On a robustness property in single-facility location in continuous space

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Abstract We consider a single–facility location problem in continuous space, here the problem of minimizing a sum or the maximum of the possibly weighted distances from a facility to a set of points of demand. The main result of this paper shows that every solution (optimal facility location) of this problem has an interesting robustness property. Any optimal facility location is the most robust in the following sense: given a suitable highest admissible cost, it allows the greatest perturbation of the locations of the demand without exceeding this highest admissible chosen cost.

Keywords Single–Facility Location · Robustness · Fermat–Weber problem · 1–center problem

Mathematics Subject Classification (2000) MSC 90B85

1 Introduction

Our work is concerned with the classical single–facility location problem on continuous space, with discrete demand and mixed norms. This problem comes from a particular approach of the more general following decision problem: one has to choose the location of a facility serving the demand of a discrete set of points. The problem is parameterized by the *data* (description of the demand) and this decision has a *cost*. For generalities and more about facility location problem, one can consult Drezner (1995) and Drezner and Hamacher (2002) and references therein.

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Perhaps the most immediate concern about this problem was/is to minimize the cost which leads to an enormous literature devoted to various aspects of this question (see above for references). We will call it the initial problem.

Particular instances of this problem are the Fermat–Weber problem (1–median problem) and the 1–center problem (see below). Recall also that the mathematical problem behind appears in Descriptive Data Summarization where the optimal facility location can be interpret as a measure of the central tendency of our data (the demand), it has been used, for example, in Cluster Analysis.

Minimizing the cost is an important purpose but other considerations may lead to different concerns. In this paper, we propose to take into account uncertainty in the *location* of the demand. In facility location problem, points of demand can be derived from aggregation, as gravity centers of some districts. Having in mind that facility location decisions are very long term decisions, it seems reasonnable to be concerned with demographic changes which may cause displacement of these gravity centers. Secondly, in descriptive data summarization, some noise in the data can be assimilated as changing in the position.

All this considerations can make interesting looking for a facility location that tolerate the greatest perturbations in the locations of the demand in the sense that the cost doesn't exceed a given highest admissible cost. This highest admissible cost can be chosen with the help of the value of the initial single–facility location problem evocated above. This is our robust counterpart of the initial problem.

Our main result is that, a bit surprisingly, in the single–facility location problem considered, both the initial and the robust counterpart problems have the same solution. In other words, the solutions of our initial single facility location problem exhibit a kind robustness property.

We situate our work in the continuation of Carrizosa and Nickel (2003) where Carrizosa and S. Nickel introduced in Locational Analysis this very natural and interesting idea of robustness, appearing yet in robust control and which has been also developed in infogap theory (see below for references). In their work, they consider a similar problem but paying attention to uncertainty on the demand.

Let us add one last motivation: as said above, the classical, old and intensively studied Fermat–Weber problem is one instance of the problem considered in our work so, our main result gives a new (essentially qualitative) property of this problem.

The plan of the paper is as follows: in the next section, the single–facility location problem in a continuous space with mixed norms is described and we give some elements about robustness which leads to consider the maximization of the *radius of robustness*. In the section 2, we give the main Theorem which asserts that maximizing the radius of robustness in the single–facility location problem with mixed norms is equivalent to the initial problem which means that the solutions of this problem present a certain robustness property.

2 Preliminaries

Consider $n \in \mathbb{N}^\star$ points of demand, numeroted from 1 to n. Let $x = (x_1, \ldots, x_n)$ represents the location of the demand (where $x_i \in \mathbb{R}^d$ is the location of the i–th point of demand), $y \in \mathbb{R}^d$ be the location of the facility. Satisfaction of the demand by the facility has a cost that takes the general form

$$C(x,y) = N(\gamma_1(x_1 - y), \dots, \gamma_n(x_n - y)) \tag{1}$$

where N is an absolute norm on $\mathbb{R}^n$ (one can consult Bauer et al. (1961) for more details about absolute norms.), i.e. a norm which satisfies:

$$\forall (v_1, \dots, v_n) \in \mathbb{R}^n, \quad N(v_1, \dots, v_n) = N(|v_1|, \dots, |v_n|),$$

and, for $i = 1, \dots, n$, γ_i is a norm on $\mathbb{R}^d$. In all the sequel, if γ is a norm on a vectorial space E, the unit ball associated with γ is $B_\gamma = \{x \in E \mid \gamma(x) \leq 1\}$ and the distance associated with γ on E, is $d_\gamma : (x,y) \to \gamma(x-y)$.

This setting is convenient to include some classical cases we are going to describe briefly now.

When $N = N_1 : (v_1, \dots, v_n) \to \sum_{i=1,\dots,n} |v_i|$, the ℓ^1–norm on $\mathbb{R}^n$, and, for $i = 1, \dots, n$, $\gamma_i = \omega_i |.|_i$, where $\omega_i > 0$ is the demand at the ith–point and the distance from the ith point is measured by the distance associated with a norm $|.|_i$ or will be approximated by the distance associated with a norm $|.|_i$ (see Brimberg and Love (1995) and references therein), the cost

$$C(x,y) = \sum_{i=1,\dots,n} \gamma_i(x_i - y) = \sum_{i=1,\dots,n} \omega_i |x_i - y|_i$$

can be seen as a total transportation cost. It is a cost associated with an idea of *efficacy/efficiency*.

When $N = N_\infty : (v_1, \dots, v_n) \to \max_{i=1,\dots,n} |v_i|$, the ℓ^∞–norm on $\mathbb{R}^n$, the cost

$$C(x,y) = \max_{i=1,\dots,n} \gamma_i(x_i - y)$$

is the maximal distance of a point of demand from the facility. It is a cost associated with an idea of *equity*,

In all the sequel, the norms γ_i, $i = 1, \dots, n$, are fixed and the norm N in the cost (1) equals N_1 or N_∞.

If we set

$$W \overset{\text{not.}}{=} N \circ \gamma_1 \times \cdots \times \gamma_n : (v_1, \dots, v_n) \to N(\gamma_1(v_1), \dots, \gamma_n(v_n)),$$

using the fact that N is absolute, W is a norm on $(\mathbb{R}^d)^n$ and the cost given by (1) is, for $x = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$, $y \in \mathbb{R}^d$ and setting $\hat{y} = (y, \dots, y) \in (\mathbb{R}^d)^n$,

$$C(x,y) = W(x - \hat{y}).$$

As said in the introduction, our initial problem is

$$\text{Minimize } \; C(x,y) = W(x - \hat{y}), \quad y \in \mathbb{R}^d. \tag{$\mathscr{P}_x$}$$

As noted in Carrizosa and Fliege (2002), it is interesting to note that solving ($\mathscr{P}_x$) is equivalent to solve the following problem

$$\text{Minimize } \; W(x - y), \quad y \in \Delta, \tag{$\mathscr{P}'_x$}$$

where $\Delta = \{(z,...,z) \,|\, z \in \mathbb{R}^d\} \subset (\mathbb{R}^d)^n$. So the problem $(\mathscr{P}_x)$ is equivalent to a *projection problem* of the data x on a subspace Δ of $(\mathbb{R}^d)^n$ using the distance associated with the norm $W = N \circ \gamma_1 \times \cdots \times \gamma_n$.

One can refer to Plastria (1995), Drezner et al. (2002), Hansen et al. (1980), Durier (1992), Durier (1995) and references therein for more details about the single–facility location problem considered here.

Now suppose that we want to take into account some (deterministic) uncertainty about the locations of the demand described by x. More precisely, suppose that, once the facility location y chosen on the basis of locations of the demand x, there may be some variations on the locations of the demand (from the reference $\overline{x}$). *A priori*, just choosing y minimizing the criterion $y \to C(\overline{x}, y)$ seems unsatisfying.

Without going into too much detail, to better understand the approach considered in Carrizosa and Nickel (2003), we find instructive to develop some general considerations around robustness.

The matter discussed until the end of this section being largely independent of our facility location decision problem and because it implies only very light modifications, we adopt an abstract framework in order to both show the real scope of the concepts and ideas introduced, and better situate the different approaches of robustness. The decision problem is the following:

$(\mathscr{D}_x)$ Given *data* x in a set X, choose an element y in a set Y knowing that it generates a *cost* $C(x,y) \in \overline{\mathbb{R}}_+$.

Of course, we will seek a decision satisfying some properties that we consider desirable. Probably one of the most classical procedure to solve $(\mathscr{D}_x)$ and choose y consists in minimizing the cost:

$$\text{Minimize } C(x,y), \quad y \in Y. \qquad (\mathscr{P}_x)$$

But in the decision problem $(\mathscr{D}_x)$, you can choose just to *control* the cost.

Given a maximal admissible cost $\alpha \in \mathbb{R}_+$, in presence of uncertainty, supposing that (X,d) is a metric space and that the *true* data x' belongs to the open ball $\mathring{B}(x,r) = \{z \in X \,|\, d(x,z) < r\}$ for some $r > 0$, this naturally leads to the *robust* parameterized feasibility problem

$$\text{Find } y \in Y \text{ such that } \mathring{B}(x,r) \subset [C(.,y) \leq \alpha] \qquad (2)$$

where $[C(.,y) \leq \alpha] = \{x' \in X \,|\, C(x',y) \leq \alpha\}$.

Now you can introduce optimization in the problem (2) by two very natural ways.

A first approach consists in optimizing the maximal cost. This leads to the following problem, fixing $x \in X$, $r > 0$:

$$\text{Minimize } \alpha, \quad (y,\alpha) \in Y \times \overline{\mathbb{R}}_+ \text{ satisfying } \mathring{B}(x,r) \subset [C(.,y) \leq \alpha] \qquad (3)$$

It is easily seen that $(\overline{y}, \overline{\alpha})$ is a solution of (3) if and only if $\overline{y}$ is solution of

$$\text{Minimize } \sup_{x' \in \mathring{B}(x,r)} C(x',y), \quad y \in Y, \qquad (4)$$

and $\overline{\alpha} = \sup_{x' \in \mathring{B}(x,r)} C(x', \overline{y})$.

This is probably the most classical procedure to solve $(\mathscr{D}_x)$ *robustly* and choose y in this context consists in minimizing the cost in *the worst case*. Allowing a more general uncertainty set then an open ball leads to a problem sometimes called the *robust counterpart* of the initial problem $(\mathscr{P}_x)$.

This method, often named *robust optimization*, has been intensely studied for many continuous optimization problems, see Ben-Tal et al. (2009), Kouvelis and Yu (1997) and references therein.

Another way to introduce optimization in the problem (2) is, for data $x \in X$, to fix a maximal cost $\alpha \in \mathbb{R}_+$ and to maximize the radius of the perturbation.

This leads to the following optimization problem:

$$\text{Maximize } r, \quad (y,r) \in Y \times \mathbb{R}_+ \text{ satisfying } \mathring{B}(x,r) \subset [C(.,y) \leq \alpha] \tag{5}$$

and it is easy to check that $(\overline{y}, \overline{r})$ is a solution of (5) if and only if $\overline{y}$ is a solution of

$$\text{Maximize } \rho(y;x,\alpha) = d(x, [C(.,y) > \alpha]), \quad y \in Y, \tag{$\mathscr{R}_x$}$$

and $\overline{r} = \rho(\overline{y}; x, \alpha)$.

This is the approach considered in Carrizosa and Nickel (2003). As far as we know, it is a particular case of a general method known on one hand as *infogap theory* and developped essentially by Y. Ben-Haim since the end of the 90's (see Ben-Haim (2001) and references therein), and on the other hand, some similar considerations have been made in *robust control*, beginning with the problem of maximizing the *radius of stability* of linear systems (see Hinrichsen et al. (1990), Hinrichsen and Pritchard (1990)). So, this way to consider robustness leads to the introduction of what we call *the radius of robustness* $\rho(y;x,\alpha) = d(x, [C(.,y) > \alpha])$ called itself *robustness* in Carrizosa and Nickel (2003), *robustness function* or *stability radius* in previous works around this method.

In the sequel, the problem $(\mathscr{R}_x)$ will be our robust counterpart of the initial problem $(\mathscr{P}_x)$.

In Carrizosa and Nickel (2003), they consider uncertainty on the weights $\omega = (\omega_1, ..., \omega_n) \in \mathbb{R}^n_+$ ($Y = S \subset \mathbb{R}^2$, $X = \mathbb{R}^n_+$). They calculate the radius of robustness

$$\rho(y;\omega,\alpha) = \frac{(\alpha - TC(\omega,y))^+}{\|(d_1(y), ..., d_n(y))\|^\circ}$$

where, $s^+ = \max(0,s)$ for $s \in \mathbb{R}$, for $i = 1, ..., n$, $d_i(y)$ is the distance from y to the ith point of demand, $TC(\omega,y) = \sum_{i=1,...,n} \omega_i d_i(y)$ and $\|.\|^\circ$ is the dual norm of an absolute norm $\|.\|$ chosen to measure the distances between the weights on $\mathbb{R}^n$. Moreover, they consider the problem $(\mathscr{R}_x)$ for some classical norms on $\mathbb{R}^2$ and S convex.

To conclude this section, one can consult the surveys Snyder (2004), Drezner and Guyse (1999) for some considerations about robustness in facility location problems, and Beyer and Sendhoff (2007), Roy (2010) for more general considerations about robustness in Optimization, Operation Research and Aid Decision.

3 Maximizing the radius of robustness for the single–facility location problem in a continuous space

As said before, the purpose of this paper is to show that for the facility location problem considered, optimizing the minisum or minimax criterion is equivalent to optimizing the radius of robustness.

So let us clarify our robust counterpart of the problem $(\mathscr{P}_x)$. On one hand, the cost in the problem $(\mathscr{P}_x)$ is based on the choice of a certain norm W on $(\mathbb{R}^d)^n$. As we have seen, for our problem, the norms of most interest are $W = N_1 \circ \gamma_1 \times \cdots \times \gamma_n$ or $W = N_\infty \circ \gamma_1 \times \cdots \times \gamma_n$.

On the other hand, in order to consider the radius of robustness, we have to choose a distance on the space of data $(\mathbb{R}^d)^n$. Due to the interpretation of the problem, it seems natural to consider the distance $d_{\tilde{W}}$ associated with the norm $\tilde{W} = N_\infty \circ \gamma_1 \times \cdots \times \gamma_n$, i.e. for $x = (x_1,...,x_n)$, $x' = (x'_1,...,x'_n) \in (\mathbb{R}^d)^n$,

$$d_{\tilde{W}}(x,x') = N_\infty(\gamma_1(x_1 - x'_1),...,\gamma_n(x_n - x'_n)).$$

Nevertheless, as in Carrizosa and Nickel (2003), one can use $\tilde{W} = \tilde{N} \circ \gamma_1 \times \cdots \times \gamma_n$ where $\tilde{N}$ is any absolute norm on $\mathbb{R}^n$. If $y \in \mathbb{R}^d$, $\alpha > 0$,

$$[C(.,y) > \alpha] = [W(. - \hat{y}) > \alpha] = B_W(\hat{y},\alpha)^c,$$

where $B_W(\hat{y},\alpha)$ is the closed ball centered at $\hat{y}$ of radius α associated with W and $B_W(\hat{y},\alpha)^c = \{z \in (\mathbb{R}^d)^n \,|\, z \notin B_W(\hat{y},\alpha)\}$ is the complementary set of $B_W(\hat{y},\alpha)$. So, for $x = (x_1,...,x_n) \in (\mathbb{R}^d)^n$, the radius of robustness is

$$\rho(y;x,\alpha) = d_{\tilde{W}}(x, B_W(\hat{y},\alpha)^c) = d_{\tilde{W}}(x - \hat{y}, \alpha B_W^c) = \alpha d_{\tilde{W}}\left(\frac{x-\hat{y}}{\alpha}, B_W^c\right). \tag{6}$$

It follows that our robust counterpart of the problem $(\mathscr{P}_x)$ is:

$$\text{Maximize } d_{\tilde{W}}(x - \hat{y}, \alpha B_W^c), \quad y \in \mathbb{R}^d. \tag{$\mathscr{R}_x$}$$

To begin with, we consider the *trivial* case $W = \tilde{W}$. Since for $z \in (\mathbb{R}^d)^n$,

$$d_W(z, B_W^c) = (1 - W(z))^+,$$

so for $y \in \mathbb{R}^d$, $x \in (\mathbb{R}^d)^n$:

$$d_W(x - \hat{y}, \alpha B_W^c) = (\alpha - W(x - \hat{y}))^+. \tag{7}$$

This leads easily to the following result concerning the minimax criterion, where $v(\mathscr{P}_x) = \inf_{y\in\mathbb{R}^d} W(x - \hat{y})$ is the value of the initial problem $(\mathscr{P}_x)$:

Theorem 1 *Let $x = (x_1,...,x_n) \in (\mathbb{R}^d)^n$. For any $\alpha > v(\mathscr{P}_x)$, a point $y \in \mathbb{R}^d$ is a solution of the 1–center problem $(\mathscr{P}_x)$ (where $N = N_\infty$) if and only if it is a solution of $(\mathscr{R}_x)$ (where $N = N_\infty$ too) and measuring the distance between the data using $N_\infty \circ \gamma_1 \times \cdots \times \gamma_n$. Moreover, the radius of robustness is*

$$\rho(y;x,\alpha) = \alpha - N_\infty(\gamma_1(x_1 - y),...,\gamma_n(x_n - y)) = \alpha - v(\mathscr{P}_x).$$

Proof One always has

$$\sup_{y\in\mathbb{R}^d} \rho(y;x,\alpha) = \sup_{y\in\mathbb{R}^d} (\alpha - W(x-\hat{y}))^+ = (\alpha - \inf_{y\in\mathbb{R}^d} W(x-\hat{y}))^+ = (\alpha - v(\mathscr{P}_x))^+ = \alpha - v(\mathscr{P}_x). \quad (8)$$

Now if y is a 1–center then for all $y' \in \mathbb{R}^d$, $W(x-\hat{y}') \geq W(x-\hat{y})$ so one obtains

$$\rho(y;x,\alpha) = (\alpha - W(x-\hat{y}))^+ \geq (\alpha - W(x-\hat{y}'))^+ = \rho(y';x,\alpha).$$

If y maximizes the radius of robustness then, due to (8),

$$(\alpha - W(x-\hat{y}))^+ = \alpha - v(\mathscr{P}_x) > 0,$$

so $\alpha - W(x-\hat{y}) \geq 0$ and one obtains, for all $y' \in \mathbb{R}^d$:

$$\alpha - W(x-\hat{y}) = \alpha - v(\mathscr{P}_x) \geq \alpha - W(x-\hat{y}')$$

which implies that y is a 1–center.

Note that this theorem still holds if we replace N_∞ by any absolute norm on $\mathbb{R}^n$, in particular N_1.

Let us now consider the case when $W = N_1 \circ \gamma_1 \times \cdots \times \gamma_n$ and $\tilde{W} = N_\infty \circ \gamma_1 \times \cdots \times \gamma_n$. Due to the special relation between N_1 and N_∞ and the polyhedrality of N_∞, we shall establish a formula similar to (7) leading to a theorem analogous to the Theorem 1. This is the main result of our paper. Its proof is based on several technical lemmas. We begin by describing the setting and some notations. One can consult Hiriart-Urruty and Lemaréchal (2001) for the well known results of Convex Analysis we used in the sequel.

Let E be a finite dimensional real vector space, provided with a *duality* $\langle .,. \rangle$. We provide $E = \mathbb{R}^n$ with its usual duality

$$((x_1,...,x_n),(y_1,...,y_n)) \longrightarrow x_1y_1 + \cdots + x_ny_n$$

and $E = (\mathbb{R}^d)^n$ with its usual duality too,

$$((x_1,...,x_n),(y_1,...,y_n)) \longrightarrow \langle x_1,y_1\rangle + \cdots + \langle x_n,y_n\rangle.$$

If C is a convex set of E, we denote by $\mathrm{ext}(C)$ the set of extreme points of C. Now if W is a norm on E, we denote by W°, the dual norm of W, defined by $W^\circ(x) = \sup_{x'\in B_W} \langle x,x'\rangle$, for $x \in E$.

We will need the following direct consequence of a immediate generalization of the *Ascoli formula*, suppose that E is provided with a norm W (see e.g., in the case of a norm, Singer (2006), Corollary 1.4 and references therein, or, in the case of a gauge, Plastria and Carrizosa (2001) Theorem 1.1). If $z,x \in E$, $z \neq 0$, then the distance from x to the half space $[\langle .,z\rangle > \alpha] = \{z' \in E \mid \langle z',z\rangle > \alpha\}$ is given by

$$d_W(x,[\langle .,z\rangle > \alpha]) = \frac{(\alpha - \langle z,x\rangle)^+}{W^\circ(z)}, \quad (9)$$

and using (9), if W, $\tilde{W}$ are two norms on E, the for all $z \in E$,

$$d_{\tilde{W}}(z,B_W^c) = \inf_{z'\in \mathrm{ext}(B_{W^\circ})} \frac{(1-\langle z',z\rangle)^+}{\tilde{W}^\circ(z')}. \quad (10)$$

Lemma 1 (Carrizosa and Fliege (2002)) *Let $W = N \circ \gamma_1 \times \cdots \times \gamma_n$ where N is an absolute norm on $\mathbb{R}^n$. Then*

$$W^\circ = N^\circ \circ \gamma_1^\circ \times \cdots \times \gamma_n^\circ.$$

Lemma 2 (Dowling and Saejung (2008)) *Let $W = N \circ \gamma_1 \times \cdots \times \gamma_n$ where N is an absolute norm on $\mathbb{R}^n$, if $x = (x_1, \ldots, x_n)$ is an extreme point of B_W then $(\gamma_1(x_1), \ldots, \gamma_n(x_n))$ is an extreme point of B_N.*

Lemma 3 *Let $W = N_1 \circ \gamma_1 \times \cdots \times \gamma_n$, $\tilde{W} = \tilde{N} \circ \gamma_1 \times \cdots \times \gamma_n$, where $\tilde{N}$ is an absolute norm on $\mathbb{R}^n$. Then, for all $z \in (\mathbb{R}^d)^n$,*

$$d_{\tilde{W}}(z, B_W^c) = \frac{(1 - W(z))^+}{\tilde{N}^\circ(1, \ldots, 1)}.$$

Proof We know from (10) that:

$$d_{\tilde{W}}(z, B_W^c) = \inf_{z' \in \mathrm{ext}(B_{W^\circ})} \frac{(1 - \langle z', z\rangle)^+}{\tilde{W}^\circ(z')}.$$

Due to lemma 1, $W^\circ = (N_1 \circ \gamma_1 \times \cdots \times \gamma_n)^\circ = N_\infty \circ \gamma_1^\circ \times \cdots \times \gamma_n^\circ$. If $z' = (z_1', \ldots, z_n') \in \mathrm{ext}(B_{W^\circ})$ then, due to lemma 2, $(\gamma_1^\circ(z_1), \ldots, \gamma_n^\circ(z_n')) \in \mathrm{ext}(B_{N_\infty})$. But

$$\mathrm{ext}(B_{N_\infty}) = \Big\{ (s_1, \ldots., s_n), \, s_1, \ldots, s_n = \pm 1 \Big\},$$

so $\gamma_1^\circ(z_1') = \ldots = \gamma_n^\circ(z_n) = 1$. Now, using lemma 1 again, $\tilde{W}^\circ = \tilde{N}^\circ \circ \gamma_1^\circ \times \cdots \times \gamma_n^\circ$ so for all $z' \in \mathrm{ext}(B_{W^\circ})$, knowing that $\gamma_1^\circ(z_1') = \ldots = \gamma_n^\circ(z_n') = 1$, one obtains $\tilde{W}^\circ(z') = \tilde{N}^\circ(1, \ldots, 1)$ and we conclude, using that

$$\inf_{z' \in \mathrm{ext}(B_{W^\circ})} (1 - \langle z', z\rangle)^+ = \Big(1 - \sup_{z' \in B_{W^\circ}} \langle z', z\rangle \Big)^+ = (1 - W^{\circ\circ}(z))^+.$$

□

Theorem 2 *Let $x = (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$. For any $\alpha > v(\mathscr{P}_x)$, a point $y \in \mathbb{R}^d$ is a solution of the generalized Fermat–Weber problem ($\mathscr{P}_x$) (where $N = N_1$) if and only if it is a solution of ($\mathscr{R}_x$) (with $N = N_1$ too) and measuring the distance between the data using $\tilde{N} \circ \gamma_1 \times \cdots \times \gamma_n$, where $\tilde{N}$ is an absolute norm on $\mathbb{R}^d$. Moreover, the radius of robustness is*

$$\rho(y; x, \alpha) = \frac{\alpha - N_1(\gamma_1(x_1 - y), \ldots, \gamma_n(x_n - y))}{\tilde{N}^\circ(1, \ldots, 1)} = \frac{\alpha - v(\mathscr{P}_x)}{\tilde{N}^\circ(1, \ldots, 1)}.$$

Proof The Theorem is a direct consequence of the Lemma 3. Using this Lemma 3 and (6), one has for every $z \in \mathbb{R}^d$, $\alpha > 0$, $x = (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$,

$$\rho(z; x, \alpha) = \frac{(\alpha - N_1(\gamma_1(x_1 - z), \ldots, \gamma_n(x_n - z)))^+}{\tilde{N}^\circ(1, \ldots, 1)}.$$

One concludes as in Theorem 1. □

The following result may be proved in much the same way as Theorem 2, supposing that $\tilde{N}$ is symmetric, which means that for all $(s_1,...,s_n) \in \mathbb{R}^n$ and permutation σ of $\{1,...,n\}$,

$$\tilde{N}(s_1,...,s_n) = \tilde{N}(s_{\sigma(1)},...,s_{\sigma(n)}),$$

using the facts that $\tilde{N}^\circ$ is also symmetric and

$$\text{ext}(B_{N_1}) = \Big\{(s_1,....,s_n),\, \exists i \in \{1,...,n\}, s_i = \pm 1 \text{ and } s_j = 0, \forall j \neq i\Big\}.$$

For example, the ℓ_p–norm on $\mathbb{R}^n$, $p \in [1,+\infty]$, are symmetric.

Theorem 3 *Let $x = (x_1,...,x_n) \in (\mathbb{R}^d)^n$. For any $\alpha > v(\mathscr{P}_x)$, a point $y \in \mathbb{R}^d$ is a solution of the 1–center problem ($\mathscr{P}_x$) (where $N = N_\infty$) if and only if it is a solution of ($\mathscr{R}_x$) (with $N = N_\infty$ too) and measuring the distance between the data using $\tilde{N} \circ \gamma_1 \times \cdots \times \gamma_n$, where $\tilde{N}$ is an absolute norm on $\mathbb{R}^d$. Moreover, the radius of robustness is*

$$\rho(y;x,\alpha) = \frac{\alpha - N_\infty(\gamma_1(x_1-y),...,\gamma_n(x_n-y))}{\tilde{N}^\circ(1,0,...,0)} = \frac{\alpha - v(\mathscr{P}_x)}{\tilde{N}^\circ(1,0,...,0)}.$$

4 Conclusion

In this work, we have established a robustness property of solutions of two classical single–facility location problems with mixed norms on continuous space. We have proved that, in a sense, the solutions of the classical Weber problem (Minisum problem) and Minimax problem (1–center problem) are the most robust relatively to perturbations of the locations of the demand. Moreover, we give some quantitative information about this robustness by an explicit calculus of the radius of robustness of the classical solutions, calculation which requires only the easy computation of the value of the convex initial problem.

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