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Hyperbolic traveling waves driven by growth

Emeric Bouin[*], Vincent Calvez[†‡], Grégoire Nadin[§]

November 21, 2016

Abstract

We perform the analysis of a hyperbolic model which is the analog of the Fisher-KPP equation. This model accounts for particles that move at maximal speed ε^{-1} ($\varepsilon > 0$), and proliferate according to a reaction term of monostable type. We study the existence and stability of traveling fronts. We exhibit a transition depending on the parameter ε: for small ε the behaviour is essentially the same as for the diffusive Fisher-KPP equation. However, for large ε the traveling front with minimal speed is discontinuous and travels at the maximal speed ε^{-1}. The traveling fronts with minimal speed are linearly stable in weighted L^2 spaces. We also prove local nonlinear stability of the traveling front with minimal speed when ε is smaller than the transition parameter.

Key-words: traveling waves; Fisher-KPP equation; telegraph equation; nonlinear stability.
AMS classification: 35B35; 35B40; 35L70; 35Q92

1 Introduction

We consider the problem of traveling fronts driven by growth (*e.g.* cell division) together with cell dispersal, where the motion process is given by a hyperbolic equation. This is motivated by the occurence of traveling pulses in populations of bacteria swimming inside a narrow channel [1, 38]. It has been demonstrated that kinetic models are well adapted to this problem [39]. We will focus on the following model introduced by Dunbar and Othmer [13] (see also Hadeler [26]) and Fedotov [16, 17, 18]

$$\varepsilon^2\partial_{tt}\rho_\varepsilon(t,x) + \left(1-\varepsilon^2 F'(\rho_\varepsilon(t,x))\right)\partial_t\rho_\varepsilon(t,x) - \partial_{xx}\rho_\varepsilon(t,x) = F(\rho_\varepsilon(t,x))\,, \quad t>0\,, \quad x\in\mathbb{R}\,. \tag{1.1}$$

The cell density is denoted by $\rho_\varepsilon(t,x)$. The parameter $\varepsilon > 0$ is a scaling factor. It accounts for the ratio between the mean free path of cells and the space scale. The growth function F is subject to the following assumptions (the so-called monostable nonlinearity)

$$\begin{cases} F\in\mathcal{C}^3([0,1])\,, \quad F \text{ is uniformly strictly concave}: \ \inf_{[0,1]}(-F'') =: \alpha > 0\,, \\ F(0)=F(1)=0\,, \quad F(\rho)>0 \ \text{ if } \ \rho\in(0,1)\,. \end{cases} \tag{1.2}$$

For the sake of clarity we will sometimes take as an example the logistic growth function $F(\rho) = \rho(1-\rho)$.

Equation (1.1) is equivalent to the hyperbolic system

$$\begin{cases} \partial_t\rho_\varepsilon + \varepsilon^{-1}\partial_x\left(j_\varepsilon\right) = F(\rho_\varepsilon) \\ \varepsilon\partial_t j_\varepsilon + \partial_x\rho_\varepsilon = -\varepsilon^{-1}j_\varepsilon\,. \end{cases} \tag{1.3}$$

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The expression of j_ε can be computed explicitly in terms of ρ_ε as follows,

$$j_\varepsilon(t,x) = -\frac{1}{\varepsilon}\int_0^t \partial_x \rho_\varepsilon(s,x)\exp\left(\frac{s-t}{\varepsilon^2}\right) ds + j_\varepsilon(0,x)\,, \tag{1.4}$$

but this expression will not be directly used afterwards. We will successively use the formulation (1.1) or the equivalent formulation (1.3).

Since the pioneering work by Fisher [21] and Kolmogorov-Petrovskii-Piskunov [29], dispersion of biological species has been usually modelled by mean of reaction-diffusion equations. The main drawback of these models is that they allow infinite speed of propagation. This is clearly irrelevant for biological species. Several modifications have been proposed to circumvent this issue. It has been proposed to replace the linear diffusion by a nonlinear diffusion of porous-medium type [40, 31, 35]. This is known to yield propagation of the support at finite speed [32, 33]. The density-dependent diffusion coefficient stems for a pressure effect among individuals which influences the speed of diffusion. Pressure is very low when the population is sparse, whereas it has a strong effect when the population is highly densified. Recently, this approach has been developped for the invasion of glioma cells in the brain [7]. Alternatively, some authors have proposed to impose a limiting flux for which the nonlinearity involves the gradient of the concentration [3, 8, 4].

The diffusion approximation is generally acceptable in ecological problems where space and time scales are large enough. However, kinetic equations have emerged recently to model self-organization in bacterial population at smaller scales [2, 34, 14, 30, 36, 38, 39]. These models are based on velocity-jump processes. It is now standard to perform a drift-diffusion limit to recover classical reaction-diffusion equations [27, 10, 14, 28]. However it is claimed in [39] that the diffusion approximation is not suitable, and the full kinetic equation has to be handled with. Equation (1.1) can be reformulated as a kinetic equation with two velocities only $v = \pm\varepsilon^{-1}$ (see (2.1) below). This provides a clear biological interpretation of equation (1.1) as a simple model for bacteria colonies where bacteria reproduce themselves, and move following a run-and-tumble process.

Hyperbolic models coupled with growth have already been studied in [13, 26, 23, 11]. In [26] it is required that the nonlinear function in front of the time first derivative $\partial_t\rho_\varepsilon$ is positive (namely here, $1-\varepsilon^2F'(\rho)>0$). Indeed, this enables to perform a suitable change of variables in order to reduce to the classical Fisher-KPP problem. In our context this is equivalent to $\varepsilon^2F'(0)<1$ since F is concave. In [23] this nonlinear contribution is replaced by 1: the authors study the following equation (damped hyperbolic Fisher-KPP equation),

$$\varepsilon^2\partial_{tt}\rho_\varepsilon(t,x) + \partial_t\rho_\varepsilon(t,x) - \partial_{xx}\rho_\varepsilon(t,x) = F(\rho_\varepsilon(t,x))\,.$$

We also refer to [11] where the authors analyse a kinetic model more general than (1.1). They develop a perturbative approach, close to the diffusive regime $\varepsilon \ll 1$.

It is worth recalling some basic results related to reaction-diffusion equations. First, as $\varepsilon\to 0$ the density ρ_ε solution to (1.1) formally converges to a solution of the Fisher-KPP equation [11]:

$$\partial_t\rho_0(t,x) - \partial_{xx}\rho_0(t,x) = F(\rho_0(t,x))\,.$$

The long time behaviour of such equation is well understood since the pioneering works by Kolmogorov-Petrovsky-Piskunov [29] and Aronson-Weinberger [5]. For nonincreasing initial data with sufficient decay at infinity the solution behaves asymptotically as a traveling front moving at the speed $s = 2\sqrt{F'(0)}$. Moreover the traveling front solution with minimal speed is stable in some L^2 weighted space [22].

In this work we prove that analogous results hold true in the *parabolic regime* $\varepsilon^2F'(0)<1$. Namely there exists a continuum of speeds $[s^*(\varepsilon),\varepsilon^{-1})$ for which (1.1) admits smooth traveling fronts. The minimal speed is given by [16]

$$s^*(\varepsilon) = \frac{2\sqrt{F'(0)}}{1+\varepsilon^2F'(0)}\,, \quad \text{if } \varepsilon^2F'(0)<1\,. \tag{1.5}$$

Obviously we have $s^*(\varepsilon) \leq \min(2\sqrt{F'(0)}, \varepsilon^{-1})$. There also exists *supersonic* traveling fronts, with speed $s > \varepsilon^{-1}$. This appears surprising at first glance since the speed of propagation for the hyperbolic equation (1.1) is ε^{-1} (see formulation (1.3) and Section 2). These fronts are essentially driven by growth, since they travel faster than the maximum speed of propagation. The results are summarized in the following Theorem.

Theorem 1 (Parabolic regime)**.** *Assume that $\varepsilon^2 F'(0) < 1$. The following alternatives hold:*

(a) *There exists no smooth or weak traveling front of speed $s \in [0, s^*(\varepsilon))$.*

(b) *For all $s \in [s^*(\varepsilon), \varepsilon^{-1})$, there exists a smooth traveling front solution of* (1.1) *with speed s.*

(c) *For $s = \varepsilon^{-1}$ there exists a weak traveling front.*

(d) *For all $s \in (\varepsilon^{-1}, \infty)$ there also exists a smooth traveling front of speed s.*

We also obtain that the minimal speed traveling front is nonlinearly locally stable in the parabolic regime $\varepsilon^2 F'(0) < 1$ (see Section 5, Theorem 15).

There is a transition occuring when $\varepsilon^2 F'(0) = 1$. In the *hyperbolic regime* $\varepsilon^2 F'(0) \geq 1$ the minimal speed speed becomes:

$$s^*(\varepsilon) = \varepsilon^{-1}\,, \quad \text{if } \varepsilon^2 F'(0) \geq 1\,. \tag{1.6}$$

On the other hand, the front traveling with minimal speed $s^*(\varepsilon)$ is discontinuous as soon as $\varepsilon^2 F'(0) > 1$. In the critical case $\varepsilon^2 F'(0) = 1$ there exists a continuous but not smooth traveling front with minimal speed $s^* = \sqrt{F'(0)}$.

Theorem 2 (Hyperbolic regime)**.** *Assume that $\varepsilon^2 F'(0) \geq 1$. The following alternatives hold:*

(a) *There exists no smooth or weak traveling front of speed $s \in [0, s^*(\varepsilon))$.*

(b) *There exists a weak traveling front solution of* (1.1) *with speed $s^*(\varepsilon) = \varepsilon^{-1}$. The wave is discontinuous if $\varepsilon^2 F'(0) > 1$.*

(c) *For all $s \in (\varepsilon^{-1}, \infty)$ there exists a smooth traveling front of speed s.*

We conclude this introduction by giving the precise definition of traveling fronts (smooth and weak) that will be used throughout the paper.

Definition 3. *We say that a function $\rho(t,x)$ is a smooth traveling front solution with speed s of equation* (1.1) *if it can be written $\rho(t,x) = \nu(x - st)$, where $\nu \in \mathcal{C}^2(\mathbb{R})$, $\nu \geq 0$, $\nu(-\infty) = 1$, $\nu(+\infty) = 0$ and ν satisfies*

$$(\varepsilon^2 s^2 - 1)\nu''(z) - \big(1 - \varepsilon^2 F'(\nu(z))\big)\, s\nu'(z) = F(\nu(z))\,, \quad z \in \mathbb{R}\,. \tag{1.7}$$

We say that ρ is a weak traveling front with speed s if it can be written $\rho(t,x) = \nu(x - st)$, where $\nu \in L^\infty(\mathbb{R})$, $\nu \geq 0$, $\nu(-\infty) = 1$, $\nu(+\infty) = 0$ and ν satisfies (1.7) *in the sense of distributions:*

$$\forall \varphi \in \mathcal{D}(\mathbb{R})\,, \quad \int_{\mathbb{R}} \Big((\varepsilon^2 s^2 - 1)\nu\varphi'' + \big(\nu - \varepsilon^2 F(\nu)\big)\, s\varphi' - F(\nu)\varphi \Big) dx = 0\,.$$

In the following Section 2 we show some numerical simulations in order to illustrate our results. Section 3 is devoted to the proof of existence of the traveling fronts in the various regimes (resp. parabolic, hyperbolic, and supersonic). Finally, in Section 4 and Section 5 we prove the stability of the traveling fronts having minimal speed $s^*(\varepsilon)$. We begin with linear stability (Section 4) since it is technically better tractable, and it let us discuss the case of the hyperbolic regime. We prove the full nonlinear stability in the range $\varepsilon \in (0, 1/\sqrt{F'(0)})$ (parabolic regime) in Section 5.

2 Numerical simulations

In this Section we perform numerical simulations of (1.1). We choose a logistic reaction term: $F(\rho) = \rho(1-\rho)$. We first symmetrize the hyperbolic system (1.3) by introducing $f^+ = \frac{1}{2}(\rho + j)$ and $f^- = \frac{1}{2}(\rho - j)$. This results in the following system:

$$\begin{cases} \partial_t f^+(t,x) + \varepsilon^{-1}\partial_x f^+(t,x) = \dfrac{\varepsilon^{-2}}{2}\left(f^-(t,x) - f^+(t,x)\right) + \dfrac{1}{2}F(\rho(t,x)) \\ \partial_t f^-(t,x) - \varepsilon^{-1}\partial_x f^-(t,x) = \dfrac{\varepsilon^{-2}}{2}\left(f^+(t,x) - f^-(t,x)\right) + \dfrac{1}{2}F(\rho(t,x)) \,. \end{cases} \tag{2.1}$$

In other words, the population is split into two subpopulations: $\rho = f^+ + f^-$, where the density f^+ denotes particles moving to the right with velocity ε^{-1}, whereas f^- denotes particles moving to the left with the opposite velocity.

We discretize the transport part using a finite volume scheme. Since we want to catch discontinuous fronts in the hyperbolic regime $\varepsilon^2 F'(0) > 1$, we aim to avoid numerical diffusion. Therefore we use a nonlinear flux-limiter scheme [25, 12]. The reaction part is discretized following the Euler explicit method.

$$f^+_{n+1,i} = f^+_{n,i} - \varepsilon^{-1}\frac{\Delta t}{\Delta x}\left(f^+_{n,i} + p_i\frac{\Delta x}{2} - f^+_{n,i-1} - p_{i-1}\frac{\Delta x}{2}\right) + \varepsilon^{-2}\frac{\Delta t}{2}\left(f^-_{n,i} - f^+_{n,i}\right) + \frac{\Delta t}{2}F(\rho_{n,i})\,.$$

The non-linear reconstruction of the slope is given by

$$p_i = \text{minmod}\left(\frac{f^+_{n,i} - f^+_{n,i-1}}{\Delta x}, \frac{f^+_{n,i+1} - f^+_{n,i}}{\Delta x}\right), \quad \text{minmod}\,(p,q) = \begin{cases} 0 & \text{if sign}\,(p) \neq \text{sign}\,(q) \\ \min(|p|,|q|)\text{sign}\,(p) & \text{if sign}\,(p) = \text{sign}\,(q) \end{cases}$$

We compute the solution on the interval (a,b) with the following boundary conditions: $f^+(a) = 1/2$ and $f^-(b) = 0$. The discretization of the second equation for f^- (2.1) is similar. The CFL condition reads $\Delta t < \varepsilon\Delta x$. It degenerates when $\varepsilon \searrow 0$, but we are mainly interested in the hyperbolic regime when ε is large enough. Other strategies should be used in the diffusive regime $\varepsilon \ll 1$, e.g. asymptotic-preserving schemes (see [19, 9] and references therein).

Results of the numerical simulations in various regimes (parabolic and hyperbolic) are shown in Figure 1.

3 Traveling wave solutions: Proof of Theorems 1 and 2

3.1 Characteristic equation

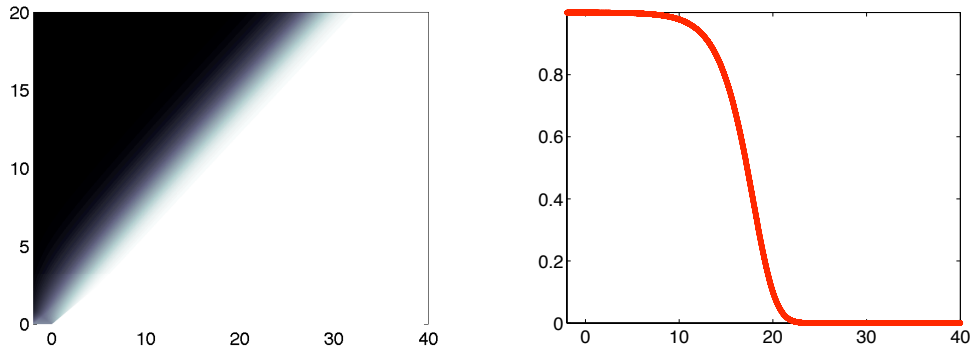
We begin with a careful study of the linearization of (1.7) around $\nu \approx 0$. We expect an exponential decay $e^{-\lambda z}$ as $z \to +\infty$. The characteristic equation reads as follows,

$$(\varepsilon^2 s^2 - 1)\lambda^2 + (1 - \varepsilon^2 F'(0))s\lambda - F'(0) = 0\,. \tag{3.1}$$

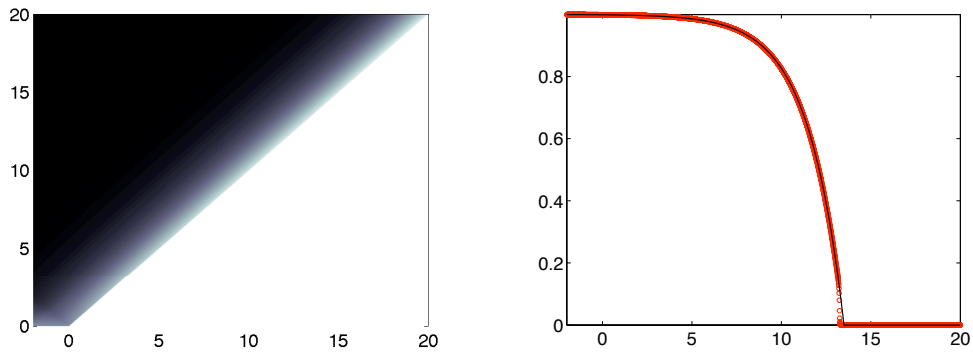
The discriminant is $\Delta = \left(\varepsilon^2 F'(0) + 1\right)^2 s^2 - 4F'(0)$. Hence we expect an oscillatory behaviour in the case $\Delta < 0$, i.e. $s < s^*(\varepsilon)$. We assume henceforth $s \geq s^*(\varepsilon)$. In the case $s < \varepsilon^{-1}$ (*subsonic fronts*) we have to distinguish between the *parabolic regime* $\varepsilon^2 F'(0) < 1$ and the *hyperbolic regime* $\varepsilon^2 F'(0) > 1$. In the former regime equation (3.1) possesses two positive roots, accounting for a damped behaviour. In the latter regime equation (3.1) possesses two negative roots. In the case $s > \varepsilon^{-1}$ (*supersonic fronts*) we get two roots having opposite signs.

Next we investigate the linear behaviour close to $\nu \approx 1$. We expect an exponential relaxation $1 - e^{\lambda' z}$ as $z \to -\infty$. The characteristic equation reads as follows,

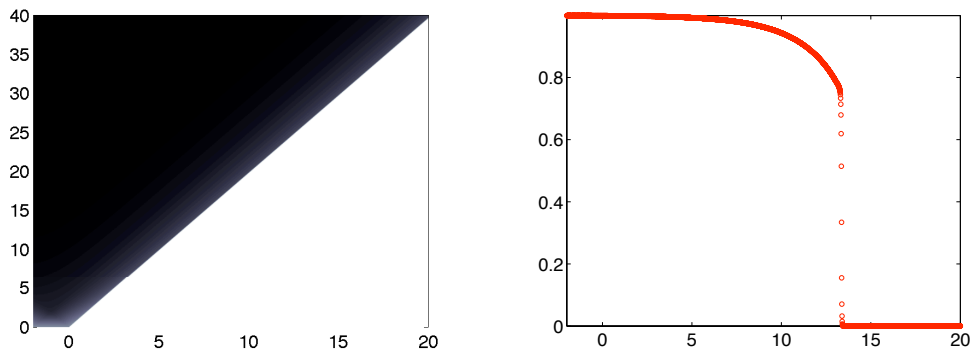
$$(\varepsilon^2 s^2 - 1)\lambda'^2 - (1 - \varepsilon^2 F'(1))s\lambda' - F'(1) = 0\,. \tag{3.2}$$



(a) $\varepsilon = 1/2$



(b) $\varepsilon = 1$



(c) $\varepsilon = 2$

Figure 1: Numerical simulations of the equation (1.1) for $F(\rho) = \rho(1-\rho)$ and for different values of $\varepsilon = 0.5, 1, 2$. Numerical method is described in Section 2. The initial data is the step function $f^+(x < 0) = 1$, $f^+(x > 0) = 0$, and $f^- \equiv 0$. For each value of ε we plot the density function $\rho = f^+ + f^-$ in the (x, t) space, and the density $\rho(t_0, \cdot)$ at some chosen time t_0. We clearly observe in every cases a front traveling asymptotically at speed $s^*(\varepsilon)$ as expected. We also observe the transition between a smooth front and a discontinuous one. The transition occurs at $\varepsilon = 1$. In the case $\varepsilon = 1$ we have superposed the expected profile $\nu(z) = \left(1 - e^{z/2}\right)_+$ in black, continous line, for the sake of comparison.

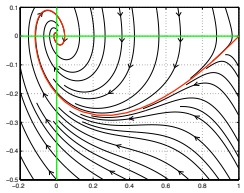
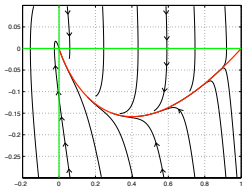
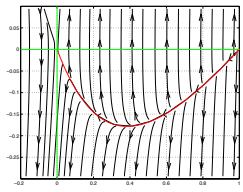
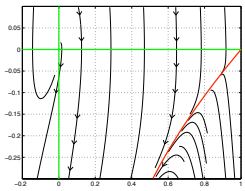
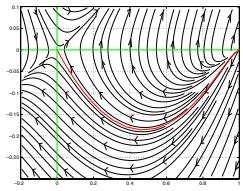
	$s < \varepsilon^{-1}$		$s > \varepsilon^{-1}$
parabolic	if $s < s^*(\varepsilon)$, NO 	if $s \geq s^*(\varepsilon)$, YES 	YES 
hyperbolic	NO 		YES 

Table 1: Phase plane dynamics depending on the regime (parabolic vs. hyperbolic) and the value of the speed with respect to $s^*(\varepsilon)$ and ε^{-1}. In every picture the red line represents the traveling front trajectory, and the green lines are the axes $\{u = 0\}$ and $\{v = 0\}$. We do not consider the case $s = \varepsilon^{-1}$ since the dynamics are singular in this case and should be considered separately (see Section 3.5).

We have $\Delta' = [\varepsilon^2 F'(1) + 1]^2 s^2 - 4F'(1) > 0$. In the case $s < \varepsilon^{-1}$ equation (3.2) possesses two roots having opposite signs. In the case $s > \varepsilon^{-1}$ it has two positive roots.

We summarize our expectations about the possible existence of nonnegative traveling fronts in Table 1.

3.2 Proof of Theorems 1.(a) and 2.(a): Obstruction for $s < s^*(\varepsilon)$

In this section we prove that no traveling front solution exists if the speed is below $s^*(\varepsilon)$.

Proposition 4. *There exists no traveling front with speed s for $s < s^*(\varepsilon)$, where $s^*(\varepsilon)$ is given by* (1.5)-(1.6).

Remark 5. *Note that the proof below works in both cases $\varepsilon^2 F'(0) < 1$ and $\varepsilon^2 F'(0) \geq 1$.*

Proof. We argue by contradiction. The obstruction comes from the exponential decay at $+\infty$. Assume that there exists such a traveling front $\nu(z)$. As $s < s^*(\varepsilon)$, one has $s < \varepsilon^{-1}$ in the parabolic as well as in the hyperbolic regime. Hence, as ν is bounded and satisfies the elliptic equation (1.7) in the sense of distributions, classical regularity estimates show that ν is smooth. It is necessarily decreasing as soon at it is below 1. Otherwise, it would reach a local minimum at some point $z_0 \in \mathbb{R}$, for which $\nu(z_0) < 1$, $\nu'(z_0) = 0$ and $\nu''(z_0) \geq 0$. It would then follow from (1.7) that $F(\nu(z_0)) \leq 0$ and thus $\nu(z_0) = 0$. As $F \in \mathcal{C}^1([0,1])$, the Cauchy-Lipschitz theorem would imply $\nu \equiv 0$, a contradiction.

Next, we define the exponential rate of decay at $+\infty$:

$$\lambda := \liminf_{z \to +\infty} \frac{-\nu'(z)}{\nu(z)} \geq 0\,.$$

Consider a sequence $z_n \to +\infty$ such that $-\nu'(z_n)/\nu(z_n) \to \lambda$ and define the renormalized shift:

$$\nu_n(z) := \frac{\nu(z + z_n)}{\nu(z_n)}\,.$$

This function is locally bounded by classical Harnack estimates. It satisfies

$$(\varepsilon^2 s^2 - 1)\nu_n''(z) + \left(\varepsilon^2 F'(\nu(z+z_n)) - 1\right) s\nu_n'(z) = \frac{1}{\nu(z_n)} F(\nu(z_n)\nu_n(z)), \quad z \in \mathbb{R}.$$

As $F \in \mathcal{C}^1([0,1])$, $F(0) = 0$ and F is concave, the functions $z \mapsto \left(\varepsilon^2 F'(\nu(z+z_n)) - 1\right) s$ and $z \mapsto \frac{1}{\nu(z_n)} F(\nu(z_n)\nu_n(z))$ are uniformly bounded, uniformly in n. Hence, Schauder elliptic regularity estimates yield that the sequence $(\nu_n)_n$ is locally bounded in the Hölder space $\mathcal{C}^\alpha(K)$ for any compact subset $K \subset \mathbb{R}$ and any $\alpha \in (0,1)$. The Ascoli theorem and a diagonal extraction process give an extraction, that we still denote $(\nu_n)_n$, such that $(\nu_n)_n$ converges to some function ν_∞ in $\mathcal{C}^\alpha(K)$ for any compact subset $K \subset \mathbb{R}$ and any $\alpha \in (0,1)$. The limiting function is a solution in the sense of distributions of

$$(\varepsilon^2 s^2 - 1)\nu_\infty''(z) + \left(\varepsilon^2 F'(0) - 1\right) s\nu_\infty'(z) = F'(0)\nu_\infty(z), \quad z \in \mathbb{R}. \tag{3.3}$$

As this equation is linear, one has $\nu_\infty \in \mathcal{C}^\infty(\mathbb{R})$. If $\nu_\infty(z_0) = 0$, then as ν_∞ is nonnegative, one would get $\nu_\infty'(z_0) = 0$ and thus $\nu_\infty \equiv 0$ by uniqueness of the Cauchy problem, which would be a contradiction since $\nu_\infty(0) = \lim_{n\to+\infty} \nu_n(0) = 1$. Thus ν_∞ is positive.

Define $V = \nu_\infty'/\nu_\infty$. The definition of λ yields $\min_{\mathbb{R}} V = V(0) = -\lambda$. Thus $V'(0) = 0$. Hence we deduce from (3.3) that $\nu_\infty(z) = \nu_\infty(0)e^{-\lambda z}$. Plugging this into (3.3), we obtain that λ satisfies the following second order equation,

$$(\varepsilon^2 s^2 - 1)\lambda^2 - (\varepsilon^2 F'(0) - 1)s\lambda - F'(0) = 0.$$

We know from Section 3.1 that, both in the parabolic and hyperbolic regimes, there is no real root in the case $s < s^*(\varepsilon)$. □

3.3 Proof of Theorem 1.(b): Existence of smooth traveling fronts in the parabolic regime $s \in [s^*(\varepsilon), \varepsilon^{-1})$

In [26] the author proves the existence of traveling front, by reducing the problem to the classical Fisher-KPP problem. It is required that the nonlinear function $1 - \varepsilon^2 F'(\rho)$ remains positive, which reads exactly $\varepsilon^2 F'(0) < 1$ in our context. We present below a direct proof based on the method of sub- and supersolutions, following the method developed by Berestycki and Hamel in [6].

3.3.1 The linearized problem

Proposition 6. *Let λ_s be the smallest (positive) root of the characteristic polynomial* (3.1)*. Then $\overline{\nu}(z) = \min\{1, e^{-\lambda_s z}\}$ is a supersolution of (1.7).*

Proof. Let $r(z) = e^{-\lambda_s z}$. Then as r is decreasing and F is concave, it is easy to see that r is a supersolution of (1.7). On the other hand, the constant function 1 is clearly a solution of (1.7). We conclude since the minimum of two supersolutions is a supersolution. □

3.3.2 Resolution of the problem on a bounded interval

Proposition 7. *For all $a > 0$ and $\tau \in \mathbb{R}$, there exists a solution $\nu_{a,\tau}$ of*

$$\begin{cases} (\varepsilon^2 s^2 - 1)\nu_{a,\tau}'' + (\varepsilon^2 F'(\nu_{a,\tau}) - 1)s\nu_{a,\tau}' = F(\nu_{a,\tau}) \text{ in } (-a,a), \\ \nu_{a,\tau}(-a) = \overline{\nu}(-a+\tau), \\ \nu_{a,\tau}(a) = \overline{\nu}(a+\tau). \end{cases} \tag{3.4}$$

Moreover, this function is nonincreasing over $(-a,a)$ and it is unique in the class of nonincreasing functions.

In order to prove this result, we consider the following sequence of problems:

- $\nu_0(z) = \overline{\nu}(z+\tau)$
- ν_{n+1} is solution to

$$\begin{cases} (\varepsilon^2 s^2 - 1)\nu_{n+1}'' + (\varepsilon^2 F'(\nu_n) - 1)s\nu_{n+1}' + M\nu_{n+1} = F(\nu_n) + M\nu_n \text{ in } (-a,a), \\ \nu_{n+1}(-a) = \overline{\nu}(-a+\tau), \\ \nu_{n+1}(a) = \overline{\nu}(a+\tau), \end{cases} \tag{3.5}$$

where $\overline{\nu}$ is defined in Proposition 6 and $M > \frac{s^2}{2}\big(\varepsilon^2 F'(0) - 1\big)$ is large enough so that $s \mapsto F(s) + Ms$ is increasing.

Lemma 8. *The sequence $(\nu_n)_n$ is well-defined. The functions $z \mapsto \nu_n(z)$ are nonincreasing and for all $z \in (-a,a)$, the sequence $(\nu_n(z))_n$ is nonincreasing.*

Proof. We prove this Lemma by induction. Clearly, ν_0 is nonincreasing. First, one can find a unique weak solution $\nu_1 \in \mathcal{C}^0([-a,a])$ of

$$\begin{cases} (\varepsilon^2 s^2 - 1)\nu_1'' + (\varepsilon^2 F'(\nu_0) - 1)s\nu_1' + M\nu_1 = F(\nu_0) + M\nu_0 \text{ in } (-a,a), \\ \nu_1(-a) = \overline{\nu}(-a+\tau), \\ \nu_1(a) = \overline{\nu}(a+\tau), \end{cases} \tag{3.6}$$

using the Lax-Milgram theorem and noticing that the underlying operator is coercive since $M > \frac{s^2}{2}\big(\varepsilon^2 F'(0) - 1\big)$ and $s < \varepsilon^{-1}$.

Let $w_0 = \nu_1 - \nu_0$. As ν_0 is a supersolution of equation (1.7), one has

$$\begin{cases} (\varepsilon^2 s^2 - 1)w_0'' + (\varepsilon^2 F'(\nu_0) - 1)sw_0' + Mw_0 \le 0 \text{ in } (-a,a), \\ w_0(-a) = w_0(a) = 0. \end{cases}$$

As $M > 0$, the weak maximum principle gives $w_0 \le 0$, that is, $\nu_1 \le \nu_0$.

Define the constant function $\underline{\nu} = \overline{\nu}(a+\tau)$. It satisfies

$$(\varepsilon^2 s^2 - 1)\underline{\nu}'' + (\varepsilon^2 F'(\nu_0) - 1)s\underline{\nu}' + M\underline{\nu} = M\underline{\nu} \le F(\underline{\nu}) + M\underline{\nu} \le F(\nu_0) + M\nu_0$$

in $(-a,a)$ since $s \mapsto F(s) + Ms$ is increasing and $\nu_0(z) = \overline{\nu}(z+\tau) \ge \overline{\nu}(a+\tau) = \underline{\nu}$ by monotonicity of $\overline{\nu}$. The same arguments as above lead to $\nu_1 \ge \underline{\nu}$.

Assume that Lemma 8 is true up to rank n. The existence and the uniqueness of ν_{n+1} follow from the same arguments as that of ν_1. Let $w_n = \nu_{n+1} - \nu_n$. As F is concave and $\nu_{n-1} \ge \nu_n$, we know that $F'(\nu_{n-1}) \le F'(\nu_n)$. As ν_n is nonincreasing, we thus get

$$\begin{cases} (\varepsilon^2 s^2 - 1)w_n'' + (\varepsilon^2 F'(\nu_n) - 1)sw_n' + Mw_n \le 0 \text{ in } (-a,a), \\ w_n(-a) = w_n(a) = 0. \end{cases}$$

Hence, $w_n \le 0$ and thus $\nu_{n+1} \le \nu_n$. Similarly, one easily proves that $\nu_{n+1} \ge \underline{\nu}$ in $(-a,a)$.

Differentiating (3.5) and denoting $v = \nu_{n+1}'$, one gets

$$(\varepsilon^2 s^2 - 1)v'' + \big(\varepsilon^2 F'(\nu_0) - 1\big)sv' + \big(M + \varepsilon^2 F''(\nu_0)\nu_0'\big)v = \big(F'(\nu_0) + M\big)\nu_0' \le 0 \text{ in } (-a,a)$$

since $s \mapsto F(s) + Ms$ is increasing and ν_0 is nonincreasing. As F is concave, the zeroth-order term is positive and thus the elliptic maximum principle ensures that v reaches its maximum at $z = -a$ or at $z = a$. But as $\overline{\nu}(a+\tau) \le \nu_{n+1}(z) \le \overline{\nu}(z+\tau)$ for all $z \in (-a,a)$, one has

$$v(-a) \le \limsup_{z \to -a^+} \frac{\nu_{n+1}(z) - \nu_{n+1}(-a)}{z+a} \le \limsup_{z \to -a^+} \frac{\overline{\nu}(z+\tau) - \overline{\nu}(-a+\tau)}{z+a} \le 0$$

and similarly $v(a) \le 0$. Thus $v \le 0$, meaning that ν_{n+1} is nonincreasing. □

Proof of Proposition 7. As the sequence $(\nu_n)_n$ is decreasing and bounded from below, it admits a limit $\nu_{a,\tau}$ as $n \to +\infty$. It easily follows from the classical regularity estimates that $\nu_{a,\tau}$ satisfies the properties of Proposition 7.

If ν_1 and ν_2 are two nondecreasing solutions of (3.4), then the same arguments as before give that $\nu_1^\mu < \nu_1$ in Σ_μ for all $\mu \in (0, 2a)$. Hence, $\nu_1 \leq \nu_2$ and a symmetry argument gives $\nu_1 \equiv \nu_2$. □

Lemma 9. *For all $a > 0$, there exists $\tau_a \in \mathbb{R}$ such that $\nu_{a,\tau_a}(0) = \frac{1}{2}$.*

Proof. Define $I(\tau) := \nu_{a,\tau}(0)$. It follows from the classical regularity estimates and from the uniqueness of $\nu_{a,\tau}$ that I is a continuous function. Moreover, as $\nu_{a,\tau}$ is nonincreasing, one has

$$\overline{\nu}(a + \tau) \leq I(\tau) \leq \overline{\nu}(-a + \tau),$$

where $\overline{\nu}$ is defined in Proposition 6. As $\overline{\nu}(\cdot + \tau) \to 0$ as $\tau \to +\infty$ and $\overline{\nu}(\cdot + \tau) \to 1$ as $\tau \to -\infty$ locally uniformly on $\mathbb{R}$, one has $I(-\infty) = 1$ and $I(+\infty) = 0$. The conclusion follows. □

3.3.3 Existence of traveling fronts with speeds $s \in [s^*(\varepsilon), \varepsilon^{-1})$

We conclude by giving the proof of Theorem 1 as a combination of the above results.

Proof of Theorem 1. Consider a sequence $(a_n)_n$ such that $\lim_{n\to+\infty} a_n = +\infty$ and define $\nu_n(z) := \nu_{a_n,\tau_{a_n}}$ for all $z \in [-a_n, a_n]$. This function is decreasing and satisfies $\nu_n(0) = 1/2$, $0 \leq \nu_n \leq 1$ and

$$(\varepsilon^2 s^2 - 1)\nu_n'' + (\varepsilon^2 F'(\nu_n) - 1)s\nu_n' = F(\nu_n) \text{ in } (-a_n, a_n).$$

Similar arguments as in the proof of Proposition 4 yield that the sequence $(\nu_{a_n,\tau_{a_n}})_n$ converges in $\mathcal{C}^0_{loc}(\mathbb{R})$ as $n \to +\infty$ to a function ν, up to extraction. Then ν satisfies

$$(\varepsilon^2 s^2 - 1)\nu'' + (\varepsilon^2 F'(\nu) - 1)s\nu' = F(\nu) \text{ in } \mathbb{R},$$

it is nonincreasing, $0 \leq \nu \leq 1$ and $\nu(0) = 1/2$.

Define $\ell_\pm := \lim_{z\to\pm\infty} \nu(z)$. Passing to the (weak) limit in the equation satisfied by ν, one gets $F(\ell_\pm) = 0$. As $0 \leq \ell_\pm \leq 1$, the hypotheses on F give $\ell_\pm \in \{0, 1\}$. On the other hand, as ν is nonincreasing, one has

$$\ell_+ \leq \nu(0) = 1/2 \leq \ell_-.$$

We conclude that $\ell_- = \nu(-\infty) = 1$ and $\ell_+ = \nu(+\infty) = 0$. □

The following classical inequality satisfied by the traveling profile will be required later.

Lemma 10. *The traveling profile ν satisfies: $\forall z\ \nu'(z) + \lambda\nu(z) \geq 0$, where λ is the smallest positive root of* (3.1).

Proof. We introduce $\varphi(z) = -\frac{\nu'(z)}{\nu(z)}$. It is nonnegative, and it satisfies the following first-order ODE with a source term

$$\left(\varepsilon^2 s^2 - 1\right)\left(-\varphi'(z) + \varphi(z)^2\right) + \left(1 - \varepsilon^2 F'(\nu(z))\right) s\varphi(z) = \frac{F(\nu(z))}{\nu(z)}.$$

Since F is concave, φ satisfies the differential inequality

$$\left(1 - \varepsilon^2 s^2\right)\varphi'(z) \leq \left(1 - \varepsilon^2 s^2\right)\varphi(z)^2 - \left(1 - \varepsilon^2 F'(0)\right) s\varphi(z) + F'(0).$$

The right-hand-side is the characteristic polynomial of the linearized equation (3.1). Moreover the function φ verifies $\lim_{z\to-\infty} \varphi(z) = 0$. Hence a simple ODE argument shows that $\forall z\ \varphi(z) \leq \lambda$. □

3.4 Proof of Theorem 1.(c): Existence of weak traveling fronts of speed $s = \varepsilon^{-1}$ in the parabolic regime

The aim of this Section is to prove that in the parabolic regime $\varepsilon^2 F'(0) < 1$, there still exists traveling fronts in the limit case $s = \varepsilon^{-1}$ but in the weak sense.

Proposition 11. *Assume that $\varepsilon^2 F'(0) < 1$. Then there exists a weak traveling front of speed $s = \varepsilon^{-1}$.*

Proof. Let $s_n = \varepsilon^{-1} - 1/n$ for all n large enough so that $s_n \geq s^*(\varepsilon)$. We know from the previous Section that we can associate with the speed s_n a smooth traveling front ν_n and that we can assume, up to translation, that $\nu_n(0) = 1/2$. Multiplying equation (1.7) by ν_n' and integrating by parts over $\mathbb{R}$, one gets

$$\begin{aligned} s_n\big(1 - \varepsilon^2 F'(0)\big) \int_{\mathbb{R}} \nu_n'(z)^2 dz &\leq s_n \int_{\mathbb{R}} \Big(1 - \varepsilon^2 F'\big(\nu_n(z)\big)\Big) \nu_n'(z)^2 dz \\ &= -\int_{\mathbb{R}} F\big(\nu_n(z)\big)\nu_n'(z) dz \\ &= -\int_0^1 F(u) du. \end{aligned}$$

Hence, as $\varepsilon^2 F'(0) < 1$, the sequence $(\nu_n')_n$ is bounded in $L^2(\mathbb{R})$ and one can assume, up to extraction, that it admits a weak limit V_∞ in $L^2(\mathbb{R})$. It follows that the sequence $(\nu_n)_n$ converges locally uniformly to $\nu_\infty(z) := \int_0^z V_\infty(z')dz' + 1/2$. Passing to the limit in (1.7), we get that this function is a weak solution of

$$-\big(1 - \varepsilon^2 F'(\nu_\infty(z))\big)\, s\nu_\infty'(z) = F(\nu_\infty(z))\,, \quad z \in \mathbb{R}\,,$$

which ends the proof. □

3.5 Proof of Theorem 2.(b): Existence of weak traveling fronts of speed $s = \varepsilon^{-1}$ in the hyperbolic regime

In this Section we investigate the existence of traveling fronts with critical speed $s = \varepsilon^{-1}$ in the hyperbolic regime $\varepsilon^2 F'(0) = 1$.

Proof of Theorem 2. The function $G(\rho) := \varepsilon^2 F(\rho) - \rho$ is concave, and vanishes when $\rho = 0$. Furthermore, $G(1) < 0$ and $G'(0) = \varepsilon^2 F'(0) - 1 \geq 0$. We now distinguish between the two cases $\varepsilon^2 F'(0) > 1$ and $\varepsilon^2 F'(0) = 1$.

1. **First case:** $\varepsilon^2 F'(0) > 1$**.** As G' is decreasing, there exists a unique $\theta_\varepsilon \in (0,1)$ such that G vanishes.

2. **Second case:** $\varepsilon^2 F'(0) = 1$**.** The only root of G is $\rho = 0$. In this case we set $\theta_\varepsilon = 0$.

For both cases, we have $G'(\rho) < 0$ for all $\rho > \theta_\varepsilon$ since G is strictly concave and $G(0) = G(\theta_\varepsilon) = 0$. Hence, $\varepsilon^2 F'(\rho) < 1$ for all $\rho > \theta_\varepsilon$. Set ν the maximal solution of

$$\begin{cases} \nu'(z) = \dfrac{\varepsilon F(\nu(z))}{\varepsilon^2 F'(\nu(z)) - 1}, \\ \nu(0) = \dfrac{1 + \theta_\varepsilon}{2} > \theta_\varepsilon. \end{cases} \tag{3.7}$$

Let I be the (maximal) interval of definition of ν, with $0 \in I$, and

$$z_0 = \sup\{z \in I, \nu(z) > \theta_\varepsilon\}.$$

1- Conclusion of the argument in the first case: $\varepsilon^2 F'(0) > 1$.

Since $\theta_\varepsilon > 0$, we have necessarily $z_0 < +\infty$. From (3.7), ν is decreasing on $(-\infty, z_0)$. Thus, we have $\nu(z) \to \theta_\varepsilon$ as $z \to z_0^-$. Moreover, one easily gets $\nu(-\infty) = 1$.

We set $\nu(z_0) = \theta_\varepsilon$ and we extend ν by 0 over (z_0, ∞). We observe that ν is a weak solution, in the sense of distributions, of

$$\left(\varepsilon^2 F(\nu) - \nu\right)' = \varepsilon F(\nu) \text{ on } \mathbb{R}$$

since $\varepsilon^2 F(0) = 0$ and $\varepsilon^2 F(\theta_\varepsilon) = \theta_\varepsilon$.

Up to space shifting $z - z_0$, we may assume that the discontinuity arises at $z = 0$.

Example: the case $F(\rho) = \rho(1-\rho)$ and $\varepsilon > 1$. The traveling profile solves

$$\nu'(z) = \frac{\varepsilon \nu(z)(1-\nu(z))}{\varepsilon^2 - 1 - 2\varepsilon^2 \nu(z)},$$

or equivalently

$$\nu(z)^{\varepsilon^2 - 1} \left(1 - \nu(z)\right)^{\varepsilon^2 + 1} = k e^{\varepsilon z}.$$

The constant k is determined by the condition $\nu(0) = \theta_\varepsilon = 1 - \varepsilon^{-2}$. Finally the traveling profile $\nu(z)$ satisfies the following implicit relation:

$$\nu(z)^{\varepsilon^2 - 1} \left(1 - \nu(z)\right)^{1+\varepsilon^2} = \left(1 - \varepsilon^{-2}\right)^{\varepsilon^2 - 1} \left(\varepsilon^{-2}\right)^{\varepsilon^2 + 1} e^{\varepsilon z} = \left(\varepsilon^2 - 1\right)^{\varepsilon^2 - 1} e^{\varepsilon z + 2\varepsilon^2 \log \varepsilon^2}. \tag{3.8}$$

2- Conclusion of the argument in the second case: $\varepsilon^2 F'(0) = 1$.

The difference here is that $\theta_\varepsilon = 0$. To conclude the proof as previously, we just need to check that z_0 is finite. We argue by contradiction. Assume $z_0 = +\infty$. Linearizing the r.h.s. of (3.7) near $\nu = 0$, we get

$$\nu'(z) = \frac{F'(0)}{\varepsilon F''(0)} + o(\nu(z)), \quad \text{as } z \to +\infty \tag{3.9}$$

We get a contradiction because $\varepsilon^{-1} F'(0)/F''(0) < 0$.

Finally, we create a continuous front with the same extension idea as for the first case.

Example: the case $F(\rho) = \rho(1-\rho)$ and $\varepsilon = 1$. The traveling profile reads (3.8):

$$\nu(z) = \left(1 - e^{z/2}\right)_+.$$

□

3.6 Proof of Theorem 1.(d) and Theorem 2.(c): Existence of supersonic traveling fronts $s > \varepsilon^{-1}$

In this Section we investigate the existence of supersonic traveling fronts with speeds above the maximal speed of propagation $s > \varepsilon^{-1}$. These fronts are essentially driven by growth. The existence of such "unrealistic" fronts is motivated by the extreme case $\varepsilon \to +\infty$ for which we have formally $\partial_t \rho = F(\rho)$ (1.3). There exist traveling fronts of arbitrary speed which are solutions to $-s\nu' = F(\nu)$.

Proposition 12. *Given any speed $s > \varepsilon^{-1}$ there exists a smooth traveling front $\nu(x - st)$ with this speed.*

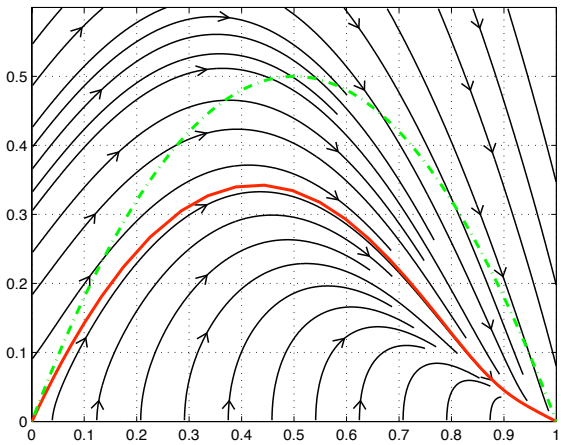


Figure 2: Supersonic traveling front in the phase plane (V, V') for the nonlinearity $F(\rho) = \rho(1-\rho)$, and parameters $\varepsilon = \sqrt{2}$, $s = 1 > \varepsilon^{-1}$. Be aware of the time reversal $\nu(z) = V(-z)$, which is the reason why $V' \geq 0$. The red line represents the traveling profile, and the green line represents the supersolution $kF(v)$.

Proof. We sketch the proof. We give below the key arguments derived from phase plane analysis. The same procedure as developped in Section 3.3 based on sub- and supersolutions could be reproduced based on the following ingredients.

We learn from simple phase plane considerations associated to (1.7) that the situation is reversed in comparison to the classical Fisher-KPP case (or $\varepsilon^2 F'(0) < 1$ and $s \in [s^*(\varepsilon), \varepsilon^{-1})$). Namely the point $(0,0)$ is a saddle point (instead of a stable node) whereas $(1,0)$ is an unstable node (instead of saddle point). This motivates "time reversal": $V(z) = \nu(-z)$. Equation (1.7) becomes

$$(\varepsilon^2 s^2 - 1)V''(z) - \left(\varepsilon^2 F'\left(V(z)\right) - 1\right) sV'(z) = F\left(V(z)\right), \quad z \in \mathbb{R}.$$

We make the classical phase-plane transformation $V' = P$ [29, 20]. We end up with the implicit ODE with Dirichlet boundary conditions for P:

$$(\varepsilon^2 s^2 - 1)P'(v) - \left(\varepsilon^2 F'(v) - 1\right) s = \frac{F(v)}{P(v)}, \quad P(0) = P(1) = 0.$$

The unstable direction is given by $P(v) = \lambda v$ where λ is the positive root of

$$(\varepsilon^2 s^2 - 1)\lambda - \left(\varepsilon^2 F'(0) - 1\right) s = \frac{F'(0)}{\lambda}. \tag{3.10}$$

Since F is concave we deduce that $P(v) = \lambda v$ is a supersolution as in Proposition 6. In fact, denoting $Q(v) = P(v) - \lambda v$ we have

$$\begin{aligned}(\varepsilon^2 s^2 - 1)Q'(v) = (\varepsilon^2 s^2 - 1)(P'(v) - \lambda) &\leq s\varepsilon^2 \left(F'(v) - F'(0)\right) + \frac{F(v)}{P(v)} - \frac{F'(0)}{\lambda}, \\ &\leq F'(0)v\left(\frac{1}{P(v)} - \frac{1}{\lambda v}\right) \\ &\leq -\frac{F'(0)}{\lambda P(v)} Q(v).\end{aligned}$$

Hence the trajectory leaving the saddle point $(0,0)$ in the phase plane (V, V') remains below the line $V' \leq \lambda V$.

On the other hand it is straightforward to check that $kF(v)$ is a supersolution where $k = \varepsilon^2 s/(\varepsilon^2 s^2 - 1)$. We denote $R(v) = P(v) - kF(v)$. We have $ks > 1$ and

$$(\varepsilon^2 s^2 - 1)R'(v) = (\varepsilon^2 s^2 - 1)(P'(v) - kF'(v)) = \varepsilon^2 sF'(v) - s + \frac{F(v)}{P(v)} - (\varepsilon^2 s^2 - 1)kF'(v)$$
$$= -s + \frac{1}{k} - \frac{R(v)}{kP(v)} < -\frac{R(v)}{kP(v)}\,.$$

We also show that initially (as $v \to 0$) we have $kF'(0) > \lambda$. This proves that $R(v) \leq 0$ for all $v \in (0,1)$. Indeed, we plug $kF'(0)$ in place of λ into (3.10) and we get

$$(\varepsilon^2 s^2 - 1)kF'(0) - \left(\varepsilon^2 F'(0) - 1\right) s = s > \frac{1}{k} = \frac{F'(0)}{kF'(0)}\,.$$

As a conclusion the trajectory leaving the saddle node at $(0,0)$ is trapped in the set $\{0 \leq v \leq 1\,,\ 0 \leq p \leq kF(v)\}$ (see Fig. 2). By the Poincaré-Bendixon Theorem it necessarily converges to the stable node at $(1,0)$. This heteroclinic trajectory is the traveling front in the supersonic case. □

4 Linear stability of traveling front solutions

In this Section we investigate the linear stability of the traveling front having minimal speed $s = s^*(\varepsilon)$ in both the parabolic and the hyperbolic regime. We seek stability in some weighted L^2 space. The important matter here is to identify the weight e^{ϕ}. The same weight shall be used crucially for the nonlinear stability analysis (Section 5).

We recall that the minimal speed is given by

$$s^*(\varepsilon) = \begin{cases} \dfrac{2\sqrt{F'(0)}}{1+\varepsilon^2 F'(0)} & \text{if } \varepsilon^2 F'(0) < 1 \\ \varepsilon^{-1} & \text{if } \varepsilon^2 F'(0) \geq 1 \end{cases}$$

The profile of the wave has the following properties in the case $\varepsilon^2 F'(0) < 1$:

$$\forall z \quad \nu(z) \geq 0\,, \quad \partial_z \nu(z) \leq 0\,, \quad \partial_z \nu(z) + \lambda \nu(z) \geq 0\,,$$

where the decay exponent λ is

$$\lambda = \frac{s(1-\varepsilon^2 F'(0))}{2(1-\varepsilon^2 s^2)} = \frac{1+\varepsilon^2 F'(0)}{1-\varepsilon^2 F'(0)}\,.$$

We will use in this Section the formulation (1.3) of our system. The linearized system around the stationary profile ν in the moving frame $z = x - st$ reads

$$\begin{cases} (\partial_t - s\partial_z)u + \partial_z\left(\dfrac{v}{\varepsilon}\right) = F'(\nu)u \\ \varepsilon(\partial_t - s\partial_z)v + \partial_z u = -\dfrac{v}{\varepsilon}\,. \end{cases} \tag{4.1}$$

Proposition 13. *Let $\varepsilon > 0$. In the hyperbolic regime $\varepsilon^2 F'(0) \geq 1$ assume in addition that the initial perturbation has the same support as the wave. There exists a function $\phi_\varepsilon(z)$ such that the minimal speed traveling front is linearly stable in the weighted $L^2(e^{2\phi_\varepsilon(z)}dz)$ space. More precisely the following Lyapunov identity holds true for solutions of the linear system* (4.1),

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}} \left(|u|^2 + |v|^2\right) e^{2\phi_\varepsilon(z)}\, dz\right) \leq 0\,.$$

Proof. We denote $\phi = \phi_\varepsilon$ for the sake of clarity. We multiply the first equation by $ue^{2\phi}$, and the second equation by $ve^{2\phi}$, where ϕ is to be determined. We get

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}} |u|^2 e^{2\phi(z)}\,dz\right) + \frac{s}{2}\int_{\mathbb{R}} |u|^2 \partial_z e^{2\phi(z)}\,dz + \int_{\mathbb{R}} \partial_z\left(\frac{v}{\varepsilon}\right) ue^{2\phi(z)}\,dz = \int_{\mathbb{R}} F'(\nu)|u|^2 e^{2\phi(z)}\,dz\,,$$
$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}} |v|^2 e^{2\phi(z)}\,dz\right) + \frac{s}{2}\int_{\mathbb{R}} |v|^2 \partial_z e^{2\phi(z)}\,dz + \int_{\mathbb{R}} \partial_z\left(\frac{u}{\varepsilon}\right) ve^{2\phi(z)}\,dz = -\frac{1}{\varepsilon^2}\int_{\mathbb{R}} |v|^2 e^{2\phi(z)}\,dz\,.$$

Summing the two estimates we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}} \left(|u|^2 + |v|^2\right) e^{2\phi(z)}\,dz\right)$$
$$+\int_{\mathbb{R}} \left(s\partial_z\phi(z) - F'(\nu)\right)|u|^2 e^{2\phi(z)}\,dz + \int_{\mathbb{R}} \left(s\partial_z\phi(z) + \frac{1}{\varepsilon^2}\right)|v|^2 e^{2\phi(z)}\,dz - \frac{2}{\varepsilon}\int_{\mathbb{R}} \left(\partial_z\phi(z)\right) uve^{2\phi(z)}\,dz = 0\,.$$

We seek an energy dissipation estimate, see (4.3) below. Therefore we require that the last quadratic form acting on (u, v) is nonnegative. This is guaranteed if $\partial_z\phi \geq 0$ and the following discriminant is nonpositive:

$$\begin{aligned}\Delta(z) &= \frac{4}{\varepsilon^2}\left(\partial_z\phi(z)\right)^2 - 4\left(s\partial_z\phi(z) - F'(\nu)\right)\left(s\partial_z\phi(z) + \frac{1}{\varepsilon^2}\right)\\ &= \frac{4}{\varepsilon^2}\left(\left(1-\varepsilon^2 s^2\right)\left(\partial_z\phi(z)\right)^2 - s\left(1-\varepsilon^2 F'(\nu)\right)\partial_z\phi(z) + F'(\nu)\right)\,. \end{aligned} \tag{4.2}$$

The rest of the proof is devoted to finding such a weight $\phi(z)$ satisfying this sign condition. We distinguish between the parabolic and the hyperbolic regime.

1- The parabolic regime.

In the case $\varepsilon^2 F'(0) < 1$ we have $\varepsilon^2 s^2 < 1$. Hence the optimal choice for $\partial_z\phi$ is:

$$\partial_z\phi(z) = \frac{s\left(1-\varepsilon^2 F'(\nu)\right)}{2\left(1-\varepsilon^2 s^2\right)} = \lambda\frac{1-\varepsilon^2 F'(\nu)}{1-\varepsilon^2 F'(0)} \geq 0\,.$$

Notice that $\partial_z\phi \to \lambda$ as $z \to +\infty$. We check that the discriminant is indeed nonpositive:

$$\begin{aligned}\varepsilon^2\Delta(z) &= -4\left(1-\varepsilon^2 s^2\right)\left(\partial_z\phi(z)\right)^2 + 4F'(\nu)\\ &= \frac{1}{\left(1-\varepsilon^2 s^2\right)}\left(-s^2\left(1+\varepsilon^2 F'(\nu)\right)^2 + 4F'(\nu)\right)\\ &= \frac{1}{\left(1-\varepsilon^2 F'(0)\right)^2}\left(-4F'(0)\left(1+\varepsilon^2 F'(\nu)\right)^2 + 4F'(\nu)\left(1+\varepsilon^2 F'(0)\right)^2\right)\\ &= \frac{-4}{\left(1-\varepsilon^2 F'(0)\right)^2}\left(F'(0) - F'(\nu)\right)\left(1-\varepsilon^4 F'(0)F'(\nu)\right)\,.\end{aligned}$$

We have $\Delta(z) \leq 0$ since $\forall z\ F'(\nu(z)) \leq F'(0)$ and $\varepsilon^2 F'(0) < 1$. Since the quadratic form is nonnegative, we may control it by a sum of squares. This is the purpose of the next computation. We have

$$\left|\frac{2}{\varepsilon}\int_{\mathbb{R}} \left(\partial_z\phi(z)\right) uve^{2\phi(z)}\,dz\right|$$
$$\leq \int_{\mathbb{R}} \left(s\partial_z\phi(z) - F'(\nu) - A(z)\right)|u|^2 e^{2\phi(z)}\,dz + \int_{\mathbb{R}} \left(s\partial_z\phi(z) + \frac{1}{\varepsilon^2} - A(z)\right)|v|^2 e^{2\phi(z)}\,dz\,,$$

where $A(z)$ is solution of

$$4\left(s\partial_z\phi(z) - F'(\nu) - A(z)\right)\left(s\partial_z\phi(z) + \frac{1}{\varepsilon^2} - A(z)\right) = \frac{4}{\varepsilon^2}\left(\partial_z\phi(z)\right)^2\,.$$

A straightforward computation gives

$$
\begin{aligned}
2A(z) &= \left(2s\partial_z\phi(z) - F'(\nu) + \frac{1}{\varepsilon^2}\right) - \left(\left(F'(\nu) + \frac{1}{\varepsilon^2}\right)^2 + \frac{4}{\varepsilon^2}\left(\partial_z\phi(z)\right)^2\right)^{1/2} \\
&= \frac{1-\varepsilon^2 F'(\nu)}{\varepsilon^2\left(1-\varepsilon^2 s^2\right)} - \frac{1}{\varepsilon^2\left(1-\varepsilon^2 s^2\right)}\left(\left(1-\varepsilon^2 s^2\right)^2\left(1+\varepsilon^2 F'(\nu)\right)^2 + \varepsilon^2 s^2\left(1-\varepsilon^2 F'(\nu)\right)^2\right)^{1/2} \\
&= \frac{1-\varepsilon^2 F'(\nu)}{\varepsilon^2\left(1-\varepsilon^2 s^2\right)}\left(1 - \left(\left(\frac{1-\varepsilon^2 F'(0)}{1+\varepsilon^2 F'(0)}\right)^4\left(\frac{1+\varepsilon^2 F'(\nu)}{1-\varepsilon^2 F'(\nu)}\right)^2 + \frac{4\varepsilon^2 F'(0)}{\left(1+\varepsilon^2 F'(0)\right)^2}\right)^{1/2}\right) \\
&= \frac{1-\varepsilon^2 F'(\nu)}{\varepsilon^2\left(1-\varepsilon^2 s^2\right)}\left(1 - \left(1 + \left(\frac{1-\varepsilon^2 F'(0)}{1+\varepsilon^2 F'(0)}\right)^4\left(\frac{1+\varepsilon^2 F'(\nu)}{1-\varepsilon^2 F'(\nu)}\right)^2 - \frac{\left(1-\varepsilon^2 F'(0)\right)^2}{\left(1+\varepsilon^2 F'(0)\right)^2}\right)^{1/2}\right).
\end{aligned}
$$

We clearly have $A(z) \geq 0$ since

$$
\forall z \quad \frac{1+\varepsilon^2 F'(\nu)}{1-\varepsilon^2 F'(\nu)} \leq \frac{1+\varepsilon^2 F'(0)}{1-\varepsilon^2 F'(0)}.
$$

Finally we obtain in the case $\varepsilon^2 F'(0) < 1$,

$$
\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}}\left(|u|^2 + |v|^2\right)e^{2\phi(z)}\,dz\right) + \int_{\mathbb{R}} A(z)\left(|u|^2 + |v|^2\right)e^{2\phi(z)}\,dz \leq 0. \tag{4.3}
$$

2- The hyperbolic regime.

We assume for simplicity that the support of the traveling profile is $\operatorname{Supp}\nu = (-\infty, 0]$.
In the hyperbolic regime we have $s = \varepsilon^{-1}$, so the discriminant equation (4.2) reduces to

$$
\Delta(z) = \frac{4}{\varepsilon^2}\left(-s\left(1-\varepsilon^2 F'(\nu)\right)\partial_z\phi(z) + F'(\nu)\right).
$$

We naturally choose

$$
\partial_z\phi(z) = \frac{\varepsilon F'(\nu)}{1-\varepsilon^2 F'(\nu)}.
$$

Within this choice for ϕ we get,

$$
\frac{d}{dt}\left(\frac{1}{2}\int_{z\leq 0}\left(|u|^2 + |v|^2\right)e^{2\phi(z)}\,dz\right) + \int_{z\leq 0} A(z)\left(\varepsilon^2 F'(\nu(z))u - v\right)^2 e^{2\phi(z)}\,dz = 0,
$$

where the additional weight in the dissipation writes:

$$
A(z) = \frac{1}{\varepsilon^2\left(1-\varepsilon^2 F'(\nu(z))\right)}.
$$

In the case $\varepsilon^2 F'(0) > 1$ we have $1 - \varepsilon^2 F'(\nu(z)) > 0$ on $\operatorname{Supp}\nu$ (see Section 3.5). Notice that the monotonicity of ϕ may change on $\operatorname{Supp}\nu$ since $F'(\nu(z))$ may change sign. We observe that $A(z)$ is uniformly bounded from below on $\operatorname{Supp}\nu$.

In the transition case $\varepsilon^2 F'(0) = 1$, we have $\partial_z\phi(z) \to +\infty$ as $z \to 0^-$. We observe that $A(z) \to +\infty$ as $z \to 0^-$ too.

□

Example: the case $F(\rho) = \rho(1-\rho)$, and $\varepsilon = 1$. We can easily compute from Section 3.5

$$
\phi(z) = -\frac{z}{2} - \log\left(1 - e^{z/2}\right).
$$

Remark 14 (Lack of coercivity)**. 1- The parabolic regime.** *We directly observe that $A(z) \to 0$ as $z \to +\infty$ in the Lyapunov identity* (4.3)*. This corresponds to the lack of coercivity of the linear operator. It has been clearly identified for the classical Fisher-KPP equation [22, 23]. This lack of coercivity is a source of complication for the next question,* i.e. *nonlinear stability (see Section 5).* **2- The hyperbolic regime.** *The situation is more degenerated here: the dissipation provides information about the relaxation of v towards $\varepsilon^2 F'(\nu)u$ only.*

5 Nonlinear stability of traveling front solutions in the parabolic regime $\varepsilon^2 F'(0) < 1$

In this Section we investigate the stability of the traveling profile having minimal speed $s = s^*(\varepsilon)$ in the parabolic regime. We seek stability in the energy class. Energy methods have been successfully applied to reaction-diffusion equations [22, 23, 37, 24]. We follow the strategy developped in [23] for a simpler equation, namely the damped hyperbolic Fisher-KPP equation.

Before stating the theorem we give some useful notations. The perturbation is denoted by $u(t,z) = \rho(t,z) - \nu(z)$ where $z = x - st$ is the space variable in the moving frame. We also need some weighted perturbation $w = e^{\phi} u$, where ϕ is an explicit weight to be precised later (5.11).

Theorem 15. *For all $\varepsilon \in \left(0, 1/\sqrt{F'(0)}\right)$ there exists a constant $c(\varepsilon)$ such that the following claim holds true: let u^0 be any compactly supported initial perturbation which satisfies*

$$\|u^0\|^2_{H^1(\mathbb{R})} + \|w^0\|^2_{H^1(\mathbb{R})} \leq c(\varepsilon)\,,$$

then there exists $z_0 \in \mathbb{R}$ such that

$$\sup_{t>0} \left(\|\partial_z u(t,\cdot)\|_2^2 + \int_{z<z_0} |u(t,z)|^2\, dz + \|w(t,\cdot)\|^2_{H^1} \right) \leq c(\varepsilon)\,,$$

remains uniformly small for all time $t > 0$, and the perturbation is globally decaying in the following sense:

$$\left(\|\partial_z u\|_2^2 + \int_{z<z_0} |u|^2\, dz + \|\partial_z w\|_2^2 + \int_{z>z_0} e^{-\phi(z)} |w|^2\, dz \right) \in L^2(0,+\infty)\,.$$

Remark 16.

1. *The additional weight $e^{-\phi(z)}$ in the last contribution (weighted L^2 space) is specific to the lack of coercivity in the energy estimates.*
2. *The constant $c(\varepsilon)$ that we obtain degenerates as $\varepsilon \to 1/\sqrt{F'(0)}$, due to the transition from a parabolic to an hyperbolic regime.*
3. *We restrict ourselves to compactly supported initial perturbations u^0 to justify all integration by parts. Indeed the solution $u(t,z)$ remains compactly supported for all $t > 0$ because of the finite speed of propagation (see the kinetic formulation* (2.1) *and [15, Chapter 12]). The result would be the same if we were assuming that u^0 decays sufficiently fast at infinity.*

Proof. We proceed in several steps.

1- Derivation of the energy estimates. The equation satisfied by the perturbation u writes

$$\begin{aligned} \varepsilon^2 \left(\partial_{tt} u - 2s\partial_{tz} u + s^2 \partial_{zz} u\right) + \left(1 - \varepsilon^2 F'(\nu + u)\right)\left(\partial_t u - s\partial_z u\right) - \partial_{zz} u \\ + \varepsilon^2 (F'(\nu+u) - F'(\nu)) s\partial_z \nu = F(\nu+u) - F(\nu)\,. \end{aligned} \tag{5.1}$$

We write the nonlinearities as follows:

$$\begin{aligned} F'(\nu+u) &= F'(\nu) + K_1(z;u)u\,, \\ F'(\nu+u) - F'(\nu) &= F''(\nu)u + K_2(z;u)u^2\,, \\ F(\nu+u) - F(\nu) &= F'(\nu)u + K_3(z;u)u^2\,. \end{aligned}$$

where the functions K_i are uniformly bounded in $L^\infty(\mathbb{R})$. More precisely we have

$$K_1(z;u) = \int_0^1 F''(\nu+tu)dt\,,\ K_2(z;u) = \int_0^1 (1-t)F'''(\nu+tu)dt\,,\ K_3(z;u) = \int_0^1 (1-t)F''(\nu+tu)dt\,.$$

Thus we can decompose equation (5.1) into linear and nonlinear contributions:

$$\begin{aligned}\varepsilon^2\left(\partial_{tt}u - 2s\partial_{tz}u + s^2\partial_{zz}u\right) + \left(1-\varepsilon^2F'(\nu)\right)\left(\partial_t u - s\partial_z u\right) - \partial_{zz}u + \left(s\varepsilon^2F''(\nu)\partial_z\nu - F'(\nu)\right)u \\ = \varepsilon^2K_1(z;u)u\left(\partial_t u - s\partial_z u\right) + \left(K_3(z;u) - s\varepsilon^2K_2(z;u)\partial_z\nu\right)u^2\,. \quad (5.2)\end{aligned}$$

Testing equation (5.2) against $\partial_t u - s\partial_z u$ yields our first energy estimate (hyperbolic energy):

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{\varepsilon^2}{2}\int_{\mathbb{R}}\left|\partial_t u - s\partial_z u\right|^2\,dz + \frac{1}{2}\int_{\mathbb{R}}\left|\partial_z u\right|^2\,dz + \frac{1}{2}\int_{\mathbb{R}}\left(s\varepsilon^2F''(\nu)\partial_z\nu - F'(\nu)\right)|u|^2\,dz\right\}\\ &\qquad + \int_{\mathbb{R}}\left(1-\varepsilon^2F'(\nu)\right)\left|\partial_t u - s\partial_z u\right|^2 + \frac{s}{2}\int_{\mathbb{R}}\partial_z\left(s\varepsilon^2F''(\nu)\partial_z\nu - F'(\nu)\right)|u|^2\,dz\\ &= \varepsilon^2\int_{\mathbb{R}}K_1(z;u)u\left|\partial_t u - s\partial_z u\right|^2\,dz + \int_{\mathbb{R}}\left(K_3(z;u) - se^2K_2(z;u)\partial_z\nu\right)u^2\left(\partial_t u - s\partial_z u\right)\,dz\,. \quad (5.3)\end{aligned}$$

We are lacking coercivity with respect to H^1 norm in the energy dissipation. Testing equation (5.2) against u yields our second energy estimate (parabolic energy):

$$\begin{aligned}&\frac{d}{dt}\left\{\varepsilon^2\int_{\mathbb{R}}u\left(\partial_t u - s\partial_z u\right)\,dz + \frac{1}{2}\int_{\mathbb{R}}\left(1-\varepsilon^2F'(\nu)\right)|u|^2\,dz\right\}\\ &\qquad - \varepsilon^2\int_{\mathbb{R}}\left|\partial_t u - s\partial_z u\right|^2dz + \int_{\mathbb{R}}\left|\partial_z u\right|^2dz + \int_{\mathbb{R}}\left(\frac{s\varepsilon^2}{2}F''(\nu)\partial_z\nu - F'(\nu)\right)|u|^2\,dz\\ &\qquad = \varepsilon^2\int_{\mathbb{R}}K_1(z;u)u^2\left(\partial_t u - s\partial_z u\right)\,dz + \int_{\mathbb{R}}\left(K_3(z;u) - se^2K_2(z;u)\partial_z\nu\right)u^3\,dz\,. \quad (5.4)\end{aligned}$$

We introduce the following notations for the two energy contributions and the respective quadratic dissipations (5.3),(5.4):

$$\begin{aligned}E_1^u(t) &= \frac{\varepsilon^2}{2}\int_{\mathbb{R}}\left|\partial_t u - s\partial_z u\right|^2\,dz + \frac{1}{2}\int_{\mathbb{R}}\left|\partial_z u\right|^2\,dz + \frac{1}{2}\int_{\mathbb{R}}\left(s\varepsilon^2F''(\nu)\partial_z\nu - F'(\nu)\right)|u|^2\,dz\,,\\ E_2^u(t) &= \varepsilon^2\int_{\mathbb{R}}u\left(\partial_t u - s\partial_z u\right)\,dz + \frac{1}{2}\int_{\mathbb{R}}\left(1-\varepsilon^2F'(\nu)\right)|u|^2\,dz\,,\\ Q_1^u(t) &= \int_{\mathbb{R}}\left(1-\varepsilon^2F'(\nu)\right)\left|\partial_t u - s\partial_z u\right|^2 + \frac{s}{2}\int_{\mathbb{R}}\partial_z\left(s\varepsilon^2F''(\nu)\partial_z\nu - F'(\nu)\right)|u|^2\,dz\,,\\ Q_2^u(t) &= -\varepsilon^2\int_{\mathbb{R}}\left|\partial_t u - s\partial_z\right|^2dz + \int_{\mathbb{R}}\left|\partial_z u\right|^2dz + \int_{\mathbb{R}}\left(\frac{s\varepsilon^2}{2}F''(\nu)\partial_z\nu - F'(\nu)\right)|u|^2\,dz\,.\end{aligned}$$

The delicate issue is to control the zeroth-order terms. In particular we define the weights

$$\begin{aligned}A_1(z) &= s\varepsilon^2F''(\nu)\partial_z\nu - F'(\nu)\,,\\ A_2(z) &= \frac{s\varepsilon^2}{2}F''(\nu)\partial_z\nu - F'(\nu)\,.\end{aligned}$$

They change sign over $\mathbb{R}$. More precisely we have

$$\begin{aligned}\lim_{z\to-\infty}A_1(z) &= \lim_{z\to-\infty}A_2(z) = -F'(1) > 0\,,\\ \lim_{z\to+\infty}A_1(z) &= \lim_{z\to+\infty}A_2(z) = -F'(0) < 0\,.\end{aligned}$$

To circumvent this issue we introduce $w(t,z) = e^{\phi(z)}u(t,z)$ as in [23] and the previous Section 4, where $\phi(z)$ is a weight to be determined later (5.10). The new function $w(t,z)$ satisfies the following equation:

$$\begin{aligned}
&\varepsilon^2\partial_{tt}w - 2\varepsilon^2 s\partial_{tz}w + \left(2\varepsilon^2 s\partial_z\phi + 1 - \varepsilon^2 F'(\nu)\right)\partial_t w + \left(-s\left(1-\varepsilon^2F'(\nu)\right) - 2(\varepsilon^2 s^2 - 1)\partial_z\phi\right)\partial_z w \\
&+ (\varepsilon^2 s^2 - 1)\partial_{zz}w + \left(s\left(1-\varepsilon^2F'(\nu)\right)\partial_z\phi + (\varepsilon^2 s^2 - 1)\left(-\partial_{zz}\phi + |\partial_z\phi|^2\right) + \varepsilon^2 sF''(\nu)\partial_z\nu - F'(\nu)\right)w \\
&\qquad = \varepsilon^2 K_1(z;u)u\left(\partial_t w - s\partial_z w\right) + \left(K_3(z;u) - \varepsilon^2 sK_2(z;u)\partial_z\nu + \varepsilon^2 sK_1(z;u)\partial_z\phi\right)uw\,. \qquad (5.5)
\end{aligned}$$

We denote the prefactors of $\partial_t w$, $\partial_z w$ and w as A_3, A_4 and A_5 respectively:

$$\begin{aligned}
A_3(z) &= 2\varepsilon^2 s\partial_z\phi + 1 - \varepsilon^2 F'(\nu)\,, \\
A_4(z) &= -s\left(1-\varepsilon^2F'(\nu)\right) - 2(\varepsilon^2 s^2 - 1)\partial_z\phi\,, \qquad (5.6)\\
A_5(z) &= s\left(1-\varepsilon^2F'(\nu)\right)\partial_z\phi + (\varepsilon^2 s^2 - 1)\left(-\partial_{zz}\phi + |\partial_z\phi|^2\right) + \varepsilon^2 sF''(\nu)\partial_z\nu - F'(\nu)\,.
\end{aligned}$$

Testing (5.5) against $\partial_t w$, we obtain our third energy estimate:

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{\varepsilon^2}{2}\int_{\mathbb{R}}|\partial_t w|^2\,dz + \frac{1-\varepsilon^2 s^2}{2}\int_{\mathbb{R}}|\partial_z w|^2\,dz + \frac{1}{2}\int_{\mathbb{R}}A_5(z)|w|^2\,dz\right\} \\
&\qquad\qquad + \int_{\mathbb{R}}A_3(z)|\partial_t w|^2\,dz + \int_{\mathbb{R}}A_4(z)\partial_t w\partial_z w\,dz \\
&= \varepsilon^2\int_{\mathbb{R}}K_1(z;u)u\left(|\partial_t w|^2 - s\partial_t w\partial_z w\right)dz + \int_{\mathbb{R}}\left(K_3(z;u) - \varepsilon^2 sK_2(z;u)\partial_z\nu + \varepsilon^2 sK_1(z;u)\partial_z\phi\right)uw\partial_t w\,dz\,,
\end{aligned}$$

Testing (5.5) against w we obtain our last energy estimate:

$$\begin{aligned}
&\frac{d}{dt}\left\{\varepsilon^2\int_{\mathbb{R}}w\partial_t w\,dz + \frac{1}{2}\int_{\mathbb{R}}A_3(z)|w|^2\,dz\right\} \\
&-\varepsilon^2\int_{\mathbb{R}}|\partial_t w|^2\,dz + (1-\varepsilon^2 s^2)\int_{\mathbb{R}}|\partial_z w|^2\,dz + 2s\varepsilon^2\int_{\mathbb{R}}\partial_t w\partial_z w\,dz + \int_{\mathbb{R}}\left(A_5(z) - \frac{\partial_z A_4(z)}{2}\right)|w|^2\,dz \\
&= \int_{\mathbb{R}}\left(K_3(z;u) - \varepsilon^2 sK_2(z;u)\partial_z\nu + \varepsilon^2 sK_1(z;u)\partial_z\phi\right)uw^2\,dz + \int_{\mathbb{R}}\varepsilon^2 K_1(z;u)uw\left(\partial_t w - s\partial_z w\right)dz\,,
\end{aligned}$$

We introduce again useful notations for the two energy contributions and the associated quadratic dissipations:

$$E_1^w(t) = \frac{\varepsilon^2}{2}\int_{\mathbb{R}}|\partial_t w|^2\,dz + \frac{1-\varepsilon^2 s^2}{2}\int_{\mathbb{R}}|\partial_z w|^2\,dz + \frac{1}{2}\int_{\mathbb{R}}A_5(z)|w|^2\,dz\,, \qquad (5.7)$$

$$E_2^w(t) = \varepsilon^2\int_{\mathbb{R}}w\partial_t w\,dz + \frac{1}{2}\int_{\mathbb{R}}A_3(z)|w|^2\,dz\,, \qquad (5.8)$$

$$\begin{aligned}
Q_1^w(t) &= \int_{\mathbb{R}}A_3(z)|\partial_t w|^2\,dz + \int_{\mathbb{R}}A_4(z)\partial_t w\partial_z w\,dz\,. \\
Q_2^w(t) &= -\varepsilon^2\int_{\mathbb{R}}|\partial_t w|^2\,dz + (1-\varepsilon^2 s^2)\int_{\mathbb{R}}|\partial_z w|^2\,dz + 2\varepsilon^2 s\int_{\mathbb{R}}\partial_t w\partial_z w\,dz + \int_{\mathbb{R}}\left(A_5(z) - \frac{\partial_z A_4(z)}{2}\right)|w|^2\,dz\,. \qquad (5.9)
\end{aligned}$$

To determine $\phi(z)$ we examinate (5.7)–(5.8). We first require the natural condition $\partial_z\phi(z) \geq 0$. This clearly ensures $A_3(z) \geq 1 - \varepsilon^2 F'(0)$. We examinate the condition $A_5(z) - \frac{1}{2}\partial_z A_4(z) \geq 0$ (5.9) in order to fully determine the weight $\phi(z)$:

$$\begin{aligned}
A_5(z) - \frac{\partial_z A_4(z)}{2} &= s\left(1-\varepsilon^2F'(\nu)\right)\partial_z\phi + (\varepsilon^2 s^2 - 1)\left(-\partial_{zz}\phi + |\partial_z\phi|^2\right) + \varepsilon^2 sF''(\nu)\partial_z\nu - F'(\nu) \\
&\quad - \frac{1}{2}\varepsilon^2 sF''(\nu)\partial_z\nu + (\varepsilon^2 s^2 - 1)\partial_{zz}\phi \\
&= (\varepsilon^2 s^2 - 1)\,|\partial_z\phi|^2 + s\left(1-\varepsilon^2F'(\nu)\right)\partial_z\phi + \frac{1}{2}\varepsilon^2 sF''(\nu)\partial_z\nu - F'(\nu)\,.
\end{aligned}$$

This is a second-order equation in the variable $\partial_z\phi$. Maximization of this quantity is achieved when

$$\partial_z\phi = \frac{s\left(1-\varepsilon^2F'(\nu)\right)}{2(1-\varepsilon^2s^2)} \geq 0\,. \tag{5.10}$$

We notice that this is equivalent to setting $A_4(z)=0$ (5.6). Then we obtain

$$\begin{aligned}
A_5(z) &= \frac{s^2\left(1-\varepsilon^2F'(\nu)\right)^2}{4(1-\varepsilon^2s^2)} + \frac{1}{2}\varepsilon^2sF''(\nu)\partial_z\nu - F'(\nu)\\
&= \frac{1}{4(1-\varepsilon^2s^2)}\left(s^2\left(1+\varepsilon^2F'(\nu)\right)^2 - 4F'(\nu)\right) + \frac{1}{2}\varepsilon^2sF''(\nu)\partial_z\nu\\
&= \frac{1}{4\left(1-\varepsilon^2F'(0)\right)^2}\left(4F'(0)\left(1+\varepsilon^2F'(\nu)\right)^2 - 4F'(\nu)\left(1+\varepsilon^2F'(0)\right)^2\right) + \frac{1}{2}\varepsilon^2sF''(\nu)\partial_z\nu\\
&= \frac{1}{\left(1-\varepsilon^2F'(0)\right)^2}\left(F'(0)-F'(\nu)\right)\left(1-\varepsilon^4F'(0)F'(\nu)\right) + \frac{1}{2}\varepsilon^2sF''(\nu)\partial_z\nu\,.
\end{aligned}$$

We check that $A_5(z)\geq 0$ since $\forall z\ F'(\nu(z))\leq F'(0)$, $\varepsilon^2F'(0)<1$, $\forall z\ F''(\nu(z))\leq 0$ and $\forall z\ \partial_z\ \nu(z)\leq 0$.

We recall that the exponential decay of ν at $+\infty$ is given by the eigenvalue $\lambda>0$, where

$$\lambda = \frac{s(1-\varepsilon^2F'(0))}{2(1-\varepsilon^2s^2)}\,.$$

Therefore we can rewrite

$$\partial_z\phi = \lambda\frac{1-\varepsilon^2F'(\nu)}{1-\varepsilon^2F'(0)}\,. \tag{5.11}$$

Remark 17. *As far as we are concerned with linear stability, the energies E_1^w and E_2^w contain enough information. However proving nonlinear stability requires an additional control of u in L^∞ which can be obtained using E_1^u and E_2^u [23].*

2- Combination of the energy estimates. We first examinate the energies E_1^u and E_2^u. We clearly have

$$\begin{aligned}
E_2^u(t) &\geq -\frac{\varepsilon^4}{1-\varepsilon^2F'(0)}\left\|\partial_tu - s\partial_zu\right\|_2^2 - \frac{1-\varepsilon^2F'(0)}{4}\|u\|_2^2 + \frac{1-\varepsilon^2F'(0)}{2}\|u\|_2^2\\
&\geq -\frac{\varepsilon^4}{1-\varepsilon^2F'(0)}\left\|\partial_tu - s\partial_zu\right\|_2^2 + \frac{1-\varepsilon^2F'(0)}{4}\|u\|_2^2
\end{aligned}$$

We set

$$\delta = \frac{1-\varepsilon^2F'(0)}{2\varepsilon^2}\,.$$

We have on the one hand

$$E_1^u(t)+\delta E_2^u(t) \geq \frac{1}{2}\left\|\partial_zu\right\|_2^2\,dz + \int_{\mathbb{R}} A_6(z)|u|^2\,dz\,,$$

where $A_6(z)$ is defined as

$$A_6(z) = \frac{1}{2}A_1(z) + \delta\frac{1-\varepsilon^2F'(0)}{4}\,.$$

We have on the other hand,

$$Q_1^u(t)+\delta Q_2^u(t) \geq \frac{1-\varepsilon^2F'(0)}{2}\left\|\partial_tu - s\partial_zu\right\|_2^2 + \delta\|\partial_zu\|_2^2 + \int_{\mathbb{R}} A_7(z)|u|^2\,dz\,,$$

where $A_7(z)$ is defined as

$$A_7(z) = \frac{s}{2}\partial_z \left(s\varepsilon^2 F''(\nu)\partial_z \nu - F'(\nu)\right) + \delta A_2(z)\,.$$

We have both $\lim_{z\to-\infty} A_6(z) > 0$ and $\lim_{z\to-\infty} A_7(z) > 0$. Accordingly there exists $\alpha > 0$ and $z_0 \in \mathbb{R}$ such that

$$\forall z < z_0 \quad \min(A_6(z), A_7(z)) > \alpha\,.$$

In order to control the zeroth-order terms over $(z_0, +\infty)$ we shall use the last two energy estimates. First we observe that $\forall z > z_0$ $|u(z)| = |e^{-\phi(z)}w(z)| \le e^{-\phi(z_0)}|w(z)|$ since ϕ is increasing. We set $\phi(z_0) = 0$ without loss of generality. This determines completely ϕ together with the condition (5.10). We have

$$E_1^u(t) + \delta E_2^u(t) \ge \frac{1}{2}\,\|\partial_z u\|_2^2\,dz + \alpha \int_{z<z_0} |u|^2\,dz - \left\|A_6 e^{-2\phi}\mathbf{1}_{z>z_0}\right\|_\infty \int_{z>z_0} |w|^2\,dz\,,$$

$$Q_1^u(t) + \delta Q_2^u(t) \ge \frac{1-\varepsilon^2 F'(0)}{2}\,\|\partial_t u - s\partial_z u\|_2^2 + \delta\|\partial_z u\|_2^2 + \alpha\int_{z<z_0} |u|^2\,dz - \left\|\frac{A_7 e^{-2\phi}}{A_5}\mathbf{1}_{z>z_0}\right\|_\infty \int_{z>z_0} A_5(z)|w|^2\,dz\,.$$

Lemma 18. *We have* $\dfrac{A_7 e^{-2\phi}}{A_5} \in L^\infty(z_0, +\infty)$ *and* $\dfrac{e^{-\phi}}{A_5} \in L^\infty(z_0, +\infty)$.

Proof. The first claim is clearly a consequence of the second claim since $A_7 e^{-\phi} \in L^\infty(z_0, +\infty)$. First we have

$$F'(0) - F'(\nu) \ge \left(\inf_{[0,1]}(-F'')\right)\nu = \alpha\nu\,,$$

where $\alpha > 0$ is the coercivity constant of $-F$ (1.2). As a consequence, $A_5 \ge \left(\frac{(1+\varepsilon^2 F'(0))\alpha}{1-\varepsilon^2 F'(0)}\right)\nu$. Second we recall $\partial_z\nu + \lambda\nu \ge 0$ (Lemma 10), so that $\forall z > z_0$, $\nu(z) \ge \nu(z_0)e^{-\lambda(z-z_0)}$. Finally we have $\forall z$, $\partial_z\phi \ge \lambda$ (5.11), thus

$$\forall z > z_0, \quad e^{-\phi(z)} \le e^{-\phi(z_0)}e^{-\lambda(z-z_0)} \le \frac{e^{-\phi(z_0)}}{\nu(z_0)}\nu(z) \le \frac{e^{-\phi(z_0)}}{\nu(z_0)}\left(\frac{1-\varepsilon^2 F'(0)}{(1+\varepsilon^2 F'(0))\,\alpha}\right)A_5(z)\,.$$

□

We now focus on the second series of energy estimates. We clearly have

$$\begin{aligned} E_2^w(t) &\ge -\frac{\varepsilon^4}{1-\varepsilon^2 F'(0)}\,\|\partial_t w\|_2^2 - \frac{1-\varepsilon^2 F'(0)}{4}\|w\|_2^2 + \frac{1-\varepsilon^2 F'(0)}{2}\|w\|_2^2 \\ &\ge -\frac{\varepsilon^4}{1-\varepsilon^2 F'(0)}\,\|\partial_t w\|_2^2 + \frac{1-\varepsilon^2 F'(0)}{4}\|w\|_2^2\,, \end{aligned}$$

and

$$\begin{aligned} Q_2^w(t) &\ge -\varepsilon^2\|\partial_t w\|_2^2 + (1-\varepsilon^2 s^2)\|\partial_z w\|_2^2 - \frac{2\varepsilon^4 s^2}{1-\varepsilon^2 s^2}\|\partial_t w\|_2^2 - \frac{1-\varepsilon^2 s^2}{2}\|\partial_z w\|_2^2 + \int_{\mathbb{R}} A_5(z)|w|^2\,dz \\ &\ge -\varepsilon^2\frac{1+\varepsilon^2 s^2}{1-\varepsilon^2 s^2}\|\partial_t w\|_2^2 + \frac{1-\varepsilon^2 s^2}{2}\|\partial_z w\|_2^2 + \left\|\frac{e^{-\phi}}{A_5}\mathbf{1}_{z>z_0}\right\|_\infty^{-1}\int_{z>z_0} e^{-\phi}|w|^2\,dz\,. \end{aligned}$$

We set

$$\delta' = \frac{1-\varepsilon^2 F'(0)}{2\varepsilon^2}\cdot\frac{1-\varepsilon^2 s^2}{1+\varepsilon^2 s^2} < \delta\,.$$

We have on the one hand

$$E_1^w(t) + \delta' E_2^w(t) \ge \frac{1-\varepsilon^2 s^2}{2}\,\|\partial_z w\|_2^2\,dz + \int_{\mathbb{R}} A_8(z)|w|^2\,dz\,,$$

where $A_8(z)$ is defined as

$$A_8(z) = \frac{1}{2}A_5(z) + \delta' \frac{1-\varepsilon^2 F'(0)}{4} \geq \delta' \frac{1-\varepsilon^2 F'(0)}{4}\,.$$

We have on the other hand,

$$Q_1^w(t) + \delta' Q_2^w(t) \geq \frac{1-\varepsilon^2 F'(0)}{2}\,\|\partial_t w\|_2^2 + \delta' \frac{1-\varepsilon^2 s^2}{2}\|\partial_z w\|_2^2 + \delta' \left\| \frac{e^{-\phi}}{A_5}\mathbf{1}_{z>z_0}\right\|_\infty^{-1} \int_{z>z_0} h|w|^2\,dz\,.$$

Combining all these estimates, we define $E(t) = E_1^w(t) + \delta' E_2^w(t) + \delta''\left(E_1^u(t) + \delta E_2^u(t)\right)$ and $Q(t) = Q_1^w(t) + \delta' Q_2^w(t) + \delta''\left(Q_1^u(t) + \delta Q_2^u(t)\right)$, where $\delta'' > 0$ is defined such as the following condition holds true

$$\delta'' < \delta' \min\left(\frac{1-\varepsilon^2 F'(0)}{4}\left\|A_6 e^{-2\phi}\mathbf{1}_{z>z_0}\right\|_\infty^{-1}, \left\| \frac{A_7 e^{-2\phi}}{A_5}\mathbf{1}_{z>z_0}\right\|_\infty^{-1}\right)\,.$$

finally obtain our main estimate,

$$\begin{aligned}\frac{d}{dt}E(t) + Q(t) \leq{}& \mathcal{O}\left(\int_{\mathbb{R}} |u||\partial_t u - s\partial_z u|^2\,dz + \int_{\mathbb{R}} |u|^3\,dz\right)\\ &+ \mathcal{O}\left(\int_{\mathbb{R}} e^{-\phi}|w||\partial_t w|^2\,dz + \int_{\mathbb{R}} e^{-\phi}|w||\partial_z w|^2\,dz + \int_{\mathbb{R}} e^{-\phi}|w|^3\,dz\right)\,, \quad (5.12)\end{aligned}$$

where

$$\begin{aligned}E(t) &\geq \mathcal{O}\left(\|\partial_z u\|_2^2 + \int_{z<z_0}|u|^2\,dz + \|\partial_z w\|_2^2 + \|w\|_2^2\right)\,,\\ Q(t) &\geq \mathcal{O}\left(\|\partial_t - s\partial_z u\|_2^2 + \|\partial_z u\|_2^2 + \int_{z<z_0}|u|^2\,dz + \|\partial_t w\|_2^2 + \|\partial_z w\|_2^2 + \int_{z>z_0} e^{-\phi}|w|^2\,dz\right)\,.\end{aligned}$$

3- Control of the nonlinear contributions. Our goal is to control the size of the perturbation u in L^∞. For this purpose we use the embeddings of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$ and $L^4(\mathbb{R})$:

$$\begin{aligned}\|u\|_\infty &\leq C\|u\|_2^{1/2}\|\partial_z u\|_2^{1/2}\,,\\ \|u\|_4 &\leq C\|u\|_2^{3/4}\|\partial_z u\|_2^{1/4}\,.\end{aligned}$$

We examinate successively the nonlinear contributions. We recall $u = e^{-\phi}w$. First we have

$$\begin{aligned}\int_{\mathbb{R}} |u||\partial_t u - s\partial_z u|^2\,dz &\leq \|u\|_\infty \|\partial_t - s\partial_z u\|_2^2\\ &\leq \mathcal{O}\left(E(t)^{1/2}Q(t)\right)\,,\end{aligned}$$

and similar estimates can be derived for all the contributions in the r.h.s. of (5.12) except for the last one. Second we have

$$\begin{aligned}\int_{\mathbb{R}} e^{-\phi}|w|^3\,dz &= \int_{z<z_0} e^{-\phi}|w|^3\,dz + \int_{z>z_0} e^{-\phi}|w|^3\,dz\\ &\leq \|u\|_{L^\infty(-\infty,z_0)}\|w\|^2_{L^2(-\infty,z_0)} + \|e^{-\phi/2}w\|^2_{L^4(z_0,+\infty)}\|w\|_{L^2(z_0,+\infty)}\\ &\leq C\|u\|^{1/2}_{L^2(-\infty,z_0)}\|\partial_z u\|^{1/2}_{L^2(-\infty,z_0)}\|w\|^2_{L^2(-\infty,z_0)}\\ &\quad + C\|e^{-\phi/2}w\|^{3/2}_{L^2(z_0,+\infty)}\left\|\partial_z\left(e^{-\phi/2}w\right)\right\|^{1/2}_{L^2(z_0,+\infty)}\|w\|_{L^2(z_0,+\infty)}\,.\end{aligned}$$

We have

$$\|w\|^2_{L^2(-\infty,z_0)} \le \|u\|^2_{L^2(-\infty,z_0)}\,,$$
$$\|e^{-\phi/2}w\|^2_{L^2(z_0,+\infty)} = \int_{z>z_0} e^{-\phi}|w|^2\,dz\,,$$
$$\left\|\partial_z\left(e^{-\phi/2}w\right)\right\|^2_{L^2(z_0,+\infty)} \le 2\int_{z>z_0}\left(e^{-\phi}|\partial_z w|^2 + \frac{1}{4}|\partial_z\phi|^2 e^{-\phi}|w|^2\right)dz \le 2\int_{z>z_0}|\partial_z w|^2\,dz + C\int_{z>z_0} e^{-\phi}|w|^2\,dz\,.$$

Consequently we obtain

$$\int_{\mathbb{R}} e^{-\phi}|w|^3\,dz \le \mathcal{O}\left(E(t)^{1/2}Q(t)\right)\,.$$

Finally we get

$$\frac{d}{dt}E(t) + Q(t) \le \mathcal{O}\left(E(t)^{1/2}Q(t)\right)\,.$$

This estimate ensures that the energy is nonincreasing provided that it is initially small enough. Indeed there exists a constant C such that $\frac{d}{dt}E(t) + Q(t) \le CE(t)^{1/2}Q(t)$. We set $c = C^{-2}/2$. If initially $E^0 \le c$ then the previous differential inequality guarantees that $E(t)$ is decaying and remains below the level c. Therefore $E(t)$ is positive decaying, and the dissipation $Q(t)$ is integrable. This concludes the proof of Theorem 15. □

References

[1] J. Adler, *Chemotaxis in bacteria*, Science **153** (1966), 708–716.

[2] W. Alt, *Biased random walk models for chemotaxis and related diffusion approximations*, J. Math. Biol. **9** (1980), 147–177.

[3] F. Andreu, V. Caselles, J. Mazón, S. Moll, *Finite propagation speed for limited flux diffusion equations*, Arch. Ration. Mech. Anal. **182** (2006), 269–297.

[4] F. Andreu, J. Calvo, J.M. Mazón, J. Soler, *On a nonlinear flux-limited equation arising in the transport of morphogens*, J. Differential Equations **252** (2012), 5763–5813.

[5] D.G. Aronson, H.F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. in Math. **30** (1978), 33–76.

[6] Berestycki H., Hamel F., *Front propagation in periodic excitable media*, Comm. Pure Appl. Math. **55** (2002), 949–1032.

[7] C. Deroulers, M. Aubert, M. Badoual, B. Grammaticos, *Modeling tumor cell migration: from microscopic to macroscopic models*, Phys. Rev. E **79** (2009).

[8] J. Campos, P. Guerrero P., O. Sánchez, J. Soler, *On the analysis of travelling waves to a nonlinear flux limited reaction-diffusion equation*, preprint 2011.

[9] J.A. Carrillo, Th. Goudon, P. Lafitte, F. Vecil, *Numerical schemes of diffusion asymptotics and moment closures for kinetic equations*, J. Sci. Comput. **36** (2008), 113–149.

[10] F.A.C.C. Chalub, P. Markowich, B. Perthame and C. Schmeiser, *Kinetic models for chemotaxis and their drift-diffusion limits*, Monatsh. Math., **142** (2004), 123–141.

[11] C. Cuesta, S. Hittmeir, C. Schmeiser, *Traveling waves of a kinetic transport model for the KPP-Fisher equation*, preprint 2011.

[12] B. Després, F. Lagoutière, *Un schéma non linéaire anti-dissipatif pour l'équation d'advection linéaire*, C. R. Acad. Sci. Paris **328** (1999), 939–944.

[13] S.R. Dunbar, H.G. Othmer, *On a nonlinear hyperbolic equation describing transmission lines, cell movement, and branching random walks*, Nonlinear oscillations in biology and chemistry (Salt Lake City, Utah, 1985), 274–289, Lecture Notes in Biomath. **66**, Springer, Berlin, 1986.

[14] R. Erban, H.G. Othmer, *From individual to collective behavior in bacterial chemotaxis*, SIAM J. Appl. Math. **65** (2004), 361–391.

[15] L. C. Evans, *Partial Dierential Equations*, American Mathematical Society, Providence, RI, 2002.

[16] S. Fedotov, *Traveling waves in a reaction-diffusion system: diffusion with finite velocity and Kolmogorov-Petrovskii-Piskunov kinetics*, Phys. Rev. E **58** (1998), 5143–5145.

[17] S. Fedotov, *Wave front for a reaction-diffusion system and relativistic Hamilton-Jacobi dynamics*, Phys. Rev. E **59** (1999), 5040–5044.

[18] S. Fedotov, A. Iomin, *Probabilistic approach to a proliferation and migration dichotomy in tumor cell invasion*, Phys. Rev. E **77** (2008), 031911.

[19] F. Filbet, S. Jin, *A class of asymptotic-preserving schemes for kinetic equations and related problems with stiff sources*, J. Comput. Phys. **229** (2010), 7625–7648.

[20] P.C. Fife, J.B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Ration. Mech. Anal. **65** (1977), 335–361.

[21] R.A. Fisher, *The advance of advantageous genes*, Ann. Eugenics **65**(1937), 335–369.

[22] Th. Gallay, *Local stability of critical fronts in nonlinear parabolic partial differential equations*, Nonlinearity **7** (1994), 741–764.

[23] Th. Gallay, G. Raugel, *Stability of travelling waves for a damped hyperbolic equation*, Z. Angew. Math. Phys. **48** (1997), 451–479.

[24] Th. Gallay, R. Joly, *Global stability of travelling fronts for a damped wave equation with bistable nonlinearity*, Ann. Sci. c. Norm. Supr. **42** (2009), 103–140.

[25] E. Godlewski, P.-A. Raviart, *Hyperbolic systems of conservation laws*, Mathématiques & Applications, Ellipses, Paris, 1991.

[26] K.P. Hadeler, *Hyperbolic travelling fronts*, Proc. Edinburgh Math. Soc. **31** (1988), 89–97.

[27] T. Hillen, H.G. Othmer, *The diffusion limit of transport equations derived from velocity-jump processes*, SIAM J. Appl. Math. **61** (2000), 751–775.

[28] H.J. Hwang, K. Kang, A. Stevens, *Global existence of classical solutions for a hyperbolic chemotaxis model and its parabolic limit*, Indiana Univ. Math. J. **55** (2006), 289–316.

[29] A.N. Kolmogorov, I.G. Petrovsky, N.S. Piskunov, *Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Moskow Univ. Math. Bull. **1** (1937), 1–25.

[30] B.C. Mazzag, I.B. Zhulin, A. Mogilner, *Model of bacterial band formation in aerotaxis*, Biophys. J. **85** (2003), 3558–74.

[31] J.D. Murray, Mathematical biology. I. An introduction. Third edition. Springer-Verlag, New York, 2002.

[32] T. Nagai, M. Mimura, *Some nonlinear degenerate diffusion equations related to population dynamics*, J. Math. Soc. Japan **35** (1983), 539–562.

[33] T. Nagai, M. Mimura, *Asymptotic behavior for a nonlinear degenerate diffusion equation in population dynamics*, SIAM J. Appl. Math. **43** (1983), 449–464.

[34] H.G. Othmer, S.R. Dunbar, W. Alt, *Models of dispersal in biological systems*, J. Math. Biol. **26** (1988), 263–298.

[35] K.J. Painter, Th. Hillen, *Volume-filling and quorum-sensing in models for chemosensitive movement*, Can. Appl. Math. Q. **10** (2002), 501–543.

[36] B. Perthame, Transport Equations in Biology, *Frontiers in Mathematics*, Birkhäuser Verlag, Basel, 2007.

[37] E. Risler, *Global convergence toward traveling fronts in nonlinear parabolic systems with a gradient structure*, Ann. Inst. H. Poincar Anal. Non Linaire **25** (2008), 381–424.

[38] J. Saragosti, V. Calvez, N. Bournaveas, A. Buguin, P. Silberzan, B. Perthame, *Mathematical description of bacterial traveling pulses*, PLoS Comput Biol 6 (2010), e1000890.

[39] J. Saragosti, V. Calvez, N. Bournaveas, B. Perthame, A. Buguin, P. Silberzan, *Directional persistence of chemotactic bacteria in a traveling concentration wave*, PNAS **108** (2011), 16235–40.

[40] N. Shigesada, *Spatial distribution of dispersing animals*, J. Math. Biol. **9** (1980), 85–96.