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Multidegree for bifiltered D -modules and hypergeometric systems

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Abstract

In that paper, we recall the notion of the multidegree for D -modules, as exposed in a previous paper[2], with a slight simplification. A particular emphasis is given on hypergeometric systems, used to provide interesting and computable examples.

Introduction

This paper is an introduction to the theory of the multidegree for D -modules, as exposed in a previous paper[2]. The multidegree has been defined by E. Miller[10]: that is a generalization in multigraded algebra of the usual degree known in projective geometry. In our previous paper[2] we adapted it to the setting of bifiltered modules over the ring D of linear partial differential operators. Here the definition of the multidegree is slightly simplified: it becomes the identical counterpart, in the category of bifiltered D -modules, of the definition of Miller. We give detailed examples from the theory of A -hypergeometric systems of I.M. Gelfand, A.V. Zelevinsky and M.M. Kapranov[3].

The paper is organized as follows. In Section 1 we recall the definition of the objects we are interested in, up to the definition of (F, V) -bifiltered free resolutions of D -modules, following T. Oaku and N. Takayama[11]. Then we define the multidegree for bifiltered D -modules in Section 2. Some examples of A -hypergeometric systems are discussed in Section 3, leading to open questions which generalize known facts about the holonomic rank of A -hypergeometric systems. In Section 4 we give details on the simplification of the definition of the multidegree we give in Section 2, which consists in proving that a D -module M and its homogenization $\mathcal{R}_V(M)$ with respect to the V -filtration have same codimension.

1 Bifiltered free resolution of D -modules

Let $D = \mathbb{C}[x_1, \dots, x_n, t_1, \dots, t_p] \langle \partial_{x_1}, \dots, \partial_{x_n}, \partial_{t_1}, \dots, \partial_{t_p} \rangle$ denote the Weyl algebra in $n + p$ variables. Denoting the monomial

$$x_1^{\alpha_1} \dots x_n^{\alpha_n} t_1^{\mu_1} \dots t_p^{\mu_p} \partial_{x_1}^{\beta_1} \dots \partial_{x_n}^{\beta_n} \partial_{t_1}^{\nu_1} \dots \partial_{t_p}^{\nu_p}$$

by $x^\alpha t^\mu \partial_x^\beta \partial_t^\nu$, every element P in D is written uniquely as a finite sum with complex coefficients

$$P = \sum a_{\alpha, \beta, \mu, \nu} x^\alpha t^\mu \partial_x^\beta \partial_t^\nu. \quad (1)$$

The fundamental relations are $\partial_{x_i} x_i = x_i \partial_{x_i} + 1$ for $i = 1, \dots, n$ and $\partial_{t_i} t_i = t_i \partial_{t_i} + 1$ for $i = 1, \dots, p$.

We describe two important filtrations on D . For $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{N}^r$, let $|\lambda| = \lambda_1 + \dots + \lambda_r$. Let $\text{ord}^F(P)$ (resp. $\text{ord}^V(P)$) be the maximum of $|\beta| + |\nu|$ (resp. $|\nu| - |\mu|$) over the monomials $x^\alpha t^\mu \partial_x^\beta \partial_t^\nu$ appearing in (1). Then we define the filtrations

$$F_d(D) = \{P \in D, \text{ord}^F(P) \leq d\}$$

for $d \in \mathbb{Z}$ (with $F_d(D) = \{0\}$ for $d < 0$) and

$$V_k(D) = \{P \in D, \text{ord}^V(P) \leq k\}$$

for $k \in \mathbb{Z}$. The filtration $(F_d(D))$ is the most classical one, and the filtration $(V_k(D))$ is the so-called V -filtration along $t_1 = \dots = t_p = 0$ of Kashiwara-Malgrange.

The graded ring $\text{gr}^F(D) = \bigoplus_d F_d(D)/F_{d-1}(D)$ is a commutative polynomial ring, whereas the graded ring $\text{gr}^V(D) = \bigoplus_k V_k(D)/V_{k-1}(D)$ is isomorphic to D .

Let M be a left finitely generated D -module. We will define the notion of a good F -filtration of M . For $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r$, let $D^r[\mathbf{n}]$ denote the free module D^r endowed with the filtration

$$F_d(D^r[\mathbf{n}]) = \bigoplus_{i=1}^r F_{d-i}(D).$$

A *good* F -filtration of M is a sequence $(F_d(M))_{d \in \mathbb{Z}}$ of sub-vector spaces of M such that there exists a presentation

$$M \xrightarrow{\phi} \frac{D^r}{N},$$

with N a sub- D -module of D^r , and a vector shift $\mathbf{n} \in \mathbb{N}^r$, such that

$$F_d(M) \xrightarrow{\phi} \frac{F_d(D^r[\mathbf{n}]) + N}{N}.$$

The module $\text{gr}^F(M) = \bigoplus_d F_d(M)/F_{d-1}(M)$ is a graded finitely generated module over $\text{gr}^F(D)$. It is proved that the radical of the annihilator of $\text{gr}^F(M)$ does not depend on the good filtration of M . Then $\text{codim} M$ is defined as the codimension of the ring $\text{gr}^F(D)/\text{Ann}(\text{gr}^F(M))$, which does not depend on the

good filtration. A fundamental fact is that $\text{codim} M \leq n + p$ if $M \neq 0$. When $\text{codim} M = n + p$, the module M is said to be holonomic.

Now we introduce the bifiltration

$$F_{d,k}(D) = F_d(D) \cap V_k(D)$$

for $d, k \in \mathbb{Z}$, and we define the notion of a good (F, V) -bifiltration of M . For $\mathbf{n} = (n_1, \dots, n_r)$ and $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$, let $D^r[\mathbf{n}][\mathbf{m}]$ denote the free D -module D^r endowed with the bifiltration

$$F_{d,k}(D^r) = \bigoplus_{i=1}^r F_{d-i, k-i}(D).$$

A *good* (F, V) -bifiltration of M is a sequence $(F_{d,k}(M))_{d,k \in \mathbb{Z}}$ of sub-vector spaces of M such that there exists a presentation

$$M \underset{\sim}{\simeq} \frac{D^r}{N},$$

with N a sub- D -module of D^r , and vector shifts $\mathbf{n}, \mathbf{m} \in \mathbb{N}^r$, such that

$$F_{d,k}(M) \underset{\sim}{\simeq} \frac{F_{d,k}(D^r[\mathbf{n}][\mathbf{m}]) + N}{N}.$$

We are now in position to define the notion of a bifiltered free resolution of a module M endowed with a good bifiltration $(F_{d,k}(M))_{d,k}$. Such a resolution is an exact sequence

$$0 \rightarrow D^{r_s}[\mathbf{n}^{(\delta)}][\mathbf{m}^{(\delta)}] \xrightarrow{\phi_s} \dots \xrightarrow{\phi_1} D^{r_0}[\mathbf{n}^{(0)}][\mathbf{m}^{(0)}] \xrightarrow{\phi_0} M \rightarrow 0$$

such that for any $d, k \in \mathbb{Z}$, we have an exact sequence

$$0 \rightarrow F_{d,k}(D^{r_s}[\mathbf{n}^{(\delta)}][\mathbf{m}^{(\delta)}]) \xrightarrow{\phi_s} \dots \xrightarrow{\phi_1} F_{d,k}(D^{r_0}[\mathbf{n}^{(0)}][\mathbf{m}^{(0)}]) \xrightarrow{\phi_0} F_{d,k}(M) \rightarrow 0.$$

Example 1.1. Let $D = \mathbb{C}[t_1, t_2]\langle \partial_{t_1}, \partial_{t_2} \rangle$. Let I be the ideal generated by $\partial_{t_1} - \partial_{t_2}$ and $t_1 \partial_{t_1} + t_2 \partial_{t_2}$, and $M = D/I$ endowed with the good bifiltration $F_{d,k}(M) = (F_{d,k}(D) + I)/I$. That is the hypergeometric system $M_A(0, 0)$ associated with $A = \begin{pmatrix} 1 & 1 \end{pmatrix}$, see Section 3. We have a bifiltered free resolution:

$$0 \rightarrow D^1[2][1] \xrightarrow{\phi_2} D^2[1, 1][1, 0] \xrightarrow{\phi_1} D^1[0][0] \rightarrow M \rightarrow 0$$

with

$$\begin{aligned} \phi_1(1, 0) &= \partial_{t_1} - \partial_{t_2}, \\ \phi_1(0, 1) &= t_1 \partial_{t_1} + t_2 \partial_{t_2}, \\ \phi_2(1) &= (t_1 \partial_{t_1} + t_2 \partial_{t_2}, -(\partial_{t_1} - \partial_{t_2})). \end{aligned}$$

Bifiltered free resolutions have been introduced by T. Oaku and N. Takayama[11]. The same authors[12] show that such resolutions are at the heart of the computation of important objects in D -module theory: the restriction of a D -module

along a smooth subvariety, the algebraic local cohomology, the tensor product and localization.

However, the F -filtration they use is slightly different from ours: that is the filtration by the total order (i.e. x_i, t_i, ∂_{x_i} and ∂_{t_i} are given the weight 1 for all i), which allows them to define the notion of a *minimal* bifiltered free resolution. The ranks r_i , called Betti numbers, and the shifts $\mathbf{n}^{(i)}, \mathbf{m}^{(i)}$ do not depend on the minimal bifiltered free resolution, but do depend on the bifiltration $(F_{d,k}(M))_{d,k}$. Our choice for the filtration F comes from the common use of it, together with the V -filtration, in the theory of slopes and irregularity for D -modules, see e.g. the course of Y. Laurent[7]. In our setting the notion of a *minimal* bifiltered free resolution no longer makes sense. However, we will derive from the Betti numbers and shifts of an arbitrary bifiltered free resolution an invariant of the D -module M : the multidegree.

2 Multidegree for bifiltered D -modules

The multidegree has been introduced by E. Miller[10] in a commutative multi-graded context. It is a generalization of the notion of the degree known in projective geometry. We adapt it to the setting of bifiltered D -modules. First, we define the K -polynomial of a bifiltered D -module. Let

$$0 \rightarrow D^{r_\delta}[\mathbf{n}^{(\delta)}][\mathbf{m}^{(\delta)}] \xrightarrow{\phi_\delta} \dots \xrightarrow{\phi_1} D^{r_0}[\mathbf{n}^{(0)}][\mathbf{m}^{(0)}] \xrightarrow{\phi_0} M \rightarrow 0$$

be a bifiltered free resolution of M .

Definition 2.1. The K -polynomial of $D^r[\mathbf{n}][\mathbf{m}]$ with respect to (F, V) is defined by

$$K_{F,V}(D^r[\mathbf{n}][\mathbf{m}]; T_1, T_2) = \sum_{i=1}^r T_1^{\mathbf{n}_i} T_2^{\mathbf{m}_i} \in \mathbb{Z}[T_1, T_2, T_1^{-1}, T_2^{-1}].$$

The K -polynomial of M with respect to (F, V) is defined by

$$K_{F,V}(M; T_1, T_2) = \sum_{i=0}^{\delta} (-1)^i K_{F,V}(D^{r_i}[\mathbf{n}^{(i)}][\mathbf{m}^{(i)}]; T_1, T_2) \in \mathbb{Z}[T_1, T_2, T_1^{-1}, T_2^{-1}].$$

The definition of $K_{F,V}(M; T_1, T_2)$ does not depend on the bifiltered free resolution (Proposition 3.2 of our previous paper[2]), thus that is an invariant of the bifiltered module $(M, (F_{d,k}(M))_{d,k})$ very close to the data of the Betti numbers and shifts.

Example 2.1 (Continuation of Example 1.1). From the bifiltered free resolution

$$0 \rightarrow D^1[2][1] \xrightarrow{\phi_2} D^2[1, 1][1, 0] \xrightarrow{\phi_1} D^1[0][0] \rightarrow M \rightarrow 0,$$

we compute the K -polynomial $K(M; T_1, T_2) = 1 - (T_1 T_2 + T_1) + T_1^2 T_2$.

But $K_{F,V}(M; T_1, T_2)$ depends on the bifiltration chosen, as shown in the following example.

Example 2.2. Let $D = \mathbb{C}[t]\langle \partial_t \rangle$ and $M = D$ endowed with the good bifiltration $F_{d,k}(M) = F_{d,k}(D)$. Then M admits the bifiltered free resolution

$$0 \rightarrow D[0][0] \rightarrow M \rightarrow 0$$

and thus $K(M; T_1, T_2) = 1$. Now let

$$M' = \frac{D^2}{D \cdot (1, 1)}.$$

We have an isomorphism $M \simeq M'$, given by $1 \mapsto \overline{(1, 0)}$. We endow M' with the bifiltration

$$F_{d,k}(M') = \frac{F_{d,k}(D^2[1, 0][0, 1]) + D \cdot (1, 1)}{D \cdot (1, 1)}.$$

Then we have the following bifiltered free resolution:

$$0 \rightarrow D[1][1] \xrightarrow{\phi} D^2[1, 0][0, 1] \rightarrow M' \rightarrow 0,$$

with $\phi(1) = (1, 1)$. Then $K(M', T_1, T_2) = T_1 + T_2 - T_1 T_2$.

We now define the multidegree.

Definition 2.2. $K_{F,V}(M; 1 - T_1, 1 - T_2)$ is a well-defined power series in T_1, T_2 . We denote by $\mathcal{C}_{F,V}(M; T_1, T_2)$ the sum of the terms whose total degree in T_1, T_2 equals $\text{codim } M$ in the expansion of $K_{F,V}(M; 1 - T_1, 1 - T_2)$. This is called the *multidegree* of M with respect to (F, V) .

Example 2.3 (Continuation of Example 2.1). We have $\text{codim } M = 2$ and

$$\begin{aligned} K(M; 1 - T_1, 1 - T_2) &= 1 - ((1 - T_1)(1 - T_2) + (1 - T_1)) + ((1 - T_1)^2(1 - T_2)) \\ &= (T_1^2 + T_1 T_2) - T_1^2 T_2, \end{aligned}$$

thus $\mathcal{C}(M; T_1, T_2) = T_1^2 + T_1 T_2$.

The multidegree $\mathcal{C}_{F,V}(M; T_1, T_2)$ is a coarser invariant than $K(M; T_1, T_2)$, but its advantage is that it does not depend on the good bifiltration. In Example 2.2, we have $\mathcal{C}_{F,V}(M; T_1, T_2) = \mathcal{C}_{F,V}(M'; T_1, T_2) = 1$.

Theorem 2.1. $\mathcal{C}_{F,V}(M; T_1, T_2)$ does not depend on the good bifiltration of M .

Proof. This theorem is similar to Theorem 3.1 of our previous paper[2], proved using an argument from Y. Laurent and T. Monteiro-Fernandes[8]. But our definition of the multidegree is slightly simpler than that given in our previous paper[2]. Let \mathbb{K} denote the fraction field of $\mathbb{C}[x]$ and $\mathcal{R}_V(M)$ denote the Rees module associated with M considered as a V -filtered module. $\mathcal{R}_V(M)$ is naturally endowed with a F -filtration. In our previous paper[2], we have

defined the multidegree as the sum of the terms whose total degree equals $\text{codim}(\mathbb{K} \otimes \text{gr}^F(\mathcal{R}_V(M)))$ in the expansion of $K_{F,V}(M; 1 - T_1, 1 - T_2)$. Here for the sake of simplicity we no longer use \mathbb{K} : we define the multidegree as the sum of the terms whose total degree equals $\text{codim}(\text{gr}^F(\mathcal{R}_V(M)))$. The proof of the invariance also works.

The remaining problem, so as to be in accordance with Definition 2.2, is to prove that $\text{codim}(\text{gr}^F(\mathcal{R}_V(M))) = \text{codim } M$. We postpone the proof of it to Section 4. \square

3 Examples from the theory of hypergeometric systems

Let us consider a class of D -modules introduced by I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky[3], generalizing the Gauss hypergeometric equations, called A -hypergeometric systems. They are constructed as follows. In this section $D = \mathbb{C}[x_1, \dots, x_n] \langle \partial_1, \dots, \partial_n \rangle$. Let A be a $d \times n$ integer matrix and β_1, \dots, β_d be complex numbers. We assume that the abelian group generated by the columns a_1, \dots, a_n of A is equal to \mathbb{Z}^d . One defines first the toric ideal I_A : that is the ideal of $\mathbb{C}[\partial_1, \dots, \partial_n]$ generated by the elements $\partial^u - \partial^v$ with $u, v \in \mathbb{N}^n$ such that $A \cdot u = A \cdot v$. Then one defines the hypergeometric ideal $H_A(\beta)$: that is the ideal of D generated by I_A and the elements $\sum_j a_{i,j} x_j \partial_j - \beta_i$ for $i = 1, \dots, d$. Finally the A -hypergeometric system $M_A(\beta)$ is defined by the quotient $D/H_A(\beta)$. A. Adolphson[1] proved (in the general case) that these D -modules are holonomic. As do M. Schulze and U. Walther[14], we assume that the columns of A lie in a single open halfspace. The book of M. Saito, B. Sturmfels and N. Takayama[13] is a complete reference on (homogeneous) A -hypergeometric systems.

Our purpose in that section is to give some calculations of multidegree for hypergeometric systems. We will make the V -filtration of D vary, but the module $M_A(\beta)$ will always be endowed with the bifiltration $F_{d,k}(M_A(\beta)) = (F_{d,k}(D) + H_A(\beta))/H_A(\beta)$ once the V -filtration of D is chosen. The computations are done using the computer algebra systems Singular[5] and Macaulay2[4].

3.1 V -filtration along the origin

At first we consider the V -filtration along $x_1 = \dots = x_n = 0$.

Example 3.1. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

Then I_A is generated by $\partial_2 \partial_4 - \partial_3^2, \partial_1 \partial_4 - \partial_2 \partial_3, \partial_1 \partial_3 - \partial_2^2$. The ideal $H_A(\beta)$ is generated by I_A and the elements $x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + x_4 \partial_4 - \beta_1$; $x_2 \partial_2 + 2x_3 \partial_3 + 3x_4 \partial_4 - \beta_2$. For all β , we have

$$\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = 3T_1^4 + 6T_1^3 T_2 + 3T_1^2 T_2^2 = 3T_1^2 (T_1 + T_2)^2.$$

Let $\text{vol}(A)$ denote the normalized volume (with $\text{vol}([0, 1]^d) = d!$) of the convex hull in \mathbb{R}^d of the set $\{0, a_1, \dots, a_n\}$. Then we have in Example 3.1:

$$\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = \text{vol}(A) \cdot T_1^d (T_1 + T_2)^{n-d}. \quad (2)$$

Let us take a homogenizing variable h and let $H(I_A) \subset \mathbb{C}[\partial, h]$ denote the homogenization of I_A with respect to the total order in $\partial_1, \dots, \partial_n$. We proved the following (Theorem 5.2 of our previous paper[2]): if $\mathbb{C}[\partial, h]/H(I_A)$ is a Cohen-Macaulay ring and the parameters β_1, \dots, β_d are generic, then the formula (2) holds.

Question 3.1. Does the formula (2) hold if β_1, \dots, β_d are generic, but without the Cohen-Macaulay assumption?

For a holonomic module M , let us write $\mathcal{C}_{F,V}(M; T_1, T_2) = b_0(M)T_1^n + b_1(M)T_1^{n-1}T_2 + \dots + b_n(M)T_2^n$. From (2) we have $b_0(M_A(\beta)) = \text{vol}(A)$. We claim that the latter equality holds for all A , if β_1, \dots, β_d are generic. Indeed if M is any holonomic D -module we have $b_0(M) = \text{rank}(M)$ (Remark 5.1 of our previous paper[2]), where $\text{rank}(M)$, the holonomic rank of M , is the dimension of the vector space of local holomorphic solutions of M (considered as a system of linear partial differential equations) at a generic point. Let us remark that the niceness assumption in Remark 5.1 of our previous paper[2] can be dropped because of Proposition 4.1. Now, by Adolphson[1], $\text{rank}(M_A(\beta)) = \text{vol}(A)$ for generic β_1, \dots, β_d , which concludes to prove our claim.

Example 3.2 (from Saito-Sturmfels-Takayama[13]). Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$

Then I_A is generated by $\partial_2\partial_4^2 - \partial_3^3, \partial_1\partial_4 - \partial_2\partial_3, \partial_1\partial_3^2 - \partial_2^2\partial_4, \partial_1^2\partial_3 - \partial_2^3$. Here $\mathbb{C}[\partial, h]/H(I_A)$ is not Cohen-Macaulay. However for $(\beta_1, \beta_2) \neq (1, 2)$, we have

$$\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = 4T_1^4 + 8T_1^3T_2 + 4T_1^2T_2^2 = 4T_1^2(T_1 + T_2)^2,$$

which agrees with the formula (2).

For $(\beta_1, \beta_2) = (1, 2)$, we have

$$\mathcal{C}_{F,V}(M_A(\beta); T_1, T_2) = 5T_1^4 + 12T_1^3T_2 + 10T_1^2T_2^2 + 4T_1T_2^3 + T_2^4.$$

Let us remark that $(T_1 + T_2)^2$ is still a factor, indeed

$$5T_1^4 + 12T_1^3T_2 + 10T_1^2T_2^2 + 4T_1T_2^3 + T_2^4 = (T_1 + T_2)^2(5T_1^2 + 2T_1T_2 + T_2^2).$$

3.2 V -filtration along coordinate hyperplanes

Let us reconsider Examples 3.1 and 3.2, with the V -filtration along $x_i = 0$ for some fixed i .

Example 3.3 (continuation of Example 3.1).

- V -filtration along $x_1 = 0$: for any β_1, β_2 ,

$$\mathcal{C}_{F,V}(M_A(\beta)) = 3T_1^4 + 2T_1^3T_2.$$

- V -filtration along $x_2 = 0$: for any β_1, β_2 ,

$$\mathcal{C}_{F,V}(M_A(\beta)) = 3T_1^4 + 3T_1^3T_2.$$

- V -filtration along $x_3 = 0$: same as V -filtration along $x_2 = 0$.
- V -filtration along $x_4 = 0$: same as V -filtration along $x_1 = 0$.

Example 3.4 (Continuation of Example 3.2). Let us take the V -filtration along $x_4 = 0$. Then if $(\beta_1, \beta_2) \neq (1, 2)$, then

$$\mathcal{C}_{F,V}(M_A(\beta)) = 4T_1^4 + 3T_1^3T_2.$$

For $(\beta_1, \beta_2) = (1, 2)$, we have

$$\mathcal{C}_{F,V}(M_A(\beta)) = 5T_1^4 + 4T_1^3T_2.$$

3.3 Dependency of the multidegree on the parameters

Studying the dependency of the multidegree on the parameters β_1, \dots, β_d is a natural problem. A basic known fact is that the multidegree remains constant outside of an algebraic hypersurface of \mathbb{C}^d (analogous to Proposition 1.5 of our previous paper[2]). The dependency of the holonomic rank of $M_A(\beta)$ has been deeply studied by several authors, for instance in Saito-Sturmfels-Takayama[13] and L.F. Matusevich, E. Miller and U. Walther[9]. In particular it has been proved in the latter paper that the holonomic rank of $M_A(\beta)$ is upper semi-continuous as a function on the parameter $(\beta_1, \dots, \beta_d)$, which means that the holonomic rank at an exceptionnal parameter is greater than the holonomic rank at a generic parameter. Thus the coefficient $b_0(M_A(\beta))$ is an upper semi-continuous function on β_1, \dots, β_d . Nobuki Takayama pointed out the problem of that semi-continuity for the other coefficients of the multidegree, as also suggest Examples 3.2 and 3.4.

Question 3.2. Let us fix the matrix A , the V -filtration on D and $0 \leq i \leq n$. Is the coefficient $b_i(M_A(\beta))$ a upper semi-continuous function on β_1, \dots, β_d ?

3.4 Positivity

The following positivity problem is due to an observation by Michel Granger.

Question 3.3. Let us fix A , β_1, \dots, β_d , the V -filtration on D and $0 \leq i \leq n$. Do we always have $b_i(M_A(\beta)) \geq 0$?

That is the case in all the examples we considered. Moreover, since $b_0(M_A(\beta)) = \text{rank}(M_A(\beta))$, the positivity is true for $b_0(M_A(\beta))$.

4 Proof of Theorem 2.1

Here we complete the proof of Theorem 2.1. Again $D = \mathbb{C}[x, t]\langle \partial_x, \partial_t \rangle$. We recall the definition of the Rees ring $\mathcal{R}_V(D) = \bigoplus V_k(D)T^k$ endowed with the filtration $F_d(\mathcal{R}_V(D)) = \bigoplus_k F_{d,k}(D)T^k$. We have a ring isomorphism $\mathcal{R}_V(D) \simeq D[\theta]$ given by $x_i T^0 \mapsto x_i$; $t_i T^{-1} \mapsto t_i$; $T^1 \mapsto \theta$; $\partial_{x_i} T^0 \mapsto \partial_{x_i}$ and $\partial_{t_i} T^1 \mapsto \partial_{t_i}$. The F -filtration on $D[\theta]$ induced by this isomorphism is given by assigning the weights $(\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, 0)$ to $(\mathbf{x}, \mathbf{t}, \partial_{\mathbf{x}}, \partial_{\mathbf{t}}, \theta)$. Suppose that, as a bifiltered D -module, $M = D^r[\mathbf{n}][\mathbf{m}]/N$. We endow M with the V -filtration $V_k(M) = V_k(D^r[\mathbf{m}]/N)$.

Then we associate with M a Rees module $\mathcal{R}_V(M) = \bigoplus V_k(M)T^k$ endowed with the F -filtration $F_d(\mathcal{R}_V(M)) = \bigoplus_k F_{d,k}(M)T^k$. We have an isomorphism of graded modules

$$\frac{D[\theta]^r[\mathbf{m}]}{H^V(N)} \simeq \mathcal{R}_V(M)$$

given by $\bar{e}_i \mapsto \bar{e}_i T^{m_i}$, with (e_i) the canonical base either of $D[\theta]^r$ or of D^r , and $H^V(N)$ the homogenization of N with respect to V . Furthermore it is an isomorphism of F -filtered modules

$$\frac{D[\theta]^r[\mathbf{n}]}{H^V(N)} \simeq \mathcal{R}_V(M).$$

We denote by $\text{codim} \mathcal{R}_V(M)$ the codimension of the module $\text{gr}^F(\mathcal{R}_V(M))$. In fact it is allowed to replace the weight vector $(\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, 0)$ defining the F -filtration by any non-negative weight vector G , giving rise to a filtration on $D[\theta]$ also denoted by G , such that $\text{gr}^G(D[\theta])$ is a commutative ring (see Proposition 5.1 of our previous paper[2]).

Proposition 4.1. $\text{codim} \mathcal{R}_V(M) = \text{codim} M$.

Lemma 4.1. $\text{codim} \mathcal{R}_V(M) \leq \text{codim} M$.

Proof. We make use of the characterization of the codimension by means of extension groups (see R. Hotta, K. Takeuchi and T. Tanisaki[6], Theorem D.4.3):

$$\begin{aligned} \text{codim} M &= \inf \{i : \text{Ext}_D^i(M, D) \neq 0\}, \\ \text{codim} \mathcal{R}_V(M) &= \inf \{i : \text{Ext}_{\mathcal{R}_V(D)}^i(\mathcal{R}_V(M), \mathcal{R}_V(D)) \neq 0\}. \end{aligned}$$

Let

$$\cdots \rightarrow \mathcal{L}_1 \xrightarrow{\phi_1} \mathcal{L}_0 \xrightarrow{\phi_0} M \rightarrow 0$$

be a V -adapted free resolution of M , with $\mathcal{L}_i = D^{r_i}[\mathbf{m}^{(i)}]$. It induces a graded free resolution

$$\cdots \rightarrow \mathcal{R}_V(\mathcal{L}_1) \xrightarrow{\mathcal{R}_V(\phi_1)} \mathcal{R}_V(\mathcal{L}_0) \xrightarrow{\mathcal{R}_V(\phi_0)} \mathcal{R}_V(M) \rightarrow 0,$$

with $\mathcal{R}_V(\mathcal{L}_i) \simeq D[\theta]^{r_i}[\mathbf{m}^{(i)}]$.

If N is a left D -module, let $N^* = \text{Hom}_D(N, D)$. If N is a left $\mathcal{R}_V(D)$ -module, let $N^* = \text{Hom}_{\mathcal{R}_V(D)}(N, \mathcal{R}_V(D))$.

The complex \mathcal{L}^* :

$$0 \rightarrow \mathcal{L}_0^* \xrightarrow{\phi_0^*} \mathcal{L}_1^* \xrightarrow{\phi_1^*} \dots$$

gives the groups $\text{Ext}_D^i(M, D)$. The complex $\mathcal{R}_V(\mathcal{L})^*$:

$$0 \rightarrow \mathcal{R}_V(\mathcal{L}_0)^* \xrightarrow{\mathcal{R}_V(\phi_0)^*} \mathcal{R}_V(\mathcal{L}_1)^* \xrightarrow{\mathcal{R}_V(\phi_1)^*} \dots$$

gives the groups $\text{Ext}_{\mathcal{R}_V(D)}^i(\mathcal{R}_V M, \mathcal{R}_V D)$.

If N is a left D -module endowed with a good V -filtration, then we endow N^* by the exhaustive filtration

$$V_k(N^*) = \{u : N \rightarrow D \text{ such that } \forall j, u(V_j(N)) \subset V_{j+k}(D)\}.$$

Let us consider $\phi_i : \mathcal{L}_i \rightarrow \mathcal{L}_{i-1}$ together with $\phi_i^* : \mathcal{L}_{i-1}^* \rightarrow \mathcal{L}_i^*$. We endow $\ker(\phi_i^*)$ with the induced V -filtration. We claim that

$$\mathcal{R}_V(\ker(\phi_i^*)) \simeq \ker(\mathcal{R}_V(\phi_i)^*).$$

Indeed, let $\mathcal{L} = D^r[\mathbf{m}]$ be a V -filtered free module. We have a bijection $\mathcal{R}_V(\mathcal{L}^*) \simeq (\mathcal{R}_V \mathcal{L})^*$, by mapping $\oplus u_k T^k$, with $u_k \in V_k(\mathcal{L}^*)$, to $\sum \mathcal{R}_V(u_k) \in (\mathcal{R}_V \mathcal{L})^*$. Under that bijection, $\mathcal{R}_V(\ker(\phi_i^*)) \subset \mathcal{R}_V(\mathcal{L}_{i-1}^*)$ is seen as a subset of $(\mathcal{R}_V \mathcal{L}_{i-1})^*$ and is equal to $\ker(\mathcal{R}_V(\phi_i)^*)$.

On the other hand, let us endow $\text{Im}(\phi_i^*)$ with the induced V -filtration. Using the identification $\mathcal{R}_V(\mathcal{L}_i^*) \simeq (\mathcal{R}_V \mathcal{L}_i)^*$, we have

$$\text{Im}(\mathcal{R}_V(\phi_i)^*) \subset \mathcal{R}_V(\text{Im}(\phi_i^*)).$$

Then if $H_i(\mathcal{L}^*) = \ker(\phi_{i+1}^*)/\text{Im}(\phi_i^*)$ is endowed with the quotient V -filtration, we have

$$H_i(\mathcal{R}_V(\mathcal{L})^*) = \frac{\ker(\mathcal{R}_V(\phi_{i+1})^*)}{\text{Im}(\mathcal{R}_V(\phi_i)^*)} \rightarrow \frac{\mathcal{R}_V(\ker(\phi_{i+1}^*))}{\mathcal{R}_V(\text{Im}(\phi_i^*))} = \mathcal{R}_V(H_i(\mathcal{L}^*)).$$

Thus if $H_i(\mathcal{R}_V(\mathcal{L})^*) = 0$, then $\mathcal{R}_V(H_i(\mathcal{L}^*)) = 0$ which implies that $H_i(\mathcal{L}^*) = 0$. The lemma follows. \square

Lemma 4.2. $\text{codim} M \leq \text{codim} \mathcal{R}_V(M)$.

Proof. Here we make use of the theory of Gelfand-Kirillov dimension, see e.g. G.G. Smith[15]. Let now F denote the Bernstein filtration on D , i.e. each variable $x_i, \partial_{x_i}, t_i, \partial_{t_i}$ has weight 1. We endow M with the good (F, V) -bifiltration, still denoted by $F_{d,k}(M)$, given by the quotient $D^r[\mathbf{0}][\mathbf{m}]/N$.

Let $\phi : \mathbb{N} \rightarrow \mathbb{R}$. Let $\gamma(\phi) = \inf\{i : f(d) \leq d^i \text{ for } d \text{ large enough}\}$. By Gelfand-Kirillov theory, we have $\gamma(d \mapsto \dim_{\mathbb{C}} F_d(M)) = \dim M = 2n - \text{codim} M$.

Let us define the filtration $(G_d(D[\theta]))$ by giving the weight 1 to all the variables. Then $\text{gr}^G(D[\theta])$ is commutative and for any d , $G_d(D[\theta])$ is finitely dimensional over \mathbb{C} . Then we endow $\mathcal{R}_V(M)$ with the G -filtration $G_d(\mathcal{R}_V(M)) = G_d(D[\theta]^r[\mathbf{0}]/H^V(N))$.

Let $d \in \mathbb{N}$ and E_d be the interval $[\min_i m_i - d, \max_i m_i + d]$. We have

$$G_d(\mathcal{R}_V M) \subset \bigoplus_{k \in E_d} F_{d,k}(M) T^k \subset \bigoplus_{k \in E_d} F_d(M) T^k.$$

Then $\dim_{\mathbb{C}} G_d(\mathcal{R}_V(M)) \leq (2d + c) \dim_{\mathbb{C}} F_d(M)$, with $c = \max_i m_i - \min_i m_i + 1$. Thus

$$\begin{aligned} \dim \mathcal{R}_V(M) &= \gamma(d \mapsto \dim_{\mathbb{C}} G_d(\mathcal{R}_V(M))) \\ &\leq \gamma(d \mapsto (2d + c) \dim_{\mathbb{C}} F_d(M)) \\ &= 1 + \gamma(d \mapsto \dim_{\mathbb{C}} F_d(M)) \\ &= 1 + \dim M. \end{aligned}$$

□

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