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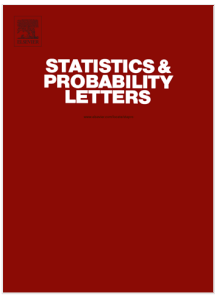

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A lower bound for the reliability function of multiple failure mode systems

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Abstract

In Multiple Failure Mode systems (MFM systems) each component may either work or experience one of $m \geq 2$ different failure modes. This work presents a lower product-type bound for the MFM system's reliability function, using the theory of stochastic ordering. Numerical calculations are also carried out to compare the new bound to other bounds that appeared in the reliability literature recently.

Keywords and phrases: Reliability bounds; Minimal cut sets; Stochastic ordering; Multiple failures

1. Introduction

In traditional reliability theory, the components of a system and the system itself are usually assumed to have two different states: on (good, operating) or off (down, failed). Because of this assumption, the structure function of the system is a binary function of binary variables, and the respective model is usually referred to as binary reliability system. However, for many engineering systems, the binary assumption may not be appropriate for describing the possible states that each of the components may experience. For example, in fluid control networks, a defective valve may be either "stuck-open" or "stuck-closed", in safety monitoring systems, a device will malfunction if it "fails to detect breakdown" or "initiates a false alarm" etc. A structure whose components experience two different modes of failure is usually referred to as three-state device.

A natural extension of the three-state devices is easily developed by assigning to each component $m \geq 2$ failure modes. The resulting structure will then be called multistate system.

During the last decades, the multistate systems have attracted substantial research interest. In the publications appeared in the literature so far, two different set-ups have been considered. In the first one, each component and the system itself is assuming a finite number of states, say $0,1,...,m,$ each corresponding to a different level of operating efficiency (key references to this subject are Barlow and Wu(1978), Ross(1979) and the recent monograph by Kuo and Zuo(2003)). The basic feature of this set-up is that the operating modes of the components (and the system) are ordinable, i.e. the $m+1$ states represent successive deterioration levels ranging from perfect functioning (state 0) down to complete failure (state m). Obviously, this is not always feasible; for example, neither the

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fluid control networks nor the safety monitoring systems mentioned earlier can be modeled by this set-up (for $m = 2$).

The second family of multistate structures does not make use of any state ordering and the description of system's failure is achieved by considering m different types of minimal cut sets. More specifically, a family of minimal cut sets $\boldsymbol{C}_s$ is assigned to each failure mode $s \in \{1, 2, ..., m\}$. Then system's breakdown comes whenever all the components contained in a cut set $C \in \boldsymbol{C}_s$ have shifted to failure mode s (for some $s \in \{1, 2, ..., m\}$). In order to distinguish between the two different set-ups we shall retain the term multistate systems for the first one and will be using the nomenclature *Multiple Failure Mode* (MFM) systems for the latter. The interest reader may refer to Barlow et al(1963), Ben-Dov(1980), Boutsikas and Koutras(2002), Koutras(1997) and Satoh et al(1993) for results related to MFM systems; note that the special case $m = 2$ is usually referred to as DFM (*Dual Failure Mode*) while for $m = 1$ we get the classical two-state or SFM (*Single Failure Mode*) structures.

The main objective of the present work is to establish a tight lower bound for the reliability function of a MFM system. The development of the bound is achieved by making beneficial use of results from the area of stochastic ordering.

The rest of the paper is organized as follows: in Section 2 we introduce all necessary notation and terminology used through the presentation. In Section 3 we firstly prove a stochastic ordering result between a set of variables describing a MFM system's operation and a set of auxiliary variables. Then we obtain a multiplicative reliability bound for the reliability function of a general MFM system. Finally, in Section 4 we carry out extensive numerical calculations and compare the new bound to several bounds that have appeared recently in the reliability literature.

2. Definitions and notations

Let $\boldsymbol{I} = \{1, 2, ..., n\}$ be the set of components of a MFM system. For each component $i \in \boldsymbol{I}$ we shall denote by 0 its working state and by $\boldsymbol{S} = \{1, 2, ..., m\}$ the set of its failure modes. At a specific time instance t, component $i \in \boldsymbol{I}$ can only be in one of the $m+1$ mutually exclusive states of $\mathbf{S} \cup \{0\}$. Therefore, if we denote by $p_i = q_{0i}$ its survival probability and by q_{si}, $s = 1, 2, ..., m$ the probability that component i experiences failure mode s, then

$$p_i + \sum_{s=1}^{m} q_{si} = \sum_{s=0}^{m} q_{si} = 1, \qquad i = 1, 2, ..., n$$

(note that, time t has been suppressed in the notation, since the study of the system will be carried out at a fixed instance). The systems considered in the present work are shifting to a non operational

state whenever specific groups of components experience simultaneously the same type of failure. To be more specific, we assume that, by analysing the system's structure, we may determine m families of minimal subsets $\boldsymbol{C}_1, \boldsymbol{C}_2, ..., \boldsymbol{C}_m$ of the power set of $\boldsymbol{I}$, such that the system fails if and only if there exists an $s \in \boldsymbol{S}$ and $C \in \boldsymbol{C}_s$ with all components in C experiencing (at time t) failure of type s. We shall refer to the last event by the term *system failure in mode* s, while the elements of $\boldsymbol{C}_s$ (subsets of $\boldsymbol{I}$) will be called *minimal cut sets of type* s.

Apparently, the state of component $i \in \boldsymbol{I}$ can be described by the aid of an integer-valued random variable V_i defined on $\boldsymbol{S} \cup \{0\}$ as follows

$$V_i = \begin{cases} 0, & \text{if component } i \text{ is working} \\ 1, & \text{if component } i \text{ is in failure mode } 1 \\ \vdots & \\ m, & \text{if component } i \text{ is in failure mode } m. \end{cases}$$

Nevertheless, it is more convenient to replace the random variable V_i by a binary random (column) vector $\boldsymbol{X}_{\cdot i} = (X_{1i}, X_{2i}, ..., X_{mi})'$ with the entry X_{si} taking on the value 0 whenever component i experiences failure mode s and 1 if not (manifestly $\boldsymbol{X}_{\cdot i} = (1, 1, ..., 1)'$ if and only if the component is working). Using this set-up we may easily describe the operation of a MFM system by the aid of binary functions similar to the well known structure functions of the traditional reliability theory.

To achieve the aforementioned goal, let us arrange the column vectors $\boldsymbol{X}_{\cdot 1}, \boldsymbol{X}_{\cdot 2}, ..., \boldsymbol{X}_{\cdot n}$ in a rectangular binary random matrix

$$\boldsymbol{X} = (X_{si})_{m \times n} = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m1} & X_{m2} & \cdots & X_{mn} \end{pmatrix},$$

and denote by $\boldsymbol{X}_{1\cdot}, \boldsymbol{X}_{2\cdot}, ..., \boldsymbol{X}_{m\cdot}$ its rows. Then the system will be in failure mode s, if and only if the (binary) structure function

$$\varphi_s(\boldsymbol{X}_{s\cdot}) = \prod_{C \in \boldsymbol{C}_s} \left(1 - \prod_{i \in C} (1 - X_{si}) \right)$$

takes on the value 0. As a consequence, the probability that the system does not experience failure mode s equals $E(\varphi(\boldsymbol{X}_{s\cdot}))$. Obviously, the last quantity is the reliability of a classical SFM (single failure mode, binary) system with components' failure probabilities $q_{s1}, q_{s2}, ..., q_{sn}$ and respective

survival probabilities $1-q_{s1}, 1-q_{s2}, ..., 1-q_{sn}$. From now on we shall use the notation $\boldsymbol{q}_{s\cdot} = (q_{s1}, q_{s2}, ..., q_{sn})$ for the vector of components' failure probabilities in mode s and

$$R_s(\boldsymbol{q}_{s\cdot}) = E(\varphi(\boldsymbol{X}_{s\cdot}))$$

for the associated reliability. Alert readers may easily realize that, a MFM system will fail if and only if $\varphi(\boldsymbol{X}_{s\cdot})$ vanishes for at least one $s \in \boldsymbol{S}$. Therefore, the binary (and coordinatewise increasing) function $\varphi : \{0,1\}^{mn} \to \{0,1\}$, defined by

$$\varphi = \varphi(\boldsymbol{X}_{1\cdot}, \boldsymbol{X}_{2\cdot}, ..., \boldsymbol{X}_{m\cdot}) = \prod_{s=1}^{m} \varphi_s(\boldsymbol{X}_{s\cdot}) = \prod_{s=1}^{m} \left(\prod_{C \in \boldsymbol{C}_s} \left(1 - \prod_{i \in C} (1 - X_{si}) \right) \right)$$

can be exploited to describe system's operation: $\varphi = 1$ if the system works while $\varphi = 0$ if the system is down (in any of the m failure modes). Moreover, the reliability of the system admits the following compact formula

$$R(\boldsymbol{q}) = E(\varphi(\boldsymbol{X}_{1\cdot}, \boldsymbol{X}_{2\cdot}, ..., \boldsymbol{X}_{m\cdot})) = E\left(\prod_{s=1}^{m} \varphi_s(\boldsymbol{X}_{s\cdot}) \right) = E\left(\prod_{s=1}^{m} \left(\prod_{C \in \boldsymbol{C}_s} \left(1 - \prod_{i \in C} (1 - X_{si}) \right) \right) \right) \quad (1)$$

where $\boldsymbol{q}$ stands for the $m \times n$ matrix

$$\boldsymbol{q} = \begin{pmatrix} \boldsymbol{q}_{1\cdot} \\ \boldsymbol{q}_{2\cdot} \\ \vdots \\ \boldsymbol{q}_{m\cdot} \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{m1} & q_{m2} & \cdots & q_{mn} \end{pmatrix}.$$

The aforementioned approach for the description of a MFM system was first used by Boutsikas and Koutras(2002). The same authors illustrated how this approach can be applied in small MFM systems in order to arrive at an explicit (sum of products) expression for the reliability $R(\boldsymbol{q})$. They also developed an alternative expression for $R(\boldsymbol{q})$ in terms of the reliability of a series structure comprising of SFM systems.

Motivated by the fact that, as n, m and the cardinalities of $\boldsymbol{C}_s$, $s \in \boldsymbol{S}$, increase, the exact reliability evaluation of MFM systems rapidly becomes tedious, Boutsikas and Koutras(2002) proceeded to the construction of computationally tractable reliability bounds and asymptotic results. In the next section we are using tools from the area of stochastic ordering to develop a new multiplicative lower bound for the reliability of a MFM system that improves the multiplicative lower bound suggested by Boutsikas and Koutras(2002).

3. A Lower reliability bound for MFM systems

The development of our reliability bound is couched on a key result of the theory of stochastic ordering of multivariate random variables. To begin with, let us recall that a (univariate) random variable Z is said to be smaller than the random variable Y in the (with respect to) usual stochastic order if

$$P(Z > u) \le P(Y > u), \text{ for all } u \in (-\infty,+\infty)$$

(or equivalently $P(Z \le u) \ge P(Y \le u)$, for all $u \in (-\infty,+\infty)$). In this case we shall write $Z \le_{st} Y$. It is not difficult to verify that the stochastic ordering condition stated about can be replaced by the condition

$$E(f(Z)) \le E(f(Y))$$

for all increasing functions f for which the expectations exist.

The last result carries over to the multivariate case as well. More specifically, if $\boldsymbol{Z}$ and $\boldsymbol{Y}$ are random vectors in R^n, then $\boldsymbol{Z}$ will be smaller than $\boldsymbol{Y}$ in the usual stochastic order (denoted by $\boldsymbol{Z} \le_{st} \boldsymbol{Y}$) if and only if

$$E(f(\boldsymbol{Z})) \le E(f(\boldsymbol{Y}))$$

for all increasing functions f for which the expectations exist.

Needless to say, confirming the stochastic ordering between two variables by the aid of the last condition is a tough task. Fortunately, a sufficient condition is available for verifying the multivariate stochastic order by means of the usual univariate stochastic order. This is described in the following theorem (see e.g. Shaked and Shanthikumar(1994) or Muller and Stoyan(2002)).

Theorem 3.1. *Let* $\boldsymbol{Z} = (Z_1, Z_2, ..., Z_n)$ *and* $\boldsymbol{Y} = (Y_1, Y_2, ..., Y_n)$ *be two n-dimensional random vectors. If* $Z_1 \le_{st} Y_1$, *and for* $s = 2,...,n$,

$$[Z_s \mid Z_1 = z_1, Z_2 = z_2, ..., Z_{s-1} = z_{s-1}] \le_{st} [Y_s \mid Y_1 = y_1, Y_2 = y_2, ..., Y_{s-1} = y_{s-1}],$$

whenever $z_j \le y_j$ *for all* $j = 1, 2, ..., s-1$, *then*

$$\boldsymbol{Z} \le_{st} \boldsymbol{Y}.$$

The symbol $[Z_s \mid Z_1 = z_1, Z_2 = z_2, ..., Z_{s-1} = z_{s-1}]$ used above denotes any univariate random variable that has its distribution the conditional distribution of Z_s given that $Z_1 = z_1, Z_2 = z_2, ..., Z_{s-1} = z_{s-1}$.

We are now ready to state and prove our first result.

Proposition 3.1. *For a MFM system with* n *components and m different failure modes, consider a fixed* $i \in \boldsymbol{I}$ *and denote by* $\boldsymbol{X}_{\cdot i} = (X_{1i}, X_{2i}, ..., X_{mi})'$ *the column vector describing the i-th component's state. If*

$$\boldsymbol{T}_{\cdot i} = (T_{1i}, T_{2i}, ..., T_{mi})'$$

is a binary (*column*) *vector consisting of m independent binary random variables defined by*

$P(T_{1i} = 0) = q_{1i},\ P(T_{1i} = 1) = 1 - q_{1i},$

$P(T_{si} = 0) = (q_{si})/(1 - \sum_{j=1}^{s-1} q_{ji})$, $P(T_{si} = 1) = (1 - \sum_{j=1}^{s} q_{ji})/(1 - \sum_{j=1}^{s-1} q_{ji}),\ \ s = 2, ..., m$

then

$$\boldsymbol{T}_{\cdot i}' \leq_{st} \boldsymbol{X}_{\cdot i}'.$$

Proof. In view of Theorem 3.1 it suffices to verify that $P(T_{1i} \leq u) \geq P(X_{1i} \leq u)$, for all $u \in (-\infty, +\infty)$ and, for $s = 2, ..., m$

$$P(T_{si} \leq u \mid T_{1i} = t_1, ..., T_{s-1,i} = t_{s-1}) \geq P(X_{si} \leq u \mid X_{1i} = x_1, ..., X_{s-1,i} = x_{s-1}), \text{for all } u \in (-\infty, +\infty)$$

whenever $t_j \leq x_j$, $j = 1, 2, ..., s-1$. Since both T_{si} and X_{si} are binary (0-1) random variables, the first condition holds true as an equality since, for $u \geq 1$ both sides become 1, for $u < 0$ both sides vanish while for $0 \leq u < 1$ we have

$$P(T_{1i} \leq u) = P(T_{1i} = 0) = q_{1i} = P(X_{1i} = 0) = P(X_{1i} \leq u).$$

As far as the second inequality is concerned, observe first that, due to the independence of the random variables $T_{1i}, T_{2i}, ..., T_{si}$ the conditional probability in the LHS equals

$$P(T_{si} \leq u) = \begin{cases} 1, & \text{if } u \geq 1 \\ \dfrac{q_{si}}{1 - \sum_{j=1}^{s-1} q_{ji}}, & \text{if } 0 \leq u < 1 \\ 0, & \text{if } u < 0. \end{cases}$$

For $u \geq 1$ both sides of the inequality we are looking at turn out to be 1, while for $u < 0$ both vanish. Therefore, the proof will be completed should we be able to prove that

$$P(X_{si} \leq u \mid X_{1i} = x_1, X_{2i} = x_2, ..., X_{s-1,i} = x_{s-1}) \leq \frac{q_{si}}{1 - \sum_{j=1}^{s-1} q_{ji}}$$

for $0 \leq u < 1$ and $t_j \leq x_j$, $j = 1, 2, ..., s-1$, or equivalently

$$P(X_{si} = 0 \mid X_{1i} = x_1, X_{2i} = x_2, ..., X_{s-1,i} = x_{s-1}) \leq \frac{q_{si}}{1 - \sum_{j=1}^{s-1} q_{ji}}$$

for all $x_j \in \{0, 1\}$, $j = 1, 2, ..., s-1$.

Taking into account that, by the definition of the random variables $X_{1i}, X_{2i}, ..., X_{mi}$ no more than one of them can take on the value 0 (with the rest of them being 1), we conclude that

a. if at least one of $x_1, x_2, ..., x_{s-1}$ equals 0, then the inequality holds true in a trivial way, since its LHS vanishes,

b. if none of $x_1, x_2, ..., x_{s-1}$ equals 0, then the LHS of the inequality, becomes

$$P(X_{si}=0 \mid X_{1i}=1,...,X_{s-1,i}=1)=\frac{P(X_{si}=0,X_{1i}=1,...,X_{s-1,i}=1)}{P(X_{1i}=1,...,X_{s-1,i}=1)}=\frac{P(X_{si}=0)}{1-\sum_{j=1}^{s-1}P(X_{ji}=0)}=\frac{q_{si}}{1-\sum_{j=1}^{s-1}q_{ji}}$$

As a consequence, the inequality we are interested at, holds also true in this case as equality. The preceding arguments guarantee that the assumptions of Theorem 3.1 are valid for the random vectors $\boldsymbol{T}_{\cdot i}^{'}$ and $\boldsymbol{X}_{\cdot i}^{'}$, thereof establishing the desired stochastic ordering for them. □

In the next proposition, we establish a lower bound for the reliability function of a MFM system which uses the reliability function of appropriate SFM systems. The core of the proof for this result is the stochastic order developed in Proposition 3.1.

Proposition 3.2. *Consider a MFM system with n independent components each one experiencing m distinct failure modes. For* $i \in \boldsymbol{I}$ *and* $s \in \boldsymbol{S}$ *let* Q_{si} *be the quantities*

$$Q_{si}=\begin{cases} q_{1i}, \text{ if } s=1 \\ \dfrac{q_{si}}{1-\sum_{j=1}^{s-1}q_{ji}}, \text{ if } s \geq 2 \end{cases} \tag{2}$$

and denote by $R_s = R_s(Q_{s1}, Q_{s2}, ..., Q_{sn})$, *the reliability function of a SFM system with component failure probabilities* Q_{si}, $i \in \boldsymbol{I}$ *and minimal cut sets all* $C \in \boldsymbol{C}_s$. *Then, the reliability* $R = R(\boldsymbol{q})$, $\boldsymbol{q} = (q_{si})_{m \times n}$ *of the MFM system is bounded from below as follows*

$$R(\boldsymbol{q}) \geq \prod_{s=1}^{m} R_s(Q_{s1}, Q_{s2}, ..., Q_{sn}).$$

Proof. Let us denote by $\psi : \{0,1\}^{mn} \to \{0,1\}$ the structure function of the *mn* dimensional vector $(\boldsymbol{X}_{\cdot 1}^{'}, \boldsymbol{X}_{\cdot 2}^{'}, ..., \boldsymbol{X}_{\cdot n}^{'})$, i.e.

$$\psi(\boldsymbol{X}_{\cdot 1}^{'}, \boldsymbol{X}_{\cdot 2}^{'}, ..., \boldsymbol{X}_{\cdot n}^{'}) = \prod_{s=1}^{m}\left(\prod_{C \in \boldsymbol{C}_s}\left(1-\prod_{i \in C}(1-X_{si})\right)\right).$$

It is straightforward that ψ is a binary coordinatewise increasing function.

Consider now for each $i \in \boldsymbol{I}$ the binary vectors $\boldsymbol{T}_{\cdot i}^{'}$ introduced in Proposition 3.1, so that the *m* entries of $\boldsymbol{T}_{\cdot i}^{'}$ are independent and the vector variables $\boldsymbol{T}_{\cdot 1}^{'}, \boldsymbol{T}_{\cdot 2}^{'}, ..., \boldsymbol{T}_{\cdot n}^{'}$ are also mutually independent. Then we shall have

$$\boldsymbol{T}_{\cdot i}^{'} \leq_{st} \boldsymbol{X}_{\cdot i}^{'}, \text{ for } i = 1, 2, ..., n$$

where $\boldsymbol{T}_{\cdot 1}^{'}, \boldsymbol{T}_{\cdot 2}^{'}, ..., \boldsymbol{T}_{\cdot n}^{'}$ is a family of independent variables and $\boldsymbol{X}_{\cdot 1}^{'}, \boldsymbol{X}_{\cdot 2}^{'}, ..., \boldsymbol{X}_{\cdot n}^{'}$ a second family of independent variables (the last assertion is immediately justified by the components' independence).

By virtue of the multivariate stochastic ordering closure property under conjunction (see e.g. Shaked and Shanthikumar(1994)) we deduce

$$\psi(\boldsymbol{T}_{\cdot 1}^{'}, \boldsymbol{T}_{\cdot 2}^{'}, ..., \boldsymbol{T}_{\cdot n}^{'}) \leq_{st} \psi(\boldsymbol{X}_{\cdot 1}^{'}, \boldsymbol{X}_{\cdot 2}^{'}, ..., \boldsymbol{X}_{\cdot n}^{'})$$

and hence

$$E(\psi(\boldsymbol{T}_{\cdot 1}^{'}, \boldsymbol{T}_{\cdot 2}^{'}, ..., \boldsymbol{T}_{\cdot n}^{'})) \leq E(\psi(\boldsymbol{X}_{\cdot 1}^{'}, \boldsymbol{X}_{\cdot 2}^{'}, ..., \boldsymbol{X}_{\cdot n}^{'}))$$

or equivalently

$$E(\varphi(\boldsymbol{T}_{1\cdot}, \boldsymbol{T}_{2\cdot}, ..., \boldsymbol{T}_{m\cdot})) \leq E(\varphi(\boldsymbol{X}_{1\cdot}, \boldsymbol{X}_{2\cdot}, ..., \boldsymbol{X}_{m\cdot})).$$

Manifestly, the RHS is the reliability $R = R(\boldsymbol{q})$ of the MFM system. On the other hand, applying formula (1) for $\boldsymbol{T}_{1\cdot}, \boldsymbol{T}_{2\cdot}, ..., \boldsymbol{T}_{m\cdot}$, we may write the LHS in the form $E(\prod_{s=1}^{m} \varphi_s(\boldsymbol{T}_{s\cdot}))$, which, in view of the independence of T_{si}'s, can be further analyzed as

$$E(\prod_{s=1}^{m} \varphi_s(\boldsymbol{T}_{s\cdot})) = \prod_{s=1}^{m} E(\varphi_s(\boldsymbol{T}_{s\cdot})).$$

The proof is now easily completed on observing that the terms of the last product are in fact the reliabilities of SFM systems with component failure probabilities Q_{si}, $i \in \boldsymbol{I}$, and minimal cut sets all $C \in \boldsymbol{C}_s$. □

Let us denote by $L(\boldsymbol{q})$ the lower bound established in Proposition 3.2, i.e.

$$L(\boldsymbol{q}) = \prod_{s=1}^{m} R_s(Q_{s1}, Q_{s2}, ..., Q_{sn})$$

with the Q_{si}'s given by (3.1). Making use of the well known Esary and Proschan bounds for the SFM systems (see e.g. Barlow and Proschan(1981)) we may bound $L(\boldsymbol{q})$ from below as follows.

Corollary 3.1. *Consider a MFM system with n independent components each one experiencing m distinct failure modes. Then its reliability function* $R(\boldsymbol{q})$ *is bounded from below as follows*

$$R(\boldsymbol{q}) \geq \prod_{s=1}^{m} (\prod_{C \in \boldsymbol{C}_s} (1 - \prod_{i \in C} Q_{si}))$$

where Q_{si} *are given by* (2).

Proof. Follows immediately from Proposition 3.2 by applying the inequality

$$R_s(Q_{s1}, Q_{s2}, ..., Q_{sn}) \geq \prod_{C \in \boldsymbol{C}_s} (1 - \prod_{i \in C} Q_{si})$$

which holds true for any SFM system (see Barlow and Proschan(1981)). □

We are thus in possession of an explicit lower bound that can be effortlessly evaluated if we have at hand the families of minimal cut sets $\boldsymbol{C}_s, s = 1, 2, ..., m$. A similar bound has been suggested by Boutsikas and Koutras(2002). More specifically, using an expression of the MFM system reliability

through the reliability functions of SFM systems and properties of associated random variables, they proved that

$$R(\boldsymbol{q}) \geq \prod_{s=1}^{m} R_s(Q'_{s1}, Q'_{s2}, ..., Q'_{sn}) = L'(\boldsymbol{q}) \tag{3}$$

where

$$Q'_{si} = \frac{q_{si}}{p_i + q_{si}}, i = 1, 2, ..., n .$$

Recalling the monotonicity property of the reliability function of SFM systems and observing that

$$Q'_{si} \geq Q_{si}, \text{ for } s = 1, 2, ..., m \text{ and } i = 1, 2, ..., n$$

it is readily ascertainable that the lower bound established in Proposition 3.2 is uniformly better than the Boutsikas and Koutras bound, that is

$$L(\boldsymbol{q}) \geq L'(\boldsymbol{q}), \text{ for all } \boldsymbol{q} = (q_{si})_{m \times n} .$$

It is noteworthy that the new multiplicative bound $L(\boldsymbol{q})$ depends on the order in which the m families of cut sets $\boldsymbol{C}_s, s = 1, 2, ..., m$ are entering in the calculations. This is transparent from the form of Q_{si} (see formula (2)) and is due to the fact that, in the proof of Proposition 3.1, we had to work with conditional distributions of a variable associated to a specific failure mode s, given the values of variables associated to the preceding failure modes $1, 2, ..., s-1$. A direct consequence of this observation is that, one could probably improve on the lower bound's value by considering all $m!$ possible bounds and choosing the one that maximizes $L(\boldsymbol{q})$. It is however unclear that the slight improvement achieved by that compensates for the additional computational effort needed.

4. Numerical Comparisons

As pointed out by Boutsikas and Koutras(2002), several classes of MFM systems could be constructed by placing the classical SFM systems in a multistate environment. For example, considering a relay circuit with a bridge structure topology and assuming that each of the five components can be either "failed-open" or "failed-short (closed)" gives birth to a typical MFM system. In the same spirit, the well known k-out-of-n, consecutive-k-out-of-n systems and their generalizations (the interested reader may refer to the excellent monograph by Kuo and Zuo(2003)) can be effortlessly adjusted to a multistate environment.

In this section we shall proceed to the numerical evaluation of the new bounds for a consecutive-$k_1, k_2, ..., k_m$-out-of-n: MFM system and compare them to other bounds that have appeared in the literature recently. A consecutive-$k_1, k_2, ..., k_m$-out-of-n: MFM system consists of n linearly arranged components, each one experiencing m different failure modes; the system enters failure mode s

$s = 1, 2, ..., m$, whenever at least k_s consecutive components have shifted to mode s. The special case m=2 was studied in detail by Koutras(1997) while bounds for the general case have been suggested by Boutsikas and Koutras(2002) and Chryssaphinou and Vaggelatou(2002). It is worth mentioning that the reliability of that system is closely related to the waiting time for a run of alike outcomes in a sequence of multistate trials (see Aki(1992) or Fu and Lou(2003)). It is straight forward that the family of minimal cut sets of type s will be given by

$$\boldsymbol{C}_s = \{\{i, i+1, ..., i+k_s-1\} : i = 1, 2, ..., n-k_s+1\}, s = 1, 2, ..., m.$$

For typographical convenience, throughout this section we shall be assuming that the n components of the system are independent and identical (iid case) with $p_i = q_{0i} = p$, $q_{si} = q_s$, $s = 1, 2, ..., m$. Moreover we shall use the symbol $R_s(q)$ to denote the reliability of an ordinary iid SFM consecutive-k_s-out-of-n system with component failure probabilities q; the evaluation of $R_s(q)$ can be easily accomplished by initiating a simple recursive scheme, exploiting exact combinatorial formulae or applying a Markov chain embedding technique (see e.g. Balakrishnan and Koutras(2002), Chao et al(1995), Fu and Lou(2003) or Kuo and Zuo(2003)).

With the notation introduced so far, the lower bound given in Proposition 3.2 and bound (3) take on the form

$$L = L(q_1, q_2, ..., q_m) = \prod_{s=1}^{m} R_s(Q_s), \quad L' = L'(q_1, q_2, ..., q_m) = \prod_{s=1}^{m} R_s(Q_s'),$$

where

$$Q_s = \begin{cases} q_1, \text{ if } s = 1 \\ \dfrac{q_s}{1 - \sum_{j=1}^{s-1} q_j}, \text{ if } s \geq 2 \end{cases} \quad \text{and} \quad Q_s' = \frac{q_s}{p + q_s}, \text{ for } s = 1, 2, ..., m$$

As already stated after Colloray 3.1, $L(\boldsymbol{q})$ is always better (larger) than $L'(\boldsymbol{q})$. In Table 1 we consider several values of the parameters involved and present the MFM system reliability $R = R(q_1, q_2, ..., q_m)$ (for m =2 the exact value was computed, while for m=3 we used simulation), the lower bounds L, L' and the relative improvement $(L - L')/R$ when L is used instead of L', for approximating R.

Two additional lower bounds for the reliability function of a MFM system include (see Boutsikas and Koutras(2002))

$$L_B = L_B(\boldsymbol{q}_{1\cdot}, \boldsymbol{q}_{2\cdot}, ..., \boldsymbol{q}_{m\cdot}) = \sum_{s=1}^{m} R_s(\boldsymbol{q}_{s\cdot}) - m + 1, \quad L_C = L_C(\boldsymbol{q}_{1\cdot}, \boldsymbol{q}_{2\cdot}, ..., \boldsymbol{q}_{m\cdot}) = \prod_{s=1}^{m} R_s(\boldsymbol{q}_{s\cdot}) - \sum_{1 \leq s < t \leq m} c_{st},$$

where

$$c_{st} = \sum_{C \in \mathbf{C}_s} \sum_{\substack{D \in \mathbf{C}_t \\ D \cap C \neq \varnothing}} \prod_{i \in C} q_{si} \prod_{j \in D} q_{tj} .$$

The fist bound follows from a direct application of the Bonferroni inequality while the second one makes use of a general covariance inequality established by Boutsikas and Koutras(2000).

It is of interest to note that, by rejecting the last term of the RHS of L_c we may gain the following product type upper bound (c.f. Boutsikas and Koutras (2002)),

$$U_C = U_C(\boldsymbol{q}_{1\cdot}, \boldsymbol{q}_{2\cdot}, ..., \boldsymbol{q}_{m\cdot}) = \prod_{s=1}^{m} R_s(\boldsymbol{q}_{s\cdot}) .$$

Note that, this bound, is exactly of the same form as the bound developed in the present article, apart from the fact that the individual component reliabilities involved, are different. Since,

$$L'(\boldsymbol{q}) \leq L(\boldsymbol{q}) \leq R(\boldsymbol{q}) \leq U_C(\boldsymbol{q}) ,$$

for all $\boldsymbol{q} = (q_{si})_{m \times n}$, and $L'(\boldsymbol{q}), U_C(\boldsymbol{q})$ tend asymptotically (under certain conditions) to the exact reliability function $R(\boldsymbol{q})$, the new bound $L(\boldsymbol{q})$ is of the correct order in the sense that it is also converging to $R(\boldsymbol{q})$. Thus one could exploit both product type bounds $L(\boldsymbol{q})$ and $U_C(\boldsymbol{q})$ to establish a tight interval estimation of the exact system reliability.

In Tables 2 and 3 we perform a numerical comparison between the new bound and L_B, L_C for several values of the parameters n, k_1, k_2, k_3 and $q_s, s = 1,2,3$ (the boldfaced entries in these tables indicate the best bound for each parameter selection). As our experimentation reveals, in most of the cases, the new bound betters the rest up-to-date available bounds, especially when at least one of the SFM systems involved in the calculation of the lower bound, is of low reliability; in this case, a rule of thumb for obtaining better bounds, is to use the family of minimal cut sets corresponding to the low reliability system, in the first term of the product $\prod_{s=1}^{m} R_s(Q_s)$.

Finally, in Table 4 we present the values of the new bound $L = L(q_1, q_2, q_3)$ for a consecutive-k_1, k_2, k_3-out-of-n: MFM system with $k_1 = k_2 = k_3 = k$ and the values of q_1, q_2, q_3 being chosen so that all possible permutations are generated; this choice offers the opportunity to observe how the rearrangement of the three families of minimal cut sets affects the lower bound L (the exact system reliability remains unchanged under these permutations).

It is worth stressing that, the new bound does not require any additional computational load, as compared to L_B, L_C; all of them use the reliabilities of SFM consecutive-k-out-of-n systems (L_C calls for the evaluation of the extra terms c_{st} as well). Should one wish to avoid the calculation of

the (exact) reliability of consecutive-k-out-of-n systems, he could exploit Corollary 3.1 thereof obtaining the following simpler lower bound

$$L^* = \prod_{s=1}^{m} (1 - Q_s^{k_s})^{n-k_s+1} .$$

For the benefit of the practical minded reader, we have also tabulated L^* in the last column of Tables 2 and 3, so that an assessment be made of the loss in accuracy created when replacing $R_s(q)$ by their Esary and Proschan lower bounds. It is of interest to note that, in some cases, L^* remains better than L_B and L_C.

In closing we mention that, both L and L^* obtain non negative values, in contrast to L_B and L_C that may become negative for specific choices of the parameters involved. It goes without saying that, in these cases, the lower reliability bound will be set equal to zero.

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Table 1. Exact reliability values and relative error improvement

n	m	k_1	k_2	k_3	q_1	q_2	q_3	R	L	L'	$(L-L')/R$ (%)
100	2	2	2	-	0.08	0.05	-	0.4348	0.4186	0.3935	5.79
		2	3	-	0.05	0.05	-	0.7800	0.7786	0.7594	2.47
		3	5	-	0.10	0.06	-	0.9152	0.9151	0.8994	1.73
		5	4	-	0.20	0.15	-	0.9355	0.8844	0.8595	2.66
100	3	2	2	2	0.10	0.05	0.05	0.2502	0.2168	0.1705	18.52
		2	3	4	0.09	0.03	0.06	0.4811	0.4726	0.4068	13.68
		3	4	4	0.15	0.05	0.05	0.7559	0.7506	0.6793	9.43
		6	2	4	0.16	0.06	0.08	0.7104	0.6151	0.5544	8.55
250	3	2	3	2	0.06	0.05	0.02	0.3705	0.3648	0.3198	12.15
		2	2	2	0.10	0.05	0.05	0.0300	0.0214	0.0117	32.34
		6	6	6	0.20	0.15	0.15	0.9853	0.9515	0.8585	9.44
		5	4	5	0.12	0.12	0.12	0.9437	0.9052	0.8433	6.56

Table 2. Reliability bounds for consecutive-2, 3, 4-out-of-n: MFM system

n	q_1	q_2	q_3	L	L_c	L_B	L^*
100	0.16	0.01	0.02	**0.10472**	**0.10472**	0.10463	0.07672
	0.20	0.02	0.02	**0.03132**	0.03122	0.03059	0.01755
	0.15	0.08	0.10	**0.12348**	0.12334	0.08232	0.09421
	0.16	0.15	0.15	**0.05532**	0.03463	-0.1839	0.03530
500	0.01	0.01	0.01	0.95130	**0.95131**	0.95129	0.95083
	0.09	0.02	0.01	**0.02343**	0.02333	0.01965	0.01719
	0.08	0.04	0.04	**0.04861**	0.04821	0.01929	0.03889
	0.09	0.09	0.08	**0.01463**	0.00394	-0.27695	0.01020
1.000	0.06	0.03	0.01	**0.03203**	0.03181	0.00724	0.02637
	0.08	0.02	0.01	**0.00253**	0.00233	-0.00525	0.00162
	0.06	0.01	0.06	**0.03248**	0.03237	0.01999	0.02675
	0.08	0.05	0.02	**0.00219**	-0.00093	-0.10941	0.00140

Table 3. Reliability bounds for consecutive-k_1,k_2,k_3-out-of-5.000: MFM system

q_1	q_2	q_3	k_1	k_2	k_3	L	L_C	L_B	L^*
0.10	0.01	0.02	3	2	5	**0.00596**	0.00469	-0.37949	0.00363
			3	4	3	**0.01038**	0.01035	-0.02751	0.00636
			5	5	6	**0.95603**	**0.95603**	**0.95603**	0.95127
0.02	0.09	0.03	2	8	7	**0.14064**	**0.14064**	0.14062	0.13533
			2	3	6	**0.00414**	-0.00076	-0.82333	0.00282
			9	6	5	0.99707	**0.99747**	**0.99747**	0.99679
0.03	0.02	0.04	2	4	4	**0.01241**	0.01238	-0.00038	0.01091
			2	3	5	**0.01207**	0.01197	-0.02631	0.01061
			4	6	9	**0.99608**	**0.99608**	**0.99608**	0.99596

Table 4. Reliability bounds for consecutive-k_1,k_2,k_3-out-of-1.000: MFM system with $k_1=k_2=k_3=k$

k=3				k=4				k=5			
q_1	q_2	q_3	L	q_1	q_2	q_3	L	q_1	q_2	q_3	L
0.10	0.05	0.01	**0.34491**	0.10	0.05	0.01	**0.90596**	0.10	0.05	0.01	**0.99058**
0.10	0.01	0.05	0.34314	0.10	0.01	0.05	0.90559	0.10	0.01	0.05	0.99055
0.05	0.10	0.01	0.31173	0.05	0.10	0.01	0.89090	0.05	0.10	0.01	0.98826
0.05	0.01	0.10	0.30190	0.05	0.01	0.10	0.88682	0.05	0.01	0.10	0.98765
0.01	0.10	0.05	0.33425	0.01	0.10	0.05	0.90236	0.01	0.10	0.05	0.99011
0.01	0.05	0.10	0.30088	0.01	0.05	0.10	0.88661	0.01	0.05	0.10	0.98764