



**HAL**  
open science

## On the estimation of the second order parameter in extreme-value theory

El Hadji Deme, Laurent Gardes, Stéphane Girard

► **To cite this version:**

El Hadji Deme, Laurent Gardes, Stéphane Girard. On the estimation of the second order parameter in extreme-value theory. 2011. hal-00634573v1

**HAL Id: hal-00634573**

**<https://hal.science/hal-00634573v1>**

Preprint submitted on 21 Oct 2011 (v1), last revised 16 Nov 2012 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the estimation of the second order parameter in extreme-value theory

El hadji DEME<sup>(1,3)</sup>, Laurent GARDES<sup>(2,\*)</sup> and Stéphane GIRARD<sup>(3)</sup>

<sup>(1)</sup> LERSTAD, Université Gaston Berger, Saint-Louis, Sénégal.

<sup>(2)</sup> Université de Strasbourg & CNRS, IRMA UMR 7501.

7 rue René Descartes, 67084 Strasbourg Cedex, France.

<sup>(3)</sup> Team Mistis, INRIA Rhône-Alpes & Laboratoire Jean Kuntzmann.

655 avenue de l'Europe, Montbonnot, 38334 Saint-Ismier Cedex, France.

\* Corresponding author, [gardes@unistra.fr](mailto:gardes@unistra.fr)

## Abstract

The extreme-value index  $\gamma$  plays an important parameter in extreme-value theory since it controls the first order behavior of the distribution tail. In the literature, numerous estimators of this parameter have been proposed especially in the case of heavy-tailed distributions, which is the situation considered here. Most of these estimators depend on the  $k$  largest observations of the underlying sample. Their bias is controlled by the second order parameter  $\rho$ . In order to reduce the bias of  $\gamma$ 's estimators or to select the best number  $k$  of observations to use, the knowledge of  $\rho$  is essential. In this paper, we propose a simple approach to estimate the second order parameter  $\rho$  leading to both existing and new estimators. We establish a general result that can be used to easily prove the asymptotic normality of a large number of estimators proposed in the literature or to compare different estimators within a given family.

**Keywords:** Extreme-value theory; Heavy-tailed distribution; Extreme-value index; Second order parameter; Asymptotic properties.

## 1 Introduction

Extreme-value theory establishes the asymptotic behavior of the largest observations in a sample. It provides methods for extending the empirical distribution function beyond the observed data. It is thus possible to estimate quantities related to the tail of a distribution such as small exceedance probabilities or extreme quantiles. We refer to [9, 22] for general accounts on extreme-value theory. More specifically, let  $X_1, \dots, X_n$  be a sequence of random variables (rv), independent and identically distributed from a cumulative distribution function (cdf)  $F$ . Extreme-value theory establishes

that the asymptotic distribution of the maximum  $X_{n,n} = \max\{X_1, \dots, X_n\}$  properly rescaled is the extreme-value distribution with cdf

$$G_\gamma(x) = \exp(-(1 + \gamma x)_+)^{-1/\gamma}$$

where  $y_+ = \max(y, 0)$ . The parameter  $\gamma \in \mathbb{R}$  is referred to as the extreme-value index. Here, we focus on the case where  $\gamma > 0$ . In such a situation,  $F$  is said to belong to the maximum domain of attraction of the Fréchet distribution. In this domain of attraction, a simple characterization of distributions is available: the quantile function  $U(x) := F^{\leftarrow}(1 - 1/x)$  can be written as

$$U(x) = x^\gamma \ell(x),$$

where  $\ell$  is a slowly varying function at infinity *i.e.* for all  $\lambda > 0$ ,

$$\lim_{x \rightarrow \infty} \frac{\ell(\lambda x)}{\ell(x)} = 1. \quad (1)$$

The distribution  $F$  is said to be heavy tailed and the extreme-value parameter  $\gamma$  governs the heaviness of the tail. The estimation of  $\gamma$  is a central topic in the analysis of such distributions. Several estimators have thus been proposed in the statistical literature and their asymptotic distributions established under a second order condition: There exist a function  $A(x) \rightarrow 0$  of constant sign for large values of  $x$  and a second order parameter  $\rho \leq 0$  such that, for every  $\lambda > 0$ ,

$$\lim_{x \rightarrow \infty} \frac{1}{A(x)} \log \left( \frac{\ell(\lambda x)}{\ell(x)} \right) = K_\rho(\lambda) := \int_1^\lambda u^{\rho-1} du. \quad (2)$$

Let us highlight that (2) implies that  $|A|$  is regularly varying with index  $\rho$ , see [14]. Hence, as the second order parameter  $\rho$  decreases, the rate of convergence in (1) increases. Thus, the knowledge of  $\rho$  can be of high interest in real problems. For example, the second order parameter is of primordial importance in the adaptive choice of the best number of upper order statistics to be considered in the estimation of the extreme-value index [21]. The estimation of  $\rho$  can also be used to proposed bias reduced estimators of the extreme value index (see for instance [4, 18, 20]) even though some bias reduction can be achieved with the canonical choice  $\rho = -1$  as suggested in [10, 19]. For the above mentioned reasons, the estimation of the second order parameter  $\rho$  has received a lot of attention in the extreme-value literature, see for instance [3, 6, 11, 15, 16, 23, 26].

In this paper, we propose a simple and general approach to estimate  $\rho$ . Let  $\mathbb{I} = {}^t(1, \dots, 1) \in \mathbb{R}^d$ . The two main ingredients of our approach are a statistic  $T_n = T_n(X_1, \dots, X_n) \in \mathbb{R}^d$  verifying the following three assumptions:

**(T1)** There exists rvs  $\omega_n, \chi_n$  and a function  $f : \mathbb{R}^- \rightarrow \mathbb{R}^d$  such that  $\omega_n^{-1}(T_n - \chi_n \mathbb{I}) \xrightarrow{\mathbb{P}} f(\rho)$ .

and a function  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$  such that

**(Ψ1)**  $\psi(x + \lambda \mathbb{I}) = \psi(x)$  for all  $x \in \mathbb{R}^d$  and  $\lambda \in \mathbb{R}$ ,

( $\Psi 2$ )  $\psi(\lambda x) = \psi(x)$  for all  $\lambda \in \mathbb{R} \setminus \{0\}$ .

Note that (**T1**) imposes that  $T_n$  properly normalized converges in probability to some function of  $\rho$ , while ( $\Psi 1$ ) and ( $\Psi 2$ ) mean that  $\psi$  is both location and shift invariant. Starting from these three assumptions, we straightforwardly obtain that

$$\psi(\omega_n^{-1}(T_n - \chi_n \mathbb{I})) = \psi(T_n) \xrightarrow{\mathbb{P}} \psi(f(\rho)),$$

under a continuity condition on  $\psi$ . Denoting by  $Z_n := \psi(T_n)$  and by  $\varphi := \psi \circ f : \mathbb{R}^- \rightarrow \mathbb{R}$ , we obtain  $Z_n \xrightarrow{\mathbb{P}} \varphi(\rho)$ . It is thus clear that, under an additional regularity assumption and assuming that both  $Z_n$  and  $\varphi$  are known,  $\rho$  can be consistently estimated thanks to  $\varphi^{-1}(Z_n)$ . This estimation principle is described more precisely in Section 2. The consistency and asymptotic normality of the proposed estimator is also established. Examples of  $T_n$  statistics are presented in Section 3. Some functions  $\psi$  are proposed in Section 4 and it is shown that the above mentioned estimators [6, 11, 15, 16] can be read as particular cases of our approach. As a consequence, this remark permits to establish their asymptotic properties in a simple and unified way. We also illustrate how a lot of new asymptotically Gaussian estimators can be derived from this framework.

## 2 Main results

Recall that  $T_n$  is a  $\mathbb{R}^d$ - random vector verifying (**T1**) and  $\psi$  is a function  $\mathbb{R}^d \rightarrow \mathbb{R}$  verifying ( $\Psi 1$ ) and ( $\Psi 2$ ). We further assume that:

( $\Psi 3$ ) There exist  $J_0 \subseteq \mathbb{R}^-$  and an open interval  $J \subset \mathbb{R}$  such that  $\varphi = \psi \circ f$  is a bijection  $J_0 \rightarrow J$ .

Under this assumption, the following estimator of  $\rho$  may be considered:

$$\hat{\rho}_n = \varphi^{-1}(Z_n) \mathbb{1}\{Z_n \in J\}. \quad (3)$$

To derive the consistency of  $\hat{\rho}_n$ , an additional regularity assumption is introduced:

( $\Psi 4$ )  $\psi$  is continuous in a neighborhood of  $f(\rho)$  and  $f$  is continuous in a neighborhood of  $\rho$ .

The proof of the next result is based on the heuristic consideration of Section 1 and is detailed in the Appendix.

**Theorem 1.** *If (**T1**) and ( $\Psi 1$ )–( $\Psi 4$ ) hold then  $\hat{\rho}_n \xrightarrow{\mathbb{P}} \rho$  as  $n \rightarrow \infty$ .*

The asymptotic normality of  $\hat{\rho}_n$  can be established under a stronger version of ( $\Psi 4$ ):

( $\Psi 5$ )  $\psi$  is continuously differentiable in a neighborhood of  $f(\rho)$  and  $f$  is continuously differentiable in a neighborhood of  $\rho$ ,

and the assumption that a normalized version of  $T_n$  is itself asymptotically Gaussian:

**(T2)** There exists two rvs  $\omega_n, \chi_n$ , a sequence  $v_n \rightarrow \infty$ , two functions  $f, m : \mathbb{R}^- \rightarrow \mathbb{R}^d$  and a  $d \times d$  matrix  $\Sigma$  such that  $v_n(\omega_n^{-1}(T_n - \chi_n \mathbb{I}) - f(\rho)) \xrightarrow{d} \mathcal{N}_d(m(\rho), \gamma^2 \Sigma)$ .

**Theorem 2.** Suppose **(T2)**, **(Ψ1)**–**(Ψ3)** and **(Ψ5)** hold. If  $\rho \in J_0$  and  $\varphi'(\rho) \neq 0$ , then

$$v_n(\hat{\rho}_n - \rho) \xrightarrow{d} \mathcal{N} \left( \frac{m_\psi(\rho)}{\varphi'(\rho)}, \frac{\sigma_\psi^2(\rho)}{(\varphi'(\rho))^2} \right),$$

with  $\varphi'(\rho) = {}^t f'(\rho) \nabla \psi(f(\rho))$  and where we have defined

$$\begin{aligned} m_\psi(\rho) &:= {}^t m(\rho) \nabla \psi(f(\rho)), \\ \sigma_\psi^2(\rho) &:= {}^t \nabla \psi(f(\rho)) \Sigma \nabla \psi(f(\rho)). \end{aligned}$$

### 3 Examples of $T_n$ statistics

Let  $X_{1,n} \leq \dots \leq X_{n,n}$  be the sample of ascending order statistics and  $k = k_n$  be an intermediate sequence *i.e.* such that  $k \rightarrow \infty$  and  $k/n \rightarrow 0$  as  $n \rightarrow \infty$ . Most extreme-value estimators are based either on the log-excesses ( $\log X_{n-j+1,n} - \log X_{n-k,n}$ ) or on the rescaled log-spacings  $j(\log X_{n-j+1,n} - \log X_{n-j,n})$  defined for  $j = 1, \dots, k$ . In the following, two examples of  $T_n$  statistics are presented basing on weighted means of the log-excesses and of the rescaled log-spacings.

The first example is based on the statistics

$$R_k(\tau) = \frac{1}{k} \sum_{j=1}^k H_\tau \left( \frac{j}{k+1} \right) j(\log X_{n-j+1,n} - \log X_{n-j,n}), \quad (4)$$

where  $H_\tau : [0, 1] \rightarrow \mathbb{R}$  is a weight function indexed by a parameter  $\tau \in (0, \infty)$ . Without loss of generality, one can assume that  $H_\tau$  integrates to one. This statistic is used for instance in [1] to estimate the extreme-value index  $\gamma$  and in [15, 23] to estimate the second order parameter  $\rho$ . It is a particular case of the kernel statistic introduced in [7]. Let us also note that, in the case where  $H_\tau(u) = 1$  for all  $u \in [0, 1]$ ,  $R_k(\tau)$  reduces to the well-known Hill estimator [24]. The asymptotic properties of  $R_k(\tau)$  require some technical condition (denoted by **(C1)**) on the weight function  $H_\tau$ . It has been first introduced in [1] and it is recalled hereafter. Introducing the operator

$$\mu : h \in L_2([0, 1]) \longrightarrow \mu(h) = \int_0^1 h(u) du \in \mathbb{R}$$

and  $I_t(u) = u^{-t}$  for  $t \leq 0$  and  $u \in (0, 1]$ , the condition can be written as

**(C1)**  $H_\tau \in L_2([0, 1])$ ,  $\mu(|H_\tau| I_{\rho+1+\varepsilon}) < \infty$  and

$$H_\tau(t) = \frac{1}{t} \int_0^t u(\nu) d\nu \text{ and}$$

for some  $\varepsilon > 0$  and for some function  $u$  satisfying for all  $j = 1, \dots, k$

$$\left| (k+1) \int_{(j-1)/(k+1)}^{j/(k+1)} u(t) dt \right| \leq g \left( \frac{j}{k+1} \right),$$

where  $g$  is a positive continuous and integrable function defined on  $(0, 1)$ . Furthermore, for  $\eta \in \{0, 1\}$ , and  $k \rightarrow \infty$ :

$$\begin{aligned} \frac{1}{k} \sum_{j=1}^k H_\tau \left( \frac{j}{k+1} \right) \left( \frac{j}{k+1} \right)^{-\eta\rho} &= \mu(H_\tau I_{\eta\rho}) + o(k^{-1/2}), \\ \max_{j \in \{1, \dots, k\}} \left| H_\tau \left( \frac{j}{k+1} \right) \right| &= o(k^{1/2}). \end{aligned}$$

It is then possible to define a statistic  $T_n^{(R)}$  on the basis of  $R_k(\tau)$  as

$$T_n^{(R)} = \left( T_{n,i}^{(R)} = (R_k(\tau_i)/\gamma)^{\theta_i}, i = 1, \dots, d \right),$$

where  $\theta_i, i = 1, \dots, d$  are arbitrary real parameters. In the next lemma, it is proved that  $T_n^{(R)}$  satisfies condition **(T2)** under a third order condition, which is a refinement of (2):

**(C2)** There exist functions  $A(x) \rightarrow 0$  and  $B(x) \rightarrow 0$  both of constant sign for large values of  $x$ , a second order parameter  $\rho \leq 0$  and a third order parameter  $\beta \leq 0$  such that, for every  $\lambda > 0$ ,

$$\lim_{x \rightarrow \infty} \frac{(\log \ell(\lambda x) - \log \ell(x)) / A(x) - K_\rho(\lambda)}{B(x)} = L_{(\rho, \beta)}(\lambda) := \int_1^\lambda s^{\rho-1} \int_1^s u^{\beta-1} dud s,$$

and the functions  $|A|$  and  $|B|$  are regularly varying functions with index  $\rho$  and  $\beta$  respectively.

This condition is the cornerstone for establishing the asymptotic normality of estimators of  $\rho$ . Let us denote by  $Y_{n-k,n}$  the  $n - k$  largest order statistics from a  $n$ -sample of standard Pareto rv.

**Lemma 1.** *Suppose **(C1)**, **(C2)** hold and let  $k = k_n$  be an intermediate sequence  $k$  such that*

$$k \rightarrow \infty, n/k \rightarrow \infty, k^{1/2}A(n/k) \rightarrow \infty, k^{1/2}A^2(n/k) \rightarrow \lambda_A \text{ and } k^{1/2}A(n/k)B(n/k) \rightarrow \lambda_B, \quad (5)$$

for  $\lambda_A \in \mathbb{R}$  and  $\lambda_B \in \mathbb{R}$ . Then, the random vector  $T_n^{(R)}$  satisfies **(T2)** with  $\omega_n^{(R)} = A(Y_{n-k,n})/\gamma$ ,  $\chi_n^{(R)} = 1$ ,  $v_n = k^{1/2}A(n/k)$ ,

$$\begin{aligned} f^{(R)}(\rho) &= (\theta_i \mu(H_{\tau_i} I_\rho), i = 1, \dots, d), \\ m^{(R)}(\rho) &= \left( \lambda_A \frac{\theta_i(\theta_i - 1)}{2\gamma} \mu^2(H_{\tau_i} I_\rho) - \lambda_B \theta_i \mu(H_{\tau_i} I_\rho K_{-\beta}); i = 1, \dots, d \right), \end{aligned}$$

and, for  $(i, j) \in \{1, \dots, d\}^2$ ,  $\Sigma_{i,j}^{(R)} = \theta_i \theta_j \mu(H_{\tau_i} H_{\tau_j})$ .

The proof is a straightforward consequence of Theorem 2 and Appendix A.5 in [15]. The second example requires some additional notations. Let us consider the operator

$$\vartheta : (h_1, h_2) \in L_2([0, 1]) \times L_2([0, 1]) \longrightarrow \vartheta(h_1, h_2) = \int_0^1 \int_0^1 \frac{h_1(u)h_2(v)}{u \wedge v} dudv - \mu(h_1)\mu(h_2) \in \mathbb{R}$$

and the two functions  $\bar{I}_t(u) = (1 - u)^{-t}$  and  $J_t(u) = (-\log u)^{-t}$  defined for  $t \leq 0$  and  $u \in [0, 1)$ .

The statistics of interest are

$$S_k(\tau, \alpha) = \frac{1}{k} \sum_{j=1}^k G_{\tau, \alpha} \left( \frac{j}{k+1} \right) (\log X_{n-j+1,n} - \log X_{n-k,n})^\alpha, \quad (6)$$

where  $G_{\tau,\alpha}$  is a positive function indexed by two positive parameters  $\alpha$  and  $\tau$ . Without loss of generality, it can be assumed that  $\mu(G_{\tau,\alpha}J_{-\alpha}) = 1$ . In [8] an estimator of  $\gamma$  based on this statistic is introduced in the particular case where  $G$  is constant. Most recently, in [6, 12, 23, 25] the statistics  $S_k(\tau, \alpha)$  is used to estimate the parameters  $\gamma$  and  $\rho$ . The asymptotic distribution of these estimators is obtained under the following assumption on the function  $G_{\tau,\alpha}$ .

**(C3)** The function  $G_{\tau,\alpha}$  is positive, non-increasing and integrable on  $(0, 1)$ . Furthermore, there exists  $\delta > 1/2$  such that  $\mu(G_{\tau,\alpha}I_{-\delta}) < \infty$  and  $\mu(G_{\tau,\alpha}\bar{I}_{-\delta}) < \infty$ .

It is then possible to define a statistic  $T_n^{(S)}$  on the basis of  $S_k(\tau, \alpha)$  as

$$T_n^{(S)} = \left( T_{n,i}^{(S)} = (S_k(\tau_i, \alpha_i)/\gamma^{\alpha_i})^{\theta_i}, i = 1, \dots, d \right).$$

The following result is the analog of Lemma 1 for the above statistics.

**Lemma 2.** *Suppose (C2), (C3) hold. If the intermediate sequence  $k$  satisfy (5) then the random vector  $T_n^{(S)}$  satisfies (T2) with  $\omega_n^{(S)} = A(n/k)/\gamma$ ,  $\chi_n^{(S)} = 1$ ,  $v_n = k^{1/2}A(n/k)$ ,*

$$\begin{aligned} f^{(S)}(\rho) &= (-\theta_i \alpha_i \mu(G_{\tau_i, \alpha_i} J_{-\alpha_i+1} K_{-\rho}); i = 1, \dots, d), \\ m^{(S)}(\rho) &= \left( \lambda_A \frac{\theta_i \alpha_i (\alpha_i - 1)}{2\gamma} \mu(G_{\tau_i, \alpha_i} J_{2-\alpha_i} K_{-\rho}^2) + \lambda_B \alpha_i \theta_i \mu(G_{\tau_i, \alpha_i} J_{1-\alpha_i} L_{-\rho, -\beta}); i = 1, \dots, d \right), \end{aligned}$$

and, for  $(i, j) \in \{1, \dots, d\}^2$ ,  $\Sigma_{i,j}^{(S)} = \theta_i \theta_j \alpha_i \alpha_j \vartheta(G_{\tau_i, \alpha_i} J_{1-\alpha_i}, G_{\tau_j, \alpha_j} J_{1-\alpha_j})$ .

The proof is a straightforward consequence of Proposition 2 and Lemma 1 in [6]. In the next section, we illustrate how the combination of the statistics  $T_n^{(R)}$  and  $T_n^{(S)}$  with some function  $\psi$  following (3) can lead to existing or new estimators of  $\rho$ .

## 4 Applications

In this section, we propose estimators of  $\rho$  based on the statistic  $T_n^{(R)}$  (subsection 4.1) and  $T_n^{(S)}$  (subsection 4.2). In both cases,  $d = 8$  and the following function  $\psi_\delta : \mathcal{D} \mapsto \mathbb{R} \setminus \{0\}$  is considered

$$\psi_\delta(x_1, \dots, x_8) = \tilde{\psi}_\delta(x_1 - x_2, x_3 - x_4, x_5 - x_6, x_7 - x_8),$$

where  $\delta \geq 0$ ,  $\mathcal{D} = \{(x_1, \dots, x_8) \in \mathbb{R}^8; x_1 \neq x_2, x_3 \neq x_4, \text{ and } (x_5 - x_6)(x_7 - x_8) > 0\}$ , and  $\tilde{\psi}_\delta : \mathbb{R}^4 \mapsto \mathbb{R}$  is given by:

$$\tilde{\psi}_\delta(y_1, \dots, y_4) = \frac{y_1}{y_2} \left( \frac{y_4}{y_3} \right)^\delta.$$

Let us highlight that  $\psi_\delta$  verifies the invariance properties **(Ψ1)** and **(Ψ2)**.

### 4.1 Estimators based on the statistic $R_k(\tau)$

Since  $d = 8$ , the statistic  $T_n^{(R)}$  depends on 16 parameters:  $\{(\theta_i, \tau_i) \in (0, \infty)^2, i = 1, \dots, 8\}$ . The following condition on these parameters is introduced. Let  $\tilde{\theta} = (\tilde{\theta}_1, \dots, \tilde{\theta}_4) \in (0, \infty)^4$  with  $\tilde{\theta}_3 \neq \tilde{\theta}_4$ .

**(C4)**  $\{\theta_i = \tilde{\theta}_{\lceil i/2 \rceil}, i = 1, \dots, 8\}$  with  $\delta = (\tilde{\theta}_1 - \tilde{\theta}_2)/(\tilde{\theta}_3 - \tilde{\theta}_4)$ . Furthermore,  $\tau_1 < \tau_2 \leq \tau_3 < \tau_4$ ,  
 $\tau_5 < \tau_6 \leq \tau_7 < \tau_8$ ,

where  $\lceil x \rceil = \inf\{n \in \mathbb{N} | x \leq n\}$ . Under this condition,  $T_n^{(R)}$  involves 12 free parameters. We also introduce the following notation:  $Z_n^{(R)} = \psi_\delta(T_n^{(R)})$  and  $\varphi_\delta^{(R)} = \psi_\delta \circ f^{(R)}$ . Note that, since  $\delta = (\tilde{\theta}_1 - \tilde{\theta}_2)/(\tilde{\theta}_3 - \tilde{\theta}_4)$ , it is easy to check that  $Z_n^{(R)}$  does not depend on the unknown parameter  $\gamma$ . We now establish the asymptotic normality of the estimator  $\hat{\rho}_n^{(R)}$  defined by (3) when the statistic  $T_n^{(R)}$  and the function  $\psi_\delta$  are used. The following additional condition is required:

**(C5)** The function  $\nu_\rho(\tau) = \mu(H_\tau I_\rho)$  is differentiable with, for all  $\rho < 0$  and all  $\tau \in \mathbb{R}$ ,  $\nu'_\rho(\tau) > 0$ .

Let us denote for  $i \in \{1, \dots, 4\}$ ,

$$m_A^{(R,i)} = \exp \left\{ (\tilde{\theta}_i - 1) \mu \left( (H_{\tau_{2i-1}} + H_{\tau_{2i}}) I_\rho \right) \right\}, \quad m_B^{(R,i)} = \exp \left\{ \frac{\mu \left( (H_{\tau_{2i-1}} - H_{\tau_{2i}}) I_\rho K_{-\beta} \right)}{\mu \left( (H_{\tau_{2i-1}} - H_{\tau_{2i}}) I_\rho \right)} \right\},$$

and for  $u \in [0, 1]$ ,

$$v^{(R,i)}(u) = \exp \left\{ \frac{H_{\tau_{2i-1}}(u) - H_{\tau_{2i}}(u)}{\mu \left( (H_{\tau_{2i-1}} - H_{\tau_{2i}}) I_\rho \right)} \right\}.$$

For the sake of simplicity, we also introduce  $m_A^{(R)} = (m_A^{(R,i)}, i = 1, \dots, 4)$ ,  $m_B^{(R)} = (m_B^{(R,i)}, i = 1, \dots, 4)$  and  $v^{(R)} = (v^{(R,i)}, i = 1, \dots, 4)$ .

**Corollary 1.** *Suppose (C1), (C2), (C4) and (C5) hold. There exist two intervals  $J$  and  $J_0$  such that for all  $\rho \in J_0$  and for a sequence  $k$  satisfying (5),*

$$k^{1/2} A(n/k) (\hat{\rho}_n^{(R)} - \rho) \xrightarrow{d} \mathcal{N} \left( \frac{\lambda_A}{2\gamma} \mathcal{AB}_1^{(R)}(\delta, \rho) - \lambda_B \mathcal{AB}_2^{(R)}(\delta, \rho, \beta), \mathcal{AV}^{(R)}(\delta, \rho) \right)$$

where

$$\begin{aligned} \mathcal{AB}_1^{(R)}(\delta, \rho) &= \frac{\varphi_\delta^{(R)}(\rho)}{[\varphi_\delta^{(R)}]'(\rho)} \log \tilde{\psi}_\delta(m_A^{(R)}), \\ \mathcal{AB}_2^{(R)}(\delta, \rho, \beta) &= \frac{\varphi_\delta^{(R)}(\rho)}{[\varphi_\delta^{(R)}]'(\rho)} \log \tilde{\psi}_\delta(m_B^{(R)}), \\ \mathcal{AV}^{(R)}(\delta, \rho) &= \left( \frac{\varphi_\delta^{(R)}(\rho)}{[\varphi_\delta^{(R)}]'(\rho)} \right)^2 \mu \left( \log^2 \tilde{\psi}_\delta(v^{(R)}) \right). \end{aligned}$$

Note that this result can be read as an extension of [15], Proposition 3, in two ways. First, we do not limit ourselves to the case  $\delta = 1$ . Second, we do not assume that the function  $\varphi_\delta^{(R)}$  is a bijection, but it is shown to be a consequence of **(C4)**. Besides, as illustrated below, the proof of Corollary 1 is very simple basing on Theorem 2 and Lemma 1.

**Proof** – Clearly,  $\psi_\delta$  satisfies **(Ψ1)** and **(Ψ2)**. Moreover, Lemma 1 shows that **(T2)** holds. To apply Theorem 2 it only remains to prove that **(Ψ3)** and **(Ψ5)** are satisfied. First remark that



under **(C4)** and **(C5)**,  $\varphi_\delta^{(R)}(\rho)$  is well defined for all  $\rho \leq 0$  since  $f^{(R)}(\rho) \in \mathcal{D}$ . Furthermore, from Lemma 1, we have for  $i = 1, \dots, 4$ ,

$$T_{n,2i-1}^{(R)} - T_{n,2i}^{(R)} = \frac{\tilde{\theta}_i A(Y_{n-k,n})}{\gamma} (\mu(H_{\tau_{2i-1}} I_\rho) - \mu(H_{\tau_{2i}} I_\rho)) (1 + o_P(1)),$$

as  $n$  goes to infinity. Hence, conditions **(C4)** and **(C5)** imply that  $T_n^{(R)} \in \mathcal{D}$ . Finally, using Lerch's Theorem (see [5], page 345), condition **(C4)** implies that there exists  $\rho_0 < 0$  such that the first derivative of  $\varphi_\rho^{(R)}$  is non zero at  $\rho_0$ . Thus, the inverse function theorem insures the existence of intervals  $J_0$  and  $J$  for which the function  $\varphi_\delta^{(R)}$  is a continuously differentiable bijection from  $J_0$  to  $J$ . In conclusion, conditions **(Ψ3)** and **(Ψ5)** are satisfied and Theorem 2 applies.  $\blacksquare$

As an example, the function  $H_\tau : u \in [0, 1] \mapsto \tau u^{\tau-1}$ ,  $\tau \geq 1$  satisfies conditions **(C1)** and **(C4)** since  $\nu_\rho(\tau) = \tau/(\tau - \rho)$ . Letting  $\tau_1 \leq \tau_5$ ,  $\tau_2 = \tau_3$ ,  $\tau_4 = \tau_8$  and  $\tau_6 = \tau_7$  leads to a simple expression of  $\varphi_\delta^{(R)}$ :

$$\varphi_\delta^{(R)}(\rho) = \omega(\delta, \tilde{\theta}) \left( \frac{\tau_4 - \rho}{\tau_1 - \rho} \right) \left( \frac{\tau_5 - \rho}{\tau_4 - \rho} \right)^\delta \text{ where } \omega(\delta, \tilde{\theta}) = \left( \frac{\tilde{\theta}_1(\tau_1 - \tau_2)}{\tilde{\theta}_2(\tau_2 - \tau_4)} \right) \left( \frac{\tilde{\theta}_4(\tau_6 - \tau_4)}{\tilde{\theta}_3(\tau_5 - \tau_6)} \right)^\delta.$$

Moreover, one also has explicit forms for  $J_0$  and  $J$  in two situations:

- (i) If  $0 \leq \delta < \delta_0 := (\tau_4 - \tau_1)/(\tau_4 - \tau_5)$  then  $\varphi_\delta^{(R)}$  is increasing from  $J_0 = \mathbb{R}^-$  to  $J = \omega(\delta, \tilde{\theta}) \bullet (1, \tilde{\psi}_\delta(\tau_4, \tau_1, \tau_4, \tau_5))$ .
- (ii) If  $\delta \geq \delta_1 := \delta_0 \tau_5 / \tau_1$  then  $\varphi_\delta^{(R)}$  is decreasing from  $J_0 = \mathbb{R}^-$  to  $J = \omega(\delta, \tilde{\theta}) \bullet (\tilde{\psi}_\delta(\tau_4, \tau_1, \tau_4, \tau_5), 1)$ .

Here,  $\bullet$  denotes the scaling operator. The case  $\delta \in [\delta_0, \delta_1]$  is not considered here, since one can show that, in this situation,  $J_0 \subsetneq \mathbb{R}^-$  and thus the condition  $\rho \in J_0$  of Corollary 1 is not necessarily satisfied. Let us now list some particular cases where the inverse function of  $\varphi_\delta^{(R)}$  is explicit.

**Example 1.** Let  $\delta = 1$  i.e.  $\tilde{\theta}_1 - \tilde{\theta}_2 = \tilde{\theta}_3 - \tilde{\theta}_4$ . The rv  $Z_n^{(R)}$  is denoted by  $Z_{n,1}^{(R)}$ . Since  $\delta_0 > 1$ , we are in situation (i) and

$$\hat{\rho}_{n,1}^{(R)} = \frac{\tau_5 \omega(1, \tilde{\theta}) - \tau_1 Z_{n,1}^{(R)}}{\omega(1, \tilde{\theta}) - Z_{n,1}^{(R)}} \mathbb{1}\{Z_{n,1}^{(R)} \in \omega(1, \tilde{\theta}) \bullet (1, \tilde{\psi}_1(\tau_4, \tau_1, \tau_4, \tau_5))\}.$$

Remark that this estimator coincides with the one proposed in [15], Lemma 1.

**Example 2.** Let  $\delta = 0$  i.e.  $\tilde{\theta}_1 = \tilde{\theta}_2$ . The rv  $Z_n^{(R)}$  is thus denoted by  $Z_{n,2}^{(R)}$ . Again, we are in situation (i) and a new estimator of  $\rho$  is obtained

$$\hat{\rho}_{n,2}^{(R)} = \frac{\tau_4 \omega(0, \tilde{\theta}) - \tau_1 Z_{n,2}^{(R)}}{\omega(0, \tilde{\theta}) - Z_{n,2}^{(R)}} \mathbb{1}\{Z_{n,2}^{(R)} \in \omega(0, \tilde{\theta}) \bullet (1, \tilde{\psi}_0(\tau_4, \tau_1, \tau_4, \tau_5))\}.$$

**Example 3.** Let  $\tau_1 = \tau_5$ . In this case  $\delta_0 = \delta_1 = 1$  and thus, we are in situation (i) if  $\delta < 1$  and in situation (ii) otherwise. In this case, the rv  $Z_n^{(R)}$  is denoted by  $Z_{n,3}^{(R)}$ . A new estimator of  $\rho$  is obtained:

$$\hat{\rho}_{n,3}^{(R)} = \frac{\tau_4 (Z_{n,3}^{(R)} / \omega(\delta, \tilde{\theta}))^{1/(\delta-1)} - \tau_1}{(Z_{n,3}^{(R)} / \omega(\delta, \tilde{\theta}))^{1/(\delta-1)} - 1} \mathbb{1}\{Z_{n,3}^{(R)} \in J\}.$$

## 4.2 Estimators based on the statistic $S_k(\tau, \alpha)$

The statistic  $T_n^{(S)}$  depends on 24 parameters:  $\{(\theta_i, \tau_i, \alpha_i) \in (0, \infty)^3, i = 1, \dots, 8\}$ . Let  $(\zeta_1, \dots, \zeta_4) \in (0, \infty)^4$  with  $\zeta_3 \neq \zeta_4$ . In the following, we assume that

**(C6)**  $\{\theta_i \alpha_i = \zeta_{\lceil i/2 \rceil}, i = 1, \dots, 8\}$  with  $\delta = (\zeta_1 - \zeta_2)/(\zeta_3 - \zeta_4)$ . Furthermore,  $(\tau_{2i-1}, \alpha_{2i-1}) \neq (\tau_{2i}, \alpha_{2i})$ , for  $i = 1, \dots, 4$  and, for  $i = 3, 4$ ,  $(\tau_{2i-1}, \alpha_{2i-1}) \leq (\tau_{2i}, \alpha_{2i})$ ,

where  $(x, y) \neq (s, t)$  means that  $x \neq s$  and/or  $y \neq t$  and  $(x, y) \leq (s, t)$  means that  $x \leq s$  and  $y \leq t$ . We introduce the notations:  $Z_n^{(S)} = \psi_\delta(T_n^{(S)})$  and  $\varphi_\delta^{(S)} = \psi_\delta \circ f^{(S)}$ . Under this condition,  $T_n^{(S)}$  involves 20 free parameters. Besides, since  $\delta = (\zeta_1 - \zeta_2)/(\zeta_3 - \zeta_4)$ , it is easy to check that  $Z_n^{(S)}$  does not depend on the unknown parameter  $\gamma$ . To establish the asymptotic distribution of the estimator  $\hat{\rho}_n^{(S)}$ , the following condition is required:

**(C7)** For all  $\rho < 0$ , the function  $\nu_\rho(\tau, \alpha) = \mu(G_{\tau, \alpha} J_{1-\alpha} K_\rho)$  is differentiable with  $\frac{\partial}{\partial \tau} \nu_\rho(\tau, \alpha) > 0$  and  $\frac{\partial}{\partial \alpha} \nu_\rho(\tau, \alpha) > 0$  for all  $\alpha > 0$  and all  $\tau \in \mathbb{R}$ .

For  $i = 1, \dots, 4$ , we introduce the notations:

$$m_A^{(S,i)} = \exp \left\{ \frac{(\alpha_{2i-1} - 1)\mu(G_{\tau_{2i-1}, \alpha_{2i-1}} J_{2-\alpha_{2i-1}} K_{-\rho}^2) - (\alpha_{2i} - 1)\mu(G_{\tau_{2i}, \alpha_{2i}} J_{2-\alpha_{2i}} K_{-\rho}^2)}{\mu(G_{\tau_{2i}, \alpha_{2i}} J_{1-\alpha_{2i}} K_{-\rho}) - \mu(G_{\tau_{2i-1}, \alpha_{2i-1}} J_{1-\alpha_{2i-1}} K_{-\rho})} \right\},$$

$$m_B^{(S,i)} = \exp \left\{ \frac{\mu(G_{\tau_{2i-1}, \alpha_{2i-1}} J_{1-\alpha_{2i-1}} L_{(-\rho, -\beta)}) - \mu(G_{\tau_{2i}, \alpha_{2i}} J_{1-\alpha_{2i}} L_{(-\rho, -\beta)})}{\mu(G_{\tau_{2i}, \alpha_{2i}} J_{1-\alpha_{2i}} K_{-\rho}) - \mu(G_{\tau_{2i-1}, \alpha_{2i-1}} J_{1-\alpha_{2i-1}} K_{-\rho})} \right\},$$

and for  $u \in [0, 1]$ ,

$$v^{(S,i)}(u) = \frac{G_{\tau_{2i-1}, \alpha_{2i-1}}(u) J_{1-\alpha_{2i-1}}(u) - G_{\tau_{2i}, \alpha_{2i}}(u) J_{1-\alpha_{2i}}(u)}{\mu(G_{\tau_{2i}, \alpha_{2i}} J_{1-\alpha_{2i}} K_{-\rho}) - \mu(G_{\tau_{2i-1}, \alpha_{2i-1}} J_{1-\alpha_{2i-1}} K_{-\rho})}.$$

Let us also consider  $m_A^{(S)} = (m_A^{(S,i)}, i = 1, \dots, 4)$  and  $m_B^{(S)} = (m_B^{(S,i)}, i = 1, \dots, 4)$ . The next result is a direct consequence of Theorem 2 and Lemma 2.

**Corollary 2.** *Suppose (C2), (C3), (C6) and (C7) hold. There exist two intervals  $J$  and  $J_0$  such that for all  $\rho \in J_0$  and for a sequence  $k$  satisfying (5),*

$$k^{1/2} A(n/k)(\hat{\rho}_n^{(S)} - \rho) \xrightarrow{d} \mathcal{N} \left( \frac{\lambda_A}{2\gamma} \mathcal{AB}_1^{(S)}(\delta, \rho) + \lambda_B \mathcal{AB}_2^{(S)}(\delta, \rho, \beta), \mathcal{AV}^{(S)}(\delta, \rho) \right)$$

where

$$\mathcal{AB}_1^{(S)}(\delta, \rho) = \frac{\varphi_\delta^{(S)}(\rho)}{[\varphi_\delta^{(S)}]'\!(\rho)} \log \tilde{\psi}_\delta(m_A^{(S)}),$$

$$\mathcal{AB}_2^{(S)}(\delta, \rho, \beta) = \frac{\varphi_\delta^{(S)}(\rho)}{[\varphi_\delta^{(S)}]'\!(\rho)} \log \tilde{\psi}_\delta(m_B^{(S)}),$$

$$\mathcal{AV}^{(S)}(\delta, \rho) = \left( \frac{\gamma \varphi_\delta^{(S)}(\rho)}{[\varphi_\delta^{(S)}]'\!(\rho)} \right)^2 \vartheta \left( v^{(S,1)} - v^{(S,2)} - \delta(v^{(S,3)} - v^{(S,4)}), v^{(S,1)} - v^{(S,2)} - \delta(v^{(S,3)} - v^{(S,4)}) \right).$$

**Proof** – The proof follows the same lines as the one of Corollary 1. It consists in remarking that, under **(C6)** and **(C7)**, one has  $f^{(S)}(\rho) \in \mathcal{D}$  and  $T_n^{(S)} \in \mathcal{D}$  since,

$$T_{n,2i-1}^{(S)} - T_{n,2i}^{(S)} = \frac{\zeta_i A(n/k)}{\gamma} \left( \mu(G_{\tau_{2i-1}, \alpha_{2i-1}} J_{1-\alpha_{2i-1}} K_\rho) - \mu(G_{\tau_{2i}, \alpha_{2i}} J_{1-\alpha_{2i}} K_\rho) \right) (1 + o_P(1)).$$

■

Let us highlight that Proposition 5, Proposition 7 and Proposition 9 of [6] are particular cases of Corollary 2 for three different value of  $\delta$  ( $\delta = 2$ ,  $\delta = 1$  and  $\delta = 0$  respectively). The asymptotic normality of the estimators proposed in [16] and in [12] can also be easily established with Corollary 2.

As an example of function  $G_{\tau, \alpha}$ , one can consider the function defined on  $[0, 1]$  by:

$$G_{\tau, \alpha}(u) = \frac{g_{\tau-1}(u)}{\int_0^1 g_{\tau-1}(x) J_{-\alpha}(x) dx} \text{ for } \tau \geq 1 \text{ and } \alpha > 0,$$

where the function  $g_\tau$  is given by

$$g_0(x) = 1, \quad g_{\tau-1}(x) = \frac{\tau}{\tau-1} (1 - x^{\tau-1}), \forall \tau > 1.$$

Clearly, the function  $G_{\tau, \alpha}$  satisfies condition **(C7)** and, under **(C6)**, the expression of  $\varphi_\rho^{(S)}$  is

$$\varphi_\delta^{(S)}(\rho) = \frac{\zeta_1}{\zeta_2} \left( \frac{\zeta_4}{\zeta_3} \right)^\delta \frac{\nu_\rho(\tau_1, \alpha_1) - \nu_\rho(\tau_2, \alpha_2)}{\nu_\rho(\tau_3, \alpha_3) - \nu_\rho(\tau_4, \alpha_4)} \left[ \frac{\nu_\rho(\tau_7, \alpha_7) - \nu_\rho(\tau_8, \alpha_8)}{\nu_\rho(\tau_5, \alpha_5) - \nu_\rho(\tau_6, \alpha_6)} \right]^\delta$$

with

$$\nu_\rho(\tau, \alpha) = \frac{1 - (1 - \rho)^{-\alpha} + (\tau - \rho)^{-\alpha} - \tau^{-\alpha}}{\alpha \rho (1 - \tau^{-\alpha-1})} \quad \text{if } \tau \neq 1 \quad \text{and} \quad \nu_\rho(1, \alpha) = \frac{1 - (1 - \rho)^\alpha - 1}{\alpha \rho (1 - \rho)^\alpha}.$$

Even if Corollary 2 ensures the existence of intervals  $J_0$  and  $J$ , they are impossible to specify in the general case. In the following, we consider several sets of parameters where these intervals can be easily exhibited and for which the inverse function  $\varphi_\delta^{(S)}$  admits an explicit form. To this end, it is assumed that  $\tau_2 = \tau_3 = \tau_5 = \tau_6 = \tau_7 = \tau_8 = \alpha_7 = 1$  and the following notation is introduced:

$$\omega^*(\delta, \zeta) = \frac{\zeta_1}{\zeta_2} \left( \frac{3\zeta_4}{\zeta_3} \right)^\delta.$$

In all the examples below,  $J_0 = \mathbb{R}^-$  and thus the condition  $\rho \in J_0$  is always satisfied. The first three examples correspond to existing estimators of the second order parameter while the two last examples give rise to new estimators.

**Example 4.** Let  $\delta = 0$  (i.e.  $\zeta_1 = \zeta_2$ ),  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ ,  $\tau_1 = \alpha_5 = \alpha_8 = 2$  and  $\tau_4 = \alpha_6 = 3$ . Denoting by  $Z_{n,4}^{(S)}$  the rv  $Z_n^{(S)}$ , the estimator of  $\rho$  is given by:

$$\hat{\rho}_{n,4}^{(S)} = \frac{6(Z_{n,4}^{(S)} + 2)}{3Z_{n,4}^{(S)} + 4} \mathbb{1}\{Z_{n,4}^{(S)} \in (-2, -4/3)\}.$$

Note that this estimator corresponds to the estimator  $\hat{\rho}_{n,k}^{[2]}$  defined in [6], Section 5.2.

**Example 5.** Let  $\delta = 0$ ,  $\alpha_1 = \alpha_3 = \alpha_4 = 1$ ,  $\tau_1 = \tau_4 = \alpha_2 = \alpha_5 = \alpha_8 = 2$  and  $\alpha_6 = 3$ . Denoting by  $Z_{n,5}^{(S)}$  the rv  $Z_n^{(S)}$ , we find back the estimator  $\hat{\rho}_{n,k}^{[3]}$  proposed in [6], Section 5.2:

$$\hat{\rho}_{n,5}^{(S)} = \frac{2(Z_{n,5}^{(S)} - 2)}{2Z_{n,5}^{(S)} - 1} \mathbb{1}\{Z_{n,5}^{(S)} \in (1/2, 2)\}.$$

**Example 6.** Let  $\delta = 0$ ,  $\tau_1 = \tau_4 = \alpha_1 = 1$ ,  $\alpha_2 = \alpha_3 = \alpha_5 = \alpha_8 = 2$  and  $\alpha_4 = \alpha_6 = 3$ . Denoting by  $Z_{n,6}^{(S)}$  the rv  $Z_n^{(S)}$ , the estimator of  $\rho$  is given by:

$$\hat{\rho}_{n,6}^{(S)} = \frac{3(Z_{n,6}^{(S)} - \omega^*(0, \zeta))}{Z_{n,6}^{(S)} - 3\omega^*(0, \zeta)} \mathbb{1}\{Z_{n,6}^{(S)} \in \omega^*(0, \zeta) \bullet (1/3, 1)\}.$$

This estimator corresponds to the one proposed in [11].

**Example 7.** Consider the case  $\delta = 1$  (i.e.  $\zeta_1 - \zeta_2 = \zeta_3 - \zeta_4$ ),  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ ,  $\tau_1 = \alpha_5 = \alpha_8 = 2$  and  $\tau_4 = \alpha_6 = 3$ . Denoting by  $Z_{n,7}^{(S)}$  the rv  $Z_n^{(S)}$ , a new estimator of  $\rho$  is given by:

$$\hat{\rho}_{n,7}^{(S)} = \frac{Z_{n,7}^{(S)} + 4/3\omega^*(1, \zeta)}{2Z_{n,7}^{(S)} + 4/3\omega^*(1, \zeta)} \mathbb{1}\{Z_{n,7}^{(S)} \in \omega^*(1, \zeta) \bullet (-4/3, -2/3)\}.$$

**Example 8.** Let  $\delta = 1$ ,  $\alpha_1 = \alpha_3 = \alpha_4 = 1$ ,  $\tau_1 = \tau_4 = \alpha_2 = \alpha_5 = \alpha_8 = 2$  and  $\alpha_6 = 3$ . Denoting by  $Z_{n,8}^{(S)}$  the rv  $Z_n^{(S)}$ , we obtain a new estimator of  $\rho$ :

$$\hat{\rho}_{n,8}^{(S)} = \frac{3Z_{n,8}^{(S)} - 4\omega^*(1, \zeta)}{Z_{n,8}^{(S)} - \omega^*(1, \zeta)} \mathbb{1}\{Z_{n,8}^{(S)} \in \omega^*(1, \zeta) \bullet (1/2, 2/3)\}.$$

To summarize, we have illustrated how Theorem 2 may be used to prove the asymptotic normality of estimators built on the statistics  $T_n^{(R)}$  and  $T_n^{(S)}$ : Corollary 1 and Corollary 2 cover a large number of estimators proposed in the literature. Four new estimators of  $\rho$  have been introduced:  $\hat{\rho}_{n,2}^{(R)}$ ,  $\hat{\rho}_{n,3}^{(R)}$ ,  $\hat{\rho}_{n,7}^{(S)}$  and  $\hat{\rho}_{n,8}^{(S)}$ . All of them are explicit and are asymptotically Gaussian. The comparison of their finite sample properties is a huge task since they may depend on their parameters  $(\theta_i, \tau_i, \alpha_i)$  as well as on the simulated distribution. This point is beyond the scope of this paper. We conclude this study by proposing a method for selecting some ‘‘asymptotic optimal’’ parameters within a family of estimators.

## 5 Asymptotic comparison

In this section, we focus on the estimators of  $\rho$  based on the statistic  $R_k(\tau_i)$  considered in Section 4.1 with kernel functions  $H_{\tau_i}(u) = \tau_i u^{\tau_i - 1}$ , for  $i = 1, \dots, 8$ . The values of the parameters  $\tau_1, \dots, \tau_8$ ,  $\theta_1$ ,  $\theta_3$  and  $\theta_4$  are taken as in [15]:  $\tau_i = \tilde{\tau}_i + 1$  with  $\tilde{\tau}_1 = 0.25$ ,  $\tilde{\tau}_2 = \tilde{\tau}_3 = 0.75$ ,  $\tilde{\tau}_4 = \tilde{\tau}_8 = 1$ ,  $\tilde{\tau}_5 = 0.5$ ,  $\tilde{\tau}_6 = \tilde{\tau}_7 = 0.75$ ,  $\tilde{\theta}_1 = 0.01$ ,  $\tilde{\theta}_3 = 0.02$  and  $\tilde{\theta}_4 = 0.04$ . For these parameters, we have  $\delta_0 = 1.5$  and  $\delta_1 = 1.8$ . Recall that  $\tilde{\theta}_2 = \tilde{\theta}_1 + \delta(\tilde{\theta}_4 - \tilde{\theta}_3)$  for  $\delta \geq 0$ . In the following, we propose to choose the remaining parameter  $\delta$  using a method similar to the one proposed in [13]. It consists in

minimizing with respect to  $\delta$  an upper bound on the asymptotic mean-squared error. The method is described in Section 5.1 and an example of application is presented in Section 5.2.

## 5.1 Controlling the asymptotic mean-squared error

As in [15], we assume that  $\rho = \beta$ . Following Corollary 1, the asymptotic bias components of  $\rho_n^{(R)}$  are respectively proportional to  $\mathcal{AB}_1^{(R)}(\delta, \rho)$  and  $\mathcal{AB}_2^{(R)}(\delta, \rho, \rho)$  while its asymptotic variance is proportional to  $\mathcal{AV}^{(R)}(\delta, \rho)$ . The asymptotic mean-squared error of  $\rho_n^{(R)}$  can be defined as

$$\mathcal{AMSE}(\delta, \gamma, \rho) = \frac{1}{kA^2(n/k)} \left( \left( \frac{\lambda_A + o(1)}{2\gamma} \mathcal{AB}_1^{(R)}(\delta, \rho) - (\lambda_B + o(1)) \mathcal{AB}_2^{(R)}(\delta, \rho, \rho) \right)^2 + \gamma^2 \mathcal{AV}^{(R)}(\delta, \rho) \right).$$

One way to choose the parameter  $\delta$  could be to minimize the above asymptotic mean-squared error. In practice, the parameters  $\gamma, \rho$  as well as the functions  $A$  and  $B$  are unknown and thus the asymptotic mean-squared error cannot be evaluated. To overcome this problem, it is possible to introduce an upper bound on  $\mathcal{AMSE}(\delta, \gamma, \rho)$ . Assuming that  $\delta \in [0, \delta_0) \cup (\delta_1, \infty)$  and  $\rho \in [\rho_{\min}, \rho_{\max}]$ , it is easy to check that  $|\mathcal{AB}_1^{(R)}(\delta, \rho)| \geq |\mathcal{AB}_1^{(R)}(\delta_1, \rho_{\max})|$ ,  $|\mathcal{AB}_2^{(R)}(\delta, \rho, \rho)| \geq |\mathcal{AB}_2^{(R)}(\delta_0, \rho_{\min}, \rho_{\min})|$  and, numerically, we can see that  $\mathcal{AV}^{(R)}(\delta, \rho) \geq \mathcal{AV}^{(R)}(1.32, -0.46)$ . We thus have:

$$\mathcal{AMSE}(\delta, \gamma, \rho) \leq \frac{C\pi(\delta, \rho)}{kA^2(n/k)},$$

with  $\pi(\delta, \rho) = (\mathcal{AB}_1^{(R)}(\delta, \rho) \mathcal{AB}_2^{(R)}(\delta, \rho, \rho))^2 \mathcal{AV}^{(R)}(\delta, \rho)$  and where the constant  $C$  does not depend on  $\delta$  and  $\rho$ . We thus consider for  $\rho < 0$  the parameter  $\delta$  minimizing the function  $\pi(\delta, \rho)$ . By numerical minimization of  $\pi(\delta, \rho)$ , it is possible to compute the optimal  $\delta$  as a function of  $\rho$ . The resulting function is depicted on Figure 1. Of course, the optimal  $\delta$  is unreachable in practice since it depends on  $\rho$ , but three interesting values of  $\delta$  appear:  $\delta = \delta_1 = 1.8$  which is approximately the optimal value of  $\delta$  when  $\rho \leq -3$ ,  $\delta = \infty$  when  $\rho$  is in the neighborhood of  $-1.7$ ,  $\delta = \delta_0 = 1.5$  in the situation  $\rho \geq -1$ .

## 5.2 Illustration on a Burr distribution

Let us illustrate the above result on a specific Pareto-type model, namely the Burr distribution with cdf  $F(x) = 1 - (\zeta/(\zeta + x^\eta))^\lambda$ ,  $x > 0$ ,  $\zeta, \lambda, \eta > 0$ . The associated extreme-value index is  $\gamma = 1/(\lambda\eta)$  and this model satisfies the second order condition **(C2)** with  $\rho = \beta = -1/\lambda$ ,  $A(x) = \gamma x^\rho/(1 - x^\rho)$  and  $B(x) = \rho x^\rho/(1 - x^\rho)$ . For such choices of functions  $A$  and  $B$ , one can check that  $k = (\lambda_A n^{-2\rho}/\gamma^2)^{2/(1-4\rho)}$  satisfies condition (5) with  $\lambda_B = \rho\lambda_A/\gamma$ . Six sets of parameters are considered:  $\zeta = 1$ ,  $\lambda = 1/\eta$ ,  $\rho = -\eta$  with  $\eta \in \{0.25, 1, 2.5, 3, 4, 6\}$ . Taking  $\lambda_A = k^{1/2} A^2(n/k)$ , the asymptotic mean-squared error of  $\hat{\rho}_n^{(R)}$  is plotted on Figure 2 as a function of  $k \in \{10, 20, \dots, 4990\}$  in the case  $n = 5000$  and for  $\delta \in \{0, 1, 1.5, 1.8, 20\}$ . Note that  $\delta = 20$  can be assimilated to  $\delta = +\infty$  appearing in the minimization of  $\pi$ , see Subsection 5.1. It appears

that the value of  $\delta$  minimizing the function  $\pi$  also provides good results in terms of asymptotic mean-squared error:

- If  $\rho \leq -4$ , the smallest  $\mathcal{AMSE}$  is obtained with  $\delta = 1.8$ .
- If  $-3 \leq \rho \leq -2.5$ , the best  $\mathcal{AMSE}$  is given by  $\delta = +\infty$ .
- If  $\rho \geq -1$ , the smallest  $\mathcal{AMSE}$  is given by  $\delta = 1.5$ .

As a conclusion, it appears on this particular case, that the set of values  $\{1.5, 1.8, +\infty\}$  obtained by minimizing the function  $\pi$  are also of interest to minimize the asymptotic mean-squared error. More generally, the minimization of  $\pi = (\mathcal{AB}_1\mathcal{AB}_2)^2\mathcal{AV}$  should permit to determine “optimal” values for the parameters of any estimator of  $\rho$  in the family (3).

## 6 Proofs

**Proof of Theorem 1.** Clearly,  $(\Psi 1)$  and  $(\Psi 2)$  entail  $Z_n = \psi(\omega_n^{-1}(T_n - \chi_n\mathbb{I}))$ . Moreover,  $(\mathbf{T}1)$  and  $(\Psi 4)$  yield  $Z_n \xrightarrow{\mathbb{P}} \psi(f(\rho)) = \varphi(\rho)$ . For all  $\varepsilon > 0$ , we have

$$\begin{aligned} \mathbb{P}(|\hat{\rho}_n - \rho| > \varepsilon) &= \mathbb{P}(\{|\hat{\rho}_n - \rho| > \varepsilon\} \cap \{Z_n \in J\}) + \mathbb{P}(\{|\hat{\rho}_n - \rho| > \varepsilon\} \cap \{Z_n \notin J\}) \\ &\leq \mathbb{P}(\{|\hat{\rho}_n - \rho| > \varepsilon\} \cap \{Z_n \in J\}) + \mathbb{P}(\{Z_n \notin J\}) \\ &= \mathbb{P}(\{|\varphi^{-1}(Z_n) - \rho| > \varepsilon\} \cap \{Z_n \in J\}) + \mathbb{P}(\{Z_n \notin J\}). \end{aligned}$$

From  $(\Psi 3)$  and  $(\Psi 4)$ ,  $\varphi^{-1}$  is also continuous in a neighborhood of  $\varphi(\rho)$ . Since  $Z_n \xrightarrow{\mathbb{P}} \varphi(\rho)$ , it follows that  $\mathbb{P}(\{|\varphi^{-1}(Z_n) - \rho| > \varepsilon\} \cap \{Z_n \in J\}) \rightarrow 0$  as  $n \rightarrow \infty$ . Besides,  $\rho \in J_0$  yields  $\varphi(\rho) \in J$  and thus

$$\mathbb{P}(\{Z_n \notin J\}) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (7)$$

As a conclusion,  $\mathbb{P}(|\hat{\rho}_n - \rho| > \varepsilon) \rightarrow 0$  as  $n \rightarrow \infty$  and the result is proved.  $\blacksquare$

**Proof of Theorem 2.** Recalling that  $Z_n = \psi(\omega_n^{-1}(T_n - \chi_n\mathbb{I}))$ , a first order Taylor expansion shows that there exists  $\varepsilon \in (0, 1)$  such that

$$v_n(Z_n - \varphi(\rho)) = {}^t(v_n\xi_n) \nabla\psi(f(\rho) + \varepsilon\xi_n),$$

where we have defined  $\xi_n = \omega_n^{-1}(T_n - \chi_n\mathbb{I}) - f(\rho)$ . Therefore,  $\xi_n \xrightarrow{\mathbb{P}} 0$  and  $(\Psi 5)$  entail that  $\nabla\psi(f(\rho) + \varepsilon\xi_n) \xrightarrow{\mathbb{P}} \nabla\psi(f(\rho))$ . Thus, taking account of  $(\mathbf{T}2)$ , we obtain that

$$v_n(Z_n - \varphi(\rho)) \xrightarrow{d} \mathcal{N}(m_\psi(\rho), \sigma_\psi^2(\rho)). \quad (8)$$

Now,  $P_n(x) := \mathbb{P}(\{v_n(\hat{\rho}_n - \rho) \leq x\})$  can be rewritten as

$$\begin{aligned} P_n(x) &= \mathbb{P}(\{v_n(\hat{\rho}_n - \rho) \leq x\} \cap \{Z_n \in J\}) + \mathbb{P}(\{v_n(\hat{\rho}_n - \rho) \leq x\} \cap \{Z_n \notin J\}) \\ &= \mathbb{P}(\{v_n(\varphi^{-1}(Z_n) - \rho) \leq x\} \cap \{Z_n \in J\}) + \mathbb{P}(\{v_n(\hat{\rho}_n - \rho) \leq x\} \cap \{Z_n \notin J\}) \\ &=: P_{1,n}(x) + P_{2,n}(x). \end{aligned}$$

Let us first note that

$$0 \leq P_{2,n}(x) \leq \mathbb{P}(\{Z_n \notin J\}) \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (9)$$

in view of (7) in the proof of Theorem 1. Focusing on  $P_{1,n}(x)$ , since  $\varphi$  is continuously differentiable in a neighborhood of  $\rho$  and  $\varphi'(\rho) \neq 0$ , it follows that  $\varphi$  is monotone in a neighborhood of  $\rho$ . Let us consider the case where  $\varphi$  is decreasing, the case  $\varphi$  increasing being similar. Writing  $J = (a, b)$ , it follows that

$$\begin{aligned} P_{1,n}(x) &= \mathbb{P}(\{a \vee \varphi(\rho + x/v_n) \leq Z_n \leq b\}) \\ &= \mathbb{P}(\{v_n(a \vee \varphi(\rho + x/v_n) - \varphi(\rho)) < v_n(Z_n - \varphi(\rho)) \leq v_n(b - \varphi(\rho))\}). \end{aligned}$$

Introducing  $G_n$  the cumulative distribution function of  $v_n(Z_n - \varphi(\rho))$ , we have

$$\begin{aligned} 1 - P_{1,n}(x) &= 1 - G_n(v_n(b - \varphi(\rho))) + G_n(v_n(a \vee \varphi(\rho + x/v_n) - \varphi(\rho))) \\ &= 1 - G_n(v_n(b - \varphi(\rho))) + G_n(v_n(a - \varphi(\rho))) \vee G_n(v_n(\varphi(\rho + x/v_n) - \varphi(\rho))) \\ &=: P_{1,1,n} + P_{1,2,n} \vee P_{1,3,n}(x). \end{aligned}$$

Let  $G$  denote the cumulative distribution function of the  $\mathcal{N}(m_\psi(\rho), \sigma_\psi^2(\rho))$  distribution. It is straightforward that

$$P_{1,1,n} \leq 1 - G(v_n(b - \varphi(\rho))) + \sup_{t \in \mathbb{R}} |G_n(t) - G(t)|.$$

Since  $\rho \in J_0$ , we have  $\varphi(\rho) \in J = (a, b)$ . In particular,  $b > \varphi(\rho)$  yields  $1 - G(v_n(b - \varphi(\rho))) \rightarrow 0$  as  $n \rightarrow \infty$ . Besides, (8) shows that  $G_n(t) \rightarrow G(t)$  for all  $t \in \mathbb{R}$  and thus  $G_n(t) \rightarrow G(t)$  uniformly, see for instance [9], p.552. As a preliminary conclusion  $P_{1,1,n} \rightarrow 0$  and, similarly,  $P_{1,2,n} \rightarrow 0$  as  $n \rightarrow \infty$ . Finally,

$$|P_{1,3,n}(x) - G(x\varphi'(\rho))| \leq |G(v_n(\varphi(\rho + x/v_n) - \varphi(\rho)) - G(x\varphi'(\rho)))| + \sup_{t \in \mathbb{R}} |G_n(t) - G(t)|$$

and, in view of **(Ψ5)**,  $v_n(\varphi(\rho + x/v_n) - \varphi(\rho)) \rightarrow x\varphi'(\rho)$  as  $n \rightarrow \infty$ , which leads to  $P_{1,3,n}(x) \rightarrow G(x\varphi'(\rho))$  as  $n \rightarrow \infty$ . We thus have shown that

$$P_{1,n}(x) \rightarrow 1 - G(x\varphi'(\rho)) = G(x|\varphi'(\rho)|) \text{ as } n \rightarrow \infty. \quad (10)$$

Collecting (9) and (10) yields

$$\mathbb{P}(\{v_n(\hat{\rho}_n - \rho) \leq x\}) \rightarrow G(x|\varphi'(\rho)|) \text{ as } n \rightarrow \infty$$

and concludes the proof. ■

## References

- [1] J. Beirlant, G. Dierckx, Y. Goegebeur and G. Matthys. Tail index estimation and an exponential regression model. *Extremes*, **2**, 177–200, 1999.
- [2] J. Beirlant, Y. Goegebeur, R. Verlaack and P. Vynckier. Burr regression and portfolio segmentation. *Insurance: Mathematics and Economics*, **23**, 231–250, 1998.
- [3] J. Cai, L. de Haan and C. Zhou. Bias correction in extreme value statistics with index around zero. *Submitted*, 2011.
- [4] F. Cairo, M.I. Gomes and D. Pestana. Direct reduction of bias of the classical Hill estimator. *REVSTAT - Statistical Journal*, **3(2)**, 113–136, 2005.
- [5] H.S. Carslaw and J.C. Jaeger. *Operational Methods in Applied Mathematics*, Oxford University Press, 1948.
- [6] G. Ciuperca and C. Mercadier. Semi-parametric estimation for heavy tailed distributions. *Extremes*, **13**, 55–87, 2010.
- [7] S. Csörgö, P. Deheuvels, and D.M. Mason. Kernel estimates of the tail index of a distribution. *Annals of Statistics*, **13**, 1050–1077, 1985.
- [8] A.L.M. Dekkers, J.H.J. Einmahl, and L. de Haan. A moment estimator for the index of an extreme-value distribution. *Annals of Statistics*, **17**, 1833–1855, 1989.
- [9] P. Embrechts, C. Klüppelberg, and T. Mikosch. *Modelling extremal events*, Springer, 1997.
- [10] A. Feuerverger and P. Hall. Estimating a tail exponent by modeling departure from a Pareto distribution. *Annals of Statistics*, **27**, 760–781, 1999.
- [11] M.I. Fraga Alves, M.I. Gomes, and L. de Haan. A new class of semi-parametric estimators of the second order parameter. *Portugaliae Mathematica*, **60(2)**, 193–213, 2003.
- [12] M.I. Fraga Alves, L. de Haan and T. Lin. Estimation of the parameter controlling the speed of convergence in extreme value theory. *Mathematical Methods of Statistics*, **12(2)**, 155–176, 2003.
- [13] L. Gardes and S. Girard. Conditional extremes from heavy-tailed distributions: An application to the estimation of extreme rainfall return levels. *Extremes*, **13(2)**:177–204, 2010.
- [14] J. Geluk and L. de Haan. *Regular Variation, Extensions and Tauberian Theorems*, CWI Tract 40, Center for Mathematics and Computer Science, Amsterdam, Netherlands, 1987.
- [15] Y. Goegebeur, J. Beirlant, and T. de Wet. Kernel estimators for the second order parameter in extreme value statistics. *Journal of Statistical Planning and Inference*, **140**, 2632–2652, 2010.



- [16] M.I. Gomes, L. de Haan and L. Peng. Semi-parametric estimation of the second order parameter in statistics of extreme. *Extremes*, **5**, 387–414, 2002.
- [17] M.I. Gomes and M.J. Martins. Generalizations of the Hill estimator - asymptotic versus finite sample behaviour. *Journal of Statistical Planning and Inference*, **93**, 161–180, 2001.
- [18] M.I. Gomes and M.J. Martins. "Asymptotically unbiased" estimators of the tail index based on external estimation of the second order parameter. *Extremes*, **5(1)**, 5–31, 2002.
- [19] M.I. Gomes and M.J. Martins. Bias reduction and explicit semi-parametric estimation of the tail index. *Journal of Statistical Planning and Inference*, **124**, 361–378, 2004.
- [20] M.I. Gomes, M.J. Martins and M. Neves. Improving second order reduced bias extreme value index estimator. *REVSTAT - Statistical Journal*, **5(2)**, 177–207, 2007.
- [21] M.I. Gomes and O. Oliveira. The bootstrap methodology in statistics of extremes Choice of the optimal sample fraction. *Extremes*, **4(4)**, 331–358, 2001.
- [22] L. de Haan and A. Ferreira. *Extreme Value Theory: An Introduction*, Springer Series in Operations Research and Financial Engineering, Springer, 2006.
- [23] P. Hall and A.H. Welsh. Adaptative estimates of parameters of regular variation. *The Annals of Statistics*, **13**, 331–341, 1985.
- [24] B.M. Hill. A simple general approach to inference about the tail of a distribution, *Annals of Statistics*, **3**, 1163–1174, 1975.
- [25] L. Peng. Asymptotic unbiased estimators for the extreme value index. *Statistics and Probability Letters*, **38**, 107–115, 1998.
- [26] J. Worms and R. Worms. Estimation of second order parameters using probability weighted moments. *ESAIM: Probability and Statistics*, to appear.

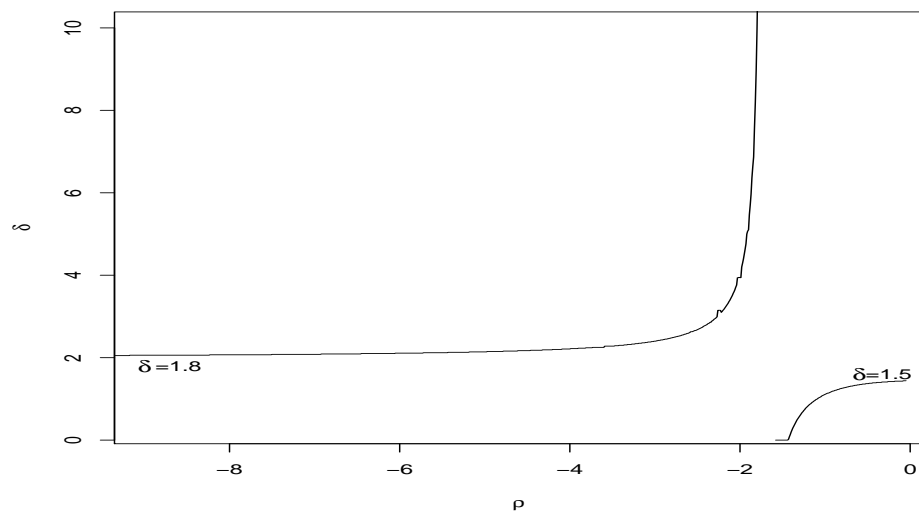


Figure 1: Optimal  $\delta$  as a function of  $\rho$

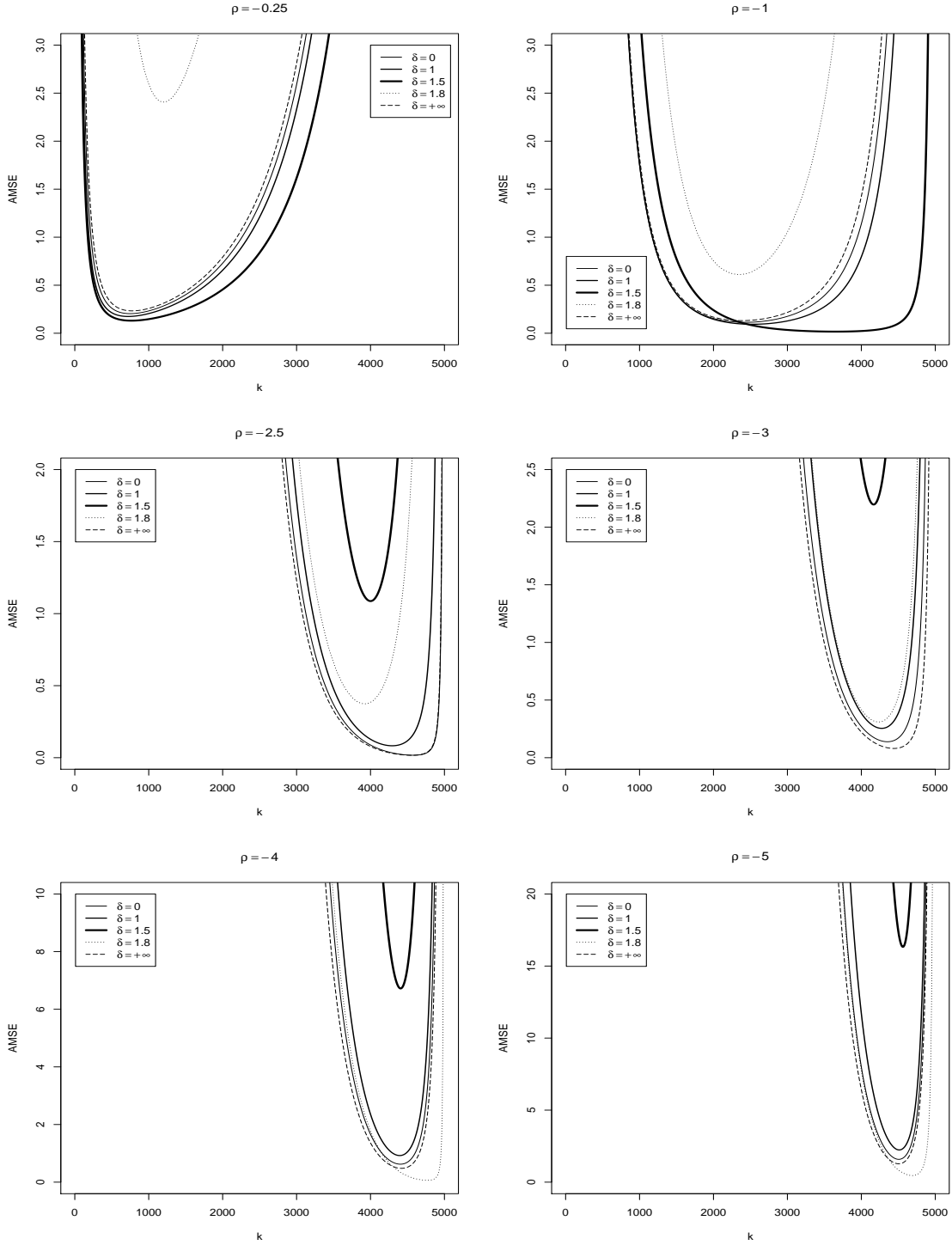


Figure 2: Asymptotic mean-squared error as a function of  $k$  for a Burr distribution.