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# AN INTERMEDIATE TARGETS METHOD FOR TIME PARALLELIZATION IN OPTIMAL CONTROL

YVON MADAY, JULIEN SALOMON, AND KAMEL RIAHI

ABSTRACT. In this paper, we present a method that enables to solve in parallel the Euler-Lagrange system associated with the optimal control of a parabolic equation. Our approach is based on an iterative update of a sequence of intermediate targets and gives rise independent sub-problems that can be solved in parallel. Numerical experiments show the efficiency of our method.

Dans cet article, on présente une méthode permettant une parallélisation en temps de la résolution des équations d'Euler-Lagrange associées à un problème de contrôle optimal dans le cas parabolique. Notre approche est basée sur une mise à jour itérative de cibles intermédiaires et donne lieu à des sous-problèmes de contrôle indépendants. Les résultats numériques prouvent l'efficacité de la méthode.

## 1. INTRODUCTION:

In the last decade, time domain decomposition has been exploited to accelerate the simulation of systems ruled by time dependent partial differential equations [1]. Among others, the parareal algorithm [5] or multi-shooting schemes [4] have shown excellent results. In the framework of optimal control, this approach has been used to control parabolic systems [2], [7]. In this paper, we introduce a new approach to tackle such problems. The strategy we follow is based on the concept of target trajectory that has been introduced in the case of hyperbolic systems in [6]. Because of the irreversibility of parabolic equations, a new definition of this trajectory is considered. It enables us to define at each bound of the time sub-domains relevant initial conditions and intermediate targets, so that the initial problem is split into independent optimization problems.

We now introduce some notations. Given  $d \in \mathbb{N}$ , we consider the optimal control problem associated with a heat equation defined on a compact set  $\Omega \subset \mathbb{R}^d$  and a time interval interval  $I = [0, T]$ . The control applies on a subset  $\Omega_c \subset \Omega$  and belongs to the Hilbert space  $\mathcal{H} := L^2(I; L^2(\Omega_c))$  whose scalar product and norm are denoted by  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  and  $\|\cdot\|_{\mathcal{H}}$  respectively. We also denote by  $\|\cdot\|$  and  $\|\cdot\|_c$  the norms associated with the spaces  $L^2(\Omega)$  and  $L^2(\Omega_c)$  respectively. Given a function  $\varphi$  defined on  $I \times \Omega$ , we denote its restriction to a sub-domain  $I' \times \Omega \subset I \times \Omega$  by  $\varphi|_{I'}$ .

The paper is organized as follows. The optimal control problem is presented in Sec. 2 ; Sec. 3 is devoted to the description of the time parallelization setting. The algorithm is given in Sec. 4. Finally, numerical results are presented in Sec. 5.

## 2. OPTIMAL CONTROL PROBLEM

Given  $\alpha > 0$ ,  $T > 0$ , a target state  $y_{target} \in L^2(\Omega)$ , we consider the optimal control problem : Find  $v^* \in \mathcal{H}$

$$v^* := \operatorname{argmin}_{v \in \mathcal{H}} J(v),$$

where  $J$  is the quadratic cost functional defined by

$$(1) \quad J(v) := \frac{1}{2} \|y(T) - y_{target}\|^2 + \frac{\alpha}{2} \|v\|_{\mathcal{H}}^2.$$

The state variable  $y$  depends linearly on the control  $v$  through the evolution equation

$$(2) \quad \partial_t y - \nu \Delta y = Bv,$$

with an initial data  $y_0 \in L^2(\Omega)$ . In this equation  $\Delta$  denotes the Laplace operator,  $\nu$  is the diffusion coefficient and  $B$  is the injection from  $L^2(\Omega_c)$  to  $L^2(\Omega)$ , so that  $Bv \in L^2(I; L^2(\Omega))$ .

The corresponding optimality system reads

$$(3) \quad \begin{cases} \partial_t y - \nu \Delta y &= Bv & \text{on } I \times \Omega \\ y(0) &= y_0, \end{cases}$$

$$(4) \quad \begin{cases} \partial_t p + \nu \Delta p &= 0 & \text{on } I \times \Omega \\ p(T) &= y(T) - y_{target}, \end{cases}$$

$$(5) \quad \alpha v + B^* p = 0,$$

where  $B^*$  is the adjoint operator of  $B$ .

Note that for any  $\alpha > 0$ , the functional  $J$  is strictly convex, so that the system (3–5) has a unique solution  $v^*$ . We denote by  $y^*$ ,  $p^*$  the associated state and adjoint state.

## 3. TIME PARALLELIZATION SETTING

We now aim at solving in parallel the coupled system corresponding to Equations (3–5). To do this, we decompose the interval  $I$  into subintervals and introduce a set of intermediate target states so that the initial problem is replaced by a set of smaller independent optimal control problems. The resolution of these problems is achieved using an inner loop, whereas the intermediate states are updated by an outer loop.

We start with the definition of the target states. Given a control  $v \in \mathcal{H}$  and its corresponding state  $y$  and adjoint state  $p$ , we define the *target trajectory* by:

$$(6) \quad \chi(v) = y(v) - p(v) \quad \text{on } I \times \Omega$$

In what follows and for the sake of simplicity, we omit the dependence of  $y$ ,  $p$  and  $\chi$  on the control  $v$  in the notations. The introduction of this trajectory is motivated by the following result.

**Lemma 1.** *We keep the previous notations. Let  $\tau \in ]0, T[$ , and the optimal control problem: Find  $w_\tau^* \in \mathcal{H}$  such that*

$$w_\tau^* := \operatorname{argmin}_{w \in \mathcal{H}} J_\tau(w),$$

where

$$J_\tau(w) := \frac{1}{2} \|y(\tau) - \chi^*(\tau)\|^2 + \frac{\alpha}{2} \|w\|_{L^2([0, \tau]; L^2(\Omega_c))}^2$$

with  $y(\tau)$  the solution of the Equation (2). We have

$$w_\tau^* = v_{|[0,\tau]}^*$$

**Proof:** Thanks to the uniqueness of the solution of the optimization problem associated to  $J_\tau$ , it is sufficient to show that  $v_{|[0,\tau]}^*$  is a solution of its optimality system. This one is given by Equation (3) (with  $v = w_\tau$ ) restricted to  $[0, \tau] \times \Omega$  and

$$(7) \quad \begin{cases} \partial_t \tilde{p} + \nu \Delta \tilde{p} & = 0 & \text{on } [0, \tau] \times \Omega \\ \tilde{p}(\tau) & = y(\tau) - \chi(\tau), \end{cases}$$

$$(8) \quad \alpha w_\tau + B^* \tilde{p} = 0.$$

First, note that  $y_{|[0,\tau]}^*$  obviously satisfies Equation (3) restricted to  $[0, \tau] \times \Omega$  with  $v = v_{|[0,\tau]}^*$ . It directly follows from the definition of  $\chi^*$  (see (6)), that:

$$p^*(\tau) = y^*(\tau) - \chi^*(\tau),$$

so that  $p_{|[0,\tau]}^*$  satisfies (7). Finally, Equation (8) is a consequence of (5). The result follows.  $\square$

Given  $N \geq 1$ , we decompose the interval  $I = [0, T]$  into subintervals  $I = \cup_{n=0}^{N-1} I_n$ , where  $I_n = [t_n, t_{n+1}]$ ,  $t_0 = 0 < t_1 < \dots < t_{N-1} < t_N = T$ . We also introduce the spaces  $\mathcal{H}_n := L^2(I_n; L^2(\Omega_c))$  and the corresponding scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{H}_n}$  and norm  $\|\cdot\|_{\mathcal{H}_n}$ . In this framework, given  $v_n \in \mathcal{H}_n$  we define  $v_n^*$  as follows

$$(9) \quad v_n^* := \operatorname{argmin}_{v_n \in \mathcal{H}_n} J_n^v(v_n),$$

with

$$(10) \quad J_n^v(v_n) := \frac{1}{2} \|y_n(t_{n+1}) - \chi(t_{n+1})\|^2 + \frac{\alpha}{2} \|v_n\|_{\mathcal{H}_n}^2,$$

where  $\chi$  is associated to  $v$  through the definition (6). In this functional, the state  $y_n$  is defined by

$$(11) \quad \begin{cases} \partial_t y_n - \nu \Delta y_n & = B v_n & \text{on } I_n \times \Omega \\ y_n(t_n) & = y(t_n). \end{cases}$$

These subproblems have the same structure as the original one and are also strictly convex. Note also that their definitions depend on the control  $v$  through the target trajectory, hence the notation  $J_n^v$ .

The optimality system associated with these minimization problems are given by Equation (11) and

$$(12) \quad \begin{cases} \partial_t p_n + \nu \Delta p_n & = 0 & \text{on } I_n \times \Omega \\ p_n(t_{n+1}) & = y(t_{n+1}) - \chi(t_{n+1}), \end{cases}$$

$$(13) \quad \alpha v_n + B^* p_n = 0.$$

**Lemma 2.** *We keep the previous notations. Denote by  $\chi^*$  the target trajectory defined by Equation (6) with  $y = y^*$  and  $p = p^*$  and by  $y_n^*, p_n^*, v_n^*$  the solutions of Equations (11–13) associated with  $v^*$ . One has:*

$$v_n^* = v_{|I_n}^*.$$

The proof of this result follows the lines of Lemma 1 and is left as an exercise to reader.

#### 4. ALGORITHM

We are now in the position to propose a time parallelized procedure to solve Equations (3)–(5). Consider an initial control  $v^0$  and assume that, at step  $k$ , a control  $v^k$  is known. The computation of  $v^{k+1}$  is achieved as follows:

- (1) Compute  $y^k$ ,  $p^k$  and the associated target trajectory  $\chi^k$  according to Equations (3), (4) and (6) respectively.
- (2) For  $n = 0, \dots, N - 1$ , solve the sub-problems (9) in parallel and denote by  $\tilde{v}_n^{k+1}$  the corresponding solutions.
- (3) Define  $\tilde{v}^{k+1}$  as the concatenation of the sequence  $(\tilde{v}_n^{k+1})_{n=0, \dots, N-1}$ .
- (4) Update the control variable by

$$(14) \quad v^{k+1} = v^k + \theta_k(\tilde{v}^{k+1} - v^k),$$

where the value  $\theta_k$  is chosen to minimize  $J(v^k + \theta_k(\tilde{v}^{k+1} - v^k))$ .

We have not detailed Step 2 as we rather aim at presenting a structure for a general approach. However, because of the strict convexity of the problems we consider, a small number of conjugate gradient method steps can be used to achieve the resolution of these steps.

#### 5. NUMERICAL EXPERIMENTS

In this section, we test the efficiency of our method. We consider a 2D example, where  $\Omega = [0, 1] \times [0, 1]$  and  $\Omega_c = [\frac{1}{3}, \frac{2}{3}] \times [\frac{1}{3}, \frac{2}{3}]$ . The parameters related to our control problem are  $T = 6.4$ ,  $\alpha = 10^{-2}$  and  $\nu = 10^{-2}$ . The time interval is discretized using a uniform step  $\delta t = 10^{-3}$ , and an Implicit-Euler solver is used to approximate the solution of Equations (3–4). For the space discretization, we use  $\mathbb{P}_1$  finite elements. Our implementation make use of the freeware FreeFem [8] and the parallelization is achieved thanks to the Message Passing Interface library. The independent optimization procedures required in Step 2 are simply carried out using one step of an optimal step gradient method. The results are presented in Figure 1.

In the first plot, we consider the evolution of the cost functional values with respect to the iterations and do not take into account the parallelization. The result reveals that our algorithm significantly accelerates the optimization process. This outcome may indicates that the splitting introduced in our approach acts as a preconditionner during the numerical optimization. This will be the purpose of some further investigation [9], in the same spirit as in [7].

In a second plot, we represent the evolution of the cost functional values with respect to the number of matrix-vector product. Parallel computations that are done in Step 2 are only counted once. When comparing with a standard optimal gradient step method, we observe speed-up approximatively equal to 3.

#### REFERENCES

- [1] J.-L. LIONS, Virtual and effective control for distributed systems and decomposition of everything *J. Anal. Math.* vol. 80 numero 257-297 , 2000.

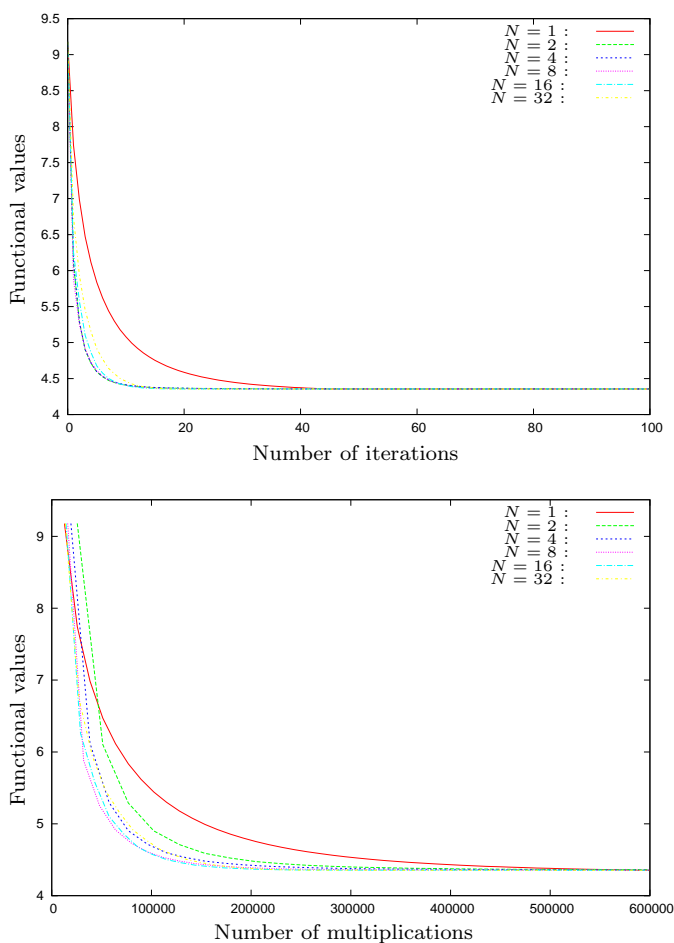


FIGURE 1. Functional values evolution, with respect to the number of iterations (top) and multiplications (bottom).

- [2] Y. MADAY, G. TURINICI, A parareal in time procedure for the control of partial differential equations , *C. R. Math. Acad. Sci. Paris* 335, vol. 4, numero 387-392, 2002.
- [3] G. BAL , Y. MADAY A parareal time discretization for non-linear PDEs with application to the pricing of an american put , *Springer, Lect Notes Comput. Sci. Eng.* , numero 189-202, 2002.
- [4] A. BELLEN, M. ZENARO Parallel algorithms for initial value problems for nonlinear vector difference and differential equations , *J. Comput. Appl. Math.*, vol. 25, numero 341-350, 1989.
- [5] J.-L. LIONS , Y. MADAY, G TURINICI, Résolution d'EDP par un schéma pararréel , *C. R. Acad. Sci. Paris*, vol. I 332 , numero 661-668, 2001.
- [6] Y. MADAY, J. SALOMON, G. TURINICI, Parareal in time control for quantum systems , *SIAM J. Num. Anal.*, vol. 45 (6), numero 2468-2482, 2007.
- [7] T. P. MATHEW, M. SARKIS, C. E. SCHAEERER, Analysis of block parareal preconditioners for parabolic optimal control problems , *SIAM J. Sci. Comp.*, vol. 32 (3), numero 1180-1200, 2010.
- [8] O. PIRONNEAU, F. HECHT, K. OHTSUKA, Free soft : FreeFem++-mpi , <http://www.freefem.org>.

[9] M.-K. RIAHI, Thèse de doctorat de l'université Pierre et Marie Curie, Paris6 , *To appear*

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