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Beta-hypergeometric distributions and random continued fractions

Claudio Ascì *, Gérard Letac †, Mauro Piccioni ‡

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Abstract. In this paper an enlargement of the beta family of distributions on $(0, 1)$ is presented. Distributions in this class are characterized as being the laws of certain random continued fractions associated to products of independent random matrices of order 2 whose entries are either constant or beta distributed. The result can be proved by a famous 1879 Thomae formula on generalized hypergeometric functions ${}_3F_2$.

Keywords: beta distributions of second kind, hypergeometric functions, Thomae identities.

I Introduction

The aim of this paper is to generalize the following well known facts about gamma and beta distributions. If the independent random variables X and G_1 are distributed as beta $\beta_{a,a'}$ and gamma $\gamma_{a+a',1}$ respectively, then $XG_1 \sim \gamma_{a,1}$. Furthermore, if the random variable G_2 is distributed as $\gamma_{a',1}$ and if it is independent of (X, G_1) , then

$$G_2 + XG_1 \sim \gamma_{a+a',1}, \quad \frac{G_2}{G_2 + XG_1} = \frac{1}{1 + X \frac{G_1}{G_2}} \sim \beta_{a',a} \quad (1.1)$$

and these two random variables are independent. Next, recall that for any $a, b > 0$ the beta law of the second kind $\beta_{b,a}^{(2)}$ given by

$$\beta_{b,a}^{(2)}(dw) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{w^{b-1}}{(1+w)^{a+b}} \mathbf{1}_{(0,\infty)}(w) dw$$

is the law of the ratio of two independent gamma variables with shape parameters b for the numerator and a for the denominator, and the same scale parameter (equivalently, it is the law of $(1-U)/U$ with $U \sim \beta_{a,b}$). Thus we have that the random variable appearing in (1.1)

$$W' \equiv \frac{G_1}{G_2} \sim \beta_{a+a',a'}^{(2)} \quad (1.2)$$

and we can reexpress (1.1) by saying that if W' has the law (1.2) and is independent of $X \sim \beta_{a,a'}$, then

$$\frac{1}{1 + W'X} \sim \beta_{a',a}.$$

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Moreover, if $W \sim \beta_{a+a',a}^{(2)}$ is independent of (X, W') , then

$$\frac{1}{1 + \frac{W}{1+W'X}} \sim X. \quad (1.3)$$

This relation (1.3) characterizes the law $\beta_{a,a'}$ (Chamayou and Letac (1991), Example 11). In this paper we generalize the result (1.3) to the general case $W \sim \beta_{b,a}^{(2)}$, $W' \sim \beta_{b,a'}^{(2)}$, with $b > 0$ not necessarily equal to $a + a'$. We prove that there is again a unique law parameterized by (a, a', b) for X , supported by the positive real line, which ensures the equality in law (1.3). In Section 2 we will explicitly present its density function and prove this characterization. In Section 3 some of the properties of this family of laws will be investigated. In Section 4 we will prove the identifiability of the parameters (a, a', b) and in Section 5 we discuss symmetry properties. Finally, in Section 6 we will give some particularly interesting cases. The appendix links our results with the Thomae (1879) formula.

II The beta-hypergeometric distribution

First recall that for any real number a the sequence of Pochhammer's symbols $\{(a)_n\}_{n=0}^{\infty}$ is defined by $(a)_0 = 1$ and $(a)_{n+1} = (a+n)(a)_n$. For positive numbers $a_1, \dots, a_p, b_1, \dots, b_q$ we consider the sum of the power series

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{n! (b_1)_n \dots (b_q)_n} z^n. \quad (2.4)$$

For $p = q + 1$, which is the case of our interest, the power series (2.4) is always convergent for $|z| < 1$ by means of the ratio criterion. If $x > 0$ then

$$\Gamma(x+n) = (x)_n \Gamma(x), \quad (2.5)$$

so that by the Stirling approximation to the gamma function $(x)_n \sim \frac{n^{x-1}n!}{\Gamma(x)}$. Hence the series (2.4) converges for $z = 1$ if and only if

$$c = b_1 + \dots + b_q - a_1 - \dots - a_{q+1} > 0 \quad (2.6)$$

since the coefficient of the general term of the series (2.4) is equivalent (up to a multiplicative constant) to n^{-1-c} . We will also need to consider (2.4) when some of the a_j is negative, in which case the series (2.4) terminates if at least one of them is a non positive integer. Otherwise, since $-k < a < -k+1$ implies $(a)_n = (a)_k(a+k)_{n-k}$ for $n \geq k$, the general term of (2.4) has constant sign eventually in n , and again we have that the series (2.4) converges for $z = 1$ if and only if $c > 0$. Next for any positive a, a' and b , define the following probability measure

$$\mu_{a,a',b}(dx) = C(a, a', b) x^{a-1} (1-x)^{b-1} {}_2F_1(a, b; a+a'; x) \mathbf{1}_{(0,1)}(x) dx, \quad (2.7)$$

where

$$C(a, a', b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b) {}_3F_2(a, a, b; a+b, a+a'; 1)}. \quad (2.8)$$

By our previous remarks ${}_3F_2(a, a, b; a+b, a+a'; 1)$ is finite and Theorem 38 in Rainville (1960), page 104, can be applied to prove that (2.7) integrates to 1. As suggested by a referee we get a mixture representation for $\mu_{a,a',b}$: if N is an integer valued random variable with

$$\mathbf{E}(z^N) = \frac{{}_3F_2(a, b, a; a+a', a+b; z)}{{}_3F_2(a, b, a; a+a', a+b; 1)}$$

and if $X|N \sim \beta_{a+N,b}$, then $X \sim \mu_{a,a',b}$. A noticeable subclass of this family is met for $b = a + a'$, since

$${}_2F_1(a, a + a'; a + a'; x) = {}_1F_0(a; -; x) = (1 - x)^{-a}.$$

Therefore $\mu_{a,a',a+a'}(dx) = \beta_{a,a'}(dx)$.

Theorem 2.1. The probability measure $\mu_{a,a',b}$ has the following property. If $X \sim \mu_{a,a',b}$ and $W' \sim \beta_{b,a'}^{(2)}$ are independent then

$$\frac{1}{1 + XW'} \sim \mu_{a',a,b}. \quad (2.9)$$

Proof. Suppose now that $X' \sim \mu_{a',a,b}$. The relation (2.9) is equivalent to $\frac{1-X'}{X'} \sim XW'$. The density of $V = \frac{1-X'}{X'}$ is

$$f_V(v) = C' \frac{v^{b-1}}{(1+v)^{a'+b}} {}_2F_1(a', b; a + a'; \frac{1}{1+v}) \mathbf{1}_{(0,\infty)}(v),$$

where $C' = C(a', a, b)$. On the other hand the density of $U = XW'$ is given by

$$\begin{aligned} f_U(u) &= C \frac{\Gamma(a' + b)}{\Gamma(a')\Gamma(b)} \frac{1}{u} \int_0^1 x^{a-1} (1-x)^{b-1} {}_2F_1(a, b; a + a'; x) \left(\frac{u}{x}\right)^b \frac{1}{(1 + \frac{u}{x})^{a'+b}} dx \\ &= C \frac{\Gamma(a' + b)}{\Gamma(a')\Gamma(b)} u^{b-1} \int_0^1 \frac{x^{a+a'-1} (1-x)^{b-1}}{(x+u)^{a'+b}} {}_2F_1(a, b; a + a'; x) dx \\ &= C \frac{\Gamma(a' + b)}{\Gamma(a')\Gamma(b)} u^{b-1} \int_0^1 \frac{t^{b-1} (1-t)^{a+a'-1}}{(1+u-t)^{a'+b}} {}_2F_1(a, b; a + a'; 1-t) dt \\ &= C \frac{\Gamma(a' + b)}{\Gamma(a')\Gamma(b)} \frac{u^{b-1}}{(1+u)^{a'+b}} \int_0^1 \frac{t^{b-1} (1-t)^{a+a'-1}}{(1 - \frac{t}{1+u})^{a'+b}} {}_2F_1(a, b; a + a'; 1-t) dt, \end{aligned}$$

where $C = C(a, a', b)$. We conclude the proof of the theorem by means of the following

Lemma 2.2. For $a, a', b > 0$ and for $0 < z < 1$ we have

$$\int_0^1 \frac{t^{b-1} (1-t)^{a+a'-1}}{(1-zt)^{a'+b}} {}_2F_1(a, b; a + a'; 1-t) dt = \frac{\Gamma(a')\Gamma(b)}{\Gamma(a' + b)} {}_2F_1(a', b; a + a'; z). \quad (2.10)$$

Proof of Lemma 2.2. Observe that

$$\begin{aligned}
 & \int_0^1 \frac{t^{b-1}(1-t)^{a+a'-1}}{(1-zt)^{a'+b}} {}_2F_1(a, b; a+a'; 1-t) dt \\
 \stackrel{(1)}{=} & \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(a+a')_n} \int_0^1 \frac{t^{b-1}(1-t)^{a+a'+n-1}}{(1-zt)^{a'+b}} dt \\
 \stackrel{(2)}{=} & \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(a+a')_n} \frac{\Gamma(b)\Gamma(a+a'+n)}{\Gamma(a+a'+b+n)} {}_2F_1(a'+b, b; a+a'+b+n; z) \\
 \stackrel{(3)}{=} & \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(a+a')_n} \frac{\Gamma(b)\Gamma(a+a'+n)}{\Gamma(a+a'+b+n)} \sum_{k=0}^{\infty} \frac{(a'+b)_k(b)_k}{k!(a+a'+b+n)_k} z^k \\
 \stackrel{(4)}{=} & \sum_{k=0}^{\infty} \frac{(a'+b)_k(b)_k}{k!} z^k \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(a+a')_n} \frac{\Gamma(b)\Gamma(a+a'+n)}{\Gamma(a+a'+b+n+k)} \\
 \stackrel{(5)}{=} & \sum_{k=0}^{\infty} \frac{(a'+b)_k(b)_k}{k!} z^k \frac{\Gamma(b)\Gamma(a+a')}{\Gamma(a+a'+b+k)} {}_2F_1(a, b; a+a'+b+k; 1) \\
 \stackrel{(6)}{=} & \sum_{k=0}^{\infty} \frac{(a'+b)_k(b)_k}{k!} z^k \frac{\Gamma(b)\Gamma(a+a')}{\Gamma(a+a'+b+k)} \frac{\Gamma(a+a'+b+k)\Gamma(a'+k)}{\Gamma(a'+b+k)\Gamma(a+a'+k)} \\
 \stackrel{(7)}{=} & \frac{\Gamma(a')\Gamma(b)}{\Gamma(a'+b)} \sum_{k=0}^{\infty} \frac{(a')_k(b)_k}{k!(a+a')_k} z^k = \frac{\Gamma(a')\Gamma(b)}{\Gamma(a'+b)} {}_2F_1(a', b; a+a'; z).
 \end{aligned}$$

In this sequence of equalities (1) and (4) come from the summation of series with positive terms under the integral (or sum) sign, (2) comes from the Gauss formula (see Rainville (1960), page 47, Theorem 16), (3) and (5) come from the definition of ${}_2F_1$, (6) come from the classical evaluation of ${}_2F_1$ in 1 (see Rainville (1960), page 49, Theorem 18) and (7) comes from (2.5) with $x = a'$, $x = a + a'$ and $x = a' + b$. \square

End of the proof of Theorem 2.1. By substituting $z = 1/(1+u)$ with $u > 0$ in (2.10) we get that the densities of U and V are equal, as their normalizing constants C and C' . \square

Remark 2.3. We actually proved that $C(a, a'; b)$ is a symmetric function of (a, a') , equivalently

$$\frac{{}_3F_2(a, a, b; a+b, a+a'; 1)}{\Gamma(a')\Gamma(a+b)} = \frac{{}_3F_2(a', a', b; a'+b, a+a'; 1)}{\Gamma(a)\Gamma(a'+b)}. \quad (2.11)$$

The following theorem is the main result of our paper. It gives a characterization of distributions of the family $\mu_{a,a',b}$ which generalizes that discussed in the introduction for the beta distribution, to which it reduces for $a + a' = b$. **Theorem 2.4.**

1. If $W \sim \beta_{b,a}^{(2)}$, $W' \sim \beta_{b,a'}^{(2)}$ and $X > 0$ are three independent random variables, then

$$X \sim \frac{1}{1 + \frac{W}{1+W'X}} \quad (2.12)$$

if and only if $X \sim \mu_{a,a',b}$.

2. If $W \sim \beta_{b,a}^{(2)}$ and $X > 0$ are two independent random variables, then

$$X \sim \frac{1}{1 + WX} \quad (2.13)$$

if and only if $X \sim \mu_{a,a',b}$.

3. Let $(W_n)_{n \geq 1}$ and $(W'_n)_{n \geq 1}$ be two independent i.i.d sequences of random variables with respective distributions $\beta_{b,a}^{(2)}$ and $\beta_{b,a'}^{(2)}$. Then $\mu_{a,a',b}$ is the distribution of the random continued fraction

$$\frac{1}{1 + \frac{W_1}{1 + \frac{W'_1}{1 + \frac{W_2}{1 + \dots}}}}. \quad (2.14)$$

Proof. By Theorem 14, page 506 of Chrystal (1964) (2.14) converges almost surely, since $\sum_n (W_n^{-1} + W'_n{}^{-1}) = \infty$ almost surely. We are in position to apply a general principle (Proposition 1 in Chamayou and Letac (1991)) to the sequence $(F_n)_{n=1}^\infty$ of random mappings of $(0, 1)$ into itself defined by

$$F_n(z) = \frac{1}{1 + \frac{W_n}{1 + W'_n z}}.$$

Since $F_1 \circ \dots \circ F_n(z)$ has almost surely a random constant limit X then the distribution of X is a stationary distribution of the Markov chain $\omega_n = F_n \circ \dots \circ F_1(z)$ and this stationary distribution is unique. Since Theorem 2.1 implies that $\mu_{a,a',b}$ is a stationary distribution of the Markov chain $(\omega_n)_{n=0}^\infty$, this proves that $X \sim \mu_{a,a',b}$, which implies the first and the third statements of the theorem. The second statement is obtained by repeating the argument with the simpler random mappings

$$G_n(z) = \frac{1}{1 + W_n z}. \quad \square$$

Remark 2.5. The results of the above theorem can be reformulated in alternative ways by recalling from the introduction that random variables with the $\beta_{b,a}^{(2)}$ law can be expressed either as a ratio Z/Y of independent $Z \sim \Gamma_{b,1}$ and $Y \sim \Gamma_{a,1}$ or as $(1 - U)/U$ with $U \sim \beta_{a,b}$.

Remark 2.6. In Pham-Gia (2004) a particular type of densities called hyperbeta densities are discussed. An hyperbeta density starts with the data of an integer $m \geq 2$ and of two sequences $\mathbf{a} = (a_1, \dots, a_{m+1})$ and $\mathbf{b} = (b_1, \dots, b_m)$ of positive numbers such that $b_1 < a_1$ and $\sum_{j>1} b_j - \sum_{i>1} a_i > 0$. Denote $p = b_1$ and $q = a_1 - b_1$ for simplicity and write \mathbf{a}' and \mathbf{b}' for the sequences \mathbf{a} and \mathbf{b} where a_1 and b_1 have been taken away. We have therefore

$$\int_0^1 {}_{m+1}F_m(\mathbf{a}; \mathbf{b}; x) \beta_{p,q}(dx) = {}_{m+2}F_m(\mathbf{a}, p; \mathbf{b}, p+q; 1) = {}_mF_{m-1}(\mathbf{a}'; \mathbf{b}'; 1).$$

The corresponding hyperbeta distribution is $\frac{{}_{m+1}F_m(\mathbf{a}; \mathbf{b}; x)}{{}_mF_{m-1}(\mathbf{a}'; \mathbf{b}'; 1)} \beta_{p,q}(dx)$. It is designed for having a hypergeometric normalizing constant ${}_mF_{m-1}(\mathbf{a}'; \mathbf{b}'; 1)$ less complex than the density itself. The simplest version is obtained for $m = 2$ and the five positive parameters a, b, c, p, q

$$\frac{\Gamma(c+a)\Gamma(c+b)}{\Gamma(c)\Gamma(a+b+c)} {}_3F_2(p+q, a, b; p, a+b+c; x) \beta_{p,q}(dx). \quad (2.15)$$

We know of no probabilistic interpretation of these hyperbeta distributions. It is easily checked that none of them is beta-hypergeometric in the sense of the present paper: they do not overlap at all with ours, except in the trivial case of beta distributions (for $p = a$ and $q = b + c$ in (2.15) $\beta_{a,c}$ is obtained).

III Some properties of beta-hypergeometric distributions

By using the classical formula (Rainville (1960) Theorem 21)

$${}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{\gamma-\alpha-\beta} {}_2F_1(\gamma-\alpha, \gamma-\beta; \gamma; z), \quad (3.16)$$

with $(\alpha, \beta, \gamma) = (a, b, a+a')$, we can rewrite (2.7) as

$$\mu_{a,a',b}(dx) = \frac{\Gamma(a+b)\Gamma(a')}{\Gamma(b)\Gamma(a+a'){}_3F_2(a, a, b; a+b, a+a'; 1)} {}_2F_1(a', a+a'-b; a+a'; x)\beta_{a,a'}(dx), \quad (3.17)$$

where

$$\beta_{a,a'}(dx) = \frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')} x^{a-1}(1-x)^{a'-1} \mathbf{1}_{(0,1)}(x) dx$$

is the beta distribution with parameters a and $a' > 0$. Hence if $d = a + a' - b = 0$, then $\mu_{a,a',b}$ coincides with $\beta_{a,a'}$. From (3.17) we get that

$$\frac{d\mu_{a,a',b}}{d\beta_{a,a'}}(x) \propto {}_2F_1(a', a+a'-b; a+a'; x), \quad x \in (0, 1).$$

For $d > 0$ the function ${}_2F_1(a', d; a+a'; x)$ is increasing in $x \in (0, 1)$. For $d < 0$ use the Gauss representation formula

$${}_2F_1(a', d; a+a'; x) = \int_0^1 (1-tx)^{-d} \beta_{a',a}(dt) \quad (3.18)$$

to check that ${}_2F_1(a', d; a+a'; x)$ is decreasing. More generally for any fixed a and $a' > 0$ the family $\{\mu_{a,a',a+a'-d}; d \in (-\infty, a+a')\}$ is seen to be increasing in the likelihood ratio order. Moreover, from (3.18) we notice that $\mu_{a,a',b}$ is the marginal law of the x -component of a joint law on $(0, 1)^2$

$$\mu(dx, dt) = \frac{\Gamma(a+b)\Gamma(a')}{\Gamma(b)\Gamma(a+a'){}_3F_2(a, a, b; a+b, a+a'; 1)} (1-tx)^{-(a+a'-b)} \beta_{a',a}(dt) \beta_{a,a'}(dx),$$

whereas the t -marginal is clearly $\mu_{a',a,b}$. The Mellin transform of $\mu_{a,a',b}$ can be easily computed from the definition; more generally if $X \sim \mu_{a,a',b}$ we can easily compute $\mathbb{E}(X^t(1-X)^s)$. But with the aid of Theorem 2.1 we can also obtain another expression for the Mellin transform of X . This is the purpose of the following

Proposition 3.1. For real t and s the integral $I_{a,a',b}(t, s) = \int_0^1 x^t(1-x)^s \mu_{a,a',b}(dx)$ converges if and only if $t > -a$ and $s > -\min(a', b)$. In this case

$$I_{a,a',b}(t, s) = \frac{\Gamma(a+b)\Gamma(a+t)\Gamma(b+s)}{\Gamma(a)\Gamma(b)\Gamma(a+b+t+s)} \frac{{}_3F_2(a+t, a, b; a+b+t+s, a+a'; 1)}{{}_3F_2(a, a, b; a+b, a+a'; 1)}. \quad (3.19)$$

Furthermore for $t > -a$

$$I_{a,a',b}(t, 0) = \int_0^1 x^t \mu_{a,a',b}(dx) = \frac{{}_3F_2(a', a'-t, b; a'+b, a+a'; 1)}{{}_3F_2(a', a', b; a'+b, a+a'; 1)}. \quad (3.20)$$

Proof. Since all terms are positive we can invert sums and integrals, whether they diverge or not:

$$\int_0^1 x^t (1-x)^s \mu_{a,a',b}(dx) = C \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (a+a')_n} \int_0^1 x^{a+t+n-1} (1-x)^{b+s-1} dx \quad (3.21)$$

$$\begin{aligned} &= C \frac{\Gamma(a+t)\Gamma(b+s)}{\Gamma(a+b+t+s)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (a+t)_n}{n! (a+a')_n (a+b+t+s)_n} \\ &= C \frac{\Gamma(a+t)\Gamma(b+s)}{\Gamma(a+b+t+s)} {}_3F_2(a+t, a, b; a+b+t+s, a+a'; 1), \end{aligned} \quad (3.22)$$

where $C = C(a, a', b)$. Note that the integrals at the right hand side of (3.21) are all finite if and only if $a+t > 0$ and $b+s > 0$. Under this condition the series of (3.22) converges if and only if $a'+s > 0$ according to the criterion of (2.6). For the second part, first apply Theorem 2.1: given two independent random variables $X \sim \mu_{a,a',b}$ and $W' \sim \beta_{b,a'}^{(2)}$, then $X' = (1 + XW')^{-1} \sim \mu_{a',a,b}$. Since $\frac{1-X'}{X'} = XW'$ and the Mellin transform of W' is given by

$$\int w^t \beta_{b,a'}^{(2)}(dw) = \frac{\Gamma(a'-t)\Gamma(b+t)}{\Gamma(a')\Gamma(b)}, \quad -b < t < a', \quad (3.23)$$

for $-b < t < a'$

$$\mathbb{E}(X^t) = \frac{1}{\mathbb{E}((W')^t)} \mathbb{E}\left(\left(\frac{1-X'}{X'}\right)^t\right).$$

Next apply the first part of the proposition by replacing (a, a', b, t, s) by $(a', a, b, -t, t)$, getting the result (3.20) for $-\min(b, a) < t < a'$. To extend it to $t > -a$, we observe that the right hand side of (3.20) is finite if and only if $t > -a$ and is a positive real analytic function of t on this interval. The principle of maximal analyticity (see e.g. Letac and Mora (1990), Theorem 3.1) for Laplace transforms therefore implies that (3.20) holds for $t > -a$. \square

Remark 3.2. Our main result has allowed us to obtain two different expressions for the Mellin transform $I(t, 0; a, b, a')$, obtained from (3.20) and (3.19). Equating these two and recalling the equality (2.11) already obtained, we obtain

$$\frac{{}_3F_2(a+t, a, b; a+b+t, a+a'; 1)}{\Gamma(a')\Gamma(a+b+t)} = \frac{{}_3F_2(a', a'-t, b; a'+b, a+a'; 1)}{\Gamma(a+t)\Gamma(a'+b)}, \quad (3.24)$$

for $t > -a$. It is readily checked that this equality (3.24) is a particular case of a result due to Thomae (1879), which is reported as Theorem A.1 in the Appendix, under the form stated by Maier (2005), page 46. To conclude this section, we prove and extend some observations of a referee about the limits in distribution of the family $\mu_{a,a',b}$.

Proposition 3.3. The following convergences in law hold:

1. $\lim_{a \rightarrow 0} \mu_{a,a',b} = \delta_0$ and $\lim_{a \rightarrow \infty} \mu_{a,a',b} = \delta_1$
2. $\lim_{a' \rightarrow 0} \mu_{a,a',b} = \delta_1$ and $\lim_{a' \rightarrow \infty} \mu_{a,a',b} = \beta_{a,b}$
3. $\lim_{b \rightarrow 0} \mu_{a,a',b} = \delta_1$, $\lim_{b \rightarrow \infty} \mu_{a,a',b} = \delta_0$ if $a \leq a'$ and $\lim_{b \rightarrow \infty} \mu_{a,a',b} = \beta_{a-a',a'}$ if $a' < a$.

Proof. Since all probabilities here are concentrated on the compact interval $[0, 1]$, from the Weierstrass uniform approximation theorem of continuous functions by sequences of polynomials all the above statements will be proved if we prove convergence of moments, or even if we prove the simple convergence of the Mellin transforms for $t > 0$. For statement 1. we use (3.20): since

$${}_3F_2(a', a' - t, b; a' + b, a + a'; 1) = \sum_{n=0}^{\infty} \frac{(a' - t)_n (b)_n}{n! (a' + b)_n} \times \frac{(a')_n}{(a + a')_n},$$

we observe that $a \mapsto \frac{(a')_n}{(a + a')_n}$ is a decreasing function of a on $(0, \infty)$. The monotone convergence or the dominated convergence theorems therefore enable us to invert sum and limit. For instance for $t > 0$ and $a \rightarrow 0$ we get

$$\lim_{a \rightarrow 0} {}_3F_2(a', a' - t, b; a' + b, a + a'; 1) = \sum_{n=0}^{\infty} \frac{(a' - t)_n (b)_n}{n! (a' + b)_n} \times \lim_{a \rightarrow 0} \frac{(a')_n}{(a + a')_n} = {}_2F_1(a' - t, b; a' + b; 1)$$

(here we have used the fact that

$$\sum_{n=0}^{\infty} \left| \frac{(a' - t)_n (b)_n}{n! (a' + b)_n} \right| < \infty,$$

from the introduction). Similarly for the denominator in (3.20) we have $\lim_{a \rightarrow 0} {}_3F_2(a', a', b; a' + b, a + a'; 1) = \infty$. This proves that the limit of the Mellin transform (3.20) when $a \rightarrow 0$ is zero for any $t > 0$, which implies $\lim_{a \rightarrow 0} \mu_{a, a', b} = \delta_0$. When $a \rightarrow \infty$ all the terms except the first go to zero in ${}_3F_2(a', a' - t, b; a' + b, a + a'; 1)$ go to zero except the first, thus the Mellin transform (3.20) tends to 1 for any $t > 0$, which implies $\lim_{a \rightarrow \infty} \mu_{a, a', b} = \delta_1$. The proof of the statement 2. requires either the use of (3.19) in the particular case $s = 0$ or equivalently it can be obtained by plugging in the relation of Theorem 2.1 the limit distributions obtained in statement 1. The proof of statement 3. is analogous to that of statement 1. For the last limit result notice that, for $a' < a$, the Mellin transform of $\beta_{a-a', a'}$ is

$$\frac{{}_2F_1(a', a' - t; a + a'; 1)}{{}_2F_1(a', a'; a + a'; 1)} = \frac{\Gamma(a)\Gamma(a - a' + t)}{\Gamma(a - a')\Gamma(a + t)}. \quad \square$$

IV Identifiability of (a, a', b)

Proposition 4.1. If (a, a', b) and (a_1, a'_1, b_1) in $(0, \infty)^3$ are such that $\mu_{a, a', b} = \mu_{a_1, a'_1, b_1}$ then $(a, a', b) = (a_1, a'_1, b_1)$.

Proof. Let $C = C(a, a', b)$ and $C_1 = C(a_1, a'_1, b_1)$. We see easily that $a = a_1$ and that $C = C_1$ since for $x \rightarrow 0$ the densities of $\mu_{a, a', b}$ and μ_{a_1, a'_1, b_1} are equivalent to Cx^{a-1} and $C_1x^{a_1-1}$, respectively. Thus writing the equality of densities leads to

$$(1 - x)^b {}_2F_1(a, b; a + a'; x) = (1 - x)^{b_1} {}_2F_1(a, b_1; a + a'_1; x)$$

for all $x \in (0, 1)$. Now use a classical formula (Rainville (1960), Theorem 20, page 60) to obtain for all $x \in (0, \frac{1}{2})$

$${}_2F_1(a', b; a + a'; -x/(1 - x)) = {}_2F_1(a'_1, b_1; a + a'_1; -x/(1 - x)),$$

hence for z in a neighborhood of 0 we have

$${}_2F_1(a', b; c; z) = {}_2F_1(a'_1, b_1; c_1; z),$$

where $c = a + a'$ and $c_1 = a + a'_1$. By equating the coefficients of z , z^2 and z^3 and taking ratios we get

$$\frac{a'b}{c} = \frac{a'_1 b_1}{c_1}, \quad \frac{(a' + 1)(b + 1)}{(c + 1)} = \frac{(a'_1 + 1)(b_1 + 1)}{(c_1 + 1)}, \quad \frac{(a' + 2)(b + 2)}{(c + 2)} = \frac{(a'_1 + 2)(b_1 + 2)}{(c_1 + 2)}.$$

From this we get $(a', b, c) = (a'_1, b_1, c_1)$ or (b_1, a'_1, c_1) . Here are the details: from the knowledge of

$$\lambda_0 = \frac{a'b}{c}, \quad \lambda_1 = \frac{(a' + 1)(b + 1)}{c + 1}, \quad \lambda_2 = \frac{(a' + 2)(b + 2)}{c + 2},$$

we get by taking suitable linear combination of both members

$$c(2\lambda_1 - \lambda_0 - \lambda_2) + 2 + 2\lambda_1 - 2\lambda_2 = 0, \quad a' + b = (c + 1)\lambda_1 - c\lambda_0 - 1, \quad a'b = c\lambda_0,$$

from which c can be uniquely obtained from the first equation (since $\lambda_i > 0$ for $i = 0, 1, 2$ such an equation cannot be undetermined) and the unordered pair $\{a', b\}$ from the remaining two. But since $c = a + a'$ we derive a' and thus b as well. Hence $a' = a_1$, $b = b_1$ and $c = c_1$ as promised. \square

V Symmetry of $\mu_{a,a',b}$

In this section we prove that $\mu_{a,a',b}$ is symmetric with respect to $1/2$ if and only if $a = a'$ and $b = a + a'$, i.e. there are no symmetric laws out of the beta family. More generally the transformation $x \mapsto 1 - x$ on a $\mu_{a,a',b}$ -distributed random variable induces a law which is not a member of the family, except in the beta case. The main argument for the proof of this result is contained in the following

Lemma 5.1. Let (α, β, γ) and $(\alpha_1, \beta_1, \gamma_1)$ be real numbers such that $\gamma > 0$ and $\gamma_1 > 0$. Consider the functions y and y_1 defined on $(0, 1)$ by

$$y(x) = {}_2F_1(\alpha, \beta; \gamma; 1 - x), \quad y_1(x) = {}_2F_1(\alpha_1, \beta_1; \gamma_1; x)$$

and assume that $Dy = y_1$ for some real constant $D \neq 0$. Then $y_1 \equiv 1$, that is $\alpha\beta = \alpha_1\beta_1 = 0$.

Proof. The functions y and y_1 are solutions of the differential equations (see Rainville (1960), page 53)

$$x(1 - x)y_1''(x) + (\gamma_1 - (\alpha_1 + \beta_1 + 1)x)y_1'(x) - \alpha_1\beta_1 y_1(x) = 0 \quad (5.25)$$

$$x(1 - x)y''(x) + (-\gamma + \alpha + \beta + 1 - (\alpha + \beta + 1)x)y'(x) - \alpha\beta y(x) = 0 \quad (5.26)$$

(the first being obtained as the second by changing x into $1 - x$). Since $Dy = y_1$ multiplying both sides of (5.26) by D we get another differential equation for y_1 which subtracted from (5.25) yields $(A - Bx)y_1'(x) = Cy_1(x)$ where A, B, C are suitable constants. By definition $y_1(0+) = 1$; now suppose that y_1 is not constantly equal to 1. Then $C \neq 0$ and $(A, B) \neq (0, 0)$. If $B = 0$ then $y_1(x) = e^{Cx/A}$, which is not a hypergeometric function as seen by its power series expansion. If $A = 0$ then $y_1(0+) = 1$ is impossible. Thus without loss of generality we assume $A = 1$ and we get $y_1(x) = (1 - Bx)^{-a}$ with $a = C/B \neq 0$. Expanding $x \mapsto (1 - Bx)^{-a}$ in power series

$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} (B)^n x^n$ shows that it is a hypergeometric function only if $B = 1$. To conclude the proof

$${}_2F_1(\alpha, \beta; \gamma; x) = y(1-x) = D^{-1}y_1(1-x) = D^{-1}x^{-a},$$

which is impossible since the function on the r.h.s. is unbounded in any interval $(0, \epsilon)$ for $a > 0$ and is zero at 0 if $a < 0$. \square

We can now prove the promised result.

Theorem 5.2. Let $X \sim \mu_{a,a',b}$. Then there exists (a_1, a'_1, b_1) in $(0, \infty)^3$ such that $X_1 = 1 - X \sim \mu_{a_1, a'_1, b_1}$ if and only if $a_1 = a'$, $a'_1 = a$ and $b_1 = b = a + a' = a_1 + a'_1$.

Proof. The if part is obvious since $\mu_{a,a',a+a'} = \beta_{a,a'}$. Conversely, suppose that $X_1 = 1 - X$ with $X \sim \mu_{a,a',b}$ and $X_1 \sim \mu_{a_1, a'_1, b_1}$. Then, for all $x \in (0, 1)$ we have:

$$Cx^{a_1-1}(1-x)^{b_1-1} {}_2F_1(a_1, b_1; a_1 + a'_1; x) = (1-x)^{a-1}x^{b-1} {}_2F_1(a, b; a + a'; 1-x)$$

or equivalently

$$C(1-x)^{b_1-a} {}_2F_1(a_1, b_1; a_1 + a'_1; x) = x^{b-a_1} {}_2F_1(a, b; a + a'; 1-x), \quad (5.27)$$

C being a constant. Introduce $d = a + a' - b$ and $d_1 = a_1 + a'_1 - b_1$. Applying (3.16) to both sides of (5.27) (specifically $(\alpha, \beta, \Gamma, z) = (a_1, b_1; a_1 + a'_1; x)$ and $(a, b; a + a'; 1-x)$) we get

$$C(1-x)^{a'_1-a} {}_2F_1(a'_1, d_1; a_1 + a'_1; x) = x^{a'-a_1} {}_2F_1(a', d; a + a'; 1-x). \quad (5.28)$$

Note for all $t \in \mathbb{R}$ we have

$$I_{a_1, a'_1, b_1}(t, -t) = \mathbb{E}\left(\left(\frac{X_1}{1-X_1}\right)^t\right) = \mathbb{E}\left(\left(\frac{1-X}{X}\right)^t\right) = I_{a, a', b}(-t, t).$$

Now application of Proposition 3.1 shows that the left hand side is finite if and only if $-a_1 < t < \min(a'_1, b_1)$ and that the right hand side is finite if and only if $-\min(a', b) < t < a$. This leads to

$$a_1 = \min(a', b), \quad a = \min(a'_1, b_1).$$

It remains to discuss several cases. If $a_1 = a' \leq b$ and $a = a'_1 \leq b_1$ then one applies Lemma 5.1 to (5.28) and we get $d = d_1 = 0$ as desired. If $a_1 = b < a'$ and $a = b_1 < a'_1$ then one applies Lemma 5.1 to (5.27) and we get the absurd statement $ab = a_1b_1 = 0$. If $a_1 = a' \leq b$ and $a = b_1 < a'_1$ then (5.27) and (3.16) we get

$$C {}_2F_1(a_1, b_1; a_1 + a'_1; x) = {}_2F_1(a', d; a + a'; 1-x),$$

leading via Lemma 5.1 to the absurd statement $a_1b_1 = 0$. The case $a = a'_1 \leq b_1$ and $a_1 = b < a'$ is similar. The result is proved. \square

From the previous result we deduce immediately the symmetry result mentioned at the beginning of the section.

Corollary 5.3. Let $X \sim \mu_{a,a',b}$ and $X' \sim \mu_{a',a,b}$. Then $X \sim 1 - X'$ if and only if $b = a + a'$ and $X \sim 1 - X$ if and only if $a = a' = b/2$.

VI Some particular examples

We list some values of (a, a', b) for which $\mu_{a,a',b}$ has a remarkable form.

1. As seen before $b = a + a'$ gives $\mu_{a,a',a+a'}(dx) = \beta_{a,a'}(dx)$. If more generally $b = a + a' + k$ where k is a non negative integer, then the hypergeometric series

$${}_2F_1(a', a + a' - b; a + a'; x)$$

terminates and is a polynomial with degree $\leq k$. Thus $\mu_{a,a',a+a'+k}(dx)$ has a simple expression from (3.17). In particular the normalizing constant and the moments of $\mu_{a,a',a+a'+k}(dx)$ are computable, since the integral of a polynomial by $\beta_{a,a'}$ is elementary. For instance

$$\mu_{a,a',a+a'+1}(dx) = \frac{a + a'}{a^2 + aa' + a'^2} (a + a'(1 - x)) \beta_{a,a'}(dx).$$

2. For $a = a' = b = 1/2$ then from Rainville (1960) page 71, exercise 19 and using Dixon's theorem in order to compute the normalizing constant, we get

$$\mu_{1/2,1/2,1/2}(dx) = \frac{8\pi}{x^{1/2}(1-x)^{1/2}\Gamma^4(\frac{1}{4})} \left\{ \int_0^{\pi/2} \frac{dt}{\sqrt{1-x\sin^2 t}} \right\} \mathbf{1}_{(0,1)}(x) dx.$$

3. For $a = a' = 1/2$ and $b = 3/2$ we can use (3.17) and Rainville (1960) page 71, exercise 20, to get

$$\mu_{1/2,1/2,3/2}(dx) = \frac{1}{{}_3F_2(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 2, 1; 1)} \frac{4}{\pi^2 x^{1/2}(1-x)^{1/2}} \int_0^{\pi/2} \sqrt{1-x\sin^2 t} dt \mathbf{1}_{(0,1)}(x) dx.$$

4. For $a = a' = b + \frac{1}{2}\epsilon$ with $\epsilon = \pm 1$ from Rainville (1960) page 70, exercise 10 we get

$$\mu_{a,a,a+\frac{1}{2}}(dx) = \frac{2^{a-1}\Gamma(3a)}{{}_2F_1(a, a; 3a; 1/2)\Gamma(a)\Gamma(2a)} (1 + \sqrt{1-x})^{1-2a} x^{a-1} (1-x)^{a-1} \mathbf{1}_{(0,1)}(x) dx$$

and for $a > \frac{1}{2}$

$$\mu_{a,a,a-\frac{1}{2}}(dx) = \frac{2^{a-1}\Gamma(3a-1)}{{}_2F_1(a, a; 3a-1; 1/2)\Gamma(a)\Gamma(2a-1)} (1 + \sqrt{1-x})^{1-2a} x^{a-1} (1-x)^{a-3/2} \mathbf{1}_{(0,1)}(x) dx,$$

where the normalizing constants have been computed by direct integration together with Gauss formula.

5. From Rainville (1960) page 71, exercise 18 we get for $a = b = 1/2, a' = 1$

$$\mu_{1/2,1,1/2}(dx) = \frac{1}{\pi {}_3F_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; 1)} \frac{\arcsin \sqrt{x}}{x\sqrt{1-x}} \mathbf{1}_{(0,1)}(x) dx$$

and for $a = b = 1, a' = 1/2$

$$\mu_{1,1/2,1}(dx) = \frac{1}{{}_3F_2(1, 1, 1; 2, \frac{3}{2}; 1)} \frac{\arcsin \sqrt{x}}{\sqrt{x}\sqrt{1-x}} \mathbf{1}_{(0,1)}(x) dx.$$

6. Using a result from the same exercise we get the elegant case $a = a' = b = 1$

$$\mu_{1,1,1}(dx) = \frac{6}{\pi^2} \log \frac{1}{1-x} \mathbf{1}_{(0,1)}(x) \frac{dx}{x}. \quad (6.29)$$

Our main result states in this case that if $(1-X)/X \sim WX$ with $X > 0$ and W independent, the latter with Pareto distribution function $1 - (1+x)^{-1}$, for $x > 0$ (alternatively if $X \sim \frac{U}{U+(1-U)X}$ for X and U independent, the latter uniform in $(0,1)$), then X has distribution (6.29). Note that when $t > -1$, $t \neq 0$, by expanding $x \mapsto \log(1-x)$ in power series, we get $\mathbb{E}(X^t) = \frac{6}{\pi^2} \frac{1}{t} \sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+t})$. When t is an integer we get $\mathbb{E}(X^t) = \frac{6}{t\pi^2} \sum_{n=1}^t \frac{1}{n}$.

The number of interesting examples is quite large, as can be seen from Abramovitz and Stegun (1964), formulas 15.1.i for $i = 3, 4, 6, 9, 10, 12 - 18$ and 15.4.i for $i = 7 - 26$.

VII Appendix: a probabilistic proof of the Thomae formula

We give here the elements of a probabilistic proof of Thomae's result that we have mentioned in Remark 3.2. It gives a second proof of (3.20), which is essentially equivalent to our Theorem 2.1.

Theorem A.1. Consider the analytic function on \mathbb{C}^5 defined by the analytic continuation of

$$(a, b, c, d, e) \mapsto {}_3F_2(a, b, c; d, e; 1) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{n! (d)_n (e)_n}.$$

Define the 5×5 matrices

$$A = I_5 + J_5 - \begin{bmatrix} J_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad 3A^{-1} = 3I_5 + J_5 - 3 \begin{bmatrix} J_3 & 0 \\ 0 & J_2 \end{bmatrix},$$

where J_k is the (k, k) matrix whose entries are 1, and define $(a, b, c, d, e) = (x, y, z, u, v)A$. Then the function

$$E(x, y, z, u, v) = \frac{1}{\Gamma(d)\Gamma(e)\Gamma(d+e-a-b-c)} {}_3F_2(a, b, c; d, e; 1)$$

is a symmetric function of (x, y, z, u, v) .

Sketch of the proof. Let $U \sim \beta_{a,b}$ and $V \sim \beta_{c,d}$ be independent; then the distribution of UV is

$$\frac{\Gamma(a+b)\Gamma(c+d)}{\Gamma(a)\Gamma(b+c+d)} {}_2F_1(b, c+d-a; b+d; 1-x) \beta_{c,b+d}(dx)$$

by an easy calculation using Gauss formula (3.18). Notice incidentally that for $a+b=1$ this has been called by Henrici (1977) the hypergeometric distribution. The Mellin transform of such a $X = 1 - UV$ is the following function of t :

$$\mathbb{E}(X^t) = \frac{\Gamma(a+b)\Gamma(c+d)\Gamma(b+d+t)}{\Gamma(a)\Gamma(b+d)\Gamma(b+c+d+t)} \times {}_3F_2(b, c+d-a, b+d+t; b+c+d+t, b+d; 1). \quad (7.30)$$

An alternative expression for this is obtained by

$$\mathbb{E}(X^t) = \mathbb{E}((1-UV)^t) = \sum_{n=0}^{\infty} \frac{(-t)_n}{n!} \mathbb{E}(U^n) \mathbb{E}(V^n) = {}_3F_2(a, c, -t; a+b, c+d; 1), \quad (7.31)$$

since $\mathbb{E}(U^n) = (a)_n/(a+b)_n$. Next check that multiplying by A^{-1} the row vectors of arguments of the functions ${}_3F_2$ appearing in the two previous expressions we get that the one in (7.30) is obtained from the one in (7.31) by the permutation $(1, 2, 3, 4, 5) \rightarrow (4, 5, 1, 2, 3)$ of the arguments. It is then easy to realize that the symmetry of the function E is guaranteed only by the invariance under this permutation. The proof is finished, since by choosing

$$a = x, b = u - x, c = y, d = v - y, t = -z$$

the vector of arguments of ${}_3F_2$ at the r.h.s. of (7.31) is precisely (x, y, z, u, v) . \square

In order to see (3.24) as a particular case of Theorem A.1 it is enough to check that the two vectors

$$\begin{aligned} & (a+t, a, b, a+b+t, a+a')3A^{-1} \\ &= (2t-b+a+a', -t-b+a+a', -t+2b-2a+a', 2t+2b+a-2a', -t-b+a+a'), \\ & (a', a'-t, b, a'+b, a+a')3A^{-1} \\ &= (2t-b+a+a', -t-b+a+a', 2t+2b+a-2a', -t+2b-2a+a', -t-b+a+a') \end{aligned}$$

are obtained the second from the first by the transposition $(3, 4)$.

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