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► **To cite this version:**

Johannes Brunnemann, David Rideout. Oriented matroidscombinatorial structures underlying loop quantum gravity. *Classical and Quantum Gravity*, 2010, 27 (20), pp.205008. 10.1088/0264-9381/27/20/205008 . hal-00634380

HAL Id: hal-00634380

<https://hal.science/hal-00634380>

Submitted on 21 Oct 2011

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Oriented Matroids – Combinatorial Structures Underlying Loop Quantum Gravity

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July 21, 2010

Abstract

We analyze combinatorial structures which play a central role in determining spectral properties of the volume operator [1] in loop quantum gravity (LQG). These structures encode geometrical information of the embedding of arbitrary valence vertices of a graph in 3-dimensional Riemannian space, and can be represented by sign strings containing relative orientations of embedded edges. We demonstrate that these signature factors are a special representation of the general mathematical concept of an oriented matroid [2, 3]. Moreover, we show that oriented matroids can also be used to describe the topology (connectedness) of directed graphs. Hence the mathematical methods developed for oriented matroids can be applied to the difficult combinatorics of embedded graphs underlying the construction of LQG. As a first application we revisit the analysis of [4, 5], and find that enumeration of all possible sign configurations used there is equivalent to enumerating all realizable oriented matroids of rank 3 [2, 3], and thus can be greatly simplified. We find that for 7-valent vertices having no coplanar triples of edge tangents, the smallest non-zero eigenvalue of the volume spectrum does not grow as one increases the maximum spin j_{\max} at the vertex, for *any* orientation of the edge tangents. This indicates that, in contrast to the area operator, considering large j_{\max} does not necessarily imply large volume eigenvalues. In addition we give an outlook to possible starting points for rewriting the combinatorics of LQG in terms of oriented matroids.

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1 Introduction

Any approach to quantum gravity is expected to exhibit a discrete nature of spacetime at distances close to the Planck scale ($\sim 10^{-35}$ m). Taking this perspective, our traditional picture of continuous spacetime geometry can only arise as a kind of (semi-)classical limit from a more fundamental combinatorial theory, linking matter and gravity. Loop quantum gravity (LQG) is supposed to give a canonical quantization of gravity, based on the initial value formulation of general relativity (GR), in a manner which is independent of any background geometry. LQG provides a mathematically rigorous way to non-perturbatively quantize geometry itself by promoting classical geometric quantities such as length, area or volume into operators in quantum theory. Remarkably these operators indeed turn out to have discrete spectra¹. During the last 20 years LQG has already produced results which are regarded as physically relevant: the absence of a cosmological Big-Bang singularity² [10], as well as the reproduction of the entropy-area law for black holes from first principles [11], just to mention a couple. Moreover for the first time there exists a proposal for implementing the Hamilton constraint, which encodes the dynamics, as an operator in quantum theory [12]. Even better, if geometry is kept dynamical, matter can be quantized without the occurrence of anomalies or UV-divergences, which plague quantum field theories constructed on a fixed background [13].

Nevertheless there are important questions regarding the construction of the physical sector of LQG which remain unanswered so far. One obstacle is the construction of the theory in terms of a projective limit over a poset of arbitrarily complicated directed graphs embedded into three dimensional Riemannian space, the Cauchy surfaces which arise in the initial value formulation of GR. Without reference to a background geometry, the characterization of such embeddings by coordinates, edge lengths or angles is meaningless. Rather one is referred to diffeomorphism invariant characterizations such as closed circuits³ in graphs or the local intersection behavior of edges at graph vertices. While the mathematical construction of the projective limit over the graph poset is well understood, the complicated combinatorics of generic elements of the poset still lack an effective characterization method which makes a study of the full theory viable.

In practical computations, the combinatorial difficulty is often circumvented by imposing simplifications to LQG. One possible simplification is based on the assumption that the underlying classical theory is not full GR, but a symmetry reduced, so-called cosmological model. This is equivalent to partially fixing a background geometry. The resulting model is quantized using techniques similar to LQG using a restricted set of graphs, adapted to the symmetry reduction. In the resulting loop quantum cosmology (LQC) (see [14, 15] and references therein) it is then possible to address questions such as the Big-Bang singularity, which is far beyond the possibilities one has in full LQG at present. Another ansatz (called Algebraic Quantum Gravity, AQG) is to keep GR, that is full background independence, but to modify the quantization procedure by restricting only to certain types

¹Thus far this is a kinematical statement. See [6, 7] for a discussion on how this property can be transferred to the physical sector of LQG. Additionally in [8, 9] a setup is given where these properties can be regarded as physical.

²In a symmetry reduced setup.

³Also called cycles in graph theory.

of graphs and embeddings in the poset [16]. This makes it possible to analyze the classical limit of the resulting model [17] as well as to construct a physical sector of LQG in a reduced phase space quantization [9].

Given the promising results of the outlined simplified models, it is highly desirable to get a better understanding of the physical implications of the underlying assumptions, as simplifications may modify dynamical properties of the analyzed model. For example, in the context of LQC it is found that the operator corresponding to the classical inverse scale factor of the universe is bounded in LQC [14], but unbounded in full LQG [18]. Moreover the Hilbert space of LQC cannot be continuously embedded into the Hilbert space obtained in LQG [19]. In AQG, on the other hand, one special graph and its embedding is fixed, in order to compute the volume spectrum. It has been suggested that only this particular embedding gives a consistent semiclassical limit of the model [20]. The mathematical tools presented here may allow us to consider this question for arbitrary embedded vertices.

In full LQG, it turns out that the global circuit structure of generic graphs and their local embedding properties are relevant for the construction of the Hamilton constraint operator of [12], which encodes the dynamics of the theory. This arises because a crucial ingredient of the Hamilton constraint operator consists in the quantum analogue to the classical volume integral of a region in three dimensional Riemannian space, called the volume operator. In [4, 5] the spectral properties of this operator due to [1] were analyzed in full LQG, and it was found that they are highly sensitive to the local graph embedding. In particular the presence of a finite smallest non-zero eigenvalue is characterized by the chosen embedding.

In order to bring the results of the outlined simplified approaches in a closer context to the full theory, it is important to continue work on full LQG, and to develop techniques to better understand its combinatorics. This paper is intended as a first step toward developing such a method. It is based on the observation that local embedded graph geometry and global graph topology have a common home in the mathematical field of (oriented) matroid theory [3, 2], which provides a precise combinatorial abstraction of simplices and directed graphs.

Oriented matroids have already been used in the physics literature. One example is quantum information theory [21]. Another is a considerable amount of work done by the author of [22]–[23] and collaborators, to investigate connections between matroid theory and supergravity [22, 24], Chern-Simons theory [25], string / M- theory [26, 27] and 2D-gravity [28]. In the context of supergravity, possible implications on the Ashtekar formalism in higher dimensions have been outlined [23]. Their approach is mainly based on the observation that certain antisymmetry properties of tensors and forms in the according contexts can be captured by oriented matroids.

We suggest a different approach here. We are going to treat the combinatorics inherent in the construction of LQG by means of oriented matroids. As we will demonstrate, the occurring combinatorial structures match in a very natural way. This opens up a new mathematical arena to study LQG, and to overcome certain technical and conceptual obstacles which, at present, make concrete analytical and numerical investigations of the full theory very difficult. In particular it becomes feasible to develop a systematic and unified treatment of global (connectedness) and local (embedding) properties of the graph poset underlying LQG.

The plan of this paper is as follows. In the next section we will describe the occurrence of geometrical and topological properties of graphs in the present formulation of LQG in more detail and characterize their combinatorics. We then introduce the concept of a matroid and an oriented matroid and show how the latter can be used in order to encode the information on global and local properties of directed graphs. Furthermore we will comment on the issue of realizability of oriented matroids, which is directly related to computing the number of possible diffeomorphic local graph embeddings, as well the number of certain graph topologies. We will finally show how this can be applied to the results of [4, 5], which will be revisited and simplified. Our findings will be discussed in section 6. There we will also comment on the notion of a semiclassical limit of the volume operator in light of the revisited numerical computation. In the summary and outlook section we outline upcoming work, and steps necessary to fully establish the oriented matroid formalism within LQG, as well as its potential impact on further studies in full LQG.

2 LQG Combinatorics

In this section we will only outline the current construction of LQG as necessary for our work. For a detailed introduction we refer to [29, 30].

LQG is based on the initial value formulation of Einstein's equations for GR, which is possible for any globally

hyperbolic spacetime. According to a theorem of Geroch [31], the spacetime is then homeomorphic⁴ to the Cartesian product of the real numbers \mathbb{R} (“time axis”) times an orientable spacelike three dimensional Cauchy surface Σ . An according 3 + 1 decomposition procedure introduced by Arnowitt, Deser and Misner (ADM) [33] then rewrites GR in terms of canonically conjugated variables. These are components of the induced spatial metric tensor on Σ and its first order (“time-”) derivatives in the \mathbb{R} -direction given by components of extrinsic curvature. In this setup the dynamics of Einstein’s equations is encoded completely in constraints (three spatial diffeomorphism and a scalar constraint) which restrict admissible initial data on a chosen Cauchy surface Σ , and assure background independence⁵. The resulting theory can then be treated using Dirac’s approach to constrained Hamiltonian systems [34], which proposes to first construct the canonical theory and second to implement the constraints formulated in terms of the canonical variables on phase space. This results in the well known Dirac algebra of Poisson brackets among the constraints.

Inspired by the work of Sen, Ashtekar realized [35] that the canonical ADM-variables of GR can be extended in terms of Lie-algebra valued densitized triads E (local orthonormal frames) and according connections A on a principal fibre bundle with basis Σ and compact structure group G . In practice G is taken to be $SU(2)$, however the construction described here works to a certain extent for general compact Lie groups G . As different triads can encode identical information on the spatial metric, another constraint, called the Gauss constraint, is added to the Dirac algebra.

This opens a way to formulate the theory in terms of holonomies and fluxes, very similar to lattice gauge theory. Consider a one dimensional embedded directed path⁶ $p \subset \Sigma$. By the holonomy map h , a connection $A \in \mathcal{A}$ can be understood as a mapping of p to the gauge group G , $A : p \xrightarrow{h} A(p) \in G$, where \mathcal{A} denotes the space of smooth connections⁷. In order to make this construction more explicit one often writes $h_p(A)$ instead of $A(p)$. The set \mathcal{P} of all paths carries a groupoid structure with respect to composition⁸. As can be shown, the set \mathcal{A} is contained in the set $\overline{\mathcal{A}} = \text{Hom}(\overline{\mathcal{P}}, G)$ of all homomorphisms (no continuity assumption) of the path groupoid $\overline{\mathcal{P}}$ to the gauge group G . The set $\overline{\mathcal{A}}$ is consequently called the space of generalized connections. $\overline{\mathcal{A}}$ is equipped with a topology as follows. One considers finite collections $E(\gamma) := \{e_1, \dots, e_N\}$ of non self-intersecting paths e_k , called edges⁹, which mutually intersect at most at their beginning $b(e_k)$ and end (final) points $f(e_k)$, called vertices $V(\gamma) := \{b(e_k), f(e_k)\}_{e_k \in E(\gamma)}$. Such a collection is called a graph γ , the set of all finite semianalytic graphs is denoted by Γ . Now one considers the subgroupoid $l(\gamma) \subset \mathcal{P}$ generated by all elements contained in $E(\gamma)$ together with their inverses and finite composition¹⁰. The characteristic function of $E(\gamma)$ is also called the *support* $\text{supp}(l(\gamma))$.

Certainly the label set $\mathcal{L} = \{l(\gamma)\}_{\gamma \in \Gamma}$ carries a partial order, that is $l(\gamma) \prec l(\gamma')$ if $l(\gamma)$ is a subgroupoid of $l(\gamma')$. By construction $X_{l(\gamma)} := \text{Hom}(l(\gamma), G)$, is understood as the set of all homomorphisms from $E(\gamma)$ to G^N . That is, one copy of G is associated to each edge via the holonomy map.

Using the partial order of the set \mathcal{L} one then constructs the projective limit $\overline{\mathcal{X}}$ among the $X_{l(\gamma)}$. It can be shown that there is bijection between $\overline{\mathcal{A}}$ and $\overline{\mathcal{X}}$. Therefore $\overline{\mathcal{A}}$ can be equipped with a topology inherited by $\overline{\mathcal{X}}$ and turns out to be compact Hausdorff by the compactness of G and Tychonoff’s theorem. This makes it possible to construct a Hilbert space $\mathcal{H}_0 = L^2(\overline{\mathcal{A}}, d\mu_0)$ as an inductive limit of Hilbert spaces $\mathcal{H}_{l(\gamma)} = L^2(X_{l(\gamma)}, d\mu_\gamma) = L^2(G_{l(\gamma)}^N, d\mu_{H^N})$

In each $\mathcal{H}_{l(\gamma)}$ one considers functions $f_\gamma : G_{l(\gamma)}^N \rightarrow \mathbb{C}$ which are called cylindrical, because their support consists of those N copies of G which are labelled by $\text{supp}(l(\gamma))$. These functions are square integrable continuous with respect to the Haar measure $d\mu_H$ on each copy of G . An orthonormal basis on \mathcal{H}_0 is given by so called spin network functions $T_{\gamma \vec{j} \vec{m} \vec{n}}$, labelled by a graph γ and irreducible representations $\vec{j} := (j_1, \dots, j_N)$ of G (“spins”) plus a pair of matrix indices $\vec{m} := (m_1, \dots, m_N)$, $\vec{n} := (n_1, \dots, n_N)$ one for each edge contained in $E(\gamma)$. This is derived from the Peter&Weyl theorem and rests on the compactness of G . Holonomies act as multiplication operators, fluxes as derivations on \mathcal{H}_0 .

⁴In fact it is even diffeomorphic to $\mathbb{R} \times \Sigma$, as proved in [32].

⁵The physical content of the theory must not depend on the choice of Cauchy surfaces.

⁶A path is an equivalence class of semianalytic curves c with respect to re-parametrization and finite semianalytic retracings. A curve is a map $c : \mathbb{R} \supset [0, 1] \ni t \rightarrow c(t) \subset \Sigma$. Here $c(0) =: b(c)$ is called the beginning point, $c(1) =: f(c)$ end point of c . Semianalyticity is needed to ensure that two paths can only have a finite number of intersections.

⁷In the context of LQG a holonomy is understood as a parallel transport of the connection A along p , where p is not necessarily a closed loop.

⁸Two paths can only be composed if they have a common point.

⁹Modulo finite retracings. Strictly speaking an edge corresponds to a path which contains a representative curve which has no self intersections.

¹⁰It follows that $l(\gamma)$ is in fact an equivalence class of graphs: $l(\gamma) = l(\gamma')$ if their edge / ground-sets $E(\gamma')$ and $E(\gamma)$ differ only by a re-labelling of elements and/or reorientation of edge directions plus finite compositions. One only considers graphs over the full n -element ground set, called maximal representatives.

Note that the construction of the measure μ_0 depends only on the topology of Σ via $\text{supp}(l(\gamma))$. It does not refer to a choice of coordinates on Σ . In this sense μ_0 is defined background independently. In particular it can be shown that finite (semi-) analytic diffeomorphisms on Σ can be unitarily implemented on \mathcal{H}_0 . Even more, μ_0 is uniquely determined if unitarity of diffeomorphisms is imposed [36, 37].

So far all these constructions are on the kinematical level. By the quantum analogue to Dirac's procedure¹¹ the classical constraint functions have to be implemented as constraint operators on \mathcal{H}_0 . Their common kernel gives the physical Hilbert space $\mathcal{H}_{\text{phys}}$. Indeed it is straightforward to obtain the set of solutions to the Gauss constraint $\mathcal{H}_{\text{Gauss}}$ as a subset of \mathcal{H}_0 .

However, one cannot in general expect $\mathcal{H}_{\text{phys}} \subseteq \mathcal{H}_0$, rather¹² elements of $\mathcal{H}_{\text{phys}}$ are likely to be distributions contained in the algebraic dual \mathcal{D}^* of \mathcal{H}_0 .

This is what happens when one constructs the space $\mathcal{H}_{\text{diff}}$ of solutions to the spatial diffeomorphism constraint. Nevertheless it is possible to rigorously construct $\mathcal{H}_{\text{diff}}$ via a rigging map construction [38]. Elements of $\mathcal{H}_{\text{diff}}$ are then labelled by equivalence classes of $\text{supp}(l(\gamma))$ with respect to finite semianalytic diffeomorphisms. As described in the introduction, each such equivalence class carries global (topological) information about the connectedness of the representatives γ , that is their circuit structure, as well as local (geometric) information about the intersection behavior of their edges at the vertices of γ .

Despite the successful solution of Gauss and diffeomorphism constraints, the scalar or Hamilton constraint, encoding the dynamics, has not been solved so far. However there is a proposal to implement it in quantum theory on \mathcal{H}_0 [12] as well as a composite operator called the master constraint on $\mathcal{H}_{\text{diff}}$ [39]. Moreover in the context of [39, 40] a rigorous program for constructing an inner product on $\mathcal{H}_{\text{phys}}$ was proposed for the first time. Completing this step is crucial in order to construct the physical sector of LQG and to make physical predictions.

A major difficulty in the treatment of the Hamilton- respectively master constraint operator in the full theory comes from the fact that both operators are graph changing, due to the presence of holonomies in the classical constraint expression which are promoted to multiplication operators as mentioned above¹³. Illustratively speaking, constructing $\mathcal{H}_{\text{phys}}$ is equivalent to finding eigenstates to graph changing operators at a projective limit.

Nevertheless operators only containing fluxes, such as the volume operator, can in principle be discussed at the level of \mathcal{H}_0 , just by evaluating their action on a function f_γ cylindrical over a graph γ . If cylindrical consistency¹⁴ can be shown, then the according operator is automatically defined on all of \mathcal{H}_0 .

In the next section we will introduce the concept of matroids and oriented matroids and show how the connectedness of graphs as well as their local geometric embedding properties can be described within this framework.

3 Matroids and Oriented Matroids

In this section we will give a brief introduction to the subject of matroids and oriented matroids. The presentation is mostly based on [41] and [3]. Partly we have also used [42, 43]. We have tried to stay close to the notation used there. Many definitions and proofs have been taken directly from these books. At some places we give page numbers. However we would like to emphasize that, in order to keep this introduction short, we have changed the manner of presentation of the subject. In the literature one starts from different axiom systems, e.g. for chirotopes, (oriented) matroids in terms of (signed) circuits, (signed) bases, etc. and proves that these are equivalent. We take this equivalence for granted and use these objects equivalently in our presentation. We have extended the explanation where we felt it would be for the benefit of the reader not familiar with (oriented) matroids.

In general matroids [41] and oriented matroids [3] can be thought of as a combinatorial abstraction of linear dependencies in a real vector space. In order to make the presentation more accessible to the reader, we will

¹¹Also referred to as refined algebraic quantization.

¹²Consider for example eigen "functions" of the position operator in quantum mechanics, given by Dirac's δ -distribution.

¹³When a cylindrical function f_γ is multiplied by a holonomy h_e supported on an edge e not contained in $E(\gamma)$ then the support of $h_e \cdot f_\gamma$ is given by $l(\gamma \cup \{e\})$.

¹⁴That is, one has to show that the action of this operator on a cylindrical function $f_{\tilde{\gamma}} \in \mathcal{H}_{l(\tilde{\gamma})}$ is equal to its action on a cylindrical function $f_\gamma \in \mathcal{H}_{l(\gamma)}$ with $\gamma \subset \tilde{\gamma}$, if $\mathcal{H}_{l(\tilde{\gamma})}$ is restricted to its subspace $\mathcal{H}_{l(\gamma)}$ supported on γ .

give some simple vector examples in section 3.1 and 3.2. However, as we will see later, the vector point of view is only one out of several possible representations for (oriented) matroids. Nevertheless for the beginning and in the subsequent definitions it is instructive to keep the vector picture in mind. This will be made more explicit in section 3.5.

First we would like to fix some notation following [41, 3]. Let E be a given finite¹⁵ set of n elements. We will also write $|E| = n$. We will denote by 2^E the set of all subsets of E . It has cardinality $|2^E| = 2^{|E|} = 2^n$, as $\sum_{k=0}^n \binom{n}{k} = 2^n$. The subsets of 2^E are called *families* and will be denoted by script capital letters, i.e. \mathcal{B} . The families can be regarded as elements of the set of subsets of 2^E , 2^{2^E} . We will call the subsets of 2^{2^E} *collections* and denote them by boldface capital letters, i.e. \mathbf{B} .

An *antichain* (also called *incomparable family*) in E is a family \mathcal{B} of subsets of E such that, for all $X, Y \in \mathcal{B}$, from $X \subseteq Y$ it follows that $X = Y$.

We will also need the notion of a *minimal* and *maximal* subset B of E with respect to a certain property *prop* that holds for B . This is achieved by ordering the subsets by set inclusion, and regarding a subset B to be maximal (minimal) in \mathcal{B} iff $\nexists B' \in \mathcal{B}$ such that $B \subset B'$ ($B \supset B'$) and also *prop* holds for B' .

3.1 Matroids

In this section we introduce the concept of a matroid. The reason for using underlined quantities as in [3] will become clear in the next section, where we introduce signed subsets of a set.

We start with the definition of a matroid in terms of its circuits. In the vector picture a circuit can be thought of as a minimal linearly dependent set of vectors.

Definition 3.1 (Matroid from Circuits [41].) A family $\underline{\mathcal{C}}$ of subsets of a set E is called the set of circuits of a matroid $\underline{\mathcal{M}} = (E, \underline{\mathcal{C}})$ on E if

(C0) *Non-emptiness:* $\emptyset \notin \underline{\mathcal{C}}$

(C2) *Incomparability:* if $\underline{\mathcal{C}}_1 \subseteq \underline{\mathcal{C}}_2$ then $\underline{\mathcal{C}}_1 = \underline{\mathcal{C}}_2 \quad \forall \underline{\mathcal{C}}_1, \underline{\mathcal{C}}_2 \in \underline{\mathcal{C}} \quad (\underline{\mathcal{C}} \text{ is an antichain})$

(C3) *Elimination:* For all $\underline{\mathcal{C}}_1, \underline{\mathcal{C}}_2 \in \underline{\mathcal{C}}$ with $\underline{\mathcal{C}}_1 \neq \underline{\mathcal{C}}_2$, and $\forall e \in E$, $\exists \underline{\mathcal{C}}_3 \in \underline{\mathcal{C}}$ such that $\underline{\mathcal{C}}_3 \subseteq (\underline{\mathcal{C}}_1 \cup \underline{\mathcal{C}}_2) \setminus \{e\}$.

Moreover we have the following definitions:

Definition 3.2 (Bases, Circuits, Hyperplanes and Cocircuits of a matroid $\underline{\mathcal{M}}$ [41].)

(i) *Bases* $\mathcal{B} := \{B \subseteq E : B \text{ is maximal, } \nexists \underline{\mathcal{C}} \in \underline{\mathcal{C}}, B \supseteq \underline{\mathcal{C}}\}$

\mathcal{B} is a family of maximal subsets B of E which contain no circuit $\underline{\mathcal{C}}$ of $\underline{\mathcal{M}}$ as a subset. \mathcal{B} is called a family of bases or basic family of $\underline{\mathcal{M}}$ and its sets B are called bases of $\underline{\mathcal{M}}$.

(ii) *Circuits* $\underline{\mathcal{C}} := \{\underline{\mathcal{C}} \subseteq E : \underline{\mathcal{C}} \text{ is minimal, } \nexists B \in \mathcal{B}, \underline{\mathcal{C}} \subseteq B\}$

Conversely to the definition of bases, the circuits $\underline{\mathcal{C}}$ of $\underline{\mathcal{M}}$ can be seen as a family of minimal subsets $\underline{\mathcal{C}}$ of E which are not contained in a basis B .

(iii) *Hyperplanes* $\underline{\mathcal{H}} := \{\underline{\mathcal{H}} \subseteq E : \underline{\mathcal{H}} \text{ is maximal, } \nexists B \in \mathcal{B}, \underline{\mathcal{H}} \supseteq B\}$

The family $\underline{\mathcal{H}}$ of hyperplanes of $\underline{\mathcal{M}}$ is given by the maximal subsets $\underline{\mathcal{H}}$ of E which contain no basis of $\underline{\mathcal{M}}$ as a subset.

(iv) *Cocircuits* $\underline{\mathcal{C}}^* = \{\underline{\mathcal{C}}^* \subseteq E : E \setminus \underline{\mathcal{C}}^* \in \underline{\mathcal{H}}\}$

The family $\underline{\mathcal{C}}^*$ consists of all subsets $\underline{\mathcal{C}}^*$ of E whose complement is a hyperplane. $\underline{\mathcal{C}}^*$ is called the family of cocircuits (also called bonds in the literature) of $\underline{\mathcal{M}}$.

¹⁵We will limit our presentation here to the finite case. However, there exists work on the infinite case e.g. [44].

We can equivalently give the definition of a matroid in terms of its bases.

Definition 3.3 (Matroid from Bases [41].) *Let E be a finite set. For a basic family \mathcal{B} of subsets of E the following axioms hold:*

- (b1) *Nontriviality:* $\mathcal{B} \neq \emptyset$.
- (b2) *Incomparability:* \mathcal{B} is an antichain of subsets in E
- (b3) *Basis-exchange axiom:* $\forall B_1, B_2 \in \mathcal{B}$ and for every $b_1 \in B_1 \setminus B_2$, $\exists b_2 \in B_2 \setminus B_1$ such that $(B_1 \setminus \{b_1\}) \cup \{b_2\} \in \mathcal{B}$. Note that this implies $(B_2 \setminus \{b_2\}) \cup \{b_1\} \in \mathcal{B}$ as well.

A pair $\underline{\mathcal{M}} = (E, \mathcal{B})$ is called a finite matroid on the ground set E .

According to [41] we will denote the collection of all basic families of E by $\mathbf{B}(E)$. In this notion of a basis one can introduce a map $\chi : 2^E \rightarrow \{0, 1\}$ indicating whether a given subset $B \subseteq E$ is a basis (a member of the basic family $\mathcal{B}(\underline{\mathcal{M}}) \in \mathbf{B}(E)$) or not:

$$\chi_{\mathcal{B}}(B) = \begin{cases} 1 & \text{iff } B \in \mathcal{B} \\ 0 & \text{else} \end{cases}$$

Moreover as a consequence of (b3) any two bases $B_1, B_2 \in \mathcal{B}$ of a matroid $\underline{\mathcal{M}} = (E, \mathcal{B})$ have the same cardinality, $|B_1| = |B_2| =: r$. The cardinality r is often called the *rank* of the matroid $\underline{\mathcal{M}}$. Finally we would like to introduce the notion of basic (co-)circuits of a matroid.

Definition 3.4 (Basic (fundamental) (co-)circuits of an element with respect to a basis [3].) *Let $\underline{\mathcal{M}} = (E, \mathcal{B})$ be a matroid on E and $B \in \mathcal{B}$ be a basis of $\underline{\mathcal{M}}$. Let $e \in E \setminus B$. Then there is a unique circuit $\underline{C}(e, B)$ of $\underline{\mathcal{M}}$ contained in $\{e\} \cup B$. The circuit $\underline{C}(e, B)$ is called the basic (or fundamental) circuit of e with respect to B . Dually if $e \in B$ then there is a unique cocircuit $\underline{C}^*(e, B)$ of $\underline{\mathcal{M}}$ which is disjoint from the hyperplane $(B \setminus e) \in \underline{\mathcal{H}}$, that is $\underline{C}^* \cap (B \setminus e) = \emptyset$ or equivalently $E \setminus \underline{C}^* = (B \setminus e) \in \underline{\mathcal{H}}$. The cocircuit $\underline{C}^*(e, B)$ is called the basic (or fundamental) cocircuit of e with respect to B .*

Notice that the opposite does not hold in general: Given a (co-)circuit, there can be several bases from which this (co-)circuit can be obtained as a fundamental (co-)circuit by adding an additional element.

Lemma 3.1 (Intersection of fundamental circuits and cocircuits [3].) *Let $\underline{\mathcal{M}}$ be a matroid. Given any circuit $\underline{C} \in \underline{\mathcal{C}}(\underline{\mathcal{M}})$ and $e, f \in \underline{C}$, $e \neq f$, then there is a cocircuit $\underline{C}^* \in \underline{\mathcal{C}}^*(\underline{\mathcal{M}})$, such that $\underline{C} \cap \underline{C}^* = \{e, f\}$.*

To see this, let B be a basis of $\underline{\mathcal{M}}$, containing $\underline{C} \setminus f$. Clearly $f \notin B$, as otherwise \underline{C} would be contained in B . By construction the basic cocircuit $\underline{C}^*(e, B)$ is such that $\underline{C} \cap \underline{C}^* = \{e, f\}$.

3.1.1 Dual Matroid

Finally we have a natural notion of duality between matroids [41].

Definition 3.5 (Dual Matroid.) *Let $\underline{\mathcal{M}} = (E, \mathcal{B})$ be a matroid. A family $\mathcal{B}^* \in \mathbf{B}(E)$ with*

$$\mathcal{B}^* := \{B^* \subseteq E : \exists B \in \mathcal{B}, B^* = E \setminus B\}$$

is called a dual basic family and gives rise to the matroid $\underline{\mathcal{M}}^ = (E, \mathcal{B}^*)$ which is called the dual matroid to $\underline{\mathcal{M}} = (E, \mathcal{B})$.*

It follows that the circuits \underline{C} of $\underline{\mathcal{M}} = (E, \mathcal{B})$ are the cocircuits \underline{C}^* of $\underline{\mathcal{M}}^* = (E, \mathcal{B}^*)$ and vice versa. To see this, notice that by definition 3.5 the map “ \star ” is bijective. That is, $\forall B \in \mathcal{B}$ there is a unique $B^* \in \mathcal{B}^*$ and vice versa, $\forall B^* \in \mathcal{B}^*$ there is a unique $B \in \mathcal{B}$ such that $(B^*)^* = B$. Then, recalling the definition of \underline{C} in terms of \mathcal{B} from definition 3.2 (ii)

$$\underline{\mathcal{C}} := \{ \underline{C} \subseteq E : \underline{C} \text{ is minimal, } \nexists B \in \mathcal{B}, \underline{C} \subseteq B \}$$

we can formally dualize it and write

$$\begin{aligned}
\mathcal{C}^* &:= \{ \underline{\mathcal{C}}^* \subseteq E : \underline{\mathcal{C}}^* \text{ minimal, } \nexists B^* \in \mathcal{B}^*, \underline{\mathcal{C}}^* \subseteq B^* \} \\
&= \{ \underline{\mathcal{C}}^* \subseteq E : \underline{\mathcal{C}}^* \text{ minimal, } \nexists B \in \mathcal{B}, \underline{\mathcal{C}}^* \subseteq E \setminus B \} \\
&= \{ \underline{\mathcal{C}}^* \subseteq E : E \setminus \underline{\mathcal{C}}^* \text{ maximal, } \nexists B \in \mathcal{B}, E \setminus \underline{\mathcal{C}}^* \supseteq B \} \\
&= \{ \underline{\mathcal{C}}^* \subseteq E : E \setminus \underline{\mathcal{C}}^* \in \underline{\mathcal{H}} \} \\
&= \underline{\mathcal{C}}^* \quad .
\end{aligned} \tag{3.1}$$

Here, from the first to the second line we have used definition 3.5 and bijectivity of “ \star ” in order to replace conditions of the form “ $\nexists B^* \in \mathcal{B}^*$ ” by “ $\nexists B \in \mathcal{B}$ ”. From the second to the third line we have used the fact from elementary set theory that for two subsets $X, Y \subseteq E$ if $X \supset E \setminus Y$ then also $E \setminus X \subset E \setminus \{E \setminus Y\}$ which is equivalent to $E \setminus X \subset Y$. From the third to the fourth line we use definition 3.2 (iii) and finally from the fourth to the fifth line we use definition 3.2 (iv).

Hence it is obvious that the circuits of $\underline{\mathcal{M}}$ are the cocircuits of $\underline{\mathcal{M}}^*$ and the cocircuits of $\underline{\mathcal{M}}$ are the circuits of $\underline{\mathcal{M}}^*$. Therefore we will drop the symbol “ \star ” in what follows, and write instead the common symbol “ \star ” for dual objects, such as circuits $\underline{\mathcal{C}}$ and cocircuits $\underline{\mathcal{C}}^*$, bases \mathcal{B} and dual bases \mathcal{B}^* , matroids $\underline{\mathcal{M}}$ and dual matroids $\underline{\mathcal{M}}^*$ etc.

Example. Consider the set $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ of 5 vectors in \mathbb{R}^3 shown in figure 1. It is given by the column vectors of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} .$$

For every 3-element subset¹⁶ $B \subset E$ we consider the map

$$\chi_{\mathcal{B}}(B) = \begin{cases} 1 & \text{iff } \det B \neq 0 \\ 0 & \text{else} \end{cases} .$$

We will abbreviate the elements \mathbf{e}_k by its label k for $k = 1 \dots, 5$, such that we can write $E = \{1, 2, 3, 4, 5\}$. We

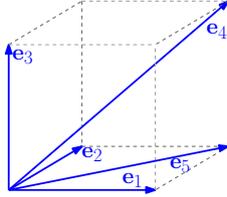


Figure 1: Vector example for a matroid.

have the non-uniform¹⁷ rank 3 vector matroid $\mathcal{M} = (E, \underline{\mathcal{C}}) = (E, \mathcal{B})$ where the sets $\underline{\mathcal{C}}$ of circuits and respectively the set \mathcal{B} of bases are given by

$$\begin{aligned}
\underline{\mathcal{C}} &= \{\{1, 2, 3, 4\}, \{1, 2, 5\}, \{3, 4, 5\}\} \\
\mathcal{B} &= \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}\}
\end{aligned}$$

Moreover we have the set of $\underline{\mathcal{H}}$ of hyperplanes and the set $\underline{\mathcal{C}}^*$ of cocircuits and dual bases \mathcal{B}^*

$$\begin{aligned}
\underline{\mathcal{H}} &= \{\{1, 2, 5\}, \{3, 4, 5\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\} \\
\underline{\mathcal{C}}^* &= \{\{3, 4\}, \{1, 2\}, \{2, 4, 5\}, \{2, 3, 5\}, \{1, 4, 5\}, \{1, 3, 5\}\} \\
\mathcal{B}^* &= \{\{4, 5\}, \{3, 5\}, \{2, 5\}, \{2, 4\}, \{2, 3\}, \{1, 5\}, \{1, 4\}, \{1, 3\}\} \quad ,
\end{aligned}$$

which can be used in order to define the rank 2 dual matroid $\mathcal{M}^* = (E, \underline{\mathcal{C}}^*) = (E, \mathcal{B}^*)$.

¹⁶This can be extended to all other subsets B' of cardinality different from 3 by setting $\chi_{\mathcal{B}}(B') = 0$.

¹⁷For this example this means that we have colinear triples of vectors in E . See section 3.3 for the definition of uniform.

3.2 Oriented Matroids

The concept of a rank r matroid $\mathcal{M} = (E, \mathcal{B})$ can be extended to the case of an oriented matroid $\mathcal{M} = (E, \mathcal{B})$. Notice however that not every matroid can be extended to an oriented matroid¹⁸.

For this we need to extend the definitions of the previous section. We label the elements of the ground set E with positive integers $1, 2, \dots, n = |E|$, and will often refer to n -tuples of subsets of E in round brackets (\cdot) (for which the ordering of the elements is important). The elements of a *sorted* set (such as a basis) are understood to be tuples in which the element labels are listed in increasing order.

Secondly we would like to extend the concept of circuits to signed circuits. For this we need the definition of *signed subsets* of E .

Definition 3.6 (Signed Subset.) See [3] p. 101 and following.

A *signed subset* X of E is a set $\underline{X} \subset E$ together with a partition (X^+, X^-) of \underline{X} into two distinguished subsets. X^+ is called the set of positive elements of X , X^- is called the set of negative elements of X , $X^+ \cap X^- = \emptyset$. The underlying set $\underline{X} = X^+ \cup X^-$ is then called the support of X . Let X, Y be signed subsets, F be an unsigned subset of E . Then the following properties hold.

- (i) *Equality:* $X = Y$ means $X^+ = Y^+$ and $X^- = Y^-$.
- (ii) *Restriction:* X is called a restriction of Y if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$
 $Y = X|_F$ is called the restriction of X to F if $Y^+ = X^+ \cap F$ and $Y^- = X^- \cap F$.
- (iii) *Composition:* $X \circ Y$ is the signed set defined by $(X \circ Y)^\pm = X^\pm \cup (Y^\pm \setminus X^\mp)$. Notice that this operation is associative, but not commutative in general.
- (iv) *Positivity:* X is called positive/negative if $X^- = \emptyset$ / $X^+ = \emptyset$. X is called empty if $X^- = X^+ = \emptyset$.
- (v) *Opposite:* $-X$ with $(-X)^\pm = X^\mp$ is called the opposite of X . The signed set ${}_F X$ is obtained from X by sign reversal or reorientation on F by $({}_F X)^\pm = (X^\pm \setminus F) \cup (X^\mp \cap F)$

For two signed sets X, Y we write $e \in X$ if $e \in \underline{X}$. We denote the cardinality of \underline{X} by $|\underline{X}| = |X|$. $X \setminus Y$ denotes the restriction of X to $\underline{X} \setminus \underline{Y}$.

Definition 3.7 (Signed Subset. Signature.) A signed set X can also be viewed as a set $\underline{X} \subset E$ together with a mapping $\text{sgn}_X : \underline{X} \rightarrow \{-1, 1\}$ such that $X^\pm = \{e \in X : \text{sgn}_X(e) = \pm 1\}$. The mapping sgn_X is called the signature of X .

Definition 3.8 (Signed Subset. Extended Signature.) The signature of X can be extended to all of E by defining the extended signature map $\text{sgn}_X : E \rightarrow \{-1, 0, 1\}^{|E|}$. Here $\text{sgn}_X(e) = \pm 1$ if $e \in X^\pm \subset E$, $\text{sgn}_X(e) = 0$ if $e \in E \setminus \underline{X}$.

X can then be identified with an element in $\{-1, 0, +1\}^{|E|} \equiv \{+, 0, -\}^{|E|}$ and we will shortly write $X \in \{-1, 0, +1\}^{|E|}$. Moreover by $\mathcal{S} \subseteq \{-1, 0, +1\}^{|E|}$ we will denote a family \mathcal{S} of signed subsets of E . For the n -element ‘ground tuple’ $E = E_n = (1, 2, \dots, n)$ we may denote X as a *sign vector* $X \in \{-1, 0, +1\}^n$. Finally we give a notion of orthogonality of signed subsets.

Definition 3.9 (Orthogonality of Signed Subsets.) Two signed subsets $X = (X^+, X^-), Y = (Y^+, Y^-)$, $X, Y \subset E$ are said to be orthogonal $X \perp Y$ if $X \cap Y = \emptyset$ or the restrictions of X and Y to their intersection are neither equal nor opposite.

That is, there are elements $e, f \in \underline{X} \cap \underline{Y}$ such that $\text{sgn}_X(e) \text{sgn}_Y(e) = -\text{sgn}_X(f) \text{sgn}_Y(f)$.¹⁹

¹⁸For a discussion of criteria for extensibility of a matroid to an oriented matroid we refer to [3].

¹⁹This is motivated from the sign properties of the scalar product of two real vectors. Consider a finite set $E = \{1, \dots, n\}$ with cardinality $|E| = n$. Let $\mathbf{u} = (u_1, \dots, u_n), \mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ be orthogonal with respect to the Euclidean inner product $\langle \cdot, \cdot \rangle$ of \mathbb{R}^n . Define $X = (X^+, X^-)$, where $X^\pm := \{i : u_i \gtrless 0\}$, and $Y = (Y^+, Y^-)$, where $Y^\pm := \{i : v_i \gtrless 0\}$. From

$$\langle u, v \rangle = \sum_{k=1}^n u_k \cdot v_k = 0$$

it follows that the non-zero terms in the sum cannot all have the same sign, that is if there are non-zero components u_k, v_k , then their signs have to be different for at least one k . Hence there exists at least one k , contained in X^+ and Y^- or X^- and Y^+ respectively. According to definition 3.9 we then say that the signed sets X and Y are orthogonal, $X \perp Y$.

With these definitions we are now ready to give the definition of an oriented matroid in terms of its signed circuits [3]:

Definition 3.10 (Oriented Matroid from Signed Circuits.) *A family \mathcal{C} of signed subsets of a set E is called the set of signed circuits of an oriented matroid $\mathcal{M} = (E, \mathcal{C})$ on E if*

- (C0) *Non-emptiness:* $\emptyset \notin \mathcal{C}$
- (C1) *Symmetry:* $\mathcal{C} = -\mathcal{C}$, that is for every $C \in \mathcal{C}$ also its opposite $-C \in \mathcal{C}$.
- (C2) *Incomparability:* if $\underline{C}_1 \subseteq \underline{C}_2$ then either $C_1 = C_2$ or $C_1 = -C_2 \quad \forall C_1, C_2 \in \mathcal{C}$
- (C3) *Elimination:* For all $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq -C_2$, if $e \in C_1^+ \cap C_2^- \exists C_3 \in \mathcal{C}$ such that $C_3^\pm \subseteq (C_1^\pm \cup C_2^\pm) \setminus \{e\}$.

Now we would like to define the oriented matroid \mathcal{M} in terms of a family \mathcal{B} of bases of the underlying rank r matroid $\underline{\mathcal{M}}$. An element $B \in \mathcal{B}$ can be written as a (sorted) r -tuple $B = (b_1, b_2, \dots, b_r)$, where each element b_k stands for an element of E . In particular there exists a permutation π , which brings all elements into lexicographic order according to E , such that $b_1 < b_2 < \dots < b_r$. Then B is called an ordered basis.

Definition 3.11 (Oriented Matroid from Oriented Bases of a Set.) *Let $\underline{\mathcal{M}} = (E, \mathcal{B})$ be a finite rank r matroid over the ground set E . An oriented matroid $\mathcal{M} = (E, \mathcal{B})$ on E is given by the bases \mathcal{B} of the underlying matroid $\underline{\mathcal{M}}$ together with a mapping $\chi_{\mathcal{B}} : E^r \ni B \rightarrow \{-1, 0, +1\}$ with*

$$\chi_{\mathcal{B}}(B) = \begin{cases} \pm 1 & \text{iff } B \in \mathcal{B} \\ 0 & \text{else} \end{cases}$$

called *basis orientation* or *chirotope*. The rank r of \mathcal{M} equals the rank of $\underline{\mathcal{M}}$ and for $\chi_{\mathcal{B}}$ it holds that

- (B1) $\chi_{\mathcal{B}}$ is alternating: for $B = (b_1, \dots, b_r)$ and $B' = (b_{\pi(1)}, \dots, b_{\pi(r)})$ consisting of the same elements as B but in different order, related to the order in B by a permutation π we have $\chi_{\mathcal{B}}(B) = \text{sgn}(\pi) \cdot \chi_{\mathcal{B}}(B')$.
- (PV) *Pivoting property:* for any two sorted bases $B, B' \in \mathcal{B}(\mathcal{M})$ with $B = (y, b_2, \dots, b_r)$ and $B' = (z, b_2, \dots, b_r)$, $y \neq z$ we have $\chi_{\mathcal{B}}(B) = -\text{sgn}_C(y) \cdot \text{sgn}_C(z) \cdot \chi_{\mathcal{B}}(B')$ where C is one of either of the two (opposite) signed circuits of \mathcal{M} whose support is contained in the set $\{y, z, b_2, \dots, b_r\}$.

We would like to remark that in general $\chi_{\mathcal{B}}$ is only fixed up to an overall sign: if $\chi_{\mathcal{B}}$ is a basis orientation for $\mathcal{M}(E, \mathcal{B})$, then also $-\chi_{\mathcal{B}}$ is a basis orientation. This must be kept in mind in subsequent computations. See also section 3.2.2.

Signed Cocircuits. With definition 3.9 we can define the set \mathcal{C}^* of *signed cocircuits* of the rank r oriented matroid \mathcal{M} as follows:

$$\mathcal{C}^* := \{C^* = (C^{*+}, C^{*-}) : \underline{C}^* \subseteq E, C^* \perp C \forall C \in \mathcal{C} \text{ (in the sense of definition 3.9)}\} . \quad (3.2)$$

Alternatively to this definition we can equip the set $\underline{\mathcal{C}}^*$ of cocircuits of the underlying matroid $\underline{\mathcal{M}}$ with a signature as follows [3]. Each support \underline{C}^* is the complement of a hyperplane $\underline{H} \in \underline{\mathcal{H}}$ as in definition 3.2, that is $\underline{H} = E \setminus \underline{C}^*$. By construction $\underline{H} = \{h_1, \dots, h_{r-1}\}$ is a maximal subset of E not containing a basis $B \in \mathcal{B}(\mathcal{M})$. Then the signature of C^* can be constructed as $(C^*)^\pm := \{e \in E \setminus \underline{H} : \chi_{\mathcal{B}}(h_1, \dots, h_{r-1}, e) = \pm 1\}$ ²⁰. In section 3.5 a geometric interpretation for \mathcal{C}^* is given.

Additionally we have the following definition.

²⁰Here it makes no difference if e.g. $e < h_{r-1}$, since this would only affect the sign of ± 1 .

If $\underline{\mathcal{M}}$ is non-uniform, it might happen that \underline{H} contains more than $r - 1$ elements. In that case take any ordered subset of \underline{H} which contains $r - 1$ elements that span \underline{H} .

Definition 3.12 (Basic (fundamental) signed (co-)circuits of an element wrt. an oriented basis.)

Let $\mathcal{M} = (E, B)$ be a given oriented matroid of rank r with underlying matroid $\underline{\mathcal{M}}$. Let $\underline{C} = (e, B)$ a fundamental circuit of $\underline{\mathcal{M}}$ as given in definition 3.4. Then the signed circuit $C \in \mathcal{C}$ supported on \underline{C} and having $\text{sgn}_C(e) = +1$ is called the basic or fundamental circuit of e with respect to $B \in \mathcal{B}$. Dually, if $e \in B$ (as in definition 3.4) let $\underline{C}^* = \underline{C}^*(e, B)$ be the unique cocircuit of $\underline{\mathcal{M}}$ disjoint from the hyperplane $B \setminus e$. Then the cocircuit $C^* \in \mathcal{C}^*$ supported on \underline{C}^* and positive on e is called the basic or fundamental cocircuit of e with respect to $B \in \mathcal{B}$.

3.2.1 Dual Oriented Matroid

Given any $B, B' \in \mathcal{B}$ with $B = (y, b_2, \dots, b_r)$, $B' = (z, b_2, \dots, b_r)$ and $y \neq z$ we may construct the basic circuit

$$\underline{C} := \underline{C}(z, B) = \underline{C}(y, B') = \{y, z, b_2, \dots, b_r\} .$$

Moreover using $B^*, B'^* \in \mathcal{B}^*$ with $B^* = E \setminus B$ and $B'^* = E \setminus B'$ we may construct the basic cocircuit

$$\underline{C}^* := \underline{C}^*(z, B) = E \setminus \{B \setminus z\} = \{E \setminus B\} \cup \{z\} = B^* \cup \{z\} = B'^* \cup \{y\} = \{E \setminus B'\} \cup \{y\} = E \setminus \{B' \setminus y\} = \underline{C}^*(y, B').$$

Here once again we see very nicely that the basic cocircuits with respect to the basic family \mathcal{B} can be regarded as basic circuits with respect to the basic family \mathcal{B}^* . Clearly we have $\underline{C} \cap \underline{C}^* = \{y, z\}$. Now choose one of the two oppositely signed circuits $\pm C \in \mathcal{C}$ and $\pm C^* \in \mathcal{C}^*$ such that for example $\text{sgn}_C(y) = \text{sgn}_{C^*}(y) = +1$.²¹ As $C \perp C^*$ by construction we get from definition 3.9 a chirotope $\chi_{\mathcal{B}^*}(B^*)$ for any $B^* \in \mathcal{B}^*$ from

$$\chi_{\mathcal{B}}(B) \cdot \chi_{\mathcal{B}}(B') = -\text{sgn}_C(y) \cdot \text{sgn}_C(z) = \text{sgn}_{C^*}(y) \cdot \text{sgn}_{C^*}(z) = -\chi_{\mathcal{B}^*}(B^*) \cdot \chi_{\mathcal{B}^*}(B'^*) . \quad (3.3)$$

In the last step we have used the pivoting property of definition 3.11 in its dualized form²². Note that this identity is independent of the choice of C, C^* above, as it only uses the relative signs of y, z with respect to C, C^* . Using (3.3) we may define the dual matroid chirotope $\chi_{\mathcal{B}^*} : E^{n-r} \rightarrow \{-1, 0, +1\}$ as follows. Consider the lexicographically sorted $E = (1, 2, \dots, n)$ and an arbitrary sorted $(n-r)$ -tuple $T^* = (t_1^*, \dots, t_{n-r}^*)$ of elements in E . Write $T = (t_1, \dots, t_r)$ for some permutation of $E \setminus T^*$. Clearly, there is a permutation π mapping $(T^*, T) := (t_1^*, \dots, t_{n-r}^*, t_1, \dots, t_r)$ to $(1, 2, \dots, n)$ with parity $\text{sgn } \pi =: \text{sgn}((T^*, T))$. Now we can define

$$\begin{aligned} \chi_{\mathcal{B}^*} : E^{n-r} &\rightarrow \{-1, 0, +1\} \\ T^* = (t_1^*, \dots, t_{n-r}^*) &\mapsto \chi_{\mathcal{B}^*}(T^*) := \chi_{\mathcal{B}}(T) \cdot \text{sgn}((T^*, T)) . \end{aligned} \quad (3.4)$$

Note that this definition is compatible with (3.3), and directly relates the values of chirotope and dual chirotope, as follows. Choose $B = (y, b_2, \dots, b_r)$, $B' = (z, b_2, \dots, b_r)$ and $B^*, B'^* \in \mathcal{B}^*$ with $B^* = E \setminus B = (z, b_2^*, \dots, b_{(n-r)}^*)$, $B'^* = E \setminus B' = (y, b_2^*, \dots, b_{(n-r)}^*)$ and C, C^* as above. Clearly

$$\begin{aligned} \chi_{\mathcal{B}^*}(B^*) &\stackrel{(3.4)}{=} \chi_{\mathcal{B}}(B) \cdot \text{sgn}(z, b_2^*, \dots, b_{(n-r)}^*, y, b_2, \dots, b_r) \\ &\stackrel{(PV)}{=} -\text{sgn}_C(y) \cdot \text{sgn}_C(z) \cdot \chi_{\mathcal{B}}(B') \cdot \text{sgn}(z, b_2^*, \dots, b_{(n-r)}^*, y, b_2, \dots, b_r) \\ &\stackrel{(3.2)}{=} \text{sgn}_{C^*}(y) \cdot \text{sgn}_{C^*}(z) \cdot \chi_{\mathcal{B}}(B') \cdot \text{sgn}(z, b_2^*, \dots, b_{(n-r)}^*, y, b_2, \dots, b_r) \\ &\stackrel{z \leftrightarrow y}{=} -\text{sgn}_{C^*}(y) \cdot \text{sgn}_{C^*}(z) \cdot \chi_{\mathcal{B}}(B') \cdot \text{sgn}(y, b_2^*, \dots, b_{(n-r)}^*, z, b_2, \dots, b_r) \\ &\stackrel{(3.4)}{=} -\text{sgn}_{C^*}(y) \cdot \text{sgn}_{C^*}(z) \cdot \chi_{\mathcal{B}^*}(B'^*) . \end{aligned} \quad (3.5)$$

Definition 3.13 (Dual Oriented Matroid.) Let $\mathcal{M} = (E, \mathcal{B})$ be an oriented rank r matroid with chirotope $\chi_{\mathcal{B}}$ over the ordered ground set $E = (1, \dots, n)$. A family $\mathcal{B}^* \in \mathbf{B}(E)$ with

$$\mathcal{B}^* := \{E \setminus B : B \in \mathcal{B}\}$$

together with the map $\chi_{\mathcal{B}^*} : E^{n-r} \ni B^* \rightarrow \{-1, 0, +1\}$ for which $B^* = E \setminus B$, $B \in E^r$

$$\chi_{\mathcal{B}^*}(B^*) = \chi_{\mathcal{B}}(B) \cdot \text{sgn}((B^*, B)) = \begin{cases} \pm 1 & \text{iff } B^* \in \mathcal{B}^* \text{ which is equivalent to } B \in \mathcal{B} \\ 0 & \text{iff } B^* \notin \mathcal{B}^* \text{ which is equivalent to } B \notin \mathcal{B} \end{cases}$$

²¹This can always be done without loss of generality, because $\pm C \in \mathcal{C}$ and $\pm C^* \in \mathcal{C}^*$ by construction.

²²Explicitly given in definition 3.13.

with $\text{sgn}((B^*, B))$ defined as above gives rise to an oriented matroid $\mathcal{M}^* = (E, \mathcal{B}^*)$, called the dual oriented matroid to $\mathcal{M} = (E, \mathcal{B})$, of rank $(n - r)$. In particular it holds for $\chi_{\mathcal{B}^*}$ that

(B1) $\chi_{\mathcal{B}^*}$ is alternating: for all $B^* = (b_1^*, \dots, b_{n-r}^*)$ and $B'^* = (b_{\pi(1)}^*, \dots, b_{\pi(n-r)}^*)$ consisting of the same elements as B^* but in different order, related to the order in B^* by a permutation π , we have $\chi_{\mathcal{B}^*}(B^*) = \text{sgn}(\pi) \cdot \chi_{\mathcal{B}^*}(B'^*)$.

(PV*) Dual Pivoting property: for any $B^*, B'^* \in \mathcal{B}^*$ with $B^* = (y, b_2^*, \dots, b_{(n-r)}^*)$ and $B'^* = (z, b_2^*, \dots, b_{(n-r)}^*)$ where $y \neq z$ we have $\chi_{\mathcal{B}^*}(B^*) = -\text{sgn}_{C^*}(y) \cdot \text{sgn}_{C^*}(z) \cdot \chi_{\mathcal{B}^*}(B'^*)$, where C^* is one of the two oppositely signed cocircuits of \mathcal{M} whose support is contained in the set $\{y, z, b_2^*, \dots, b_{(n-r)}^*\}$.

Note that again it holds that $(\mathcal{M}^*)^* = \mathcal{M}$. The signed cocircuits of \mathcal{M} are the signed circuits of \mathcal{M}^* and the signed circuits of \mathcal{M} are the signed cocircuits of \mathcal{M}^* .

Notice that (PV) and (PV*) are equivalent for a map $\chi_{\mathcal{B}}$ if \mathcal{M} is an oriented matroid due to (3.3) [3].

In practice the identity in equation (3.3) might often give an efficient way to compute the dual chirotope $\chi_{\mathcal{B}^*}$ from the chirotope $\chi_{\mathcal{B}}$.

Example continued. For the vector example at the end of section 3.1 we have $E = (1, 2, 3, 4, 5)$ and the according set of sorted bases \mathcal{B} . The set \mathcal{C} of signed circuits is given by $\mathcal{C} = \{\pm C_1, \pm C_2, \pm C_3\}$ with

	C_1	C_2	C_3
C_k^+	{1, 2, 3}	{1, 2}	{3, 5}
C_k^-	{4}	{5}	{4}

For the chirotope we consider for every 3 element subset B of E the map

$$\chi_{\mathcal{B}}(B) = \begin{cases} \text{sgn}(\det B) & \text{iff } \det B \neq 0 \\ 0 & \text{else} \end{cases} .$$

which leads to the following basis orientation²³:

B	(1, 2, 3)	(1, 2, 4)	(1, 2, 5)	(1, 3, 4)	(1, 3, 5)	(1, 4, 5)	(2, 3, 4)	(2, 3, 5)	(2, 4, 5)	(3, 4, 5)
$\chi_{\mathcal{B}}(B)$	+	+	0	-	-	-	+	+	+	0

3.2.2 Conversion from Signed Circuits to Signed Bases via Signed Basis Graphs

Here we follow [3] p. 132 and following.

The equivalence of defining an oriented matroid \mathcal{M} of rank r over a ground set E in terms of its signed circuits \mathcal{C} , as in definition 3.10, or in terms of its oriented basis \mathcal{B} , as in theorem 3.11, can be made explicit by using the so-called basis graph $BG_{\underline{\mathcal{M}}}$ of the underlying matroid $\underline{\mathcal{M}}$, which will be constructed in the sequel. Choose a family \mathcal{B} of bases of $\underline{\mathcal{M}}$. Associate to each basis $B \in \mathcal{B}$ a vertex of $BG_{\underline{\mathcal{M}}}$. Now link any two bases $B, B' \in \mathcal{B}$ by an edge if they differ by exactly one element, for example $B = (b_1, b_2, \dots, b_r)$, $B' = (b'_1, b_2, \dots, b_r)$ and $b_1 \neq b'_1$. By construction $BG_{\underline{\mathcal{M}}}$ is connected.

Now let $\mathcal{M} = (E, \mathcal{C})$ be given. Fix an ordering of the ground set E such that $E = (1, 2, \dots, n)$. Sort the elements of each basis $B \in \mathcal{B}$ according to that ordering. Now assume we have two sorted bases $B, B' \in \mathcal{B}$ with $B = (b_1, \dots, b_r)$ and $B' = (b'_1, \dots, b'_r)$ such that $B \Delta B' := (B \cup B') \setminus (B \cap B') = \{b_i, b'_j\}$. Let $C \in \mathcal{C}$ be one of

²³If $\chi_{\mathcal{B}}$ is a chirotope, then $-\chi_{\mathcal{B}}$ is also, depending of our notion of 'positive' orientation.

the two oppositely signed circuits²⁴ $C(b'_j, B)$ whose support is contained²⁵ in the set $\{b_1, \dots, b_r, b'_j\} \supseteq \underline{C}(b'_j, B)$.

Then by (B1) of definition 3.11 we have:

$$\begin{aligned}\chi_{\mathcal{B}}(b_i, b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_r) &= (-1)^{i-1} \chi_{\mathcal{B}}(B) \\ \chi_{\mathcal{B}}(b'_j, b'_1, \dots, b'_{j-1}, b'_{j+1}, \dots, b'_r) &= (-1)^{j-1} \chi_{\mathcal{B}}(B')\end{aligned}\quad (3.6)$$

Moreover by construction $(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_r) \equiv (b'_1, \dots, b'_{j-1}, b'_{j+1}, \dots, b'_r)$ and we can define

$$\begin{aligned}\eta(B, B') &:= \chi_{\mathcal{B}}(B) \cdot \chi_{\mathcal{B}}(B') = (-1)^{i+j} \cdot \chi_{\mathcal{B}}(b_i, b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_r) \cdot \chi_{\mathcal{B}}(b'_j, b'_1, \dots, b'_{j-1}, b'_{j+1}, \dots, b'_r) \\ &= (-1)^{1+i+j} \cdot \text{sgn}_C(b_i) \cdot \text{sgn}_C(b'_j)\end{aligned}\quad (3.7)$$

where we have used (PV) of definition 3.11 in the last line. Hence, starting from an arbitrarily chosen $B_0 \in \mathcal{B}$, we can construct the so-called *signed basis graph* $SBG_{\mathcal{M}}$ by attaching $\eta(B, B')$ to every edge of $BG_{\underline{\mathcal{M}}}$. Clearly $SBG_{\mathcal{M}}$ depends on the ordering chosen on the ground set E .

A chirotope $\chi : E^r \rightarrow \{-1, 0, +1\}$ can be constructed from $SBG_{\mathcal{M}}$ as follows. Choose an arbitrary $B_0 \in \mathcal{B}$. Set $\chi(B_0) = 1$. Then for any $B \in \mathcal{B}$ we have

$$\chi(B) = \prod_{i=1}^k \eta(B_{i-1}, B_i)$$

where $B_0, B_1, B_2, \dots, B_k = B$ is an arbitrary path from B_0 to B contained in $SBG_{\mathcal{M}}$. It can be shown that this definition is consistent, that is $\chi(B)$ is independent of the choice of a path, see [3] p. 132 and following.

Notice that this construction can easily be applied to cocircuits and the dual oriented matroid \mathcal{M}^* . Moreover having computed the chirotope $\chi_{\mathcal{B}}$ of $\mathcal{M} = (E, \mathcal{C})$, one can read off the chirotope for the dual matroid by using identity (3.3).

3.3 Further Definitions

Let us complete our introduction to oriented matroids by giving definitions which frequently occur in the literature [3]. Let the oriented matroid $\mathcal{M} = (E, \mathcal{C})$ be given with its set \mathcal{C} of signed circuits.

Vectors and covectors. A composition $C_1 \circ \dots \circ C_k$ of signed circuits in the sense of (iii) of definition 3.6 is called a *vector* of \mathcal{M} . The set of all vectors of \mathcal{M} is denoted by \mathcal{V} . Accordingly a composition of signed cocircuits of \mathcal{M} is called a *covector* of \mathcal{M} . The set of all covectors of \mathcal{M} is denoted by \mathcal{L} . Note that by construction vectors and covectors of \mathcal{M} are signed subsets of E . In particular any $v \in \mathcal{V}$ and any $l \in \mathcal{L}$ determines an extended signature in the sense of definition 3.8. Instead of giving \mathcal{M} in terms of its signed circuits \mathcal{C} or cocircuits \mathcal{C}^* , \mathcal{M} can also be determined in terms of its vectors \mathcal{V} or covectors \mathcal{L} [3].

Loops and coloops. Moreover an element $e \in E$ is called a *loop*²⁶ of \mathcal{M} if the signed set $(\{e\}, \emptyset) \in \mathcal{C}$. The subset of loops contained in E is denoted by E_o . If $e \notin C \forall C \in \mathcal{C}$ then e is called a *coloop* of \mathcal{M} .

Parallel elements. Two elements $e, f \in E \setminus E_o$ are called *parallel* $e \parallel f$ if $\text{sgn}_l(e) = \text{sgn}_l(f)$ or $\text{sgn}_l(e) = -\text{sgn}_l(f)$ for all covectors $l \in \mathcal{L}$.

Simple oriented matroid. An oriented matroid \mathcal{M} is called *simple* if it does not contain loops or parallel elements.

Uniform oriented matroid. An oriented matroid $\mathcal{M} = (E, \mathcal{B})$ of rank r is called *uniform* if **every** r -element subset $X \subseteq E$ is contained in the family of bases \mathcal{B} , that is $\chi_{\mathcal{B}}(X) \neq 0 \forall X \subseteq E$ such that $|X| = r$.

²⁴That is, the fundamental (basic) signed circuit or its opposite, as given by definition 3.12.

²⁵See section 3.4 for an example in which it is a proper subset.

²⁶In the picture of a vector configuration a loop corresponds to a zero vector.

3.4 Oriented Matroids from Directed Graphs

In this section we will show how the abstract framework of oriented matroids naturally arises from directed graphs. This will be done by reviewing an example from [3] in great detail.

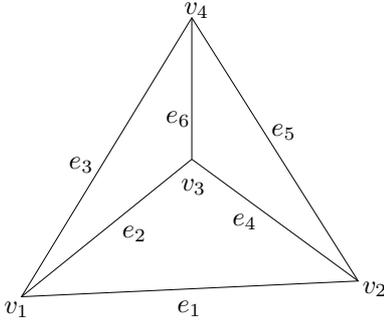


Figure 2: Undirected graph $\underline{\gamma}$.

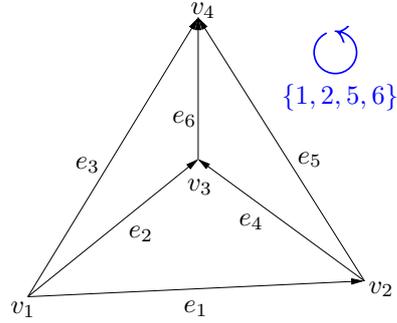


Figure 3: Directed graph γ .

Consider the graph $\underline{\gamma}$ of figure 2 with edge set $E = E(\underline{\gamma}) = \{e_1, \dots, e_6\}$ and vertex set $V(\underline{\gamma}) = \{v_1, \dots, v_4\}$. We notice that $\underline{\gamma}$ contains simple (undirected) cycles \underline{X} , for example $\{1, 2, 5, 6\}$. The set of all such cycles is denoted by $\underline{\mathcal{C}}$. Then $\underline{\mathcal{C}}$ defines the circuits of a matroid $\underline{\mathcal{M}} = (E, \underline{\mathcal{C}})$.

Now consider the directed graph γ of figure 3 now with a set $E(\gamma) = \{e_1, \dots, e_6\}$ of oriented edges and vertex set $V(\gamma) = \{v_1, \dots, v_4\}$. Consider the cycle $\underline{X} = \{1, 2, 5, 6\}$. If we introduce an anti-clockwise orientation for this cycle as indicated by \odot in figure 3, then it contains positive elements $X^+ = \{1, 5\}$ and negative elements $X^- = \{2, 6\}$. Hence $X = (X^+, X^-)$ is a signed circuit or in the wording of definition 3.6. $X = (X^+, X^-)$ is a signed set with support $\underline{X} = X^+ \cup X^-$. The set of all signed circuits obtained in this way from the oriented graph γ is denoted by \mathcal{C} . This defines the oriented matroid $\mathcal{M} = (E, \mathcal{C})$. We sometimes refer to an oriented matroid which arises from a directed graph in this way as a *graphic oriented matroid*.

We may define the extended signature sgn_X introduced in definition 3.8 with respect to the chosen orientation \odot , to write the signed circuit X as an element of $\{+, 0, -\}^{|E(\gamma)|}$: $X = \{\text{sgn}_X(e_1), \dots, \text{sgn}_X(e_6)\} = \{-, +, 0, 0, -, +\}$ or shorter (denoting e_k as simply k) $X = 1\bar{2}5\bar{6}$. In this notation the set \mathcal{C} can be written as a matrix, where each circuit is a row or the negative of a row [3]:

$$\mathcal{C} = \begin{pmatrix} + & - & 0 & + & 0 & 0 \\ + & 0 & - & 0 & + & 0 \\ 0 & + & - & 0 & 0 & + \\ 0 & 0 & 0 & - & + & - \\ + & - & 0 & 0 & + & - \\ + & 0 & - & + & 0 & + \\ 0 & + & - & - & + & 0 \end{pmatrix} \begin{array}{l} \leftarrow C_1 = 1\bar{2}4 \\ \leftarrow C_2 = 1\bar{3}5 \\ \leftarrow C_3 = 2\bar{3}6 \\ \leftarrow C_4 = \bar{4}5\bar{6} \\ \leftarrow C_5 = 1\bar{2}5\bar{6} \\ \leftarrow C_6 = 1\bar{3}46 \\ \leftarrow C_7 = 2\bar{3}45 \end{array}$$

It is instructive to check that $\mathcal{C} = \{\pm C_1, \dots, \pm C_7\}$ indeed fulfills the oriented matroid circuit axioms of definition 3.10. In particular (C2) holds.

The bases \mathcal{B} for $\underline{\mathcal{M}}$, \mathcal{M} are given by the set of spanning trees $T_{(\gamma)}$ of γ [41]. For a connected graph (or component) γ the spanning tree $T_{(\gamma)}$ is given by a collection of edges which uniquely connects any two vertices v_1, v_2 of γ by a path. If γ contains $|V(\gamma)|$ vertices, then $T_{(\gamma)}$ consists of $|V(\gamma)| - 1$ edges. Accordingly the rank of \mathcal{M}_γ is $|V(\gamma)| - 1$.

If γ is not connected but consists of N connected components $\gamma = \{\gamma_1, \gamma_2, \dots, \gamma_N\}$, then \mathcal{B} is given by the collection of trees $T_{(\gamma)} := \{T_{(\gamma_1)}, T_{(\gamma_2)}, \dots, T_{(\gamma_N)}\}$, which is also called the forest [45] of γ .

Hence in our example for γ in figure 3 we find that \mathcal{M}_γ has rank 3. By (i) in definition 3.2 of bases as maximal subsets of E containing no circuit we find that all triples except²⁷ $\underline{C}_1, \underline{C}_2, \underline{C}_3, \underline{C}_4$ of $E(\gamma) = \{1, 2, 3, 4, 5, 6\}$ are bases.

²⁷The fact that we have 3-element circuits indicates that \mathcal{M}_γ is non-uniform.

In order to compute the according chirotope, we use the signed basis graph construction (3.7) of section 3.2.2: As the chirotope is only given up to an overall sign, we choose $\chi_{\mathcal{B}}(1, 2, 3) = -1$ for the beginning and successively apply (3.7), $\chi_{\mathcal{B}}(B') = \chi_{\mathcal{B}}(B) \cdot (-1)^{1+i+j} \cdot \text{sgn}_{\mathcal{C}}(b_i) \cdot \text{sgn}_{\mathcal{C}}(b'_j)$:

$B = (b_1, b_2, b_3)$	$B' = (b'_1, b'_2, b'_3)$	$B\Delta B' = \{b_i, b'_j\}$	i	j	$B \cup B'$	$\supseteq \underline{\mathcal{C}}$	$\text{sgn}_{\mathcal{C}}(b_i)$	$\text{sgn}_{\mathcal{C}}(b'_j)$	$\chi_{\mathcal{B}}(B)$	$\chi_{\mathcal{B}}(B')$
(1, 2, 3)	(1, 2, 5)	{3, 5}	3	3	{1, 2, 3, 5}	$\underline{\mathcal{C}}_2$	-1	+1	-1	-1
(1, 2, 3)	(1, 2, 6)	{3, 6}	3	3	{1, 2, 3, 6}	$\underline{\mathcal{C}}_3$	-1	+1	-1	-1
(1, 2, 3)	(1, 3, 4)	{2, 4}	2	3	{1, 2, 3, 4}	$\underline{\mathcal{C}}_1$	-1	+1	-1	+1
(1, 2, 3)	(1, 3, 6)	{2, 6}	2	3	{1, 2, 3, 6}	$\underline{\mathcal{C}}_3$	+1	+1	-1	-1
(1, 2, 3)	(2, 3, 4)	{1, 4}	1	3	{1, 2, 3, 4}	$\underline{\mathcal{C}}_1$	+1	+1	-1	+1
(1, 2, 3)	(2, 3, 5)	{1, 5}	1	3	{1, 2, 3, 5}	$\underline{\mathcal{C}}_2$	+1	+1	-1	+1
(1, 3, 4)	(1, 4, 5)	{3, 5}	2	3	{1, 3, 4, 5}	$\underline{\mathcal{C}}_2$	-1	+1	+1	-1
(1, 2, 6)	(1, 4, 6)	{2, 4}	2	3	{1, 2, 4, 6}	$\underline{\mathcal{C}}_1$	-1	+1	-1	-1
(1, 4, 6)	(1, 5, 6)	{4, 5}	2	2	{1, 4, 5, 6}	$\underline{\mathcal{C}}_4$	-1	+1	-1	-1
(2, 3, 5)	(2, 4, 5)	{3, 4}	2	2	{2, 3, 4, 5}	$\underline{\mathcal{C}}_7$	-1	-1	+1	-1
(2, 4, 5)	(2, 4, 6)	{5, 6}	3	3	{2, 4, 5, 6}	$\underline{\mathcal{C}}_4$	+1	-1	-1	-1
(2, 4, 5)	(2, 5, 6)	{4, 6}	2	3	{2, 4, 5, 6}	$\underline{\mathcal{C}}_4$	-1	-1	-1	-1
(1, 3, 4)	(3, 4, 5)	{1, 5}	1	3	{1, 3, 4, 5}	$\underline{\mathcal{C}}_2$	+1	+1	+1	-1
(1, 3, 4)	(3, 4, 6)	{1, 6}	1	3	{1, 3, 4, 6}	$\underline{\mathcal{C}}_6$	+1	+1	+1	-1
(3, 4, 5)	(3, 5, 6)	{4, 6}	2	3	{3, 4, 5, 6}	$\underline{\mathcal{C}}_4$	-1	-1	-1	-1

Notice that in order to compute $\chi_{\mathcal{B}}(B)$ one can choose *any* $B' \in \mathcal{B}$ which differs from B in one element and the according signed circuit contained in $B \cup B'$. For instance for $B = (123)$ one could take $B' = (125)$ instead with $B\Delta B' = \{3, 5\}$, $\rightsquigarrow i = 3, j = 3$. Then one simply uses the according circuit $\mathcal{C}_2 \in \mathcal{C}$ with support $\underline{\mathcal{C}} \subseteq B \cup B' = \{1235\}$ and computes equivalently $\chi_{\mathcal{B}}(123) = (-1)^{1+3+3} \cdot \text{sgn}_{\mathcal{C}_2}(3) \cdot \text{sgn}_{\mathcal{C}_2}(5) \cdot \chi_{\mathcal{B}}(125) = (-1) \cdot (-1) \cdot (+1) \cdot (-1) = -1$.

Our findings are summarized in table 1, where we write down all triples contained in E and decide whether they are a basis element (●) or not (●). The initial choice for $\chi_{\mathcal{B}}(1, 2, 3) = -1$ is marked by a blue frame.

Basis Triple	$\chi_{\mathcal{B}}$	Basis Triple	$\chi_{\mathcal{B}}$	Basis Triple	$\chi_{\mathcal{B}}$	Basis Triple	$\chi_{\mathcal{B}}$
● 123	-1						
● 124	0						
● 125	-1						
● 126	-1						
● 134	+1	● 234	+1				
● 135	0	● 235	+1				
● 136	-1	● 236	0				
● 145	-1	● 245	-1	● 345	-1		
● 146	-1	● 246	-1	● 346	-1		
● 156	-1	● 256	-1	● 356	-1	● 456	0

Table 1: Chirotope data of the graphic oriented matroid encoded in figure 3, computed by the signed basis graph method.

We will now compute the basis chirotope $\chi_{\mathcal{B}^*}$ of the dual basis as defined in definition 3.13.

$$\mathcal{B}^* := \{E \setminus B : B \in \mathcal{B}\}$$

with its chirotope

$$\chi_{\mathcal{B}^*}(B^*) = \chi_{\mathcal{B}}(B) \cdot \text{sgn}((B^*, B)) =: \chi_{\mathcal{B}}(B) \cdot \text{sgn}(\pi(b_1^*, b_2^*, b_3^*, b_1, b_2, b_3))$$

where we have written $B = (b_1, b_2, b_3)$ and $B^* = (b_1^*, b_2^*, b_3^*)$ and the defined lexicographically sorted ground set is $E = (1, 2, 3, 4, 5, 6)$. As before $\text{sgn}(\pi(\cdot))$ denotes the parity of the permutation π in order to bring the set in the argument into lexicographic order. The result is given in table 2.

B	$\chi_{\mathcal{B}}(B)$	B^*	$\text{sgn}((B^*, B))$	$\chi_{\mathcal{B}^*}(B^*)$
123	-1	456	-1	+1
125	-1	346	-1	+1
126	-1	345	+1	-1
134	+1	256	-1	-1
136	-1	245	-1	+1
145	-1	236	-1	+1
146	-1	235	+1	-1
156	-1	234	-1	+1
234	+1	156	+1	+1
235	+1	146	-1	-1
245	-1	136	+1	-1
246	-1	135	-1	+1
256	-1	134	+1	-1
345	-1	126	-1	+1
346	-1	125	+1	-1
356	-1	124	-1	+1

non-bases			
B	$\chi_{\mathcal{B}}(B)$	B^*	$\chi_{\mathcal{B}^*}(B^*)$
124	0	356	0
135	0	246	0
236	0	145	0
456	0	123	0

Table 2: Dual chirotope data for the oriented matroid given by figure 3.

Using $\chi_{\mathcal{B}^*}$ from table 2 we can now compute the signed cocircuits \mathcal{C}^* as signed circuits of the dual matroid $\mathcal{M}^* = (E, \mathcal{B}^*)$. As $E = (1, 2, 3, 4, 5, 6)$ consists of 6 elements we know that \mathcal{M}^* has rank $r^* = 6 - r = 3$. That is, all possible cocircuits should be supported within subsets having at most 4 elements. Now consider all $\binom{6}{4} = 15$ four-element subsets of E .

1234 1235 1236 1245 1246 1256 1345 1346 1356 1456
2345 2346 2356 2456
3456

We simply compute the circuit signature induced by $\chi_{\mathcal{B}^*}$ for every tuple, using again the signed basis graph construction (3.7) in its dualized version (dual bases and cocircuits as circuits of the dual matroid).

From the dual non-bases given in table 2 we already see that

$$\begin{aligned}\underline{C}_1^* &:= \{1, 2, 3\} \subset \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\} \\ \underline{C}_2^* &:= \{1, 4, 5\} \subset \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{1, 4, 5, 6\} \\ \underline{C}_3^* &:= \{2, 4, 6\} \subset \{1, 2, 4, 6\}, \{2, 3, 4, 6\}, \{2, 4, 5, 6\} \\ \underline{C}_4^* &:= \{3, 5, 6\} \subset \{1, 3, 5, 6\}, \{2, 3, 5, 6\}, \{3, 4, 5, 6\}\end{aligned}$$

Hence we have to additionally analyze the cocircuits C_5^*, C_6^*, C_7^* whose support is equal²⁸ to $\{1, 2, 5, 6\}, \{1, 3, 4, 6\}, \{2, 3, 4, 5\}$ respectively. As an example, let us compute the signed sets for $\pm C_1^*$. First we again choose $\text{sgn}_{C_1^*}(1) = -1$. In order to compute the signs for the elements $2, 3 \in \underline{C}_1^*$ we have to choose pairs $B^*, (B^*)'$ of dual bases, such that $\underline{C}_1^* \subseteq (B^* \cup (B^*)')$ and $B^* \Delta (B^*)' = \{1, 2\}, \{1, 3\}$, for example we may choose $B^* = (1, 3, 6), (B^*)' = (2, 3, 6)$ and $B^* = (1, 2, 6), (B^*)' = (2, 3, 6)$. Then we have, using (3.7) and the dual chirotope data of table 2:

$$\begin{aligned}\text{sgn}_{C_1^*}(2) &= \text{sgn}_{C_1^*}(1) \cdot (-1)^{1+1+1} \cdot \chi_{B^*}(1, 3, 6) \cdot \chi_{(B^*)'}(2, 3, 6) = (-1) \cdot (-1) \cdot (-1) \cdot (+1) = -1 \\ \text{sgn}_{C_1^*}(3) &= \text{sgn}_{C_1^*}(1) \cdot (-1)^{1+1+2} \cdot \chi_{B^*}(1, 2, 6) \cdot \chi_{(B^*)'}(2, 3, 6) = (-1) \cdot (+1) \cdot (+1) \cdot (+1) = -1\end{aligned}$$

It follows that we can write in our compact notation $C_1^* = \overline{123}$. One can check that $C_1^* \in \mathcal{C}^*$ is orthogonal to all $C_k \in \mathcal{C}$ as by definition 3.2. We get:

	$C_1^* \cap C_k = \{a, b\}$	$\text{sgn}_{C_k}(a)$	$\text{sgn}_{C_k}(b)$	$\text{sgn}_{C_1^*}(a)$	$\text{sgn}_{C_1^*}(b)$
$C_1 = \overline{124}$	$\{1, 2\}$	+	-	-	-
$C_2 = \overline{135}$	$\{1, 3\}$	+	-	-	-
$C_3 = \overline{236}$	$\{2, 3\}$	+	-	-	-
$C_4 = \overline{456}$	\emptyset				
$C_5 = \overline{1256}$	$\{1, 2\}$	+	-	-	-
$C_6 = \overline{1346}$	$\{1, 3\}$	+	-	-	-
$C_7 = \overline{2345}$	$\{2, 3\}$	+	-	-	-

and see that the orthogonality condition is fulfilled. Similarly one can proceed with the remaining cocircuits, where we use as before $B^* \Delta (B^*)' := (B^* \cup (B^*)') \setminus (B^* \cap (B^*)') = \{b_i, b'_j\}$, $B^* = (b_1, b_2, b_3), (B^*)' = (b'_1, b'_2, b'_3)$

²⁸As these 4-element subsets are extensions of dual bases by one element, and we already have computed all cocircuits with fewer than 4 elements, the remaining cocircuits must be supported on 4 elements.

and $\underline{C}^* \subseteq \{B^* \cup (B^*)'\}$:

\underline{C}^*	choose	$\{b_i, b'_j\}$	(b_1, b_2, b_3)	(b'_1, b'_2, b'_3)	i	j	$\chi_{B^*}(B^*)$	$\chi_{B^*}((B^*)')$	$\text{sgn}_{C^*}(b_j)$
$\{1, 4, 5\}$	$\text{sgn}_{C^*}(1) = -1$	$\{1, 4\}$	$(1, 2, 5)$	$(2, 4, 5)$	1	2	-1	+1	+1
		$\{1, 5\}$	$(1, 2, 4)$	$(2, 4, 5)$	1	2	+1	+1	+1
$\{2, 4, 6\}$	$\text{sgn}_{C^*}(2) = -1$	$\{2, 4\}$	$(2, 3, 6)$	$(3, 4, 6)$	1	2	+1	+1	-1
		$\{2, 6\}$	$(2, 3, 4)$	$(3, 4, 6)$	1	3	+1	+1	+1
$\{3, 5, 6\}$	$\text{sgn}_{C^*}(3) = -1$	$\{3, 5\}$	$(2, 3, 6)$	$(2, 5, 6)$	2	2	+1	-1	-1
		$\{3, 6\}$	$(3, 4, 5)$	$(3, 4, 6)$	1	3	-1	+1	-1
$\{1, 2, 5, 6\}$	$\text{sgn}_{C^*}(1) = -1$	$\{1, 2\}$	$(1, 2, 5)$	$(2, 5, 6)$	1	1	+1	-1	-1
		$\{1, 5\}$	$(1, 2, 6)$	$(2, 5, 6)$	1	2	+1	-1	+1
		$\{1, 6\}$	$(1, 2, 5)$	$(2, 5, 6)$	1	3	-1	-1	+1
$\{1, 3, 4, 6\}$	$\text{sgn}_{C^*}(1) = -1$	$\{1, 3\}$	$(1, 4, 6)$	$(3, 4, 6)$	1	1	-1	+1	-1
		$\{1, 4\}$	$(1, 3, 6)$	$(3, 4, 6)$	1	2	-1	+1	+1
		$\{1, 6\}$	$(1, 3, 4)$	$(3, 4, 6)$	1	3	-1	+1	-1
$\{2, 3, 4, 5\}$	$\text{sgn}_{C^*}(2) = -1$	$\{2, 3\}$	$(2, 3, 4)$	$(3, 4, 5)$	1	1	+1	-1	-1
		$\{2, 4\}$	$(2, 3, 5)$	$(3, 4, 5)$	1	2	-1	-1	-1
		$\{2, 5\}$	$(2, 3, 4)$	$(3, 4, 5)$	1	3	+1	-1	-1

In this way we end up with a set of 14 cocircuits [3] p. 9, given by the following signed sets and their negatives:

$$\begin{aligned}
C_1^* &= \overline{123} \\
C_2^* &= \overline{145} \\
C_3^* &= \overline{246} \\
C_4^* &= \overline{356} \\
C_5^* &= \overline{1256} \\
C_6^* &= \overline{1346} \\
C_7^* &= \overline{2345}
\end{aligned} \tag{3.10}$$

Certainly it is also possible to directly use the fact that cocircuits are orthogonal to all circuits. Writing $\text{sgn}_{C_k}(a), \text{sgn}_{C_k}(b)$ as in (3.9) for every pair $\{a, b\}$ contained in the set \mathcal{C} of all circuits, one can compare these data for all elements of the cocircuits and obtain their partition into positive and negative parts.

3.5 Oriented Matroids from Vector Configurations

Consider a finite set $E = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of $n \geq r$ non-zero vectors, spanning the r -dimensional vector space \mathbb{R}^r . We can express the minimal linear dependencies among the elements of E by the solutions to

$$\sum_{i=1}^n \lambda_i \mathbf{v}_i = 0 \quad \text{with } \lambda_i \in \mathbb{R}.$$

A solution is given by elements of the form $\vec{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ having at least two non-zero components. For each such solution $\vec{\lambda}$ we define the set $\underline{C} = \{i : \lambda_i \neq 0\}$, called a circuit. The set of all \underline{C} then builds up the set of all circuits \mathcal{C} of the (unoriented) matroid $\underline{\mathcal{M}} = (E, \mathcal{C})$. By construction $\underline{\mathcal{M}}$ has rank r .

In order to obtain an oriented matroid, consider for each solution $\vec{\lambda}$ the signed set $C = (C^+, C^-)$ with $C^+ = \{i : \lambda_i > 0\}$ and $C^- = \{i : \lambda_i < 0\}$, giving a signed circuit. We denote the set of all C by \mathcal{C} . This defines an oriented matroid $\mathcal{M} = (E, \mathcal{C})$, also called a *vector oriented matroid*.

We may also introduce the extended signature sgn_C for all $C \in \mathcal{C}$. Then the signed circuit C can be written as an element in $\{+, 0, -\}^n$ as $C = (\text{sgn}_C(\lambda_1), \dots, \text{sgn}_C(\lambda_n))$. Notice that the linear dependencies encoded in each circuit \underline{C} are only given up to an overall scalar. Hence each element $\underline{C} \in \underline{\mathcal{C}}$ gives rise to two elements $C = (C^+, C^-)$ and $-C = (C^-, C^+)$ in \mathcal{C} .

It is also straightforward to see that the set \mathcal{B} of bases of $\mathcal{M} = (E, \mathcal{C})$ is given by all subsets B of E which contain r linearly independent vectors. The natural notion of a basis orientation $\chi_B(B)$, $B = (b_1, \dots, b_r)$, can be written in terms of an $(r \times r)$ -matrix with column vectors $(\mathbf{v}_{b_1}, \dots, \mathbf{v}_{b_r})$ by

$$\chi_B(B) = \chi_B(b_1, \dots, b_r) = \text{sgn}(\det(\mathbf{v}_{b_1}, \dots, \mathbf{v}_{b_r})) =: \text{sgn}([b_1, \dots, b_r]) \quad (3.11)$$

Here and in what follows we write $[b_1, \dots, b_r]$ to abbreviate determinant expressions like $\det(\mathbf{v}_{b_1}, \dots, \mathbf{v}_{b_r})$. It may be checked that this notion of χ_B is consistent with definition 3.11 by the determinant properties.

Moreover the set \mathcal{C}^* of signed cocircuits of $\mathcal{M} = (E, \mathcal{C})$ gets the following geometric interpretation: Consider an $(r-1)$ -subset $\{h_1, \dots, h_{r-1}\}$ of E such that $H = \{\mathbf{v}_{h_1}, \dots, \mathbf{v}_{h_{r-1}}\}$ spans an $(r-1)$ -dimensional hyperplane which cuts \mathbb{R}^n into two half spaces H^+, H^- . Equip H with an orientation and choose the labels H^+, H^- such that H^+ is the positive and H^- the negative half space with respect to the chosen orientation. Then a signed cocircuit C^* of \mathcal{M} with $E \setminus C^* = H$ is given by $C^* = (C^{*+}, C^{*-})$ where $C^{*\pm}$ is the set of vectors in E contained in the according half space H^\pm . By construction \mathcal{C}^* contains $\pm C^*$, since the orientation of H can be chosen arbitrarily without loss of generality.

Grassmann Plücker identities and determinants [3]. Given any two sorted sets²⁹ $B, B' \subset E$ where $B = (b_1, \dots, b_r)$, $B' = (b'_1, \dots, b'_r)$, the product $[b_1, \dots, b_r] \cdot [b'_1, \dots, b'_r]$ can be expressed as

$$[b_1, \dots, b_r] \cdot [b'_1, \dots, b'_r] = \sum_{k=1}^r [b'_k, b_2, \dots, b_r] \cdot [b'_1, \dots, b'_{k-1}, b_1, b'_{k+1}, \dots, b'_r] \quad (3.12)$$

one special example being the Laplace expansion for determinants³⁰. In general expressions like (3.12) are called *Grassmann Plücker relations for determinants*. Notice that the difference between the left hand side and right hand side of (3.12) gives an antisymmetric linear expression in the $(r+1)$ arguments $b_1, b'_1, b'_2, \dots, b'_r$, that is an $(r+1)$ -form on \mathbb{R}^r . This vanishes by construction [3].

Obviously (3.12) implies $[b_1, \dots, b_r] \cdot [b'_1, \dots, b'_r] \geq 0$ if $[b'_k, b_2, \dots, b_r] \cdot [b'_1, \dots, b'_{k-1}, b_1, b'_{k+1}, \dots, b'_r] \geq 0$ holds for all $k = 1 \dots r$. With the identification of signs of determinants and chirotopes given in (3.11), this gives rise to

Definition 3.14 (Grassmann Plücker relations for chirotopes [3].) *Let the oriented matroid $\mathcal{M} = (E, \mathcal{B})$ be given with the chirotope χ_B giving a basis orientation of \mathcal{B} . Let $B, B' \subset E$ be given with $B = (b_1, \dots, b_r)$, $B' = (b'_1, \dots, b'_r)$. Then*

$$(B2) \quad \text{For all } B, B' \text{ such that } \chi_B(b'_k, b_2, \dots, b_r) \cdot \chi_B(b'_1, \dots, b'_{k-1}, b_1, b'_{k+1}, \dots, b'_r) \geq 0 \quad \forall k = 1 \dots r \\ \text{it holds that } \chi_B(b_1, \dots, b_r) \cdot \chi_B(b'_1, \dots, b'_r) \geq 0$$

or equivalently

$$(B2') \quad \text{For all } B, B' \text{ such that } \chi_B(b_1, \dots, b_r) \cdot \chi_B(b'_1, \dots, b'_r) \neq 0 \quad \exists k \in \{1, \dots, r\} \text{ such that} \\ \chi_B(b'_k, b_2, \dots, b_r) \cdot \chi_B(b'_1, \dots, b'_{k-1}, b_1, b'_{k+1}, \dots, b'_r) = \chi_B(b_1, \dots, b_r) \cdot \chi_B(b'_1, \dots, b'_r).$$

The equivalence of (B2) and (B2') can be seen as follows [3]: For $r = 1$ it is trivially fulfilled. For $r \geq 2$ set $\epsilon_k := \chi_B(b'_k, b_2, \dots, b_r) \cdot \chi_B(b'_1, \dots, b'_{k-1}, b_1, b'_{k+1}, \dots, b'_r)$. Observe that under the permutation $b'_1 \leftrightarrow b'_2$ the sign tuple $(\epsilon_1, \epsilon_2, \dots, \epsilon_r)$ changes into $(-\epsilon_2, -\epsilon_1, \dots, -\epsilon_r)$ as well as $\chi_B(b_1, \dots, b_r) \cdot \chi_B(b'_1, b'_2, \dots, b'_r) = -\chi_B(b_1, \dots, b_r) \cdot \chi_B(b'_2, b'_1, \dots, b'_r)$.

²⁹Not necessarily $B, B' \in \mathcal{B}$.

³⁰Let $B, B' \in \mathcal{B}$ and (b_1, \dots, b_r) be the coordinate vectors, that is $b_1 = (1, 0, \dots, 0)^t$, $b_2 = (0, 1, \dots, 0)^t$, \dots , $b_r = (0, 0, \dots, 1)^t$ (t stands for transposed). Let (b'_1, \dots, b'_r) be another basis. Certainly, $[b_1, \dots, b_r]$ corresponds to the determinant of the unit $(r \times r)$ -matrix. Inserting this, (3.12) corresponds to the Laplace expansion of the determinant of the matrix whose column vectors are given by (b'_1, \dots, b'_r) .

Now assume (B2) holds and $\chi_{\mathcal{B}}(b_1, \dots, b_r) \cdot \chi_{\mathcal{B}}(b'_1, \dots, b'_r) > 0$. Then there must be a k such that $\epsilon_k > 0$, as otherwise (3.12) would give a contradiction. By permuting $b'_1 \leftrightarrow b'_2$ $\chi_{\mathcal{B}}(b_1, \dots, b_r) \cdot \chi_{\mathcal{B}}(b'_2, b'_1, \dots, b'_r) < 0$ and as all ϵ_k change their sign also $\epsilon_k < 0$. Hence (B2') holds.

Conversely assume (B2') holds. Then we know that whenever $\chi_{\mathcal{B}}(b_1, \dots, b_r) \cdot \chi_{\mathcal{B}}(b'_1, \dots, b'_r) \neq 0$ there is a $k \in \{1, \dots, r\}$ such that $\epsilon_k = \chi_{\mathcal{B}}(b_1, \dots, b_r) \cdot \chi_{\mathcal{B}}(b'_1, \dots, b'_r) \neq 0$. Now if $\epsilon_k \geq 0 \ \forall k$ then it must hold that $\chi_{\mathcal{B}}(b_1, \dots, b_r) \cdot \chi_{\mathcal{B}}(b'_1, \dots, b'_r) \geq 0$. By applying the permutation argument again it also holds for $\epsilon_k \leq 0 \ \forall k$ that $\chi_{\mathcal{B}}(b_1, \dots, b_r) \cdot \chi_{\mathcal{B}}(b'_1, \dots, b'_r) < 0$. The case $\chi_{\mathcal{B}}(b_1, \dots, b_r) \cdot \chi_{\mathcal{B}}(b'_1, \dots, b'_r) = 0$ is contained in both arguments.

The question arising here is whether (B2), (B2') arise only in case of a vector realization of the oriented matroid $\mathcal{M} = (E, \mathcal{B})$. The answer is given in the following

Lemma 3.2 (Grassmann-Plücker Relations and Basis Orientation [3].) *Let the oriented matroid \mathcal{M} be given in terms of the ground set E and its set of cocircuits \mathcal{C}^* , $\mathcal{M} = (E, \mathcal{C}^*)$ with basis orientation $\chi_{\mathcal{B}}$. Then its chirotope $\chi_{\mathcal{B}}$ satisfies (B2').*

To see this [3], consider $B, B' \subset E$ with $B = (b_1, \dots, b_r)$, $B' = (b'_1, \dots, b'_r)$, such that $\epsilon := \chi_{\mathcal{B}}(b_1, \dots, b_r) \cdot \chi_{\mathcal{B}}(b'_1, \dots, b'_r) \in \{-1, +1\}$. If $b_1 \in B'$ then (B2') is trivially satisfied.

If $b_1 \notin B'$ consider the basic circuit $C = (b_1, B')$ and the basic cocircuit $C^* = (b_1, B)$ defined as in definition 3.4. By construction $b_1 \in \underline{C} \cap \underline{C}^*$. By orthogonality of C, C^* there is a $k \in \{1, \dots, r\}$ such that $\text{sgn}_C(b_1) \text{sgn}_C(b'_k) = -\text{sgn}_{C^*}(b_1) \text{sgn}_{C^*}(b'_k)$. Now \mathcal{M} is an oriented matroid and hence (PV*) and (PV) are satisfied by $\chi_{\mathcal{B}}$. Therefore it holds that

$$\begin{aligned} \chi_{\mathcal{B}}(b'_1, \dots, b'_{k-1}, b_1, b'_{k+1}, \dots, b'_r) &= -\text{sgn}_C(b_1) \cdot \text{sgn}_C(b'_k) \cdot \chi_{\mathcal{B}}(b'_1, \dots, b'_{k-1}, b'_k, b'_{k+1}, \dots, b'_r) \\ \chi_{\mathcal{B}}(b'_k, b_2, \dots, b_r) &= \text{sgn}_{C^*}(b_1) \cdot \text{sgn}_{C^*}(b'_k) \cdot \chi_{\mathcal{B}}(b_1, b_2, \dots, b_r) \\ \leadsto \quad \epsilon &= \chi_{\mathcal{B}}(b'_1, \dots, b'_r) \cdot \chi_{\mathcal{B}}(b_1, \dots, b_r) = \chi_{\mathcal{B}}(b'_k, b_2, \dots, b_r) \cdot \chi_{\mathcal{B}}(b'_1, \dots, b'_{k-1}, b_1, b'_{k+1}, \dots, b'_r) \end{aligned}$$

However the existence of $k \in \{1, \dots, r\}$ such that the last line holds is precisely the statement of (B2').

We see that (B2), (B2') of definition 3.14 is not a consequence of the realization of \mathcal{M} as a vector configuration. Rather it is a general consequence from the combinatorial definition 3.11.

As an example [3], consider the vectors $E = (\mathbf{v}_1, \dots, \mathbf{v}_6)$ in \mathbb{R}^3 , given by the columns of the matrix.

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

From A we can read off the set of minimal linear dependencies between the vectors in E . For instance we have $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_5 - \mathbf{v}_6 = 0$, that is $\vec{\lambda} = (1, -1, 0, 0, 1, -1)$ and $X = \{+, -, 0, 0, +, -\}$. This can be compared to the graph example in the previous section. Here we also obtain the same list of signed circuits as in the graph example above. The column vectors of the matrix A contain the set \mathcal{B} of bases of the oriented matroid \mathcal{M} obtained in the previous section.

3.6 Oriented Matroids from Hyperplane and Sphere Arrangements

There are two additional realizations of oriented matroids, related to vector configurations, which we introduce for completeness [3, 43].

Consider the rank r vector oriented matroid $\mathcal{M} = (E, \mathcal{C})$ as defined at the beginning of section 3.5. Each element \mathbf{v}_i of ground set $E = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of \mathcal{M} can be used to define an $(r-1)$ -dimensional hyperplane H_i through the origin in \mathbb{R}^r as $H_i = \{\mathbf{x} \in \mathbb{R}^r : \langle \mathbf{v}_i, \mathbf{x} \rangle = 0\}$, where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product on \mathbb{R}^r . That is, \mathbf{v}_i is the normal vector to H_i . Obviously the orientation of \mathbf{v}_i can be used to define an orientation of H_i . Then we may define the positive side of H_i by $H_i^+ := \{\mathbf{x} \in \mathbb{R}^r : \langle \mathbf{v}_i, \mathbf{x} \rangle > 0\}$ and the negative side $H_i^- := \{\mathbf{x} \in \mathbb{R}^r : \langle \mathbf{v}_i, \mathbf{x} \rangle < 0\}$.

In this way E corresponds to an oriented arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ of n $(r-1)$ -dimensional hyperplanes. In turn \mathcal{A} gives rise to a decomposition of \mathbb{R}^r into r -dimensional cells \mathcal{W} . The interior of each such cell is uniquely described by its relative position with respect to the hyperplanes contained in \mathcal{A} . For each $H_i \in \mathcal{A}$ we can say if \mathcal{W} is on the positive or negative side of H_i . That is \mathcal{W} is uniquely described by an element of $\{+1, -1\}^n$.

Obviously the oriented vector matroid $\mathcal{M} = (E, \mathcal{B})$ can be equivalently encoded in the arrangement \mathcal{A} . To specify \mathcal{M} we need to specify the unique cell $\mathcal{W} \equiv (+1, \dots, +1)$ situated on the positive side of every hyperplane.

We may alternatively consider the intersection of \mathcal{A} with the unit sphere $S^{r-1} = \{\mathbf{x} \in \mathbb{R}^r : \|\mathbf{x}\| = 1\}$ around the origin of \mathbb{R}^r . This gives rise to an arrangement $\mathcal{S} = \{S_1, \dots, S_n\}$ of n unit $(r-2)$ -spheres S_i on S^{r-1} defined by $S_i := S^{r-1} \cap H_i$. Equivalently we define $S_i^\pm := S^{r-1} \cap H_i^\pm$. This is depicted in figure 4. The sphere arrangement $\mathcal{S} = \{S_1, \dots, S_n\}$ together with the specification of a positive and negative side S_i^\pm for every $S_i \in \mathcal{S}$ is called a signed sphere arrangement.

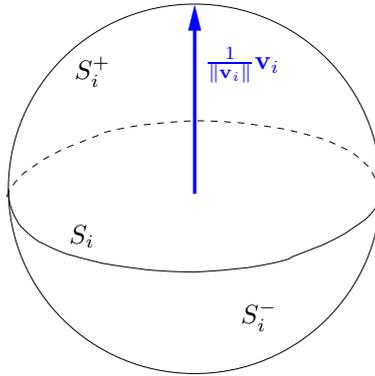


Figure 4: S^{r-1} split into two open hemispheres S_i^+ , S_i^- by an $(r-2)$ -sphere S_i , induced by the element \mathbf{v}_i of the ground set E .

Now recall the rank r vector oriented matroid $\mathcal{M} = (E, \mathcal{C})$ from the beginning of section 3.5. For a signed circuit $C \in \mathcal{C}$ it holds that³¹

$$\bigcup_{\mathbf{v}_i \in C} \overline{S_i^{\text{sgn}_C(\mathbf{v}_i)}} = S^{r-1} \quad \text{where} \quad S_i^{\text{sgn}_C(\mathbf{v}_i)} = S_i^\pm \quad \text{and} \quad \overline{S_i^\pm} \text{ denotes the closure } \overline{S_i^\pm} = S_i^\pm \cup S_i$$

We will see in section 3.7 how sphere arrangements can be used for deciding the so-called realizability problem of oriented matroids.

3.7 Realizability of Oriented Matroids

In the sections 3.5 and 3.6 we have outlined how oriented matroids can arise from different geometric setups corresponding to vector configurations. This section will be concerned with the opposite point of view: Given an oriented rank r matroid $\mathcal{M} = (E, \mathcal{B})$ defined as in section 3.2, when can it be represented as a vector configuration? This question is called the realizability problem for oriented matroids. It is of particular relevance if one wants to compute all possible vector oriented matroids of rank r constructable over an abstract ground set E . For instance one would like to compute all classes of diffeomorphic embeddings of a graph vertex as discussed in section 4.1.

The main statement [3, 43] here is that reorientation³² equivalence classes (\equiv equivalence classes under relabelling of the elements of the ground set E and a possible change of orientation for each element $e \in E$) of oriented matroids correspond to signed arrangements of modified arrangements of spheres called signed pseudosphere arrangements on $S^{r-1} \subset \mathbb{R}^r$.

³¹Notice that the definition of H_i^\pm is slightly different in [3] and [43]. In the former, H_i^\pm contains H_i and is thus closed, whereas in the latter H_i is defined as an open half space of \mathbb{R}^r . We regard the notation of [43] as more convenient, as it avoids difficulties in assigning a sign to points $x \in H_i \subset \mathbb{R}^r$, and separates H_i from H_i^\pm . In general a circuit $C \in \mathcal{C}$ corresponds to a minimal system of closed hemispheres that cover the whole unit sphere S^{r-1} .

³²See section 3.8 for the definition of reorientation.

Consider an $(r-2)$ -sphere S_i as depicted in figure 4. A subset³³ \mathcal{S}_i of S^{r-1} is called a *pseudosphere* if $\mathcal{S} = h(S_i)$ for some homeomorphism $h : S^{r-1} \rightarrow S^{r-1}$. Using h we can also define the open subspaces $\mathcal{S}_i^\pm := h(S_i^\pm)$, as depicted in figure 5. The arrangement $\mathcal{S} = \{\mathcal{S}_1, \dots, \mathcal{S}_1\}$ of pseudospheres $\mathcal{S}_i \subset S^{r-1} \subset \mathbb{R}^r$ together with a fixed choice of positive side \mathcal{S}_i^+ and negative side \mathcal{S}_i^- for each \mathcal{S}_i is called a signed pseudosphere arrangement.

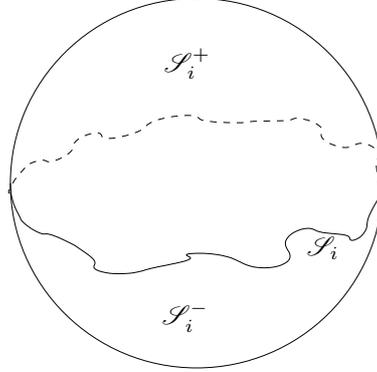


Figure 5: S^{r-1} split into two half spaces \mathcal{S}_i^+ , \mathcal{S}_i^- by an $(r-2)$ -pseudosphere \mathcal{S}_i homeomorphic to the $(r-2)$ -sphere S_i induced by the matroid element \mathbf{v}_i as shown in figure 4.

Now conversely let a signed pseudosphere arrangement $\mathcal{S} = \{\mathcal{S}_1, \dots, \mathcal{S}_n\} \subset S^{r-1} \subset \mathbb{R}^r$ be given. Choose an arbitrary n -element set $E = \{1, \dots, n\}$ as a ground set and associate to every $i \in E$ a $\mathcal{S}_i \in \mathcal{S}$. Define $\mathcal{C}(\mathcal{S})$ as the set of sign vectors $C \in \{+, -, 0\}^{|E|}$ with support $\underline{C} = \{e \in E : \text{sgn}_C(e) \neq 0\}$, which satisfy³⁴

$$(i) \bigcup_{e \in \underline{C}} \overline{\mathcal{S}_e^{\text{sgn}_C(e)}} = S^{r-1} \text{ where } \mathcal{S}_e^{\text{sgn}_C(e)} := \mathcal{S}_e^\pm \quad \text{and } \overline{\mathcal{S}_e^\pm} \text{ denotes the closure } \overline{\mathcal{S}_e^\pm} = \mathcal{S}_e^\pm \cup \mathcal{S}_e$$

(ii) The support \underline{C} is minimal with respect to (i), that is \underline{C} contains only minimal subsets of E such that the union of the closed half spaces $\overline{\mathcal{S}_e^\pm}$ covers S^{r-1} .

Then we get the following

Theorem 3.1 (Topological Representation Theorem.) *The following conditions are equivalent [3]:*

- (1) *If $\mathcal{S} = (\mathcal{S}_e)_{e \in E}$ is a signed arrangement of pseudospheres in S^{r-1} , then $\mathcal{C}(\mathcal{S})$ is the family of circuits of a rank r simple oriented matroid on E .*
- (2) *If (E, \mathcal{C}) is a rank r simple oriented matroid, then there exists a signed arrangement of pseudospheres \mathcal{S} in S^{r-1} such that $\mathcal{C} = \mathcal{C}(\mathcal{S})$.*
- (3) *$\mathcal{C}(\mathcal{S}) = \mathcal{C}(\mathcal{S}')$ for two signed arrangements \mathcal{S} and \mathcal{S}' in S^{r-1} iff $\mathcal{S}' = h(\mathcal{S})$ for some self-homeomorphism h of S^{r-1} .*

It follows that there is a one-to-one correspondence between (equivalence classes of) pseudosphere arrangements in S^{r-1} and (reorientation classes of) simple rank r oriented matroids. That is, a pseudosphere arrangement specifies an oriented matroid only up to reorientation and permutation (relabeling of the ground set).

Moreover the stretchability of the arrangement (existence of a self-homeomorphism on S^{r-1} transforming each pseudosphere \mathcal{S}_i into a proper sphere S_i) is equivalent to the realizability of the oriented matroid as a vector configuration. Note that the problem of deciding stretchability can be encoded in a system of equalities and inequalities of polynomial expressions. Finding a solution to this system is equivalent to deciding the realizability problem for an oriented matroid given by a pseudosphere arrangement [3]. For rank $r = 3$ it turns out that from $n \geq 9$ there exist oriented matroids whose corresponding pseudosphere arrangements cannot be stretched and which are thus not realizable [46]. For rank $r = 3$ results for such a counting are shown in table 3.

³³Notice that our notation distinguishes between pseudospheres and spheres, in contrast to [3].

³⁴Again notice the difference in notation between [3] and [43] as described in footnote 31. As in the case of spheres S_i we understand the \mathcal{S}_i^\pm as open subspaces in the sense of [43]. Moreover we understand $C \in \{+, -, 0\}^{|E|}$ as coming from an extended signature map, assigning a sign to every $e \in E$ in the sense of definition 3.8.

Remark. Notice that the term realizability is used in two different contexts throughout this work. In the mathematics literature such as [3] realizability of a given matroid aims at answering the question whether a given set \mathcal{C} of oriented circuits of a matroid can be realized by a vector configuration. In the context of [4, 5] the term realizability is directed towards the question whether a given set \vec{e} of sign factors in the definition of the volume operator \hat{V}_v at a vertex v in the vertex set of a graph corresponds to the chirotope of an oriented matroid of rank 3 which can be realized (in the mathematical sense) as a vector configuration. Either question of realizability not only decides whether a given family of oriented sets respectively sign factors represents an oriented matroid, but it also can be used to count the number of realizable oriented matroids of a given rank r over an n -element set E .

3.8 Operations on Oriented Matroids

Finally we would like to define certain operations which can be performed by oriented matroids. Let again be $\mathcal{M} = (E, \mathcal{C}) = (E, \mathcal{B})$ be an oriented rank r matroid over the ground set E . Then the following operations are defined [3].

Reorientation. According to definition 3.6 one can define a reorientation ${}_F\tilde{\mathcal{C}} := \tilde{\mathcal{C}}$ of a signed set \mathcal{C} on $F \subseteq E$ by $(\tilde{\mathcal{C}})^\pm = (C^\pm \setminus F) \cup (C^\mp \cap F)$, that is

- (a) $\text{sgn}_{\tilde{\mathcal{C}}}(e) := (-1)^{|F \cap \{e\}|} \cdot \text{sgn}_{\mathcal{C}}(e)$ and
- (b) ${}_F\chi_{\mathcal{B}}(e_1, \dots, e_r) = \tilde{\chi}_{\mathcal{B}}(e_1, \dots, e_r) := \chi_{\mathcal{B}}(e_1, \dots, e_r) \cdot (-1)^{|F \cap \{e_1, \dots, e_r\}|}$.

Mutation. Let \mathcal{M} furthermore be a uniform³⁵ oriented matroid. Then every r -tuple $B = (b_1, \dots, b_r)$ is a basis. An r -tuple \bar{B} yields a mutation of \mathcal{M} if the mapping $\bar{B}\chi_{\mathcal{B}}$ given by

$$\bar{B}\chi_{\mathcal{B}}(B) := \begin{cases} -\chi_{\mathcal{B}}(B) & \text{if } \bar{B} = B \\ \chi_{\mathcal{B}}(B) & \text{otherwise} \end{cases} \quad (3.13)$$

defines another oriented matroid, that is $\bar{B}\chi_{\mathcal{B}}$ satisfies the Grassmann Plücker relations $(B2) / (B2')$ of definition 3.14.

Deletion. If a subset $A \subset E$ is removed from E then the remaining matroid $\mathcal{M} \setminus A$ over the ground set $E \setminus A$ is defined by its set of signed circuits

$$\mathcal{C} \setminus A := \{C \in \mathcal{C} : C \cap A = \emptyset\} \quad (3.14)$$

Suppose $\mathcal{M} \setminus A$ has rank $s < r$. Choose an arbitrary basis $B \in \mathcal{B}$. Now take the $(r - s)$ tuple $B \cap A =: (a_1, \dots, a_{r-s}) \in A$. Then the chirotope of $\chi_{\mathcal{B} \setminus A}$ can be constructed (up to an overall sign) as follows:

$$\begin{aligned} \chi_{\mathcal{B} \setminus A} : (E \setminus A)^s &\rightarrow \{-1, 0, +1\} \\ (e_1, \dots, e_s) &\mapsto \chi_{\mathcal{B}}(e_1, \dots, e_s, a_1, \dots, a_{r-s}) \end{aligned} \quad (3.15)$$

By construction $\chi_{\mathcal{B} \setminus A}$ has all properties of a chirotope. Moreover the above definition specifies $\chi_{\mathcal{B} \setminus A}$ uniquely up to a sign [3], independent of the choice of $B \in \mathcal{B}$.

Example. As an example consider a ground set $E = (e_1, e_2, e_3)$ of 3 vectors e_1, e_2, e_3 in \mathbb{R}^2 . Let a family \mathcal{B} of bases be given by $\mathcal{B} = (B_1, B_2)$ with $B_1 = (e_1, e_3)$, $B_2 = (e_2, e_3)$, assume e_1, e_2 to be co-linear, such that $\underline{\mathcal{C}} = \{\underline{C}\}$ with $\underline{C} = \{e_1, e_2\}$. The resulting oriented matroid $\mathcal{M} = (E, \mathcal{B})$ then has rank $r = 2$. Now consider $A = \{e_3\}$. Clearly $\mathcal{M} \setminus A$ has rank $s = 1$ with bases $\mathcal{B} \setminus A = \{e_1, e_2\}$. in order to construct $\chi_{\mathcal{B} \setminus A}$, choose $B_2 \in \mathcal{B}$ with $B_2 \cap A = (e_3) =: (a_1)$. Applying (3.15) then gives $\chi_{\mathcal{B} \setminus A}(e_1) = \chi_{\mathcal{B}}(e_1, a_1)$ and $\chi_{\mathcal{B} \setminus A}(e_2) = \chi_{\mathcal{B}}(e_2, a_1)$.

³⁵See section 3.3 for the definition.

4 Oriented Matroids in Loop Quantum Gravity

Having outlined the construction of LQG in section 2, and oriented matroids in the previous section, we will now demonstrate how oriented matroids naturally enter into LQG. As we have seen, both the local geometric properties of a graph γ as well as its global topological (connectedness) properties can be described by oriented matroids.

4.1 Local Properties: Signfactors Characterizing the Volume Spectrum

As described in section 2, the measure μ_0 on the kinematical Hilbert space \mathcal{H}_0 is sensitive only to $\text{supp}(l(\gamma))$ ³⁶ of the graph γ underlying a spin network function [47] in \mathcal{H}_0 , but not to other properties of γ such as relative orientations of edges at the vertices.

In contrast to μ_0 , the flux and a composite thereof, namely the volume operator [1], are sensitive to relative orientations of edges. These relations are preserved by analytic diffeomorphisms. Even more, this sensitivity is crucial for a consistent formulation of the theory [48] and the constraint operators [12]. This is based on the fact that the spectrum of the volume operator due to [1] is characterized by signature factors resulting from the embedding of γ in Σ as shown in [4, 5]. They encode the relative orientation of all tangent vectors of N_v edges intersecting at a vertex v of γ . The classification of all possible signature factor combinations turns out to be intimately related to a central question of oriented matroid theory [3, 2], namely whether a given oriented matroid of rank 3 can be realized as a configuration of $N_v \geq 3$ vectors in \mathbb{R}^3 .

4.1.1 Construction of the Volume Operator

Within LQG, the operator corresponding to the volume of a region R in three dimensional Riemannian space plays a prominent role for the implementation of the scalar (Hamilton) constraint operator on the quantum level. In [4, 5], the spectral properties of the volume operator according to [1, 49] were analyzed on $\mathcal{H}_{\text{Gauss}}$, the gauge invariant subspace of the kinematical Hilbert space \mathcal{H}_0 of LQG. Starting from the classical volume expression rewritten in terms of Ashtekar variables³⁷

$$V(R) = \int_R d^3x \sqrt{\det q(x)} = \int_R d^3x \sqrt{\left| \frac{1}{3!} \epsilon^{ijk} \epsilon_{abc} E_i^a(x) E_j^b(x) E_k^c(x) \right|} \quad (4.1)$$

one can work out the action of $V(R)$ on a cylindrical function $f_\gamma : G^n \ni (h_{e_1}, \dots, h_{e_n}) \rightarrow f_\gamma(h_{e_1}, \dots, h_{e_n})$, which has support on the n copies of $G = SU(2)$, each labelled by one of the edges (e_1, \dots, e_n) contained in the edge set $E(\gamma)$ of a graph γ embedded into three dimensional Riemannian space. Notice that the graph γ underlying the cylindrical function f_γ is taken to be adapted to the flux operator³⁸ $E_i(S)$. That is, the elements in $E(\gamma)$, which intersect S , are subdivided, such that their subdivisions either end or start at S . With these assumptions the action of $E(S)$ on f_γ is explicitly given as follows [50]:

$$\begin{aligned} [E_i(S)f_\gamma](h_{e_1}, \dots, h_{e_n}) &= \frac{1}{2} \sum_{e \in E(\gamma)} \sum_{AB} \epsilon(e, S) \left(\delta_{e \cap S, b(e)} \frac{\tau_i}{2} h_e + \delta_{e \cap S, f(e)} h_e \frac{\tau_i}{2} \right)_{AB} \frac{\partial f_\gamma(h_{e_1}, \dots, h_{e_n})}{\partial (h_e)_{AB}} \\ &= \frac{1}{4} \sum_{e \in E(\gamma)} \epsilon(e, S) \left[(\delta_{e \cap S, b(e)} R_e^i + \delta_{e \cap S, f(e)} L_e^i) f_\gamma \right] (h_{e_1}, \dots, h_{e_n}) \quad . \end{aligned} \quad (4.2)$$

Here we denote by $[R_e^i f_\gamma](h_{e_1}, \dots, h_{e_n}) := \left. \frac{d}{dt} f_\gamma(h_{e_1}, \dots, e^{t \tau_i} h_e, \dots, h_{e_n}) \right|_{t=0}$ and respectively by $[L_e^i f_\gamma](h_{e_1}, \dots, h_{e_n}) := \left. \frac{d}{dt} f_\gamma(h_{e_1}, \dots, h_e e^{t \tau_i}, \dots, h_{e_n}) \right|_{t=0}$ the action of right / left invariant vector fields on the copy of $SU(2)$ labeled by $e \in E(\gamma)$. By τ_i we represent a basis of the Lie algebra $\mathfrak{su}(2)$, given by $\tau_i = -i \sigma_i$

³⁶ $l(\gamma)$ is defined in section 2.

³⁷Here $\epsilon_{abc}, \epsilon^{ijk}$ denote the antisymmetric symbol ($\epsilon_{123} = 1 = -\epsilon_{213}$ etc.) and we use Einstein's sum convention. Moreover $\det q(x)$ denotes the determinant of the spatial metric on the Cauchy surface Σ . $E_i^a(x)$ is an $\mathfrak{su}(2)$ -valued vector density, called a densitized triad. It holds that $E_i^a(x) E_j^b(x) \delta^{ij} = \det q(x) q^{ab}(x)$. Here δ^{ij} is the Cartan-Killing metric of $\mathfrak{su}(2)$, which is just the Euclidean metric. Moreover $q^{ab}(x)$ are the components of the inverse spatial metric in the tangent space $T_x \Sigma$ at the point $x \in \Sigma$, with respect to a given basis (e.g. a coordinate basis ∂_a).

³⁸Let $S \subset \Sigma$ be an orientable two-dimensional surface in Σ . Then the flux $E_i(S)$ is defined as the densitized triad $E_i^a(x)$ integrated over the 2-surface S , that is $E_i(S) = \int_S du dv \epsilon_{abc} \frac{\partial x^a(u,v)}{\partial u} \frac{\partial x^b(u,v)}{\partial v} E_i^c(x(u,v))$ where $x : (u,v) \mapsto x(u,v)$ denotes the embedding of the surface S into Σ parametrized by the pair (u,v) .

with σ_i being the according Pauli matrix³⁹. Moreover $\delta_{e \cap S, b(e)} = 1$ if the intersection $e \cap S$ is the (b)eginning point of e , that is e is outgoing from S and zero otherwise. Accordingly $\delta_{e \cap S, f(e)} = 1$ if the intersection $e \cap S$ is the (f)inal point of e , that is e is incoming to S and zero otherwise. Also in (4.2) each $e \in E(\gamma)$ labels a particular copy of $SU(2)$. The sum over A, B can be taken in defining a representation of $SU(2)$, but in principle any representation could be chosen⁴⁰. The orientation factor $\epsilon(e, S)$ indicates the relative orientation of the edge tangent at the intersection point and the chosen surface orientation \vec{n}_S of S .

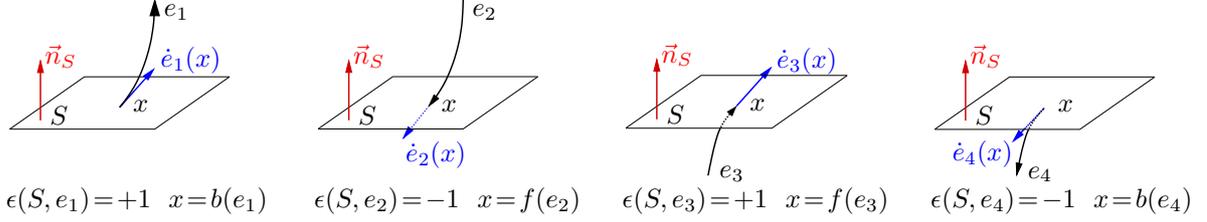


Figure 6: The four distinct configurations of the surface S and an edge e . The intersection point of S and e is denoted by x . If $S \cap e = e$ or $e \cap S = \emptyset$ then $\epsilon(S, e) = 0$.

Moreover if we set $\tilde{f}_\gamma(\dots, h, \dots) := f_\gamma(\dots, h^{-1}, \dots)$ then it holds that

$$[R_e^i \tilde{f}_\gamma](\dots, h_e, \dots) = \frac{d}{dt} f_\gamma(\dots, (\mathbb{e}^{t \tau_i} h_e)^{-1}, \dots) \Big|_{t=0} = \frac{d}{dt} f_\gamma(\dots, h_e^{-1} \mathbb{e}^{-t \tau_i}, \dots) \Big|_{t=0} = -[L_e^i f_\gamma](\dots, h_e^{-1}, \dots) \quad (4.3)$$

by the exchange property of left / right invariant vector fields. Now consider a cylindrical function $f_{e_1}(h_{e_1})$, where e_1 is given as in the first case of figure 6. Moreover $e_1^{-1} = e_2$, $l(e_1) = l(e_2)$ where e_2 is given as in the second case of figure 6. Then

$$[E_i(S) f_{e_1}](h_{e_1}) = \frac{1}{4} [R_{e_1}^i f_{e_1}](h_{e_1}) = \frac{1}{4} [L_{e_1}^i f_{e_1}](h_{e_1}^{-1}) = \frac{1}{4} [L_{e_1}^i f_{e_1}](h_{e_1}^{-1}) = \frac{1}{4} [L_{e_2}^i f_{e_2}](h_{e_2}) \quad (4.4)$$

where we have used the transformation property of the edge holonomy under edge reorientation $e \rightarrow e^{-1}$, that is $h_e^{-1} = h_{e^{-1}}$. Notice that $\epsilon(S, e_1) = -\epsilon(S, e_1^{-1}) = -\epsilon(S, e_2)$ compensates the minus sign between the action of $R_{e_1}^i$ and $L_{e_1}^i$. A similar statement holds for case 3 and 4 in figure 6. That is, the action (4.4) of the flux operator on a cylindrical function f_γ only depends on the support $l(\gamma)$, and not on the actual direction of edges contained in $E(\gamma)$.

In contrast to this, situation 1 and 3 in figure 6 have identical relative orientations $\epsilon(S, e_1) = \epsilon(S, e_3) = 1$, but

$$[E_i(S) f_{e_1}](h_{e_1}) = \frac{1}{4} [R_{e_1}^i f_{e_1}](h_{e_1}) \quad \text{and} \quad [E_i(S) f_{e_3}](h_{e_3}) = \frac{1}{4} [L_{e_3}^i f_{e_3}](h_{e_3}) \quad (4.5)$$

because $x = b(e_1) = f(e_3)$. Strictly speaking, (4.5) describes the action of the flux on two *different* cylindrical functions f_{e_1}, f_{e_3} , which have different supports $l(e_1) \neq l(e_3)$.

Now we are going to discuss the implication of this on the volume operator. In its final form, the action of the volume operator on a cylindrical function f_γ is given by [1, 49]:

$$\hat{V}(R)_\gamma f_\gamma(\cdot) = \int_R d^3x \sqrt{\widehat{\det(q(x))}_\gamma} f_\gamma(\cdot) = \int_R d^3x \hat{V}(x)_\gamma f_\gamma(\cdot) \quad (4.6)$$

where the “ $\widehat{}$ ” symbolizes the operator corresponding to the classical expression, $R \subseteq \Sigma$ denotes a region in Σ , and

$$\hat{V}(x)_\gamma = \ell_P^3 \sum_{v \in V(\gamma)} \delta^3(x, v) \hat{V}_{v, \gamma} \quad (4.7)$$

³⁹

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{with} \quad [\sigma_i, \sigma_j] = 2i \epsilon^{ijk} \sigma_k$$

⁴⁰This is a quantization ambiguity.

Here ℓ_P denotes the Planck length and $\delta^3(x, v)$ is Dirac's delta distribution. The action of $\hat{V}(R)_\gamma$ decomposes into a local action at all vertices v in the vertex set $V(\gamma)$ with valence N_v

$$\hat{V}_{v,\gamma} = \sqrt{\left| Z \sum_{I < J < K \leq N_v} \epsilon(I, J, K) \hat{q}_{IJK} \right|} \quad (4.8)$$

$$\hat{q}_{IJK} = \sum_{i,j,k=1}^3 \varepsilon_{ijk} R_I^i R_J^j R_K^k \propto \left[(R_{IJ})^2, (R_{JK})^2 \right] \quad \text{with} \quad (R_{IJ})^2 = \sum_{k=1}^3 (R_I^k + R_J^k)^2 \quad . \quad (4.9)$$

Here Z is an overall regularization constant which does not change the spectral properties modulo a possible rescaling. We will set $Z = 1$ in numerical computations⁴¹. At each vertex v one has to sum over all possible ordered triples $(e_I, e_J, e_K) \equiv (I, J, K)$, $I < J < K \leq N_v$ of outgoing edges at v . Here $\epsilon(IJK) = \text{sgn}(\det(\dot{e}_I(v), \dot{e}_J(v), \dot{e}_K(v)))$ denotes the sign of the determinant of the tangents of the three edges e_I, e_J, e_K intersecting at v . In the sequel we will denote by $\vec{\epsilon} := \{\epsilon(I, J, K)\}_{I < J < K \leq N_v}$ the set of ϵ -signfactors for all triples $I < J < K \leq N_v$.

Notice that the graph γ is adapted to $\hat{V}(R)_\gamma$ such that all vertices $v \in V(\gamma)$ with $N_v \geq 3$ are the beginning points of edges. This can be achieved by cutting each edge $e \in E(\gamma)$ into two pieces e', e'' such that e is given as a composition $e = e' \circ (e'')^{-1}$ with $b(e) = b(e')$, $f(e) = b(e'')$. This introduces a two valent vertex $v' = f(e') = f(e'')$ which does not contribute to the volume, as the action of $\hat{V}(R)_\gamma$ on two valent vertices is trivial. This adaption makes it possible to rewrite the action of $\hat{V}(R)_\gamma$ entirely in terms of right invariant vector fields R_I^k .

Moreover, if gauge invariance is imposed then the volume operator can be rewritten as

$$\hat{V}_{v,\gamma} =: \sqrt{\left| Z \cdot \sum_{I < J < K < N_v} \sigma(I, J, K) \hat{q}_{IJK} \right|} \quad (4.10)$$

where

$$\sigma(I, J, K) := \epsilon(I, J, K) - \epsilon(I, J, N) + \epsilon(I, K, N) - \epsilon(J, K, N) \quad . \quad (4.11)$$

We will accordingly denote by $\vec{\sigma} := \{\sigma(I, J, K)\}_{I < J < K < N_v}$ the set of σ -signfactors for all triples $I < J < K < N_v$.

4.1.2 Sign Factors as Chirotopes

In [4, 5] it was shown that the spectral properties of the volume operator depend strongly on the sign factors $\epsilon(I, J, K)$, respectively $\sigma(I, J, K)$. In particular the presence of a smallest non-zero eigenvalue (also referred to as a 'volume gap') depends crucially on this underlying sign configuration. In [4, 5] the set of all possible sign factors for an N_v valent vertex v was computed by a Monte-Carlo method, which sprinkles N_v points on the unit sphere and collects the occurring distinct sign factors in a table. With this method the sign configurations resulting from N_v vectors, without coplanar triples, were determined up to $N_v = 7$. However, the combinatorics of the sign configurations were not fully understood at that point.

This situation has now improved, as the problem can be viewed in the broader context of oriented matroids. The set of $\vec{\epsilon}$ sign factors resulting from a configuration of N_v vectors (without coplanar triples) can be understood as a chirotope of a (uniform) oriented matroid of rank 3 over N_v elements, represented as a set of vectors in \mathbb{R}^3 . As already pointed out in [5], the choice of a particular edge labeling has no significance, and thus the volume spectrum should be invariant under the permutation of edge labels. We thus wish to group our eigenvalue data into equivalence classes, defined by the action of permutations on the edge labels. For the case of edge spins, the action of a permutation is straightforward. Such permutations, however, act in a non-trivial way upon the chirotope. For example, if we exchange edge labels $I \leftrightarrow J$, then $\epsilon(I, J, K) \mapsto \epsilon(J, I, K) = -\epsilon(I, J, K) =: \epsilon'(I, J, K)$, $\epsilon(I, L, M) \mapsto \epsilon(J, L, M) =: \epsilon'(I, L, M)$ and $\epsilon(J, L, M) \mapsto \epsilon(I, L, M) =: \epsilon'(J, L, M)$. We thus identify chirotopes which can be transformed into each other by a permutation into equivalence classes, each identified by a canonical representative.⁴²

⁴¹Notice however that in [48] it was found that $Z = \frac{1}{48}$, which is relevant if concrete numerical values are required as e.g. in [20].

⁴²We represent each chirotope by a string of $\binom{N_v}{3}$ bits, and choose the smallest representative of each class, where the bit string is interpreted directly as a non-negative integer. Here '-' is 0, '+' 1, and the triples start at the low end with 123 at 2^0 , 124 at 2^1 , etc.

For each such representative, we compute the sigma configuration, from (4.11). For the 6- and 7-vertex, there are a small number of duplicates — multiple permutation equivalence classes of chirotopes which give rise to the same $\vec{\sigma}$ -configurations.

It turns out that a more convenient symmetry, from the perspective of the mathematics literature [46] (c.f. section 3.7), is to regard chirotopes as invariant under a reorientation of any subset of the edge vectors \hat{e}_I , in addition to any permutation of the edge labels. A classification into such reorientation equivalence classes is thus well studied, and is shown in the rightmost column of table 3.

Using the according data [51], we are now able to exactly reproduce⁴³ the permutation equivalence classes of chirotopes, that were previously computed in [4, 5] by a Monte Carlo sprinkling process for $N_v \leq 8$. This confirms the remarkable applicability of the Monte Carlo method to this problem. Having the oriented matroid data at hand, we are now sure that we had detected all uniform oriented matroids of rank 3 over a ground set of up to $N_v = 8$ elements.⁴⁴

In addition, one can write down an analytic expression which gives a lower bound to the total number $\#\vec{\epsilon}(N_v)$ of realizable $\vec{\epsilon}$ sign factors for N_v vectors [52]. For $N_v \geq 3$, it is

$$\#\vec{\epsilon}(N_v) \geq \frac{(N_v - 2)!}{4^{N_v - 2}} \prod_{s=1}^{N_v - 2} (s^3 - 2s^2 + 7s + 2). \quad (4.12)$$

Note that this expression is able to exactly reproduce $\#\vec{\epsilon}(N_v)$ for N_v up to 6. A derivation of this bound will appear in [53].

N_v	# triples	$\#\vec{\epsilon}(N_v)$ (lower bound by (4.12))	$\#\vec{\epsilon}(N_v)$ (sprinkling)	$\#\vec{\epsilon}$ permutation equivalence classes	# $\vec{\sigma}$ confi- gurations	# realizable reorientation equiv. classes
3	1	2	2	1	1	1
4	4	16	16	3	3	1
5	10	384	384	4	4	1
6	20	23 808	23 808	41	39	4
7	35	3 333 120	3 486 720	706	673	11
8	56	939 939 840	$\geq 747 735 880$	28 287		135
9	84	486 888 837 120	?	?		4 381

Table 3: Sign factor combinatorics for 3–9-valent non-coplanar vertices embedded in \mathbb{R}^3 .

As can be seen from (4.8), the volume operator evaluated at a vertex v is not only sensitive to the local edge tangent vector configuration encoded in the $\epsilon(I, J, K)$ factors, but also on whether v is the beginning or final point of the N_v edges intersecting at v . To see this, notice that the reorientation of an edge, say e_L , outgoing from v , flips the edge tangent $\hat{e}_L(v)$, and hence inverts all signs $\{\epsilon(IJL)\}$ containing the label L . However, at the same time, the action of $[R_{e_L}^i f_\gamma](\dots, h_{e_L}, \dots)$ changes to $[L_{e_L}^i f_\gamma](\dots, h_{e_L}^{-1}, \dots) = -[R_{e_L}^i \tilde{f}_\gamma](\dots, h_{e_L}, \dots)$ according to (4.3). This compensates the sign inversion of all $\epsilon(IJL)$ similarly to (4.4). This situation is depicted in figure 7 and figure 8. There e_3 is reoriented, and hence the orientation of the edge tangent $\hat{e}_3(v_2)$ is flipped compared to $\hat{e}_3(v_1)$. Although the configuration of sign factors in figures 7 and 8 differs accordingly, the volume spectrum at v_1 and v_2 is identical.

⁴³One can compute the permutation equivalence classes from the reorientation equivalences classes as follows. Choose a representative $\vec{\epsilon}_{\text{Reor}}$ of each reorientation equivalence class, and apply all 2^{N_v} reorientations to it. For each such reorientation apply each of the $N_v!$ permutations of the elements, and record the canonical representative of the resulting chirotope. The set of such canonical representatives gives the set of permutation equivalence classes of chirotopes.

⁴⁴To be completely correct we were not able to perform the sprinkling of the 8-vertex in [4, 5], because the enormous number of chirotopes overwhelmed the system memory of any machines we had available. We performed the sprinkling with $N_v = 8$ more recently, by computing the canonical representative of each chirotope as it arose, thus having to save only 706 chirotopes in place of more than 10^8 .

To rephrase this in terms of oriented matroids: If an element in the graphic matroid constructed from γ is reoriented, this induces a reorientation of the according element in the local vector oriented matroid at v . Because of the described sign compensation, the volume spectrum is thus invariant under reorientations of the graphic oriented matroid of γ .

In contrast to this, figure 9 seems to have the same sign configuration $\vec{\epsilon}$ induced from the edge tangents at v_3 as we find at v_2 . However, the support (tracing of e_3) of the underlying graph γ is different in figure 8 and figure 9. Hence by (4.5) the signs $\vec{\epsilon}$ of (4.8) containing e_3 are effectively opposite at v_2 and v_3 , and it follows that the volume spectra at v_2 and v_3 are different. This is equivalent to stating that $\hat{V}(R)$ is evaluated on spin network functions supported on non-diffeomorphic graphs.

It follows that the spectrum of the volume operator is invariant under relabellings (permutation) of edges adjacent to v .⁴⁵ It is furthermore invariant under reorientations of the adjacent edges, which induce reorientations on the vector oriented matroid encoded by the $\vec{\epsilon}$ factors (figures 7 and 8). However it is *not invariant* under an isolated reorientation of the oriented matroid encoded in the $\vec{\epsilon}$ factors alone (figure 9). Hence different volume spectra at v are characterized by permutation equivalence classes of $\vec{\epsilon}$ configurations.

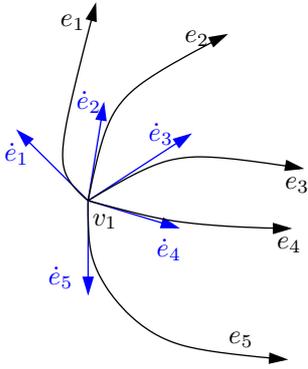


Figure 7: Vertex v_1 all edges outgoing.

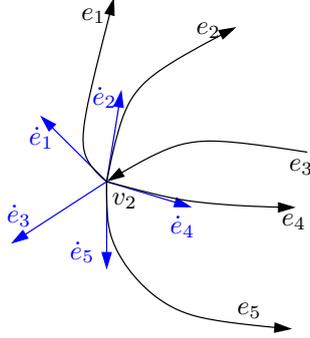


Figure 8: Vertex v_2 generated from v_1 by orientation reversal of edge e_3 .

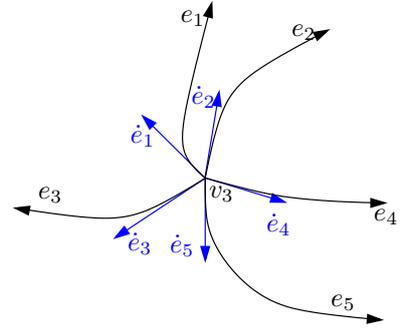


Figure 9: Vertex v_3 with identical edge tangent orientation as v_2 but with all edges outgoing.

A more general discussion, also including non-uniform oriented matroids (with coplanar edges), using methods developed in [55], will be presented in a forthcoming paper [53].

4.2 Global Properties: Gauge Invariance and Circuits on Digraphs

We have seen in section 3.4 that connectedness properties of (directed) graphs can be described by (oriented) matroids. Although originally introduced for planar graphs [41], the definition of graphic oriented matroids in terms of its signed circuits depends only on the topology of the graph (adjacency of vertices, orientation of edges), and not on any embedding. Therefore the framework of graphical oriented matroids is general enough to be applied to any abstract graph, and in particular to the graphs underlying (gauge invariant) spin network functions in LQG⁴⁶.

Given the fact that the construction of LQG as outlined in section 2 is based on partially ordered sets of graphs, it is natural to ask if we can reformulate this construction in terms of oriented matroids. This seems feasible, because it makes sense to speak of the restriction of an (oriented) matroid, if its ground set E is restricted. This can be directly seen from the deletion property in section 3.8. Hence, similarly to the partial order among the label set \mathcal{L} of section 2, the according graphic (oriented) matroids will be equipped with a partial order given by set inclusion of their ground sets. This will be further investigated in [58]. By the definition of the graphic oriented matroid in terms of circuits it seems obvious that the oriented matroid formalism should be applied at the gauge invariant level $\mathcal{H}_{\text{Gauss}}$. See also section 6 for a discussion.

⁴⁵In [49, 54, 5] the details for computing the matrix elements \hat{q}_{IJK} in (4.10) of the volume operator with respect to a basis of gauge invariant spin network functions is given. This basis is formulated in terms of so-called recoupling schemes. Permuting edge labels amounts to changing the recoupling order, which is a unitary basis transformation. This can be seen from the fact that the transformation matrix between two recoupling schemes can be written in terms of Clebsch-Gordan coefficients which are unitary. Details will be given in [53].

⁴⁶The possibility of winding numbers between edges certainly needs to be further investigated. See also [56, 57] and section 6.

In order to demonstrate the possibly new and interesting features of such a reformulation, we would like to give the following example on $\mathcal{H}_{\text{Gauss}}$:

If the Gauss constraint, and hence gauge invariance, is imposed on spin network functions, then the fundamental group of its underlying graph γ becomes relevant⁴⁷. Taking into account the properties of the measure μ_0 as described in section 2, a gauge invariant spin network function f_γ over γ is equivalent to a gauge variant spin network function over another graph $\tilde{\gamma} \subset \gamma$, the remainder of γ after the retraction to a spanning tree $T_{(\gamma)}$ of edges⁴⁸ [59]. This also holds if additionally invariance under graph automorphisms is imposed [47]. The circuit structure of a directed graph can in turn be used to define an oriented matroid [3, 2]. By definition, the oriented matroid \mathcal{M}_γ constructed from γ has the set of all spanning trees of γ as its basic family $\mathcal{B}(\mathcal{M}_\gamma)$. Then each possible edge set $E(\tilde{\gamma})$ of $\tilde{\gamma}$ obtained by removal of a $B \in \mathcal{B}(\mathcal{M}_\gamma)$ corresponds to an element B^* of the dual basis \mathcal{B}^* by definition 3.13. Hence we can reformulate the findings of [59] by saying that gauge invariant information of f_γ can be encoded in cylindrical functions $f_{\tilde{\gamma}}$, which have support only on \mathcal{B}^* , the family of bases of the dual oriented matroid \mathcal{M}_γ^* .

5 Spectrum of the Volume Operator Revisited

In [4, 5] an extensive numerical analysis of the spectrum of the volume operator was presented. For valences $N_v = 4 - 7$, and spins up to some $j_{\max}(N_v)$, for every chirotope which arose from the Monte Carlo sprinkling discussed in section 4.1.2, we presented graphs and histograms which detail various aspects of the collection of eigenvalues of the volume operator for vertices within this range of parameters.

Casting the analysis into the context of oriented matroids now allows us to better organize the collection of eigenvalues, in particular to take advantage of the permutation symmetry of the operator, as touched upon in section 4.1.2. Instead of computing the volume spectrum for all realizable chirotopes, we can restrict attention to the canonical representatives of each permutation equivalence class. Since the volume operator depends upon these chirotopes through the sigma values of (4.11) (a sigma configuration $\vec{\sigma}$), we compute this for each canonical representative of a permutation equivalence class.

When counting eigenvalues to form histograms, we follow the same practice as in [4, 5], regarding each eigenvalue from each chirotope as distinct. Thus we associate a redundancy with each permutation equivalence class of chirotopes, equal to the number of members in that class. We then add these redundancies for each chirotope which yields the same sigma configuration, to get a redundancy for each sigma configuration. The numbers of eigenvalues reported in histograms are then weighted by (half of)⁴⁹ these redundancies.

The analysis in [4, 5] considers only ‘sorted’ spin values $j_1 \leq \dots \leq j_{N_v}$, because any other assignment of spins to the edges is related to one of these by a permutation. However, this same permutation will also transform the chirotope \vec{e} to another \vec{e}' . Thus the separation of the eigenvalues into those arising from a given chirotope \vec{e} , while at the same time considering only sorted spin values $j_1 \leq \dots \leq j_{N_v}$, is ‘misleading’, in the sense that the eigenvalues which arise from the same chirotope \vec{e} , but with non-sorted spin values j_1, \dots, j_{N_v} , are found instead under a different chirotope \vec{e}' which is related to \vec{e} by a permutation which brings the spins into a sorted ordering $j_1 \leq \dots \leq j_{N_v}$.

A more ‘correct’ presentation of the eigenvalue data would combine all eigenvalues which arise from vertices which are related to each other by a permutation. We achieve this by working only with the canonical representative of each permutation equivalence class of the chirotopes, but allow *all* values of spins on the edges j_1, \dots, j_{N_v} , not just those which are sorted $j_1 \leq \dots \leq j_{N_v}$.

The resulting spectra are presented below. Remarkably it appears that the property found in [4, 5], that different

⁴⁷Consider a cylindrical function $f_\gamma(h_{e_1}, \dots, h_{e_N})$ defined over the digraph γ with edge set $E(\gamma) = \{e_1, \dots, e_N\}$ and vertex set $V(\gamma) = \{b(e_K), f(e_K)\}_{e_K \in E(\gamma)}$. In the context of LQG, gauge invariance of f_γ denotes invariance under generalized gauge transformations $\vec{\mathcal{G}}$, which consist of all mappings $g : \Sigma \ni x \mapsto g(x) \in SU(2)$ (no continuity assumption). Applying $g \in \vec{\mathcal{G}}$, a holonomy h_{e_K} transforms as $h_{e_K} \xrightarrow{g} h_{e_K}^{(g)} := g(b(e_K))h_{e_K}g(f(e_K))^{-1}$.

⁴⁸To see this, choose a spanning tree $T_{(\gamma)} = \{e_1, \dots, e_t\} \subseteq E(\gamma)$ of γ , with cardinality t . Denote $\tilde{E} := E \setminus T_{(\gamma)} = \{\tilde{e}_1, \dots, \tilde{e}_{N-t}\}$. Consider the cylindrical function $f_\gamma(h_{\tilde{e}_1}, \dots, h_{\tilde{e}_{N-t}}, h_{e_1}, \dots, h_{e_t})$ as before. Given *any* $(h_{\tilde{e}_1}, \dots, h_{\tilde{e}_{N-t}}, h_{e_1}, \dots, h_{e_t}) \in SU(2)^N$, we can choose a $g \in \vec{\mathcal{G}}$, such that $h_{e_K}^{(g)} = \mathbb{1}_{SU(2)}$ for *every* $e_K \in T_{(\gamma)}$. This implies that the gauge invariant information of f_γ can be encoded in $f_{\tilde{\gamma}}$, where $\tilde{E} = E(\tilde{\gamma})$.

⁴⁹This half stems back to the overall sign symmetry of chirotopes. In our code we consider only chirotopes which assign +1 to the basis 123. However it is interesting to note that a chirotope C will in general *not* lie in the same permutation equivalence class as its negative $-C$. However we have carefully checked that nevertheless the true redundancies as defined here are exactly double of those that arise by ignoring all chirotopes with basis 123 = -1, in spite of the separation of C from $-C$ by permutations.

$\vec{\sigma}$ -configurations have a different behavior of the smallest non-zero eigenvalue (increases, decreases, or remains constant as j_{\max} is increased) is preserved among the permutation equivalence classes for the 5-vertex. There we find precisely four permutation equivalence classes and the corresponding four different spectral properties as j_{\max} is increased: identically zero spectrum, increasing/decreasing smallest non-zero eigenvalue, and constant smallest non-zero eigenvalue. At the six vertex we also observe all four such behaviors, however at the 7-vertex the increasing smallest eigenvalue sequences seem to vanish. This may affect the notion of a semiclassical limit for the volume operator as discussed in section 6.

Our improved understanding of the action of permutations on the whole spin network vertex, including edge spins and orientation of the embedded edge tangents (as encoded in the chirotope) together, has revealed some errors in our previous analysis of [4, 5].

One such error is in the handling of the overall sign symmetry of chirotopes (that if $\chi_{\mathcal{B}}$ is a valid chirotope (equivalently $\vec{\epsilon}$ a valid sign configuration), then so $-\chi_{\mathcal{B}}$ (equivalently $-\vec{\epsilon}$). There we made a mistake in mapping this symmetry to the sigma configurations, and thus some of the sigma configurations we used in our analysis were incorrect. This affects some of the counting of sigma configurations which yield the different behaviors of the smallest non-zero eigenvalues, and some of the eigenvalues themselves. However none of the qualitative results are affected.

We were also able to utilize the permutation symmetry of the volume operator as a cross check on our code, which at the level of sigma configurations is extremely subtle. In using this we discovered a bug which affected all our results on the 7-vertex. The corrected eigenvalues are shown in section 5.4.

In addition there was a bug in our code with respect to the handling of the sigma configuration for the cubic 6-vertex. Corrected eigenvalues are shown in section 5.3.

The results from the computations shown below come from running a modified version of the code developed for references [4, 5], which is written as a ‘thorn’ within the Cactus high performance computing framework [60], on the ‘whale’ cluster of the SHARCNET grid in Southern Ontario. The computations for the 7-vertex ran efficiently on 225 cores.

5.1 Gauge Invariant 5-Vertex

In figure 10 we show histograms of the three non-trivial permutation equivalence classes of the 5-vertex, for j_{\max} up to $\frac{25}{2}$. Each is distinct yet shows similar behavior. Note the small lip at zero for the purple $\vec{\sigma} = (2, 2, 4, 0)$ histogram. This results from a non-negligible fraction of the eigenvalues descending toward zero at larger spins, c.f. figure 13. All histograms are drawn as described in [5].

Figure 11 depicts overall histograms for the 5-vertex, at various values of j_{\max} up to $25/2$. Note that the lip at zero is not really visible at this scale. Figure 12 zooms in on the small eigenvalue region. There we can discern the lip just beginning to show at $j_{\max} = 11$.

Figures 13 and 14 show the smallest and largest non-zero eigenvalues of the 5-vertex, for each $\vec{\sigma}$ and j_{\max} . For these $\vec{\sigma}$ -index 0 corresponds to $\vec{\sigma} = (2, 2, 2, 0)$, 1 to $\vec{\sigma} = (2, 0, 0, 0)$, and 2 to $\vec{\sigma} = (2, 2, 4, 0)$. Here we see the three behaviors of sigma configurations with respect to their smallest non-zero eigenvalues, as mentioned in section 5, with $\vec{\sigma}$ -index 0 increasing, 1 constant, and 2 decreasing with j_{\max} . The largest eigenvalues behave similarly for each non-trivial $\vec{\sigma}$.⁵⁰

5.2 Gauge Invariant 6-Vertex

Histograms for the 6-vertex are displayed in figures 15, 16, and 17, for j_{\max} up to $13/2$. Here there are 39 distinct sigma configurations. The beginnings of a lip at zero in the spectrum are just visible for several sigma configurations $\vec{\sigma}$, as seen in figure 16.

Figures 18 and 19 depict the smallest and largest non-zero eigenvalues for each $\vec{\sigma}$, as a function of maximum spin. Most of the sigma configurations yield smallest non-zero eigenvalues which decrease with j_{\max} , however a number of them remain constant (such as the rather degenerate $\vec{\sigma} = (2000000000)$, in which only four of the six spins play any role beyond their effect in determining the recoupling basis), and one ($\vec{\sigma} = (0222222220)$) leads to increasing smallest non-zero eigenvalues. It can also be noted, in particular in figure 18, that in general there are several sigma configurations which yield the same exact sequence of smallest non-zero eigenvalues. The largest eigenvalues all appear to increase by a power law, as expected from (4.9).

⁵⁰There is a fourth $\vec{\sigma}$ at the 5-vertex, which is all zeros, and hence leads to only zero eigenvalues. There will be such a $\vec{\sigma}$ for every valence. Zero eigenvalues are suppressed on all plots.

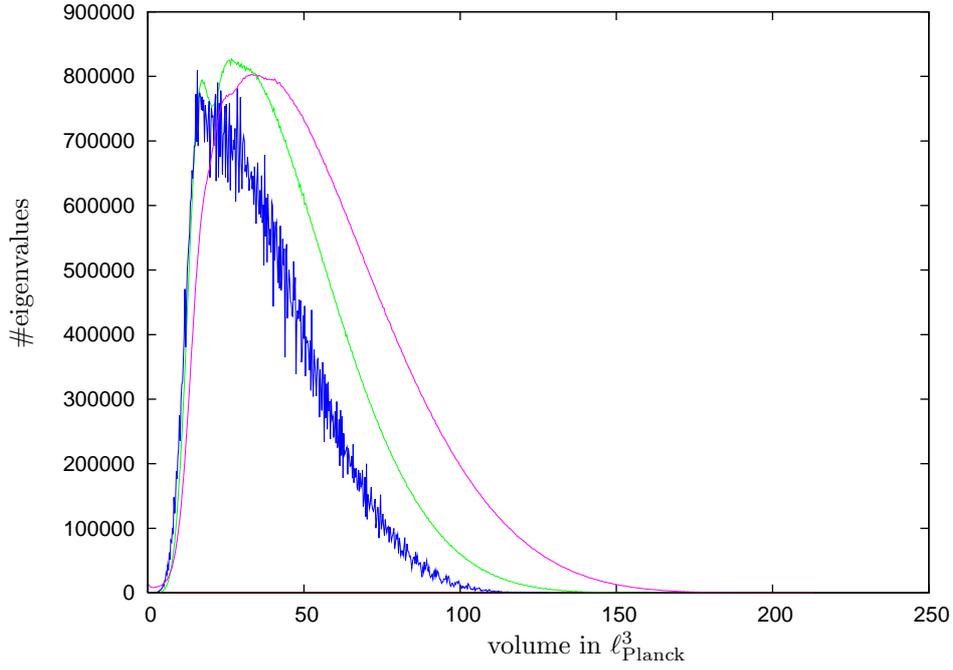


Figure 10: Histograms for each sigma configuration $\vec{\sigma}$ at the 5-vertex, up to $j_{\max} = 25/2$. The blue is for $\vec{\sigma} = (\sigma_{123}, \sigma_{124}, \sigma_{134}, \sigma_{234}) = (2, 0, 0, 0)$, the green for $\vec{\sigma} = (2, 2, 2, 0)$, and the purple for $\vec{\sigma} = (2, 2, 4, 0)$. Each histogram has 512 bins.

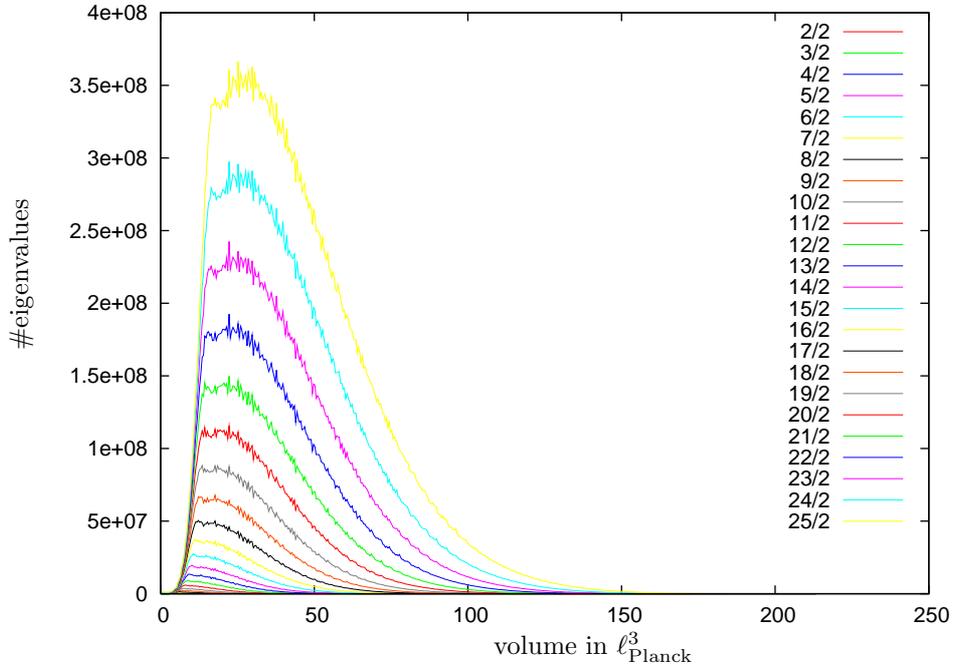


Figure 11: Histograms for the overall generic 5-vertex, up to every $j_{\max} \leq 25/2$. (By ‘generic’ we mean excluding co-planar edges, which in a sense form a set of measure zero.) Each histogram has 512 bins.

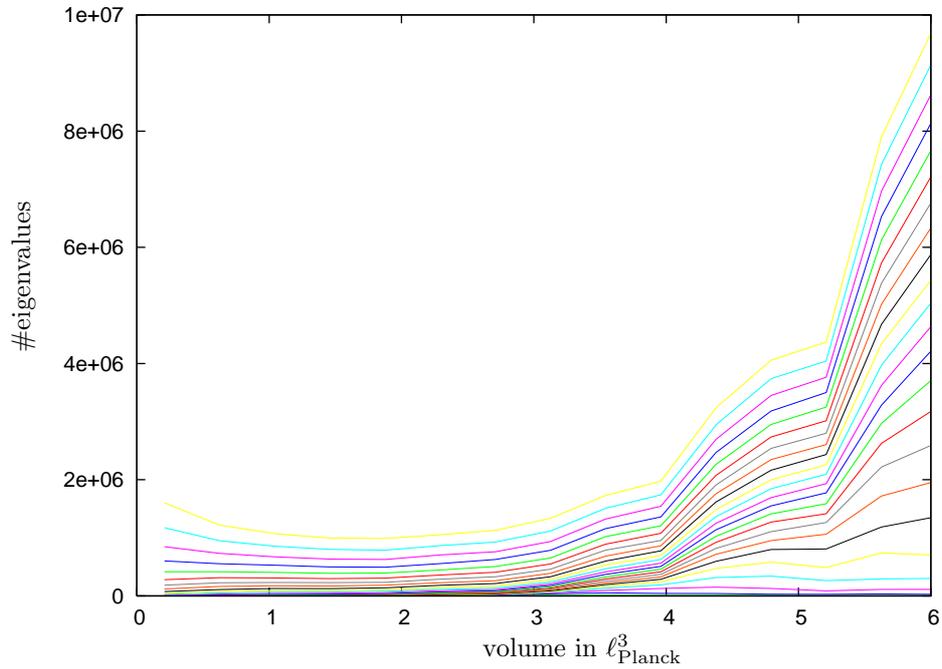


Figure 12: Histograms for the overall 5-vertex, up to every $j_{\max} \leq 25/2$, zoomed to the small eigenvalue region.

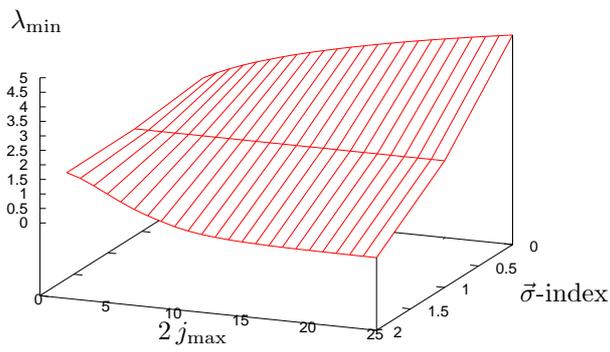


Figure 13: Smallest non-zero eigenvalues λ_{\min} at the 5-vertex.

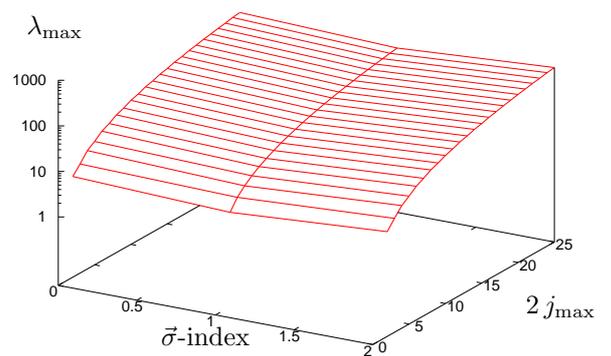


Figure 14: Largest eigenvalues λ_{\max} of the 5-vertex.

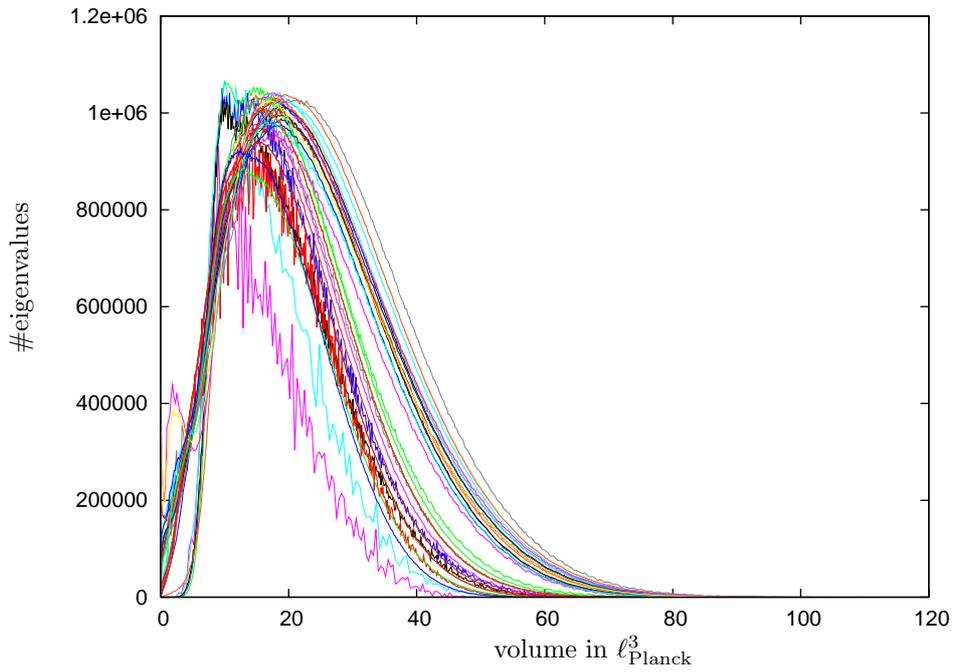


Figure 15: Histograms for each sigma configuration $\vec{\sigma}$ at the 6-vertex, up to $j_{\max} = 13/2$. Two histograms have 128 bins, all others have 512.

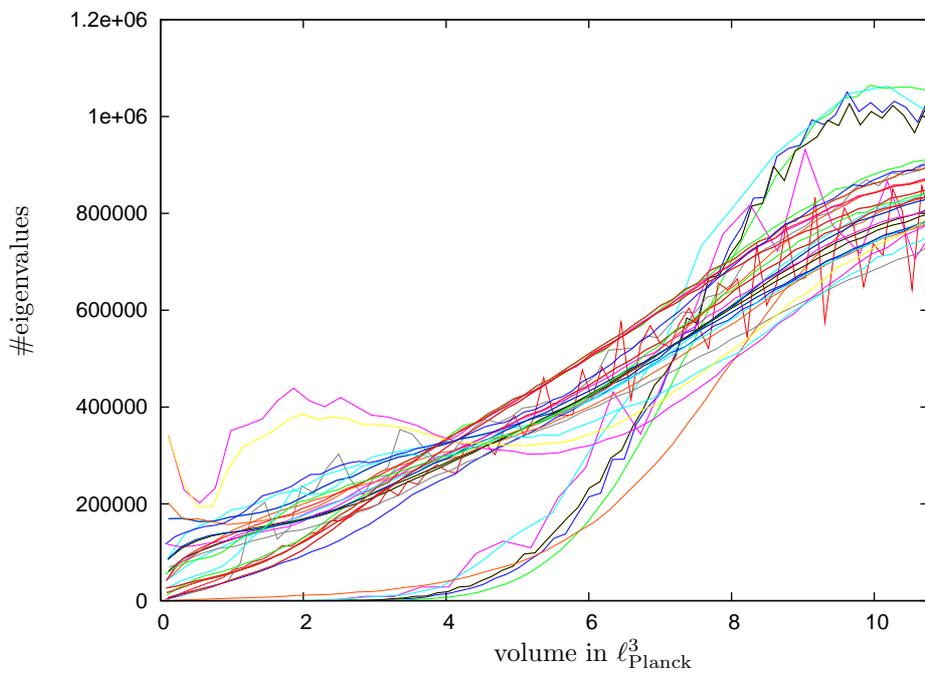


Figure 16: Histograms for each sigma configuration $\vec{\sigma}$ at the 6-vertex, up to $j_{\max} = 13/2$, zoomed to the small eigenvalue region.

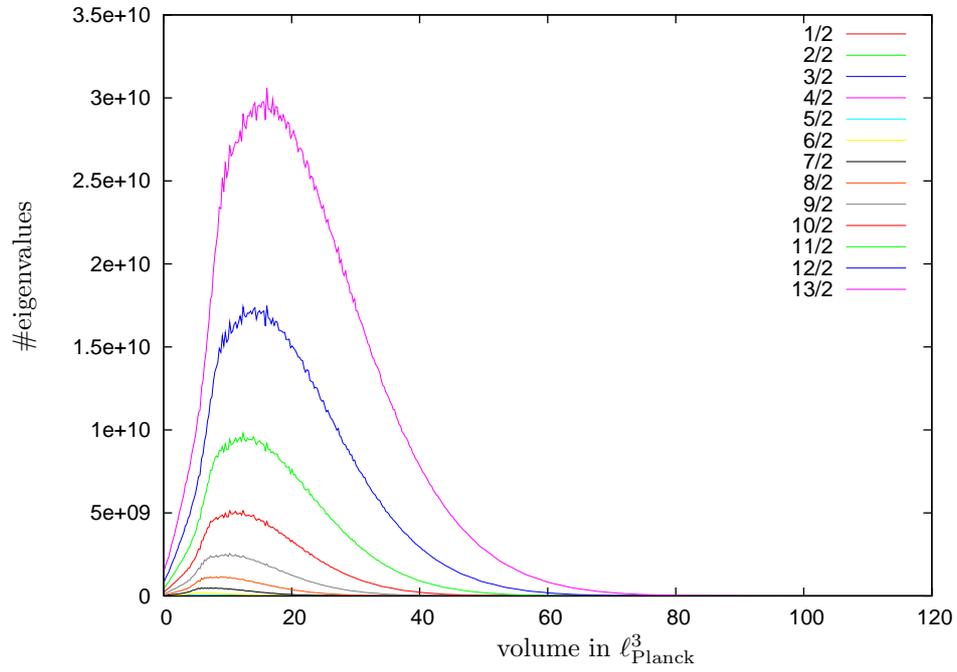


Figure 17: Histograms for the overall (generic) 6-vertex, up to every $j_{\max} \leq 13/2$. All have 512 bins.

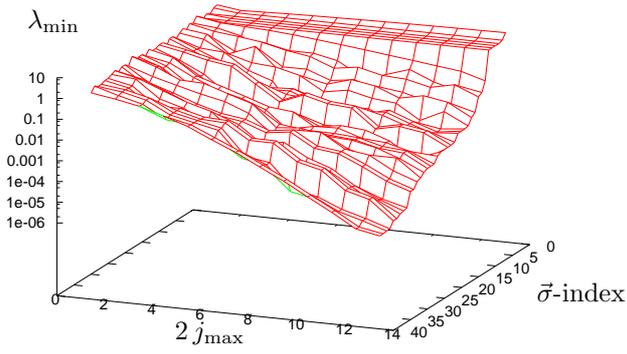


Figure 18: Smallest non-zero eigenvalues λ_{\min} at the 6-vertex.

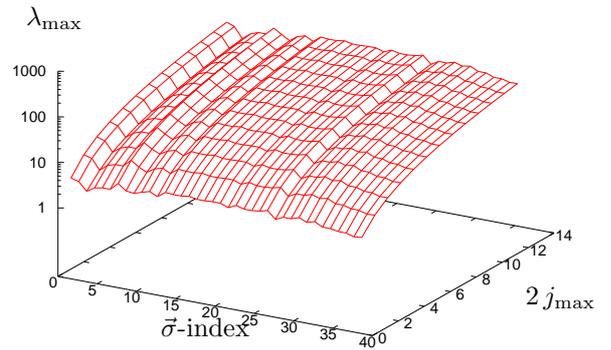


Figure 19: Largest eigenvalues λ_{\max} of the 6-vertex.

5.3 Cubic 6-Vertex

The cubic 6-vertex arises from a ‘tilation’ of \mathbb{R}^3 by cubes. I.e. it is the network of lines formed by restricting the Cartesian coordinates $x, y \in \mathbb{Z}$, with $z \in \mathbb{R}$, and likewise for $x, z \in \mathbb{Z}, y \in \mathbb{R}$, and $y, z \in \mathbb{Z}, x \in \mathbb{R}$. The edge tangents at such a vertex are of course coplanar — this is the one exception to our rule of excluding coplanar edges. The resulting chirotope and sigma configuration are detailed in [5].

In figure 20 we present histograms for the volume eigenvalues of the cubic 6-vertex, for maximum spins j_{\max} up to 5. Figure 21 displays the minimum and maximum non-zero eigenvalues, as a function of j_{\max} . One can see

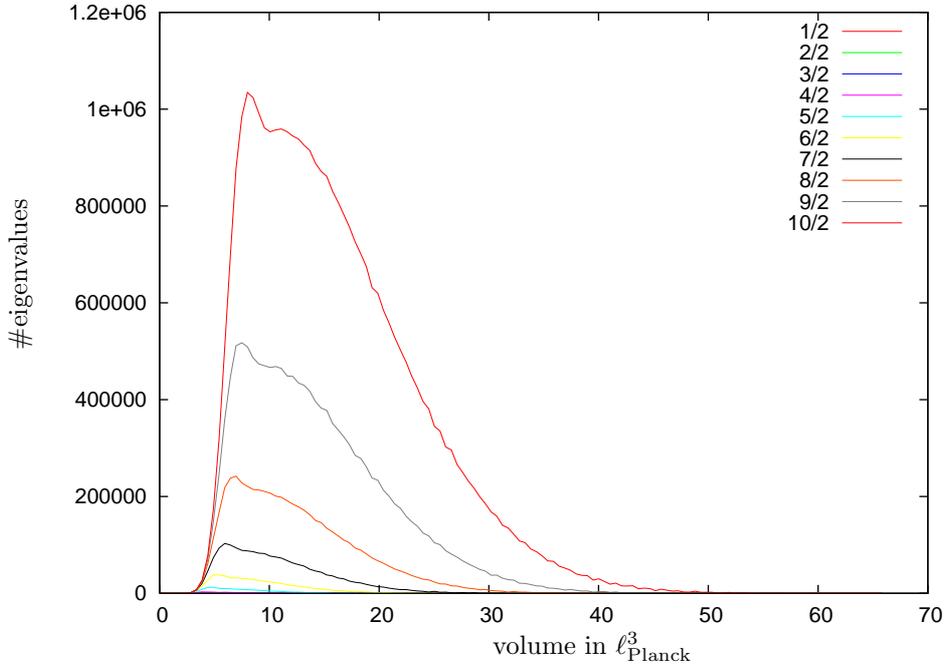


Figure 20: Histograms of volume eigenvalues for the cubic 6-vertex, one for each value of j_{\max} up to 5.

that the cubic 6-vertex is one of the rare ones for which the smallest non-zero eigenvalue increases with spin.

5.4 Gauge Invariant 7-Vertex

Histograms for the gauge invariant 7-vertex are shown in figures 22, 23, and 24, for j_{\max} up to $7/2$. There are 673 permutation equivalence classes of sigma configurations for the 7-vertex. Here there seem to be roughly two sorts of sigma configurations — those that lead to extremely jagged histograms in figure 22 (we believe that these come from degenerate $\vec{\sigma}$, which have many zeros, and thus can be effectively of lower valence), and the others which all have more-or-less the same smooth shape. $j_{\max} = 7/2$ is too small to see much evidence for a lip in the spectrum near zero volume, however there seems to be one sigma configuration which is already heading in this direction in figure 23.

Figures 25 and 26 show the extremal eigenvalues for each j_{\max} and $\vec{\sigma}$ at the 7-vertex. Here there are no $\vec{\sigma}$ for which the smallest non-zero eigenvalue increases with j_{\max} , though there are several for which this smallest eigenvalue is constant, such as $\vec{\sigma} = (22002000002000000000)$, which is one of the $\vec{\sigma}$ which yields the largest smallest non-zero eigenvalues, and the ‘4-vertex-like’ $\vec{\sigma} = (20000000000000000000)$. $\vec{\sigma} = (00000000022000000000)$ is an example of a 7-vertex sigma configuration with decreasing smallest non-zero eigenvalue. The largest eigenvalues behave as usual.

It is interesting to note that the sigma configuration $\vec{\sigma}$ in which the only non-zero sigma is $\sigma(1, 2, 3) = +2$ appears to arise at every valence, as the sigma configuration corresponding to the chirotope $\epsilon(1, 2, 3) = +1$, $\epsilon(I, J, K) = -1$ for all other triples I, J, K . This chirotope appears as the canonical representative of an equivalence class at every valence, because it corresponds to the integer 1 in our numbering scheme (and presumably because chirotopes ‘0’ and ‘1’ cannot be transformed into each other by a permutation, at any valence). It leads to a smallest non-zero eigenvalue of $\sqrt[4]{12}$ for every j_{\max} .

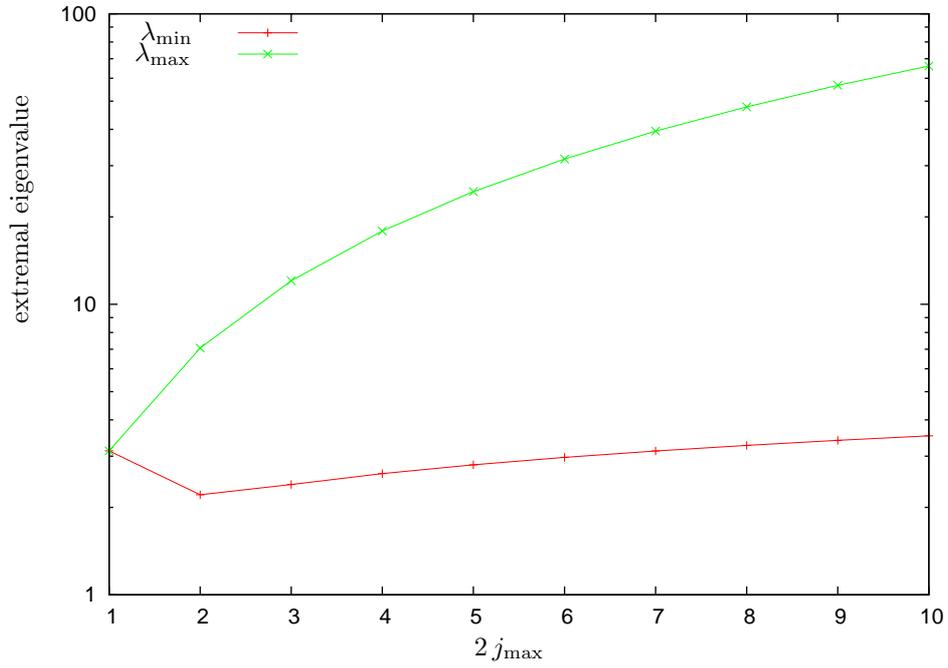


Figure 21: Smallest and largest non-zero volume eigenvalues of the cubic 6-vertex, as a function of $j_{\max} \leq 5$.

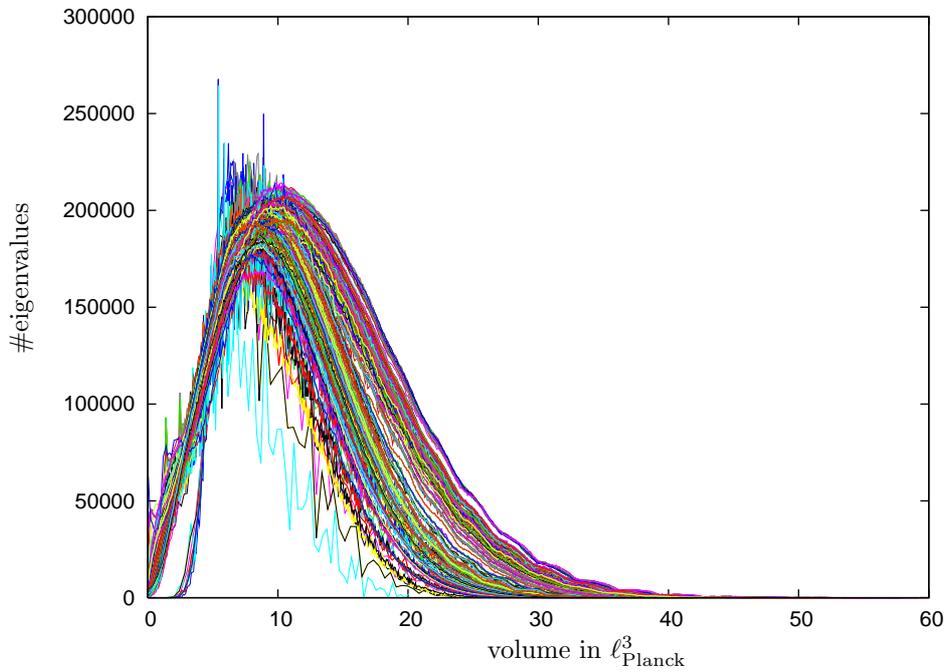


Figure 22: Histograms for each sigma configuration $\vec{\sigma}$ at the 7-vertex, up to $j_{\max} = 7/2$.

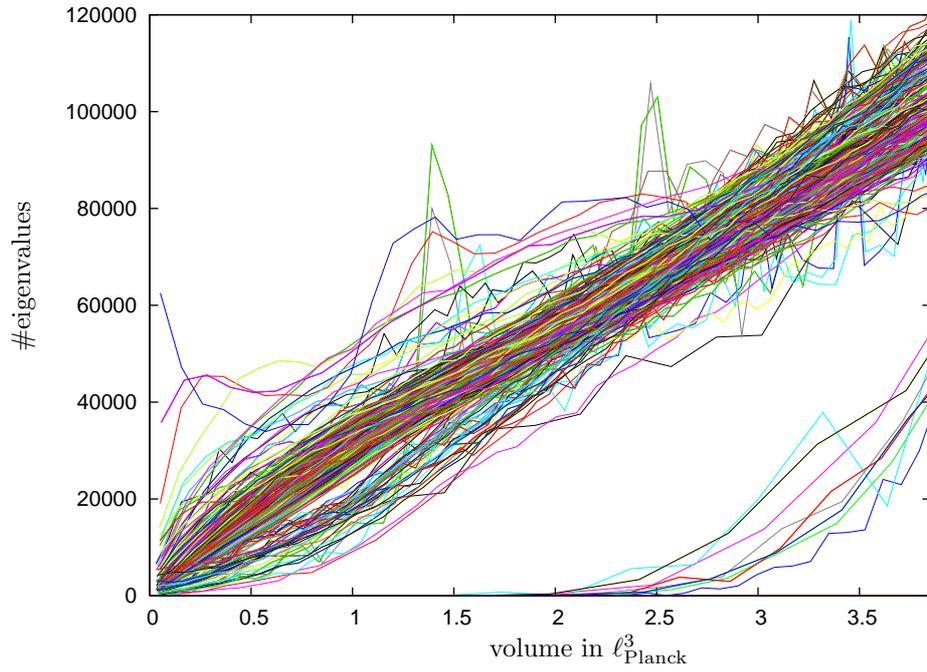


Figure 23: Histograms for each sigma configuration $\vec{\sigma}$ at the 7-vertex, up to $j_{\text{max}} = 7/2$, zoomed to the small eigenvalue region.

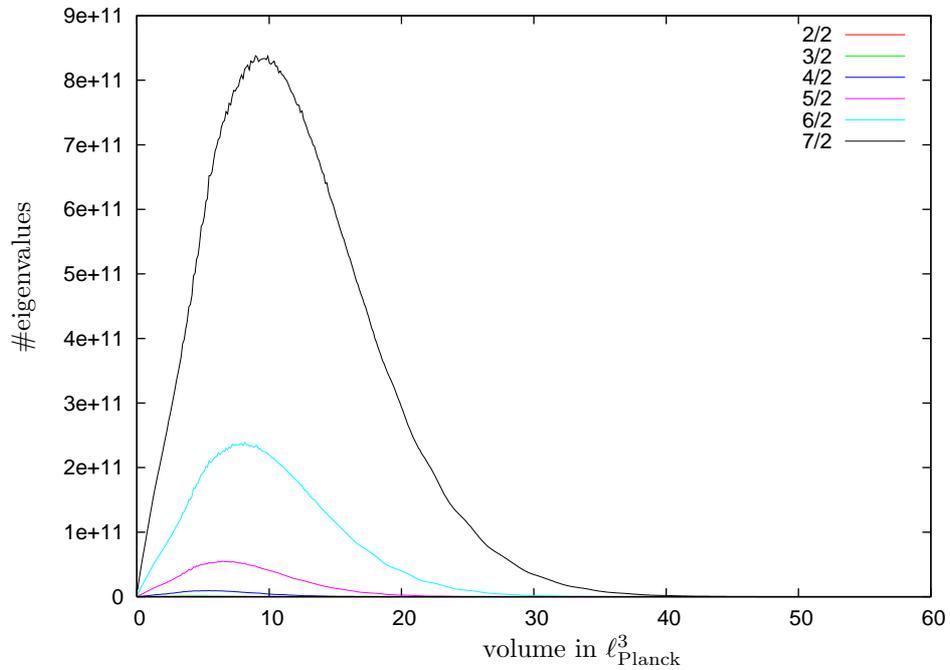


Figure 24: Histograms for the overall (generic) 7-vertex, up to every $j_{\text{max}} \leq 7/2$.

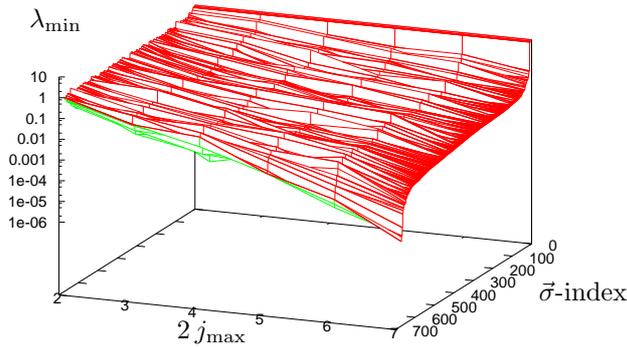


Figure 25: Smallest non-zero eigenvalues λ_{\min} at 7-vertex.

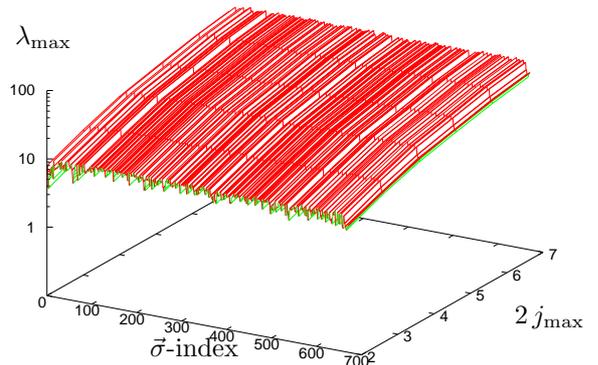


Figure 26: Largest eigenvalues λ_{\max} at 7-vertex.

6 Discussion

As we have demonstrated, the framework of oriented matroids can capture both local embedding properties of graphs as well as their global connectedness properties.

Originally developed for describing planar graphs, the framework of oriented matroids is general enough to be applied to graphs embedded in three dimensional Riemannian space, as is the case for LQG. Although the example from [3], revisited in detail in section 3.4, corresponds to a planar graph and uses a global orientation, the computation of the set \mathcal{C} of signed circuits of the according oriented matroid does not rely on this fact. Because the set \mathcal{C} of signed circuits as well as the chirotope $\chi_{\mathcal{B}}$ are symmetric with respect to a global sign reversal⁵¹, one can just take any vertex v contained in a circuit \underline{C} as a starting point and then follow \underline{C} along either direction until one comes back to v . The relative orientation of edges to the chosen direction then determines C^{\pm} .

In the outlined treatment, we certainly neglect the global knotting of edges (e.g. one edge could wind around another) of graphs embedded in three dimensional space [56]. If one wants to include these properties, one has to think about a possible extension of the oriented matroid framework for embedded digraphs. On the other hand, the knotting properties are not seen by any quantum operator corresponding to classical geometric quantities, unless they are interpreted as e.g. matter excitations as suggested e.g. in [56, 57]. Note that knotting also becomes irrelevant if automorphisms of graphs are considered [47]. Therefore we prefer to postpone considerations of knotting of edges until the connection between oriented matroids and the framework of LQG is worked out in more detail.

Regarding the local embedding of graph vertices, the occurrence of so-called moduli parameters was discussed in [61], which seems to prevent $\mathcal{H}_{\text{diff}}$ from having a countable label set. However the solution suggested there erases all local geometric information of the vertex embedding, encoded in sign factors. In contrast, we think that this information is physically relevant and should be kept, as it is crucial for the implementation of the Hamilton constraint operator [12] and a consistent formulation of LQG [48]. A solution to this issue is suggested in [40], giving $\mathcal{H}_{\text{diff}}$ a countable basis, while keeping the information on the local embedding.

The level at which the graphic oriented matroids can be introduced into the LQG formalism has to be further analyzed. At the level of \mathcal{H}_0 , one can have graphs underlying spin network functions, which have “open ends”, that is edges which have a 1-valent vertex as their beginning or final point. This difficulty could be dealt with by treating such “open ends” as effective re-tracings, however as one finally has to solve the Gauss constraint, one can directly start at the level of gauge invariant spin network functions in $\mathcal{H}_{\text{Gauss}}$, because there every edge of a graph has at least a two-valent vertex as beginning and final point. From the combinatorics side introducing graphic oriented matroids directly at the level of $\mathcal{H}_{\text{diff}}$ seems to be the most natural approach.

We have seen in section 5 that the spectral properties of the volume operator are closely related to the oriented matroid resulting from the local embedding of the vertices of a graph. Interestingly, it is not only sensitive to

⁵¹That is if $\chi_{\mathcal{B}}$ is a chirotope, then so is $-\chi_{\mathcal{B}}$. Also $\pm C \in \mathcal{C}$ for all signed circuits C . Similar statements hold for the dual oriented matroid.

those ‘local embedding chirotopes’, but also to the supports of the edges at the graph, as illustrated in figures 7–9. Rephrasing this, a reorientation in the graphic matroid (corresponding to re-directing an edge in the graph) causes a reorientation in the local vector oriented matroid. Hence, global and local graph properties are related in this way.

The different possibilities of embedding a vertex cause different behaviors of the lowest non-zero eigenvalues of the volume spectrum as the maximum spin j_{\max} at a vertex is increased. We find, in agreement with [4, 5], increasing, constant, and decreasing smallest non-zero eigenvalue sequences. In the latter two cases even very large spins can contribute microscopically small non-zero eigenvalues, and hence one cannot say that the limit $j_{\max} \rightarrow \infty$ produces only large volume eigenvalues. This is different from the area operator [62], for which large spins imply large eigenvalues. If one wants to keep this property one might regard this as a “physical” preference for vertices whose smallest non-zero eigenvalue grows with j_{\max} , such as the 4-vertex, cubic 6-vertex, or the (permutation equivalence class of) 6-vertex with $\vec{\sigma} = (022222220)$.

As a consequence we would like to point out that high valent vertices do not necessarily overcount the volume in a semiclassical analysis as presented in [20]. That is, high valence does not automatically shift the volume spectrum towards larger eigenvalues, as one might naively expect from the sum structure in (4.10). This is due to, for example, the presence of vertex embeddings which only have a small number of non-zero signs $\sigma(I, J, K)$, as defined in (4.11). In particular the number of triples (I, J, K) with $\sigma(I, J, K) \neq 0$ can be independent of the number of edges attached to the vertex. One example for such a vertex embedding is given by $\vec{\sigma} = (2, 0, 0, \dots, 0)$, as noted in section 5.4, which we find to be contained in the permutation equivalence classes of 5- up to 7-valent vertices. Such sigma configurations seem able to lead to smallest non-zero eigenvalues which are independent of both j_{\max} and valence N_v .

Given this fact, it will be instructive to analyze if one can find embedded higher valent vertices which also resemble the correct semiclassical limit in the sense of [20]. Approximating the volume of a spatial region by semiclassical coherent state techniques introduced in [63] might be a subtle mixture of the embedding of the graph supporting the coherent state, as well as choosing its topology, in particular the number of vertices and their connectedness in that region. Here the unified description of these properties in terms of oriented matroids might give us a better understanding of the possible choices one has to make in the construction of the complexifier coherent states used in [20].

7 Summary and Outlook

In this work we have demonstrated a connection between the LQG framework and the field of oriented matroids. For this we have described the combinatorial properties of LQG in section 2, and introduced matroids and oriented matroids in section 3 in an abstract way, without referring to any particular realization. In sections 3.4 and 3.5 we showed how this abstract framework can be applied in order to describe global (connectedness) and local (geometry of vertex embedding) graph properties by oriented matroids. We have also briefly discussed the issue of realizability of oriented matroids in terms of pseudosphere arrangements in section 3.7, that is the question of when an abstract oriented matroid of rank r gives rise to a configuration of vectors in \mathbb{R}^r . The obvious connection of oriented matroids to LQG is then discussed in section 4.

We see several possible benefits in this approach. First, it provides *one unified framework* to describe local and global graph combinatorics, in an abstract way. That is, in this setup we do not need to refer to the underlying manifold into which graphs are embedded. In addition the oriented matroid concept introduces the possibility of a dual description of oriented graphs in terms of vector configurations, and vice versa.

Secondly, it gives explicit ways to classify oriented matroids, and e.g. to generate representatives of equivalence classes of vector configurations under permutation. We have already taken advantage of this in section 5, where we revisit the results of [4, 5]. Using our insights from oriented matroids we are able to drastically simplify the presentation, and to confirm the vertex combinatorics used in [4, 5] by direct computation from reorientation classes of oriented matroids.

The present work will serve as a starting point to further explore this subject.

The constructions outlined in section 3.8, e.g. reorientation and deletion, can be used in order to construct an analogy to the projective limit of the graph poset in LQG, using only oriented matroids. Here the construction of matroids for infinite graphs [44] will become relevant. This question will be analyzed in [58]. We have seen

in section 4.2 gauge invariance of a cylindrical function can be described by the selection of a dual basis of the underlying graph. This points to a connection between oriented matroids and recoupling theory of $SU(2)$ representations. This is an interesting perspective for a possible extension of the oriented matroid framework by techniques from the graphical calculus of angular momentum, as for example given in [64]. For LQG it will also be crucial to describe the action of graph changing operators (holonomies) by modifications to the graphic oriented matroid. If this turns out to be possible, then it becomes feasible to cast the action of the Hamilton [12], and respectively Master constraint operator [39, 40], in LQG into the oriented matroid framework. The difficulty of finding eigenstates could then be re-formulated in terms of oriented matroids.

In section 3.2 an abstract notion of orthogonality between signed subsets is given. An interesting question is if this notion can be related to the inner product on $\mathcal{H}_{\text{diff}}$, the diffeomorphism invariant sector of LQG, where the inner product is given by a rigging map construction [38, 30]. Also, in the context of [47], it will be instructive to see if the outstanding problem of giving a basis for the automorphism invariant sector of LQG can be tackled using an oriented matroid labelling.

Besides these exciting conceptual perspectives, we have already seen that oriented matroids can be used in order to perform explicit computations in LQG. We expect the spectral analysis of the volume operator to be completed⁵² in [53], where also non-uniform oriented matroids, that is coplanar triples of tangent vectors, will be included using techniques from [55]. In this context it should also be possible to extend the analysis of [20] to all realizable vertex embeddings. Additionally the issue of taking the semiclassical limit, as discussed at the end of section 6, can be addressed in this setup.

Moreover it is now possible to classify diffeomorphic graphs⁵³, and to compute the number of diffeomorphism equivalence classes. For this the issue of realizability in section 3.7 becomes important. The development of effective algorithms for computing realizable oriented matroids for larger ground sets will be crucial for this. In the mid-term perspective, oriented matroid techniques may provide effective techniques in order to develop a computational toolkit for performing numerical simulations in full LQG.

Acknowledgments

The authors would like to thank Fernando Barbero for useful discussions and for providing the analytic counting formula (4.12) of generic sign configurations. Moreover we thank Jürgen Richter-Gebert for discussions and handing out the manuscripts [65, 66] to us and additionally Ulrich Kortenkamp and Lukas Finschi for providing code for computing reorientation classes of oriented matroids. The work of JB has been supported by the Emmy-Noether-Programm (grant FL 622/1-1) of the Deutsche Forschungsgemeinschaft. The work of DR was supported by the Perimeter Institute for Theoretical Physics. Research at the Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research & Innovation. JB would like to thank the Perimeter Institute for hospitality and financial support. DR would also like to thank the Emmy-Noether-Programm (grant FL 622/1-1) of the Deutsche Forschungsgemeinschaft for financial support. The numerical results were generated using the facilities of the Shared Hierarchical Academic Research Computing Network (SHARCNET:www.sharcnet.ca). We thank the anonymous referees for several helpful suggestions.

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⁵²In the sense that we can analyze all possible vertex embeddings, including coplanar triples. Of course, this is limited by finite computational resources.

⁵³Modulo knotting, see the discussion in section 6.

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