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Maximal probabilities of convolution powers of discrete uniform distributions

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Abstract

We prove optimal constant over root n upper bounds for the maximal probabilities of n th convolution powers of discrete uniform distributions.

Key words and phrases: Concentration functions, discrete B-spline, lattice distributions, Littlewood-Offord inequalities, Wallis product.

MSC 2000 Subject Classification: Primary 60E15, 60G50; secondary 26D15.

For $\ell, n \in \mathbb{N} := \{1, 2, 3, \dots\}$, let U_ℓ^{*n} denote the n th convolution power of the discrete uniform distribution $U_\ell := \frac{1}{\ell}(\delta_0 + \dots + \delta_{\ell-1})$. Let u_ℓ^{*n} denote the density of U_ℓ^{*n} with respect to counting measure. Thus, writing $\mathbf{1}_A(x) := 1$ if $x \in A$ and $\mathbf{1}_A(x) := 0$ otherwise, we have for $\ell \in \mathbb{N}$ and $k \in \mathbb{Z}$

$$u_\ell^{*1}(k) = \frac{1}{\ell} \mathbf{1}_{\{0, \dots, \ell-1\}}(k), \quad u_\ell^{*2}(k) = \frac{\ell - |\ell - 1 - k|}{\ell^2} \mathbf{1}_{\{0, \dots, 2(\ell-1)\}}(k) \quad (1)$$

and the general formula

$$u_\ell^{*n}(k) = \frac{1}{\ell^n} \sum_{j=0}^{\lfloor k/\ell \rfloor} (-1)^j \binom{n}{j} \binom{n+k-\ell j-1}{n-1} \quad (\ell, n \in \mathbb{N}, k \in \mathbb{Z})$$

where $\sum_{j=a}^b := 0$ if $a > b$ and where no indicator $\mathbf{1}_{\{0, \dots, n(\ell-1)\}}(k)$ is necessary on the right-hand side, for which we refer to de Moivre (1756, pp. 39–43) or Hald (1998, pp. 34–35). The purpose of this note is to provide a sharp upper bound for the maximal probabilities or concentrations

$$c_{\ell, n} := \max_{k \in \mathbb{Z}} u_\ell^{*n}(k) \quad (2)$$

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of U_ℓ^{*n} , see Remarks (d) and (h) below for possible applications. From (1), we obviously get

$$c_{\ell,1} = c_{\ell,2} = \frac{1}{\ell} \quad (\ell \in \mathbb{N}) \quad (3)$$

In what follows, we exclude the trivial case of $U_1^{*n} = \delta_0$ and hence always assume that $\ell \geq 2$.

Theorem. *Let $\ell, n \in \mathbb{N}$ with $\ell \geq 2$ and let $c_{\ell,n}$ be defined by (2). If $n \neq 2$ or $\ell \in \{2, 3, 4\}$, then*

$$c_{\ell,n} < \sqrt{\frac{6}{\pi(\ell^2 - 1)n}} \quad (4)$$

holds. If $n = 2$ and $\ell \geq 5$, then inequality (4) has to be reversed.

Remarks. (a) Let us fix $\ell \geq 2$ and denote by $\mu := (\ell - 1)/2$ and $\sigma^2 := (\ell^2 - 1)/12$ the mean and the variance of U_ℓ and let $\varphi(x) := (1/\sqrt{2\pi}) \exp(-x^2/2)$ for $x \in \mathbb{R}$. By the local central limit theorem, see e.g. Durrett (2005, p. 130), we then have $\lim_{n \rightarrow \infty} \sup_{k \in \mathbb{Z}} |\sqrt{n} u_\ell^{*n}(k) - \frac{1}{\sigma} \varphi((k - n\mu)/(\sigma\sqrt{n}))| = 0$. Since the function φ is maximal and continuous at zero, we easily get $\lim_{n \rightarrow \infty} \sqrt{n} c_{\ell,n} = \frac{1}{\sigma} \varphi(0) = \sqrt{6/(\pi(\ell^2 - 1))}$. Hence (4) is sharp for $n \rightarrow \infty$ and every ℓ , in the sense that the quotient of both sides of the inequality converges to one.

(b) A corollary to the theorem is the simpler bound

$$c_{\ell,n} < \frac{2\sqrt{2/\pi}}{\ell\sqrt{n}} \quad (\ell, n \in \mathbb{N}, \ell \geq 2) \quad (5)$$

obtained by using $\ell^2 - 1 \geq 3\ell^2/4$ in inequality (4) if $n \neq 2$, and (3) for $n = 2$. By the previous Remark (a) and by comparison with (4), it is obvious that (5) is sharp for $n \rightarrow \infty$ only if $\ell = 2$. Inequality (5) is contained in Bretagnolle (2004): His Lemme 33.4.4 a) states, in our notation,

$$c_{\ell,n} \leq \frac{2}{\ell} c_{2,n} \quad (\ell, n \in \mathbb{N}, \ell \geq 2) \quad (6)$$

which, by the standard Wallis product inequality recalled in Remark (f) below, implies (5). Further, inequality (5) results if Bretagnolle's Théorème 33.1.1 is applied to random variables each with distribution U_ℓ .

(c) The existence of some constant $A < \infty$ with

$$c_{\ell,n} < \frac{A}{\ell\sqrt{n}} \quad (\ell, n \in \mathbb{N}, \ell \geq 2) \quad (7)$$

already follows from Kesten's (1969) concentration inequality for sums of independent real-valued random variables and, alternatively, from Gamkrelidze's (1973) sharper result for the special case of identically distributed symmetric unimodal lattice random variables. In the case considered here, Gamkrelidze's result yields our inequality (4) with an additional $O(n^{-1})$ -term on the right-hand side. For a general introduction to concentration inequalities and further results, see Petrov (1995, sections 1.5 and 2.4).

(d) Bretagnolle (2004), Rogozin (1987), and Leader and Radcliffe (1994, in particular Theorem 10 and the unproved remark on p. 97) state upper bounds for concentrations of sums of independent real-valued random variables X_j in terms of concentrations of sums of certain independent Y_j with distributions U_{ℓ_j} . (Both Bretagnolle and Rogozin refer to an unpublished preprint of Bretagnolle from 1982. Leader and Radcliffe fail to give appropriate references to the probabilistic literature.) Of these authors only Bretagnolle goes on to deduce an analytically convenient and still rather sharp bound, using in particular inequality (6). Possibly the present asymptotically sharper inequality (4) could serve to improve Bretagnolle's result.

(e) Since U_{ℓ}^{*n} is a convolution of distributions unimodal on \mathbb{Z} and with some centers of symmetry, it follows from the well-known discrete Wintner theorem, see Dharmadhikari and Joag-Dev (1988, page 109, Theorem 4.7) or, more precisely, Mattner (2006, Lemma 3.3), that the density u_{ℓ}^{*n} is maximized at the one or two central points of its support $\{0, \dots, n(\ell - 1)\}$, so that we have

$$c_{\ell, n} = u_{\ell}^{*n}(\lfloor \frac{n(\ell - 1)}{2} \rfloor) = u_{\ell}^{*n}(\lceil \frac{n(\ell - 1)}{2} \rceil) \quad (8)$$

(f) For $\ell = 2$, the theorem reduces to the familiar Wallis product inequality for the maximal probabilities of symmetric binomial distributions,

$$\binom{2k}{k} 2^{-2k} < \frac{1}{\sqrt{\pi k}} \quad (k \in \mathbb{N}) \quad (9)$$

since $c_{2, 2k-1} = \binom{2k-1}{k} 2^{-(2k-1)} = \binom{2k}{k} 2^{-2k} = c_{2, 2k}$, and since the right-hand side of (4) for $\ell = 2$ and $n = 2k$ or $n = 2k - 1$ is, respectively, equal to or greater than the right-hand side of (9).

(g) A concentration bound related to the present theorem is given in Kanter (1976) and in Mattner and Roos (2007). Theorem 2.1 of the latter paper specialized to $p_j = 2/3$ for every j and the formulas (15) and (8) there yield the inequalities, sharp for $n \rightarrow \infty$,

$$\max_{k \in \mathbb{Z}} U_3^{*n}(\{k, k + 1\}) < G(2n/3) < \sqrt{\frac{3}{\pi n}} \quad (n \in \mathbb{N}) \quad (10)$$

where $G(\lambda) := e^{-\lambda}(I_0(\lambda) + I_1(\lambda))$ for $\lambda \in [0, \infty[$ and I_0, I_1 denote the usual modified Bessel functions. Since the left-hand side of (10) is $\leq 2c_{3, n}$, the inequality between the extreme members of (10) also follows from the special case $\ell = 3$ of the present theorem.

(h) A recent application of upper bounds for $c_{\ell, n}$ occurred in the construction of a two-dimensional transient but polygonally recurrent random walk by Siegmund-Schultze and von Weizsäcker (2006), who proved and used (7), see their Lemmas 6 and 1.

We will need two standard lemmas for the proof of the theorem. In what follows, we use the adjectives “positive”, “increasing” etc. in the wide sense. Thus, e.g., a function f with $0 \leq f(x) \leq f(y)$ for $x < y$ is called positive and increasing.

Lemma 1 Let $a \in]0, \infty[$ and let $f, g : [-a, a] \rightarrow \mathbb{R}$ be functions with f even, f decreasing on $[0, a]$, and g convex. Then

$$\int_{-a}^a f(x) g(x) dx \leq \frac{1}{2a} \int_{-a}^a f(x) dx \int_{-a}^a g(x) dx$$

Proof. The function h defined by $h(x) := g(x) + g(-x)$ for $x \in [-a, a]$ is even and convex. Hence on $[0, a]$, h is increasing and f is decreasing, so that the Chebyshev inequality obtained by integrating $(f(x) - f(y))(h(x) - h(y)) \leq 0$ over $[0, a] \times [0, a]$, see Mitrinović *et al.* (1993, Chapter IX) for references, yields $\int_{-a}^a fg = \int_0^a fh \leq \frac{1}{a} \int_0^a f \int_0^a h = \frac{1}{2a} \int_{-a}^a f \int_{-a}^a g$. \square

Lemma 2 For $\lambda \in]0, \infty[$, we have $\int_0^{\pi/2} \sin^\lambda(t) dt = \int_0^{\pi/2} \cos^\lambda(t) dt < \sqrt{\pi/(2\lambda)}$.

Proof. For $t \in]0, \pi/2[$, we have $\cos(t) = \exp(-\int_0^t \tan(u) du) < \exp(-t^2/2)$, since $\tan(u) > u$, so that the second integral in the claim is $< \int_0^\infty \exp(-\lambda t^2/2) dt$. \square

Proof of the theorem. Since the characteristic function \widehat{U}_ℓ of U_ℓ is given by

$$\widehat{U}_\ell(t) = \frac{1}{\ell} \sum_{k=0}^{\ell-1} e^{ikt} = \frac{e^{i\ell t} - 1}{\ell(e^{it} - 1)} = \frac{\sin(\ell t/2)}{\ell \sin(t/2)} e^{i(\ell-1)t/2} \quad (t \in \mathbb{R})$$

we get by Fourier inversion for $k \in \mathbb{Z}$

$$\begin{aligned} u_\ell^{*n}(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\widehat{U}_\ell(t))^n e^{-ikt} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\sin(\ell t/2)}{\ell \sin(t/2)} \right)^n \exp\left(i\left(\frac{n(\ell-1)}{2} - k\right)t\right) dt \\ &= \frac{2}{\pi} \int_0^{\pi/2} \left(\frac{\sin(\ell t)}{\ell \sin t} \right)^n \cos((n(\ell-1) - 2k)t) dt \end{aligned}$$

Using equality (8), we get

$$c_{\ell,n} = \frac{2}{\pi} \int_0^{\pi/2} \left(\frac{\sin(\ell t)}{\ell \sin t} \right)^n \cos(\alpha t) dt = \frac{2}{\pi} \int_0^{\pi/\ell} + \frac{2}{\pi} \int_{\pi/\ell}^{\pi/2} =: I_1 + I_2$$

with

$$\alpha := n(\ell-1) - 2 \left\lfloor \frac{n(\ell-1)}{2} \right\rfloor \in \{0, 1\}$$

To bound I_1 , we recall the power series expansion $x/\tan(x) = 1 - \sum_{k=1}^{\infty} a_k x^{2k}$ for $|x| < \pi$ with $a_k > 0$ for $k \in \mathbb{N}$, $a_1 = 1/3$, and $a_2 = 1/45$, see e.g. Burckel (1979, pp. 75–77). With $b_k := a_k/(2k)$ we get by a termwise integration

$$-\log\left(\frac{\sin x}{x}\right) = \int_0^x \left(\frac{1}{y} - \frac{1}{\tan(y)}\right) dy = \sum_{k=1}^{\infty} b_k x^{2k} \quad (|x| < \pi)$$

with $b_k > 0$ for $k \in \mathbb{N}$, $b_1 = 1/6$, and $b_2 = 1/180$. Hence, for $t \in]0, \pi/\ell[$ and with $x := \sqrt{(\ell^2 - 1)n/3}t$, we have

$$\begin{aligned} \left(\frac{\sin(\ell t)}{\ell \sin t}\right)^n &= \exp\left(n\left(\log\left(\frac{\sin(\ell t)}{\ell t}\right) - \log\left(\frac{\sin t}{t}\right)\right)\right) \\ &= \exp\left(-n\sum_{k=1}^{\infty} b_k(\ell^{2k} - 1)t^{2k}\right) \\ &\leq \exp\left(-\frac{n}{6}(\ell^2 - 1)t^2 - \frac{n}{180}(\ell^4 - 1)t^4\right) \\ &\leq e^{-x^2/2} \exp\left(-\frac{x^4}{20n}\right) \quad [\text{by } \ell^4 - 1 \geq (\ell^2 - 1)^2] \end{aligned}$$

so that, using also $\cos(\alpha t) \leq 1$ and $e^{-y} \leq 1 - y + y^2/2$ for $y \in [0, \infty[$,

$$\begin{aligned} \sqrt{\frac{\pi(\ell^2 - 1)n}{6}} I_1 &\leq \sqrt{\frac{2(\ell^2 - 1)n}{3\pi}} \int_0^{\pi/\ell} \left(\frac{\sin(\ell t)}{\ell \sin t}\right)^n dt \\ &\leq \int_0^{\pi\ell^{-1}\sqrt{(\ell^2-1)n/3}} \frac{2e^{-x^2/2}}{\sqrt{2\pi}} \exp\left(-\frac{x^4}{20n}\right) dx \\ &\leq \int_0^{\infty} \frac{2e^{-x^2/2}}{\sqrt{2\pi}} \left(1 - \frac{x^4}{20n} + \frac{x^8}{800n^2}\right) dx \\ &= 1 - \frac{3}{20n} + \frac{21}{160n^2} \end{aligned} \tag{11}$$

Now let us bound

$$I_2 = \frac{2}{\pi} \int_{\pi/\ell}^{\pi/2} \left(\frac{\sin(\ell t)}{\ell \sin t}\right)^n \cos(\alpha t) dt = \int_{\pi}^{\ell\pi/2} \sin^n(t) h(t) dt$$

where

$$h(t) := \frac{2 \cos(\alpha t/\ell)}{\pi \ell (\ell \sin(t/\ell))^n} \quad (t \in]0, \ell\pi[)$$

If n is odd, then with $m := \ell/2$ if ℓ is even, $m := (\ell - 1)/2$ if $\ell \equiv 3 \pmod{4}$, and $m := (\ell + 1)/2$ if $\ell \equiv 5 \pmod{4}$, we get

$$I_2 \leq \int_{\pi}^{m\pi} \sin^n(t) h(t) dt = \int_0^{\pi} \sin^n(t) \sum_{j=1}^{m-1} (-1)^j h(t + j\pi) dt \leq 0 \tag{12}$$

since h is positive and decreasing. If n is even, then we use $\cos x \leq 1$ and $\sin x \geq 2x/\pi$ for

$x \in [0, \pi/2]$ to get $h(t) \leq \frac{1}{\ell}(\pi/2)^{n-1}/t^n$ and hence

$$\begin{aligned}
 I_2 &\leq \frac{1}{\ell} \left(\frac{\pi}{2}\right)^{n-1} \sum_{k=1}^{\infty} \int_{k\pi}^{(k+1)\pi} \sin^n(t) \frac{1}{t^n} dt \\
 &\leq \frac{1}{\ell} \left(\frac{\pi}{2}\right)^{n-1} \sum_{k=1}^{\infty} \frac{1}{\pi} \int_{k\pi}^{(k+1)\pi} \sin^n(t) dt \int_{k\pi}^{(k+1)\pi} \frac{dt}{t^n} \\
 &\hspace{15em} [\text{Lemma 1, } t = x + (k + \frac{1}{2})\pi] \\
 &= \frac{1}{\pi \ell (n-1) 2^{n-1}} \int_0^{\pi} \sin^n(t) dt \\
 &\leq \frac{1}{\ell (n-1) 2^{n-1}} \sqrt{\frac{2}{\pi n}} \quad [\text{by Lemma 2}]
 \end{aligned} \tag{13}$$

Combining our estimates from (11), (12), (13) and using $\sqrt{\ell^2 - 1} < \ell$, we obtain

$$\sqrt{\frac{\pi(\ell^2 - 1)n}{6}} c_{\ell,n} \leq 1 - \frac{3}{20n} + \frac{21}{160n^2} + \frac{\mathbf{1}_{2\mathbb{N}}(n)}{\sqrt{3}(n-1)2^{n-1}} =: d_n \tag{14}$$

for all $\ell, n \in \mathbb{N}$ with $\ell \geq 2$. For n odd, we use $n^2 \geq n$ to get $d_n - 1 \leq \frac{1}{n}(-\frac{3}{20} + \frac{21}{160}) < 0$. For n even with $n \neq 2$, we use $n^2 \geq 4n$ and $(n-1)2^{n-1} \geq 6n$ in (14) to get

$$d_n - 1 \leq \frac{1}{n} \left(-\frac{3}{20} + \frac{21}{160} \cdot \frac{1}{4} + \frac{1}{6\sqrt{3}} \right) = \frac{1}{2n} \left(\frac{1}{3\sqrt{3}} - \frac{15}{64} \right) < 0$$

Thus for $n \neq 2$, we have $d_n < 1$, and hence inequality (4). For $n = 2$, the claim of the theorem follows from (3). \square

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