unipotent characters and discarding an eventual torsion by changing coefficients from $\overline{\mathbb{Z}}_\ell$ to $\overline{\mathbb{Q}}_\ell$, this translates into conjectures about the cohomology of $\mathbf{X}_{\mathbf{P}}$, see 10.1; these conjectures predict in particular that the image in the cohomology of our monoid M is a cyclotomic Hecke algebra.

The main feature of the ribbon categories we consider is that they are *Garside categories*. This concept has appeared in recent work to understand the ordinary and dual monoids attached to the braid groups; in the first part, we recall its basic properties and go as far as computing the centralizers of "periodic elements", which is what we need in the applications.

In the second part, we first define the parabolic Deligne-Lusztig varieties which are the aim of our study, and then go on to establish their properties. We extend to this setting in particular all the material in [BM] and [BR2].

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Part 1. Garside categories

This part collects some prerequisites on Garside categories. It is mostly self-contained apart from the next section where the proofs are omitted; we refer for them to the book [DDGKM] to appear.

2. Basic results on Garside categories

Given a category $\mathcal{C}$, we write $f \in \mathcal{C}$ to say that f is a morphism of $\mathcal{C}$, and $\mathcal{C}(x,y)$ for the set of morphisms from $x \in \mathrm{Obj}\mathcal{C}$ to $y \in \mathrm{Obj}\mathcal{C}$. We write fg for the composition of $f \in \mathcal{C}(x,y)$ and $g \in \mathcal{C}(y,z)$, and $\mathcal{C}(x)$ for $\mathcal{C}(x,x)$. By $\mathcal{S} \subset \mathcal{C}$ we mean that $\mathcal{S}$ is a set of morphisms in $\mathcal{C}$.

All the categories we consider will be left-cancellative, that is a relation $hf = hg$ implies $f = g$. We say that f left divides g, written $f \preccurlyeq g$, if there exists h such that $g = fh$.

We denote $\mathcal{C}^\times$ the set of invertible morphisms of $\mathcal{C}$, and write $f =^\times g$ if there exists $h \in \mathcal{C}^\times$ such that $fh = g$ (or equivalently there exists $h \in \mathcal{C}^\times$ such that $f = gh$).

Definition 2.1. *A Garside family in $\mathcal{C}$ is a subset $\mathcal{S} \subset \mathcal{C}$ such that;*

- *$\mathcal{S}$ together with $\mathcal{C}^\times$ generates $\mathcal{C}$.*
- *$\mathcal{C}^\times \mathcal{S} \subset \mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$.*
- *For every product fg with $f, g \in \mathcal{S} - \mathcal{C}^\times$, either $fg \in \mathcal{S}\mathcal{C}^\times$ in which case we say that (fg) is the $\mathcal{S}$-normal decomposition of fg, or we have $fg = f_1 g_1$, where $f_1 \in \mathcal{S}$, $g_1 \in \mathcal{S}\mathcal{C}^\times - \mathcal{C}^\times$ and f_1, g_1 have the property that any relation $h \preccurlyeq k f_1 g_1$ with $h \in \mathcal{S}$ implies $h \preccurlyeq k f_1$; in this case we say that (f_1, g_1) is an $\mathcal{S}$-normal decomposition of fg.*

A category is *Garside* if it has a Garside family $\mathcal{S}$ (we should talk of a *Garside system* $(\mathcal{C}, \mathcal{S})$).

In a Garside category every non-invertible element x admits a normal decomposition $(x_1, \dots, x_n)$ which means that $x = x_1 \dots x_n$ and that each (x_i, x_{i+1}) is

an $\mathcal{S}$-normal decomposition. Normal decompositions are unique up to invertibles, precisely

Lemma 2.2. *If $(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_{n'})$ are two normal decompositions of x then $n = n'$ and for any i we have $x_1 \dots x_i =^\times x'_1 \dots x'_i$.*

For $f \in \mathcal{C}$ we define $\lg_{\mathcal{S}}(f)$ to be the minimum number k of morphisms $s_1, \dots, s_k \in \mathcal{S}$ such that $s_1 \dots s_k =^\times f$, thus $\lg_{\mathcal{S}}(f) = 0$ if $f \in \mathcal{C}^\times$; if $f \notin \mathcal{C}^\times$ then $\lg_{\mathcal{S}}(f)$ is also the number of terms in a normal decomposition of f.

We have the following criterion to be Garside:

Proposition 2.3. *Let $\mathcal{S} \subset \mathcal{C}$ together with $\mathcal{C}^\times$ generate $\mathcal{C}$, and let H be a function $\mathcal{C} \xrightarrow{H} \mathcal{S}$. Consider the following properties*

(i) $\forall g \in \mathcal{C}, H(g) \preccurlyeq g$.
(ii) $\forall h \in \mathcal{S}, h \preccurlyeq g \Rightarrow h \preccurlyeq H(g)$.
(iii) $\forall f, g \in \mathcal{C}, H(fg) =^\times H(fH(g))$.
(iv) $\mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ *is closed under right-divisor.*

Then $\mathcal{S}$ is Garside if (i), (ii), (iii) hold for some H, or if (i) and (ii) hold for some H, and (iv) holds. Conversely if $\mathcal{S}$ is Garside then (iv) holds and there exists H satisfying (i) to (iii) above; such a function is called a $\mathcal{S}$-head function.

A $\mathcal{S}$-head function H computes the first term of a normal decomposition in the sense that if $(x_1, \dots, x_n)$ is a normal decomposition of x then $H(x) =^\times x_1$.

We have the following property:

Lemma 2.4. *Let H be a $\mathcal{S}$-head function, and for $x \in \mathcal{C} - \mathcal{C}^\times$ let x' be defined by $x = H(x)x'$. Then $\lg_{\mathcal{S}}(x') < \lg_{\mathcal{S}}(x)$.*

The following shows that $\mathcal{S}$ "determines" $\mathcal{C}$ up to invertibles; we say that a subset $\mathcal{C}_1$ of $\mathcal{C}$ is closed under right quotient if an equality $f = gh$ with $f, g \in \mathcal{C}_1$ implies $g \in \mathcal{C}_1$.

Lemma 2.5. *Let $\mathcal{S}$ be a Garside family in $\mathcal{C}$. Let $\mathcal{C}_1$ be a subcategory of $\mathcal{C}$ closed under right-quotient which contains $\mathcal{S}$. Then $\mathcal{C} = \mathcal{C}_1\mathcal{C}^\times$ and $\mathcal{S}$ is a Garside family in $\mathcal{C}_1$.*

gcds and lcms, Noetherianity. The existence of gcds and lcms are related when $\mathcal{C}$ is right-Noetherian, which means that there is no infinite sequence $f_0 \succcurlyeq f_1 \dots \succcurlyeq f_n \succcurlyeq \dots$ where f_i is a *proper* right divisor of f_{i+1}, that is we do not have $f_i =^\times f_{i+1}$. It means equivalently since $\mathcal{C}$ is left cancellative that there is no infinite sequence $f_0 \preccurlyeq f_1 \dots \preccurlyeq f_n \preccurlyeq \dots \preccurlyeq f$ where f_i is a proper left divisor of f_{i+1}.

We say that $\mathcal{C}$ admits local right lcms if, whenever f and g have a common right multiple, they have a right lcm. We then have:

Proposition 2.6. *If $\mathcal{C}$ is right Noetherian and admits local right lcms, then any family of morphisms of $\mathcal{C}$ with the same source has a left gcd.*

Here is a more general situation when a Garside family of a subcategory can be determined:

Lemma 2.7. *Let $\mathcal{S}$ be a Garside family in $\mathcal{C}$ assumed right-Noetherian and having local right lcms. Let $\mathcal{S}_1 \subset \mathcal{S}$ be a subfamily such that $\mathcal{S}_1\mathcal{C}^\times$ is as a subset of $\mathcal{S}\mathcal{C}^\times$ closed under right-lcm and right-quotient; then $\mathcal{S}_1$ is a Garside family in the subcategory $\mathcal{C}_1$ generated by $\mathcal{S}_1\mathcal{C}^\times$. Moreover $\mathcal{C}_1$ is a subcategory closed under right-quotient.*

The following lemma about Noetherian categories will also be useful:

Lemma 2.8. *Let $\mathcal{C}$ be a category and $\mathcal{S}$ be a set of morphisms which generates $\mathcal{C}$. Let X be a set of morphisms of $\mathcal{C}$ with same source satisfying*

(i) *X is closed under left divisor and $X = X\mathcal{C}^\times$.*
(ii) *X is a bounded and right Noetherian poset.*
(iii) *If $f \in X$, $g, h \in \mathcal{S}$ and $fg, fh \in X$ then g and h have a common right-multiple m such that $fm \in X$.*

Then X is the set of left-divisors of some morphism of C.

2.1. **Garside maps.** An important special case is when $\mathcal{S}$ is attached to a Garside map. A Garside map is a map $\mathrm{Obj}\mathcal{C} \xrightarrow{\Delta} \mathcal{C}$ where $\Delta(x) \in \mathcal{C}(x, -)$ such that $\mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ is the set of left divisors of Δ. Since by Proposition 2.3(iv) the set $\mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$ is stable by right divisor, it is also the set of right divisors of Δ.

This allows to define a functor Φ, first on objects by taking for $\Phi(x)$ the target of $\Delta(x)$, then on maps, first on morphisms $s \in \mathcal{S}$ by, if $s \in \mathcal{C}(x, -)$ defining s' by $ss' = \Delta$ (we omit the source of Δ if it is clear from the context) and then $\Phi(s)$ by $s'\Phi(s) = \Delta$. We then extend Δ by using normal decompositions; it can be shown that this is well-defined and defines a functor such that for any $f \in \mathcal{C}$ we have $f\Delta = \Delta\Phi(f)$. We then have

Proposition 2.9. *In the above context, the following are equivalent:*

- *Φ is an automorphism (that is, an invertible functor).*
- *$\mathcal{C}$ is right-cancellative.*

We will assume from now on that these properties hold. We then have the following properties:

Proposition 2.10. (i) *If $f \preccurlyeq g$ then $\lg_{\mathcal{S}}(f) \leq \lg_{\mathcal{S}}(g)$.*
(ii) *Assume $f, g, h \in \mathcal{S}$ and (f, g) is $\mathcal{S}$-normal; then $\lg_{\mathcal{S}}(fgh) \leq 2$ implies $gh \in \mathcal{S}\mathcal{C}^\times$.*

2.2. **Garside automorphisms.** An automorphism of a category $\mathcal{C}$ is a functor $F : \mathcal{C} \to \mathcal{C}$ which has an inverse.

When $\mathcal{C}$ has Garside family $\mathcal{S}$, we call *Garside automorphism* of $(\mathcal{C}, \mathcal{S})$, an automorphism F which preserves $\mathcal{S}\mathcal{C}^\times$. A typical example is the automorphism Φ of the previous subsection that we will call the *canonical Garside automorphism*.

We have the following property

Proposition 2.11. *Assume that $\mathcal{C}$ has a Garside family $\mathcal{S}$ and has no non-trivial invertible morphisms. Left F be a Garside automorphism of $\mathcal{C}$. Then the subcategory of fixed objects and morphisms $\mathcal{C}^F$ has a Garside family which consists of the fixed points $\mathcal{S}^F$.*

2.3. **An example.** An example of a Garside category is a *Garside monoid*, which is just the case where $\mathcal{C}$ has one object. A classical example is given by the *Artin monoid* $(B^+, \mathbf{S})$ associated to a Coxeter system (W, S). Then B^+ is left-cancellative, Noetherian, admits local right-lcm's and has a Garside family, the canonical lift $\mathbf{W}$ of W in B^+, which consists of the elements whose length with respect to $\mathbf{S}$ is equal to the length of their image in W.

3. Categories with automorphism

Given an automorphism F of finite order of the category $\mathcal{C}$, we define

Definition 3.1. *The semi-direct product category $\mathcal{C} \rtimes \langle F \rangle$ is the category whose objects are the objects of $\mathcal{C}$ and whose morphisms with source x are the pairs (g, F^i), which will be denoted by gF^i, where g is a morphism of $\mathcal{C}$ with source x and i is an integer. The target of this morphism is $F^{-i}(\mathrm{target}(g))$, where $\mathrm{target}(g)$ is the target of g. The composition rule is given by $gF^i \cdot hF^j = gF^i(h)F^{i+j}$ when $\mathrm{source}(h) = F^{-i}(\mathrm{target}(g))$.*

The conventions on F are such that the composition rule is natural. However, they imply that the morphism F of the semi-direct product category represents the functor F^{-1}: it is a morphism from the object $F(A)$ to the object A and we have the commutative diagram:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ A & \xrightarrow{f} & B \end{array}$$

Lemma 3.2. *If $\mathcal{S}$ is a Garside family in $\mathcal{C}$, and F a Garside automorphism of $(\mathcal{C}, \mathcal{S})$, then $\mathcal{S}$ is also a Garside family in $\mathcal{C} \rtimes \langle F \rangle$.*

Proof. A $\mathcal{S}$-normal decomposition in $\mathcal{C}$ of an element of $\mathcal{S}^2$ is still $\mathcal{S}$-normal in $\mathcal{C} \rtimes \langle F \rangle$ and since $F^i s = F^i(s)F^i$ for all $s \in \mathcal{S}$, we have as required $(\mathcal{C} \rtimes \langle F \rangle)^\times \mathcal{S} \subset \mathcal{S}(\mathcal{C} \rtimes \langle F \rangle)^\times$. □

If $(f_1, \dots f_k)$ is a normal decomposition of $f \in \mathcal{C}$ then $(f_1, \dots, f_k F^i)$ is a normal decomposition of $fF^i \in \mathcal{C} \rtimes \langle F \rangle$. Note that if $\mathcal{C}$ has no non-trivial invertible element, then the only invertibles in $\mathcal{C} \rtimes \langle F \rangle$ are $\{F^i\}_{i \in \mathbb{Z}}$. In general, if $a, b \in \mathcal{C}$ then $aF^i \preccurlyeq bF^j$ if and only if $a \preccurlyeq b$.

4. The conjugacy category

In this section we do not need $\mathcal{C}$ left-cancellative unless explicitly stated.

Definition 4.1. *Given a category $\mathcal{C}$, we define the* conjugacy category $\mathrm{Ad}\,\mathcal{C}$ *of $\mathcal{C}$ as the category whose objects are the endomorphisms of $\mathcal{C}$ and where, for $w \in \mathcal{C}(A)$ and $w' \in \mathcal{C}(B)$ we set $\mathrm{Ad}\,\mathcal{C}(w, w') = \{x \in \mathcal{C}(A, B) \mid xw' = wx\}$. We say that x* conjugates *w to w' and if $w = w'$, we call* centralizer *of w the set $\mathrm{Ad}\,\mathcal{C}(w)$. The composition of morphisms in $\mathrm{Ad}\,\mathcal{C}$ is given by the composition in $\mathcal{C}$, which is compatible with the above rules.*

Note that the formula for $\mathrm{Ad}\,\mathcal{C}(,)$ is what forces the objects of $\mathrm{Ad}\,\mathcal{C}$ to be endomorphisms of $\mathcal{C}$. We can rephrase the condition $x \in \mathrm{Ad}\,\mathcal{C}(w, -)$ as $x \preccurlyeq wx$.

If $\mathcal{C}$ is left-cancellative, then the data x and w determine w' (resp. if $\mathcal{C}$ is right-cancellative x and w' determine w). In that case we will write w^x for w' (resp. ${}^x w'$ for w). This way of writing the morphisms illustrates that our category $\mathrm{Ad}\,\mathcal{C}$ is a right conjugacy category; we could call left conjugacy category the opposed category.

A proper name for an element of $\mathrm{Ad}\,\mathcal{C}(w, w')$ should be a triple (w, x, w') (or perhaps a pair (w, x) if $\mathcal{C}$ left-cancellative), since x by itself does specify neither its source w nor its target w', but we will use just x when $\mathcal{C}$ is left-cancellative and

the context makes clear which source w is meant (or $\mathcal{C}$ is right-cancellative and the context makes clear which target is meant). We should keep in mind that the functor I which sends $w \in \mathrm{Obj}(\mathrm{Ad}\,\mathcal{C})$ to $\mathrm{source}(w)$ and (w, x, w') to x is faithful, but definitely not injective on objects. Nevertheless, the fact that I is faithful implies that $\mathrm{Ad}\,\mathcal{C}(w, -)$ identifies to a subset of $I(\mathrm{Ad}\,\mathcal{C}(w, -)) = \mathcal{C}(\mathrm{source}(w), -)$ (resp. $\mathrm{Ad}\,\mathcal{C}(-, w)$ identifies to a subset of $\mathcal{C}(-, \mathrm{source}(w))$). It follows that the category $\mathrm{Ad}\,\mathcal{C}$ inherits automatically from $\mathcal{C}$ properties such as cancellativity or Noetherianity. The functor I maps $(\mathrm{Ad}\,\mathcal{C})^\times$ surjectively to $\mathcal{C}^\times$, so in particular the subset $\mathrm{Ad}\,\mathcal{C}(w, -)$ of $\mathcal{C}(\mathrm{source}(w), -)$ is closed under multiplication by $\mathcal{C}^\times$,

Lemma 4.2.
- *Assume that $\mathcal{C}$ is left-cancellative. Then $\mathrm{Ad}\,\mathcal{C}(w, w')$ as a subset of $\mathcal{C}(\mathrm{source}(w), \mathrm{source}(w'))$ is closed under right-quotient, that is if we have an equality $y = xz$ where $y \in \mathrm{Ad}\,\mathcal{C}(w, w')$, $x \in \mathrm{Ad}\,\mathcal{C}(w, -)$ and $z \in \mathcal{C}(-, \mathrm{source}(w'))$, then $z \in \mathrm{Ad}\,C(-, w')$; and $\mathrm{Ad}\,\mathcal{C}(w, -)$ identifies to a subset of $\mathcal{C}(\mathrm{source}(w), -)$ closed under right-lcm, in the sense that if $x, y \in \mathrm{Ad}\,\mathcal{C}(w, -)$ have a right-lcm in $\mathcal{C}(\mathrm{source}(w), -)$ then this right-lcm is in $\mathrm{Ad}\,\mathcal{C}(w, -)$ and is a right-lcm of x and y in $\mathrm{Ad}\,\mathcal{C}$. In particular if $\mathcal{C}$ admits local right-lcms then so does $\mathrm{Ad}\,\mathcal{C}$.*
- *Similarly if $\mathcal{C}$ is right-cancellative, $\mathrm{Ad}\,\mathcal{C}(-, w)$ identifies to a subset of $\mathcal{C}(-, \mathrm{source}(w))$ closed under left-lcm and left-quotient.*

Proof. We assume $\mathcal{C}$ left-cancellative and show the stability by right-quotient. We have $x \preccurlyeq wx$ and $yw' = wy$. By cancellation, let us define w'' by $xw'' = wx$. Then from $xz = y \preccurlyeq wy = wxz = xw''z$ we deduce by cancellation that $z \preccurlyeq w''z$, so $z \in \mathrm{Ad}\,\mathcal{C}(w, w_1)$ where $zw_1 = w''z$. Now since $y = xz$ the equality $yw' = wy$ gives $xzw' = wxz = xw''z = xzw_1$ which shows by cancellation that $w_1 = w'$.

We now show stability by right-lcm. $x, y \in \mathrm{Ad}\,\mathcal{C}(w, -)$ means that $x \preccurlyeq wx$ and $y \preccurlyeq wy$. Suppose now that x and y have a right-lcm z in $\mathcal{C}$. Then $x \preccurlyeq wz$ and $y \preccurlyeq wz$ from which it follows that $z \preccurlyeq wz$, that is $z \in \mathrm{Ad}\,\mathcal{C}(w, -)$, and z is necessarily a right-lcm of x and y in $\mathrm{Ad}\,\mathcal{C}$.

The proof of the second part is just a mirror symmetry of the above proof. □

Proposition 4.3. *Assume that $\mathcal{C}$ is left-cancellative and that $\mathcal{S}$ is a Garside family in $\mathcal{C}$; then $I^{-1}(\mathcal{S})$ is a Garside family in $\mathrm{Ad}\,\mathcal{C}$.*

Proof. We will use Proposition 2.3 by showing that $I^{-1}(\mathcal{S}) \cup \mathcal{C}^\times$ generates $\mathrm{Ad}\,\mathcal{C}$ and exhibiting a function $H : \mathrm{Ad}\,\mathcal{C} \to I^{-1}(\mathcal{S})$ which satisfies Proposition 2.3(i), (ii) and (ii).

Let H be a $\mathcal{S}$-head function in $\mathcal{C}$. We first show that if $x \in \mathrm{Ad}\,\mathcal{C}(w, -)$ then $H(x) \in I^{-1}(\mathcal{S})$, or equivalently $H(x) \in \mathrm{Ad}\,\mathcal{C}(w, -)$. Indeed if $x \preccurlyeq wx$ then $H(x) \preccurlyeq H(wx) =^\times H(wH(x)) \preccurlyeq wH(x)$.

We now deduce by induction on the $\lg_{\mathcal{S}}$ of a morphism that $I^{-1}(\mathcal{S}) \cup \mathcal{C}^\times$ generates $\mathrm{Ad}\,\mathcal{C}$. If $x \in \mathrm{Ad}\,\mathcal{C}$ is such that $\lg_{\mathcal{S}}(x) = 1$ then $x = s\varepsilon$ with $s \in \mathcal{S}$ and $\varepsilon \in \mathcal{C}^\times$. Since $\mathrm{Ad}\,\mathcal{C}$ is closed under multiplication by $\mathcal{C}^\times$ and $s\varepsilon \in \mathrm{Ad}\,\mathcal{C}$, we have $s \in I^{-1}(\mathcal{S})$, whence the result in this case. Assume now that for $x \in \mathrm{Ad}\,\mathcal{C}$ we have $\lg_{\mathcal{S}}(x) = n$ and define x' by $x = H(x)x'$. Since we know that $H(x) \in \mathrm{Ad}\,\mathcal{C}$, we deduce by Lemma 4.2 that $x' \in \mathrm{Ad}\,\mathcal{C}$; by Lemma 2.4 we have $\lg_{\mathcal{S}}(x') < n$, whence the result.

We now claim that the restriction of H to $\mathrm{Ad}\,\mathcal{C}$ fits our purpose: it takes its values in $I^{-1}(\mathcal{S})$, and it satisfies Proposition 2.3 (i), (ii), (iii) since H does. □

Simultaneous conjugacy. A straightforward generalization of conjugacy categories is "simultaneous conjugation categories", where objects are families of morphisms $w_1, \dots, w_n$ with same source and target, and morphisms verify $x \preccurlyeq w_i x$ for all i. Most statements have a straightforward generalization to this case.

F-conjugacy. We want to consider "twisted conjugation" by an automorphism, which will be useful for applications to Deligne-Lusztig varieties, but also for internal applications, with the automorphism being the one induced by a Garside map.

Definition 4.4. *Let F be an automorphism of the category $\mathcal{C}$. We define the F-conjugacy category of $\mathcal{C}$, denoted by $F\text{-}\operatorname{Ad}\mathcal{C}$, as the category whose objects are the morphisms in some $\mathcal{C}(A, F(A))$ and where, for $w \in \mathcal{C}(A, F(A))$ and $w' \in \mathcal{C}(B, F(B))$ we set $F\text{-}\operatorname{Ad}\mathcal{C}(w, w') = \{x \in \mathcal{C} \mid xw' = wF(x)\}$. We say that x* F-conjugates *w to w' and we call* F-centralizer *of a morphism w of $\mathcal{C}$ the set $F\text{-}\operatorname{Ad}\mathcal{C}(w)$.*

Note that F-conjugacy specializes to conjugacy when $F = \mathrm{Id}$ and that the F-centralizer of x is empty unless $x \in \mathcal{C}(A, F(A))$ for some object A.

We explore now how these notions relate to conjugation in a semi-direct product category.

- Consider the application which sends $w \in \mathcal{C}(A, F(A)) \subset \operatorname{Obj}(F\text{-}\operatorname{Ad}\mathcal{C})$ to $wF \in \mathcal{C} \rtimes \langle F \rangle(A) \subset \operatorname{Obj}(\operatorname{Ad}(\mathcal{C} \rtimes \langle F \rangle))$. Since $x(w'F) = (wF)x$ is equivalent to $xw' = wF(x)$, this extends to a functor J from $F\text{-}\operatorname{Ad}\mathcal{C}$ to $\operatorname{Ad}(\mathcal{C} \rtimes \langle F \rangle)$. This functor is clearly an isomorphism onto its image.

The image $J(\operatorname{Obj}(F\text{-}\operatorname{Ad}\mathcal{C}))$ identifies via I to the set of endomorphisms in $\mathcal{C} \rtimes \langle F \rangle$ which lie in $\mathcal{C}F$. The image by J of $F\text{-}\operatorname{Ad}\mathcal{C}(w, w')$ is the set of morphisms in $\operatorname{Ad}(\mathcal{C} \rtimes \langle F \rangle)(wF, w'F)$ which, viewed via I as morphisms in $\mathcal{C} \rtimes \langle F \rangle(\mathrm{source}(wF), \mathrm{source}(w'F))$, lie in $\mathcal{C}$.

Let $\operatorname{Ad}(\mathcal{C}F)$ be the full subcategory of $\operatorname{Ad}(\mathcal{C} \rtimes \langle F \rangle)$ whose objects are morphisms in $\mathcal{C}F \subset \mathcal{C} \rtimes \langle F \rangle$. As it is a union of connected components of $\operatorname{Ad}(\mathcal{C} \rtimes \langle F \rangle)$ (in $\operatorname{Ad}(\mathcal{C} \rtimes \langle F \rangle)$, there is no morphism between two objects which do not have the same power of F) all properties we are interested in will transfer automatically from $\operatorname{Ad}(\mathcal{C} \rtimes \langle F \rangle)$ to $\operatorname{Ad}(\mathcal{C}F)$.

In particular, if $\mathcal{C}$ has a Garside family $\mathcal{S}$ and F is a Garside automorphism, then $\mathcal{S}$ is still a Garside family for $\mathcal{C} \rtimes \langle F \rangle$ by 3.2, and by the above gives rise to a Garside family $I^{-1}(\mathcal{S})$ of $\operatorname{Ad}(\mathcal{C}F)$. Now the relationship between $\operatorname{Ad}(\mathcal{C}F)$ and the image of J is as in Lemma 2.5: the image of J is a subcategory closed under right quotient (because in a relation $fg = h$ if f and h do not involve F the same must be true for g) and contains the Garside family $I^{-1}(\mathcal{S})$ of $\operatorname{Ad}(\mathcal{C}F)$.

This will allow to generally translate statements about conjugacy categories to statements about F-conjugacy categories. For example, Proposition 4.3 implies that $I^{-1}(\mathcal{S})$ is a Garside family in $\operatorname{Ad}(\mathcal{C} \rtimes \langle F \rangle)$ which implies in turn that it is a Garside family in $\operatorname{Ad}(\mathcal{C}F)$, thus $J^{-1}(I^{-1}(\mathcal{S}))$ is one for $F\text{-}\operatorname{Ad}\mathcal{C}$; this last family consists just of the $x \in F\text{-}\operatorname{Ad}\mathcal{C}(w, w')$ for various w, w' which are in $\mathcal{S}$.

If F has finite order, since $(xF)^x = Fx = (xF)^{F^{-1}}$ we see that two morphisms in $\mathcal{C}F$ are conjugate in $\mathcal{C} \rtimes \langle F \rangle$ if and only if they are conjugate by a morphism of $\mathcal{C}$.

5. The cyclic conjugacy category

A restricted form of conjugation called "cyclic conjugacy" will be important in applications. In particular, it turns out that two periodic braids are conjugate if and only if they are cyclically conjugate.

Definition 5.1. *We define the cyclic conjugacy category* $\operatorname{cyc}\mathcal{C}$ *of* $\mathcal{C}$ *as the subcategory of* $\operatorname{Ad}\mathcal{C}$ *generated by* $\{x \in \operatorname{Ad}\mathcal{C}(w,w') \mid x \preccurlyeq w\}$.

That is, $\operatorname{cyc}\mathcal{C}$ has the same objects as $\operatorname{Ad}\mathcal{C}$ but contains only the products of elementary conjugations of the form $w = xy \to w' = yx$. Note that if $\mathcal{C}$ is left- and right-cancellative, then $\cup_w\{x \in \operatorname{Ad}\mathcal{C}(w,w') \mid x \preccurlyeq w\} = \cup_w\{x \in \operatorname{Ad}\mathcal{C}(w,w') \mid w' \succcurlyeq x\}$ so cyclic conjugacy "from the left" and "from the right" are the same in this case. To be more precise, the functor which is the identity on objects, and when $w = xy$ and $w' = yx$, sends $x \in \operatorname{cyc}\mathcal{C}(w,w')$ to $y \in \operatorname{cyc}\mathcal{C}(w',w)$, is an isomorphism between $\operatorname{cyc}\mathcal{C}$ and its opposed category.

Proposition 5.2. *let* $\mathcal{C}$ *be a left-cancellative and right-Noetherian category that admits local right-lcm's; if* $\mathcal{S}$ *a Garside family in* $\mathcal{C}$ *then the set* $\mathcal{S}_1 = \cup_w\{x \in \operatorname{Ad}\mathcal{C}(w,-) \mid x \preccurlyeq w$ *and* $x \in \mathcal{S}\}$ *is a Garside family in* $\operatorname{cyc}\mathcal{C}$.

Proof. We first observe that $\mathcal{S}_1\mathcal{C}^\times$ generates $\operatorname{cyc}\mathcal{C}$. Indeed if $x \preccurlyeq w$ and we choose a decomposition $x = s_1 \dots s_n$ as a product of morphisms in $\mathcal{S}\mathcal{C}^\times$ it is clear that each s_i is in $\operatorname{cyc}\mathcal{C}$, so is in $\mathcal{S}_1$.

We use Lemma 2.7.

The lemma applies to $\operatorname{cyc}\mathcal{C}$, and gives the proposition, since $\mathcal{S}_1\mathcal{C}^\times$ is closed under right-divisor and right-lcm; this is obvious for right-divisor and for right-lcm results from the facts that $\mathcal{S}$ as a Garside family is closed under right-lcm and that a right-lcm of two divisors of w is a divisor of w. □

We also see by Lemma 2.7 that $\operatorname{cyc}\mathcal{C}$ is closed under right-quotient.

We now prove that if without assuming that $\mathcal{C}$ has a Garside family anyway $\operatorname{cyc}\mathcal{C}$ has a Garside family associated to a Garside map; when $\mathcal{C}$ has a Garside family this Garside family is larger than the Garside family $\mathcal{S}_1$ of Proposition 5.2, since it contains all left divisors of w even if w is not in $\mathcal{S}$.

Proposition 5.3. *If* $\mathcal{C}$ *is a left cancellative and right Noetherian category which admits local right-lcms then the set* $\mathcal{S}' = \cup_{w\in\mathcal{C}}\{x \in \operatorname{Ad}\mathcal{C}(w,-) \mid x \preccurlyeq w\}$ *is a Garside family in* $\operatorname{cyc}\mathcal{C}$ *associated to the Garside map* Δ *such that* $\Delta(w) = w \in \operatorname{cyc}\mathcal{C}(w)$*; the corresponding* Φ_Δ *is the identity functor.*

Proof. The set $\mathcal{S}'$ generates $\operatorname{cyc}\mathcal{C}$ by definition of $\operatorname{cyc}\mathcal{C}$. It is closed under right-divisors since $xy \preccurlyeq w$ implies $x \preccurlyeq w$ so that w^x is defined and $y \preccurlyeq w^x$; since $\mathcal{C}$ is right Noetherian and admits local right-lcms, any two morphisms of $\mathcal{C}$ with same source have a gcd by Proposition 2.6. We define a function $H : \operatorname{cyc}\mathcal{C} \to \mathcal{S}'$ by letting $H(x)$ be an arbitrarily chosen left-gcd of x and w if $x \in \operatorname{cyc}\mathcal{C}(w,-)$. Since $\operatorname{cyc}\mathcal{C}$ is closed under right-divisor, the function H satisfies properties Proposition 2.3 (i), (ii) and (iv), so $\mathcal{S}'$ is a Garside family for $\operatorname{cyc}\mathcal{C}$. The set of morphisms in $\mathcal{S}'$ with source w has an lcm which is w. Moreover if v is a right-divisor of $\Delta(w) = w$ in $\operatorname{cyc}\mathcal{C}$, which implies that $w = v'v$, then $v' \in \operatorname{cyc}\mathcal{C}(w, vv')$ thus the source of v is vv' and v divides vv', so $v \in \mathcal{S}'$; all conditions of Proposition 2.3 are fulfilled, and Δ is a Garside map since $\mathcal{S}'(w,-)$ is the set of left divisors of $\Delta(w)$. The equation $xw^x = wx$ shows that Φ_Δ is the identity. □

Proposition 5.4. *If $\mathcal{C}$ is a right-Noetherian category which admits local right-lcms, the subcategory* $\operatorname{cyc}\mathcal{C}$ *of* $\operatorname{Ad}\mathcal{C}$ *is closed under left-gcd (that is, a gcd in* $\operatorname{Ad}\mathcal{C}$ *of two morphisms in* $\operatorname{cyc}\mathcal{C}$ *is in* $\operatorname{cyc}\mathcal{C}$*).*

Proof. Let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_m)$ be normal decompositions respectively of $x \in \operatorname{cyc}\mathcal{C}(w, -)$ and $y \in \operatorname{cyc}\mathcal{C}(w, -)$ with respect to the Garside family $\mathcal{S}'$ of Proposition 5.3.

We first prove that if $\gcd(x_1, y_1) =^\times 1$ then $\gcd(x, y) =^\times 1$ (here we consider left-gcds in $\operatorname{Ad}\mathcal{C}$). We proceed by induction on $\inf\{m, n\}$. We write Δ for $\Delta(w)$ when there is no ambiguity on the source w. We have that $\gcd(x, y)$ divides

$$\begin{aligned}\gcd(x_1 \dots x_{n-1}\Delta, y_1 \dots y_{m-1}\Delta) &=^\times \gcd(\Delta x_1 \dots x_{n-1}, \Delta y_1 \dots y_{m-1}) \\ &=^\times \Delta \gcd(x_1 \dots x_{n-1}, y_1 \dots y_{m-1}) =^\times \Delta = w,\end{aligned}$$

where the first equality uses that Φ_Δ is the identity and the one before last results from the induction hypothesis. So we get that $\gcd(x, y)$ divides w in $\operatorname{Ad}\mathcal{C}$, so $\gcd(x, y) \in \mathcal{S}'$; thus $\gcd(x, y)$ divides x_1 and y_1, so is trivial.

We now prove the proposition. If $\gcd(x_1, y_1) =^\times 1$ then $\gcd(x, y) =^\times 1$ thus is in $\operatorname{cyc}\mathcal{C}$ and we are done. Otherwise let d_1 be a gcd of x_1 and y_1 and let $x^{(1)}, y^{(1)}$ be defined by $x = d_1 x^{(1)}$, $y = d_1 y^{(1)}$. Similarly let d_2 be a gcd of the first terms of a normal decomposition of $x^{(1)}$, $y^{(1)}$ and let $x^{(2)}$, $y^{(2)}$ be the remainders, etc... Since $\mathcal{C}$ is right-Noetherian the sequence $d_1, d_1 d_2, \dots$ of increasing divisors of x must stabilize at some stage k, which means that the corresponding remainders $x^{(k)}$ and $y^{(k)}$ have first terms of their normal decomposition coprime, so by the first part are themselves coprime. Thus $\gcd(x, y) =^\times d_1 \dots d_k \in \operatorname{cyc}\mathcal{C}$. □

We now give a quite general context where cyclic conjugacy is the same as conjugacy.

Proposition 5.5. *Let $\mathcal{C}$ be a right Noetherian category with a Garside map Δ, and let x be an endomorphism of $\mathcal{C}$ such that for n large enough we have $\Delta \preccurlyeq x^n$. Then for any y we have* $\operatorname{cyc}\mathcal{C}(x, y) = \operatorname{Ad}\mathcal{C}(x, y)$*.*

Proof. We first show that the property $\Delta \preccurlyeq x^n$ is stable by conjugacy (up to changing n). Indeed, if $u \in \operatorname{Ad}\mathcal{C}(x, -)$ then there exists k such that $u \preccurlyeq \Delta^k$. Then $(x^u)^{n(k+1)} = (x^{n(k+1)})^u = (u^{-1}x^{n(k+1)})u$ is divisible by Δ since $\Delta^{k+1} \preccurlyeq x^{n(k+1)}$.

It follows that it is sufficient to prove that if $f \in \operatorname{Ad}\mathcal{C}(x, y)$, $f \notin \mathcal{C}^\times$, then $\gcd(f, x) \notin \mathcal{C}^\times$. Indeed write $f = uf_1$ where $u = \gcd(f, x)$ then since $u \in \operatorname{cyc}\mathcal{C}(x, x^u)$ it is sufficient to prove that $f_1 \in \operatorname{Ad}\mathcal{C}(x^u, y)$ is actually in $\operatorname{cyc}\mathcal{C}(x^u, y)$, which we do by induction since $\mathcal{C}$ is Noetherian and x^u still satisfies the same condition.

Since as observed any $u \in \operatorname{Ad}\mathcal{C}(x, -)$ divides some power of x (x^{nk} if $u \preccurlyeq \Delta^k$) it is enough to show that if $u \in \operatorname{Ad}\mathcal{C}(x, -)$, $u \notin \mathcal{C}^\times$ and $u \preccurlyeq x^n$, then $\gcd(u, x) \notin \mathcal{C}^\times$. We do this by induction on n. From $u \in \operatorname{Ad}\mathcal{C}(x, -)$ we have $u \preccurlyeq xu$, and from $u \preccurlyeq x^n$ we deduce $u \preccurlyeq x \gcd(u, x^{n-1})$. If $\gcd(u, x^{n-1}) \in \mathcal{C}^\times$ then $u \preccurlyeq x$ and we are done: $\gcd(x, u) = u$. Otherwise let $u_1 = \gcd(u, x^{n-1})$. We have $u_1 \preccurlyeq xu_1$, $u_1 \notin \mathcal{C}^\times$ and $u_1 \preccurlyeq x^{n-1}$ thus we are done by induction. □

The F-cyclic conjugacy. Let F be a finite order automorphism of the category $\mathcal{C}$. We define $F\text{-}\operatorname{cyc}\mathcal{C}$ as the subcategory of $F\text{-}\operatorname{Ad}\mathcal{C}$ generated by $\{x \in F\text{-}\operatorname{Ad}\mathcal{C}(w, w') \mid x \preccurlyeq w\}$, or equivalently, if $\mathcal{C}$ is left- and right-cancellative, by $\{x \in \operatorname{Ad}\mathcal{C}(w, w') \mid w' \succcurlyeq F(x)\}$. By the functor J, the morphisms in $F\text{-}\operatorname{cyc}\mathcal{C}(w, w')$ identify to the

morphisms in $\mathrm{cyc}(\mathcal{C} \rtimes \langle F\rangle)(wF, w'F)$ which lie in $\mathcal{C}$. To simplify notation, we will denote by $\mathrm{cyc}\,\mathcal{C}(wF, w'F)$ this last set of morphisms. If $\mathcal{C}$ is right-Noetherian and admits local right-lcms, then $\mathcal{C} \rtimes \langle F\rangle$ also. If $\mathcal{S}$ is a Garside family in $\mathcal{C}$ and F is a Garside automorphism, and we translate Proposition 5.2 to the image of J and then to $F\text{-}\,\mathrm{cyc}\,\mathcal{C}$, we get that $\cup_w\{x \in F\text{-}\,\mathrm{Ad}\,\mathcal{C}(w,-) \mid x \preccurlyeq w \text{ and } x \in \mathcal{S}\}$ is a Garside family in $F\text{-}\,\mathrm{cyc}\,\mathcal{C}$.

Similarly Proposition 5.3 says that the set $\cup_w\{x \in F\text{-}\,\mathrm{Ad}\,\mathcal{C}(w,-) \mid x \preccurlyeq w\}$ is a Garside family in $F\text{-}\,\mathrm{cyc}\,\mathcal{C}$ associated to the Garside map Δ which sends the object w to the morphism $w \in F\text{-}\,\mathrm{cyc}\,\mathcal{C}(w, F(w))$; the associated Φ_Δ is the functor F.

Finally Proposition 5.4 says that under the assumptions of Proposition 5.3 the subcategory $F\text{-}\,\mathrm{cyc}\,\mathcal{C}$ of $F\text{-}\,\mathrm{Ad}\,\mathcal{C}$ is closed under left-gcd.

6. An example: ribbon categories

The "ribbon" category that we consider in this section occurs in particular in the study of the normalizer of a parabolic submonoid of an Artin monoid (the submonoid generated by a part of the atoms), which has been done by Paris and Godelle. The morphisms in this category will correspond to parabolic Deligne-Lusztig varieties, and the conjugacy category to endomorphisms of these varieties.

We place ourselves in the context of Artin monoids as in the example of Subsection 2.3 and want to study the category whose objects are the conjugates of a parabolic submonoid generated by $\mathbf{I} \subset \mathbf{S}$.

Most results work in the more general situation of a category and a subcategory satisfying certain assumptions that we state now.

Definition 6.1. *Let $\mathcal{C}$ be a left-cancellative right-Noetherian category which admits local right lcm's. We say that a subcategory $\mathcal{C}'$ is* parabolic *if it is closed by left-divisor and right-lcm.*

Lemma 6.2. *The above assumption is satisfied when we take for $\mathcal{C}$ an Artin monoid B^+ and for $\mathcal{C}'$ the "parabolic" submonoid $B^+_{\mathbf{I}}$ generated by $\mathbf{I} \subset \mathbf{S}$.*

Proof. We first show that $B^+_{\mathbf{I}}$ is closed by left-divisors. Since both sides of each defining relation for B^+ involve the same generators, two equivalent words involve the same generators. Hence if $xy = z$ with $z \in B^+_{\mathbf{I}}$ then x has an expression involving only elements in $\mathbf{I}$ so is in $B^+_{\mathbf{I}}$. This implies also that if two elements have a right-lcm in $B^+_{\mathbf{I}}$, this right-lcm is a right-lcm in B^+ since it is divisible by that right-lcm. It remains to show that two elements which have a common multiple in B^+ have a common multiple (hence a right-lcm) in $B^+_{\mathbf{I}}$. Taking heads we see that it is sufficient to prove that two elements of $\mathbf{W_I}$ which have a common right-multiple in $\mathbf{W}$ have a common multiple in $\mathbf{W_I}$. This is true since any element of $\mathbf{W}$ can be written uniquely as vw with $v \in \mathbf{W_I}$ and w not divisible by any element of $\mathbf{I}$. □

Lemma 6.3. *Let $\mathcal{C}'$ be a parabolic subcategory of $\mathcal{C}$. Then any $u \in \mathcal{C}$ has a maximal left-divisor $\alpha_{\mathcal{C}'}(u)$ in $\mathcal{C}'$.*

Proof. The set $X = \{x \in \mathcal{C}' \mid x \preccurlyeq u\}$ is a subset of $\mathcal{C}'$ which satisfies the assumptions of Lemma 2.8: it is closed under left-divisor, it is right-Noetherian and if xg and xh are in X with $g, h \in \mathcal{C}'$, then $\mathrm{lcm}(g,h)$ exists, since g and h left-divide $x^{-1}u$, hence $x\,\mathrm{lcm}(g,h)$ is in X since it divides u and $\mathrm{lcm}(g,h) \in \mathcal{C}'$. Thus X is the set of divisors of some morphism $\alpha_{\mathcal{C}'}(u)$. □

The simultaneous conjugacy category. We now consider a parabolic subcategory generated by a subset of the atoms. Let $\mathbf{S}$ be the set of atoms of $\mathcal{C}$, let $\mathbf{I} \subset \mathbf{S}$ and let $\mathcal{C}_\mathbf{I}$ be the subcategory generated by $\mathbf{I}$.

Assumption 6.4. *We assume that for $\mathbf{s} \in \mathbf{S}$ any conjugate $\mathbf{t}$ in $\mathcal{C}$ of $\mathbf{s}$ is in $\mathbf{S}$ (that is, if $\mathbf{s}f = f\mathbf{t}$ with f and $\mathbf{t}$ in $\mathcal{C}$ then $\mathbf{t} \in \mathbf{S}$).*

The above assumption is automatic if $\mathcal{C}$ has homogeneous relations, or equivalently has an additive length function with atoms of length 1. This is clearly the case for Artin monoids.

It follows that if $\mathcal{I}$ is the set of conjugates of $\mathbf{I}$, for any $\mathbf{J} \in \mathcal{I}$ we have $\mathbf{J} \subset \mathbf{S}$. We assume further that for any $\mathbf{J} \in \mathcal{I}$, the category $\mathcal{C}_\mathbf{J}$ is parabolic.

Let $\mathrm{Ad}\,\mathcal{C}(\mathcal{I})$ be the connected component of the simultaneous conjugacy category whose objects are the elements of $\mathcal{I}$. A morphism in $\mathrm{Ad}\,\mathcal{C}(\mathcal{I})$ is a $\mathbf{b} \in \mathcal{C}$ such that for each $\mathbf{s} \in \mathbf{I}$ we have $\mathbf{s}^\mathbf{b} \in \mathcal{C}$, which implies $\mathbf{s}^\mathbf{b} \in \mathbf{S}$. We denote such a morphism in $\mathrm{Ad}\,\mathcal{C}(\mathcal{I})(\mathbf{I}, \mathbf{J})$ by $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$ where $\mathbf{J} = \{\mathbf{s}^\mathbf{b} \mid \mathbf{b} \in \mathbf{I}\}$, and in this situation we write $\mathbf{J} = \mathbf{I}^\mathbf{b}$.

If $\mathcal{S}$ is a Garside family in $\mathcal{C}$, by Proposition 4.3 the set $\{\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \mid \mathbf{I}, \mathbf{J} \in \mathcal{I} \text{ and } \mathbf{b} \in \mathcal{S}\}$ is a Garside family in the category $\mathrm{Ad}\,\mathcal{C}(\mathcal{I})$.

The ribbon category. In our context we will just write $\alpha_\mathbf{I}$ for $\alpha_{\mathcal{C}_\mathbf{I}}$ and denote by $\omega_\mathbf{I}(\mathbf{b})$ the morphism defined by $\mathbf{b} = \alpha_\mathbf{I}(\mathbf{b})\omega_\mathbf{I}(\mathbf{b})$. We say that $\mathbf{b} \in \mathcal{C}$ is $\mathbf{I}$-reduced if it is divisible on the left by no element of $\mathbf{I}$, or equivalently if $\alpha_\mathbf{I}(\mathbf{b}) = 1$.

Definition 6.5. *We define the ribbon category $\mathcal{C}(\mathcal{I})$ as the subcategory of $\mathrm{Ad}\,\mathcal{C}(\mathcal{I})$ generated by the morphisms $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$ such that $\mathbf{b}$ is $\mathbf{I}$-reduced.*

Statements (i) and (ii) in the next proposition are a motivation for restricting to the $\mathbf{I}$-reduced morphisms: (i) shows that we "lose nothing" in doing so, and (ii) shows that being reduced is compatible with the product in $\mathrm{Ad}\,\mathcal{C}(\mathcal{I})$.

Proposition 6.6. (i) *$\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \in (\mathrm{Ad}\,\mathcal{C}(\mathcal{I}))$ if and only if $\mathbf{I} \xrightarrow{\alpha_\mathbf{I}(\mathbf{b})} \mathbf{I} \in (\mathrm{Ad}\,\mathcal{C}(\mathcal{I}))$ and $\mathbf{I} \xrightarrow{\omega_\mathbf{I}(\mathbf{b})} \mathbf{J} \in (\mathcal{C}(\mathcal{I}))$.*

(ii) *If $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \in (\mathrm{Ad}\,\mathcal{C}(\mathcal{I}))$ then for any $\mathbf{b}' \in \mathcal{C}$ we have $\alpha_\mathbf{J}(\mathbf{b}') = \alpha_\mathbf{I}(\mathbf{b}\mathbf{b}')^\mathbf{b}$. In particular if $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \in (\mathcal{C}(\mathcal{I}))$ and $\mathbf{J} \xrightarrow{\mathbf{b}'} \mathbf{K} \in (\mathrm{Ad}\,\mathcal{C}(\mathcal{I}))$ then $\mathbf{I} \xrightarrow{\mathbf{b}\mathbf{b}'} \mathbf{K} \in (\mathcal{C}(\mathcal{I}))$ if and only if $\mathbf{J} \xrightarrow{\mathbf{b}'} \mathbf{K} \in (\mathcal{C}(\mathcal{I}))$.*

(iii) *By Lemma 4.2 two morphisms $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$ and $\mathbf{I} \xrightarrow{\mathbf{b}'} \mathbf{J}'$ of $\mathrm{Ad}\,\mathcal{C}(\mathcal{I})$ have a right-lcm $\mathbf{I} \xrightarrow{\mathbf{c}} \mathbf{I}^\mathbf{c}$ where $\mathbf{c}$ is the right-lcm in $\mathcal{C}$ of $\mathbf{b}$ and $\mathbf{b}'$; if $\mathbf{b}$ and $\mathbf{b}'$ are $\mathbf{I}$-reduced, then $\mathbf{c}$ is also.*

Proof. Let us prove (i). We prove by induction on the length of $\mathbf{b}$ that if $\mathbf{s} \in \mathbf{I}$ and $\mathbf{s}^\mathbf{b} \in \mathcal{C}$ then $\mathbf{s}^{\alpha_\mathbf{I}(\mathbf{b})} \in \mathbf{I}$. This will prove (i) in one direction. The converse is obvious.

By Assumption 6.4 we have $\mathbf{sb} = \mathbf{bt}$ for some $\mathbf{t} \in \mathbf{S}$. If $\mathbf{s} \preccurlyeq \mathbf{b}$ we write $\mathbf{b} = \mathbf{sb}'$ so that $\mathbf{sb}' = \mathbf{b}'\mathbf{t}$. We have $\alpha_\mathbf{I}(\mathbf{b}) = \mathbf{s}\alpha_\mathbf{I}(\mathbf{b}')$ and we are done by induction. If $\mathbf{s}$ does not divide $\mathbf{b}$ then the lcm of $\mathbf{s}$ and $\alpha_\mathbf{I}(\mathbf{b})$ divide $\mathbf{sb} = \mathbf{bt}$ and this lcm can be written $\mathbf{sv} = \alpha_\mathbf{I}(\mathbf{b})\mathbf{u}$, with $\mathbf{v}$ and $\mathbf{u}$ in $\mathcal{C}_\mathbf{I}$ since $\mathcal{C}_\mathbf{I}$ is closed by right-lcm. We get then that $\mathbf{v}$ divides $\mathbf{b}$, so divides $\alpha_\mathbf{I}(\mathbf{b})$; thus $\alpha_\mathbf{I}(\mathbf{b})\mathbf{u} = \mathbf{vau}$ for some $\mathbf{a} \in \mathcal{C}$. By Assumption 6.4 we have that $\mathbf{au} \in \mathbf{S}$, thus $\mathbf{a} = 1$ and $\mathbf{u} \in \mathbf{S}$, hence $\mathbf{u} \in \mathbf{I}$ which is the result.

Let us prove (ii). For $\mathbf{s} \in \mathbf{I}$ let $\mathbf{s}' = \mathbf{s}^{\mathbf{b}} \in \mathbf{J}$. Assume first that $\mathbf{s} \not\preccurlyeq \mathbf{b}$. Then $\mathbf{bs}' = \mathbf{sb}$ is a common multiple of $\mathbf{s}$ and $\mathbf{b}$ which has to be their lcm since $\mathbf{s}'$ is an atom. So for $\mathbf{s} \in \mathbf{I}$ we have $\mathbf{s} \preccurlyeq \mathbf{bb}'$ if and only if $\mathbf{bs}' \preccurlyeq \mathbf{bb}'$, that is, $\mathbf{s}^{\mathbf{b}} \preccurlyeq \mathbf{b}'$ whence the result. Now if $\mathbf{s} \preccurlyeq \mathbf{b}$ we write $\mathbf{b} = \mathbf{s}^k\mathbf{b}_1$ with $\mathbf{s} \not\preccurlyeq \mathbf{b}_1$; we have $\mathbf{s}' = \mathbf{s}^{\mathbf{b}_1}$ and the above proof, with $\mathbf{b}_1$ instead of $\mathbf{b}$, applies.

To prove (iii) we will actually show the stronger statement that if for $\mathbf{b}, \mathbf{c} \in \mathcal{C}$ we have $\mathbf{b} \preccurlyeq \mathbf{c}$, $\mathbf{I}^{\mathbf{b}} \subset \mathbf{S}$ then $\alpha_{\mathbf{I}}(\mathbf{b}) \preccurlyeq \alpha_{\mathbf{I}}(\mathbf{c})$ (which is obvious) and $\omega_{\mathbf{I}}(\mathbf{b}) \preccurlyeq \omega_{\mathbf{I}}(\mathbf{c})$ (then in the situation of (iii) we get that $\omega_{\mathbf{I}}(\mathbf{c})$ is a common multiple of $\mathbf{b}$ and $\mathbf{b}'$, thus $\mathbf{c} \preccurlyeq \omega_{\mathbf{I}}(\mathbf{c})$, which is impossible unless $\alpha_{\mathbf{I}}(\mathbf{c}) = 1$). By dividing $\mathbf{b}$ and $\mathbf{c}$ by $\alpha_{\mathbf{I}}(\mathbf{b})$ we may as well assume that $\alpha_{\mathbf{I}}(\mathbf{b}) = 1$ since $\mathbf{I}^{\omega_{\mathbf{I}}(\mathbf{b})} \subset \mathbf{S}$ by (i). We write $\mathbf{c} = \mathbf{bb}_1$ and $\mathbf{J} = \mathbf{I}^{\mathbf{b}}$. By (ii) we have $\alpha_{\mathbf{I}}(\mathbf{c})^{\mathbf{b}} = \alpha_{\mathbf{J}}(\mathbf{b}_1)$, whence $\alpha_{\mathbf{I}}(\mathbf{c})\mathbf{b} = \mathbf{b}\alpha_{\mathbf{J}}(\mathbf{b}_1) \preccurlyeq \mathbf{bb}_1 = \mathbf{c} = \alpha_{\mathbf{I}}(\mathbf{c})\omega_{\mathbf{I}}(\mathbf{c})$. Left-canceling $\alpha_{\mathbf{I}}(\mathbf{c})$ we get $\mathbf{b} \preccurlyeq \omega_{\mathbf{I}}(\mathbf{c})$ which is what we want since $\mathbf{b} = \omega_{\mathbf{I}}(\mathbf{b})$. □

By Proposition 6.6 items (6.6(ii)) and (6.6(iii)) the subcategory $\mathcal{C}(\mathcal{I})$ of $\mathrm{Ad}\,\mathcal{C}(\mathcal{I})$ is closed under right-quotient and right-lcm. Next proposition shows that $\mathcal{S} \cap \mathcal{C}(\mathcal{I})$ generates $\mathcal{C}(\mathcal{I})$. By Lemma 2.7 this implies that $\mathcal{S} \cap \mathcal{C}(\mathcal{I})$ is a Garside family in $\mathcal{C}(\mathcal{I})$.

Proposition 6.7. *Let $\mathbf{b}$ be a morphism of $\mathcal{C}(\mathcal{I})$; then all the terms of the normal decomposition in $\mathrm{Ad}\,\mathcal{C}(\mathcal{I})$ of $\mathbf{b}$ are in $\mathcal{C}(\mathcal{I})$.*

Proof. Let $\mathbf{b} = \mathbf{w}_1 \dots \mathbf{w}_k$ be the normal decomposition in $\mathrm{Ad}\,\mathcal{C}(\mathcal{I})$ of $\mathbf{b} \in \mathcal{C}(\mathcal{I})(\mathbf{I}, \mathbf{J})$ (it is also the normal decomposition in $\mathcal{C}$ since the Garside family of $\mathrm{Ad}\,\mathcal{C}(\mathcal{I})$ consists of the morphisms $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J}$ where $\mathbf{w} \in \mathcal{S}$). As $\mathbf{w}_i \in \mathrm{Ad}\,\mathcal{C}(\mathcal{I})$, the source of $\mathbf{w}_i$ is $\mathbf{I}_i = {}^{\mathbf{w}_i \dots \mathbf{w}_k}\mathbf{J} \subset \mathbf{S}$. Now, ${}^{\mathbf{w}_1 \dots \mathbf{w}_{i-1}}\alpha_{\mathbf{I}_i}(\mathbf{w}_i) \in \mathcal{C}_{\mathbf{I}}$ and

$$ {}^{\mathbf{w}_1 \dots \mathbf{w}_{i-1}}\alpha_{\mathbf{I}_i}(\mathbf{w}_i) \preccurlyeq \mathbf{w}_1 \dots \mathbf{w}_{i-1}\alpha_{\mathbf{I}_i}(\mathbf{w}_i) \preccurlyeq \mathbf{w}_1 \dots \mathbf{w}_{i-1}\mathbf{w}_i \preccurlyeq \mathbf{b} $$

so divides $\alpha_{\mathbf{I}}(\mathbf{b})$, thus this element has to be 1, whence the result. □

As explained above the proposition we then get

Corollary 6.8. *The set $\mathcal{S} = \{\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J} \in \mathrm{Ad}\,\mathcal{C}(\mathcal{I}) \mid \mathbf{w} \in \mathcal{S}$ and $\alpha_{\mathbf{I}}(\mathbf{w}) = 1\}$ is a Garside family for $\mathcal{C}(\mathcal{I})$.*

We now describe, in the case of Artin monoids, the atoms of $\mathcal{C}(\mathcal{I})$, using the work of [BH].

In that case we have $\mathcal{S} = \{\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J} \in \mathrm{Ad}\,\mathcal{C}(\mathcal{I}) \mid \mathbf{w} \in \mathbf{W}$ and $\alpha_{\mathbf{I}}(\mathbf{w}) = 1\}$. Note that a morphism $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J} \in \mathcal{S}$ is determined by the image (I, w, J) in W of the triple.

If I is such that W_I is finite, we denote by w_I the longest element of W_I. With these notations, we have

Proposition 6.9. *The atoms of $\mathcal{C}(\mathcal{I})$ are the triples ${}^{v(s,I)}I, v(s, I), I)$ where $s \in S - I$ is such that, if K is the connected component of s in the Coxeter diagram of $W_{\{s\}\cup I}$, then W_K is finite, and where $v(s, I) = w_K w_{K-\{s\}}$.*

Proof. We first show that $\mathcal{S}$ identifies to a subset of the groupoid $\mathcal{G}$ defined before [BH, Proposition 2.3]: precisely $\mathcal{S}$ has the same objects and morphisms as $\mathcal{G}$, but a product of two morphisms (I, w, J) and (J, w', K) is defined in $\mathcal{S}$ (and equal to the value (I, ww', K) it takes in $\mathcal{G}$) if and only if $l(ww') = l(w) + l(w')$. Since this last condition is the condition for the lift to $\mathbf{W}$ of ww' to be $\mathbf{ww}'$, where $\mathbf{w}$ and $\mathbf{w}'$ are the respective lifts of $\mathbf{w}$ and $\mathbf{w}'$, it is enough to show that if $w \in W$ is I-reduced and

$I^w \subset S$ then $\mathbf{I}^{\mathbf{w}} \subset \mathbf{S}$ where $\mathbf{w}$ is the lift in $\mathbf{W}$ of w. Take $s \in I$, then $s^w = t \in S$ that is $sw = wt$. Since w is I-reduced we have $l(sw) = 1 + l(w)$ thus the lift of sw is $\mathbf{sw}$. Since $sw = wt$ we also have $l(wt) = l(w) + 1$ thus the lift of wt is $\mathbf{wt}$, and we have in $\mathcal{C}$ the equality $\mathbf{sw} = \mathbf{wt}$, that is $\mathbf{s}^{\mathbf{w}} \in \mathbf{S}$.

$\mathcal{G}$ is generated by the morphisms $({}^{v(s,I)}I, v(s,I), I)$ of the statement by [BH, Proposition 2.3], and by the property on the length given in that proposition this is true also for $\mathcal{S}$.

Finally, for proving that the elements $({}^{v(s,I)}I, v(s,I), I)$ are atoms we show that if $({}^{v(t,I)}I, v(t,I), I)$ is a right-divisor of $({}^{v(s,I)}I, v(s,I), I)$ then $s = t$. Let L be the connected component of $\{t\} \cup I$ containing t. If we denote by $R(W)$ the set of reflections of a Coxeter group W, we know by [BH, lemma 3.2] that $({}^{v(t,I)}I, v(t,I), I)$ is a right-divisor of $({}^{v(s,I)}I, v(s,I), I)$ if and only if $R(W_L) - R(W_{L-\{t\}}) \subset R(W_K) - R(W_{K-\{s\}})$. Since t is in the left-hand side, it has to be in K, hence $L = K$. The condition becomes then $R(W_{K-\{s\}}) \subset R(W_{K-\{t\}})$, so that $K - \{s\} \subset K - \{t\}$ which means that $s = t$ as required. □

The spherical case. We show now that $\mathcal{C}(\mathcal{I})$ is a Garside category when $\mathbf{W}$ is finite. We recall that in that case $\mathcal{C} = B^+$ has a Garside element $\mathbf{w_S}$. We denote by Φ the involution on $\mathbf{S}$ given by conjugation by $\mathbf{w_S}$. This extends naturally to involutions on $\mathcal{I}$, on $\mathcal{C}$, and finally on $\mathcal{C}(\mathcal{I})$ that we denote in the same way. We define a natural transformation $\Delta_{\mathcal{I}}$ from the identity to Φ as the collection of morphisms $\mathbf{J} \xrightarrow{\mathbf{w_J^{-1}w_S}} \Phi(\mathbf{J})$. The properties which must be satisfied by $\Delta_{\mathcal{I}}$ to be a Garside map are easily checked.

Proposition 6.10. *When W is finite the category $\mathcal{C}(\mathcal{I})$ is a Garside category with Garside map $\Delta_{\mathcal{I}}$ as defined above.*

7. Periodic elements

Definition 7.1. *Let $\mathcal{C}$ be a Garside category with Garside natural transformation Δ; then an endomorphism x of the enveloping groupoid $\mathcal{G}$ is said to be* (d,p)-periodic *if $x^d \in \Delta^p \mathcal{C}^\times$ for some non-zero integers d, p.*

Note that if x is (d,p)-periodic it is also (nd, np)-periodic for any non-zero integer n. We call d/p the *period* of x. Note that if Φ is of finite order, then a conjugate of a periodic element is periodic of the same period (though the minimal pair (d,p) may change). It can be shown that, up to cyclic conjugacy, the notion of being (d,p)-periodic depends only on the fraction d/p; it results from Proposition 5.5 that two periodic morphisms of same period are conjugate if and only if they are cyclically conjugate; our main interest is to be able to describe the centralizers of periodic morphisms.

We deal here with the case $p = 2$. We show by elementary computations that a $(d,2)$-periodic element of $\mathcal{C}$ is the same up to cyclic conjugacy as a $(d/2, 1)$-periodic element when d is even, and get a related characterization when d is odd.

We denote by $\mathcal{S}$ the Garside family of $\mathcal{C}$ and by Φ the canonical Garside automorphism Φ corresponding to the natural transformation Δ.

Lemma 7.2. *Let f be an endomorphism in $\mathcal{C}$ such that $f^d \in \Delta^2 \mathcal{C}^\times$, and let $e = \lfloor \frac{d}{2} \rfloor$. Then there exists $g \in Obj(\mathrm{cyc}\,\mathcal{C})$ such that $\mathrm{cyc}\,\mathcal{C}(f,g) \neq \emptyset$ and $g^e \in \mathcal{S}\mathcal{C}^\times$ and $g^d \in \Delta^2 \mathcal{C}^\times$.*

Further, if g is as in the conclusion above, that is $g^d \in \Delta^2\mathcal{C}^\times$ and $g^e \in \mathcal{SC}^\times$, then if d is even we have $g^e \in \Delta\mathcal{C}^\times$, and if d is odd there exists $h \in \mathcal{SC}^\times$ such that $g = h\Phi(h)\varepsilon$ and $g^e h = \Delta$, where $\varepsilon \in \mathcal{C}^\times$ is defined by $g^d = \Delta^2\varepsilon$.

Proof. We will prove by increasing induction on i that for $i \leq d/2$ there exists $v \in \operatorname{cyc}\mathcal{C}$ such that $(f^v)^i \in \mathcal{SC}^\times$ and $(f^v)^d \in \Delta^2\mathcal{C}^\times$. We start the induction with $i = 0$ where the result holds trivially with $v = 1$.

We consider now the general step: assuming the result for i such that $i + 1 \leq d/2$, we will prove it for $i + 1$. We thus have a v for step i, thus replacing if needed f by f^v we may assume that $f^i \in \mathcal{SC}^\times$ and $f^d \in \Delta^2\mathcal{C}^\times$; we will conclude by finding $v \in \mathcal{S}$ such that $v \preccurlyeq f$ and $(f^v)^{i+1} \in \mathcal{SC}^\times$ and $(f^v)^d \in \Delta^2\mathcal{C}^\times$. If $f^{i+1} \preccurlyeq \Delta$ we have the desired result with $v = 1$. We may thus assume that $\lg_{\mathcal{S}}(f^{i+1}) \geq 2$. Since $f^{i+1} \preccurlyeq \Delta^2$ we have actually $\lg_{\mathcal{S}}(f^{i+1}) = 2$ (see Proposition 2.10(i)); let $(f^i v, w)$ be a normal decomposition of f^{i+1} where $f^i v \in \mathcal{S}$ and $w \in \mathcal{SC}^\times$. As $f^i vw(f^i v) \preccurlyeq f^i vw(f^i vw) = f^{2(i+1)} \preccurlyeq f^d =^\times \Delta^2$, we still have $2 = \lg_{\mathcal{S}}((f^i v)w(f^i v)) = \lg_{\mathcal{S}}((f^i v)w)$. By Proposition 2.10(ii) we thus have $w(f^i v) \in \mathcal{SC}^\times$. Then $\mathcal{SC}^\times \ni w(f^i v) = w((vw)^i)v = (f^v)^{i+1}$ and $v \preccurlyeq f$.

So v will do if we can show $(f^v)^d \in \Delta^2\mathcal{C}^\times$. Since $f^d = \Delta^2\varepsilon$ with $\varepsilon \in \mathcal{C}^\times$, we have that f commutes with $\Delta^2\varepsilon$, thus f^{i+1} also, that is $\Phi^2(f^{i+1})\varepsilon = \varepsilon f^{i+1}$ or equivalently $\Phi^2(f^i v)\Phi^2(w)\varepsilon = \varepsilon f^i vw$. Now $(\Phi^2(f^i v), \Phi^2(w)\varepsilon)$ is an $\mathcal{SC}^\times$-normal decomposition and since $(f^i v, w)$ is a normal decomposition, by Lemma 2.2 there exists $\varepsilon' \in \mathcal{C}^\times$ such that $\Phi^2(f^i v)\varepsilon' = \varepsilon f^i v$. We have $f^i\Delta^2\Phi^2(v)\varepsilon' = \Delta^2\Phi^2(f^i v)\varepsilon' = \Delta^2\varepsilon f^i v = f^i\Delta^2\varepsilon v$, the last equality sincef^i commutes with $\Delta^2\varepsilon$. Canceling Δ^2 we get $\Phi^2(v)\varepsilon' = \varepsilon v$. We have then $v(f^v)^d = f^d v = \Delta^2\varepsilon v = \Delta^2\Phi^2(v)\varepsilon' = v\Delta^2\varepsilon'$ whence the result by canceling v on the left.

We prove now the second part. From $g^e \in \mathcal{SC}^\times$ we get that there exists $h \in \mathcal{SC}^\times$ such that $g^e h = \Delta$. If $g^d = \Delta^2\varepsilon$ with $\varepsilon \in \mathcal{C}^\times$ we get $g^e h\Delta\varepsilon = \Delta^2\varepsilon = g^d$, whence by cancellation $h\Delta\varepsilon = g^e g^a$ with $a = 1$ if d is odd and $a = 0$ if d is even. We deduce $g^e g^a = h\Delta\varepsilon = \Delta\Phi(h)\varepsilon = g^e h\Phi(h)\varepsilon$, thus $h\Phi(h)\varepsilon = g^a$.

If d is odd we get the statement of the lemma, and if d is even we get $h\Phi(h) \in \mathcal{C}^\times$, so $h \in \mathcal{C}^\times$ so $g^e \in \Delta\mathcal{C}^\times$. □

F-periodic elements. Let us apply Lemma 7.2 to the case of a semi-direct product category $\mathcal{C} \rtimes \langle F\rangle$ with F a Garside automorphism of finite order, where $\mathcal{C}$ has no non-trivial invertible element and the Garside family $\mathcal{S}$ of $\mathcal{C} \rtimes \langle F\rangle$ is in $\mathcal{C}$. Then a map $yF \in \mathcal{C}F$ is (d, p) periodic if and only if $\operatorname{target}(y) = F(\operatorname{source}(y))$ and $(yF)^d = \Delta^p F^d$.

From the lemma we can deduce the following.

Corollary 7.3. *Assume $\Phi^2 = \mathrm{Id}$ and that $yF \in \mathcal{C}F$ satisfies $(yF)^d = \Delta^2 F^d$. Then*

(i) *If $d = 2e$ is even, there exists x such that $\operatorname{cyc}\mathcal{C}(yF, xF) \neq \emptyset$ and $(xF)^e = \Delta F^e$. The centralizer of xF in $\mathcal{C}$ identifies to $\operatorname{cyc}\mathcal{C}(xF)$. Further, we may compute these endomorphisms in the category of fixed points $(\operatorname{cyc}\mathcal{C})^{\Phi F^e}$ since the morphisms in $\operatorname{cyc}\mathcal{C}(xF)$ are ΦF^e-stable.*

(ii) *If $d = 2e+1$ is odd, there exists x such that $\operatorname{cyc}\mathcal{C}(yF, xF) \neq \emptyset$ and $(xF)^d = \Delta^2 F^d$ and $(xF)^e F^{-e} \preccurlyeq \Delta$. The element s defined by $(xF)^e F^{-e} s = \Delta$ is such that, in the category $\mathcal{C} \rtimes \langle\Lambda\rangle$ with $\Lambda = \Phi^{-1}F^{-e}$, we have $x\Lambda^2 = (s\Lambda)^2$ and $(s\Lambda)^d = \Delta\Lambda^d$. The centralizer of xF in $\mathcal{C}$ identifies to $\operatorname{cyc}\mathcal{C}(s\Lambda)$. Further, we may compute these endomorphisms in the category of fixed points $(\operatorname{cyc}\mathcal{C})^{F^d}$ since $\operatorname{cyc}\mathcal{C}(s\Lambda)$ is stable by F^d.*

Proof. The lemma shows that y is cyclically F-conjugate to an x such that $(xF)^e \in \mathcal{S}F^e$ and $(xF)^d = \Delta^2 F^d$.

If d is even then $(xF)^e = \Delta F^e$.

If d is odd from $(xF)^e h = \Delta$ we get $h = sF^{-e}$ with $s \in \mathcal{S}$, and $xF = sF^{-e}\Phi(sF^{-e})F^d = s\Lambda(s)F$. As xF is F^d-stable, and $F^d\Lambda^2 = F$, this can be rewritten $x\Lambda = (s\Lambda)^2$. Since the elements of the centralizer of xF commute to F^d the centralizer is the same as that of $x\Lambda$, and from $(xF)^e s = \Delta F^e$ centralizes $s\Lambda$. From $x\Lambda = (s\Lambda)^2$ we get that the centralizer of $s\Lambda$ is the same as that of xF.

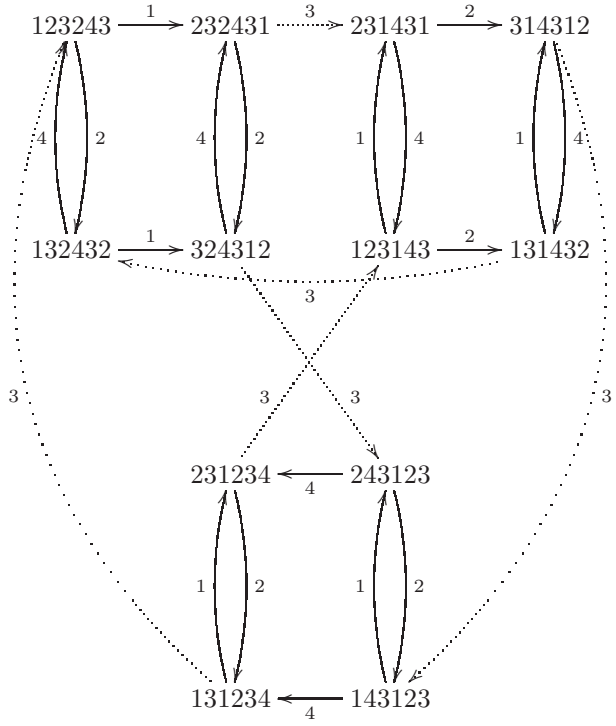
We get the corollary if we know that the centralizer of xF, for d even (resp. $s\Lambda$, for d odd) is the same as $\mathrm{cyc}\,\mathcal{C}(xF)$ (resp. $\mathrm{cyc}\,C(s\Lambda)$). But this is an immediate consequence of Proposition 5.5. □

Conjugacy of periodic elements. We recall the result of David Bessis on periodic elements. Two periodic elements of same period x and y in the classical Artin monoid are also periodic and have equal periods in the dual monoid. By the results of Bessis ([B1, 11.21]), such elements are conjugate in the dual monoid, so are conjugate in the Artin group, hence are conjugate in the classical monoid. By Proposition 5.5 they are cyclically conjugate in the classical monoid. We conjecture that the same results extend to the case of F-conjugacy.

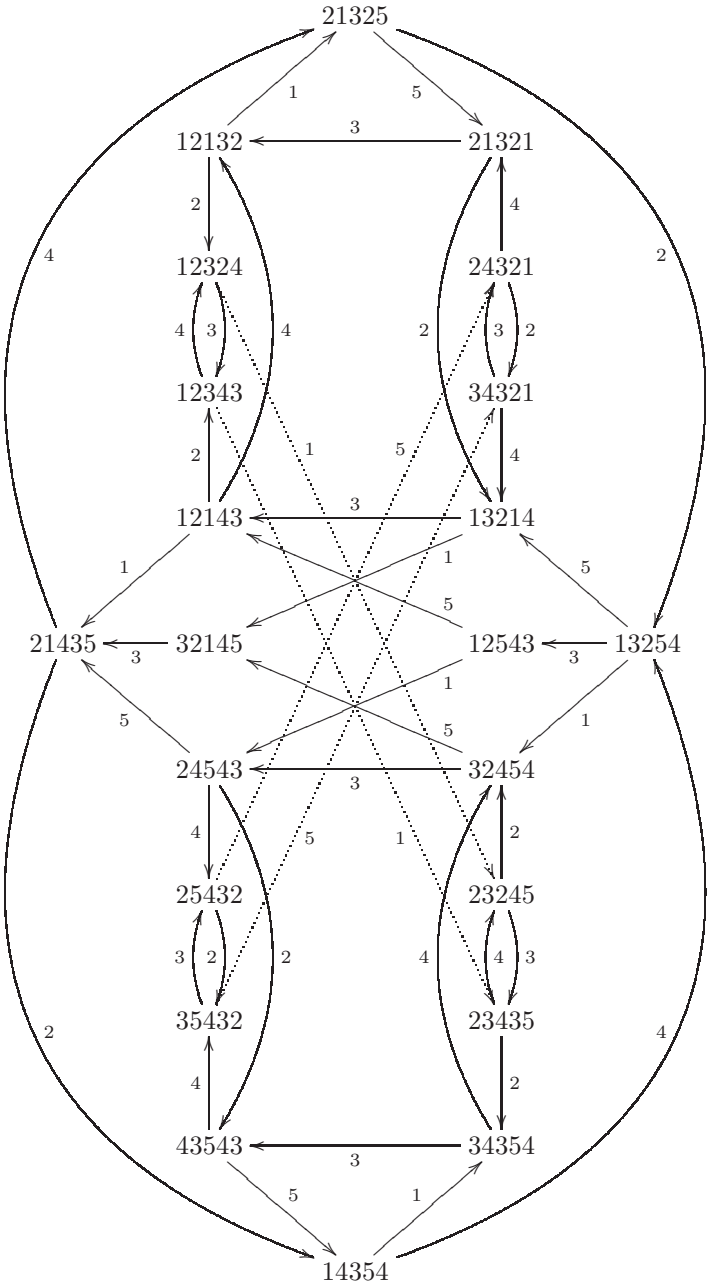
We conjecture that similarly, all periodic elements in $\mathcal{C}(\mathcal{I})$ of a given period are conjugate (thus cyclically conjugate); and that this extends also to the case of F-conjugacy.

7.1. **Two examples.** In two examples we show a picture of the category associated to the centralizer of a braid (a root of $\boldsymbol{\pi}$ as will be considered in Section 9).

We first look at the case of a $\mathbf{w}$ in the braid monoid $\mathcal{C} = B^+(W(D_4))$ such that $\mathbf{w}^2 = \Delta$, and describe the category $\mathrm{cyc}\,\mathcal{C}(\mathbf{w})^\Phi$.



We now look at the case of a $\mathbf{w}$ in the braid monoid $\mathcal{C} = B^+(W(A_5))$ such that $\mathbf{w}^3 = \Delta$, and describe the category $\operatorname{cyc}\mathcal{C}(\mathbf{s}\Phi^{-1})$ where $\mathbf{s}$ is such that $\mathbf{w} = \mathbf{s}\Phi(\mathbf{s})$.



Part 2. Deligne-Lusztig varieties and eigenspaces

In this part, we first study the Deligne-Lusztig varieties which give rise to a Lusztig induction functor $R_{\mathbf{L}}^{\mathbf{G}}(\mathrm{Id})$; we then generalize these varieties to varieties attached to morphisms in a braid category.

In the next section we consider the morphisms in this braid category which correspond to varieties which play a role in the Broué conjectures.

We finish with a section which spells out the particular form that the conjectures take on our varieties.

8. Parabolic Deligne-Lusztig varieties

Let $\mathbf{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_p$, and let F be an isogeny on $\mathbf{G}$ such that some power F^δ is a Frobenius for a split $\mathbb{F}_{q^\delta}$-structure (this defines a positive real number q such that q^δ is an integral power of p).

Let $\mathbf{L}$ be an F-stable Levi subgroup of a (non-necessarily F-stable) parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ and let $\mathbf{P} = \mathbf{L}\mathbf{V}$ be the corresponding Levi decomposition of $\mathbf{P}$. Let

$$\begin{aligned}\mathbf{X}_{\mathbf{V}} &= \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g\mathbf{V} \cap F(g\mathbf{V}) \neq \emptyset\} = \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g^{-1}{}^F g \in \mathbf{V}^F\mathbf{V}\}\\ &\simeq \{g \in \mathbf{G} \mid g^{-1}{}^F g \in {}^F\mathbf{V}\}/(\mathbf{V} \cap {}^F\mathbf{V})\end{aligned}$$

$\mathbf{X}_{\mathbf{V}}$ is a $\mathbf{G}^F$-variety-$\mathbf{L}^F$.

We choose a prime number $\ell \neq p$. Then the virtual $\mathbf{G}^F$-module-$\mathbf{L}^F$ given by $M = \sum_i (-1)^i H_c^i(\mathbf{X}_{\mathbf{V}}, \overline{\mathbb{Q}}_\ell)$ defines the cohomological induction $R_{\mathbf{L}}^{\mathbf{G}}$ which by definition maps $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$ to $M \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{L}^F} \lambda$.

The map $g\mathbf{V} \mapsto g\mathbf{P}$ makes $\mathbf{X}_{\mathbf{V}}$ an $\mathbf{L}^F$-torsor over

$$\begin{aligned}\mathbf{X}_{\mathbf{P}} &= \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g\mathbf{P} \cap F(g\mathbf{P}) \neq \emptyset\} = \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}{}^F g \in \mathbf{P}^F\mathbf{P}\}\\ &\simeq \{g \in \mathbf{G} \mid g^{-1}{}^F g \in {}^F\mathbf{P}\}/(\mathbf{P} \cap {}^F\mathbf{P}),\end{aligned}$$

a $\mathbf{G}^F$-variety such that $R_{\mathbf{L}}^{\mathbf{G}}(\mathrm{Id}) = \sum_i (-1)^i H_c^i(\mathbf{X}_{\mathbf{P}}, \overline{\mathbb{Q}}_\ell)$. The variety $\mathbf{X}_{\mathbf{P}}$ is the prototype of the varieties we want to study.

Let $\mathbf{T} \subset \mathbf{B}$ be a pair of an F-stable maximal torus and an F-stable Borel subgroup of $\mathbf{G}$. To this choice is associated a basis Π of the root system Φ of $\mathbf{G}$ with respect to $\mathbf{T}$, and a Coxeter system (W, S) for the Weyl group $W = W_{\mathbf{G}}(\mathbf{T})$. Let $X_{\mathbb{R}} = X(\mathbf{T}) \otimes \mathbb{R}$; on the vector space $X_{\mathbb{R}}$, the isogeny F acts as $q\phi$ where ϕ is of order δ and stabilizes the positive cone $\mathbb{R}^+\Pi$; we will still denote by ϕ the induced automorphism of (W, S).

To a subset $I \subset \Pi$ corresponds a subgroup $W_I \subset W$, a parabolic subgroup $\mathbf{P}_I = \coprod_{w \in W_I} \mathbf{B}w\mathbf{B}$, and the Levi subgroup $\mathbf{L}_I$ of $\mathbf{P}_I$ which contains $\mathbf{T}$.

Given any $\mathbf{P} = \mathbf{L}\mathbf{V}$ as above where $\mathbf{L}$ is F-stable, there exists $I \subset \Pi$ such that $(\mathbf{L}, \mathbf{P})$ is $\mathbf{G}$-conjugate to $(\mathbf{L}_I, \mathbf{P}_I)$; if we choose the conjugating element such that it conjugates a maximally split torus of $\mathbf{L}$ to $\mathbf{T}$ and a rational Borel subgroup of $\mathbf{L}$ containing this torus to $\mathbf{B} \cap \mathbf{L}_I$, then this element conjugates $(\mathbf{L}, \mathbf{P}, F)$ to $(\mathbf{L}_I, \mathbf{P}_I, \dot{w}F)$ where $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$ is such that ${}^{w\phi}I = I$, where w is the image of $\dot{w}$ in W.

It will be convenient to consider I as a subset of S instead of a subset of Π; the condition on w must then be stated as "$I^w = {}^\phi I$ and w is I-reduced". Via the above conjugation, the variety $\mathbf{X}_{\mathbf{P}}$ is isomorphic to the variety

$$\mathbf{X}(I, w\phi) = \{g\mathbf{P}_I \in \mathbf{G}/\mathbf{P}_I \mid g^{-1}{}^F g \in \mathbf{P}_I w^F \mathbf{P}_I\}.$$

We will denote by $\mathbf{X}_{\mathbf{G}}(I, w\phi)$ this variety when there is a possible ambiguity on the group. If we denote by $\mathbf{U}_I$ the unipotent radical of $\mathbf{P}_I$, we have $\dim \mathbf{X}(I, w\phi) = \dim \mathbf{U}_I - \dim(\mathbf{U}_I \cap {}^{wF}\mathbf{U}_I) = l(w)$. The ℓ-adic cohomology of the variety $\mathbf{X}(I, w\phi)$ gives rise to the Lusztig induction from $\mathbf{L}_I^{\dot{w}F}$ to $\mathbf{G}^F$ of the trivial representation; to avoid ambiguity on the isogenies involved, we will sometimes denote this Lusztig induction by $R^{\mathbf{G},F}_{\mathbf{L}_I,\dot{w}F}(\mathrm{Id})$.

Definition 8.1. *For two parabolic subgroups $\mathbf{P}, \mathbf{Q}$ we say that the pair $(\mathbf{P}, \mathbf{Q})$ is in relative position (I, w, J), where $I, J \subset S$ and $w \in W$, if $(\mathbf{P}, \mathbf{Q})$ is $\mathbf{G}$-conjugate to $(\mathbf{P}_I, {}^w\mathbf{P}_J)$. We denote this as $\mathbf{P} \xrightarrow{I,w,J} \mathbf{Q}$.*

Since any pair $(\mathbf{P}, \mathbf{Q})$ of parabolic subgroups share a common maximal torus, it has a relative position (I, w, J) where I, J is uniquely determined as well as the double coset $W_I w W_J$.

Let $\mathcal{P}_I$ be the variety of parabolic subgroups conjugate to $\mathbf{P}_I$ (isomorphic to $\mathbf{G}/\mathbf{P}_I$). Via the map $g\mathbf{P}_I \mapsto {}^g\mathbf{P}_I$ we have an isomorphism

$$\mathbf{X}(I, w\phi) \simeq \{\mathbf{P} \in \mathcal{P}_I \mid \mathbf{P} \xrightarrow{I,w,{}^\phi I} {}^F\mathbf{P}\};$$

it is a variety over $\mathcal{P}_I \times \mathcal{P}_{\phi I}$ by the first and second projection.

The parabolic braid category $B^+(\mathcal{I})$. Let B^+ (resp. B) denote the Artin-Tits monoid (resp. Artin-Tits group) of W, and let $\mathbf{S}$ be its generating set, which is in canonical bijection with S. To $I \subset S$ corresponds $\mathbf{I} \subset \mathbf{S}$ and the submonoid $B^+_{\mathbf{I}}$ generated by $\mathbf{I}$. By Lemma 6.3 every element of $\mathbf{b} \in B^+$ has a unique longest divisor $\alpha_{\mathbf{I}}(\mathbf{b})$ in $B^+_{\mathbf{I}}$. As in Definition 6.5 we define:

Definition 8.2. *Let $\mathcal{I}$ be the set of conjugates of some $\mathbf{I}$ in $\mathbf{S}$. Then $B^+(\mathcal{I})$ is the category whose objects are the elements of $\mathcal{I}$ and the morphisms from $\mathbf{J}$ to $\mathbf{K}$ are the $\mathbf{b} \in B^+$ such that $\mathbf{J}^{\mathbf{b}} = \mathbf{K}$ and $\alpha_{\mathbf{J}}(\mathbf{b}) = 1$.*

If $\mathbf{b} \in B^+$ determines an element of $\mathrm{Hom}_{B^+(\mathcal{I})}(\mathbf{I}, \mathbf{J})$ for some objects $\mathbf{I}, \mathbf{J}$ of $\mathcal{I}$, we will denote by $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$ this morphism to lift ambiguity on its source and target.

It is shown in Proposition 6.6 that the above definition makes sense, that is if we have a composition $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \xrightarrow{\mathbf{c}} \mathbf{K}$ in $B^+(\mathcal{I})$, then $\alpha_{\mathbf{I}}(\mathbf{bc}) = 1$. When $I = \emptyset$, $B^+(\mathcal{I})$ reduces to the Artin-Tits monoid B^+.

The canonical lift $W \xrightarrow{\sim} \mathbf{W}$ of W in B^+ is denoted by $w \mapsto \mathbf{w}$. Conversely for $\mathbf{w} \in \mathbf{W}$ we denote w its image in W. It is shown after Proposition 6.6 that $B^+(\mathcal{I})$ has a Garside family $\mathcal{S}$ consisting of the morphisms $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J}$ where $\mathbf{w} \in \mathbf{W}$ and a Garside map $\Delta_{\mathcal{I}}$ given on the object $\mathbf{I}$ by the morphism $\mathbf{I} \xrightarrow{\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0} \mathbf{I}^{\mathbf{w}_0}$ where we denote by $\mathbf{w}_{\mathbf{I}}$ the lift to $\mathbf{W}$ of the longest element of W_I, and write $\mathbf{w}_0$ for $\mathbf{w}_{\mathbf{S}}$. This includes the following:

Lemma 8.3. (i) *$\mathcal{S}$ generates $B^+(\mathcal{I})$; specifically, if $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J} \in B^+(\mathcal{I})$ and $(\mathbf{w}_1, \dots, \mathbf{w}_k)$ is the normal form of $\mathbf{w}$, there exist subsets $\mathbf{I}_i$ with $\mathbf{I}_1 = \mathbf{I}$, $\mathbf{I}_{k+1} = \mathbf{J}$ such that for all i we have $\mathbf{I}_{i+1} = \mathbf{I}_i^{\mathbf{w}_i}$; thus $\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \to \cdots \to \mathbf{I}_k \xrightarrow{\mathbf{w}_k} \mathbf{J}$ is a decomposition of $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J}$ in $B^+(\mathcal{I})$ as a product of elements of $\mathcal{S}$.*

(ii) *The relations $(\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{J} \xrightarrow{\mathbf{w}_2} \mathbf{K}) = (\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{K})$ when $\mathbf{w} = \mathbf{w}_1\mathbf{w}_2 \in \mathbf{W}$ form a presentation of $B^+(\mathcal{I})$.*

We set $\alpha = \alpha_{\mathbf{S}}$; it is an $\mathcal{S}$-head funtion. The above implies:

Lemma 8.4. *For $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{I}' \in B^+(\mathcal{I})$ and $\mathbf{v} \in B^+_{\mathbf{I}}$ we have $\alpha(\mathbf{vw}) = \alpha(\mathbf{v})\alpha(\mathbf{w})$.*

Proof. We have $\alpha(\mathbf{vw}) = \alpha(\mathbf{v}\alpha(\mathbf{w})) = \alpha(\alpha(\mathbf{w})\mathbf{v}^{\alpha(\mathbf{w})}) = \alpha(\alpha(\mathbf{w})\alpha(\mathbf{v}^{\alpha(\mathbf{w})}))$, the first and last equalities from the general properties of α. Since by Lemma 8.3(i) $\mathbf{I}^{\alpha(\mathbf{w})} \subset \mathbf{S}$, we have $\alpha(\mathbf{v}^{\alpha(\mathbf{w})}) = \alpha(\mathbf{v})^{\alpha(\mathbf{w})}$, so that $\alpha(\mathbf{vw}) = \alpha(\alpha(\mathbf{w})\alpha(\mathbf{v})^{\alpha(\mathbf{w})}) = \alpha(\alpha(\mathbf{v})\alpha(\mathbf{w}))$. Since $\alpha(\mathbf{w})$ is $\mathbf{I}$-reduced we have $\alpha(\mathbf{v})\alpha(\mathbf{w}) \in \mathbf{W}$, hence $\alpha(\alpha(\mathbf{v})\alpha(\mathbf{w})) = \alpha(\mathbf{v})\alpha(\mathbf{w})$. □

We now look at the compatibility of morphisms in $B^+(\mathcal{I})$ with "parabolic" subcategories. We denote by $B^+_{\mathbf{I}}(\mathcal{J})$ the "parabolic" subcategory of $B^+(\mathcal{I})$ where $\mathbf{S}$ and $\mathcal{I}$ are replaced respectively by $\mathbf{I}$ and $\mathcal{J}$, where $\mathcal{J}$ is the set of $B_{\mathbf{I}}$-conjugates of $\mathbf{J} \subset \mathbf{I} \subset \mathbf{S}$. We have

Proposition 8.5. *If $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{I}'$ is a morphism in $B^+(\mathcal{I})$ and $\mathbf{J} \xrightarrow{\mathbf{v}} \mathbf{J}'$ is a morphism in $B^+_{\mathbf{I}}(\mathcal{J})$ and if the normal form of $\mathbf{vw}$ is $(\mathbf{u}_1, \dots, \mathbf{u}_k)$ then for each i we have $\mathbf{u}_i = \mathbf{v}_i\mathbf{w}_i$ where $(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k)$ and $(\mathbf{v}_1, {}^{\mathbf{w}_1}\mathbf{v}_2, {}^{\mathbf{w}_1\mathbf{w}_2}\mathbf{v}_3, \dots)$ are the normal forms of $\mathbf{w}$ and $\mathbf{v}$ respectively with perhaps some added 1's at the end.*

Proof. We proceed by induction on k. By Lemma 8.4, we have $\mathbf{u}_1 = \alpha(\mathbf{v})\alpha(\mathbf{w}) = \mathbf{v}_1\mathbf{w}_1$, so that $\mathbf{u}_2 \dots \mathbf{u}_k = \omega(\mathbf{v})^{\alpha(\mathbf{w})}\omega(\mathbf{w})$. Let $\mathbf{I}' = \mathbf{I}^{\alpha(\mathbf{w})}$ and $\mathcal{J}'$ be the set of conjugates of $\mathbf{J}^{\alpha(\mathbf{w})}$ in $\mathbf{I}'$; the induction hypothesis applied to $\omega(\mathbf{v})^{\alpha(\mathbf{w})} \in B^+_{\mathbf{I}'}(\mathcal{J}')$ and $\omega(\mathbf{w}) \in B^+(\mathcal{I})$ gives the result. □

The varieties $\mathcal{O}$ attached to $B^+(\mathcal{I})$. In this subsection, we shall define a representation of $B^+(\mathcal{I})$ into the bicategory $\mathbf{X}$ of varieties over $\mathcal{P}_I \times \mathcal{P}_J$, where I, J vary over $\mathcal{I}$. The bicategory $\mathbf{X}$ has 0-cells which are the elements of $\mathcal{I}$, has 1-cells with domain $\mathbf{I}$ and codomain $\mathbf{J}$ which are the $\mathcal{P}_I \times \mathcal{P}_J$-varieties and has 2-cells which are isomorphisms of $\mathcal{P}_I \times \mathcal{P}_J$-varieties. We denote by $V(\mathbf{I}, \mathbf{J})$ the category whose objects (resp. morphisms) are the 1-cells with domain $\mathbf{I}$ and codomain $\mathbf{J}$ (resp. the 2-cells between them); in other words, $V(\mathbf{I}, \mathbf{J})$ is the category of $\mathcal{P}_I \times \mathcal{P}_J$-varieties endowed with the isomorphisms of $\mathcal{P}_I \times \mathcal{P}_J$-varieties. The horizontal composition bifunctor $V(\mathbf{I}, \mathbf{J}) \times V(\mathbf{J}, \mathbf{K}) \to V(\mathbf{I}, \mathbf{K})$ is given by the fibered product over $\mathcal{P}_J$. The vertical composition is given by the composition of isomorphisms.

Here by "representation into a bicategory" we mean a morphism of bicategories between $\mathcal{C}$ viewed as a trivial bicategory into the given bicategory. This amounts to give a map T from $\mathrm{Obj}(\mathcal{C})$ to the 0-cells, and for $f \in \mathcal{C}$ of source I and target J, an element $T(f) \in V(T(I), T(J))$ together with for each composable pair (f, g) an isomorphism $T(f)T(g) \xrightarrow{\sim} T(fg)$ such that the resulting square

$$\begin{array}{ccc} T(f)T(f')T(f'') & \xrightarrow{\sim} & T(ff')T(f'') \\ \downarrow \sim & & \downarrow \sim \\ T(f)T(f'f'') & \xrightarrow{\sim} & T(ff'f'') \end{array} \tag{8.6}$$

commutes.

We first give a representation in $\mathbf{X}$ of the Garside family $\mathcal{S}$ which is the same excepted that the above square is restricted to the case where f, ff' and $ff'f''$ are in $\mathcal{S}$, (which implies $f', f'', f'f'' \in \mathcal{S}$ since in our case $\mathcal{S}$ is closed under right divisors). We will then extend our representation to the whole category $B^+(\mathcal{I})$ by a general theorem on Garside categories.

Definition 8.7. *For* $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J} \in \mathcal{S}$ *we define*

$$\mathcal{O}(\mathbf{I}, \mathbf{w}) = \{(\mathbf{P}, \mathbf{P}') \in \mathcal{P}_I \times \mathcal{P}_J \mid \mathbf{P} \xrightarrow{I,w,J} \mathbf{P}'\}$$

where I, w, J *denote the images in* W *of* $\mathbf{I}$, $\mathbf{w}$, $\mathbf{J}$ *respectively.*

For $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J} \in \mathcal{S}$ we set $T(\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J}) = \mathcal{O}(\mathbf{I}, \mathbf{w})$. The following lemma constructs the isomorphism $T(f)T(g) \xrightarrow{\sim} T(fg)$ when $f, g, fg \in \mathcal{S}$:

Lemma 8.8. *Let* $(\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \xrightarrow{\mathbf{w}_2} \mathbf{J}) = (\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J})$ *where* $\mathbf{w} = \mathbf{w}_1\mathbf{w}_2 \in \mathbf{W}$ *be a defining relation of* $B^+(\mathcal{I})$. *Then* $(p', p'') : \mathcal{O}(I, \mathbf{w}_1) \times_{\mathcal{P}_{I_2}} \mathcal{O}(I_2, \mathbf{w}_2) \xrightarrow{\sim} \mathcal{O}(I, \mathbf{w}_1\mathbf{w}_2)$ *is an isomorphism, where* p' *and* p'' *are respectively the first and last projections..*

Proof. First notice that for two parabolic subgroups $(\mathbf{P}', \mathbf{P}'') \in \mathcal{P}_I \times \mathcal{P}_J$ we have $\mathbf{P}' \xrightarrow{I,w,J} \mathbf{P}''$ if and only if the pair $(\mathbf{P}', \mathbf{P}'')$ is conjugate to a pair containing termwise the pair $(\mathbf{B}, {}^w\mathbf{B})$. This shows that if $\mathbf{P}' \xrightarrow{I,w_1,I_2} \mathbf{P}_1$ and $\mathbf{P}_1 \xrightarrow{I_2,w_2,J} \mathbf{P}''$ then $\mathbf{P}' \xrightarrow{I,w_1w_2,J} \mathbf{P}''$, so (p', p'') goes to the claimed variety.

Conversely, we have to show that given $\mathbf{P}' \xrightarrow{I,w,J} \mathbf{P}''$ there is a unique $\mathbf{P}_1$ such that $\mathbf{P}' \xrightarrow{I,w_1,I_2} \mathbf{P}_1 \xrightarrow{I_2,w_2,J} \mathbf{P}''$. The image of $(\mathbf{B}, {}^w\mathbf{B})$ by the conjugation which sends $(\mathbf{P}_I, {}^w\mathbf{P}_J)$ to $(\mathbf{P}', \mathbf{P}'')$ is a pair of Borel subgroups $(\mathbf{B}' \subset \mathbf{P}', \mathbf{B}'' \subset \mathbf{P}'')$ in position w. Since $l(w_1) + l(w_2) = l(w)$, there is a unique Borel subgroup $\mathbf{B}_1$ such that $\mathbf{B}' \xrightarrow{w_1} \mathbf{B}_1 \xrightarrow{w_2} \mathbf{B}''$. The unique parabolic subgroup of type I_2 containing $\mathbf{B}_1$ has the desired relative positions, so $\mathbf{P}_1$ exists. And any other parabolic subgroup $\mathbf{P}_1'$ which has the desired relative positions contains a Borel subgroup $\mathbf{B}_1'$ such that $\mathbf{B}' \xrightarrow{w_1} \mathbf{B}_1' \xrightarrow{w_2} \mathbf{B}''$ (take for $\mathbf{B}_1'$ the image of ${}^{w_1}\mathbf{B}$ by the conjugation which maps $(\mathbf{P}_I, {}^{w_1}\mathbf{P}_{I_2})$ to $(\mathbf{P}', \mathbf{P}_1')$), which implies that $\mathbf{B}_1' = \mathbf{B}_1$ and thus $\mathbf{P}_1' = \mathbf{P}_1$. Thus our map is bijective on points. To show it is an isomorphism, it is sufficient to check that its target is a normal variety, which is given by

Lemma 8.9. *For* $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J} \in \mathcal{S}$ *the variety* $\mathcal{O}(\mathbf{I}, \mathbf{w})$ *is smooth.*

Proof. Consider the locally trivial fibrations with smooth fibers given by $\mathbf{G} \times \mathbf{G} \xrightarrow{p} \mathcal{P}_I \times \mathcal{P}_J : (g_1, g_2) \mapsto ({}^{g_1}\mathbf{P}_I, {}^{g_2w}\mathbf{P}_J)$ and $\mathbf{G} \times \mathbf{G} \xrightarrow{q} \mathbf{G} : (g_1, g_2) \mapsto g_1^{-1}g_2$. It is easy to check that $\mathcal{O}(\mathbf{I}, \mathbf{w}) = p(q^{-1}({}^w\mathbf{P}_J))$ thus by for example [DMR, 2.2.3] it is smooth. □

□

From the above lemma we see also that the square 8.6 commutes for elements of $\mathcal{S}$, since the isomorphism "forgetting the middle parabolic" has clearly the corresponding property. We have thus defined a representation T of $\mathcal{S}$ in $\mathbf{X}$.

We will extend T to the whole of $B^+(\mathcal{I})$ by associating to a composition $\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \to \cdots \to \mathbf{I}_k \xrightarrow{\mathbf{w}_k} \mathbf{J}$ where $\mathbf{w}_i \in \mathbf{W}$ the variety

$$\mathcal{O}(\mathbf{I}, \mathbf{w}_1) \times_{\mathcal{P}_{I_2}} \ldots \times_{\mathcal{P}_{I_k}} \mathcal{O}(\mathbf{I}_k, \mathbf{w}_k) = \{(\mathbf{P}_1, \ldots, \mathbf{P}_{k+1}) \mid \mathbf{P}_i \xrightarrow{I_i,w_i,I_{i+1}} \mathbf{P}_{i+1}\},$$

where $I_1 = I$ and $I_{k+1} = J$. It is a $\mathcal{P}_I \times \mathcal{P}_J$-variety via the first and last projections mapping respectively $(\mathbf{P}_1, \ldots, \mathbf{P}_{k+1})$ to $\mathbf{P}_1$ and $\mathbf{P}_{k+1}$, and Lemma 8.8 shows that up to isomorphism it does not depend on the chosen decomposition of $\mathbf{I} \xrightarrow{\mathbf{w}_1 \ldots \mathbf{w}_k} \mathbf{J}$. We have to show that actually there is a unique isomorphism between the various models attached to different decompositions.

This will result from a general theorem on Garside categories which generalizes a result of Deligne [D, 1.11].

Theorem 8.10. *Let $\mathcal{C}$ be a right Noetherian Garside category with Garside family $\mathcal{S}$ and which admits local right lcm's. Then any representation of $\mathcal{S}$ into a bicategory extends uniquely to a representation of $\mathcal{C}$ into the same bicategory.*

Proof. The proof goes exactly as in [D], in that what must been proven is a simple connectedness property for the set of decompositions as a product of elements of $\mathcal{S}$ of an arbitrary morphism in $\mathcal{C}$— this generalizes [D, 1.7] and is used in the same way. In his context, Deligne shows more, the contractibility of the set of decompositions; on the other hand our proof, which follows a suggestion by Serge Bouc to use a version of [Bouc, lemma 6], is simpler and holds in our more general context.

Fix $g \in \mathcal{C}$ with $g \notin \mathcal{C}^\times$. We denote by $E(g)$ the set of decompositions of g into a product of elements of $\mathcal{S} - \mathcal{C}^\times$.

Then $E(g)$ is a poset, the order being defined by

$$(g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_n) > (g_1, \dots, g_{i-1}, a, b, g_{i+1}, \dots, g_n)$$

if $ab = g_i \in \mathcal{S}$.

We recall the definition of homotopy in a poset E (a translation of the corresponding notion in a simplicial complex isomorphic as a poset to E). A path from x_1 to x_k in E is a sequence $x_1 \dots x_k$ where each x_i is comparable to x_{i+1}. The composition of paths is defined by concatenation. Homotopy, denoted by $\sim$, is the finest equivalence relation on paths compatible with concatenation and generated by the two following elementary relations: $xyz \sim xz$ if $x \leq y \leq z$ and both $xyx \sim x$ and $yxy \sim y$ when $x \leq y$. Homotopy classes form a groupoid, as the composition of a path with source x and of the inverse path is homotopic to the constant path at x. For $x \in E$ we denote by $\Pi_1(E, x)$ the fundamental group of E with base point x, which is the group of homotopy classes of loops starting from x.

A poset E is said to be *simply connected* if it is connected (there is a path linking any two elements of E) and if the fundamental group with some (or any) base point is trivial.

Note that a poset with a smallest or largest element x is simply connected since any path $(x, y, z, t, \dots, x)$ is homotopic to $(x, y, x, z, x, t, x, \dots, x)$ which is homotopic to the trivial loop.

Proposition 8.11. *The set $E(g)$ is simply connected.*

Proof. First we prove a version of a lemma from [Bouc] on order preserving maps between posets. For a poset E we put $E_{\geq x} = \{x' \in E \mid x' \geq x\}$, which is a simply connected subposet of E since it has a smallest element. If $f : X \to Y$ is an order preserving map it is compatible with homotopy (it corresponds to a continuous map between simplicial complexes), so it induces a homomorphism $f^* : \Pi_1(X, x) \to \Pi_1(Y, f(x))$.

Lemma 8.12. *(Bouc) Let $f : X \to Y$ an order preserving map between two posets. We assume that Y is connected and that for any $y \in Y$ the poset $f^{-1}(Y_{\geq y})$ is connected and non empty. Then f^* is surjective. If moreover $f^{-1}(Y_{\geq y})$ is simply connected for all y then f^* is an isomorphism.*

Proof. Let us first show that X is connected. Let $x, x' \in X$; we choose a path $y_0 \dots y_n$ in Y from $y_0 = f(x)$ to $y_n = f(x')$. For $i = 0, \dots, n$, we choose $x_i \in$

$f^{-1}(Y_{\geq y_i})$ with $x_0 = x$ and $x_n = x'$. Then if $y_i \geq y_{i+1}$ we have $f^{-1}(Y_{\geq y_i}) \subset f^{-1}(Y_{\geq y_{i+1}})$ so that there exists a path in $f^{-1}(Y_{\geq y_{i+1}})$ from x_i to x_{i+1}; otherwise $y_i < y_{i+1}$, which implies $f^{-1}(Y_{\geq y_i}) \supset f^{-1}(Y_{\geq y_{i+1}})$ and there exists a path in $f^{-1}(Y_{\geq y_i})$ from x_i to x_{i+1}. Concatenating these paths gives a path connecting x and x'.

We fix now $x_0 \in X$. Let $y_0 = f(x_0)$. We prove that $f^* : \Pi_1(X, x_0) \to \Pi_1(Y, y_0)$ is surjective. Let $y_0 y_1 \dots y_n$ with $y_n = y_0$ be a loop in Y. We lift arbitrarily this loop into a loop $x_0 — \cdots — x_n$ in X as above, (where $x_i — x_{i+1}$ stands for a path from x_i to x_{i+1} which is either in $f^{-1}(Y_{\geq y_i})$ or in $f^{-1}(Y_{\geq y_{i+1}})$. Then the path $f(x_0 — x_1 — \cdots — x_n)$ is homotopic to $y_0 \dots y_n$; this can be seen by induction: let us assume that $f(x_0 — x_1 \cdots — x_i)$ is homotopic to $y_0 \dots y_i f(x_i)$; then the same property holds for $i+1$: indeed $y_i y_{i+1} \sim y_i f(x_i) y_{i+1}$ as they are two paths in a simply connected set which is either $Y_{\geq y_i}$ or $Y_{\geq y_{i+1}}$; similarly we have $f(x_i) y_{i+1} f(x_{i+1}) \sim f(x_i — x_{i+1})$. Putting things together gives

$$\begin{aligned} y_0 \dots y_i y_{i+1} f(x_{i+1}) &\sim y_0 y_1 \dots y_i f(x_i) y_{i+1} f(x_{i+1}) \\ &\sim f(x_0 — \cdots — x_i) y_{i+1} f(x_{i+1}) \\ &\sim f(x_0 — \cdots — x_i — x_{i+1}). \end{aligned}$$

We now prove injectivity of f^* when all $f^{-1}(Y_{\geq y})$ are simply connected.

We first prove that if $x_0 — \cdots — x_n$ and $x'_0 — \cdots — x'_n$ are two loops lifting the same loop $y_0 \dots y_n$, then they are homotopic. Indeed, we get by induction on i that $x_0 — \cdots — x_i — x'_i$ and $x'_0 — \cdots — x'_i$ are homotopic paths, using the fact that x_{i-1}, x_i, x'_{i-1} and x'_i are all in the same simply connected sub-poset, namely either $f^{-1}(Y_{\geq y_{i-1}})$ or $f^{-1}(Y_{\geq y_i})$.

It remains to prove that we can lift homotopies, which amounts to show that if if we lift as above two loops which differ by an elementary homotopy, the liftings are homotopic. If $yy'y \sim y$ is an elementary homotopy with $y < y'$ (resp. $y > y'$), then $f^{-1}(Y_{\geq y'}) \subset f^{-1}(Y_{\geq y})$ (resp. $f^{-1}(Y_{\geq y}) \subset f^{-1}(Y_{\geq y'})$) and the lifting of $yy'y$ constructed as above is in $f^{-1}(Y_{\geq y})$ (resp. $f^{-1}(Y_{\geq y'})$) so is homotopic to the trivial path. If $y < y' < y''$, a lifting of $yy'y''$ constructed as above is in $f^{-1}(Y_{\geq y})$ so is homotopic to any path in $f^{-1}(Y_{\geq y})$ with the same endpoints. □

We now prove Proposition 8.11 by contradiction. If it fails we choose $g \in \mathcal{C}$ minimal for proper right divisibility such that $E(g)$ is not simply connected.

Let L be the set of elements of $\mathcal{S} - \mathcal{C}^\times$ which are left divisors of g. For any $I \subset L$, since the category admits local right lcm and is right Noetherian, the elements of I have an lcm Δ_I. Let $E_I(g) = \{(g_1, \dots, g_n) \in E(g) \mid \Delta_I \preccurlyeq g_1\}$. We claim that $E_I(g)$ is simply connected for $I \neq \emptyset$. In the following, if $\Delta_I \preccurlyeq a$, we denote by a^I the element such that $a = \Delta_I a^I$. We apply Lemma 8.12 to the map $f : E_I(g) \to E(g^I)$ defined by

$$(g_1, \dots, g_n) \mapsto \begin{cases} (g_2, \dots, g_n) & \text{if } g_1 = \Delta_I \\ (g_1^I, g_2, \dots g_n) & \text{otherwise} \end{cases}.$$

This map preserves the order and any set $f^{-1}(Y_{\geq (g_1, \dots, g_n)})$ has a least element, namely $(\Delta_I, g_1, \dots, g_n)$, so is simply connected. As by minimality of g the set $E(g^I)$ is simply connected Lemma 8.12 implies that $E_I(g)$ is simply connected.

Let Y be the set of non-empty subsets of L. We now apply Lemma 8.12 to the map $f : E(g) \to Y$ defined by $(g_1, \dots, g_n) \mapsto \{s \in L \mid s \preccurlyeq g_1\}$, where Y is ordered by inclusion. This map is order preserving since $(g_1, \dots, g_n) < (g'_1, \dots, g'_n)$

implies $g_1 \preccurlyeq g_1'$. We have $f^{-1}(Y_{\geq I}) = E_I(g)$, so this set is simply connected. Since Y, having a greatest element, is simply connected, 8.12 gives that $E(g)$ is simply connected, whence the proposition. □

□

Definition 8.13. *Since $B^+(\mathcal{I})$ satisfies the assumptions of Theorem 8.10, this theorem enables us to attach to any $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \in B^+(\mathcal{I})$ a (well-defined) variety $\mathcal{O}(\mathbf{I}, \mathbf{b})$ which for any decomposition $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} = \mathcal{O}(\mathbf{I}, \mathbf{w}_1) \times_{\mathcal{P}_{I_2}} \ldots \times_{\mathcal{P}_{I_k}} \mathcal{O}(\mathbf{I}_k, \mathbf{w}_k)$ in elements of $\mathcal{S}$ is isomorphic to*

$$\{(\mathbf{P}_1, \ldots, \mathbf{P}_{k+1}) \in \mathcal{P}_{I_1} \times \ldots \times \mathcal{P}_{I_{k+1}} \mid \mathbf{P}_i \xrightarrow{I_i, w_i, I_{i+1}} \mathbf{P}_{i+1}\},$$

where we have set $I_1 = I$ and $I_{k+1} = J$.

The Deligne-Lusztig varieties attached to $B^+(\mathcal{I})$. The automorphism ϕ lifts naturally to an automorphism of B^+ which stabilizes $\mathbf{S}$, which we will still denote by ϕ, by abuse of notation. If $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$ is a simple morphism, then $\mathbf{X}(I, w\phi)$ is the intersection of $\mathcal{O}(\mathbf{I}, \mathbf{w})$ with the graph of F, (that is, points whose image under (p', p'') has the form $(\mathbf{P}, {}^F\mathbf{P})$). More generally,

Definition 8.14. *Let $\mathbf{I} \xrightarrow{\mathbf{b}} {}^{\phi}\mathbf{I}$ be any morphism of $B^+(\mathcal{I})$; we define the variety $\mathbf{X}(\mathbf{I}, \mathbf{b}\phi)$ as the intersection of $\mathcal{O}(\mathbf{I}, \mathbf{b})$ with the graph of F.*

To any decomposition $\mathbf{I}_1 \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \to \cdots \xrightarrow{\mathbf{w}_k} {}^{\phi}\mathbf{I}$ of $\mathbf{I} \xrightarrow{\mathbf{b}} {}^{\phi}\mathbf{I}$ in elements of $\mathcal{S}$ corresponds a model of $\mathbf{X}(\mathbf{I}, \mathbf{b}\phi)$ as sequences of parabolics $(\mathbf{P}_1, \ldots, \mathbf{P}_{k+1})$ such that $\mathbf{P}_{k+1} = F(\mathbf{P}_1)$. Such a model can be interpreted as an "ordinary" parabolic Deligne-Lusztig variety in a group which is a descent of scalars:

Proposition 8.15. *Let $\mathbf{I} = \mathbf{I}_1 \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \to \cdots \to \mathbf{I}_k \xrightarrow{\mathbf{w}_k} {}^{\phi}\mathbf{I}$ be a decomposition into elements of $\mathcal{S}$ of $\mathbf{I} \xrightarrow{\mathbf{b}} {}^{\phi}\mathbf{I} \in B^+(\mathcal{I})$, let F_1 be the isogeny of $\mathbf{G}^k$ defined by $F_1(g_1, \ldots, g_k) = (g_2, \ldots, g_k, F(g_1))$ and let ϕ_1 be the corresponding automorphism of W^k.*

Then $\mathbf{X}_{\mathbf{G}}(\mathbf{I}, \mathbf{b}\phi) \simeq \mathbf{X}_{\mathbf{G}^k}(I_1 \times \ldots \times I_k, (w_1, \ldots, w_k)\phi_1)$. By this isomorphism the action of F^δ corresponds to that of $F_1^{k\delta}$ and the action of $\mathbf{G}^F$ corresponds to that of $(\mathbf{G}^k)^{F_1}$.

Proof. An element $\mathbf{P}_1 \times \ldots \times \mathbf{P}_k \in \mathbf{X}_{\mathbf{G}^k}(I_1 \times \ldots \times I_k, (w_1, \ldots, w_k)\phi_1)$ by definition satisfies

$$\mathbf{P}_1 \times \ldots \times \mathbf{P}_k \xrightarrow{I_1 \times \ldots I_k, (w_1, \ldots, w_k), I_2 \times \ldots I_k \times {}^{\phi}I_1} \mathbf{P}_2 \times \ldots \times \mathbf{P}_k \times {}^F\mathbf{P}_1$$

thus is equivalently given by a sequence $(\mathbf{P}_1, \ldots, \mathbf{P}_{k+1})$ such that $\mathbf{P}_i \xrightarrow{I_i, w_i, I_{i+1}} \mathbf{P}_{i+1}$ with $\mathbf{P}_{k+1} = {}^F\mathbf{P}_1$ and $I_{k+1} = {}^{\phi}I_1$, which is the same as an element

$$(\mathbf{P}_1, \ldots, \mathbf{P}_{k+1}) \in \mathcal{O}(\mathbf{I}_1, \mathbf{w}_1) \times_{\mathcal{P}_{I_2}} \mathcal{O}(\mathbf{I}_2, \mathbf{w}_2) \ldots \times_{\mathcal{P}_{I_{k-1}}} \mathcal{O}(\mathbf{I}_k, \mathbf{w}_k)$$

such that $\mathbf{P}_{k+1} = {}^F\mathbf{P}_1$. But, by construction, we have

$$\mathcal{O}(\mathbf{I}, \mathbf{b}) \simeq \mathcal{O}(\mathbf{I}_1, \mathbf{w}_1) \times_{\mathcal{P}_{I_2}} \mathcal{O}(\mathbf{I}_2, \mathbf{w}_2) \ldots \times_{\mathcal{P}_{I_{k-1}}} \mathcal{O}(\mathbf{I}_k, \mathbf{w}_k),$$

and thus via this isomorphism we get an element of $\mathcal{O}(\mathbf{I}, \mathbf{b})$ which is in $\mathbf{X}_{\mathbf{G}}(\mathbf{I}, \mathbf{b}\phi)$.

One checks easily that this sequence of identifications is compatible with the actions of F^δ and $\mathbf{G}^F$ as described by the proposition. □

Proposition 8.16. *The variety* $\mathbf{X}(\mathbf{I}, \mathbf{b}\phi)$ *is irreducible if and only if* $\mathbf{I} \cup c(\mathbf{b})$ *meets all the orbits of* ϕ *on* $\mathbf{S}$*, where* $c(\mathbf{b})$ *is the set of elements of* $\mathbf{S}$ *which appear in a decomposition of* $\mathbf{b}$*.*

Proof. This is, using Proposition 8.15, an immediate translation in our setting of the result [BR, Theorem 2] of Bonnafé-Rouquier. □

The varieties $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$. The conjugation which transforms $\mathbf{X}_{\mathbf{P}}$ into $\mathbf{X}(I, w\phi)$ maps $\mathbf{X}_{\mathbf{V}}$ to the $\mathbf{G}^F$-variety-$\mathbf{L}_I^{\dot{w}F}$ given by

$$\tilde{\mathbf{X}}(I, \dot{w}F) = \{g\mathbf{U}_I \in \mathbf{G}/\mathbf{U}_I \mid g^{-1\,F} g \in \mathbf{U}_I \dot{w}^F \mathbf{U}_I\},$$

where $\dot{w}$ is a representative of w (any representative can be obtained by choosing an appropriate conjugation). The map $g\mathbf{U}_I \mapsto g\mathbf{P}_I$ makes $\tilde{\mathbf{X}}(I, \dot{w}F)$ a $\mathbf{L}_I^{\dot{w}F}$-torsor over $\mathbf{X}(I, w\phi)$. We will sometimes write $\tilde{\mathbf{X}}(I, \dot{w}.F)$ to separate the Frobenius endomorphism from the representative of the Weyl group element. This will be especially useful when the ambient group is a Levi subgroup with Frobenius endomorphism of the form $\dot{x}F$.

In this section, we define a variety $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$ which generalizes $\tilde{\mathbf{X}}(I, \dot{w}F)$ by replacing $\dot{w}$ by elements of the braid group. Since $\dot{w}$ represents a choice of a lift of w to $N_{\mathbf{G}}(\mathbf{T})$, we have to make uniformly such choices for all elements of the braid group, which we do by using a "Tits homomorphism".

First, we need, when $\mathbf{w} \in \mathbf{W}$, to define a variety $\tilde{\mathcal{O}}(I, \dot{w})$ "above" $\mathcal{O}(\mathbf{I}, \mathbf{w})$ such that $\tilde{\mathbf{X}}(I, \dot{w}F)$ is the intersection of $\tilde{\mathcal{O}}(I, w)$ with the graph of F, and then we extend this construction to $\mathcal{B}^+(\mathcal{I})$.

Definition 8.17. *Let* $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J} \in \mathcal{S}$*, and let* $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$ *be a representative of* w*. We define* $\tilde{\mathcal{O}}(I, \dot{w}) = \{(g\mathbf{U}_I, g'\mathbf{U}_J) \in \mathbf{G}/\mathbf{U}_I \times \mathbf{G}/\mathbf{U}_J \mid g^{-1}g' \in \mathbf{U}_I \dot{w} \mathbf{U}_J\}$*.*

We can prove an analogue of Lemma 8.8.

Lemma 8.18. *Let* $(\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \xrightarrow{\mathbf{w}_2} \mathbf{J}) = (\mathbf{I} \xrightarrow{\mathbf{w}_1\mathbf{w}_2} \mathbf{J})$ *where* $\mathbf{w}_1\mathbf{w}_2 \in \mathbf{W}$ *be a defining relation of* $B^+(\mathcal{I})$*, and let* $\dot{w}_1, \dot{w}_2$ *be representatives of the images of* $\mathbf{w}_1$ *and* $\mathbf{w}_2$ *in* W*. Then* $(p', p'') : \tilde{\mathcal{O}}(I, \dot{w}_1) \times_{\mathbf{G}/\mathbf{U}_{I_2}} \tilde{\mathcal{O}}(I_2, \dot{w}_2) \xrightarrow{\sim} \tilde{\mathcal{O}}(I, \dot{w}_1\dot{w}_2)$ *is an isomorphism where* p' *and* p'' *are the first and last projections.*

Proof. We first note that if $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J} \in B^+(\mathcal{I})$ and $\dot{w}$ is a representative in $N_{\mathbf{G}}(\mathbf{T})$ of the image of $\mathbf{w}$ in W, then $\mathbf{U}_I \dot{w} \mathbf{U}_J$ is isomorphic by the product morphism to the direct product of varieties $(\mathbf{U}_I \cap {}^w\mathbf{U}_J^-)\dot{w} \times \mathbf{U}_J$, where $\mathbf{U}_J^-$ is the unipotent radical of the parabolic subgroup opposed to $\mathbf{P}_J$ containing $\mathbf{T}$. We now use the lemma:

Lemma 8.19. *Under the assumptions of Lemma 8.18, the product gives an isomorphism* $(\mathbf{U}_I \cap {}^{\dot{w}_1}\mathbf{U}_{I_2}^-)\dot{w}_1 \times (\mathbf{U}_{I_2} \cap {}^{\dot{w}_2}\mathbf{U}_J^-)\dot{w}_2 \xrightarrow{\sim} (\mathbf{U}_I \cap {}^{\dot{w}_1\dot{w}_2}\mathbf{U}_J^-)\dot{w}_1\dot{w}_2$*.*

Proof. As a product of root subgroups, we have $\mathbf{U}_I \cap {}^w\mathbf{U}_J^- = \prod_{-\alpha \in {}^wN(w)} \mathbf{U}_\alpha$, where $N(w) = \{\alpha \in \Phi^+ \mid {}^w\alpha \in \Phi^-\}$. The lemma is then a consequence of the equality $N(w_1)^{w_2} \coprod N(w_2) = N(w_1w_2)$ when $l(w_1) + l(w_2) = l(w_1w_2)$. □

The lemma proves in particular that if $g_1^{-1}g_2 \in \mathbf{U}_I \dot{w}_1 \mathbf{U}_{I_2}$ and $g_2^{-1}g_3 \in \mathbf{U}_{I_2} \dot{w}_2 \mathbf{U}_J$ then $g_1^{-1}g_3 \in \mathbf{U}_I \dot{w}_1 \mathbf{U}_{I_2} \dot{w}_2 \mathbf{U}_J = (\mathbf{U}_I \cap {}^{\dot{w}_1}\mathbf{U}_{I_2}^-)\dot{w}_1(\mathbf{U}_{I_2} \cap {}^{\dot{w}_2}\mathbf{U}_J^-)\dot{w}_2\mathbf{U}_J = (\mathbf{U}_I \cap {}^{\dot{w}_1\dot{w}_2}\mathbf{U}_J^-)\dot{w}_1\dot{w}_2\mathbf{U}_J = \mathbf{U}_I \dot{w}_1\dot{w}_2\mathbf{U}_J$, so the image of the morphism (p', p'') in Lemma 8.18 is indeed in the variety $\tilde{\mathcal{O}}(I, \dot{w}_1\dot{w}_2)$.

Conversely, we have to show that given $(g_1\mathbf{U}_I, g_3\mathbf{U}_j) \in \tilde{\mathcal{O}}(I, \dot{w}_1\dot{w}_2)$, there exists a unique $g_2U_{I_2}$ such that $(g_1\mathbf{U}_I, g_2\mathbf{U}_{I_2}) \in \tilde{\mathcal{O}}(I, \dot{w}_1)$ and $(g_2\mathbf{U}_{I_2}, g_3\mathbf{U}_{I_3}) \in \tilde{\mathcal{O}}(I_2, \dot{w}_2)$. The varieties involved being invariant by left translation by $\mathbf{G}$, it is enough to solve the problem when $g_1 = 1$. Then we have $g_3 \in \mathbf{U}_I\dot{w}_1\dot{w}_2\mathbf{U}_J$, and the conditions for $g_2\mathbf{U}_{I_2}$ is that $g_2\mathbf{U}_{I_2} \subset \mathbf{U}_I\dot{w}_1\mathbf{U}_{I_2}$. Any such coset has then a unique representative in $(\mathbf{U}_I \cap {}^{\dot{w}_1}\mathbf{U}^-_{I_2})\dot{w}_1$ and we will look for such a representative g_2. But we must have $g_2^{-1}g_3 \in \mathbf{U}_{I_2}\dot{w}_2\mathbf{U}_J = (\mathbf{U}_{I_2} \cap {}^{\dot{w}_2}\mathbf{U}^-_J)\dot{w}_2\mathbf{U}_J$ and since by the lemma the product gives an isomorphism between $(\mathbf{U}_I \cap {}^{\dot{w}_1}\mathbf{U}^-_{I_2})\dot{w}_1 \times (\mathbf{U}_{I_2} \cap {}^{\dot{w}_2}\mathbf{U}^-_J)\dot{w}_2\mathbf{U}_J$ and $\mathbf{U}_I\dot{w}_1\dot{w}_2\mathbf{U}_J$, the element g_3 can be decomposed in one and only one way in a product $g_2(g_2^{-1}g_3)$ satisfying the conditions. □

We will now use a Tits homomorphism, which is a homomorphism $B \xrightarrow{t} N_{\mathbf{G}}(\mathbf{T})$ which factors the projection $B \to W$ (their existence is proved in [T]). Theorem 8.10 implies that, setting $T(\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J}) = \tilde{\mathcal{O}}(I, t(\mathbf{w}))$ for $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{J} \in \mathcal{S}$ and replacing Lemma 8.8 by Lemma 8.18, we can define a representation of $B^+(\mathcal{I})$ in the bicategory $\tilde{\mathbf{X}}$ of varieties above $\mathbf{G}/\mathbf{U}_I \times \mathbf{G}/\mathbf{U}_J$ for $I, J \in \mathcal{I}$.

Definition 8.20. *The above representation defines, for any $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \in B^+(\mathcal{I})$, a variety $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$ which, for any decomposition into elements of $\mathcal{S}$ given by $(\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}) = (\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \to \ldots \to \mathbf{I}_k \xrightarrow{\mathbf{w}_k} \mathbf{J})$ is isomorphic to $\tilde{\mathcal{O}}(I, t(\mathbf{w}_1)) \times_{\mathbf{G}/\mathbf{U}_{I_2}} \ldots \times_{\mathbf{G}/\mathbf{U}_{I_k}} \tilde{\mathcal{O}}(I_k, t(\mathbf{w}_k))$.*

Proposition 8.21. *There exists a Tits homomorphism t which is F-equivariant, that is such that $t(\phi(\mathbf{b})) = F(t(\mathbf{b}))$.*

Proof. To any simple reflection $s \in S$ is associated a quasi-simple subgroup $\mathbf{G}_s$ of rank 1 of $\mathbf{G}$, generated by the root subgroups $\mathbf{U}_{\alpha_s}$ and $\mathbf{U}_{-\alpha_s}$; the 1-parameter subgroup of $\mathbf{T}$ given by $\mathbf{T} \cap \mathbf{G}_s$ is a maximal torus of $\mathbf{G}_s$. By [T, Theorem 4.4] if for any $s \in S$ we choose a representative $\dot{s}$ of s in $\mathbf{G}_s$, then these representatives satisfy the braid relations, which implies that $\mathbf{s} \mapsto \dot{s}$ induces a well defined Tits homomorphism. We claim that if s is fixed by some power ϕ^d of ϕ then there exists $\dot{s} \in \mathbf{G}_s$ fixed by F^d; we then get an F-equivariant Tits homomorphism by choosing arbitrarily $\dot{s}$ for one s in each orbit of ϕ. If s is fixed by ϕ^d then $\mathbf{G}_s$ is stable by F^d; the group $\mathbf{G}_s$ is isomorphic to either SL_2 or PSL_2 and F^d is a Frobenius endomorphism of this group. In either case the simple reflection s of $\mathbf{G}_s$ has an F^d-stable representative in $N_{\mathbf{G}_s}(\mathbf{T} \cap \mathbf{G}_s)$. □

Notation 8.22. *We assume now that we have chosen, once and for all, an F-equivariant Tits homomorphism t which is used to define the varieties $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$. For $w \in W$ we will write $\dot{w}$ for $t(\mathbf{w})$ where $\mathbf{w} \in \mathbf{W}$ is the canonical lift of w.*

Definition 8.23. *For any morphism $(\mathbf{I} \xrightarrow{\mathbf{b}} {}^{\phi}\mathbf{I}) \in B^+(\mathcal{I})$ we define $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{b}\phi) = \{x \in \tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b}) \mid p''(x) = F(p'(x))\}$.*

When $\mathbf{w} \in \mathbf{W}$ we have $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) = \tilde{\mathbf{X}}(I, \dot{w}F)$ (the variety defined at the beginning of this section).

Lemma 8.24. *For any $(\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}) \in B^+(\mathcal{I})$, there is a natural projection $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \xrightarrow{\pi} \mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$ which makes $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$ a $\mathbf{L}_I^{t(\mathbf{w})F}$-torsor over $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$.*

Proof. Let $\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \to \cdots \to \mathbf{I}_r \xrightarrow{\mathbf{w}_r} {}^{\phi}\mathbf{I}$ be a decomposition into elements of $\mathcal{S}$ of $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$, so that $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$ identifies to the set of sequences $(g_1\mathbf{U}_I, g_2\mathbf{U}_{I_2}, \dots, g_r\mathbf{U}_{I_r})$ such that $g_j^{-1}g_{j+1} \in \mathbf{U}_{I_j}t(\mathbf{w}_j)\mathbf{U}_{I_{j+1}}$ for $j < r$ and $g_r^{-1\,F}g_1 \in \mathbf{U}_{I_r}t(\mathbf{w}_r)\mathbf{U}_{\phi I}$. We define π by $g_j\mathbf{U}_{I_j} \mapsto {}^{g_j}\mathbf{P}_{I_j}$. It is easy to check that the morphism π thus defined commutes with an "elementary morphism" in the bicategories of varieties $\tilde{\mathbf{X}}$ or $\mathbf{X}$ consisting of passing from the decomposition $(\mathbf{w}_1, \dots, \mathbf{w}_i, \mathbf{w}_{i+1}, \dots, \mathbf{w}_r)$ to $(\mathbf{w}_1, \dots, \mathbf{w}_i\mathbf{w}_{i+1}, \dots, \mathbf{w}_r)$ when $\mathbf{I}_i \xrightarrow{\mathbf{w}_i\mathbf{w}_{i+1}} \mathbf{I}_{i+2} \in \mathcal{S}$. Thus by 8.6 the morphism π is well-defined independently of the decomposition chosen of $\mathbf{w}$. We claim that π makes $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$ a $\mathbf{L}^{t(\mathbf{w})F}$-torsor over $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$. Indeed, the fiber $\pi^{-1}(({}^{g_1}\mathbf{P}_I, {}^{g_2}\mathbf{P}_{I_2}, \dots, {}^{g_r}\mathbf{P}_{I_r}))$ consists of the $(g_1l_1\mathbf{U}_I, \dots, g_rl_r\mathbf{U}_{I_r}) \in \tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$ with $l_i \in \mathbf{L}_{I_i}$, that is such that

$$\begin{aligned}\text{for } j < r \text{ we have } g_j^{-1}g_{j+1} &\in (\mathbf{U}_{I_j}t(\mathbf{w}_j)\mathbf{U}_{I_{j+1}}) \cap l_i(\mathbf{U}_{I_j}t(\mathbf{w}_j)\mathbf{U}_{I_{j+1}})l_{i+1}^{-1}\\ \text{and } g_r^{-1\,F}g_1 &\in (\mathbf{U}_{I_r}t(\mathbf{w}_r)\mathbf{U}_{\phi I}) \cap l_r(\mathbf{U}_{I_r}t(\mathbf{w}_r)\mathbf{U}_{\phi I})^F l_1^{-1}.\end{aligned}$$

Now

$$(\mathbf{U}_{I_j}t(\mathbf{w}_j)\mathbf{U}_{I_{j+1}}) \cap l_i(\mathbf{U}_{I_j}t(\mathbf{w}_j)\mathbf{U}_{I_{j+1}})l_{i+1}^{-1} = (\mathbf{U}_{I_j}t(\mathbf{w}_j)\mathbf{U}_{I_{j+1}}) \cap \mathbf{U}_{I_j}t(\mathbf{w}_j)\mathbf{U}_{I_{j+1}}l_i^{t(\mathbf{w}_j)}l_{i+1}^{-1}$$

and the intersection is non-empty if and only if $\mathbf{U}_{I_j}^{t(\mathbf{w}_j)} \cap \mathbf{U}_{I_{j+1}}l_i^{t(\mathbf{w}_j)}l_{i+1}^{-1} \neq \emptyset$, which, since $\mathbf{P}_{I_i}^{t(\mathbf{w}_j)}$ and $\mathbf{P}_{I_{i+1}}$ are two parabolic subgroups with the same Levi subgroup, occurs only if $l_i^{t(\mathbf{w}_i)} = l_{i+1}$. Similarly we get $l_r^{t(\mathbf{w}_r)} = {}^F l_1$, so in the end the fiber is given by the l_1 such that $l_1 = {}^{t(\mathbf{w})F}l_1$. □

We end this subsection with the analogue of Proposition 8.15 for $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{b}\phi)$.

Proposition 8.25. *Let $\mathbf{I} = \mathbf{I}_1 \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \to \cdots \to \mathbf{I}_k \xrightarrow{\mathbf{w}_k} {}^{\phi}\mathbf{I}$ be a decomposition into elements of $\mathcal{S}$ of $\mathbf{I} \xrightarrow{\mathbf{b}} {}^{\phi}\mathbf{I} \in B^+(\mathcal{I})$, let F_1 be the isogeny of $\mathbf{G}^k$ as in Proposition 8.15.*

Then $\tilde{\mathbf{X}}_{\mathbf{G}}(\mathbf{I}, \mathbf{b}\phi) \simeq \tilde{\mathbf{X}}_{\mathbf{G}^k}(I_1 \times \ldots \times I_k, (\dot{w}_1, \dots, \dot{w}_k)F_1)$. By this isomorphism the action of F^δ corresponds to that of $F_1^{k\delta}$, the action of $\mathbf{G}^F$ corresponds to that of $(\mathbf{G}^k)^{F_1}$, and the action of $\mathbf{L}_I^{t(\mathbf{b})F}$ corresponds to that of $(\mathbf{L}_{I_1} \times \cdots \times \mathbf{L}_{I_k})^{(\dot{w}_1, \dots, \dot{w}_k)F_1}$.

Proof. An element $x_1\mathbf{U}_{I_1} \times \ldots \times x_k\mathbf{U}_{I_k} \in \tilde{\mathbf{X}}_{\mathbf{G}^k}(I_1 \times \ldots \times I_k, (\dot{w}_1, \dots, \dot{w}_k)F_1)$ by definition satisfies $(x_i\mathbf{U}_{I_i}, x_{i+1}\mathbf{U}_{I_{i+1}}) \in \tilde{\mathcal{O}}(I_i, \dot{w}_i)$ for $i = 1, \dots, k$, where we have put $I_{k+1} = {}^F I_1$ and $x_{k+1}\mathbf{U}_{I_k+1} = {}^F(x_1\mathbf{U}_{I_1})$. This is the same as an element in the intersection of $\tilde{\mathcal{O}}(\mathbf{I}_1, \mathbf{w}_1) \times_{\mathbf{G}/\mathbf{U}_{I_2}} \tilde{\mathcal{O}}(\mathbf{I}_2, \mathbf{w}_2) \dots \times_{\mathbf{G}/\mathbf{U}_{I_{k-1}}} \tilde{\mathcal{O}}(\mathbf{I}_k, \mathbf{w}_k)$ with the graph of F. Since, by definition, we have

$$\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b}) \simeq \tilde{\mathcal{O}}(\mathbf{I}_1, \mathbf{w}_1) \times_{\mathbf{G}/\mathbf{U}_{I_2}} \tilde{\mathcal{O}}(\mathbf{I}_2, \mathbf{w}_2) \dots \times_{\mathbf{G}/\mathbf{U}_{I_{k-1}}} \tilde{\mathcal{O}}(\mathbf{I}_k, \mathbf{w}_k),$$

via this last isomorphism we get an element of $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$ which is in $\tilde{\mathbf{X}}_{\mathbf{G}}(\mathbf{I}, \mathbf{b}\phi)$.

One checks easily that this sequence of identifications is compatible with the actions of F^δ, of $\mathbf{G}^F$ and of $\mathbf{L}_I^{t(\mathbf{b})F}$ as described by the proposition. □

Cohomology. We start with an isomorphism which reflects the transitivity of Lusztig's induction.

Proposition 8.26. *Let $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I} \in B^+(\mathcal{I})$. Then if w is the image of $\mathbf{w}$ in W, the automorphism $w\phi$ lifts to an automorphism that we will still denote $w\phi$ of $B_{\mathbf{I}}^+$. For $\mathbf{J} \subset \mathbf{I}$, let $\mathbf{J} \xrightarrow{\mathbf{v}} {}^{w\phi}\mathbf{J} \in B_{\mathbf{I}}^+(\mathcal{J})$. Then*

(i) *We have an isomorphism* $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)\times_{\mathbf{L}_I^{t(\mathbf{w})F}}\tilde{\mathbf{X}}_{\mathbf{L}_I}(\mathbf{J},\mathbf{v}w\phi)\xrightarrow{\sim}\tilde{\mathbf{X}}(\mathbf{J},\mathbf{vw}\phi)$ *of* $\mathbf{G}^F$*-varieties-*$\mathbf{L}_J^{t(\mathbf{vw})F}$*. This isomorphism is compatible with the action of* F^n *for any* n *such that* $\mathbf{I}$*,* $\mathbf{J}$*,* $\mathbf{v}$ *and* $\mathbf{w}$ *are* ϕ^n*-stable.*

(ii) *Through the quotient by* $\mathbf{L}_J^{t(\mathbf{vw})F}$ *(see Lemma 8.24) we get an isomorphism of* $\mathbf{G}^F$*-varieties*

$$\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)\times_{\mathbf{L}_I^{t(\mathbf{w})F}}\mathbf{X}_{\mathbf{L}_I}(\mathbf{J},\mathbf{v}w\phi)\xrightarrow{\sim}\mathbf{X}(\mathbf{J},\mathbf{vw}\phi).$$

Proof. We first look at the case $\mathbf{w},\mathbf{v}\in\mathbf{W}$ (which implies $\mathbf{vw}\in\mathbf{W}$), in which case the isomorphism we seek is

$$\tilde{\mathbf{X}}(I,\dot{w}F)\times_{\mathbf{L}_I^{\dot{w}F}}\tilde{\mathbf{X}}_{\mathbf{L}_I}(J,\dot{v}.\dot{w}F)\xrightarrow{\sim}\tilde{\mathbf{X}}(J,\dot{v}\dot{w}F)$$

where v is the image of $\mathbf{v}$ in W. This is the content of Lusztig's proof of the transitivity of his induction (see [Lu, lemma 3]), that we recall and detail in our context. We claim that $(g\mathbf{U}_I,l\mathbf{V}_J)\mapsto g\mathbf{U}_Il\mathbf{V}_J=gl\mathbf{U}_J$ where $\mathbf{V}_J=\mathbf{L}_I\cap\mathbf{U}_J$ induces the isomorphism we want. We have

$$\mathbf{U}_J\dot{v}\dot{w}^F\mathbf{U}_J=\mathbf{U}_I\mathbf{V}_J\dot{v}\dot{w}^F\mathbf{V}_J{}^F\mathbf{U}_I=\mathbf{U}_I\mathbf{V}_J\dot{v}^{\dot{w}F}\mathbf{V}_J\dot{w}^F\mathbf{U}_I.$$

Since $\mathbf{V}_J\dot{v}^{\dot{w}F}\mathbf{V}_J$ is in $\mathbf{L}_I$, so normalizes $\mathbf{U}_I$ we get finally

$$\mathbf{U}_J\dot{v}\dot{w}^F\mathbf{U}_J=\mathbf{V}_J\dot{v}^{\dot{w}F}\mathbf{V}_J\mathbf{U}_I\dot{w}^F\mathbf{U}_I.$$

Hence if $(g\mathbf{U}_I,l\mathbf{V}_J)\in\tilde{\mathbf{X}}(I,\dot{w}F)\times\tilde{\mathbf{X}}_{\mathbf{L}_I}(\dot{v}\dot{w}\phi)$, we have

$$\begin{aligned}(gl)^{-1\,F}(gl)\in l^{-1}\mathbf{U}_I\dot{w}^F\mathbf{U}_I{}^Fl&=l^{-1}\mathbf{U}_I{}^{\dot{w}F}l\dot{w}^F\mathbf{U}_I\\&=l^{-1\,\dot{w}F}l\mathbf{U}_I\dot{w}^F\mathbf{U}_I\subset\mathbf{V}_J\dot{v}^{\dot{w}F}\mathbf{V}_J\mathbf{U}_I\dot{w}^F\mathbf{U}_I=\mathbf{U}_J\dot{v}\dot{w}^F\mathbf{U}_J.\end{aligned}$$

Hence we have defined a morphism $\tilde{\mathbf{X}}(I,\dot{w}F)\times\tilde{\mathbf{X}}_{\mathbf{L}_I}(\dot{v}.\dot{w}\phi)\to\tilde{\mathbf{X}}(J,\dot{v}\dot{w}F)$ of $\mathbf{G}^F$-varieties-$\mathbf{L}_J^{\dot{v}\dot{w}F}$. We show now that it is surjective. The product $\mathbf{L}_I.(\mathbf{U}_I\dot{w}^F\mathbf{U}_I)$ is direct: a computation shows that this results from the unicity in the decomposition $\mathbf{P}_I\cap{}^{\dot{w}F}\mathbf{U}_I=\mathbf{L}_I.(\mathbf{U}_I\cap{}^{\dot{w}F}\mathbf{U}_I)$. Hence an element $x^{-1\,F}x\in\mathbf{U}_J\dot{v}\dot{w}^F\mathbf{U}_J$ defines unique elements $l\in\mathbf{V}_J\dot{v}^{\dot{w}F}\mathbf{V}_J$ and $u\in\mathbf{U}_I\dot{w}^F\mathbf{U}_I$ such that $x^{-1\,F}x=lu$. If, using Lang's theorem, we write $l=l'^{-1\,\dot{w}F}l'$ with $l'\in\mathbf{L}_I$, the element $g=xl'^{-1}$ satisfies $g^{-1\,F}g=l'x^{-1\,F}x^Fl'^{-1}={}^{\dot{w}F}l'u^Fl'^{-1}\in{}^{\dot{w}F}l'\mathbf{U}_I\dot{w}^F\mathbf{U}_I{}^Fl'^{-1}=\mathbf{U}_I\dot{w}^F\mathbf{U}_I$. Hence $(g\mathbf{U}_I,l'\mathbf{V}_J)$ is a preimage of $x\mathbf{U}_J$ in $\tilde{\mathbf{X}}(I,\dot{w}F)\times\tilde{\mathbf{X}}_{\mathbf{L}_I}(J,\dot{v}\dot{w}\phi)$.

Let us look now at the fibers of the above morphism. If $g'\mathbf{U}_Il'\mathbf{V}_J=g\mathbf{U}_Il\mathbf{V}_J$ then $g'^{-1}g\in\mathbf{P}_I$ so up to $\mathbf{U}_I$ we may assume $g'=g\lambda$ with $\lambda\in\mathbf{L}_I$; we have then $\lambda l'\mathbf{U}_J=l\mathbf{U}_J$, so that $l^{-1}\lambda l'\in\mathbf{U}_J\cap\mathbf{L}_I=\mathbf{V}_J$; moreover if $g\lambda\mathbf{U}_I\in\tilde{\mathbf{X}}(I,\dot{w}F)$ with $\lambda\in\mathbf{L}_I$, then $\lambda^{-1}\mathbf{U}_I\dot{w}^F\mathbf{U}_I{}^F\lambda=\mathbf{U}_I\dot{w}^F\mathbf{U}_I$ which implies $\lambda\in\mathbf{L}_I^{\dot{w}F}$. Conversely, the action of $\lambda\in\mathbf{L}_I^{\dot{w}F}$ given by $(g\mathbf{U}_I,l\mathbf{V}_J)\mapsto(g\lambda\mathbf{U}_I,\lambda^{-1}l\mathbf{V}_J)$ preserves the subvariety $\tilde{\mathbf{X}}(I,\dot{w}F)\times\tilde{\mathbf{X}}_{\mathbf{L}_I}(\dot{v}w\phi)$, of $\mathbf{G}/\mathbf{U}_I\times\mathbf{L}_I/\mathbf{V}_J$. Hence the fibers are the orbits under this action of $\mathbf{L}_I^{\dot{w}F}$.

Now the morphism $j:(g\mathbf{U}_I,l\mathbf{V}_J)\mapsto gl\mathbf{U}_J$ is an isomorphism $\mathbf{G}/\mathbf{U}_I\times_{\mathbf{L}_I}\mathbf{L}_I/\mathbf{V}_J\simeq\mathbf{G}/\mathbf{U}_J$ since $g\mathbf{U}_J\mapsto(g\mathbf{U}_I,\mathbf{V}_J)$ is its inverse. By what we have seen above the restriction of j to the closed subvariety $\tilde{\mathbf{X}}(I,\dot{w}F)\times_{\mathbf{L}_I^{\dot{w}F}}\tilde{\mathbf{X}}_{\mathbf{L}_I}(J,\dot{v}\dot{w}\phi)$ maps this variety surjectively on the closed subvariety $\tilde{\mathbf{X}}(J,\dot{v}\dot{w}F)$ of $\mathbf{G}/\mathbf{U}_J$, hence we get the isomorphism we want.

We now consider the case of generalized varieties. Let k be the number of terms of the normal form of $\mathbf{vw}$ and let $\mathbf{I}\xrightarrow{\mathbf{w}_1}\mathbf{I}_2\xrightarrow{\mathbf{w}_2}\mathbf{I}_3\to\cdots\to\mathbf{I}_r\xrightarrow{\mathbf{w}_r}{}^\phi\mathbf{I}$ be the normal form of $\mathbf{I}\xrightarrow{\mathbf{w}}{}^\phi\mathbf{I}$, perhaps extended by some identity morphisms. We have

$\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi) \simeq \tilde{\mathbf{X}}(I_1\times I_2\times\cdots\times I_k,(t(\mathbf{w}_1),\dots,t(\mathbf{w}_k))F_1)$, where F_1 is as in Proposition 8.15. Let us write $(\mathbf{v}_1\mathbf{w}_1,\dots,\mathbf{v}_k\mathbf{w}_k)$ for the normal form of $\mathbf{vw}$, with same notation as in Proposition 8.5. Let $J_1 = J$ and $J_{j+1} = J_j^{v_jw_j} \subset I_{j+1}$ for $j = 1,\dots,k-1$. We apply the first part of the proof to the group $\mathbf{G}^k$ with isogeny F_1 with I, J, w, and v replaced respectively by $I_1\times\cdots\times I_k$, $J_1\times\cdots\times J_k$, $(w_1\dots,w_k)$ and $(v_1,\dots,v_k)$. Using the isomorphisms from Proposition 8.25;

$$\tilde{\mathbf{X}}_{\mathbf{G}^k}(J_1\times\cdots J_k,(\dot v_1\dot w_1,\dots,\dot v_k\dot w_k)\phi_1) \simeq \tilde{\mathbf{X}}(\mathbf{J},\mathbf{vw}\phi)$$

and

$$\tilde{\mathbf{X}}_{\mathbf{L}_{I_1\times\cdots\times I_k}}(J_1\times\cdots\times J_k,(v_1,\dots,v_k).(t(\mathbf{w}_1),\dots,t(\mathbf{w}_k))F_1) \simeq \tilde{\mathbf{X}}_{\mathbf{L}_I}(\mathbf{J},\mathbf{v}w\phi),$$

we get (i). Now (ii) is immediate from (i) taking the quotient on both sides by $\mathbf{L}_J^{t(\mathbf{vw})F}$. □

If π is the projection of Lemma 8.24, the sheaf $\pi_!\overline{\mathbb{Q}}_\ell$ decomposes into a direct sum of sheaves indexed by the irreducible characters of $\mathbf{L}_\mathbf{I}^{t(\mathbf{w})F}$. We will denote by $\mathbf{St}$ the subsheaf indexed by the Steinberg character of $\mathbf{L}_\mathbf{I}^{t(\mathbf{w})F}$.

In the particular case where $\mathbf{I} = \emptyset$ we write $\mathbf{X}(\mathbf{w}\phi)$ for $\mathbf{X}(\mathbf{I},\mathbf{w}\phi)$. Quite a few theorems are known about the ℓ-adic cohomology of these varieties (see [DMR]). The following corollary of Proposition 8.26 relates the cohomology of a general variety to this particular case; its part (ii) is a refinement of Corollary 8.42.

Corollary 8.27. *Let* $\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi\mathbf{I} \in B^+(\mathcal{I})$.

(i) *For all* $\mathbf{v} \in B_\mathbf{I}^+$ *and all* i *we have the following inclusions of* $\mathbf{G}^F \rtimes \langle F^\delta\rangle$*-modules:*

$$H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell) \subset H_c^{i+2l(\mathbf{v})}(\mathbf{X}(\mathbf{vw}\phi),\overline{\mathbb{Q}}_\ell)(-l(\mathbf{v}))$$

and

$$H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\mathbf{St}) \subset H_c^{i+l(\mathbf{v})}(\mathbf{X}(\mathbf{vw}\phi),\overline{\mathbb{Q}}_\ell)$$

(ii) *For all* i *we have the following equality of* $\mathbf{G}^F \rtimes \langle F^\delta\rangle$*-modules:*

$$H_c^i(\mathbf{X}(\underline{w}_I\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell) = \sum_{j+2k=i} H_c^j(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)\otimes\overline{\mathbb{Q}}_\ell^{\,n_{I,k}}(k)$$

where $n_{I,k} = |\{v \in W_I \mid l(v) = k\}|$*, where* w_I *is the longest element of* W_I *and the variety* $\mathbf{X}(\underline{w}_I\mathbf{w}\phi)$ *is the union* $\bigcup_{\mathbf{v}\in\mathbf{W_I}}\mathbf{X}(\mathbf{vw}\phi)$ *as defined in* [DMR, 2.3.2].

Proof. For getting (i), we apply the Künneth formula to the isomorphism of Proposition 8.26 when $\mathbf{J} = \emptyset$. If we decompose the equality given by the Künneth formula according to the characters of $\mathbf{L}_\mathbf{I}^{t(\mathbf{w})F}$, we get

$$\oplus_{\chi\in\mathrm{Irr}(\mathbf{L}_\mathbf{I}^{t(\mathbf{w})F})}\oplus_j H_c^{i-j}(\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)_\chi \otimes_{\mathbf{L}_\mathbf{I}^{t(\mathbf{w})F}} H_c^j(\mathbf{X}_{\mathbf{L_I}}(\mathbf{v}w\phi),\overline{\mathbb{Q}}_\ell)_{\overline{\chi}} \simeq H_c^i(\mathbf{X}(\mathbf{vw}\phi),\overline{\mathbb{Q}}_\ell).$$

We now use that $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell) = H_c^i(\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)_{\mathrm{Id}}$, and $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\mathbf{St}) = H_c^i(\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)_{\mathrm{St}}$ where Id and St denote the identity and Steinberg characters of $\mathbf{L}_\mathbf{I}^{t(\mathbf{w})F}$, and the facts that

- the only j such that $H_c^j(\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}w\phi),\overline{\mathbb{Q}}_\ell)_{\mathrm{Id}}$ is non-trivial is $j = 2l(\mathbf{v})$ and in that case the cohomology group has dimension 1 and $t(\mathbf{w}F)$ acts by $q^{l(\mathbf{v})}$ (see [DMR, 3.3.14]).

- the only j such that $H_c^j(\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}w\phi),\overline{\mathbb{Q}}_\ell)_{\mathrm{St}}$ is non-trivial is $j=l(\mathbf{v})$ and that isotypic component is of multiplicity one, with trivial action of $t(\mathbf{w}F)$ (see [DMR, 3.3.15]).

Hence we have

$$\oplus_j H_c^{i-j}(\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)_{\mathrm{Id}}\otimes H_c^j(\mathbf{X}_{\mathbf{L_I}}(\mathbf{v}w\phi),\overline{\mathbb{Q}}_\ell)_{\mathrm{Id}}=H_c^{i-2l(\mathbf{v})}(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)(l(\mathbf{v})),$$

and similarly

$$\oplus_j H_c^{i-j}(\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)_{\mathrm{St}}\otimes H_c^j(\mathbf{X}_{\mathbf{L_I}}(\mathbf{v}w\phi),\overline{\mathbb{Q}}_\ell)_{\mathrm{St}}=H_c^{i-l(\mathbf{v})}(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell).$$

We now prove (ii). Let $\mathcal{B}_I$ be the variety of Borel subgroups of $\mathbf{L}_I$, identified to $\mathbf{L}_I/\mathbf{B}_I$. We first prove that we have an isomorphism $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)\times_{\mathbf{L}_I^{w\phi}}\mathcal{B}_I\xrightarrow{\sim}\mathbf{X}(\underline{w}_I\mathbf{w}\phi)$. The variety $\mathbf{X}(\underline{w}_I\mathbf{w}\phi)$ is the union $\bigcup_{\mathbf{v}\in\mathbf{W}_I}\mathbf{X}(\mathbf{v}\mathbf{w}\phi)$. The variety $\mathcal{B}_I$ is the union of the varieties $\mathbf{X}_{\mathbf{L}_I}(\mathbf{v}w\phi)$ when $\mathbf{v}$ runs over $\mathbf{W_I}$. The isomorphisms given by Proposition 8.26 when $\mathbf{J}=\emptyset$ and $\mathbf{v}$ running over $\mathbf{W_I}$ can be glued together since they are defined by a formula independent of $\mathbf{v}$. We thus get a bijective morphism $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)\times_{\mathbf{L}_I^{w\phi}}\mathcal{B}_I\to\mathbf{X}(\underline{w}_I\mathbf{w}\phi)$ which is an isomorphism since $\mathbf{X}(\underline{w}_I\mathbf{w}\phi)$ is normal (see [DMR, 2.3.5]). We now get (ii) from the fact that $H_c^k(\mathcal{B}_I,\overline{\mathbb{Q}}_\ell)$ is 0 if k is odd and if $k=2k'$ is a trivial $\mathbf{L}_I^{t(\mathbf{w})F}$-module of dimension $n_{I,k'}$, where F acts by the scalar $q^{k'}$; this results for example from the cellular decomposition into affine spaces given by the Bruhat decomposition and the fact that the action of $\mathbf{L}_I^{t(\mathbf{w})F}$ extends to the connected group $\mathbf{L}_I$. □

Corollary 8.28. (i) *The $\mathbf{G}^F$-module $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)$ is unipotent. The eigenvalues of F^δ on an irreducible $\mathbf{G}^F$-submodule ρ of $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)$ are in $q^{\delta\mathbb{N}}\lambda_\rho\omega_\rho$, where λ_ρ is a root of unity and ω_ρ is an element of $\{1,q^{\delta/2}\}$ which are both independent of i and $\mathbf{w}$.*

(ii) *We have $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)=0$ unless $l(\mathbf{w})\le i\le 2l(\mathbf{w})$.*

(iii) *The eigenvalues of F^δ on $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)$ are of absolute value less than $q^{\delta i/2}$.*

(iv) *The Steinberg representation does not occur in any cohomology group of $\mathbf{X}(\mathbf{I},\mathbf{w}\phi)$ unless $\mathbf{I}=\emptyset$ in which case it occurs with multiplicity 1 in $H_c^{l(\mathbf{w})}(\mathbf{X}(\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)$, associated to the eigenvalue 1 of F^δ.*

(v) *The trivial representation occurs with multiplicity 1 in $H_c^{2l(\mathbf{w})}(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)$, associated to the eigenvalue $q^{\delta l(\mathbf{w})}$ of F^δ, and does not occur in any other cohomology group of $\mathbf{X}(\mathbf{I},\mathbf{w}\phi)$.*

Proof. (i) is a straightforward consequence of Corollary 8.27(i) since the result is known for $H_c^j(\mathbf{X}(\mathbf{v}\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)$ (see [DMR, 3.3.4] and [DMR, 3.3.10 (i)]).

(ii) and (iii) are similarly a straightforward consequence of Corollary 8.27(i) applied with $\mathbf{v}=1$ and of [DMR, 3.3.22] and [DMR, 3.3.10(i)].

For (iv), we first note that by Corollary 8.27(i) applied with $\mathbf{v}=1$ and [DMR, 3.3.15] the Steinberg representation has multiplicity at most 1 in $H_c^{l(\mathbf{w})}(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)$, associated to the eigenvalue 1 of F^δ, and does not occur in any other cohomology group of $\mathbf{X}(\mathbf{I},\mathbf{w}\phi)$. To see when it does occur, it is enough then to use Proposition 8.40 and the Lefschetz formula. The only $U_{\tilde{\chi}}$ such that the Steinberg representation has a non-zero scalar product with $\mathrm{Sh}_{F^m/F}U_{\tilde{\chi}}$ is the Steinberg representation, and

for the corresponding $\tilde{\chi}$ we have

$$\tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}}\phi) = \begin{cases} (-1)^{l(\mathbf{w})} & \text{if } \mathbf{I} = \emptyset \\ 0 & \text{otherwise} \end{cases}.$$

(v) is similarly a consequence of Corollary 8.27(i), [DMR, 3.3.14], 8.40, the Lefschetz formula, and that if $\tilde{\chi}_{q^m}$ corresponds to the trivial representation we have $\tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}}\phi) = q^{ml(\mathbf{w})}$. □

Endomorphisms of parabolic Deligne-Lusztig varieties — the conjugacy category $\mathcal{D}^+(\mathcal{I})$.

Definition 8.29. *Given any morphism $\mathbf{I} \xrightarrow{\mathbf{v}} \mathbf{J} \in B^+(\mathcal{I})$ which is a left divisor of $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$ we define morphisms of varieties:*

(i) *$D_{\mathbf{v}} : \mathbf{X}(\mathbf{I}, \mathbf{w}\phi) \to \mathbf{X}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v})$ as the restriction of the morphism*

$$(a,b) \mapsto (b, {}^F a) : \mathcal{O}(\mathbf{I}, \mathbf{w}) = \mathcal{O}(\mathbf{I}, \mathbf{v}) \times_{\mathcal{P}_J} \mathcal{O}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}) \to \mathcal{O}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}) \times_{\mathcal{P}_{\phi_I}} \mathcal{O}({}^{\phi}\mathbf{I}, {}^{\phi}\mathbf{v}) = \mathcal{O}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}^{\phi}\mathbf{v}).$$

(ii) *$\tilde{D}_{\mathbf{v}} : \tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \to \tilde{\mathbf{X}}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v})$ as the restriction of the morphism*

$$(a,b) \mapsto (b, {}^F a) : \tilde{\mathcal{O}}(\mathbf{I}, \mathbf{w}) = \tilde{\mathcal{O}}(\mathbf{I}, \mathbf{v}) \times_{\mathbf{G}/\mathbf{U}_J} \tilde{\mathcal{O}}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}) \to \tilde{\mathcal{O}}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}) \times_{\mathbf{G}/\mathbf{U}_{\phi_I}} \tilde{\mathcal{O}}({}^{\phi}\mathbf{I}, {}^{\phi}\mathbf{v}) = \tilde{\mathcal{O}}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}^{\phi}\mathbf{v}).$$

Note that the existence of well-defined decompositions as above of $\mathcal{O}(\mathbf{I}, \mathbf{w})$ and of $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{w})$ are consequences of Theorem 8.10. We have written $\mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v}$ for $\mathbf{v}^{-1}\mathbf{w}^{\phi}\mathbf{v}\phi$.

Note that when $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v}^{-1}\mathbf{w}^{\phi}\mathbf{v}$ are in $\mathbf{W}$ the endomorphism $D_{\mathbf{v}}$ maps $g\mathbf{P}_I \in \mathbf{X}(I, w\phi)$ to $g'\mathbf{P}_J \in \mathbf{X}(J, v^{-1}w\phi v)$ such that $g^{-1}g' \in \mathbf{P}_I v \mathbf{P}_J$ and $g'^{-1\,F}g \in \mathbf{P}_J v^{-1} w^F \mathbf{P}_I$ and similarly for $\tilde{D}_{\mathbf{v}}$.

Note also that $D_{\mathbf{v}}$ and $\tilde{D}_{\mathbf{v}}$ are equivalences of étale sites; indeed, the proof of [DMR, 3.1.6] applies without change in our case.

The definition of $\tilde{D}_{\mathbf{v}}$ and $D_{\mathbf{v}}$ shows the following property:

Lemma 8.30. *The following diagram is commutative:*

$$\begin{array}{ccc} \tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) & \xrightarrow{\tilde{D}_{\mathbf{v}}} & \tilde{\mathbf{X}}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v}) \\ \downarrow & & \downarrow \\ \mathbf{X}(\mathbf{I}, \mathbf{w}\phi) & \xrightarrow{D_{\mathbf{v}}} & \mathbf{X}(\mathbf{J}, \mathbf{v}^{-1}\mathbf{w}\phi\mathbf{v}) \end{array}$$

where the vertical arrows are the respective quotients by $\mathbf{L}_I^{t(\mathbf{w})F}$ and $\mathbf{L}_J^{t(\mathbf{v}^{-1}\mathbf{w}^{\phi}\mathbf{v})F}$ as explained in Lemma 8.24.

Definition 8.31. *We denote by $\mathcal{D}^+(\mathcal{I})$ the category $\phi\text{-}\mathrm{cyc}\, B^+(\mathcal{I})$; that is the objects of $\mathcal{D}^+(\mathcal{I})$ are the morphisms in $B^+(\mathcal{I})$ of the form $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$ and the morphisms are generated by the "simple" morphisms that we will denote $\mathrm{ad}\,\mathbf{v}$, for $\mathbf{v} \preccurlyeq \mathbf{w}$; such a morphism, more formally denoted by $\mathbf{I} \xrightarrow{\mathrm{ad}\,\mathbf{v}} \mathbf{J}$, where $\mathbf{J} = \mathbf{I}^{\mathbf{v}}$, goes from $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$ to $\mathbf{J} \xrightarrow{\mathbf{v}^{-1}\mathbf{w}^{\phi}\mathbf{v}} {}^{\phi}\mathbf{J}$. The relations are given by the equalities $\mathrm{ad}\,\mathbf{v}_1 \dots \mathrm{ad}\,\mathbf{v}_k = \mathrm{ad}\,\mathbf{v}'_1 \dots \mathrm{ad}\,\mathbf{v}'_{k'}$ whenever $\mathrm{ad}\,\mathbf{v}_i$ are simple and $\mathbf{v}_1 \dots \mathbf{v}_k = \mathbf{v}'_1 \dots \mathbf{v}'_{k'}$ in B^+.*

We will denote $\mathbf{I} \xrightarrow{\operatorname{ad}\mathbf{v}} \mathbf{J}$ a general morphism of $\mathcal{D}^+(\mathbf{I})$ (where $\mathbf{v} = \mathbf{v}_1 \dots \mathbf{v}_k$ with the $\operatorname{ad}\mathbf{v}_i$ simple).

As a further consequence of Theorem 8.10, the map which sends a simple morphism $\operatorname{ad}\mathbf{v}$ to $D_{\mathbf{v}}$ gives a natural morphism of monoids $\operatorname{End}_{\mathcal{D}^+(\mathcal{I})}(\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}) \to \operatorname{End}_{\mathbf{G}^F}(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi))$, whose image consists of equivalences of étale sites.

By Proposition 5.2 the category $\mathcal{D}^+(\mathcal{I})$ has a Garside family consisting of the simple morphisms. Those of source $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$ correspond to the set of $\mathbf{v} \preccurlyeq \mathbf{w}$ such that $\mathbf{I}^{\mathbf{v}} \subset \mathbf{S}$. For $J \subset I \subset S$ we will denote by $\mathcal{D}_I^+(\mathcal{J})$ the analogous category where S is replaced by I and I by J.

Proposition 8.32. *With same assumptions and notation as in Proposition 8.26, let $\mathbf{J} \xrightarrow{\mathbf{x}} \mathbf{J}^{\mathbf{x}} \in B_I^+(\mathcal{J})$ be a left divisor of $\mathbf{J} \xrightarrow{\mathbf{v}} {}^{\mathbf{w}\phi}\mathbf{J}$. The following diagram is commutative:*

$$\begin{array}{ccc} \tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \times_{\mathbf{L}_I^{\dot{w}F}} \tilde{\mathbf{X}}_{\mathbf{L}_I}(\mathbf{J}, \mathbf{v} \cdot w\phi) & \xrightarrow{\sim} & \tilde{\mathbf{X}}(\mathbf{J}, \mathbf{v}\mathbf{w}\phi) \\ \downarrow{\scriptstyle \mathrm{Id} \times \tilde{D}_{\mathbf{x}}} & & \downarrow{\scriptstyle \tilde{D}_{\mathbf{x}}} \\ \tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi) \times_{\mathbf{L}_I^{\dot{w}F}} \tilde{\mathbf{X}}_{\mathbf{L}_I}(\mathbf{J}^{\mathbf{x}}, \mathbf{x}^{-1}(\mathbf{v} \cdot w\phi)\mathbf{x}) & \xrightarrow{\sim} & \tilde{\mathbf{X}}(\mathbf{J}^{\mathbf{x}}, \mathbf{x}^{-1}\mathbf{v}\mathbf{w}\phi\mathbf{x}) \end{array}$$

Proof. Decomposing $\mathbf{x}$ into a product of simples in $\mathcal{D}_I^+(\mathcal{J})$ the definitions show that it is sufficient to prove the result for $\mathbf{x} \in \mathbf{W}$. We use then Proposition 8.25 to reduce the proof to the case where $\mathbf{vw}$ and $\mathbf{v}^{-1}\mathbf{w}^{\phi}\mathbf{v}$ are in $\mathbf{W}$ (in which case $\mathbf{w}$ and $\mathbf{v}^{-1\,w\phi}\mathbf{w}$ are in $\mathbf{W}$ too). We can make this reduction if we know that the isomorphism of Proposition 8.25 is compatible with the action of $D_{\mathbf{x}}$ for $\mathbf{x} \in \mathbf{W}$ (we will then use this fact in $\mathbf{G}$ and in $\mathbf{L}_I$). Take $(\mathbf{I}, \mathbf{y}, {}^{\phi}\mathbf{I}) \in B^+(\mathcal{I})$ and $\mathbf{x} \in \mathbf{W}$ such that $\mathbf{I} \xrightarrow{\mathbf{x}} \mathbf{I}^x$ is a left divisor of $\mathbf{I} \xrightarrow{\mathbf{y}} {}^{\phi}\mathbf{I}$. Let $\mathbf{y} = \mathbf{y}_1 \dots \mathbf{y}_k$ be a decomposition of $\mathbf{y}$ as a product of elements of $\mathbf{W}$ such that $\mathbf{x} = \mathbf{y}_1$. The endomorphism $D_{\mathbf{x}}$ maps the sequence $(g_1\mathbf{U}_1, \dots, g_k\mathbf{U}_k)$ such that $g_i^{-1}g_{i+1} \in \mathbf{U}_i\dot{y}_i\mathbf{U}_{i+1}$ and $g_k^{-1\,F}g_1 \in \mathbf{U}_k\dot{y}_k{}^F\mathbf{U}_1$ to the sequence $(g_2\mathbf{U}_2, \dots, g_k\mathbf{U}_k, {}^Fg_1{}^F\mathbf{U}_1)$. On the other hand, via the isomorphism of Proposition 8.25, using the decomposition $(\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k, 1)$ of $\mathbf{y}$, the sequence $(g_1\mathbf{U}_1, \dots, g_k\mathbf{U}_k)$ corresponds to $((g_1, \dots, g_k, {}^Fg_1)(\mathbf{U}_1, \dots, \mathbf{U}_k, {}^F\mathbf{U}_1) \in \tilde{X}_{\mathbf{G}^{k+1}}(I_1 \times \dots \times I_k \times {}^FI_1, (\dot{y}_1, \dots, \dot{y}_k, 1)F_1)$. This element is mapped by $D_{(y_1,1,\dots,1)}$ to the element $(g_2, g_2, \dots, g_k, {}^Fg_1)(\mathbf{U}_2, \mathbf{U}_2, \dots, \mathbf{U}_k, {}^F\mathbf{U}_1)$ which is in $\tilde{X}_{\mathbf{G}^{k+1}}(I_2 \times I_2 \times I_3 \times \dots \times I_k \times {}^FI_1, (1, \dot{y}_2, \dots, \dot{y}_k, {}^Fy_1)F_1)$. Since this last element corresponds by the isomorphism of Proposition 8.25 to $(g_2\mathbf{U}_2, \dots, g_k\mathbf{U}_k, {}^Fg_1{}^F\mathbf{U}_1)$, we have proved the compatibility we want.

Assume now $\mathbf{vw}$ and $\mathbf{v}^{-1}\mathbf{w}^{\phi}\mathbf{v}$ in $\mathbf{W}$. We start with $(g\mathbf{U}_I, l\mathbf{V}_J) \in \tilde{\mathbf{X}}(I, \dot{w}F) \times \tilde{\mathbf{X}}_{\mathbf{L}_I}(J, vw\phi)$. This element is mapped by the top isomorphism of the diagram to $gl\mathbf{U}_J$. As we have seen above Lemma 8.30 it is mapped by $\mathrm{Id} \times D_{\mathbf{x}}$ to $(g\mathbf{U}_I, l'\mathbf{V}_{J^x})$ where $l^{-1}l' \in \mathbf{V}_J x \mathbf{V}_{J^x}$ and $l'^{-1\,\dot{w}F}l \in \mathbf{V}_{J^x}x^{-1}v^{wF}\mathbf{V}_J$. This element is mapped to $gl'\mathbf{U}_{J^x}$ by the bottom isomorphism of the diagram. We have to check that $gl'\mathbf{U}_{J^x} = D_{\mathbf{x}}(gl\mathbf{U}_J)$. But $(gl)^{-1}gl' = l^{-1}l'$ is in $\mathbf{V}_J x \mathbf{V}_{J^x} \subset \mathbf{U}_J x \mathbf{U}_{J^x}$ and

$$\begin{aligned}(gl')^{-1\,F}(gl) = l'^{-1}g^{-1\,F}g^Fl \in l'^{-1}\mathbf{U}_I\dot{w}^F\mathbf{U}_I{}^Fl = \mathbf{U}_I l'^{-1\,\dot{w}F}l\dot{w}^F\mathbf{U}_I \\ \subset \mathbf{U}_I\mathbf{V}_{J^x}x^{-1}vw^F\mathbf{V}_J{}^F\mathbf{U}_I = \mathbf{U}_{J^x}x^{-1}vw^F\mathbf{U}_J,\end{aligned}$$

so that $(gl'\mathbf{U}_{J^x}) = D_{\mathbf{x}}(gl\mathbf{U}_J)$. □

Using Proposition 8.26(ii) and Lemma 8.30 we get

Corollary 8.33. *The following diagram is commutative:*

$$\begin{array}{ccc} \tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)\times_{\mathbf{L}_I^{\dot{w}F}}\mathbf{X}_{\mathbf{L}_I}(\mathbf{J},\mathbf{v}\cdot w\phi) & \xrightarrow{\sim} & \mathbf{X}(\mathbf{J},\mathbf{v}\mathbf{w}\phi) \\ \downarrow{\scriptstyle \mathrm{Id}\times D_{\mathbf{x}}} & & \downarrow{\scriptstyle D_{\mathbf{x}}} \\ \tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)\times_{\mathbf{L}_I^{\dot{w}F}}\mathbf{X}_{\mathbf{L}_I}(\mathbf{J}^{\mathbf{x}},\mathbf{x}^{-1}(\mathbf{v}\cdot w\phi)\mathbf{x}) & \xrightarrow{\sim} & \mathbf{X}(\mathbf{J}^{\mathbf{x}},\mathbf{x}^{-1}\mathbf{v}\mathbf{w}\phi\mathbf{x}) \end{array}$$

In the following example we can describe explicitly $\mathrm{End}_{\mathcal{D}^+(\mathcal{I})}\mathbf{I}\xrightarrow{\mathbf{w}}{}^{\phi}\mathbf{I}$.

Definition 8.34. *We define $\boldsymbol{\pi}=\mathbf{w}_0^2$ (it is a generator of the center of the pure braid group) and similarly for $I\subset S$ we define $\boldsymbol{\pi}_{\mathbf{I}}=\mathbf{w}_{\mathbf{I}}^2$.*

An example. Let $\mathbf{w}=\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}=\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0\cdot\mathbf{w}_0\mathbf{w}_{\mathbf{I}}^{-1}$. Then any $\mathbf{v}\in\mathrm{Hom}_{B^+(\mathcal{I})}(\mathbf{I},\mathbf{I})^{\phi}$ gives endomorphisms $\tilde{D}_{\mathbf{v}}$ and $D_{\mathbf{v}}$ of $\tilde{\mathbf{X}}(\mathbf{I},(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})\phi)$ and $\mathbf{X}(\mathbf{I},(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})\phi)$ respectively. This comes from Lemma 8.3 since if the i-th term of the normal form of $\mathbf{I}\xrightarrow{\mathbf{v}}{}^{\phi}\mathbf{I}$ is $\mathbf{I}_i\xrightarrow{\mathbf{w}_i}\mathbf{I}_{i+1}$ then $\mathbf{w}_i$ divides $\boldsymbol{\pi}/\boldsymbol{\pi}_{I_i}$ and conjugates it to $\boldsymbol{\pi}/\boldsymbol{\pi}_{I_{i+1}}$.

Let $\mathbf{w}\in\mathrm{Hom}_{B^+(\mathcal{I})}(\mathbf{I},{}^{\phi}\mathbf{I})$ be such that $(\mathbf{w}\phi)^d=(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})\phi^d$. The map $g\mathbf{U}_I\mapsto(g\mathbf{U}_I,{}^F(g\mathbf{U}_I),{}^{F^2}(g\mathbf{U}_I),\dots,{}^{F^{d-1}}(g\mathbf{U}_I))$ identifies $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)$ with the subvariety of $\tilde{\mathbf{X}}(\mathbf{I},(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})\phi^d)$ defined by the equation $\tilde{D}_{\mathbf{w}}x=Fx$. This implies that the intersection of the centralizer of $\mathbf{w}\phi$ with $\mathrm{Hom}_{B^+(\mathcal{I})}(\mathbf{I},\mathbf{I})$ acts on $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)$.

Similarly the map $\mathbf{P}\mapsto(\mathbf{P},{}^F\mathbf{P},{}^{F^2}\mathbf{P},\dots,{}^{F^{d-1}}\mathbf{P})$ identifies $\mathbf{X}(\mathbf{I},\mathbf{w}\phi)$ with the subvariety of $\mathbf{X}(\mathbf{I},(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})\phi^d)$ defined by the equation $D_{\mathbf{w}}x=Fx$ and the intersection of the centralizer of $\mathbf{w}\phi$ with $\mathrm{Hom}_{B^+(\mathcal{I})}(\mathbf{I},\mathbf{I})$ acts on $\mathbf{X}(\mathbf{I},\mathbf{w}\phi)$. □

We now give a more general case where we can describe $\mathrm{End}_{\mathcal{D}^+(\mathcal{I})}(\mathbf{I}\xrightarrow{\mathbf{w}}{}^{\phi}\mathbf{I})$.

Theorem 8.35. *Assume that some power of $\mathbf{w}\phi$ is divisible on the left by $\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0$. Then $\mathrm{End}_{\mathcal{D}^+(\mathcal{I})}\mathbf{I}\xrightarrow{\mathbf{w}}{}^{\phi}\mathbf{I}$ consists of the morphisms $\mathbf{I}\xrightarrow{\mathrm{ad}\,\mathbf{b}}\mathbf{I}$ where $\mathbf{b}$ runs over the submonoid $B^+_{\mathbf{w}}$ of $C_{B^+}(\mathbf{w}\phi)$ consisting of the elements such that ${}^{\mathbf{b}}\mathbf{I}=\mathbf{I}$ and $\alpha_{\mathbf{I}}(\mathbf{b})=1$.*

Proof. This is an immediate translation of Proposition 5.5, since the Garside map of $B^+(\mathcal{I})$ is $\mathbf{I}\xrightarrow{\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0}\mathbf{I}^{\mathbf{w}_0}$; the submonoid $B^+_{\mathbf{w}}$ is the centralizer of the morphism $\mathbf{I}\xrightarrow{\mathbf{w}}{}^{\phi}\mathbf{I}$ of $B^+(\mathbf{I})$. □

Note that if k is the smallest power such that ${}^{\phi^k}\mathbf{I}=\mathbf{I}$ and ${}^{\phi^k}\mathbf{w}=\mathbf{w}$, then $\mathbf{w}^{(k)}:=\mathbf{w}{}^{\phi}\mathbf{w}\dots{}^{\phi^{k-1}}\mathbf{w}$ is in $B^+_{\mathbf{w}}$. Since $\mathbf{I}\xrightarrow{\mathrm{ad}\,\mathbf{w}}{}^{\phi}\mathbf{I}$ is the Garside map of $\mathcal{D}^+(\mathbf{I})$ described in Proposition 5.3, it follows that under the assumptions of Theorem 8.35 every element of $B^+_{\mathbf{w}}$ divides a power of $\mathbf{w}^{(k)}$. In particular, in the case $\mathbf{I}=\emptyset$, the group $C_B(\mathbf{w}\phi)$ is generated as a monoid, with the notations of [DM2, 2.1], by $\mathrm{End}_{\mathcal{D}^+}(\mathbf{w})$ and $(\mathbf{w}^{(k)})^{-1}$. Thus Theorem 8.35 in this particular case gives a positive answer to conjecture [DM2, 2.1].

Affineness. Until the end of the text, we will consider varieties which satisfy the assumption of Theorem 8.35. They have many nice properties. We show in this subsection that they are affine, by adapting the proof of Bonnafé and Rouquier [BR2] to our case; we use the existence of the varieties $\tilde{\mathcal{O}}(\mathbf{I},\mathbf{b})$ and $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{b}\phi)$ to replace doing a quotient by $\mathbf{L}_I$ by doing a quotient by $\mathbf{L}_I^F$.

Proposition 8.36. *Assume the morphism* $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J} \in B^+(\mathcal{I})$ *is left-divisible by* $\Delta_{\mathcal{I}}$. *Then the variety* $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{b})$ *is affine.*

Proof. By assumption there exists a decomposition into elements of $\mathcal{S}$ of $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$ of the form $\mathbf{I} \xrightarrow{\mathbf{w_I}^{-1}\mathbf{w}_0} \mathbf{I}_1 \xrightarrow{\mathbf{v}_1} \mathbf{I}_2 \xrightarrow{\mathbf{v}_2} \mathbf{I}_3 \to \cdots \to \mathbf{I}_r \xrightarrow{\mathbf{v}_r} \mathbf{J}$. We show that the map φ defined by:

$$\begin{aligned}\mathbf{G} \times \prod_{i=1}^{i=r} (\mathbf{U}_{I_i} \cap {}^{v_i}\mathbf{U}^-_{I_{i+1}})\dot{v}_i \to \\ \tilde{\mathcal{O}}(I, \dot{w}_I^{-1}\dot{w}_0) \times_{\mathbf{G}/\mathbf{U}_{I_1}} \tilde{\mathcal{O}}(I_1, \dot{v}_1) \dots \times_{\mathbf{G}/\mathbf{U}_{I_r}} \tilde{\mathcal{O}}(I_r, \dot{v}_r) \\ (g, h_1, \dots, h_r) \mapsto \\ (g\mathbf{U}_I, g\dot{w}_I^{-1}\dot{w}_0\mathbf{U}_{I_1}, g\dot{w}_I^{-1}\dot{w}_0 h_1 \mathbf{U}_{I_2}, \dots, g\dot{w}_I^{-1}\dot{w}_0 h_1 \dots h_r \mathbf{U}_J)\end{aligned}$$

is an isomorphism; since the first variety is a product of affine varieties this will prove our claim.

Since $\mathbf{U}_{I_i}\dot{v}_i\mathbf{U}_{I_{i+1}}$ is isomorphic to $(\mathbf{U}_{I_i} \cap {}^{v_i}\mathbf{U}^-_{I_{i+1}})\dot{v}_i \times \mathbf{U}_{I_{i+1}}$, by composition with the first projection we get a morphism $\eta_i : \mathbf{U}_{I_i}\dot{v}_i\mathbf{U}_{I_{i+1}} \to (\mathbf{U}_{I_i} \cap {}^{v_i}\mathbf{U}^-_{I_{i+1}})\dot{v}_i$ for $i = 1, \dots, r$, where $I_{r+1} = J$. For $x = (g\mathbf{U}_I, g_1\mathbf{U}_{I_1}, g_2\mathbf{U}_{I_2}, \dots, g_r\mathbf{U}_{I_r}, g_{r+1}\mathbf{U}_J)$ in $\tilde{\mathcal{O}}(I, \dot{w}_I^{-1}\dot{w}_0) \times_{\mathbf{G}/\mathbf{U}_{I_1}} \tilde{\mathcal{O}}(I_1, \dot{v}_1) \dots \times_{\mathbf{G}/\mathbf{U}_{I_r}} \tilde{\mathcal{O}}(I_r, \dot{v}_r)$ we put $\psi(x) = g\eta(g^{-1}g_1)$, $\psi_1(x) = \psi(x)\dot{w}_0$, $\psi_i(x) = \eta_i((\psi(x)\psi_1(x)\dots\psi_{i-1}(x))^{-1}g_i)$. We claim that the maps ψ (resp. ψ_i) are well defined, that is do not depend on the representative g (resp. g_i) chosen; the morphism $x \mapsto (\psi(x), \psi_1(x), \dots, \psi_r(x))$ is then clearly inverse to φ. Since $\eta_i(hu) = \eta_i(h)$ for all $h \in \mathbf{U}_{I_i}\dot{v}_i\mathbf{U}_{I_{i+1}}$ and all $u \in \mathbf{U}_{I_{i+1}}$, we get that all ψ_i are well-defined. Since moreover $\eta(uh) = u\eta(h)$ for all $h \in \mathbf{U}_I\dot{w}_I^{-1}\dot{w}_0\mathbf{U}_{I_1}$ and all $u \in \mathbf{U}_I$, we get that ψ also is well-defined, whence our claim. □

Proposition 8.37. *Assume that we are under the assumptions of Theorem 8.35, that is* $(\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}) \in B^+(\mathcal{I})$ *has some power divisible by* $\Delta_{\mathcal{I}}$, *or equivalently some power of* $\mathbf{w}\phi$ *is divisible on the left by* $\mathbf{w}_\mathbf{I}^{-1}\mathbf{w}_0$. *Assume further that the Tits homomorphism* t *has been chosen* F*-equivariant. Then* $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$ *is affine.*

Proof. Let us, similarly to after Theorem 8.35, define k as the smallest power such that ${}^{\phi^k}\mathbf{I} = \mathbf{I}$, ${}^{\phi^k}\mathbf{w} = \mathbf{w}$ and $\mathbf{w}_\mathbf{I}^{-1}\mathbf{w}_0 \preccurlyeq \mathbf{w}^{(k)}$, where $\mathbf{w}^{(k)} := \mathbf{w}^{\phi}\mathbf{w} \dots {}^{\phi^{k-1}}\mathbf{w}$.

We will embed $\tilde{\mathbf{X}}(\mathbf{I}, \mathbf{w}\phi)$ as a closed subvariety in $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{w}^{(k)})$, which will prove it is affine.

Let $\mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \xrightarrow{\mathbf{w}_2} \mathbf{I}_3 \to \cdots \to \mathbf{I}_r \xrightarrow{\mathbf{w}_r} {}^{\phi}\mathbf{I}$ be a decomposition of $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$ into elements of $\mathcal{S}$, so that $\tilde{\mathcal{O}}(\mathbf{I}, \mathbf{w}^{(k)})$ identifies to the set of sequences

$$\begin{aligned}(&g_{1,1}\mathbf{U}_I, g_{1,2}\mathbf{U}_{I_2}, \dots, g_{1,r}\mathbf{U}_{I_r}, \\ &g_{2,1}\mathbf{U}_{\phi I}, g_{2,2}\mathbf{U}_{\phi I_2}, \dots, g_{2,r}\mathbf{U}_{\phi I_r}, \\ &\dots, \\ &g_{k,1}\mathbf{U}_{\phi^{k-1}I}, g_{k,2}\mathbf{U}_{\phi^{k-1}I_2}, \dots, g_{k,r}\mathbf{U}_{\phi^{k-1}I_r}, \\ &g_{k+1,1}\mathbf{U}_I)\end{aligned}$$

such that for $j < r$ we have $g_{i,j}^{-1}g_{i,j+1} \in \mathbf{U}_{\phi^{i-1}I_j}{}^{F^{i-1}}\dot{w}_j\mathbf{U}_{\phi^{i-1}I_{j+1}}$ and $g_{i,r}^{-1}g_{i+1,1} \in \mathbf{U}_{\phi^{i-1}I_r}{}^{F^{i-1}}\dot{w}_r\mathbf{U}_{\phi^i I}$; note that we have used the F-equivariance of t to write ${}^{F^i}\dot{w}_j$ for $t({}^{\phi^i}\mathbf{w}_j)$.

Similarly $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)$ identifies to the set of sequences $(g_1\mathbf{U}_I, g_2\mathbf{U}_{I_2},\dots,g_r\mathbf{U}_{I_r})$ such that $g_j^{-1}g_{j+1}\in\mathbf{U}_{I_j}\dot{w}_j\mathbf{U}_{I_{j+1}}$ for $j<r$ and $g_r^{-1}{}^F g_1\in\mathbf{U}_{I_r}\dot{w}_r\mathbf{U}_{\phi I}$. It is thus clear that the map

$$\begin{aligned}(g_1\mathbf{U}_I, g_2\mathbf{U}_{I_2},\dots,g_r\mathbf{U}_{I_r})\mapsto (&g_1\mathbf{U}_I, g_2\mathbf{U}_{I_2},\dots,g_r\mathbf{U}_{I_r},\\ &{}^F g_1\mathbf{U}_{\phi I}, {}^F g_2\mathbf{U}_{\phi I_2},\dots,{}^F g_r\mathbf{U}_{\phi I_r},\\ &\dots,\\ &{}^{F^{k-1}} g_1\mathbf{U}_{\phi^{k-1}I},\dots,{}^{F^{k-1}} g_r\mathbf{U}_{\phi^{k-1}I_r}, {}^{F^k} g_1\mathbf{U}_I)\end{aligned}$$

identifies $\tilde{\mathbf{X}}(\mathbf{I},\mathbf{w}\phi)$ to the closed subvariety of $\tilde{\mathcal{O}}(\mathbf{I},\mathbf{w}^{(k)})$ defined by $g_{i+1,j}\mathbf{U}_{\phi^i I_j} = {}^F(g_{i,j}\mathbf{U}_{\phi^{i-1}I_j})$ for all i,j. □

Corollary 8.38. *Assume that we are under the assumptions of Theorem 8.35, that is* $(\mathbf{I}\xrightarrow{\mathbf{w}}{}^{\phi}\mathbf{I})\in B^+(\mathcal{I})$ *has some power divisible by* $\Delta_{\mathcal{I}}$*, or equivalently some power of* $\mathbf{w}\phi$ *is divisible on the left by* $\mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0$*. Then* $\mathbf{X}(\mathbf{I},\mathbf{w}\phi)$ *is affine.*

Proof. Indeed, by Proposition 8.37 and Lemma 8.24, it is the quotient of an affine variety by a finite group, so is affine. □

Shintani descent identity. In this subsection we give a formula for the Leftschetz number of a variety $\mathbf{X}(\mathbf{I},\mathbf{w}F)$ which we deduce from a "Shintani descent identity".

Let m be a multiple of δ, and let $\mathcal{H}_{q^m}(W) := \mathrm{End}_{\mathbf{G}^{F^m}}(\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{B})^{F^m}])$. Let $e_{\mathbf{B}} = |\mathbf{B}^{F^m}|^{-1}|\sum_{b\in\mathbf{B}^{F^m}} b$; the $\mathbf{G}^{F^m}$-module $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{B})^{F^m}]$ identifies with $\overline{\mathbb{Q}}_\ell[\mathbf{G}^{F^m}]e_{\mathbf{B}}$ and $\mathcal{H}_{q^m}(W)$ with $e_{\mathbf{B}}\overline{\mathbb{Q}}_\ell[\mathbf{G}^{F^m}]e_{\mathbf{B}}$ acting by right multiplication. It has a basis consisting of the operators $T_w = |\mathbf{B}^{F^m}\cap{}^w\mathbf{B}^{F^m}|\sum_{g\in\mathbf{B}^{F^m}w\mathbf{B}^{F^m}} g = e_{\mathbf{B}}we_{\mathbf{B}}$ for $w\in W$, since W is a set of representatives of $\mathbf{B}^{F^m}\backslash\mathbf{G}/\mathbf{B}^{F^m}$ (see [Bou] IV, §2 exercice 22). If we identify $\mathbf{G}/\mathbf{B}$ to the variety $\mathcal{B}$ of the Borel subgroups of $\mathbf{G}$, the operator T_w becomes

$$T_w : \mathbf{B}'\mapsto \sum_{\{\mathbf{B}''\in\mathcal{B}^{F^m}\mid \mathbf{B}''\xrightarrow{w}\mathbf{B}'\}}\mathbf{B}''.$$

Similarly the algebra $\mathcal{H}_{q^m}(W,W_I) := \mathrm{End}_{\mathbf{G}^{F^m}}(\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}])$ has a $\overline{\mathbb{Q}}_\ell$-basis consisting of the operators $X_w = |\mathbf{P}_I^{F^m}\cap{}^w\mathbf{P}_I^{F^m}|\sum_{g\in\mathbf{P}_I^{F^m}w\mathbf{P}_I^{F^m}} g = e_{\mathbf{P}_I}we_{\mathbf{P}_I}$ where $e_{\mathbf{P}_I} = |\mathbf{P}_I^{F^m}|^{-1}\sum_{p\in\mathbf{P}_I^{F^m}} p$ and w runs over a set of representatives of the double cosets $\mathbf{P}_I^{F^m}\backslash\mathbf{G}^{F^m}/\mathbf{P}_I^{F^m}\simeq W_I\backslash W/W_I$. Identifying $\mathbf{G}/\mathbf{P}_I$ to the variety $\mathcal{P}_I$ of the parabolic subgroups $\mathbf{G}$-conjugate to $\mathbf{P}_I$ we have

$$X_w : \mathbf{P}\mapsto \sum_{\{\mathbf{P}'\in\mathcal{P}_I^{F^m}\mid \mathbf{P}'\xrightarrow{I,w,I}\mathbf{P}\}}\mathbf{P}',$$

The multiplication by the idempotent $X_1 = e_{\mathbf{P}_I} = \sum_{v\in W_I}|\mathbf{B}^{F^m}\cap{}^v\mathbf{B}^{F^m}|^{-1}T_v$ makes $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]$ into a direct factor of $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{B})^{F^m}]$ and the equality $X_w = X_1T_wX_1$ is compatible with this inclusion. Note that this inclusion maps a parabolic $\mathbf{P}$ conjugate to $\mathbf{P}_I$ in $\mathbf{G}^{F^m}$ to the sum of all F^m-stable Borel subgroups of $\mathbf{P}$.

We may define a $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{End}_{B^+(\mathcal{I})}(\mathbf{I})$ on $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]$ by sending $\mathbf{I}\xrightarrow{\mathbf{w}}\mathbf{I}$ to the operator $X_{\mathbf{w}}\in\mathcal{H}(W,W_I)$ defined by

$$X_{\mathbf{w}}(\mathbf{P}) = \sum_{\{x\in\mathcal{O}(\mathbf{I},\mathbf{w})^{F^m}\mid p''(x)=\mathbf{P}\}} p'(x).$$

In the particular case where $I = \emptyset$ we get an operator denoted by $T_{\mathbf{w}}$, defined for any $\mathbf{w}$ in B^+. For $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{I} \in \mathrm{End}_{B^+(\mathcal{I})}(\mathbf{I})$ the operator $X_{\mathbf{w}}$ identifies to $X_1 T_{\mathbf{w}} X_1 = X_1 T_{\mathbf{w}}$ since $\mathbf{I}^{\mathbf{w}} = \mathbf{I}$.

Similarly, if $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$ is a morphism of $B^+(\mathcal{I})$, we may associate to it an endomorphism $X_{\mathbf{w}\phi}$ of $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]$ by the formula

$$X_{\mathbf{w}\phi}(\mathbf{P}) = \sum_{\{x \in \mathcal{O}(\mathbf{I},\mathbf{w})^{F^m} \mid p''(x) = F(\mathbf{P})\}} p'(x).$$

When $\phi(I) = I$ we have $X_{\mathbf{w}\phi} = X_{\mathbf{w}}\phi$. In general we have $X_{\mathbf{w}\phi} = X_1 T_{\mathbf{w}}\phi$ on $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]$ seen as a subspace of $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{B})^{F^m}]$: on the latter representation one can separate the action of F; the operator F sends the submodule $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]$ to $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_{\phi(I)})^{F^m}]$ which is sent back to $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]$ by $X_1 T_{\mathbf{w}}$. The endomorphism $X_{\mathbf{w}\phi}$ commutes with $\mathbf{G}^{F^m}$ like F, hence normalizes $\mathcal{H}_{q^m}(W, W_I)$; its action identifies to the conjugation action of $T_{\mathbf{w}}\phi$ on $\mathcal{H}_{q^m}(W, W_I)$ inside $\mathcal{H}_{q^m}(W) \rtimes \langle\phi\rangle$.

Recall that $\mathrm{Sh}_{F^m/F}$ denotes the "norm" map which maps the F-class of $g' = h.{}^F h^{-1} \in \mathbf{G}^{F^m}$ to the class of $g = h^{-1}.{}^{F^m}h \in \mathbf{G}^F$.

Proposition 8.39 (Shintani descent identity)**.** *Let $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$ be a morphism of $B^+(\mathcal{I})$, and let m be a multiple of δ. Then*

$$(g \mapsto |\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m}|) = \mathrm{Sh}_{F^m/F}(g' \mapsto \mathrm{Trace}(g' X_{\mathbf{w}\phi} \mid \overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]).$$

Proof. Let $g = h^{-1}.{}^{F^m}h$ and $g' = h.{}^F h^{-1}$, so that the class of g is $\mathrm{Sh}_{F^m/F}$ of the F-class of g'; we have $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m} = \{x \in \mathcal{O}(\mathbf{I}, \mathbf{w}) \mid {}^{F^m h}x = {}^h x \text{ and } p''({}^h x) = {}^{g'F}p'({}^h x)\}$. Taking ${}^h x$ as a variable in the last formula we get that $|\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m}| = |\{x \in \mathcal{O}(\mathbf{I}, \mathbf{w})^{F^m} \mid p''(x) = {}^{g'F}p'(x)\}|$. Putting $\mathbf{P} = p'(x)$ this last number becomes $\sum_{\mathbf{P} \in \mathcal{P}_I^{F^m}} |\{x \in \mathcal{O}(\mathbf{I}, \mathbf{w})^{F^m} \mid p'(x) = \mathbf{P} \text{ and } p''(x) = {}^{g'F}\mathbf{P}\}|$. On the other hand the trace of $g' X_{\mathbf{w}\phi}$ is the sum over $\mathbf{P} \in \mathcal{P}_I^{F^m}$ of the coefficient of $\mathbf{P}$ in $\sum_{\{x \in \mathcal{O}(\mathbf{I},\mathbf{w})^{F^m} \mid p''(x) = F(\mathbf{P})\}} g' p'(x)$. This coefficient is equal to $|\{x \in \mathcal{O}(\mathbf{I}, \mathbf{w})^{F^m} \mid g' p'(x) = \mathbf{P} \text{ and } p''(x) = {}^F\mathbf{P}\}| = |\{x \in \mathcal{O}(\mathbf{I}, \mathbf{w})^{F^m} \mid p'(x) = \mathbf{P} \text{ and } p''(x) = {}^{g'F}\mathbf{P}\}|$, this last equality by changing $g'x$ into x. □

Corresponding to the specialization $q^{m/2} \mapsto 1 : \mathcal{H}_{q^m}(W) \to \overline{\mathbb{Q}}_\ell W$, there is a bijection $\chi \mapsto \chi_{q^m} : \mathrm{Irr}(W) \to \mathrm{Irr}(\mathcal{H}_{q^m}(W))$. If we choose an extension $\tilde{\chi}$ to $W \rtimes \langle\phi\rangle$ of each character in $\mathrm{Irr}(W)^\phi$, we get a corresponding extension $\tilde{\chi}_{q^m} \in \mathrm{Irr}(\mathcal{H}_{q^m}(W) \rtimes \langle\phi\rangle)$, and if $U_\chi \in \mathrm{Irr}(\mathbf{G}^{F^m})$ is the corresponding character of $\mathbf{G}^{F^m}$, we get a corresponding extension $U_{\tilde{\chi}}$ of U_χ to $\mathbf{G}^{F^m} \rtimes \langle F\rangle$ (see [DM1, III théorème 1.3]). With these notations, the Shintani descent identity becomes

Proposition 8.40.

$$(g \mapsto |\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m}|) = \sum_{\chi \in \mathrm{Irr}(W)^\phi} \tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}}\phi)\, \mathrm{Sh}_{F^m/F}\, U_{\tilde{\chi}}$$

and the only characters χ in that sum which give a non-zero contribution are those which are a component of $\mathrm{Ind}_{W_I}^W \mathrm{Id}$.

Proof. We have $\mathrm{Trace}(g' X_{\mathbf{w}\phi} \mid \overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]) = \mathrm{Trace}(g' X_1 T_{\mathbf{w}}\phi \mid \overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{B})^{F^m}])$ since X_1 is the projector onto $\overline{\mathbb{Q}}_\ell[(\mathbf{G}/\mathbf{P}_I)^{F^m}]$. Hence $(g \mapsto |\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m}|) = \sum_{\chi \in \mathrm{Irr}(W)^\phi} \tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}}\phi)\, \mathrm{Sh}_{F^m/F}\, U_{\tilde{\chi}}$. Since X_1 acts by 0 on the representation of character χ if χ is not a component of $\mathrm{Ind}_{W_I}^W \mathrm{Id}$, we get the second assertion. □

Finally, if λ_ρ is the root of unity attached to $\rho \in \mathcal{E}(\mathbf{G}^F, 1)$ as in [DMR, 3.3.4], the above formula translates, using [DMR, 3.3.7] as

Corollary 8.41.

$$|\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)^{gF^m}| = \sum_{\rho \in \mathcal{E}(\mathbf{G}^F,1)} \lambda_\rho^{m/\delta} \rho(g) \sum_{\{\chi \in \mathrm{Irr}(W)^\phi \mid \langle \mathrm{Res}^W_{W_I} \chi, \mathrm{Id}\rangle_{W_I} \neq 0\}} \tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}}\phi)\langle \rho, R_{\tilde{\chi}}\rangle_{\mathbf{G}^F}$$

where $R_{\tilde{\chi}} = |W|^{-1} \sum_{w \in W} \tilde{\chi}(w\phi) R^{\mathbf{G}}_{\mathbf{T}_w}(\mathrm{Id})$, and, using the Lefschetz formula and taking the "limit for $m \to 0$" (see for example [DMR, 3.3.8]) we get the equality of virtual characters

Corollary 8.42.

$$\sum_i (-1)^i H^i_c(\mathbf{X}(\mathbf{I}, \mathbf{w}\phi), \overline{\mathbb{Q}}_\ell) = \sum_{\{\chi \in \mathrm{Irr}(W)^\phi \mid \langle \mathrm{Res}^W_{W_I} \chi, \mathrm{Id}\rangle_{W_I} \neq 0\}} \tilde{\chi}(x_1 w\phi) R_{\tilde{\chi}},$$

where w is the image of $\mathbf{w}$ in W and $x_1 = |W_I|^{-1} \sum_{v \in W_I} v$.

9. Eigenspaces and roots of $\boldsymbol{\pi/\pi_{\mathbf{I}}}$

Let $\ell \neq p$ be a prime such that the ℓ-Sylow S of $\mathbf{G}^F$ is abelian.

Then "generic block theory" (see [BMM]) associates to ℓ a root of unity ζ and some $w\phi \in W\phi$ such that its ζ-eigenspace in V in $X := X_{\mathbb{R}} \otimes \mathbb{C}$ is non-zero and maximal among ζ-eigenspaces of elements of $W\phi$; for any such ζ, there exists a unique minimal subtorus $\mathbf{S}$ of $\mathbf{T}$ such that $V \subset X(\mathbf{S}) \otimes \mathbb{C}$. If the coset $W\phi$ is rational $X(\mathbf{S}) \otimes \mathbb{C}$ is the kernel of $\Phi_d(w\phi)$, where d is the order of ζ. Otherwise, in the "very twisted" cases ${}^2B_2, {}^2F_4$ (resp. ${}^2G_2$) we have to replace Φ_d by the irreducible cyclotomic polynomial Φ over $\mathbb{Q}(\sqrt{2})$ (resp. $\mathbb{Q}(\sqrt{3})$) of which ζ is a root. The torus $\mathbf{S}$ is then called a Φ-Sylow; we have $|\mathbf{S}^F| = \Phi(q)^{\dim V}$.

The relationship with ℓ is that S is a subgroup of $\mathbf{S}^F$, and thus that $|\mathbf{G}^F|/|\mathbf{S}^F|$ is prime to ℓ; we have $N_{\mathbf{G}^F}(S) = N_{\mathbf{G}^F}(\mathbf{S}) = N_{\mathbf{G}^F}(\mathbf{L})$ where $\mathbf{L} := C_{\mathbf{G}}(\mathbf{S})$ is a Levi subgroup of $\mathbf{G}$ whose Weyl group is $C_W(V)$. Conversely, any maximal ζ-eigenspace for any ζ determines some primes ℓ with abelian Sylow, those which divide $\Phi(q)^{\dim V}$ and no other cyclotomic factor of $|\mathbf{G}^F|$.

The classes $C_W(V)w\phi$, where $V = \mathrm{Ker}(w\phi - \zeta)$ is maximal, form a single orbit under W-conjugacy [see eg. [Br, 5.6(i)]]; the maximality implies that all elements of $C_W(V)w\phi$ have same ζ-eigenspace.

We will see in Theorem 9.1(i) that up to conjugacy we may assume that $C_W(V)$ is a standard parabolic group W_I; then the Broué conjectures predict that for an appropriate choice of coset $C_W(V)w\phi$ in its $N_W(W_I)$-conjugacy class the cohomology complex of the variety $\mathbf{X}(\mathbf{I}, \mathbf{w}\phi)$ should be a tilting complex realizing a derived equivalence between the unipotent parts of the ℓ-principal blocks of $\mathbf{G}^F$ and of $N_{\mathbf{G}^F}(\mathbf{S})$. We want to describe explicitly what should be a "good" choice of w.

Since it is no more effort to have a result in the context of any finite real reflection group than for a context which includes the Ree and Suzuki groups, we give a more general statement.

In what follows we look at real reflection cosets $W\phi$ of finite order, that is W is a finite reflection group acting on the real vector space $X_{\mathbb{R}}$ and ϕ is an element of $N_{\mathrm{GL}(X_{\mathbb{R}})}(W)$, such that $W\phi$ is of finite order δ, that is δ is the smallest integer such that $(W\phi)^\delta = W$ (equivalently ϕ is of finite order). Since W is transitive on the chambers of the real hyperplane arrangement it determines, one can always

choose ϕ so that it preserves a chamber of this arrangement. Such elements ϕ are the 1-regular elements of the coset, thus are of order δ.

Theorem 9.1. *Let $W\phi \subset \mathrm{GL}(X_{\mathbb{R}})$ be a finite order real reflection coset, such that ϕ preserves a chamber of the hyperplane arrangement on $X_{\mathbb{R}}$ determined by W, thus induces an automorphism of the Coxeter system (W,S) determined by this chamber. We call again ϕ the induced automorphism of the braid group B of W, and denote by $\mathbf{S},\mathbf{W}$ the lifts of S,W to B (see around Definition 8.2).*

Let $\zeta_d = e^{2i\pi/d}$ and let V be a subspace of $X := X_{\mathbb{R}} \otimes \mathbb{C}$ on which some element of $W\phi$ acts by ζ_d. Then we may choose V in its W-orbit such that:

(i) *$C_W(V) = W_I$ for some $I \subset S$.*
(ii) *If $W_I w\phi$ is the W_I-coset of elements which act by ζ_d on V, where w is I-reduced, then when $d \neq 1$ we have $l(w) = (2/d)(l(w_0 w_I^{-1}))$ and $l((w\phi)^i \phi^{-i}) = il(w)$ if $2i \leq d$.*

Further, when $d \neq 1$ the lift $\mathbf{w} \in \mathbf{W}$ of a w as in (ii) satisfies ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$ and $(\mathbf{w}\phi)^d = \phi^d \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$, where $\mathbf{I} \subset \mathbf{S}$ is the lift of I.

Finally note that if $d = 1$ then $w = 1$ in (ii) and we may lift it to $\mathbf{w} := \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$ and we still have ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$ and $(\mathbf{w}\phi)^d = \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}\phi^d$.

Note that in particular, for the w in (ii) we have $(w\phi)^d = \phi^d$.

Proof. Since $W\langle\phi\rangle$ is finite, we may find a scalar product on $X_{\mathbb{R}}$ (extending to an Hermitian product of X) invariant by W and ϕ. The subspace $X'_{\mathbb{R}}$ of $X_{\mathbb{R}}$ on which W acts non-trivially (the subspace spanned by the root lines of W) identifies to the reflection representation of the Coxeter system (W,S) (see for example [Bou, chap. 5, §3]). We will use the root system Φ on $X'_{\mathbb{R}}$ consisting of the vectors of length 1 for this scalar product along the root lines of W, which is thus preserved by $W\langle\phi\rangle$. The strategy for the proof of (i) will be, rather than change V, to choose an order on Φ such that the corresponding basis makes $C_W(V)$ a standard parabolic subgroup of W.

Let $v \in V$ be a generic vector, that is such that $C_W(v) = C_W(V)$. Multiplying v if needed by a complex number of absolute value 1, we may assume that for any $\alpha \in \Phi$ we have $\Re\langle v,\alpha\rangle = 0$ if and only if $\langle v,\alpha\rangle = 0$. Then there exists an order on Φ such that $\Phi^+ \subset \{\alpha \in \Phi \mid \Re(\langle v,\alpha\rangle) \geq 0\}$. Let Π be the corresponding basis and let $I = \{\alpha \in \Pi | \Re(\langle v,\alpha\rangle) = 0\}$. Then for $\alpha \in \Phi$ we have $\alpha \in \Phi_I$ if and only if $\langle v,\alpha\rangle = 0$, thus $C_W(V) = C_W(v) = W_I$. This proves (i).

We prove now (ii). The element $w\phi$ sends v to $\zeta_d v$, thus preserves Φ_I, and since we chose w to be I-reduced we have ${}^{w\phi}I = I$.

Note that $(w\phi)^d = \phi^d$. Indeed $(w\phi)^d$ fixes v, thus preserves the sign of any root not in Φ_I; as ${}^{w\phi}I = I$, it also preserves the sign of roots in Φ_I. It is thus equal to the only element ϕ^d of $W\phi^d$ which preserves the signs of all roots. We get also that ${}^{\phi^d}I = I$.

Since $\langle v, {}^{(w\phi)^m}\alpha\rangle = \langle {}^{(w\phi)^{-m}}v, \alpha\rangle = \zeta_d^{-m}\langle v,\alpha\rangle$, we get that all orbits of $w\phi$ on $\Phi - \Phi_I$ have cardinality a multiple of d; it is thus possible by partitioning suitably those orbits, to get a partition of $\Phi - \Phi_I$ in subsets $\mathcal{O}$ of the form $\{\alpha, {}^{w\phi}\alpha, \ldots, {}^{(w\phi)^{d-1}}\alpha\}$; and the numbers $\{\langle v,\beta\rangle \mid \beta \in \mathcal{O}\}$ for a given $\mathcal{O}$ form the vertices of a regular d-gon centered at $0 \in \mathbb{C}$; the action of $w\phi$ is the rotation by $-2\pi/d$ of this d-gon. Looking at the real parts of the vertices of this d-gon, we see that for $m \leq d/2$, exactly m positive roots in $\mathcal{O}$ are sent to negative roots by $(w\phi)^m$. Since this holds for all $\mathcal{O}$,

we get that for $m \leq d/2$ we have $l(\phi^{-m}(w\phi)^m) = \frac{m|\Phi-\Phi_I|}{d}$; thus if $\mathbf{w}$ is the lift of w to $\mathbf{W}$ we have $(\mathbf{w}\phi)^i \in \mathbf{W}\phi^i$ if $2i \leq d$.

If $d = 1$ since $w\phi = \phi$ we have $w = 1$ so we may lift it to $\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$ as stated. Otherwise we finish with the following

Lemma 9.2. *Assume that ${}^{w\phi}W_I = W_I$, that w is I-reduced, that $(w\phi)^d = \phi^d$ and that $l((w\phi)^i\phi^{-i}) = (2i/d)l(w_0w_I^{-1})$ if $2i \leq d$. Then if $\mathbf{w}$ is the lift of w to $\mathbf{W}$ we have ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$ and if $d \neq 1$ we have $(\mathbf{w}\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$.*

Proof. Since w is I-reduced and $w\phi$ normalizes W_I we get that $w\phi$ stabilizes I, which lifts to the braid group as ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$.

Assume first d even and let $d = 2d'$ and $x = \phi^{-d'}(w\phi)^{d'}$. Then $l(x) = (1/2)l(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}) = l(w_0) - l(w_I)$ and since x is reduced-I it is equal to the only reduced-I element of that length which is $w_0w_I^{-1}$. Since the lengths add we can lift the equality $(w\phi)^{d'} = \phi^{d'}w_0w_I^{-1}$ to the braid monoid as $(\mathbf{w}\phi)^{d'} = \phi^{d'}\mathbf{w}_0\mathbf{w}_I^{-1}$. By a similar reasoning using that $(w\phi)^{d'}\phi^{-d'}$ is the unique I-reduced element of its length, we get also $(\mathbf{w}\phi)^{d'} = \mathbf{w}_I^{-1}\mathbf{w}_0\phi^{d'}$. Thus $(\mathbf{w}\phi)^d = \mathbf{w}_I^{-1}\mathbf{w}_0\phi^{d'}\phi^{d'}\mathbf{w}_0\mathbf{w}_I^{-1} = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$, where the last equality uses that $\phi^d = (w\phi)^d$ preserves $\mathbf{I}$, whence the lemma in this case.

Assume now that $d = 2d' + 1$; then $(w\phi)^{d'}\phi^{-d'}$ is I-reduced and $\phi^{-d'}(w\phi)^{d'}$ is reduced-I. Using that any reduced-$\mathbf{I}$ simple of B^+ is a right divisor of $\mathbf{w}_0\mathbf{w}_I^{-1}$ (resp. any $\mathbf{I}$-reduced simple is a left divisor of $\mathbf{w}_I^{-1}\mathbf{w}_0$), we get that there exists simples $\mathbf{t}, \mathbf{u}$ such that $\phi^{d'}\mathbf{w}_I^{-1}\mathbf{w}_0 = \mathbf{t}(\mathbf{w}\phi)^{d'}$ and $\mathbf{w}_0\mathbf{w}_I^{-1}\phi^{d'} = (\mathbf{w}\phi)^{d'}\mathbf{u}$. Thus $\phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}} = \mathbf{w}_0\mathbf{w}_I^{-1}\phi^d\mathbf{w}_I^{-1}\mathbf{w}_0 = (\mathbf{w}\phi)^{d'}\mathbf{u}\phi\mathbf{t}(\mathbf{w}\phi)^{d'}$, the first equality since ${}^{\phi^d}I = I$. The image in $W\phi^d$ of the left-hand side is ϕ^d, and $(w\phi)^d = \phi^d$. We deduce that the image in $W\phi$ of $\mathbf{u}\phi\mathbf{t}$ is $w\phi$. If $d \neq 1$ then $d' \neq 0$ and we have $l(\mathbf{u}) = l(\mathbf{t}) = l(\mathbf{w})/2$; thus $\mathbf{u}\phi\mathbf{t} = \mathbf{w}\phi$ and $(\mathbf{w}\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$. □

□

Note that Theorem 9.1 only handles the case of eigenspaces for the eigenvalue ζ_d, and not for another primitive d-th root of unity ζ_d^k. However, note that if the coset $W\phi$ preserves a $\mathbb{Q}$-structure on $X_{\mathbb{R}}$ (which is the case for cosets associated to finite reductive groups, excepted for the "very twisted" cases ${}^2B_2, {}^2G_2$ and ${}^2F_4$), then if ζ_d^k is an eigenvalue of $w\phi$, the Galois conjugate ζ_d is also an eigenvalue, for a Galois conjugate eigenspace. In general, since we assume $W\phi$ real, we may assume $2k \leq d$ since if ζ_d^k is an eigenvalue of $w\phi$ the complex conjugate ζ_d^{d-k} is also an eigenvalue, for the complex conjugate eigenspace. In this last case we may say the following (here we assume $d \neq 1$):

Corollary 9.3. *In the situation of Theorem 9.1, let $\zeta = \zeta_d^k$ with k prime to d and to the order δ of ϕ with $2k \leq d$ and let V be a subspace of X on which some element of $W\phi$ acts by ζ. Then we may choose V in its W-orbit such that:*

(i) *$C_W(V) = W_I$ for some $I \subset S$.*
(ii) *If $W_Iw\phi$ is the W_I-coset of elements which act by ζ on V, and w is the unique I-reduced element of that coset, then $l(w) = (2k/d)(l(w_0w_I^{-1}))$ and $l((w\phi)^i\phi^{-i}) = il(w)$ if $2ik \leq d$.*

Further, if $\mathbf{w}$ is the lift of w as in (ii) to $\mathbf{W}$ and $\mathbf{I} \subset \mathbf{S}$ is the lift of I, then ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$ and $(\mathbf{w}\phi)^d = \phi^d(\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}})^k$.

Proof. The proof of (i) in Theorem 9.1 does not use that the eigenvalue is ζ_d, so still applies.

Let k' be the multiplicative inverse of $k \pmod{\operatorname{lcm}(d,\delta)}$, and let $w_1\phi_1 = (w\phi)^{k'}$, where $\phi_1 = \phi^{k'}$. Then $w_1\phi_1$ has V as a ζ_d-eigenspace, so we may apply Theorem 9.1 to it. The same argument as at the beginning of the proof of Theorem 9.1(ii) shows that $(w\phi)^d = \phi^d$ and ${}^{\phi^d}I = I$. Thus the order of $w\phi$ is $\operatorname{lcm}(d,\delta)$ and thus $w\phi = (w_1\phi_1)^k$; we have also $W_I w\phi = (W_I w_1\phi_1)^k$, whence (ii).

Finally, by Theorem 9.1 the lift $\mathbf{w}_1$ of w_1 to B satisfies ${}^{\mathbf{w}_1\phi_1}\mathbf{I} = \mathbf{I}$ and $(\mathbf{w}_1\phi_1)^d = \phi_1^d\boldsymbol{\pi}/\boldsymbol{\pi}_I$, thus if we define $\mathbf{w}$ by $(\mathbf{w}_1\phi_1)^k = \mathbf{w}\phi$, then $\mathbf{w}$ is the lift of w and satisfies the last part of the corollary, using ${}^{\phi^d}I = I$. □

The condition above that k is prime to δ seems a limitation but we do not know of an example where $w\phi$ has an eigenvalue ζ_d^k with k prime to d but not prime to δ and does not have another Galois-conjugate eigenvalue $\zeta_d^{k'}$ with k' prime to δ.

We give now a converse.

Theorem 9.4. *Let (W,S), ϕ, $X_{\mathbb{R}}$, X, $\mathbf{S}, B, B^+$ be as in Theorem 9.1 For $d \in \mathbb{N}$, let $\mathbf{w} \in B^+$ and $\mathbf{I} \subset \mathbf{S}$ be such that:*

(i) ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$.
(ii) $(\mathbf{w}\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$.

Denote by w and I the images in W of $\mathbf{w}$ and $\mathbf{I}$, let $\zeta_d = e^{2i\pi/d}$, let $V \subset X$ be the ζ_d-eigenspace of $w\phi$, and let X^{W_I} be the fixed point space of W_I; then $W_I = C_W(X^{W_I} \cap V)$, in particular $C_W(V) \subset W_I$.

Further, the following two assertions are equivalent:

(iii) *$\mathbf{w}$ is maximal with respect to (i) and (ii) above, that is, there do not exist $\mathbf{J} \subsetneq \mathbf{I}$ and $\mathbf{v} \in B^+_{\mathbf{I}}$ such that ${}^{\mathbf{v}\mathbf{w}\phi}\mathbf{J} = \mathbf{J}$ and $(\mathbf{v}\mathbf{w}\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}$.*
(iv) *No element of the coset $W_I w\phi$ has a non-zero ζ_d-eigenvector on the subspace spanned by the root lines of W_I.*

Proof. We first notice that conditions (i) and (ii) are equivalent to require that in the category $B^+(\mathcal{I})$ the morphism $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}\phi$ is a d-th root of $\Delta^2_{\mathcal{I}}$. In our setting Lemma 7.2 thus reduces to the following generalization of [BM, lemme 6.9]

Lemma 9.5. *Let $\mathbf{w} \in B^+$ and $\mathbf{I} \subset \mathbf{S}$ be such that ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$ and $(\mathbf{w}\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$. Then there exists $\mathbf{v} \in (B^+)^{\phi^d}$ such that $(\mathbf{w}\phi)^{\mathbf{v}} \in B^+\phi$, $\mathbf{I}^{\mathbf{v}} \subset \mathbf{S}$ and $((\mathbf{w}\phi)^{\mathbf{v}})^{\lfloor \frac{d}{2} \rfloor} \in \mathbf{W}\phi^{\lfloor \frac{d}{2} \rfloor}$. Further, $\operatorname{ad}\mathbf{v}$ defines a morphism in $\mathcal{D}^+(\mathcal{I})^{\phi^d}$ (that is, the conjugation is by "ϕ^d-stable cyclic permutations").*

Thus if we define $\mathbf{w}'$ and $\mathbf{J}$ by $(\mathbf{w}\phi)^{\mathbf{v}} = \mathbf{w}'\phi$ and $\mathbf{I}^{\mathbf{v}} = \mathbf{J}$, we have $(\mathbf{w}'\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}$ and ${}^{\mathbf{w}'\phi}\mathbf{J} = \mathbf{J}$.

As the result to prove in Theorem 9.4 is invariant by a conjugacy in B which sends $\mathbf{w}\phi$ to $B^+\phi$ and $\mathbf{I}$ to another subset of $\mathbf{S}$, we may replace $(\mathbf{w}\phi, \mathbf{I})$ by a conjugate as in Lemma 9.5, thus assume that w and I satisfy the assumptions of the next lemma.

To state the next lemma we extend the length function from W to $W \rtimes \langle\phi\rangle$ by setting $l(w\phi^i) = l(w)$.

Lemma 9.6. *Let $w \in W, I \subset S$ be such that $(w\phi)^d = \phi^d$, ${}^{w\phi}I = I$ and such that $l((w\phi)^i) = \frac{2i}{d}l(w_I^{-1}w_0)$ for any $i \leq d/2$. We have*

(i) *If Φ be a ϕ-stable root system for W (as in the proof of Theorem 9.1), then $\Phi - \Phi_I$ is the disjoint union of sets of the form $\{\alpha, {}^{w\phi}\alpha, \dots, {}^{(w\phi)^{d-1}}\alpha\}$ with $\alpha, {}^{w\phi}\alpha, \dots, {}^{(w\phi)^{\lfloor d/2\rfloor -1}}\alpha$ of same sign and ${}^{(w\phi)^{\lfloor d/2\rfloor}}\alpha, \dots, {}^{(w\phi)^{d-1}}\alpha$ of the opposite sign.*

(ii) *The order of $w\phi$ is $\mathrm{lcm}(d, \delta)$.*

(iii) *If $d > 1$, then $W_I = C_W(X^{W_I} \cap \ker(w\phi - \zeta_d))$.*

Proof. The statement is empty for $d = 1$ so in the following proof we assume $d > 1$.

For $x \in W \rtimes \langle\phi\rangle$ let $N(x) = \{\alpha \in \Phi^+ \mid {}^x\alpha \in \Phi^-\}$; it is well known that for $x \in W$ we have $l(x) = |N(x)|$. This still holds for $x = w\phi^i \in W \rtimes \langle\phi\rangle$ since $N(w\phi^i) = {}^{\phi^{-i}}N(w)$. It follows that for $x, y \in W \rtimes \langle\phi\rangle$ we have $l(xy) = l(x) + l(y)$ if and only if $N(xy) = N(y) \coprod {}^{y^{-1}}N(x)$. In particular $l((w\phi)^i) = il(w\phi)$ for $i \leq d/2$ implies ${}^{(w\phi)^{-i}}N(w\phi) \subset \Phi^+$ for $i \leq d/2 - 1$.

Let us partition each $w\phi$-orbit in $\Phi - \Phi_I$ into "pseudo-orbits" of the form $\{\alpha, {}^{w\phi}\alpha, \dots, {}^{(w\phi)^{k-1}}\alpha\}$, where k is minimal such that ${}^{(w\phi)^k}\alpha = {}^{\phi^k}\alpha$ (then k divides d); a pseudo-orbit is an orbit if $\phi = 1$. The action of $w\phi$ defines a cyclic order on each pseudo-orbit. The previous paragraph shows that when there is a sign change in a pseudo-orbit, at least the next $\lfloor d/2\rfloor$ roots for the cyclic order have the same sign. On the other hand, as ϕ^k preserves Φ^+, each pseudo-orbit contains an even number of sign changes. Thus if there is at least one sign change we have $k \geq 2\lfloor d/2\rfloor$. Since k divides d, we must have $k = d$ for pseudo-orbits which have a sign change, and then they have exactly two sign changes. As the total number of sign changes is $2l(w) = 2|\Phi - \Phi_I|/d$, there are $|\Phi - \Phi_I|/d$ pseudo-orbits with sign changes; their total cardinality is $|\Phi - \Phi_I|$, thus there are no other pseudo-orbits and up to a cyclic permutation we may assume that each pseudo-orbit consists of $\lfloor d/2\rfloor$ roots of the same sign followed by $d - \lfloor d/2\rfloor$ of the opposite sign. We have proved (i).

Let $d' = \mathrm{lcm}(d, \delta)$. The proof of (i) shows that the order of $w\phi$ is a multiple of d. Since the order of $(w\phi)^d = \phi^d$ is d'/d, we get (ii).

We now prove (iii). Let $V = \ker(w\phi - \zeta_d)$. Since $W\langle\phi\rangle$ is finite, we may find a scalar product on X invariant by W and ϕ. We have then $X^{W_I} = \Phi_I^\perp$. The map $p = \frac{1}{d'}\sum_{i=0}^{d'-1} \zeta_d^{-i}(w\phi)^i$ is the (unique up to scalar) $w\phi$-invariant projector on V, thus is the orthogonal projector on V.

We claim that $p(\alpha) \notin < \Phi_I >$ for any $\alpha \in \Phi - \Phi_I$. As $p((w\phi)^i\alpha) = \zeta_d^i p(\alpha)$ it is enough to assume that α is the first element of a pseudo-orbit; replacing if needed α by $-\alpha$ we may even assume $\alpha \in \Phi^+$. Looking at imaginary parts, we have $\Im(\zeta_d^i) \geq 0$ for $0 \leq i < \lfloor d/2\rfloor$, and $\Im(\zeta_d^i) < 0$ for $\lfloor d/2\rfloor \leq i < d$. Let λ be a linear form such that λ is 0 on Φ_I and is real strictly positive on $\Phi^+ - \Phi_I$; we have $\lambda({}^{(w\phi)^i}\alpha) > 0$ for $0 \leq i < \lfloor d/2\rfloor$, and $\lambda({}^{(w\phi)^i}\alpha) < 0$ for $\lfloor d/2\rfloor \leq i < d$; it follows that $\Im(\lambda(\zeta_d^i {}^{(w\phi)^i}\alpha)) > 0$ for all elements of the pseudo-orbit. If $d' = d$ we have thus $\Im(\lambda(p(\alpha))) > 0$, in particular $p(\alpha) \notin < \Phi_I >$. If $d' > d$, since $\phi^d\alpha$ is also a positive root and the first term of the next pseudo-orbit the same computation applies to the other pseudo-orbits and we conclude the same way.

Now $C_W(X^{W_I} \cap V)$ is generated by the reflections whose root is orthogonal to $X^{W_I} \cap V$, that is whose root is in $< \Phi_I > + V^\perp$. If α is such a root we have $p(\alpha) \in < \Phi_I >$, whence $\alpha \in \Phi_I$ by the above claim. This proves that $C_W(X^{W_I} \cap V) \subset W_I$. Since the reverse inclusion is true, we get (iii). □

We return to the proof of Theorem 9.4. Assertion (iii) of Lemma 9.6 gives the first assertion of the theorem. We now show ¬(iii)⇒ ¬(iv). If $\mathbf{w}$ is not maximal, there exists $\mathbf{J} \subsetneq \mathbf{I}$ and $\mathbf{v} \in B_{\mathbf{I}}^+$ such that ${}^{\mathbf{vw}\phi}\mathbf{J} = \mathbf{J}$ and $(\mathbf{vw}\phi)^d = \phi^d \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}$. If we denote ψ the automorphism of $B_{\mathbf{I}}$ induced by the automorphism $\mathbf{w}\phi$ of $\mathbf{I}$, we have ${}^{\mathbf{v}\psi}\mathbf{J} = \mathbf{J}$ and $(\mathbf{v}\psi)^d = \psi^d \boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}}$. Let X_I be the subspace of X spanned by Φ_I. It follows from the first part of the theorem applied with X, ϕ, $\mathbf{w}$ and w respectively replaced with X_I, ψ, $\mathbf{v}$ and v that $v\psi = vw\phi$ has a non-zero ζ_d-eigenspace in X_I, since if V' is the ζ_d-eigenspace of $vw\phi$ we get $C_{W_I}(V') \subset W_J \subsetneq W_I$; this contradicts (iv).

We show finally that ¬(iv)⇒ ¬(iii). If some element of $W_I\psi$ has a non-zero ζ_d-eigenvector on X_I, by Theorem 9.1 applied to $W_I\psi$ acting on X_I we get the existence of $\mathbf{J} \subsetneq \mathbf{I}$ and $\mathbf{v} \in B_{\mathbf{I}}^+$ satisfying ${}^{\mathbf{v}\psi}\mathbf{J} = \mathbf{J}$ and $(\mathbf{v}\psi)^d = \psi^d \boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}}$. Using that $(\mathbf{w}\phi)^d = \phi^d \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}$, it follows that $(\mathbf{vw}\phi)^d = (\mathbf{w}\phi)^d \boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}} = \phi^d \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}} \cdot \boldsymbol{\pi}_{\mathbf{I}}/\boldsymbol{\pi}_{\mathbf{J}} = \phi^d \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{J}}$ so $\mathbf{w}$ is not maximal. □

The assumption (iv) above can be replaced by two others, thanks to the following lemma which holds for any complex reflection coset and any ζ.

Lemma 9.7. *Let W be finite a (pseudo)-reflection group on the complex vector space X and let ϕ be an automorphism of X of finite order which normalizes W. Let V be the ζ-eigenspace of an element $w\phi \in W\phi$. Assume that W' is a parabolic subgroup of W which is $w\phi$-stable and such that $C_W(V) \subset W'$, and let X' denote the subspace of X spanned by the root lines of W'. Then the condition*

(i) $V \cap X' = 0$.

is equivalent to

(ii) $C_W(V) = W'$.

While the stronger condition

(iv) *No element of the coset $W'w\phi$ has a non-zero ζ-eigenvector on X'.*

is equivalent to the conjunction of (ii) and

(iii) *the space V is maximal among the ζ-eigenspaces of elements of $W\phi$.*

Proof. Since $W\langle\phi\rangle$ is finite we may endow X with a $W\langle\phi\rangle$-invariant scalar product, which we shall do.

We show (i) ⇔ (ii). Assume (i); since $w\phi$ has no non-zero ζ-eigenvector in X' and X' is $w\phi$-stable, we have $V \perp X'$, so that $W' \subset C_W(V)$, whence (ii) since the reverse inclusion is true by assumption. Conversely, (ii) implies that $V \subset X'^{\perp}$ thus $V \cap X' = 0$.

We show (iv) ⇒ (iii). There exists an element of $W\phi$ whose ζ-eigenspace V_1 is maximal with $V \subset V_1$. Then $C_W(V_1) \subset C_W(V) \subset W'$ and the $C_W(V_1)$-coset of elements of $W\phi$ which act by ζ on V_1 is a subset of the coset $C_W(V)w\phi$ of elements which act by ζ on V. Thus this coset is of the form $C_W(V_1)vw\phi$ for some $v \in W'$. By (i) ⇒ (ii) applied with $w\phi$ replaced by $vw\phi$ we get $C_W(V_1) = W'$. Since $v \in W'$ this implies that $vw\phi$ and $w\phi$ have same action on V_1 so that $w\phi$ acts by ζ on V_1, thus $V_1 \subset V$.

Conversely, assume that (ii) and (iii) are true. If there exists $v \in W'$ such that $vw\phi$ has a non-zero ζ-eigenvector in X', then since v acts trivially on V by (ii), the element $vw\phi$ acts by ζ on V and on a non-zero vector of X' so has a ζ-eigenspace strictly larger that V, contradicting (iii). □

Let us give now examples which illustrate the need for the conditions in Theorem 9.4 and Lemma 9.7.

We first show that if $\mathbf{w}\phi$ satisfies Theorem 9.4 (i) and (ii) but is not maximal in the sense of Theorem 9.4(iii) then $\ker(w\phi - \zeta)$ need not be maximal: Let us take $W = W(A_3)$, $\phi = 1$, $d = 2$, $\zeta = -1$, $\mathbf{I} = \{\mathbf{s}_2\}$ (where the conventions for the generators of W are as in the appendix, see 11.2), $\mathbf{w} = \mathbf{w}_\mathbf{I}^{-1}\mathbf{w}_0$. We have $\mathbf{w}^2 = \boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I}$ but $\ker(w+1)$ is not maximal: it is of dimension 1 and a 2-dimensional -1-eigenspace is obtained for $\mathbf{w} = \mathbf{w}_0$.

In the above example we still have $C_W(V) = W_I$ but even this need not happen; at the same time we illustrate that the maximality of $V = \ker(w\phi - \zeta)$ does not imply the maximality of $\mathbf{w}$ if $C_W(V) \subsetneq W_I$; we take $W = W(A_3)$, $\phi = 1$, $d = 2$, $\zeta = -1$, but this time $I = \{\mathbf{s}_1, \mathbf{s}_3\}$, $\mathbf{w} = \mathbf{w}_I^{-1}\mathbf{w}_0$. We have $\mathbf{w}^2 = \boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I}$ and $\ker(w+1)$ is maximal (w is conjugate to w_0, thus -1-regular) but $\mathbf{w}$ is not maximal. In this case $C_W(V) = \{1\}$.

The smallest example with a maximal $\mathbf{w}\phi$ and non-trivial $\mathbf{I}$ is for $W = W(A_4)$, $\phi = 1$, $d = 3$, $\mathbf{w} = \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_3\mathbf{s}_2$ and $\mathbf{I} = \{\mathbf{s}_3\}$. Then $\mathbf{w}^3 = \boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I}$; this corresponds to the smallest example with a non-regular eigenvalue: ζ_3 is not regular in A_4.

Lemma 9.8. *Let $W\phi$ be a complex reflection coset and let V be the ζ-eigenspace of $w\phi \in W\phi$; then*

(i) $N_W(V) = N_W(C_W(V)w\phi)$.

(ii) *If $W\phi$ is real, and $C_W(V) = W_I$ where (W, S) is a Coxeter system and $I \subset S$, and w is I-reduced, then the subgroup $\{v \in C_W(w\phi) \cap N_W(W_I) \mid v$ is I-reduced$\}$ is a section of $N_W(V)/C_W(V)$ in W.*

Proof. Let W_1 denote the parabolic subgroup $C_W(V)$. All elements of $W_1 w\phi$ have the same ζ-eigenspace V, so $N_W(W_1 w\phi)$ normalizes V; conversely, an element of $N_W(V)$ normalizes W_1 and conjugates $w\phi$ to an element $w'\phi$ with same ζ-eigenspace, thus w and w' differ by an element of W_1, whence (i).

For the second item, $N_W(W_I w\phi)/W_I$ admits as a section the set of I-reduced elements, and such an element will conjugate $w\phi$ to the element of the coset $W_I w\phi$ which is I-reduced, so will centralize $w\phi$. □

If we have a Garside category C with Garside natural transformation Δ, on which we are given an automorphism ϕ, we can consider the semi-direct product of C by ϕ (see Definition 3.1). We will then call (p, q)-periodic a morphism $\mathbf{w}\phi \in C\phi$ such that $\text{target}(\mathbf{w}) = \phi(\text{source}(\mathbf{w}))$ and $(\mathbf{w}\phi)^p = \Delta^q\phi^p$. An element satisfying (i) and (ii) of Theorem 9.4 is thus a $(d, 2)$-periodic element of $B^+(\mathcal{I})\phi$, since $\Delta_\mathcal{I}^2$ starting from the object $\mathbf{I}$ is $\mathbf{I} \xrightarrow{\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I}} \mathbf{I}$. Lemma 9.5 shows that such an element is cyclically conjugate to an element which satisfies in addition $(\mathbf{w}\phi)^{d'} \in \mathbf{W}\phi^{d'}$, where $d' = \lfloor \frac{d}{2} \rfloor$. We will call *good* a periodic element which satisfies the above condition.

The following proposition, which rephrases Corollary 7.3 in our setting, shows that it makes sense to write a period of the form $(d, 2)$ as a fraction $d/2$, since it shows that when $2|d$, a good $(d, 2)$-periodic element such that $(\mathbf{w}\phi)^d = \Delta_\mathcal{I}^2$ satisfies $(\mathbf{w}\phi)^{d/2} = \Delta_\mathcal{I}$. We will thus call such elements $d/2$-periodic. In [DDGKM] the analogous statement is shown for a general p/q.

Proposition 9.9. *Assume the morphism $\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi\mathbf{I}$ is good $d/2$-periodic (which means that $\mathbf{w} \in B^+$ satisfies ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$, $(\mathbf{w}\phi)^d = \phi^d\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I}$ and that in addition $(\mathbf{w}\phi)^{d'} \in \mathbf{W}\phi^{d'}$, where $d' = \lfloor \frac{d}{2} \rfloor$). Then if d is even we have $(\mathbf{w}\phi)^{d'} = \mathbf{w}_\mathbf{I}^{-1}\mathbf{w}_0\phi^{d'}$,*

and if d is odd there exists $\mathbf{u} \in \mathbf{W}^{\Phi^d}$ with $\mathbf{I}^{\mathbf{u}} \subset \mathbf{S}$ such that $\mathbf{w}\phi = \mathbf{u}\phi \cdot {}^{\mathbf{w}_0\phi^{d'}}\mathbf{u}$ and $(\mathbf{w}\phi)^{d'}\mathbf{u} = \mathbf{w}_{\mathbf{I}}^{-1}\mathbf{w}_0\phi^{d'}$.

Let us define the *ζ-rank* of a (complex) reflection coset $W\phi \subset \mathrm{GL}(X)$ as the maximal dimension of a ζ-eigenspace of an element of $W\phi$, and the ζ-rank of an element of $W\phi$ as the dimension of its ζ-eigenspace.

Let us say that a periodic element of $B^+(\mathcal{I})\phi$ is maximal if it is maximal in the sense of Theorem 9.4(iii). Another way to state the maximality of a periodic element is to require that $|\mathbf{I}|$ be not more than the rank of the centralizer of a maximal ζ_d-eigenspace: indeed if $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$ is not maximal there exists $\mathbf{J}$ and $\mathbf{v}$ as in Theorem 9.4(iii) and, since Theorem 9.4(iii) implies Lemma 9.7(iii), the element $vw\phi$ has maximal ζ_d-rank, and the centralizer of its ζ_d-eigenspace has rank $|\mathbf{J}| < |\mathbf{I}|$.

A particular case of Theorems 9.1 and 9.4 is

Corollary 9.10. *Let V' be the ζ_d-eigenspace of an element of $W\phi$ of maximal ζ_d-rank. Then there is a W-conjugate V of V' such that $C_W(V) = W_I$ and the $w\phi$ defined in Theorem 9.1(ii) induces a $d/2$-periodic $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$ which is maximal. Conversely, for a $d/2$-periodic maximal $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$ the image $w\phi$ in $W\phi$ has maximal ζ_d-rank.*

Lemma 9.11. *Let $W\phi \subset \mathrm{GL}(X_{\mathbb{R}})$ be a finite order real reflection coset such that ϕ preserves the chamber of the corresponding hyperplane arrangement determining the Coxeter system (W, S).*

For $w \in W\phi$ let $\mathbf{w} \in \mathbf{W}$ be the lift of w; for $I \subset S$, the existence of a morphism $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I} \in B^+(\mathcal{I})$ is equivalent to:

(i) *${}^{w\phi}I = I$ and w is I-reduced.*

Then, for $d > 1$, the morphism $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$ is good $d/2$-periodic if and only if the following two conditions are satisfied.

(ii) *$l((w\phi)^i\phi^{-i}) = \frac{2i}{d}l(w_I^{-1}w_0)$ for $0 < i \leq \lfloor \frac{d}{2} \rfloor$.*

(iii) *$(w\phi)^d = \phi^d$.*

If, moreover,

(iv) *$W_I w\phi$ has ζ_d-rank 0 on the subspace spanned by the root lines of W_I,*

then $w\phi$ is maximal in the sense of Theorem 9.4(iv).

Proof. By definition, if $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I}$ is good $d/2$-periodic then (i), (ii), (iii) are satisfied. Conversely, Lemma 9.2 shows that the lift of a w satisfying (i), (ii), (iii) is good $d/2$-periodic.

Property (iv) means that no element $vw\phi$ with $v \in W_I$ has an eigenvalue ζ_d on the subspace spanned by the root lines of W_I which is exactly the equivalent Definition 9.4(iv) of a maximal element. □

Note that d and I in the above assumptions (i), (ii), (iii) are uniquely determined by w since d is the smallest power of $w\phi$ which is a power of ϕ and I is given uniquely by $(\mathbf{w}\phi)^d = \boldsymbol{\pi}/\boldsymbol{\pi}_{\mathbf{I}}\phi^d$.

Definition 9.12. *We say that $w\phi \in W\phi$ is* ζ_d-good *(relative to $W\phi$ and I) if it satisfies (i), (ii), (iii) in Lemma 9.11.*

We say $w\phi$ is ζ_d-good maximal *if it satisfies in addition (iv) in Lemma 9.11.*

In particular, ζ_d-good elements lift to good $d/2$-periodic elements, and ζ_d-good maximal elements lift to good maximal $d/2$-periodic elements.

The ζ_d-good maximal elements belong to a single conjugacy class of W. The following lemma applied with $\zeta = \zeta_d$ gives a characterization of this class.

Lemma 9.13. *Let $W\phi$ be a finite order real reflection coset such that ϕ preserves a chamber of the corresponding hyperplane arrangement. The elements of $W\phi$ which have a ζ-eigenspace V of maximal dimension and among those, have the largest dimension of fixed points, are conjugate.*

Proof. Let w and V be as in the lemma. Since, by [S, Theorem 3.4(iii) and Theorem 6.2(iii)], the maximal ζ-eigenspaces are conjugate, we may fix V. Since $C_W(V)$ is a parabolic subgroup of the Coxeter group W normalized by $w\phi$, the coset $C_W(V)w\phi$ is a real reflection coset; in this coset there are 1-regular elements, which are those which preserve a chamber of the corresponding real hyperplane arrangement; the 1-regular elements have maximal 1-rank, that is have the largest dimension of fixed points, and they form a single $C_W(V)$-orbit under conjugacy, whence the lemma. □

Lemma 9.14. *Let $w\phi$ be a ζ_d-good maximal element, let I be as in Lemma 9.11 and let V_1 be the fixed point subspace of $w\phi$ in the space spanned by the root lines of W_I; then $w\phi$ is regular in the coset $C_W(V_1)w\phi$.*

Proof. Let $W' = C_W(V_1)$; we first note that since $w\phi$ normalizes V_1 it normalizes also W', so $W'w\phi$ is indeed a reflection coset. We have thus only to prove that $C_{W'}(V)$ is trivial, where V is the ζ_d-eigenspace of $w\phi$. This last group is generated by the reflections with respect to roots both orthogonal to V and to V_1, which are the roots of $W_I = C_W(V)$ orthogonal to V_1. Since $w\phi$ preserves a chamber of W_I, the sum v of the positive roots of W_I with respect to the order defined by this chamber is in V_1 and is in the chamber: this is well known for a true root system; here we have taken all the roots to be of length 1 but the usual proof (see [Bou, Chapitre VI §1, Proposition 29]) is still valid. Since no root is orthogonal to a vector v inside a chamber, W_I has no root orthogonal to V_1, hence $C_{W'}(V) = \{1\}$. □

Note that the map $C_{W'}(w\phi) = N_{W'}(V) \to N_W(V)/C_W(V)$ in the above proof is injective, but not always surjective: if W of type E_7, if $\phi = \mathrm{Id}$ and $\zeta = i$, a fourth root of unity, then $N_W(V)/C_W(V)$ is the complex reflection group G_8, while W' is of type D_4 and $N_{W'}(V)/C_{W'}(V)$ is the complex reflection group $G(4,2,2)$. However, we will see in appendix 1 that there are only 4 such cases for irreducible groups W; to see in the other cases that $C_{W'}(w\phi) \simeq N_W(V)/C_W(V)$ it is sufficient to check that they have same reflection degrees, which is a simple arithmetic check on the reflection degrees of W and W'.

10. Conjectures

The following conjectures extends those of [DM2, 2.1].

Conjecture 10.1. *Let $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I} \in B^+(\mathcal{I})$ be a maximal $d/2$-periodic morphism. Then*

(i) *The group $B_{\mathbf{w}}$ generated by the monoid $B^+_{\mathbf{w}}$ of Theorem 8.35 is isomorphic to the braid group of the complex reflection group $W_w := N_W(W_I w\phi)/W_I$.*

(ii) *The natural morphism* $\mathrm{End}_{\mathcal{D}^+(\mathcal{I})}\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I} \to \mathrm{End}_{\mathbf{G}^F}(\mathbf{X}(\mathbf{I},\mathbf{w}\phi))$ *(see below Definition 8.31) gives rise to a morphism* $B_{\mathbf{w}} \to \mathrm{End}_{\mathbf{G}^F} H_c^*(\mathbf{X}(\mathbf{I},\mathbf{w}\phi))$ *which factors through a special representation of a* ζ_d*-cyclotomic Hecke algebra* $\mathcal{H}_{\mathbf{w}}$ *for* W_w*.*

(iii) *The odd and even* $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi))$ *are disjoint, and the above morphism extends to a surjective morphism* $\overline{\mathbb{Q}}_\ell[B_{\mathbf{w}}] \to \mathrm{End}_{\mathbf{G}^F}(H_c^*(\mathbf{X}(\mathbf{I},\mathbf{w}\phi)))$*.*

Lemma 10.2. *Let* $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I} \in B^+(\mathcal{I})$ *be a maximal* $d/2$*-periodic morphism and assume conjectures 10.1; then for any* $i \neq j$ *the* $\mathbf{G}^F$*-modules* $H_c^i(\mathbf{X}(\mathbf{I},\mathbf{w}\phi))$ *and* $H_c^j(\mathbf{X}(\mathbf{I},\mathbf{w}\phi))$ *are disjoint.*

Proof. Since the image of the morphism of Conjecture 10.1(ii) consists of equivalences of étale sites, it follows that the action of $\mathcal{H}_{\mathbf{w}}$ on $H_c^*(\mathbf{X}(\mathbf{I},\mathbf{w}\phi))$ preserves individual cohomology groups. The surjectivity of the morphism of (iii) implies that for $\rho \in \mathrm{Irr}(\mathbf{G}^F)$, the ρ-isotypic part of $H_c^*(\mathbf{X}(\mathbf{I},\mathbf{w}\phi))$ affords an irreducible $\mathcal{H}_{\mathbf{w}}$-module; this would not be possible if this ρ-isotypic part was spread over several distinct cohomology groups. □

We will now explore the information given by the Shintani descent identity on the above conjectures

Lemma 10.3. *Let* $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I} \in B^+(\mathcal{I})$ *be a* $d/2$*-periodic morphism. With the notations of Proposition 8.40, we have* $\tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}}\phi) = q^{m\frac{l(\boldsymbol{\pi})-l(\boldsymbol{\pi}_\mathbf{I})-a_\chi-A_\chi}{d}}\tilde{\chi}(e_I w F)$ *for* $\chi \in \mathrm{Irr}(W)^\phi$*, where* a_χ *(resp.* A_χ*) is the valuation (resp. the degree) of generic degree of* χ *and* $e_I = |W_I|^{-1}\sum_{v\in W_I} v$*.*

Proof. We have $(X_1 T_{\mathbf{w}}\phi)^d = X_1 T_{\boldsymbol{\pi}}/T_{\boldsymbol{\pi}_\mathbf{I}}\phi^d = q^{-l(\boldsymbol{\pi}_\mathbf{I})} X_1 T_{\boldsymbol{\pi}}\phi^d$ since X_1 commutes with $T_{\mathbf{w}}\phi$ and since for any $v \in W_I$ we have $X_1 T_v = q^{l(v)} T_v$. Since $T_{\boldsymbol{\pi}}$ acts on the representation of character χ_{q^m} as the scalar $q^{m(l(\boldsymbol{\pi})-a_\chi-A_\chi)}$, it follows that all the eigenvalues of $X_1 T_{\mathbf{w}}\phi$ on this representation are equal to $q^{m\frac{l(\boldsymbol{\pi})-l(\boldsymbol{\pi}_\mathbf{I})-a_\chi-A_\chi}{d}}$ times a root of unity. To compute the sum of these roots of unity, we may use the specialization $q^{m/2} \mapsto 1$, whence $\tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}}\phi)$ specializes to $\tilde{\chi}(e_I w\phi)$. □

Proposition 10.4. *Let* $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{I} \in B^+(\mathcal{I})$ *be a* $d/2$*-periodic morphism. For any* m *multiple of* δ*, we have*

$$|\mathbf{X}(\mathbf{I},\mathbf{w}\phi)^{gF^m}| = \sum_{\rho\in\mathcal{E}(\mathbf{G}^F,1)} \lambda_\rho^{m/\delta} q^{m\frac{l(\boldsymbol{\pi})-l(\boldsymbol{\pi}_\mathbf{I})-a_\rho-A_\rho}{d}} \langle \rho, R^{\mathbf{G},F}_{\mathbf{L}_I,\dot{w}F}\,\mathrm{Id}\rangle_{\mathbf{G}^F}\rho(g).$$

Proof. We start with Corollary 8.41, whose statement reads, using the value of $\tilde{\chi}_{q^m}(X_1 T_{\mathbf{w}}\phi)$ given by Lemma 10.3:

$$|\mathbf{X}(\mathbf{I},\mathbf{w}\phi)^{gF^m}| = \sum_{\rho\in\mathcal{E}(\mathbf{G}^F,1)} \lambda_\rho^{m/\delta}\rho(g) \sum_{\chi\in\mathrm{Irr}(W)^\phi} q^{m\frac{l(\boldsymbol{\pi})-l(\boldsymbol{\pi}_\mathbf{I})-a_\chi-A_\chi}{d}}\tilde{\chi}(e_I w\phi)\langle\rho, R_{\tilde{\chi}}\rangle_{\mathbf{G}^F}.$$

Using that for any ρ such that $\langle\rho, R_{\tilde{\chi}}\rangle_{\mathbf{G}^F} \neq 0$ we have $a_\rho = a\chi$ and $A_\rho = A_\chi$ the right-hand side can be rewritten

$$\sum_{\rho\in\mathcal{E}(\mathbf{G}^F,1)} \lambda_\rho^{m/\delta} q^{m\frac{l(\boldsymbol{\pi})-l(\boldsymbol{\pi}_\mathbf{I})-a_\rho-A_\rho}{d}}\rho(g)\langle\rho, \sum_{\chi\in\mathrm{Irr}(W)^\phi}\tilde{\chi}(e_I w\phi)R_{\tilde{\chi}}\rangle_{\mathbf{G}^F}.$$

The proposition is now just a matter of observing that

$$\sum_{\chi\in\mathrm{Irr}(W)^\phi}\tilde\chi(e_Iw\phi)R_{\tilde\chi}=|W_I|^{-1}\sum_{v\in W_I}\sum_{\chi\in\mathrm{Irr}(W)^\phi}\tilde\chi(vw\phi)R_{\tilde\chi}=$$
$$|W_I|^{-1}\sum_{v\in W_I}R^{\mathbf{G}}_{\mathbf{T}_{vw}}(\mathrm{Id})=R^{\mathbf{G}}_{\mathbf{L}_I}(\mathrm{Id}).$$

Where the last equality is obtained by transitivity of $R^{\mathbf{G}}_{\mathbf{L}}$ and the equality $\mathrm{Id}_{\mathbf{L}_I^{\dot wF}}=|W_I|^{-1}\sum_{v\in W_I}R^{\mathbf{L}_I,\dot wF}_{\mathbf{T}_{vw}}(\mathrm{Id})$, a torus $\mathbf{T}$ of $\mathbf{L}_I$ of type v for the isogeny $\dot wF$ being conjugate to $\mathbf{T}_{vw}$ in $\mathbf{G}$. □

Corollary 10.5. *Let $\mathbf{I}\xrightarrow{\mathbf{w}}{}^\phi\mathbf{I}\in B^+(\mathcal{I})$ be a maximal $d/2$-periodic morphism and assume conjectures 10.1; then for any $\rho\in\mathrm{Irr}(\mathbf{G}^F)$ such that $\langle\rho,R^{\mathbf{G}}_{\mathbf{L}_I}(\mathrm{Id})\rangle_{\mathbf{G}^F}\neq0$ the isogeny F^δ has a single eigenvalue on the ρ-isotypic part of $H^*_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi))$, equal to $\lambda_\rho q^{\delta\frac{l(\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I})-a_\rho-A_\rho}{d}}$.*

Proof. This follows immediately, in view of Lemma 10.2, from the comparison between Proposition 10.4 and the Lefschetz formula:

$$|\mathbf{X}(\mathbf{I},\mathbf{w}\phi)^{gF^m}|=\sum_i(-1)^i\,\mathrm{Trace}(gF^m\mid H^i_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi),\overline{\mathbb{Q}}_\ell)).$$

□

In view of Corollary 8.28(i) it follows that if $\langle\rho,R^{\mathbf{G}}_{\mathbf{L}_I}(\mathrm{Id})\rangle_{\mathbf{G}^F}\neq0$ then if $\omega_\rho=1$ then $\frac{l(\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I})-a_\rho-A_\rho}{d}\in\mathbb{N}$, and if $\omega_\rho=\sqrt{q^\delta}$ then $\frac{l(\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I})-a_\rho-A_\rho}{d}\in\mathbb{N}+1/2$.

Assuming conjectures 10.1, we choose once and for all a specialization $q^{1/a}\mapsto\zeta^{1/a}$, where $a\in\mathbb{N}$ is large enough such that $\mathcal{H}_\mathbf{w}\otimes\overline{\mathbb{Q}}_\ell[q^{1/a}]$ is split. This gives a bijection $\varphi\mapsto\varphi_q:\mathrm{Irr}(W_w)\to\mathrm{Irr}(\mathcal{H}_\mathbf{w})$, and the conjectures give a further bijection $\varphi\mapsto\rho_\varphi$ between $\mathrm{Irr}(W_w)$ and the set $\{\rho\in\mathrm{Irr}(\mathbf{G}^F)\mid\langle\rho,R^{\mathbf{G}}_{\mathbf{L}_I}(\mathrm{Id})\rangle_{\mathbf{G}^F}\neq0\}$, which is such that $\langle\rho_\varphi,R^{\mathbf{G}}_{\mathbf{L}_I}(\mathrm{Id})\rangle_{\mathbf{G}^F}=\varphi(1)$.

Corollary 10.6. *Under the assumptions of Corollary 10.5, if ω_φ is the central character of φ, then*

$$\lambda_{\rho_\varphi}=\omega_\varphi((w\phi)^\delta)\zeta^{-\delta\frac{l(\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I})-a_{\rho_\varphi}-A_{\rho_\varphi}}{d}}.$$

Proof. We first note that it makes sense to apply φ to $(w\phi)^\delta$, since $(w\phi)^\delta\in W_w$. Actually $(\mathbf{w}\phi)^\delta$ is a central element of $B_\mathbf{w}$ and maps by the morphism of Conjecture 10.1(iii) to F^δ, thus the eigenvalue of F^δ on the ρ_φ-isotypic part of $H^*_c(\mathbf{X}(\mathbf{I},\mathbf{w}\phi))$ is equal to $\omega_{\varphi_q}((\mathbf{w}\phi)^\delta)$; thus $\omega_{\varphi_q}((\mathbf{w}\phi)^\delta)=\lambda_{\rho_\varphi}q^{\delta\frac{l(\boldsymbol{\pi}/\boldsymbol{\pi}_\mathbf{I})-a_{\rho_\varphi}-A_{\rho_\varphi}}{d}}$. The statement follows by applying the specialization $q^{1/a}\mapsto\zeta^{1/a}$ to this equality. □

11. Appendix 1: good ζ_d-maximal elements in reductive groups

We will describe, in a reductive group $\mathbf{G}$, for each d, a ζ_d- good maximal element $w\phi$ relative to $W\phi$ and some $I\subset S$. Thus the variety $\mathbf{X}(\mathbf{I},\mathbf{w}\phi)$ will be the one whose cohomology should be a tilting complex for the Broué conjectures for an ℓ dividing $\Phi_d(q)$.

Since such an element depends only on the Weyl group, we may assume that $\mathbf{G}$ is semi-simple and simply connected. Now, a semi-simple and simply connected group is a direct product of restrictions of scalars of simply connected quasi-simple

groups. A ζ_d-good (resp. maximal) element in a direct product is the product of a ζ_d-good (resp. maximal) element in each component. So we reduce immediately to the case of restriction of scalars.

11.1. **Restrictions of scalars.** A restriction of scalars is a group of the form $\mathbf{G}^n$, with an isogeny F_1 such that $F_1(x_0,\dots,x_{n-1}) = (x_1,\dots,x_{n-1},F(x_0))$. Thus $(\mathbf{G}^n)^{F_1} \simeq \mathbf{G}^F$.

If F induces ϕ on the Weyl group W of G then $(\mathbf{G}^n, F_1)$ corresponds to the reflection coset $W^n \cdot \sigma$, where $\sigma(x_1,\dots,x_n) = (x_2,\dots,x_n,\phi(x_1))$.

In the first two propositions of this section, we will study such a "restriction of scalars" for arbitrary complex reflection cosets. Thus we start with a reflection coset $W\phi$, with $W \subset \mathrm{GL}(V)$ a complex reflection group where $V = \mathbb{C}^r$, and $\phi \in N_{\mathrm{GL}(V)}(W)$. We denote by δ the order of $W\phi$ (the minimal i such that $(W\phi)^i = W$). We want to study the eigenvalues of elements in the coset $W^n \cdot \sigma \subset \mathrm{GL}(V^n)$, where $\sigma(x_1,\dots,x_n) = (x_2,\dots,x_n,\phi(x_1))$; we call this coset the *restriction of scalars* of the coset $W\phi$.

Recall that, if S_W is the coinvariant algebra of W (the quotient of the symmetric algebra of V^* by the ideal generated by the W-invariants of positive degree), for any W-module X the graded vector space $(S_W \otimes X^*)^W$ admits a homogeneous basis formed of eigenvectors of ϕ. The degrees of the elements of this basis are called the X-exponents of W and the corresponding eigenvalues of ϕ the X-factors of $W\phi$. For $X = V$, the V-exponents n_i satisfy $n_i = d_i - 1$ where the d_i's are the reflection degrees of W, and the V-factors ε_i are equal to the factors of $W\phi$. For $X = V^*$, the $n_i - 1$ where n_i are the V^*-exponents are called the codegrees d_i^* of W and the corresponding V^*-factors ε_i^* are called the cofactors of $W\phi$. By Springer [S, 6.4], the ζ-rank of $W\phi$ is equal to $|\{i \mid \zeta^{d_i} = \varepsilon_i\}|$. By analogy with the ζ-rank, we define the ζ-corank of $W\phi$ as $|\{i \mid \zeta^{d_i^*} = \varepsilon_i^*\}|$. By for example [Br, 5.19.2] an eigenvalue is regular if it has same rank and corank.

Proposition 11.1. *Let $W^n \cdot \sigma$ be a restriction of scalars of the complex reflection coset $W\phi$. Then the ζ-rank (resp. corank) of $W^n \cdot \sigma$ is equal to the ζ^n-rank (resp. corank) of $W\phi$.*

In particular, ζ is regular for $W^n \cdot \sigma$ if and only if ζ^n is regular for $W \cdot \phi$.

Proof. The pairs of a reflection degree and the corresponding factor of σ for the coset $W^n \cdot \sigma$ are the pairs $(d_i, \zeta_n^j \sqrt[n]{\varepsilon_i})$, where $i \in \{1,\dots,r\}, j \in \{1,\dots,n\}$ and where $\sqrt[n]{\varepsilon_i}$ represents an n-th root of ε_i (that we choose arbitrarily for each i).

Similarly, the pairs of a reflection codegree and the corresponding cofactor are $(d_i^*, \zeta_n^j \sqrt[n]{\varepsilon_i^*})$.

In particular the ζ-rank of $W^n \cdot \sigma$ is $|\{(i,j) \mid \zeta^{d_i} = \zeta_n^j \sqrt[n]{\varepsilon_i}\}|$ and the ζ-corank is $|\{(i,j) \mid \zeta^{d_i^*} = \zeta_n^j \sqrt[n]{\varepsilon_i^*}\}|$.

Given $a \in \mathbb{N}$, there is at most one j such that the equality $\zeta^a = \zeta_n^j \sqrt[n]{\varepsilon_i}$ holds, and there is one j if and only if $\zeta^{na} = \varepsilon_i$. Thus we have

$$|\{(i,j) \mid \zeta^{d_i} = \zeta_n^j \sqrt[n]{\varepsilon_i}\}| = |\{i \mid \zeta^{nd_i} = \varepsilon_i\}|$$

and similarly for the corank, whence the two assertions of the statement. □

We assume now that $\zeta = \zeta_d$; note that ζ_d^n is a d/k-th root of unity, where $k = \gcd(n,d)$, but it is not a distinguished root of unity. We have however the following:

Proposition 11.2. *Let $W^n \cdot \sigma$ be a restriction of scalars of the complex reflection coset $W\phi$ and for $d \in \mathbb{N}$ let $k = \gcd(n, d)$; then there exists m such that $m(n/k) \equiv 1 \pmod{d/k}$ and $\gcd(m, \delta) = 1$, and for such an m the ζ_d-rank (resp. corank) of $W^n \cdot \sigma$ is equal to the $\zeta_{d/k}$-rank (resp. corank) of $W \cdot \phi^m$.*

Proof. We first show that m exists. Choose an m such that $m(n/k) \equiv 1 \pmod{d/k}$. Since m is prime to d/k it is prime to $\gcd(d/k, \delta)$. By adding to m a multiple of d/k we can add modulo δ any multiple of $\gcd(d/k, \delta)$, thus we can reach a number prime to δ, using the general fact that for any divisor δ' of δ, the natural projection $\mathbb{Z}/\delta\mathbb{Z} \xrightarrow{\theta} \mathbb{Z}/\delta'\mathbb{Z}$ is such that $\theta((\mathbb{Z}/\delta\mathbb{Z})^\times) \supset (\mathbb{Z}/\delta'\mathbb{Z})^\times$.

By Proposition 11.1, the ζ_d-rank (resp. corank) of $W^n \cdot \sigma$ is equal to the $\zeta_d^n = \zeta_{d/k}^{n/k}$-rank (resp. corank) of $W \cdot \phi$. Now ε_i^m (resp. ε_i^{*m}) are the factors (resp. cofactors) of $W \cdot \phi^m$ and since m is prime to δ and $\varepsilon_i^\delta = 1$, we have $|\{i \mid \zeta^{d_i} = \varepsilon_i\}| = |\{i \mid (\zeta^m)^{d_i} = \varepsilon_i^m\}|$, (similarly for d_i^*, ε_i^*); thus the $\zeta_{d/k}^{n/k}$-rank (resp. corank) of $W \cdot \phi$ is equal in turn to the $\zeta_{d/k}^{m \cdot n/k}$-rank (resp. corank) of $W \cdot \phi^m$. Now, since $m(n/k) \equiv 1 \pmod{d/k}$, we have $\zeta_{d/k}^{m \cdot n/k} = \zeta_{d/k}$. □

We now assume, until the end of the subsection, that $W\phi$ is a real reflection coset of order δ, that ϕ preserves a chamber corresponding to the Coxeter system (W, S), and that $\zeta = \zeta_d$ is a distinguished root of unity. We will use the criteria of Lemma 9.11 to check that an element is ζ_d-good (resp. maximal).

Proposition 11.3. *Under the assumptions of Proposition 11.2, let $v\phi^m$ be a $\zeta_{d/k}$-good element relative to $W\phi^m$ and I. Then*

- *If either $k = 1$ or d/k is even, define $w = (w_0, \dots, w_{n-1}) \in W^n$ by $w_{ik} = \phi^{im}(v)$, and $w_j = 1$ if $j \not\equiv 0 \pmod{k}$*
- *If d/k is odd and $k \neq 1$, by Corollary 7.3 there exists $v_1, v_2 \in W$ such that $v\phi^m = v_1\phi^m v_2$ and $(v\phi^m)^{(\frac{d}{k}-1)/2} v_1 = w_I^{-1} w_0 \phi^{m(\frac{d}{k}-1)/2}$; define $w = (w_0, \dots, w_{n-1}) \in W^n$ by*

$$w_j = \begin{cases} \phi^{im}(v_2) & \text{if } j = ik \\ \phi^{(i+1)m}(v_1) & \text{if } j = ik + \lfloor \frac{k}{2} \rfloor \\ 1 & \text{if } j \not\equiv 0, \lfloor \frac{k}{2} \rfloor \pmod{k} \end{cases}$$

In each case $w\sigma$ is a ζ_d-good element relative to $W^n\sigma$ and $\underline{I}$ where $\underline{I} = (I_0, \dots, I_{n-1}) \subset S^n$ with $I_j = {}^{w_j w_{j+1} \dots w_{n-1}\phi} I$ and we have $N_{W^n}(W_{\underline{I}} w\sigma)/W_{\underline{I}} \simeq N_W(W_I v\phi^m)/W_I$.

If moreover $v\phi^m$ is maximal then $w\sigma$ is also maximal.

Proof. To lighten the notation, we set $n' = n/k$ and $d' = d/k$.

We recall that $v\phi^m$ being $\zeta_{d'}$-good means ${}^{v\phi^m} I = I$ and v is I-reduced, $(v\phi^m)^{d'} = \phi^{md'}$, and $l((v\phi^m)^i \phi^{-im}) = 2i/d' \cdot l(w_I^{-1} w_0)$ for $0 \leq i \leq \lfloor \frac{d'}{2} \rfloor$. We have to show the same conditions for $w\sigma$, that is

(i) ${}^{w\sigma}(I_0, \dots, I_{n-1}) = (I_0, \dots, I_{n-1})$ and w is $(I_0, \dots, I_{n-1})$-reduced.
(ii) $(w\sigma)^d = \sigma^d$.
(iii) $l((w\sigma)^i \sigma^{-i}) = \frac{2in}{d} l(w_I^{-1} w_0)$ for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

We first note:

Lemma 11.4. *$\phi^{d'}$ stabilizes v and I (thus $\phi^{\gcd(d', \delta)}$ also).*

Proof. As $(v\phi^m)^{d'} = \phi^{md'}$, we find that $\phi^{md'}$ stabilizes $v\phi^m$ and I, thus v and I. Since m is invertible modulo δ, we get that $\phi^{d'}$ stabilizes v and I. □

We first check that $\underline{I} \subset S^n$. In the case d' even, each I_j is of the form ${}^{\phi^{im}(v)\phi^{(i+1)m}(v)\dots\phi^{(n'-1)m}(v)\phi}I$ (where ik is the smallest multiple of k greater than j). If d' is odd I_j is either as above or of the form ${}^{\phi^{(i-1)m}(v_2)\phi^{im}(v)\phi^{(i+1)m}(v)\dots\phi^{(n'-1)m}(v)\phi}I$. In the first case, since $1 - mn' \equiv 0 \pmod{d'}$ and $\phi^{d'}$ stabilizes I, by Lemma 11.4, we can write

$$I_j = {}^{\phi^{im}(v)\phi^{(i+1)m}(v)\dots\phi^{(n'-1)m}(v)\phi^{mn'}}I = {}^{\phi^{im}(v\phi^m)^{n'-i}}I = {}^{\phi^{im}}I \subset S.$$

In the second case, if we put $J = I^{v_1} = {}^{v_2\phi^m}I$, a subset of S by Corollary 7.3, we get $I_j = {}^{\phi^{(i-1)m}(v_2)\phi^{im}}I = {}^{\phi^{(i-1)m}}J$.

We now check (i). The verification of ${}^{w\sigma}(I_0,\dots,I_{n-1}) = (I_0,\dots,I_{n-1})$ reduces to ${}^{w_0w_1\dots w_{n-1}\phi}I = I$, which itself reduces to ${}^{v\phi^m(v)\dots\phi^{(n'-1)m}(v)\phi}I = I$, which is true by the case $i = 0$ of the above computation. Similarly, checking that $w\sigma$ is $(I_0,\dots,I_{n-1})$-reduced reduces to the check that for each j the element w_j is I_j-reduced, where $I_j = {}^{w_jw_{j+1}\dots w_{n-1}\phi}I = I^{w_0\dots w_{j-1}}$, or equivalently that $w_0\dots w_{j-1}$ is I-reduced. Thus in the d' even case we have to check that $v\phi^m(v)\dots\phi^{im}(v)$ is I-reduced for $0 \le i < n'$. This results from the fact that v is I-reduced and that $v\phi^m$ normalizes I. In the d' odd case we have also to check that $v\phi^m(v)\dots\phi^{(i-1)m}(v)\phi^{im}(v_1)$ is I-reduced, which follows from the fact that v is I-reduced, that $v\phi^m$ normalizes I and that v_1 is also I-reduced, which we know by Corollary 7.3.

For checking (ii) and (iii) we compute $(w\sigma)^i$. For any $(w_0,\dots,w_{n-1}) \in W^n$ we have $\sigma(w_0,\dots,w_{n-1}) = (w_1,\dots,w_{n-1},\phi(w_0))\sigma$, thus we find that if we define for all j the element $w_j = \phi^{\frac{j-j_0}{n}}(w_{j_0}) = \phi^{\lfloor\frac{j}{n}\rfloor}(w_{j_0})$ where $j_0 \equiv j \pmod n$ and $0 \le j_0 < n$, we have

$$(w\sigma)^i = (w_0\dots w_{i-1}, w_1\dots w_i, \dots, w_{n-1}\dots w_{i+n-2})\sigma^i.$$

Each product $w_uw_{u+1}\dots w_{u+i-1}$ appearing in the above expression is, up to applying a power of ϕ, of the form $(v\phi^m)^j\phi^{-mj}$ or in the d' odd case additionally of one of the forms $(v_2\phi^mv_1)^j\phi^{-mj}$, $(v_1\phi^mv_2)^jv_1\phi^{-mj}$ or $v_2(v_1\phi^mv_2)^j\phi^{-mj}$, for some j which depends on u and i. If i is a multiple of k the last two forms do not appear and $j = i/k$. In particular if $i = d$ we get either $(v\phi^m)^{d/k}\phi^{-md/k}$ or $(v_2\phi^mv_1)^{d/k}\phi^{-md/k}$. Since $(v\phi^m)^{d'} = \phi^{d'm}$ we have also $(v_2\phi^mv_1)^{d'} = v_1^{-1}(v\phi^m)^{d'}v_1 = \phi^{md'}$, since v, hence v_1, is $\phi^{md'}$-stable, whence (ii).

To check (iii) it is enough check it for $i = 1$, which is clear since $l(w) = n'l(v) = \frac{2in}{d}l(w_I^{-1}w_0)$ and $l(v) = l(v_1) + l(\phi^m(v_2))$ (by Corollary 7.3) and to check that in a product $w_uw_{u+1}\dots w_{u+i-1}$ the lengths add for all $i \le \lfloor\frac{d}{2}\rfloor$: the lengths will then add in $(w\sigma)^i$ for $i \le \lfloor\frac{d}{2}\rfloor$ which gives (iii). In the d' even case this is a result of the lengths adding for $(v\phi^m)^j\phi^{-mj}$. In the d' odd case, we know by Corollary 7.3 that the lengths add in a product of at most d' terms of the form ${v_1}^{\phi^m}{v_2}^{\phi^m}v_1\dots$ or of the form ${v_2}^{\phi^m}{v_1}^{\phi_m}v_2\dots$. We claim that to get more than d' non-trivial terms in the product $w_uw_{u+1}\dots w_{u+i-1}$ we need $i > \lfloor\frac{d}{2}\rfloor$. The maximal number of non-trivial terms is obtained when the first or the last term is non trivial. To get $d'+1$ non-trivial terms we need $i \ge \frac{d'+1}{2}k + \lfloor\frac{k}{2}\rfloor$, since $d'+1$ is even. But $\frac{d'+1}{2}k + \lfloor\frac{k}{2}\rfloor = \lfloor\frac{kd'}{2}\rfloor + k > \frac{d}{2}$, whence our claim.

Computing now $N_{W^n}(W_{\underline{I}}w\sigma)$, we find that $(g_0,\dots,g_{n-1})$ normalizes $(W_{\underline{I}}w\sigma)$ if and only if:

$$\begin{aligned} g_0W_{I_0} &= W_{I_0}{}^{w_0}g_1\\ \dots &= \dots\\ g_{n-2}W_{I_{n-2}} &= W_{I_{n-2}}{}^{w_{n-2}}g_{n-1}\\ g_{n-1}W_{I_{n-1}} &= W_{I_{n-1}}{}^{w_{n-1}\phi}g_0 \end{aligned}$$

which, using the value $I_j = {}^{w_j\dots w_{n-1}\phi}I = I^{w_0\dots w_{j-1}}$ becomes

$$\begin{aligned} g_0W_I &= W_I{}^{w_0}g_1\\ \dots &= \dots\\ {}^{w_0\dots w_{n-3}}g_{n-2}W_I &= W_I{}^{w_0\dots w_{n-2}}g_{n-1}\\ {}^{w_0\dots w_{n-2}}g_{n-1}W_I &= W_I{}^{w_0\dots w_{n-1}\phi}g_0 \end{aligned}$$

We now notice that an equality $aW_I = W_Ib$ is equivalent to: a normalizes W_I, and $aW_I = bW_I$. Thus our equations are equivalent to: g_0 normalizes W_I, the cosets $W_Ig_0,\dots,W_I{}^{w_0\dots w_{n-2}}g_{n-1}$ are equal (thus determined by g_0) and $W_Ig_0 = W_I{}^{w_0\dots w_{n-1}\phi}g_0$. The last equality means that g_0 normalizes $W_Iw_0\dots w_{n-1}\phi$; we find $N_{W^n}(W_{\underline{I}}w\sigma)/W_{\underline{I}} \simeq N_W(W_Iw_0\dots w_{n-1}\phi)/W_I = N_W(W_I(v\phi^m)^{n'}\phi^{1-mn'})/W_I$. Since $1-mn' \equiv 0 \pmod{d'}$, by Lemma 11.4 $\phi^{1-mn'}$ commutes with $v\phi^m$, thus $((v\phi^m)^{n'}\phi^{1-mn'})^m = (v\phi^m)^{n'm}\phi^{m-mn'm}$. Let us write $n'm = ad'+1$; using that $(v\phi^m)^{d'} = \phi^{md'}$ we get $(v\phi^m)^{n'm}\phi^{m-mn'm} = (v\phi^m)^{ad'+1}\phi^{-amd'} = v\phi^m$, thus the above coset has same normalizer as $W_Iv\phi^m$.

Assume now that $v\phi^m$ is maximal, that is $W_Iv\phi^m$ has $\zeta_{d'}$-rank equal to 0. We prove the same for $w\sigma$, that is $(W_{I_0}\times\dots\times W_{I_{n-1}})w\sigma$ has ζ_d-rank 0. Identifying I_j to I via $w_j\dots w_{n-1}\phi$, the coset $(W_{I_0}\times\dots\times W_{I_{n-1}})w\sigma$ identifies to $W_I^n\sigma'$ where

$$\begin{aligned} \sigma'(x_0,\dots,x_{n-1}) &= (x_1,\dots,x_{n-1},(w_0\dots w_{n-1}\phi)(x_0))\\ &= (x_1,\dots,x_{n-1},((v\phi^m)^{n'}\phi^{1-mn'})(x_0)), \end{aligned}$$

since in each case we have $w_0\dots w_{n-1}\phi = ((v\phi^m)^{n'}\phi^{1-mn'})(x_0))$. Now by Proposition 11.2 the ζ_d-rank of this last coset is equal to the $\zeta_{d'}$-rank of the coset $W((v\phi^m)^{n'}\phi^{1-mn'})^m$. But we have checked above that $((v\phi^m)^{n'}n\phi^{1-mn'})^m = v\phi^m$, thus the sought $\zeta_{d'}$-rank is the same as the $\zeta_{d'}$-rank of $W_Iv\phi^m$ which is 0 by assumption. □

11.2. **Case of irreducible Coxeter cosets.** We now look at the case of quasi-simple simply connected reductive groups $\mathbf{G}$, or equivalently at the case of irreducible Coxeter cosets $W\phi$. We will look at any real Coxeter coset $W\phi$ since it is not much more effort than to look just at the rational ones.

We use the classification. We are going to give, for each irreducible type and each possible d, a representative $w\phi$ of the ζ_d-good maximal elements, describing the corresponding I; since conjecturally for a given d all such elements are conjugate in the braid group, this describes all the ζ_d-good maximal elements. We also describe the relative complex reflection group $W(w\phi) := N_W(V)/C_W(V)$, where V is the ζ_d-eigenspace of $w\phi$. In the cases where the injection $C_{W'}(w\phi) \to N_W(V)/C_W(V) = W(w\phi)$ of the remark after Lemma 9.14, is surjective, where $W' = C_W(V_1)$ and V_1 is the fixed point subspace of $w\phi$ in the space spanned by the root lines of W_I,

we use it to deduce $W(w\phi)$ from $W' = C_W(V_1)$ since the centralizers of regular elements are known (see [BM, Annexe 1]).

Types A_n and ${}^2A_n$. $\underset{s_1}{\bigcirc}\!\!-\!\!\underset{s_2}{\bigcirc}\cdots\underset{s_n}{\bigcirc}$. ${}^2A_n$ is defined by the diagram automorphism ϕ which exchanges s_i and s_{n+1-i}.

For any integer $1 < d \leq n+1$, we define

$$v_d = s_1 s_2 \dots s_{n-\lfloor \frac{d}{2} \rfloor} s_n s_{n-1} \dots s_{\lfloor \frac{d+1}{2} \rfloor} \text{ and } J_d = \{s_i \mid \lfloor \frac{d+1}{2} \rfloor + 1 \leq i \leq n - \lfloor \frac{d}{2} \rfloor\}.$$

If d is odd we have $v_d = v'_d{}^{\phi} v'_d$, where $v'_d = s_1 s_2 \dots s_{n-\lfloor \frac{d}{2} \rfloor}$.

Now, for $1 < d \leq n+1$, let kd be the largest multiple of d less than or equal to $n+1$, so that $\frac{n+1}{2} < kd \leq n+1$ and $k = \lfloor \frac{n+1}{d} \rfloor$. We then define $w_d = v_{kd}^k$, $I_d = J_{kd}$ and if d is odd we define w'_d by

$$w'_d\phi = \begin{cases} (v'_{kd}\phi)^k & \text{if } k \text{ is odd,} \\ v_{kd}^{k/2}\,\phi & \text{if } k \text{ is even,} \end{cases}$$

Theorem 11.5. *For $W = W(A_n)$, ζ_d-good maximal elements exist for $1 < d \leq n+1$; a representative is w_d, with $I = I_d$ and $W(w_d) = G(d, 1, \lfloor \frac{n+1}{d} \rfloor)$.*

For $W\phi$, ζ_d-good maximal elements exist for the following d with representatives as follows:

- *$d \equiv 0 \pmod 4$, $1 < d \leq n+1$; a representative is $w_d\phi$ with $I = I_d$ and $W(w_d\phi) = G(d, 1, \lfloor \frac{n+1}{d} \rfloor)$.*
- *$d \equiv 2 \pmod 4$, $1 < d \leq 2(n+1)$; a representative is $w'_{d/2}\phi$ with $I = I_{d/2}$ and $W(w'_{d/2}\phi) = G(d/2, 1, \lfloor \frac{2(n+1)}{d} \rfloor)$.*
- *d odd, $1 < d \leq \frac{n+1}{2}$. If $d \neq 1$ a representative is $w_{2d}^2\phi$ with $I = I_{2d}$ and $W(w_{2d}^2\phi) = G(2d, 1, \lfloor \frac{n+1}{2d} \rfloor)$.*

Proof. We identify the Weyl group of type A_n as usual with $\mathfrak{S}_{n+1}$ by $s_i \mapsto (i, i+1)$; the automorphism ϕ maps to the exchange of i and $n+2-i$. An easy computation shows that the element v_d maps to the d-cycle $(1, 2, \dots, \lfloor \frac{d+1}{2} \rfloor, n+1, n, \dots, n+2-\lfloor \frac{d}{2} \rfloor)$ and that for d odd v'_d maps to the cycle $(1, 2, \dots, n - \frac{d-3}{2})$.

Lemma 11.6. *If d is even v_d and w_d are ϕ-stable. If d is odd we have $w_d = w'_d.{}^{\phi}w'_d$.*

Proof. That d is even implies $\lfloor \frac{d+1}{2} \rfloor = \lfloor \frac{d}{2} \rfloor$, thus in the above cycle ϕ exchanges the two sequences $1, 2, \dots, \lfloor \frac{d+1}{2} \rfloor$ and $n+1, n, \dots, n+2-\lfloor \frac{d}{2} \rfloor$, thus v_d is ϕ-stable. The same follows for w_d, with $k = \lfloor \frac{n+1}{d} \rfloor$, since kd is even if d is even.

For d odd we have

$$w'_d.{}^{\phi}w'_d = (w'_d\phi)^2 = \begin{cases} (v'_{kd}\phi)^{2k} & \text{if } k \text{ is odd,} \\ v_{kd}^{k/2}.{}^{\phi}(v_{kd}^{k/2}) & \text{if } k \text{ is even.} \end{cases}$$

If k is odd we have $(v'_{kd}\phi)^{2k} = (v'_{kd}{}^{\phi}v'_{kd})^k = v_{kd}^k = w_d$. If k is even then v_{kd} is ϕ-stable thus $v_{kd}^{k/2}.{}^{\phi}(v_{kd}^{k/2}) = v_{kd}^k = w_d$. □

Lemma 11.7. *For $1 < d \leq n+1$,*

- *the element v_d is J_d-reduced and stabilizes J_d.*
- *the element w_d is I_d-reduced and stabilizes I_d.*
- *for d odd, the element v'_d is J_d-reduced and $v'_d\phi$ stabilizes J_d.*
- *for d odd, the element w'_d is I_d-reduced and $w'_d\phi$ stabilizes I_d.*

Proof. The property for w_d (resp. w'_d) follows from that for v_d (resp. v'_d) and the definitions since being I_d-reduced and stabilizing I_d are properties stable by taking a power.

It is clear on the expression of v_d as a cycle that it fixes i and $i+1$ if $s_i \in J_d$ thus it fixes the simple roots corresponding to J_d, whence the lemma for v_d.

For d odd, $1 < d \leq n+1$, an easy computation shows that $v'_d = (1, 2, \dots, n - \frac{d-3}{2})$, and that $v'_d\phi$ preserves the simple roots corresponding to J_d. □

Lemma 11.8. *For $1 < d \leq n+1$ and for $0 < i \leq \lfloor \frac{d}{2} \rfloor$, we have*

- $l(v_d^i) = \frac{2i}{d} l(w_{J_d}^{-1} w_0)$ *and* $l(w_d^i) = \frac{2i}{d} l(w_{I_d}^{-1} w_0)$
- *(for d odd)* $l((v'_d\phi)^i \phi^{-i}) = \frac{i}{d} l(w_{J_d}^{-1} w_0)$ *and* $l((w'_d\phi)^i \phi^{-i}) = \frac{i}{d} l(w_{I_d}^{-1} w_0)$.

Proof. It is straightforward to see that the result for w_d (resp. w'_d) results from the result for v_d (resp. v'_d or v_d) and the definitions.

Note that the group W_{J_d} is of type A_{n-d}, thus $l(w_{J_d}^{-1} w_0) = \frac{n(n+1)}{2} - \frac{(n-d)(n-d+1)}{2} = \frac{(2n-d+1)d}{2}$.

We first prove the result for v_d and v'_d when $i = 1$. For odd d we have by definition $l(v'_d) = n - \frac{d-1}{2} = \frac{2n-d+1}{2}$ which is the formula we want for v'_d. To find the length of v_d one can use that $s_n s_{n-1} \dots s_{\lfloor \frac{d+1}{2} \rfloor}$ is $\{s_1, s_2, \dots, s_{n-1}\}$-reduced, thus adds to $s_1 s_2 \dots s_{n-\lfloor \frac{d}{2} \rfloor}$, which gives $l(v_d) = 2n - d + 1$, the result for v_d.

We now show by direct computation that when d is even $v_d^{d/2} = w_{J_d}^{-1} w_0$. Raising the cycle $(1, 2, \dots, \frac{d}{2}, n+1, n, \dots, n+2-\frac{d}{2})$ to the $d/2$-th power we get $(1, n+1)(2, n) \dots (\frac{d}{2}, n+2-\frac{d}{2})$ which gives the result since $w_{J_d} = (\frac{d}{2}+1, n+1-\frac{d}{2}) \dots (\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n+1}{2} \rfloor)$. The lemma follows for v_d with d even since its truth for $i = 1$ and $i = \frac{d}{2}$ implies its truth for all i between these values.

We show now similarly that for odd d we have $(v'_d\phi)^d = w_{J_d}^{-1} w_0 \phi^d$. Since ϕ acts on W by the inner automorphism given by w_0, this is the same as $(v'_d w_0)^d = w_{J_d}$. We find that $(1, 2, \dots, n - \frac{d-3}{2}) w_0 = (1, n+1, 2, n, 3, n-1 \dots, n - \frac{d-5}{2}, \frac{d+1}{2})(\frac{d+3}{2}, n - \frac{d-3}{2}) \dots (\lfloor \frac{n+3}{2} \rfloor, \lfloor \frac{n+4}{2} \rfloor)$ as a product of disjoint cycles, which gives the result since $(1, n+1, 2, n, 3, n-1, \dots, n - \frac{d-5}{2}, \frac{d+1}{2})$ is a d-cycle and $(\frac{d+3}{2}, n - \frac{d-3}{2}) \dots (\lfloor \frac{n+3}{2} \rfloor, \lfloor \frac{n+4}{2} \rfloor) = w_{J_d}$. This proves the lemma for w'_d by interpolating the other values of i as above.

It remains the case of v_d for odd d. We then have $v_d = (v'_d\phi)^2$ where the lengths add, and we deduce the result for v_d from the result for v'_d. □

Lemma 11.9. *The following elements are ζ_d-good*

- *For $1 < d \leq n+1$, the elements v_d and w_d.*
- *For $d \equiv 0 \pmod 4, d \leq n+1$ the elements $v_d\phi$ and $w_d\phi$.*
- *For $d \equiv 2 \pmod 4, d \leq 2(n+1)$ the elements $v'_{d/2}\phi$ and $w'_{d/2}\phi$.*
- *For d odd, $d \leq \frac{n+1}{2}$ the elements $v_{2d}^2\phi$ and $w_{2d}^2\phi$.*

Proof. In view of the previous lemmas, the only thing left to check is that in each case, the chosen element x in W (resp. $W\phi$) satisfies $x^d = 1$ (resp. $(x\phi)^d = \phi^d$). Once again, it is easy to check that the property for w_d (resp. w'_d) results from that for v_d (resp. v'_d or v_d) and the definitions.

It is clear that $v_d^d = 1$ since then it is a d-cycle, from which it follows that when $d \equiv 2 \pmod 4$ we have $(v'_{d/2}\phi)^d = v_{d/2}^{d/2} = 1$. The other cases are obvious. □

To prove the theorem, it remains to check that:

• The possible d for which the ζ_d-rank of W (resp. $W\phi$) is non-zero are as described in the theorem. In the untwisted case they are the divisors of one of the degrees, which are $2, \dots, n+1$. In the twisted case the pairs of degrees and factors are $(2,1), \dots, (i,(-1)^i), \dots, (n+1,(-1)^{n+1})$ and we get the given list by the formula for the ζ_d-rank recalled above Proposition 11.1.

• The coset $W_I w\phi$ has ζ_d-rank 0 on the subspace spanned by the root lines of W_I. For this we first have to describe the type of the coset, which is a consequence of the analysis we did to show that $w\phi$ stabilizes I. We may assume I non-empty.

Let us look first at the untwisted case. We found that w_d acts trivially on I_d, so the coset is of untwisted type A_{n-kd} where $k = \lfloor \frac{n+1}{d} \rfloor$. Since $1 + n - kd < d$ by construction, this coset has ζ_d-rank 0.

In the twisted case, if $d \equiv 0 \pmod 4$, the coset is $W_{I_d} w_d \phi$, which since w_d acts trivially on I_d and ϕ acts by the non-trivial diagram automorphism, is of type ${}^2A_{n-kd}$ where $k = \lfloor \frac{n+1}{d} \rfloor$. Since $n - kd = n - \lfloor \frac{n+1}{d} \rfloor d < d - 1$, this coset has ζ_d-rank 0.

If d is odd, the coset is $W_{I_{2d}} w_{2d}^2 \phi$, which since w_{2d} acts trivially on I_{2d} and ϕ acts by the non-trivial diagram automorphism, is of type ${}^2A_{n-2kd}$ where $k = \lfloor \frac{n+1}{2d} \rfloor$. Since $n - 2kd = n - \lfloor \frac{n+1}{2d} \rfloor 2d < 2d$, this coset has ζ_d-rank 0.

Finally, if $d \equiv 2$ modulo 4, the coset is $W_{I_{d/2}} w'_{d/2} \phi$. Let $k = \lfloor \frac{2(n+1)}{d} \rfloor$; then $W_{I_{d/2}}$ is of type $A_{n-kd/2}$. If k is even then $w'_{d/2} = w_{kd/2}^{k/2}$ and the coset is of type ${}^2A_{n-kd/2}$. Since $n - kd/2 < d/2 - 1$, this coset has ζ_d-rank 0. Finally if k is odd $w'_{d/2}\phi = (w'_{kd/2}\phi)^k$. Since $kd/2$ is odd, we found that $w'_{kd/2}\phi$ acts trivially on $I_{d/2}$ so the coset is of type $A_{n-kd/2}$, and has also has ζ_d-rank 0.

• Determine the group $W(w\phi)$ (resp. $W(w)$) in each case, We first give V_1 and the coset $C_W(V_1)w\phi$ or $C_W(V_1)w$. In the untwisted case w_d acts trivially on the roots of W_{I_d}, hence V_1 is spanned by these roots and $C_W(V_1)$ is generated by the reflection with respect to the roots orthogonal to those, which gives that $C_W(V_1)$ is of type $A_{d\lfloor \frac{n+1}{d} \rfloor - 1}$ if $d \not| n$ and A_n otherwise. In the twisted case if $d \equiv 0 \pmod 4$ since w_d acts trivially on the roots of W_{I_d} the space V_1 is spanned by the sums of the orbits of the roots under ϕ which is the non-trivial automorphism of that root system. Hence the type of the coset $C_W(V_1)w_d\phi$ is ${}^2A_{d\lfloor \frac{n+1}{d} \rfloor - 1}$ if n is odd and ${}^2A_{d\lfloor \frac{n+1}{d} \rfloor}$ if n is even. If d is odd a similar computation gives that the type of the coset $C_W(V_1)w_{2d}^2\phi$ is ${}^2A_{2d\lfloor \frac{n+1}{2d} \rfloor - 1}$ if n is odd and ${}^2A_{2d\lfloor \frac{n+1}{2d} \rfloor}$ if n is even. If $d \equiv 2 \pmod 4$ $w'_{d/2}\phi$ acts also by the non-trivial automorphism on $W_{I_{d/2}}$ and we get that the coset $C_W(V_1)w'_{d/2}\phi$ is of type ${}^2A_{\frac{d}{2}\lfloor \frac{2(n+1)}{d} \rfloor}$ if n and $\lfloor \frac{2(n+1)}{d} \rfloor$ have the same parity and ${}^2A_{\frac{d}{2}\lfloor \frac{2(n+1)}{d} \rfloor - 1}$ otherwise.

Knowing the type of the coset in each case, we deduce the group $W(w\phi)$ (resp. $W(w)$) as in the remark at the beginning of subsection 11.2. □

Type B_n 

. For d even, $2 \le d \le 2n$ we define

$$v_d = s_{n+1-d/2} \dots s_2 s_1 s_2 \dots s_n \text{ and } J_d = \{s_i \mid 1 \le i \le n - d/2\}.$$

Note that v_{2n} is the Coxeter element $s_1 s_2 \dots s_n$. Now for $1 \le d \le 2n$, that we require even if $d > n$, we define w_d as follows: let kd be the largest even multiple of d less than or equal to $2n$ so that $k = \lfloor \frac{2n}{d} \rfloor$ if d is even and $k = 2\lfloor \frac{n}{d} \rfloor$ is d is odd. We define $w_d = v_{kd}^k$ and $I_d = J_{kd}$.

Theorem 11.10. *For $W = W(B_n)$, ζ_d-good maximal elements exist for odd d less than or equal to n and even d less than or equal to $2n$. A representative is w_d, with $I = I_d$; we have $W(w_d) = G(d, 1, \lfloor \frac{2n}{d} \rfloor)$ if d is even and $W(w_d) = G(2d, 1, \lfloor \frac{n}{d} \rfloor)$ if d is odd.*

Proof. We identify as usual the Weyl group of type B_n to the group of signed permutations on $\{1, \dots n\}$ by $s_i \mapsto (i-1, i)$ for $i \geq 2$ and $s_1 \mapsto (1, -1)$. The element v_d maps to the d-cycle (or signed $d/2$-cycle) given by $(n+1-d/2, n+2-d/2, \dots, n-1, n, d/2-n-1, d/2-n-2, \dots, -n)$. This element normalizes J_d and acts trivially on the corresponding roots, so is J_d-reduced. The same is thus true for w_d and I_d, since these properties carry to powers.

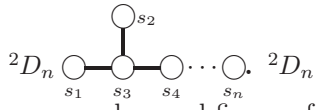
Lemma 11.11. *For $0 < i \leq \lfloor \frac{d}{2} \rfloor$ we have $l(v_d^i) = \frac{2i}{d} l(w_{J_d}^{-1} w_0)$ and $l(w_d^i) = \frac{2i}{d} l(w_{I_d}^{-1} w_0)$.*

Proof. As in Lemma 11.8 it is sufficient to prove the lemma for v_d, which we do now. To find the length of v_d we note that $s_1 s_2 \dots s_n$ is $\{s_2, s_3, \dots, s_n\}$-reduced so that the lengths of $s_{n+1-d/2} \dots s_2$ and of $s_1 s_2 \dots s_n$ add, whence $l(v_d) = 2n - d/2$. Since $l(w_0) = n^2$ and $l(w_{J_d}) = (n - d/2)^2$ we have $l(w_{I_d}^{-1} w_0) = nd - d^2/4$, which gives the result for $i = 1$. Written as permutations w_0 is the product of all sign changes and w_{I_d} is the product of all sign changes on the set $\{1, \dots, n - d/2\}$; a direct computation shows that $v_d^{d/2}$ is the product of all sign changes on $\{n+1-d/2, \dots, n\}$, hence $v_d^{d/2} = w_{I_d}^{-1} w_0$. The lemma follows for the other values of d. □

Since $v_d^{d/2} = w_{I_d}^{-1} w_0$ we have $v_d^d = 1$, so the same property is true for w_d, thus the above lemma shows that v_d and w_d are ζ_d-good elements.

Note also that Theorem 11.10 describes all d such that W has non-zero ζ_d-rank since the degrees of $W(B_n)$ are all the even integers from 2 to $2n$. We prove now the maximality property 9.11(iv) for w_d. If k is as in the definition of w_d, the group W_{I_d} is a Weyl group of type $B_{n-kd/2}$ and w_d acts trivially on I_d. Since $n - kd/2 < d$ the ζ_d-rank of $W_{I_d} w_d$ is zero on the subspace spanned by the roots corresponding to I_d.

It remains to get the type of $W(w_d)$. Since w_d acts trivially on I_d the space V_1 of Lemma 9.14 is spanned by the root lines of W_{I_d} and $C_W(V_1)$ is spanned by the roots orthogonal to those, so is of type $B_{kd/2}$. We then deduce the group $W(w_d)$ as in the remark at the beginning of subsection 11.2, as the centralizer of a ζ_d-regular element in a group of type $B_{kd/2}$. □

Types D_n and ${}^2D_n$ 

. ${}^2D_n$ is defined by the diagram automorphism ϕ which exchanges s_1 and s_2 and fixes s_i for $i > 2$.

For d even, $2 \leq d \leq 2(n-1)$ we define

$$v_d = s_{n+1-d/2} \dots s_3 s_2 s_1 s_3 \dots s_n \text{ and } J_d = \begin{cases} \emptyset \text{ if } d = 2(n-1) \\ \{s_i \mid 1 \leq i \leq n - d/2\} \text{ otherwise.} \end{cases}$$

Note that $v_{2(n-1)}$ is a Coxeter element. Then for $1 \leq d \leq 2(n-1)$, that we require even if $d > n$, we let kd be the largest even multiple of d less than $2n$, so that $k = \lfloor \frac{2n-2}{d} \rfloor$ if d is even and $k = 2\lfloor \frac{n-1}{d} \rfloor$ if d is odd, and define $w_d = v_{kd}^k$ and $I_d = J_{kd}$.

Note that v_d, and thus w_d, are ϕ-stable.

Theorem 11.12.
- *For $W = W(D_n)$ there exist ζ_d-good maximal elements for odd d less than or equal to n and even d less than or equal to $2(n-1)$. When d does not divide n a representative is w_d, with $I = I_d$; in this case, if d is odd $W(w_d) = G(2d, 1, \lfloor \frac{n-1}{d} \rfloor)$ and if d is even $W(w_d) = G(d, 1, \lfloor \frac{2n-2}{d} \rfloor)$.*

  If $d|n$ a representative is $w_n^{n/d}$ where $w_n = s_1 s_2 s_3 \dots s_n s_3 s_4 \dots s_{n-1}$. In this case $I = \emptyset$ and $W(w_n^{n/d}) = G(2d, 2, n/d)$.
- *For $W\phi$ there exist ζ_d-good maximal elements for odd d less than n, for even d less than $2(n-1)$ and for $d = 2n$. Except in the case when d divides $2n$ and $2n/d$ is odd a representative is $w_d\phi$, with $I = I_d$ and $W(w_d\phi) = G(2d, 1, \lfloor \frac{n-1}{d} \rfloor)$ if d is odd and $W(w_d\phi) = G(d, 1, \lfloor \frac{2n-2}{d} \rfloor)$ if d is even. In the excluded case a representative is $(w_{2n}\phi)^{2n/d}$ where $w_{2n} = s_1 s_3 s_4 \dots s_n$. In this case $I = \emptyset$ and $W((w_{2n}\phi)^{2n/d}) = G(d, 2, 2n/d)$.*

Proof. The cases D_n with $d|n$ or ${}^2D_n$ with $d|2n$ and $2n/d$ odd involve regular elements, so are dealt with in [BM]. We thus consider only the other cases.

We identify the Weyl group of type D_n to the group of signed permutations on $\{1, \dots n\}$ with an even number of sign changes, by mapping s_i to $(i-1, i)$ for $i \neq 2$ and s_2 to $(1, -2)(-1, 2)$. For d even v_d maps to $(1, -1)(n+1-d/2, n+2-d/2, \dots, n-1, n, d/2-n-1, \dots, 1-n, -n)$. This element normalizes J_d: when $J_d \neq \emptyset$, it exchanges the simple roots corresponding to s_1 and s_2 and acts trivially on the other simple roots indexed by J_d, so it is J_d-reduced. It follows that w_d normalizes I_d and is I_d-reduced.

Lemma 11.13. *For $0 < i \leq \lfloor \frac{d}{2} \rfloor$ we have $l(v_d^i) = \frac{2i}{d} l(w_{J_d}^{-1} w_0)$ and $l(w_d^i) = \frac{2i}{d} l(w_{I_d}^{-1} w_0)$.*

Proof. As in Lemma 11.8 it is sufficient to prove the lemma for v_d. To find the length of v_d we note that $s_2 s_1 s_3 s_4 \dots s_n$ is $\{s_3, \dots, s_n\}$-reduced so that the lengths of $s_{n+1-d/2} \dots s_3$ and of $s_2 s_1 s_3 \dots s_n$ add, whence $l(v_d) = 2n - 1 - d/2$. Since $l(w_0) = n^2 - n$ and $l(w_{J_d}) = (n - d/2)^2 - (n - d/2)$, we have $l(w_{J_d}^{-1} w_0) = d/2(2n - 1 - d/2)$. which gives the result for $i = 1$. Written as permutations $w_0 = (1, -1)^n (2, -2) \dots (n, -n)$ and $w_{J_d} = (1, -1)^{n-d/2} (2, -2) \dots (n-d/2, d/2-n)$; a direct computation shows that $v_d^{d/2} = (1, -1)^{d/2} (n+1-d/2, d/2-n-1) \dots (n, -n)$, hence $v_d^{d/2} = w_{J_d}^{-1} w_0$. The lemma follows for smaller i. □

Since $v_d^{d/2} = w_{J_d}^{-1} w_0$ and J_d is w_0 stable we have $v_d^d = 1$, so the same property follows for w_d which shows that v_d and w_d are ζ_d-good elements.

We also note that the theorem describes all d such that the ζ_d-rank is not zero, since the degrees of $W(D_n)$ are all the even integers from 2 to $2n-2$ and n, and in the twisted case the factor associated to the degree n is -1 and the other factors are equal to 1.

Since w_d is ϕ-stable the element $w_d\phi$ is also ζ_d-good.

We now check Lemma 9.11(iv), that is that the ζ_d-rank of $W_{I_d} w_d$ in the untwisted case, resp. $W_{I_d} w_d \phi$ in the twisted case is 0 on the subspace spanned by the roots corresponding to I_d. This property is clear if $I_d = \emptyset$. Otherwise:

• In the untwisted case the type of the coset is $D_{n-kd/2}$ if k is even and ${}^2D_{n-kd/2}$ if k is odd, where k is as in the definition of w_d. In both cases the set of values i

such that the ζ_i-rank is not 0 consists of the even i less than $2n-kd$, the odd i less than $n-kd/2$ and in the twisted case (k odd) $i=2n-kd$. Since if d is even we have $2n-kd\leq d$ and if d is odd we have $n-kd/2\leq d$, the only case where d could be in this set is k odd and $d=2n-kd$, which means that $\frac{k+1}{2}d=n$. But d is assumed not to divide n, so this case does not happen.

• In the twisted case the type of the coset is $D_{n-kd/2}$ if k is odd and ${}^2D_{n-kd/2}$ if k is even. In both cases the set of values i such that the ζ_i-rank is not 0 consists of the even i less than $2n-kd$, the odd i less than $n-kd/2$ and in the twisted case (k even) $i=2n-kd$. Since if d is even we have $2n-kd\leq d$ and if d is odd we have $n-kd/2\leq d$, the only case where d could be in this set is k even and $d=2n-kd$, which means that $(k+1)d=2n$. But this is precisely the excluded case.

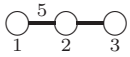
We now give $C_W(V_1)$, where V_1 is as in Lemma 9.14, in each case where I is not empty. In the untwisted case, if d is odd the group $C_W(V_1)$ is of type $D_{d\lfloor\frac{n-1}{d}\rfloor}$; if d is even the group $C_W(V_1)$ is of type $D_{\frac{d}{2}\lfloor\frac{2n-2}{d}\rfloor+1}$ if $\lfloor\frac{2n-2}{d}\rfloor$ is odd and $D_{\frac{d}{2}\lfloor\frac{2n-2}{d}\rfloor}$ if $\lfloor\frac{2n-2}{d}\rfloor$ is even. In the twisted case, if d is odd the coset $C_W(V_1)w\phi$ is of type ${}^2D_{d\lfloor\frac{n-1}{d}\rfloor+1}$ and if d is even the coset is of type ${}^2D_{\frac{d}{2}\lfloor\frac{2n-2}{d}\rfloor+1}$ if $\lfloor\frac{2n-2}{d}\rfloor$ is even and $D_{\frac{d}{2}\lfloor\frac{2n-2}{d}\rfloor}$ if $\lfloor\frac{2n-2}{d}\rfloor$ is odd. In all cases except if d is even and $\lfloor\frac{2n-2}{d}\rfloor$ is even (resp. odd) in the untwisted case (resp. twisted case) we then deduce the group $W(w\phi)$ (resp. $W(w)$) as in the remarks at the beginning of subsection 11.2 and after Lemma 9.14, since in these cases the centralizer of the regular element $w\phi$ (resp. w) in the parabolic subgroup $W'=C_W(V_1)$ has the (known) reflection degrees of $W(w\phi)$ (resp. $W(w)$). In the excluded cases the group $C_{W'}(w\phi)$ or $C_{W'}(w)$ is isomorphic to $G(d,2,\lfloor\frac{2n-2}{d}\rfloor)$ which does not have the reflection degrees of $W(w\phi)$, resp. $W(w)$. This means that the morphism of the remark after Lemma 9.14 is not surjective. We can prove in this case that $W(w\phi)$ or $W(w)$ is $G(d,1,\lfloor\frac{2n-2}{d}\rfloor)$ since it is an irreducible complex reflection group by [Br, 5.6.6] and it is the only one which has the right reflection degrees apart from the exceptions in low rank given by $G_5,G_{10},G_{15},G_{18},G_{26}$; we can exclude these since they do not have $G(d,2,\lfloor\frac{2n-2}{d}\rfloor)$ as a reflection subgroup. □

Types $I_2(n)$ and ${}^2I_2(n)$. All eigenvalues ζ such that the ζ-rank is non-zero are regular, so this case can be found in [BM].

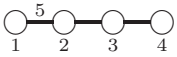
Exceptional types. Below are tables for exceptional finite Coxeter groups giving information on ζ_d-good maximal elements for each d. They were obtained with the `GAP` package `Chevie`: first, the conjugacy class of good ζ_d-maximal elements as described in Lemma 9.13 was determined; then we determined I for an element of that class, which gave $l(w_I)$. The next step was to determine the elements of the right length $2(l(w_0)-l(w_I))/d$ in that conjugacy class; this required care in large groups like E_8. The best algorithm is to start from an element of minimal length in the class (known by [GP]) and conjugate by Coxeter generators until all elements of the right length are reached.

In the following tables, we give for each possible d and each possible I for that d a representative good $w\phi$, and give the number of possible $w\phi$. We then describe the coset $W_Iw\phi$ by giving, if $I\neq\emptyset$, in the column I the permutation induced by $w\phi$ of the nodes of the Coxeter diagram indexed by I. Then we describe the isomorphism type of the complex reflection group $N_W(W_Iw\phi)/W_I=N_W(V)/C_W(V)$, where V is the ζ_d-eigenspace of $w\phi$. Finally, in the cases where $I\neq\emptyset$, we give the

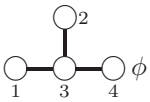
isomorphism type of $W' = C_W(V_1)$, where V_1 is the 1-eigenspace of $w\phi$ on the subspace spanned by the root lines of I. We note that there are 4 cases where $N_{W'}(V)/C_{W'}(V) \lneq N_W(V)/C_W(V)$: for $d = 5$ in ${}^2E_6$, for $d = 4$ or 5 in E_7 and for $d = 9$ in E_8.

H_3:  The reflection degrees are $2, 6, 10$.

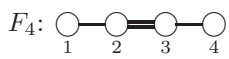
d	representative w	#good w	$C_W(w)$
10	$w_{10} = 123$	4	Z_{10}
6	$w_6 = 32121$	6	Z_6
5	w_{10}^2	4	Z_{10}
3	w_6^2	6	Z_6
2	w_0	1	H_3
1	$\cdot$	1	H_3

H_4:  The reflection degrees are $2, 12, 20, 30$.

d	representative w	#good w	$C_W(w)$
30	$w_{30} = 1234$	8	Z_{30}
20	$w_{20} = 432121$	12	Z_{20}
15	w_{30}^2	8	Z_{30}
12	$w_{12} = 2121432123$	22	Z_{12}
10	w_{30}^3 or w_{20}^2	24	G_{16}
6	w_{30}^5 or w_{12}^2	40	G_{20}
5	w_{30}^6 or w_{20}^4	24	G_{16}
4	w_{20}^5 or w_{12}^3	60	G_{22}
3	w_{30}^{10} or w_{12}^4	40	G_{20}
2	w_0	1	H_4
1	$\cdot$	1	H_4

${}^3D_4$:  ϕ does the permutation $(1, 2, 4)$. The reflection degrees are $2, 4, 4, 6$ with corresponding factors $1, \zeta_3, \zeta_3^2, 1$.

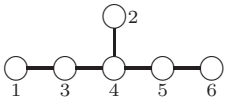
d	representative $w\phi$	#good $w\phi$	$C_W(w\phi)$
12	$w_{12}\phi = 13\phi$	6	Z_4
6	$w_6\phi = 1243\phi$	8	G_4
3	$w_6^2\phi$	8	G_4
2	$w_0\phi$	1	G_2
1	ϕ	1	G_2

F_4:  The reflection degrees are $2, 6, 8, 12$.

d	representative w	#good w	$C_W(w)$
12	$w_{12} = 1234$	8	Z_{12}
8	$w_8 = 214323$	14	Z_8
6	w_{12}^2	16	G_5
4	w_{12}^3 or w_8^2	12	G_8
3	w_{12}^4	16	G_5
2	w_0	1	F_4
1	$\cdot$	1	F_4

${}^2F_4$: ϕ does the permutation $(1,4)(2,3)$. The factors, in increasing order of the degrees, are $1, -1, 1, -1$.

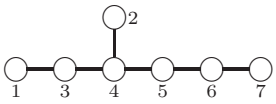
d	representative $w\phi$	#good $w\phi$	$C_W(w\phi)$
24	$w_{24}\phi = 12\phi$	6	Z_{12}
12	$w_{12}\phi = 3231\phi$	10	Z_6
8	$(w_{24}\phi)^3$	12	G_8
4	$(w_{12}\phi)^3$	24	G_{12}
2	$w_0\phi$	1	$I_2(8)$
1	ϕ	1	$I_2(8)$

E_6:  The reflection degrees are $2, 5, 6, 8, 9, 12$.

d	representative w	#good w	I	$N_W(W_I w)/W_I$	$C_W(V_1)$
12	$w_{12} = 123654$	8		Z_{12}	
9	$w_9 = 12342654$	24		Z_9	
8	$w_8 = 123436543$	14		Z_8	
6	w_{12}^2	16		G_5	
5	24231454234565	8	(3)	Z_5	A_5
	12435423456543	8	(4)		
	12314235423654	8	(5)		
4	w_8^2 or w_{12}^3	12		G_8	
3	w_{12}^4 or w_9^3	80		G_{25}	
2	w_0	1		F_4	
1	$\cdot$	1		E_6	

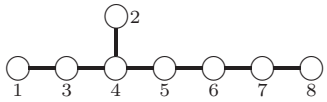
${}^2E_6$: ϕ does the permutation $(1,6)(3,5)$. The factors, in increasing order of the degrees, are $1,-1,1,1,-1,1$.

d	representative $w\phi$	#good$w\phi$	I	$N_W(W_I w\phi)/W_I$	$C_W(V_1)w\phi$
18	$w_{18}\phi = 1234\phi$	24		Z_9	
12	$w_{12}\phi = 123654\phi$	8		Z_{12}	
10	2431543ϕ	8	(3)	Z_5	${}^2A_5$
	5423145ϕ	8	(4)		
	3143542ϕ	8	(5)		
8	$w_8\phi = 123436543\phi$	14		Z_8	
6	$(w_{18}\phi)^3$	80		G_{25}	
4	$(w_{12}\phi)^3$	12		G_8	
3	$w_{12}^4\phi$	16		G_5	
2	$w_0\phi$	1		E_6	
1	ϕ	1		F_4	

E_7: 

The reflection degrees are $2,6,8,10,12,14,18$.

d	representative w	#good w	I	$N_W(W_I w)/W_I$	$C_W(V_1)$
18	$w_{18} = 1234567$	64		Z_{18}	
14	$w_{14} = 123425467$	160		Z_{14}	
12	$w_{12} = 1342546576$	8	$(2,5,7)$	Z_{12}	E_6
10	$w_{10a} = 134254234567$	8	$(2,4)$	Z_{10}	D_6
	$w_{10b} = 243154234567$	8	$(3,4)$		
	$w_{10c} = 124354265437$	8	$(4,5)$		
9	w_{18}^2	64		Z_{18}	
8	134234542346576	14	$(2)(5,7)$	Z_8	D_5
7	w_{14}^2	160		Z_{14}	
6	w_{18}^3 or w_{12}^2	800		G_{26}	
5	w_{10a}^2	8	$(2)(4)$	Z_{10}	A_5
	w_{10b}^2	8	$(3)(4)$		
	w_{10c}^2	8	$(4)(5)$		
4	$1314234542314654234567654234 56$	12	$(2)(5)(7)$	G_8	D_4
3	w_{18}^6 or w_{12}^4	800		G_{26}	
2	w_0	1		E_7	
1	$\cdot$	1		E_7	

E_8:  The reflection degrees are $2, 8, 12, 14, 18, 20, 24, 30$.

d	representative w	#good w	I	$N_W(W_I w)/W_I$	$C_W(V_1)$
30	$w_{30} = 12345678$	128		Z_{30}	
24	$w_{24} = 1234254678$	320		Z_{24}	
20	$w_{20} = 123425465478$	624		Z_{20}	
18	$w_{18a} = 1342542345678$	16	$(2,4)$	Z_{18}	E_7
	$w_{18b} = 2431542345678$	16	$(3,4)$		
	$w_{18c} = 1243542654378$	16	$(4,5)$		
15	w_{30}^2	128		Z_{30}	
14	$w_{14a} = 13423454234565768$	128	(2)	Z_{14}	E_7
	$w_{14b} = 24231454234565768$	88	(3)		
	$w_{14c} = 12435423456543768$	108	(4)		
	$w_{14d} = 12342543654276548$	68	(5)		
12	w_{24}^2	2696		G_{10}	
10	w_{30}^3 or w_{20}^2	3370		G_{16}	
9	w_{18a}^2	16	$(2)(4)$	Z_{18}	E_6
	w_{18b}^2	16	$(3)(4)$		
	w_{18c}^2	16	$(4)(5)$		
8	w_{24}^3	7748		G_9	
7	w_{14a}^2	128	(2)	Z_{14}	E_7
	w_{14b}^2	88	(3)		
	w_{14c}^2	108	(4)		
	w_{14d}^2	68	(5)		
6	w_{30}^5 or w_{24}^4	4480		G_{32}	
5	w_{30}^6 or w_{20}^4	3370		G_{16}	
4	w_{24}^6 or w_{20}^5	15120		G_{31}	
3	w_{30}^{10} or w_{24}^8	4480		G_{32}	
2	w_0	1		E_8	
1	$\cdot$	1		E_8	

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LAMFA, CNRS and Université de Picardie-Jules Verne
E-mail address: digne@u-picardie.fr

IMJ, Université Paris VII
E-mail address: jmichel@math.jussieu.fr