Asymptotic behavior of the number of Eulerian orientations of graphs

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Abstract

We consider the class of simple graphs with large algebraic connectivity (the second-smallest eigenvalue of the Laplacian matrix). For this class of graphs we determine the asymptotic behavior of the number of Eulerian orientations. In addition, we establish some new properties of the Laplacian matrix, as well as an estimate of a conditionality of matrices with the asymptotic diagonal predominance

1. Введение

The eulerian orientation of the graph G is the orientation of its edges such that for every vertex the number of incoming edges and outgoing edges are equal. We denote the number of Eulerian orientations $EO(G)$. It is easy clear that $EO(G) = 0$, if the degree of at least one vertex of G is odd. Eulerian orientations of the complete graph K_n are called regular tournaments.

In [7] it is shown that for even $n \to \infty$

$$EO(K_n) = \left(\frac{2^{n+1}}{\pi n}\right)^{(n-1)/2} n^{1/2} e^{-1/2} \left(1 + O(n^{-1/2+\varepsilon})\right) \tag{1.1}$$

for any $\varepsilon > 0$.

Undirected graphs without loops and multiple edges are called simple.

The problem of counting the number of the Eulerian orienations of an undirected simple graph is complete for the class $\#P$. Thus this problem is difficult in terms of complexity theory. The problem of counting Eulerian orienations can be reduced to counting perfect matching for a class of graphs for which it can be done approximately with high probability in polynomial time, see [4].

For the simple graph G we define $n \times n$ matrix Q by

$$Q_{jk} = \begin{cases} -1, & \{v_j, v_k\} \in EG, \\ d_j, & j = k, \\ 0, & \text{в остальных случаях}, \end{cases} \tag{1.2}$$

rwhere $n = |VG|$ and d_j denotes the degree of $v_j \in VG$. The matrix $Q = Q(G)$ is called the Laplacian matrix of the graph G. The eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ of the matrix Q are always non-negative real numbers and $\lambda_1 = 0$. The eigenvalue λ_2 is called the algebraic connectivity of the graph G. (For more information about the spectral properties of the Laplace matrix see, for example, [2] and [5].)

According to the Kirchhoff's Matrix-Tree-Theorem, see [3], we have that

$$t(G) = \frac{1}{n}\lambda_2\lambda_3\cdots\lambda_n, \tag{1.3}$$

where $t(G)$ denotes the number of spanning trees of the graph G.

In the present work we generalize approach of [7]. We determine the asymptotic behavior of the number of Eulerian orientations of simple graphs with large algebraic connectivity, see Theorem 2.1. In Section 2 we give conventions and notations and formulate the main result.

In Section 3 we prove some basic properties of the Laplacian matrix. Some statements seem to be of independent interest, for example, we obtain an estimate of a conditionality of matrices with the asymptotic diagonal predominance, see Lemma 3.2. Also, we prove the following property of simple graphs with large algebraic connectivity: when you remove the vertex the number of spanning trees decreases by no more than cn times, for some $c > 0$ depending only on λ_2/n, see. Corollary 3.3.

We prove the main result in Section 4. We express $EO(G)$ in terms of an n-dimensional integral using Cauchy's formula. The derivation of asymptotic estimation of this integral uses three lemmas (Lemma 4.1, 4.2, 4.3), whose proofs are given in Section 5.

2. Main result

Let $p \geq 1$ be a real number and $\mathbf{x} \in \mathbb{R}^n$. We use notation

$$\|\mathbf{x}\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}. \tag{2.1}$$

For $p = \infty$ we have the maximum norm

$$\|\mathbf{x}\|_\infty = \max_j |x_j|. \tag{2.2}$$

The matrix norm corresponding to the p-norm for vectors is

$$\|A\|_p = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}. \tag{2.3}$$

If A is the matrix of self-adjoint operator (symmetric matrix) then the norm $\|A\|_2$ is equal to the largest module of eigenvalue of A and the following inequality holds:

$$\|A\|_p \geq \|A\|_2. \tag{2.4}$$

For invertible matrices one can define the condition number.

$$\mu_p(A) = \|A\|_p \cdot \left\|A^{-1}\right\|_p \geq \left\|AA^{-1}\right\|_p = 1. \tag{2.5}$$

If f is bounded both above and below by g asymptotically, we use the notation

$$f(n) = \Theta_{k_1,k_2}(g(n)), \tag{2.6}$$

which implies as $n \to \infty$, eventually

$$k_1|g(n)| \leq |f(n)| \leq k_2|g(n)|. \tag{2.7}$$

When functions f and g depend not only on n, but also on other parameters $\boldsymbol{\xi}$, we use notation (2.6) meaning that condition (2.7) holds uniformly for all possible values of $\boldsymbol{\xi}$.

The main result of the present work is the following theorem.

Theorem 2.1. *Let G be simple graph with n vertices having even degree and the algebraic connectivity $\lambda_2 \geq \gamma n$ for some $\gamma > 0$. Then as $n \to \infty$*

$$EO(G) = \Theta_{k_1,k_2}\left(2^{|EG|+\frac{n-1}{2}}\pi^{-\frac{n-1}{2}}\Big/\sqrt{t(G)}\right), \tag{2.8}$$

where $t(G)$ enotes the number of spanning trees of the graph G and constants $k_1, k_2 > 0$ depend only on σ.

Remark 2.1. Taking into account (1.3), the value $t(G)$ can be represented as the principal minor of the Laplacian matrix Q.
Remark 2.2. For the complete graph $\lambda_1 = n$, $EK_n = \frac{n(n-1)}{2}$ and $t(K_n) = n^{n-2}$. The result of Theorem 2.1 for this case is in agreement with asymptotic formula (1.1).
Remark 2.3. There is the result on the asymptotic behavior of Eulerian circuits analogous to Theorem 2.1, see [1].

3. Some basic properties of the Laplacian matrix

In what follows we suppose that

$$G \text{ is a simple graph.} \tag{3.1}$$

The Laplacian matrix Q of the graph G has the eigenvector $[1, 1, \ldots, 1]^T$, corresponding to the eigenvalue $\lambda_0 = 0$. We use notation $\hat{Q} = Q + J$, where J denotes the matrix with every entry 1. Note that Q and $\hat{Q}$ have the same set of eigenvectors and eigenvalues, except for the eigenvalue corresponding to the eigenvector $[1, 1, \ldots, 1]^T$, which equals 0 for Q and n for $\hat{Q}$.

Using (2.4), we get that

$$\lambda_n = ||Q||_2 \leq ||\hat{Q}||_2 \leq ||\hat{Q}||_1 = \max_j \sum_{k=1}^{n} |\hat{Q}_{jk}| = n. \tag{3.2}$$

We denote by G_r the graph which arises from G by removing vertices $v_1, v_2, \ldots, v_r$ and all adjacent edges.

Lemma 3.1. *Let condition (3.1) holds for graph G with n vertices. Then*

$$\lambda_2(G) \leq \frac{n}{n-1} \min_j d_j, \tag{3.3}$$

$$\lambda_2(G_r) \geq \lambda_2(G) - r, \tag{3.4}$$

where $\lambda_1(G)$ is the algebraic connectivity of G and d_j is the degree of the vertex $v_j \in VG$.

The proof of Lemma 3.1 can be found in [2].

Lemma 3.2. *Let $a > 0$ and I be identity $n \times n$ matrix. Then for any n and $n \times n$ symmetric matrix X such that the matrix $I + X$ is nonsingular and $|X_{ij}| \leq a/n$,*

$$\mu_2(I + X) \leq \mu_\infty(I + X) \leq C\mu_2(I + X), \tag{3.5}$$

where C depends only on a. (does not depend on n)

Proof of Lemma 3.2. The left-hand side of (3.5) follows from (2.4). We order the eigenvalues of $I + X$ modulo

$$|\chi_1| \leq |\chi_2| \leq \ldots \leq |\chi_n|. \tag{3.6}$$

Using (2.4), we get that

$$|\chi_n| = \|I + X\|_2 \leq \|I + X\|_\infty \leq \|I\|_\infty + \|X\|_\infty \leq 1 + a. \tag{3.7}$$

We consider $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $\|\mathbf{x}\|_\infty = 1$. For simplicity, we assume that $x_1 = \|\mathbf{x}\|_\infty = 1$. We denote by $\mathcal{J} = \left\{ j \mid x_j > \dfrac{1}{2a} \right\}$.

Case 1. $|\mathcal{J}| < \dfrac{n}{4a}$. Estimating the first coordinate of $(I + X)\,\mathbf{x}$, we get that

$$\|(I + X)\mathbf{x}\|_\infty \geq x_1 - \frac{a}{n}\left(\sum_{j \in \mathcal{J}} |x_j| + \sum_{j \notin \mathcal{J}} |x_j|\right) \geq 1 - \frac{a}{n}\left(\frac{n}{4a} \cdot 1 + n \cdot \frac{1}{2a}\right) = \frac{1}{4}\,\|\mathbf{x}\|_\infty\,. \tag{3.8}$$

Case 2. $|\mathcal{J}| \geq \dfrac{n}{4a}$. Note that

$$\sqrt{n\,\|(I + X)\mathbf{x}\|_\infty^2} \geq \|(I + X)\mathbf{x}\|_2 \geq |\chi_1| \cdot \|\mathbf{x}\|_2 \geq |\chi_1| \cdot \sqrt{|\mathcal{J}| \cdot \frac{1}{4a^2}}\,\|\mathbf{x}\|_\infty\,. \tag{3.9}$$

Then

$$\|(I + X)\mathbf{x}\|_\infty \geq \frac{|\chi_1|}{4a^{3/2}}\,\|\mathbf{x}\|_\infty\,. \tag{3.10}$$

Combining(3.8) и (3.10), we get that at least one of the following inequalities holds.

$$\left\|(I + X)^{-1}\right\|_\infty \leq 4 \quad \text{или} \quad \left\|(I + X)^{-1}\right\|_\infty \leq \frac{4a^{3/2}}{|\chi_1|}. \tag{3.11}$$

From (3.7) we have that

$$\|I + X\|_\infty \leq 1 + a. \tag{3.12}$$

Taking into account $|\chi_n| \leq 1 + a$ and $\mu_2(I + X) = \dfrac{|\chi_n|}{|\chi_1|} \geq 1$, we obtain (3.5) ∎

The proofs of Lemma 4.1, Lemma 4.2 and Lemma 4.3 are based on the following property of the Laplacian matrix.

Corollary 3.1. *Let G be a simple graph with n vertices and algebraic connectivity of the graph $\lambda_2 \geq \gamma n$ for some $\gamma > 0$. Then there is some constant $c_\infty > 0$, depending only on γ, such that*

$$||\hat{Q}^{-1}||_1 = ||\hat{Q}^{-1}||_\infty \leq \frac{c_\infty}{n}. \tag{3.13}$$

Proof of Corollary 3.1. Using (3.3), we get that

$$d_j \geq \lambda_2 \frac{n-1}{n} \geq \gamma(n-1) \geq \gamma n/2. \tag{3.14}$$

Taking into account (3.2), all eigenvalues of $\hat{Q}$ are in the interval $[\gamma n; n]$. Inequality (3.13) follows easily from the assertion of Lemma 3.2 for the matrix $\Omega^T \hat{Q} \Omega$, where

$$\Omega_{jk} = \begin{cases} \frac{1}{\sqrt{d_j+1}}, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases} \tag{3.15}$$

■

The following lemma will be applied to estimate the determinant of a matrix close to the identity matrix I.

Lemma 3.3. *Let X be an $n \times n$ matrix such that $\|X\|_2 < 1$. Then for fixed $m \geq 2$*

$$\det(I + X) = \exp\left(\sum_{r=1}^{m-1} \frac{(-1)^{r+1}}{r} tr(X^r) + E_m(X) \right), \tag{3.16}$$

where tr is the trace function and

$$|E_m(X)| \leq \frac{n}{m} \frac{\|X\|_2^m}{1 - \|X\|_2}. \tag{3.17}$$

The proof of Lemma 3.3 is based on evaluating the trace of the matrix $\ln(I + X)$, using the representation as a convergent series. Lemma 3.3 was also formulated and proved in [6].

Lemma 3.4. *Let G be a simple graph with n vertices and algebraic connectivity of the graph $\lambda_2 \geq \gamma n$ for some $\gamma > 0$. et G_1 be the graph which arises from G by removing vertex v_1 and all adjacent edges. Then there is a constant $c > 0$ depending only on γ such that*

$$\det \hat{Q}_1 \geq \frac{\det \hat{Q}}{cn}. \tag{3.18}$$

Proof of Lemma 3.4. Note that the matrix M_{11} that results from deleting the first row and the first column of $\hat{Q}$ coincides with the matrix $\hat{Q}_1$ with the exception of the diagonal elements. Let Ω be a diagonal matrix such that

$$\Omega_{jj} = \begin{cases} 1, & \text{if } \{v_1, v_j\} \in EG, \\ 0, & \text{otherwise.} \end{cases} \tag{3.19}$$

Define $n \times n$ matrix X by

$$X_{jk} = \begin{cases} \frac{1}{d_1+1}, & \text{если } \{v_1, v_j\} \notin EG, \{v_1, v_k\} \notin EG, \text{ and } j, k \neq 1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.20}$$

After performing one step of the Gaussian elimination for $\hat{Q} + \Omega + X$, we obtain that

$$\det(\hat{Q} + \Omega + X) = (d_1 + 1) \det \hat{Q}_1, \tag{3.21}$$

Using (2.4), (3.14), we have that

$$||\Omega + X||_2 \leq ||\Omega||_2 + ||X||_2 \leq ||\Omega||_2 + ||X||_1 \leq 1 + \frac{n}{d_1 + 1} \leq \frac{3}{\gamma}. \tag{3.22}$$

Since the algebraic connectivity $\lambda_2 \geq \gamma n$

$$||(\Omega + X)\hat{Q}^{-1}||_2 \leq ||\Omega + X||_2 ||\hat{Q}^{-1}||_2 \leq \frac{3}{\gamma\lambda_2} \leq \frac{3}{\gamma^2 n} \tag{3.23}$$

Combining Lemma 3.3 and (3.23), we get that as $n \to \infty$

$$\begin{aligned} \det\left(I + (\Omega + X)\hat{Q}^{-1}\right) = \exp\left(tr\left((\Omega + X)\hat{Q}^{-1}\right) + E_2\left((\Omega + X)\hat{Q}^{-1}\right)\right) \geq \\ \geq \exp\left(-n\frac{3}{\gamma^2 n} + O(n^{-1})\right). \end{aligned} \tag{3.24}$$

Using (3.21) and (3.24) we get that as $n \to \infty$

$$(d_1 + 1)\det\hat{Q}_1 = \det\left(I + (\Omega + X)\hat{Q}^{-1}\right)\det\hat{Q} \geq \det\hat{Q}\exp\left(-3/\gamma^2 + O(n^{-1})\right). \tag{3.25}$$

Since $d_1 + 1 \leq n$, we obtain (3.18). ∎

Corollary 3.2. *Let the assumptions of Lemma 3.4 hold. Let G_r be the graph which arises from G by removing vertices $v_1, v_2, \ldots, v_r$ and all adjacent edges. Then there is a constant $c_1 > 0$ depending only on γ such that*

$$\det\hat{Q}(G_r) \geq \frac{\det\hat{Q}(G)}{(c_1 n)^r} \tag{3.26}$$

for any $r \leq \gamma n/2$.

Proof of Corollary 3.2. From (3.4) we have that

$$\lambda_2(G_r) \geq \gamma n - r \geq \gamma n/2. \tag{3.27}$$

Using r times the assertion of the Lemma 3.4, we obtain (3.26). ∎

According to (1.3)

$$t(G) = \frac{1}{n}\lambda_2\lambda_2\cdots\lambda_{n-1} = \frac{\det\hat{Q}}{n^2}, \tag{3.28}$$

then the following proposition holds.

Corollary 3.3. *Let the assumptions of Lemma 3.4 hold. Then there is a constant $c > 0$ depending only on γ such that*

$$t(G_1) \geq \frac{t(G)}{cn}, \tag{3.29}$$

where $t(G)$ denotes the number of spanning trees of the graph G.

Lemma 3.5. *Let $a > 0$ and the assumptions of Lemma 3.4 hold. Then for any set $A \subset VG$ such, that $|A| \geq an$, there is a function $h : VG \to \mathbb{N}_0$, having following properties:*

$$h(v) = 0, \ \textit{если } v \in A, \quad h(v) \leq H, \ \textit{для любых } v \in VG, \tag{3.30}$$

$$\left|\{w \in VG \mid (w, v) \in EG \ u \ h(w) < h(v)\}\right| \geq \alpha n, \ \textit{если } v \notin A, \tag{3.31}$$

where constants $H, \alpha > 0$ depend only on a and γ.

Proof of Лемммы 3.5. At first, we construct the set $A_1 = \{v \in VG \mid h(v) = 1\}$, having property (3.31).

If $|A| > n - \gamma n/4$, then let $A_1 = \{v \in VG \mid v \notin A\}$. Taking into account (3.14), we get that property (3.31) hold for $\alpha = \gamma/4$. In this case $H = 1$.

For $|A| \le n - \gamma n/4$ define $\mathbf{x} \in \mathbb{R}^n$ such that

$$x_j = \begin{cases} 1 - |A|/n, & v_j \in A, \\ -|A|/n, & v_j \notin A. \end{cases} \tag{3.32}$$

Since $x_1 + x_2 + \ldots + x_n = 0$

$$\mathbf{x}^T Q \mathbf{x} = \mathbf{x}^T \hat{Q} \mathbf{x} \ge \lambda_2 \|\mathbf{x}\|_2^2 \ge \lambda_2 |A| \left(\frac{n - |A|}{n}\right)^2 \ge \gamma n \, an \, (\gamma/4)^2 = \frac{a\gamma^3 n^2}{16}. \tag{3.33}$$

On the other hand,

$$\mathbf{x}^T Q \mathbf{x} = \sum_{\{v_j, v_k\} \in EG} (x_j - x_k)^2, \tag{3.34}$$

which is equal to the number of edges $(v, w) \in EG$, where $v \in A, w \notin A$. We denote A_1 the set of vertices $w \notin A$, having at least αn adjacent vertices in A, where $\alpha = \frac{1}{32} a\gamma^3$.

$$\mathbf{x}^T Q \mathbf{x} \le n|A_1| + \alpha n |VG|. \tag{3.35}$$

Combining (3.33) and (3.35), we get that $|A_1| \ge \alpha n$.

We make further construction of the function h inductively, using for the k-th step the set $A^{(k)} = A \cup A_1 \cup \ldots \cup A_k$. The number of steps does not exceed $1/\alpha$ as $|A_k| \ge \alpha n$ for each step, perhaps with the exception of the last one. ∎

4. Proof of Theorem 2.1

In a similar way as in [7] (see the proof of Theorem 3.1) we note that the function

$$\prod_{\{v_j, v_k\} \in EG} ({x_j}^{-1} x_k + {x_k}^{-1} x_j) \tag{4.1}$$

is the generating function the number of orientations of graph G by the differences in the numbers of incoming and outgoing edges at each vertex. The value $EO(G)$ is the constant term, which we can extract via Cauchy's Theorem using the unit circle as a contour for each variable. Making the substitution $x_j = e^{i\theta_j}$ for each j, we find that

$$EO(G) = 2^{|EG|} \pi^{-n} S, \qquad S = \int\limits_{U_n(\pi/2)} \prod_{\{v_j, v_k\} \in EG} \cos \Delta_{jk} \, d\boldsymbol{\theta}, \tag{4.2}$$

where $\Delta_{jk} = \theta_j - \theta_k$,

$$U_n(\rho) = \{(x_1, x_2, \ldots, x_n) \mid |x_j| \le \rho \text{ для всех } j\}, \tag{4.3}$$

and using the fact that for graphs with vertices of even degree the integrand is unchanged by the substitutions $\theta_j \to \theta_j + \pi$.

Let's start the evaluation S from the part that makes a major contribution to the integral We fix some sufficiently small constant $\varepsilon > 0$. Let

$$V_0 = \{\boldsymbol{\theta} \in U_n(\pi/2) \mid |\theta_j - \bar{\theta}| \,(\mathrm{mod}\,\pi) \le n^{-1/2+\varepsilon}, \text{ где } \bar{\theta} = \frac{\theta_1 + \ldots + \theta_n}{n}\}. \tag{4.4}$$

By Taylor's theorem we have that for $\boldsymbol{\theta} \in V_0$

$$\prod_{\{v_j,v_k\}\in EG} \cos\Delta_{jk} = \exp\left(-\frac{1}{2}\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^2 - \frac{1}{12}\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^4 + O(n^{-1+6\varepsilon})\right). \tag{4.5}$$

We denote by S_0 the contribution to S in the integration over the region V_0.

Lemma 4.1. *Let G be a simple graph with n vertices and the algebraic connectivity $\lambda_2 \ge \gamma n$ for some $\gamma > 0$. Then for any $a, b > 0$ as $n \to \infty$*

$$\int\limits_{V_0} \exp\left(-a\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^2 - b\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^4\right) d\boldsymbol{\theta} = \Theta_{k_1,k_2}\left(n\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta}\right), \tag{4.6}$$

where constants $k_1, k_2 > 0$ depend only on a, b and γ.

Lemma 4.1 follows from Lemma 8.3 of [1]. The proof is given in Section 5.

Combining (3.28), (4.2), (4.5), (4.6) and

$$\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta} = \pi^{n/2}a^{-n/2}\Big/\sqrt{\det\hat{Q}}, \tag{4.7}$$

we get that

$$S_0 = \Theta_{k_1,k_2}\left(2^{\frac{n-1}{2}}\pi^{\frac{n+1}{2}}\Big/\sqrt{t(G)}\right), \tag{4.8}$$

where constants $k_1, k_2 > 0$ depend only on γ.

Thus it remains to show that the other parts are negligible One can show that

$$|\cos(x)| \le \exp(-\frac{1}{2}x^2) \quad \text{for} \quad |x| \le \frac{9}{16}\pi. \tag{4.9}$$

Divide the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ mod π into 32 equal intervals $H_0, \ldots, H_{31}$ such that $H_0 = [-\frac{1}{64}\pi, \frac{1}{64}\pi]$. For each j, define the region $W_j \subseteq U_n(\pi/2)$ as the set of points $\boldsymbol{\theta} \in U_n(\pi/2)$, having at least $\frac{1}{32}n$ coordinates in H_j. Clearly, the W_j's cover $U_n(\pi/2)$ and also each W_j can be mapped to W_0 by a uniform translation of the θ_j mod π. This mapping preserves the integrand of (4.2), and also maps V_0 to itself, so we have that

$$\int\limits_{U_n(\pi/2)-V_0}\ \prod_{\{v_j,v_k\}\in EG}\cos\Delta_{jk}\ d\boldsymbol{\theta} \le 32Z, \tag{4.10}$$

where

$$Z = \int\limits_{W_0-V_0}\ \prod_{\{v_j,v_k\}\in EG}|\cos\Delta_{jk}|\ d\boldsymbol{\theta}. \tag{4.11}$$

We proceed by defining integrals S_1, S_2, S_3 in such a way that Z бis obviously bounded by their sum. We then show that each of them is negligible. Let

$$\begin{aligned} V_1 &= \{\boldsymbol{\theta} \in W_0 \mid |\theta_j| \geq \frac{1}{32}\pi \text{ for fewer than } n^{\varepsilon} \text{ values of } j\}, \\ V_2 &= \{\boldsymbol{\theta} \in V_1 \mid |\theta_j| \geq \frac{1}{16}\pi \text{ for at least one value of } j\}. \end{aligned} \tag{4.12}$$

Then our three integrals can be defined as

$$\begin{aligned} S_1 &= \int\limits_{W_0 - V_1} \prod_{\{v_j, v_k\} \in EG} |\cos \Delta_{jk}| \, d\boldsymbol{\theta}, \\ S_2 &= \int\limits_{V_2} \prod_{\{v_j, v_k\} \in EG} |\cos \Delta_{jk}| \, d\boldsymbol{\theta}, \\ S_3 &= \int\limits_{V_1 - V_2 - V_0} \prod_{\{v_j, v_k\} \in EG} |\cos \Delta_{jk}| \, d\boldsymbol{\theta}. \end{aligned} \tag{4.13}$$

We begin with S_1. Let h be the function from Lemma 3.5 for the set $A = \{v_j \mid |\theta_j| \leq \frac{1}{64}\pi\}$. We denote l_{min} such natural number that inequality

$$|\theta_j| \geq \frac{1}{64}\pi(1 + l/H) \tag{4.14}$$

holds for at least n^{ε}/H indices of the set $\{j \mid h(v_j) = l\}$. Existence of l_{min} follows from the definition of the region V_1. If θ_j and θ_k are such that

$$|\theta_j| \geq \frac{1}{64}\pi(1 + l_{min}/H) \;\text{ and }\; |\theta_k| \leq \frac{1}{64}\pi(1 + (l_{min} - 1)/H) \tag{4.15}$$

or vice versa, we have that $|\cos \Delta_{jk}| \leq \cos(\frac{1}{64}\pi/H)$. This includes at least $(\alpha n - n^{\varepsilon})n^{\varepsilon}/H$ edges $\{v_j, v_k\} \in EG$. Using (3.2) and (3.28), we get that as $n \to \infty$

$$S_1 \leq \pi^n \left(\cos \frac{\pi}{64H}\right)^{(\alpha n - n^{\varepsilon})n^{\varepsilon}/H} = O\Big(\exp(-cn^{1+\varepsilon})\Big) \, 2^{\frac{n-1}{2}} \pi^{\frac{n+1}{2}} \Big/ \sqrt{t(G)} \tag{4.16}$$

for some constant $c > 0$, depending only on γ.

For $1 \leq r \leq n^{\varepsilon}$ let $S_2(r)$ denote the contribution to S_2 of those $\theta \in V_2$ such that $|\theta_j| \geq \frac{1}{16}\pi$ for exactly r values of j. If $|\theta_j| \leq \frac{1}{32}\pi$ and $|\theta_k| \geq \frac{1}{16}\pi$ or vice versa, we have that

$$|\cos \Delta_{jk}| \leq \cos\left(\frac{1}{32}\pi\right) \tag{4.17}$$

This includes at least $r(\gamma n/2 - n^{\varepsilon})$ edges $\{v_j, v_k\} \in EG$, because the degree of any vertex of the graph G is at least $\gamma n/2$, see (3.14). For pairs (j, k) such that $|\theta_j|, |\theta_k| \leq \frac{1}{16}\pi$, we use (4.9). We put $\boldsymbol{\theta'} = (\theta_1, \ldots, \theta_{n-r})$.

$$S_2(r) \leq \pi^r \left(\cos \frac{\pi}{32}\right)^{r(\gamma n/2 - n^{\varepsilon})} \sum_{G_r} \int\limits_{U_{n-r}(\pi/2)} \exp\left(-\frac{1}{2} \sum_{\{v_j, v_k\} \in EG_r} \Delta_{jk}^2\right) d\boldsymbol{\theta'}, \tag{4.18}$$

where the first sum is over graphs, arises from G by removing all possible sets of r vertices.

Lemma 4.2. *Let the assumptions of Lemma 4.1 hold. Then*

$$\int\limits_{U_n(\pi/2)} \exp\left(-\frac{1}{2}\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^2\right)d\boldsymbol{\theta} \le \frac{2^{\frac{n-1}{2}}\pi^{\frac{n+1}{2}}n}{\sqrt{\det\hat{Q}}}. \tag{4.19}$$

Lemma 4.2 was formulated and proved in [1] (see Lemma 6.1). The proof is given also in Section 5.

Using Lemma 3.1 and Corollary 3.2, we get that

$$\lambda_2(G_r) \ge \gamma n - n^{\varepsilon}, \tag{4.20}$$

$$\det\hat{Q}(G_r) \ge \frac{\det\hat{Q}}{(c_1 n)^r}. \tag{4.21}$$

According to Lemma 4.2 we have that

$$\int\limits_{U_{n-r}(\pi/2)} \exp\left(-\frac{1}{2}\sum_{\{v_j,v_k\}\in EG_r}\Delta_{jk}^2\right)d\boldsymbol{\theta'} \le \frac{2^{\frac{n-r-1}{2}}\pi^{\frac{n-r+1}{2}}n}{\sqrt{\det\hat{Q}(G_r)}}. \tag{4.22}$$

Combining (4.18) with (4.21), (4.22) and allowing n^r for the choice of the set of r vertices for G_r, we get that

$$S_2(r) \le 2^{\frac{n-r-1}{2}}\pi^{\frac{n+r+1}{2}}n^{r+1}\left(\cos\frac{\pi}{32}\right)^{r(\gamma n/2-n^{\varepsilon})}\frac{(c_1 n)^{r/2}}{\sqrt{\det\hat{Q}}}. \tag{4.23}$$

Using (3.2) and summing over $0 \le r \le n^{\varepsilon}$, we find that

$$S_2 = \sum_{r=1}^{n^{\varepsilon}} S_2(r) = O(c^{-n})\,2^{\frac{n-1}{2}}\pi^{\frac{n+1}{2}}\Big/\sqrt{t(G)} \tag{4.24}$$

for some constant $c > 1$, depending only on γ.

Note that $\Delta_{jk} \le \frac{1}{8}\pi$ for $\boldsymbol{\theta} \in V_1 - V_2$, thus

$$V_1 - V_2 \subset V_3 = \{\boldsymbol{\theta} \in U_n(\pi/2) \mid |\theta_j - \bar{\theta}|\,(\mathrm{mod}\,\pi) \le \pi/8\}, \tag{4.25}$$

where

$$\bar{\theta} = \frac{\theta_1 + \ldots + \theta_n}{n}. \tag{4.26}$$

Since the integrand is invariant under uniform translation of all the θ_j's mod π as well as V_0 and V_3 are mapped into itself, we can fix $\bar{\theta} = 0$ and multiply it by the ratio of its range π to the length $n^{-1/2}$ of the vector $\frac{1}{n}[1,1,\ldots,1]^T$. Thus we get that

$$S_3 \le \pi n^{1/2} \int\limits_{L\cap U_n(\pi/8)\setminus V_0} \prod_{\{v_j,v_k\}\in EG} |\cos\Delta_{jk}|dL, \tag{4.27}$$

where L denotes the orthogonal complement to the vector $[1,1,\ldots,1]^T$. In a similar way as in (4.18) we find that

$$S_3 \le \pi n^{1/2} \int\limits_{L\cap U_n(\pi/8)\setminus V_0} \exp\left(-\frac{1}{2}\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^2\right)dL. \tag{4.28}$$

Lemma 4.3. *Let the assumptions of Lemma 4.1 hold. Then as* $n \to \infty$

$$\int\limits_{L\setminus U_n(n^{-1/2+\varepsilon})} \exp\left(-\frac{1}{2}\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^2\right) dL = O\left(\exp(-cn^{2\varepsilon})\right)\frac{2^{\frac{n-1}{2}}\pi^{\frac{n-1}{2}}n^{1/2}}{\sqrt{\det\hat{Q}}} \tag{4.29}$$

for some $c > 0$*, depending only on* γ*.*

Lemma 4.3 was formulated and proved in [1] (see Lemma 6.2). The proof is given also in Section 5.

Combining (3.28), (4.29) and (4.28), we get that as $n \to \infty$

$$S_3 = O\left(\exp(-cn^{2\varepsilon})\right)\frac{2^{\frac{n-1}{2}}\pi^{\frac{n+1}{2}}n}{\sqrt{\det\hat{Q}}} = O\left(\exp(-cn^{2\varepsilon})\right)\, 2^{\frac{n-1}{2}}\pi^{\frac{n+1}{2}}\Big/\sqrt{t(G)} \tag{4.30}$$

for some $c > 0$, depending only on γ. Combining (4.8), (4.16), (4.24) and (4.30), we obtain (2.8) as well as the following lemma.

Lemma 4.4. *Let the assumptions of Lemma 4.1 hold. Then as* $n \to \infty$

$$S = \left(1 + O\left(\exp(-cn^{2\varepsilon})\right)\right) S_0 \tag{4.31}$$

for some $c > 0$*, depending only on* γ*.*

5. Proofs of Lemmas 4.1 - 4.3

In this section we always assume that the assumptions of Lemma 4.1 hold. Let

$$\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{\theta}) = [\phi_1(\boldsymbol{\theta}), \ldots, \phi_n(\boldsymbol{\theta})]^T = \hat{Q}\boldsymbol{\theta}. \tag{5.1}$$

Let $P(\boldsymbol{\theta})$ be the orthogonal projection $\boldsymbol{\theta}$ onto the space L, where L denotes the orthogonal complement to the vector $[1, 1, \ldots, 1]^T$. Note that

$$Q\boldsymbol{\theta} = QP(\boldsymbol{\theta}). \tag{5.2}$$

For any $a > 0$, we have that:

$$\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}} d\boldsymbol{\theta} = \pi^{n/2}a^{-n/2}\Big/\sqrt{\det\hat{Q}} \tag{5.3}$$

and

$$\int\limits_{L} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}} dL = \int\limits_{L} e^{-a\,\boldsymbol{\theta}^T Q\boldsymbol{\theta}} dL = \pi^{\frac{n-1}{2}}a^{-\frac{n-1}{2}}n^{1/2}\Big/\sqrt{\det\hat{Q}}. \tag{5.4}$$

Proof of Lemma 4.2. Note that

$$\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^2 = \boldsymbol{\theta}^T Q\boldsymbol{\theta}. \tag{5.5}$$

The diagonal of the cube $U_n(\pi/2)$ is equal to $\pi n^{1/2}$. Using (5.2), we find that

$$\int\limits_{U_n(\pi/2)} \exp\left(-\frac{1}{2}\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^2\right)d\boldsymbol{\theta} \le \pi n^{1/2}\int\limits_{L} e^{-\frac{1}{2}\boldsymbol{\theta}^T Q\boldsymbol{\theta}}dL. \tag{5.6}$$

Combining (5.4) and (5.6), we obtain (4.19). ∎

Note that for some function $g_1(\boldsymbol{\theta}) = g_1(\theta_2,\dots,\theta_n)$

$$\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} = \frac{\phi_1(\boldsymbol{\theta})^2}{d_1+1} + g_1(\boldsymbol{\theta}). \tag{5.7}$$

We recall that (see (3.14))

$$\min_j d_j \ge \gamma n/2. \tag{5.8}$$

Combining (5.7) and (5.8), we get that as $n\to\infty$

$$\begin{aligned}\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta} &= \int\limits_{-\infty}^{+\infty}\dots\int\limits_{-\infty}^{+\infty} e^{-a\,g_1(\theta_2,\dots,\theta_n)}\left(\int\limits_{-\infty}^{+\infty} e^{-a\,\frac{\phi_1(\boldsymbol{\theta})^2}{d_1+1}}d\theta_1\right)d\theta_2\dots d\theta_n\\ &= \left(1+O\left(\exp(-\tilde{c}n^{2\varepsilon})\right)\right)\int\limits_{|\phi_1(\boldsymbol{\theta})|\le\frac{1}{2}c_\infty^{-1}n^{1/2+\varepsilon}} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta}\end{aligned} \tag{5.9}$$

for some $\tilde{c}>0$, depending obly on γ and a, where c_∞ is the constant of (3.13). Combining similar to (5.9) expressions for $\phi_1,\dots\phi_n$, we find that as $n\to\infty$

$$\int\limits_{||\boldsymbol{\phi}(\boldsymbol{\theta})||_\infty\le\frac{1}{2}c_\infty^{-1}n^{1/2+\varepsilon}} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta} = \left(1+O\left(\exp(-cn^{2\varepsilon})\right)\right)\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta} \tag{5.10}$$

for some $c>0$, depending only on γ и a. Using (5.10) and (3.13), we get that as $n\to\infty$

$$\int\limits_{U_n(\frac{1}{2}n^{-1/2+\varepsilon})} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta} = \left(1+O\left(\exp(-cn^{2\varepsilon})\right)\right)\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta}. \tag{5.11}$$

Proof of Lemma 4.3. Note that

$$||P(\boldsymbol{\theta})||_\infty = ||\boldsymbol{\theta}-\bar{\theta}[1,1,\dots,1]^T||_\infty \le 2||\boldsymbol{\theta}||_\infty, \tag{5.12}$$

where

$$\bar{\theta} = \frac{\theta_1+\theta_2+\dots\theta_n}{n}. \tag{5.13}$$

Thus

$$U_n(\frac{1}{2}n^{-1/2+\varepsilon}) \subset \left\{\boldsymbol{\theta} \mid P(\boldsymbol{\theta})\in U_n(n^{-1/2+\varepsilon})\right\} \tag{5.14}$$

Using (5.2), (5.5) and (5.14), we find that

$$\int\limits_{L\cap U_n(n^{-1/2+\varepsilon})} \exp\left(-\frac{1}{2}\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^2\right)dL = \int\limits_{L\cap U_n(n^{-1/2+\varepsilon})} e^{-\frac{1}{2}\,\boldsymbol{\theta}^T Q\boldsymbol{\theta}}dL =$$
$$= \int\limits_{P(\boldsymbol{\theta})\in U_n(n^{-1/2+\varepsilon})} e^{-\frac{1}{2}\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta}\Big/\int\limits_{-\infty}^{+\infty} e^{-\frac{1}{2}nx^2}dx \geq \frac{n^{1/2}}{\sqrt{2\pi}}\int\limits_{U_n(\frac{1}{2}n^{-1/2+\varepsilon})} e^{-\frac{1}{2}\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta}. \tag{5.15}$$

Combining (5.3), (5.11) and (5.15), we obtain (4.29). ∎

Proposition 5.1. *As $n\to\infty$*

$$\int\limits_{U_n(n^{-1/2+\varepsilon})} \exp\left(-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} - b\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^4\right)d\boldsymbol{\theta} =$$
$$= \Theta_{k_1,k_2}\left(\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta}\right), \tag{5.16}$$

where constants $k_1, k_2 > 0$ depend only on a, b and γ.

In thy present paper we give only the scheme of the proof of Proposition 5.1. The detailed proof can be found in [1] (see Lemma 8.3).

Scheme of the proof of Proposition 5.1. Since $a, b > 0$

$$\int\limits_{U_n(n^{-1/2+\varepsilon})} \exp\left(-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} - b\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^4\right)d\boldsymbol{\theta} \leq \int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta}. \tag{5.17}$$

Let

$$R_k(\boldsymbol{\theta}) = 8bn\sum_{j=k}^{n}\theta_j^4, \quad 1\leq k\leq n. \tag{5.18}$$

Using the representation of the integral as an iterated integral, one can show that

$$\int\limits_{U_n(n^{-1/2+\varepsilon})} \phi_j^4 e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}-R_k(\boldsymbol{\theta})}d\boldsymbol{\theta} \leq c'n^2\int\limits_{U_n(n^{-1/2+\varepsilon})} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}-R_k(\boldsymbol{\theta})}d\boldsymbol{\theta} +$$
$$+O\left(\exp(-cn^{2\varepsilon})\right)\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta} \tag{5.19}$$

for some constants $c, c' > 0$, depending only on a, b and γ.

Using (5.19), one can get that:

$$\int\limits_{U_n(n^{-1/2+\varepsilon})} \theta_j^4 e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}-R_k(\boldsymbol{\theta})}d\boldsymbol{\theta} \leq \frac{c_1'}{n^2}\int\limits_{U_n(n^{-1/2+\varepsilon})} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}-R_k(\boldsymbol{\theta})}d\boldsymbol{\theta} +$$
$$+O\left(\exp(-c_1n^{2\varepsilon})\right)\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta}; \tag{5.20}$$

$$\int\limits_{U_n(n^{-1/2+\varepsilon})} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}-R_k(\boldsymbol{\theta})}d\boldsymbol{\theta} \geq \left(1+\frac{c_2'}{n}\right)\int\limits_{U_n(n^{-1/2+\varepsilon})} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}-R_{k+1}(\boldsymbol{\theta})}d\boldsymbol{\theta}+ \\ +O\left(\exp(-c_2n^{2\varepsilon})\right)\int\limits_{\mathbb{R}^n} e^{-a\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta}, \tag{5.21}$$

for some constants $c_1, c_1', c_2, c_2' > 0$, depending only on a, b and γ. Note that

$$b\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^4 \leq 8bn\sum_{j=1}^n\theta_j^4 = R_1(\boldsymbol{\theta}). \tag{5.22}$$

Using several times inequality (5.21) for $k = 1, 2, \ldots, n-1$ and combining with (5.22), (5.17), we obtain (5.16). ∎

Proof of Lemma 4.1. Let

$$F(\boldsymbol{\theta}) = \exp\left(-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} - b\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^4\right). \tag{5.23}$$

Note that for $\theta \in L$,

$$F(\boldsymbol{\theta}) = \exp\left(-a\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^2 - b\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^4\right). \tag{5.24}$$

Since the integrand of (4.6) is invariant under uniform translation of all the θ_j's mod π as well as V_0 are mapped into itself, we can fix $\bar{\theta} = 0$ and multiply it by the ratio of its range π to the length $n^{-1/2}$ of the vector $\frac{1}{n}[1, 1, \ldots, 1]^T$. Thus we get that

$$\int\limits_{V_0}\exp\left(-a\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^2 - b\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^4\right)d\boldsymbol{\theta} = \\ = \pi n^{1/2}\int\limits_{L\cap U_n(n^{-1/2+\varepsilon})} F(\boldsymbol{\theta})dL. \tag{5.25}$$

In a similar way as in (5.15) we find that

$$\int\limits_{L\cap U_n(n^{-1/2+\varepsilon})} F(\boldsymbol{\theta})dL = \int\limits_{P(\boldsymbol{\theta})\in U_n(n^{-1/2+\varepsilon})} F(\boldsymbol{\theta})\,d\boldsymbol{\theta}\Big/\int\limits_{-\infty}^{+\infty} e^{-anx^2}dx = \\ = \left(\frac{\pi}{an}\right)^{-1/2}\int\limits_{P(\boldsymbol{\theta})\in U_n(n^{-1/2+\varepsilon})} F(\boldsymbol{\theta})\,d\boldsymbol{\theta}. \tag{5.26}$$

Using (5.11) and (5.14), we get that

$$\int\limits_{P(\boldsymbol{\theta})\in U_n(n^{-1/2+\varepsilon})} F(\boldsymbol{\theta})\,d\boldsymbol{\theta} = \int\limits_{U_n(n^{-1/2+\varepsilon})} F(\boldsymbol{\theta})\,d\boldsymbol{\theta} + O\left(\exp(-cn^{2\varepsilon})\right)\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}. \tag{5.27}$$

Combining (5.16), (5.25), (5.26) and (5.27), we obtain (4.6). ∎

6. Final remarks

Combining (4.2), (4.5), (4.6), (4.31), (5.25), (5.26), (5.27), we find that:

Proposition 6.1. *Let the assumtions of Theorem 2.1 hold. Then*

$$EO(G) = \left(1 + O\left(n^{-1+6\varepsilon}\right)\right) 2^{|EG|-1/2}\pi^{-n+1/2} n\, Int, \tag{6.1}$$

where

$$Int = \int\limits_{U_n(n^{-1/2+\varepsilon})} \exp\left(-\frac{1}{2}\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} - \frac{1}{12}\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^4\right) d\boldsymbol{\theta}, \tag{6.2}$$

$$\hat{Q} = Q + J, \quad \Delta_{jk} = \theta_j - \theta_k,$$

where $Q = Q(G)$ denotes the Laplcian matrix and J is the matrix with every entry 1.

Integral(6.2) can be evaluated more precisely for specific classes of graphs in order to get asymptotic formulas for $EO(G)$ similar to (1.1). For example, for even $n \to \infty$

$$EO(K_{n,n}) = e^{-1}\frac{2^{n^2+n-\frac{1}{2}}}{\pi^{n-\frac{1}{2}}n^{n-1}}\left(1 + O(n^{-1/2+\varepsilon})\right) \tag{6.3}$$

for any $\varepsilon > 0$, where $K_{n,n}$ denotes the complete bipartite graph with n vertices in each part.

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