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ON PÓLYA MIXTURES OF MULTIVARIATE GAUSSIAN DISTRIBUTIONS

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Abstract We define a multivariate extension of the Prentice z -distribution, as a Pólya-type (infinitely divisible) mixture of multivariate Gaussian distribution. The paper obtains representations of the Lévy measure and the probability density of such mixtures in terms of series of Bessel functions.

Keywords: Mixtures of multivariate Gaussian distribution; Pólya distributions; z -distribution; Infinitely divisible distribution.

1 Introduction

The well known z -distributions, introduced in Prentice (1975) and having density functions

$$f(x) = \frac{\exp\left\{\beta_1 \left(\frac{x-\mu}{\alpha}\right)\right\}}{\alpha B(\beta_1, \beta_2) \left(1 + \exp\left\{\frac{x-\mu}{\alpha}\right\}\right)^{\beta_1+\beta_2}}, \quad x \in \mathbb{R}^1, \quad \alpha_1, \beta_1, \beta_2 > 0, \quad \mu \in \mathbb{R}^1,$$

where $B(\beta_1, \beta_2)$ denote the Euler's beta function, are important in statistical modelling (see, e.g., Barndorff-Nielsen et al.). Observing that z -distributions are particular (Pólya) mixtures of multivariate Gaussian, the paper introduce a class of general Pólya mixtures of multivariate Gaussian. Because mixing Pólya distributions, as infinite convolutions of exponential distributions, are self-decomposable, the mixtures are infinitely divisible. Using the results in Rogozin (1965) on subordinated Lévy processes and the expansion of the mixing density, obtained in Kent (1980), we derive representation of the Lévy measure and the probability density function of such mixtures in terms of series of modified Bessel functions. As a particular case we describe a class of mixtures with z -distributed marginals.

2 Pólya distributions

Let we are given a sequence $\{\lambda_k, k = 0, 1, \dots\}$ of positive numbers, $\sum_{k=0}^{\infty} \lambda_k^{-1} < \infty$, and a sequence $\{\varepsilon_k, k \geq 0\}$ of independent exponentially distributed random variables with $E\varepsilon_k = 1, k \geq 0$. Probability laws $\mathcal{L}(\sum_{k=0}^{\infty} \lambda_k^{-1} \varepsilon_k)$ we call the Pólya distributions (see Schoenberg (1951)). Because

$$Ee^{-u\varepsilon_k} = \frac{1}{1+u}, \quad u > 0,$$

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then

$$\begin{aligned} E \left[\exp \left\{ -u \sum_{k=0}^{\infty} \lambda_k^{-1} \varepsilon_k \right\} \right] &= \prod_{k=0}^{\infty} \left(\frac{\lambda_k}{\lambda_k + u} \right) \\ &= \exp \left\{ \sum_{k=0}^{\infty} \log \left(\frac{\lambda_k}{\lambda_k + u} \right) \right\} = \exp \left\{ \int_0^{\infty} (e^{-uv} - 1) \nu(\mathrm{d}u) \right\}, \quad u > 0, \end{aligned} \quad (1)$$

where

$$\nu(\mathrm{d}v) = \frac{1}{v} \sum_{k=0}^{\infty} e^{-\lambda_k v} \mathrm{d}v, \quad v > 0. \quad (2)$$

Observing that

$$v \frac{\nu(\mathrm{d}v)}{\mathrm{d}v} = \sum_{k=0}^{\infty} e^{-\lambda_k v}, \quad v > 0,$$

is a decreasing function of v , we conclude that Pólya distributions are self-decomposable and all Lévy processes $X(t)$, $t \geq 0$ with $\mathcal{L}(X(1)) = \mathcal{L}(\sum_{k=0}^{\infty} \lambda_k^{-1} \varepsilon_k)$ are subordinators, having zero drift and the Lévy measure (2) (for used terminology see, e.g., Sato (1999)).

Denote

$$\nu_k = \lambda_k \prod_{j \neq k} \left(\frac{\lambda_j}{\lambda_j - \lambda_k} \right), \quad k = 0, 1, \dots$$

It was proved in Kent (1980) that if $0 < \lambda_0 < \lambda_1 < \dots$, $\sum_{k=0}^{\infty} \lambda_k^{-1} < \infty$ and for each $c > 0$

$$|\nu_k| = O(e^{c\lambda_k}), \quad \text{as } k \rightarrow \infty, \quad (3)$$

the density function h of the Pólya distribution $\mathcal{L}(\sum_{k=0}^{\infty} \lambda_k^{-1} \varepsilon_k)$ equals

$$h(v) = \sum_{k=0}^{\infty} \nu_k e^{-\lambda_k v}, \quad v > 0. \quad (4)$$

In the special case, when $\lambda_k = \frac{1}{2}(\varkappa + k)^2 - \chi$, $\chi < \frac{1}{2}\varkappa^2$, after elementary calculations (cf. Barndorff-Nielsen et al. (1982)) we find that

$$\nu_k(\varkappa, \chi) =: \lambda_k \prod_{j \neq k} \left(\frac{\lambda_j}{\lambda_j - \lambda_k} \right) = \frac{(-1)^k \Gamma(2\varkappa + k)(\varkappa + k)}{k! \Gamma(\varkappa + \sqrt{2\chi}) \Gamma(\varkappa - \sqrt{2\chi})}$$

and

$$P \left\{ \sum_{k=0}^{\infty} \lambda_k^{-1} \varepsilon_k \in C \right\} = \int_C h_{\varkappa, \chi}(s) \mathrm{d}s, \quad C \in \mathcal{B}(R_+),$$

where $\mathcal{B}(R_+)$ is the σ -algebra of the Borel subsets of $[0, \infty)$ and

$$h_{\varkappa, \chi}(s) = \sum_{k=0}^{\infty} \nu_k(\varkappa, \chi) e^{-[\frac{1}{2}(\varkappa+k)^2 - \chi]s}, \quad s > 0.$$

Proposition 1. (cf. Barndorff-Nielsen et al. (1982)). *For all $\chi, \varkappa > 0$, $-\varkappa < \theta < \varkappa$ and $\mu \in R^1$*

$$\begin{aligned} &\int_0^{\infty} \frac{1}{\sqrt{2\pi s \alpha^2}} \exp \left\{ -\frac{(x - \theta \alpha s - \mu)^2}{2s \alpha^2} \right\} h_{\varkappa, \frac{1}{2}\theta^2}(s) \mathrm{d}s \\ &= \frac{\exp \left\{ (\theta + \varkappa) \left(\frac{x - \mu}{\alpha} \right) \right\}}{\alpha B(\varkappa + \theta, \varkappa - \theta) \left(1 + \exp \left\{ \frac{x - \mu}{\alpha} \right\} \right)^{2\varkappa}}, \quad x \in R^1, \end{aligned} \quad (5)$$

Proof. Let $K_\gamma(u)$ be the modified Bessel function of the third type, i.e.

$$K_\gamma(u) = \frac{1}{2} \left(\frac{u}{2}\right)^\gamma \int_0^\infty e^{-v-\frac{u^2}{4v}} v^{-\gamma-1} dv, \quad u > 0, \gamma \in \mathbb{R}^1. \quad (6)$$

Then, assuming for simplicity $\mu = 0$,

$$\begin{aligned} f(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi s \alpha^2}} \exp\left\{-\frac{(x-\theta \alpha s)^2}{2s \alpha^2}\right\} h_{\varkappa, \frac{1}{2}\theta^2}(s) ds \\ &= \sum_{k=0}^\infty \nu_k(\varkappa, \frac{1}{2}\theta^2) \int_0^\infty \frac{1}{\sqrt{2\pi s \alpha^2}} \exp\left\{-\frac{x^2 - 2\theta \alpha s x + \theta^2 \alpha^2 s^2}{2\alpha^2 s} - \frac{1}{2}[(\varkappa + k)^2 - \theta^2] s\right\} ds \\ &= \sum_{k=0}^\infty \nu_k(\varkappa, \frac{1}{2}\theta^2) \frac{1}{\sqrt{2\pi} \alpha} e^{\frac{\theta x}{\alpha}} \int_0^\infty s^{-\frac{1}{2}} \exp\left\{-\frac{x^2}{2\alpha^2 s} - \frac{1}{2}(\varkappa + k)^2 s\right\} ds \end{aligned} \quad (7)$$

Taking $u = \frac{x(\varkappa+k)}{\alpha}$, $\gamma = -\frac{1}{2}$, $v = \frac{1}{2}(\varkappa + k)^2 s$ and noting that (see Janke et al. (1960))

$$K_{-\frac{1}{2}}(u) = K_{\frac{1}{2}}(u) = \sqrt{\frac{\pi}{2u}} e^{-u},$$

from (6) and (7) we find that

$$\begin{aligned} f(x) &= \sum_{k=0}^\infty \nu_k(\varkappa, \frac{1}{2}\theta^2) \frac{1}{\sqrt{2\pi} \alpha} e^{\frac{\theta x}{\alpha}} \frac{\sqrt{2}}{\varkappa + k} 2\sqrt{\frac{u}{2}} K_{\frac{1}{2}}(u) \\ &= e^{\frac{\theta x}{\alpha}} \sum_{k=0}^\infty \nu_k(\varkappa, \frac{1}{2}\theta^2) \frac{1}{\alpha(\varkappa + k)} e^{-u} \\ &= \frac{1}{\alpha} e^{\frac{\theta x}{\alpha}} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{\Gamma(2\varkappa + k)}{\Gamma(\varkappa + \theta)\Gamma(\varkappa - \theta)} e^{-\frac{x(\varkappa+k)}{\alpha}} \\ &= \frac{e^{\frac{\theta - \varkappa}{\alpha} x}}{\alpha B(\varkappa + \theta, \varkappa - \theta)} \cdot \frac{1}{(e^{-\frac{x}{\alpha}} + 1)^{2\varkappa}} \\ &= \frac{\exp\{(\theta + x)\varkappa\}}{\alpha B(\varkappa + \theta, \varkappa - \theta)(1 + e^{\frac{x}{\alpha}})^{2\varkappa}}. \end{aligned}$$

□

Remark, that the formula (5), when $\varkappa = 1$ and $\theta = 0$, was proved in Andrews and Mallows (1974).

3 Pólya mixtures of the multivariate Gaussian distributions

Let $a = (a_1, \dots, a_d) \in \mathbb{R}^d$, $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{R}^d$, $A = \|a_{jk}\|_{j,k=1}^d$ be a symmetric positive definite matrix, $g_{a,A}$ be a d -dimensional Gaussian density function with a mean vector a and a covariance matrix A , i.e.,

$$g_{a,A}(x) = \frac{1}{\sqrt{|A|}(2\pi)^{d/2}} \exp\left\{-\frac{1}{2}((x-a)A^{-1}, x-a)\right\}, \quad x \in \mathbb{R}^d,$$

where $|A| = \det A$.

Definition 1. A d -dimensional distribution with the density function

$$f_d(x) = \int_0^\infty g_{va+\mu, vA}(x)h(v)dv, \quad x \in R^d, \quad (8)$$

where h is the density function of $\mathcal{L}\left(\sum_{k=0}^\infty \frac{\varepsilon_k}{\lambda_k}\right)$, is called the Pólya mixture of the multivariate Gaussian distributions.

Denote

$$h_j = \int_0^\infty v^j h(v)dv, \quad j = 1, 2, \dots$$

It is easy to check that the mean vector m of the mixture equals

$$m =: \int_{R^d} x f_d(x)dx = \int_0^\infty (av + \mu)h(v)dv = ah_1 + \mu$$

and the covariance matrix Σ of the mixture equals

$$\begin{aligned} \Sigma &=: \int_{R^d} (x - m)^T(x - m)f_d(x)dx \\ &= \int_0^\infty \left(\int_{R^d} (x - m)^T(x - m)g_{va+\mu, vA}(x)dx \right) h(v)dv \\ &= Ah_1 + a^T a(h_2 - h_1^2), \end{aligned}$$

where a^T is the transpose of a .

Proposition 2. A probability distribution, having the density function (8), is infinitely divisible with the characteristic function

$$\hat{f}_d(z) = e^{i(\mu, z)} \prod_{k=0}^\infty \left(\frac{\lambda_k}{\lambda_k + \frac{1}{2}(zA, z) - i(a, z)} \right), \quad z \in R^d, \quad (9)$$

and a triplet of the Lévy characteristics $(\gamma_d, 0, \Pi_d)$, where

$$\gamma_d = \mu + \int_{|x| \leq 1} x \Pi_d(dx), \quad (10)$$

$$\begin{aligned} \Pi_d(B) &= \int_B \frac{2 \exp\{(aA^{-1}, x)\}}{\sqrt{|A|}(2\pi)^{d/2}} \sum_{k=0}^\infty \left[\frac{2\lambda_k + (aA^{-1}, a)}{(xA^{-1}, x)} \right]^{d/4} \times \\ &\times K_{\frac{d}{2}} \left([(2\lambda_k + (aA^{-1}, a))(xA^{-1}, x)]^{1/2} \right) dx, \quad B \in \mathcal{B}(R^d \setminus \{0\}), \end{aligned} \quad (11)$$

where $\mathcal{B}(R^d \setminus \{0\})$ is the σ -algebra of the Borel subsets of $R^d \setminus \{0\}$.

Proof. We have that

$$\int_{R^d} e^{i(z, x)} g_{va+\mu, vA}(x)dx = \exp \left\{ i(\mu, z) + v \left[i(z, a) - \frac{1}{2}(zA, z) \right] \right\}, \quad z \in R^d. \quad (12)$$

Infinite divisibility of the distribution, having the density function (8), and formula (9) follow from the well-known properties of the infinitely divisible distributions and formulas (1), (2), (12).

Using the results of Rogozin (1965) (see also Sato (1999)) on subordinated Lévy processes, we find that

$$\begin{aligned}
\Pi_d(B) &= \int_B \int_0^\infty g_{va, vA}(x) \nu(dv) dx = \\
&= \int_B \int_0^\infty \frac{1}{\sqrt{|A|}(2\pi v)^{d/2}} \exp\left\{-\frac{1}{2v}((x-va)A^{-1}, x-va)\right\} \nu(dv) dx = \\
&= \int_B \frac{\exp\{(aA^{-1}, x)\}}{\sqrt{|A|}(2\pi)^{d/2}} \int_0^\infty v^{-\frac{d}{2}-1} \exp\left\{-\frac{1}{2v}(xA^{-1}, x)\right\} \sum_{k=0}^\infty \exp\left\{-v\left(\lambda_k + \frac{1}{2}(aA^{-1}, a)\right)\right\} dv, \quad B \in \mathcal{B}(R^d \setminus \{0\}), \tag{13}
\end{aligned}$$

because

$$((x-va)A^{-1}, x-va) = (xA^{-1}, x) - 2v(aA^{-1}, x) + v^2(aA^{-1}, a).$$

Formulas (10), (11) now follow from (13) after elementary transforms with application of the classical formula (6). \square

Proposition 3. *Let the assumption (3) holds. Then the density f_d of the Pólya mixture of the multivariate Gaussian distributions is equal*

$$\begin{aligned}
f_d(x) &= \frac{2 \exp\{(aA^{-1}, x - \mu)\}}{(2\pi)^{d/2} \sqrt{|A|}} \sum_{k=0}^\infty \nu_k \left(\frac{2\lambda_k + (aA^{-1}, a)}{((x - \mu)A^{-1}, x - \mu)} \right)^{\frac{d-2}{4}} \times \\
&\quad \times K_{\frac{d-2}{2}} \left(\left([2\lambda_k + (aA^{-1}, a)]((x - \mu)A^{-1}, x - \mu) \right)^{1/2} \right), \quad x \in R^d. \tag{14}
\end{aligned}$$

Proof. Indeed, using formulas (4) and (6), we find that

$$\begin{aligned}
f_d(x) &= \int_0^\infty g_{\mu+va, vA}(x) h(v) dv = \\
&= \int_0^\infty \frac{1}{\sqrt{|A|}(2\pi v)^{d/2}} \exp\left\{-\frac{1}{2v}((x-\mu-va)A^{-1}, x-\mu-va)\right\} \sum_{k=0}^\infty \nu_k e^{-\lambda_k v} dv = \\
&= \frac{\exp\{(aA^{-1}, x - \mu)\}}{\sqrt{|A|}(2\pi)^{d/2}} \sum_{k=0}^\infty \nu_k \int_0^\infty v^{-\frac{d}{2}-1} \times \\
&\quad \times \exp\left\{-\frac{1}{2v}((x-\mu)A^{-1}, x-\mu) - v\left(\lambda_k + \frac{1}{2}(aA^{-1}, a)\right)\right\} dv = \\
&= \frac{\exp\{(aA^{-1}, x - \mu)\}}{\sqrt{|A|}(2\pi)^{d/2}} \sum_{k=0}^\infty \nu_k \left(\lambda_k + \frac{1}{2}(aA^{-1}, a)\right)^{\frac{d-2}{2}} \int_0^\infty v^{-\frac{d}{2}-1} \times
\end{aligned}$$

$$\begin{aligned}
& \times \exp \left\{ -v - \frac{1}{4v}((x - \mu)A^{-1}, x - \mu)(2\lambda_k + (aA^{-1}, a)) \right\} dv = \\
& = \frac{2 \exp\{(aA^{-1}, x - \mu)\}}{\sqrt{|A|}(2\pi)^{d/2}((x - \mu)A^{-1}, x - \mu)^{\frac{d-2}{4}}} \sum_{k=0}^{\infty} \nu_k \left(2\lambda_k + (aA^{-1}, a) \right)^{\frac{d-2}{4}} \times \\
& \times K_{\frac{d-2}{2}} \left([2\lambda_k + (aA^{-1}, a)((x - \mu)A^{-1}, x - \mu)]^{\frac{1}{2}} \right).
\end{aligned}$$

□

4 On d -dimensional z -distributions

Definition 2. An infinitely divisible d -dimensional distribution with z -distributed one-dimensional marginals we call a d -dimensional z -distribution.

Trivial examples of such distributions provide the probability laws of random vectors with independent z -distributed components.

Denote $\hat{a} = (\sqrt{a_{11}}, \dots, \sqrt{a_{dd}})$.

Proposition 4. A probability distribution, defined by the density function

$$f_d(x) = \int_0^{\infty} g_{v\theta\hat{a}+\mu, vA}(x) h_{\varkappa, \frac{1}{2}\theta^2}(v) dv, \quad x \in R^d, \quad |\theta| < \varkappa, \quad \varkappa > 0, \quad (15)$$

is a d -dimensional z -distribution, having the characteristic function

$$\hat{f}_d(z) = e^{i(\mu, z)} \frac{\Gamma(\varkappa - \sqrt{\theta^2 - (zA, z) + 2i\theta(\hat{a}, z)})\Gamma(\varkappa + \sqrt{\theta^2 - (zA, z) + 2i\theta(\hat{a}, z)})}{\Gamma(\varkappa - \theta)\Gamma(\varkappa + \theta)}, \quad (16)$$

$$z \in R^d,$$

and a triplet of the Lévy characteristics, given by the formulas (10), (11) with $a = \theta\hat{a}$, $\lambda_k = \frac{1}{2}[(\varkappa + k)^2 - \theta^2]$, $k = 0, 1, \dots$, $|\theta| < \varkappa$, $\varkappa > 0$.

The density function (15) is representable by means of the formula (14) with $\nu_k = \nu_k(\varkappa, \frac{1}{2}\theta^2)$, $a = \theta\hat{a}$ and $\lambda_k = \frac{1}{2}[(\varkappa + k)^2 - \theta^2]$, $k = 0, 1, \dots$, $|\theta| < \varkappa$, $\varkappa > 0$.

Proof. These statements follow directly from the Propositions 1, 3 and 4, observing that the one-dimensional marginals $f_d^{(j)}$ of a probability distribution, defined by (15), are equal

$$f_d^{(j)}(x_j) = \int_0^{\infty} g_{v\theta\sqrt{a_{jj}}+\mu_j, va_{jj}}(x_j) h_{\varkappa, \frac{1}{2}\theta^2}(v) dv, \quad j = 1, \dots, d,$$

and

$$\begin{aligned}
& \prod_{k=0}^{\infty} \left(\frac{\frac{1}{2}((\varkappa + k)^2 - \theta^2)}{\frac{1}{2}((\varkappa + k)^2 - \theta^2) + \frac{1}{2}(zA, z) - i\theta(\hat{a}, z)} \right) = \\
& \prod_{k=0}^{\infty} \frac{(k + \varkappa - \theta)(k + \varkappa + \theta)}{(k + \varkappa - \sqrt{\theta^2 - (zA, z) + 2i\theta(\hat{a}, z)})(k + \varkappa + \sqrt{\theta^2 - (zA, z) + 2i\theta(\hat{a}, z)}}.
\end{aligned}$$

Now it remains to apply formulas (1), (9) and the well-known formula:

$$\prod_{k=0}^{\infty} \frac{(k+b_1)(k+b_2)}{(k+c_1)(k+c_2)} = \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(b_2)},$$

where $b_1 + b_2 = c_1 + c_2$. □

For example, taking $d = 3$, $a = 0$, $\lambda_k = \frac{1}{2}(\varkappa + k)^2$ and having in mind that

$$K_{\frac{3}{2}}(u) = \left(\frac{\pi}{2u}\right)^{1/2} e^{-u} \left(1 + \frac{1}{u}\right), \quad u > 0,$$

we find after elementary summation that

$$\begin{aligned} \Pi_3(B) &= \frac{2}{(2\pi)^{3/2}\sqrt{|A|}} \int_B \sum_{k=0}^{\infty} (\varkappa + k)^{3/2} (\Delta(x))^{-\frac{3}{2}} K_{\frac{3}{2}}((\varkappa + k)\Delta(x)) dx = \\ &= \frac{1}{2\pi\sqrt{|A|}} \int_B \left[(\Delta(x))^{-3} e^{-\varkappa\Delta(x)} (1 - e^{-\Delta(x)})^{-1} + \right. \\ &\quad \left. + (\Delta(x))^{-2} e^{-\varkappa\Delta(x)} (\varkappa + (1 - \varkappa)e^{-\Delta(x)})(1 - e^{-\Delta(x)})^{-2} \right] dx, \quad B \in \mathcal{B}(R^3 \setminus \{0\}), \end{aligned}$$

where $\Delta(x) = \sqrt{(xA^{-1}, x)}$.

In this case from (14) it follows that

$$\begin{aligned} f_3(x) &= \frac{1}{(2\pi)^{3/2}\sqrt{|A|}} \sum_{k=0}^{\infty} \nu_k(\varkappa, 0) \left(\frac{k + \varkappa}{\Delta(x - \mu)}\right)^{1/2} \left(\frac{\pi}{2(k + \varkappa)\Delta(x - \mu)}\right)^{1/2} e^{-(\varkappa+k)\Delta(x-\mu)} \\ &= \frac{1}{4\pi\sqrt{|A|}\Delta(x - \mu)} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(2\varkappa + k)(\varkappa + k)}{k!\Gamma^2(\varkappa)} e^{-(\varkappa+k)\Delta(x-\mu)} \\ &= \frac{\varkappa\Gamma(2\varkappa)}{4\pi\sqrt{|A|}\Gamma^2(\varkappa)\Delta(x - \mu)} \frac{e^{(\varkappa+1)\Delta(x-\mu)} - e^{\varkappa\Delta(x-\mu)}}{(1 + e^{\Delta(x-\mu)})^{2\varkappa+1}}, \quad x \in R^3. \end{aligned}$$

Remark 1. The probability laws of random vectors with independent z -distributed components are not contained in the class of z -distributions, having the characteristic function (16). The problem of characterization of all d -dimensional z -distributions in terms of their Lévy characteristics is still open.

Remark 2. Let $X(t)$, $t \geq 0$, be a Lévy process in R^d such that $\mathcal{L}(X(1))$ is a d -dimensional z -distribution. Following Grigelionis (2001), the distributions $\{\mathcal{L}(X(t)), t > 0\}$ we call the multivariate generalized z -distributions.

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References

Andrews, D.F., Mallows, C.L., 1974. Scale mixtures of normal distributions. *J. R. Statist. Soc.*, B 36, 99–102.

- Barndorff-Nielsen, O., Kent, J., Sorensen, M., 1982. Normal variance-mean mixtures and z -distributions. *Intern. Statist. Review* 90, 141–153.
- Grigelionis, B., 2001. Generalized z -distribution and related stochastic processes. *Liet. Matem. Rink.* 41, Nr. 3, 303–319.
- Janke, E., Emde, F., Lösch, F., 1960. *Tafeln Höherer Funktionen*, Teubner, Stuttgart.
- Kent, J.T., 1980. Eigenvalue expansion for diffusion hitting times. *Z. Wahr. verw. Gebiete* 52, 309–319.
- Prentice, R.L., 1975. Discrimination among some parametric models. *Biometrika* 62, 607–614.
- Rogozin, B.A., 1965. On some classes of processes with independent increments. *Theory Probab. Appl.* 10, 479–483.
- Sato, K., 1999. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge.
- Schoenberg, I.J., 1951. On Pólya frequency functions I ; the totally positive functions and their Laplace transforms. *J. Anal. Math. Jerusalem* 1, 331–374.