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# Computational Aspects of Modular Forms and Galois Representations

Jean-Marc Couveignes, Bas Edixhoven

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# Computational aspects of modular forms and Galois representations



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# Computational aspects of modular forms and Galois representations

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*how one can compute in polynomial time  
the value of Ramanujan's tau at a prime*

Jean-Marc Couveignes and  
Bas Edixhoven, editors

PRINCETON UNIVERSITY PRESS  
PRINCETON AND OXFORD

Let  $\tau: \mathbb{N} \rightarrow \mathbb{Z}$  be defined by:

$$\sum_{n \geq 0} \tau(n)q^n = q \prod_{n \geq 1} (1 - q^n)^{24} \quad \text{in } \mathbb{Z}[[q]].$$

Then we have, in  $\mathbb{Z}/19\mathbb{Z}$ :

$$\begin{aligned}\tau(10^{1000} + 1357) &= \pm 4, \\ \tau(10^{1000} + 7383) &= \pm 2, \\ \tau(10^{1000} + 21567) &= \pm 3, \\ \tau(10^{1000} + 27057) &= 0, \\ \tau(10^{1000} + 46227) &= 0, \\ \tau(10^{1000} + 57867) &= 0, \\ \tau(10^{1000} + 64749) &= \pm 7, \\ \tau(10^{1000} + 68367) &= 0, \\ \tau(10^{1000} + 78199) &= \pm 8, \\ \tau(10^{1000} + 128647) &= 0.\end{aligned}$$

Section 7.5, Lemma 7.4.1, and [Magma], [SAGE] and [PARI].

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## *Preface*

This is a book about computational aspects of modular forms and the Galois representations attached to them. The main result is the following: Galois representations over finite fields attached to modular forms of level one can, in almost all cases, be computed in polynomial time in the weight and the size of the finite field. As a consequence, coefficients of modular forms can be computed fast via congruences, as in Schoof's algorithm for the number of points of elliptic curves over finite fields. The most important feature of the proof of the main result is that exact computations involving systems of polynomial equations in many variables are avoided by approximations and height bounds, i.e., bounds for the accuracy that is necessary to derive exact values from the approximations.

The book's authors are the two editors, Jean-Marc Couveignes and Bas Edixhoven, together with Johan Bosman, Robin de Jong, and Franz Merkl. Each chapter has its own group of authors.

Chapter 1 gives an introduction to the subject, precise statements of the main results, and places these in a somewhat wider context. Chapter 2 provides the necessary background concerning modular curves and modular forms. Chapter 3 gives a first, informal description of the algorithms. These first three chapters should allow readers without much background in arithmetic geometry to still get a good idea of what happens in the book, skipping, if necessary, some parts of Chapter 2.

Chapters 4 and 5 provide the necessary background on heights and Arakelov theory, and on algorithmic aspects of the computation with a desired accuracy of the roots of complex polynomials and power series.

Chapters 6 and 7 are concerned with some real computations of Galois representations attached to modular forms, and end with a table dealing with all cases of forms of weight at most 22 and finite fields of characteristic at

most 23.

The main ingredients for the proof of the main result are established in Chapters 8, 9, 10, 11, 12, and 13. The topics dealt with are, respectively: construction of suitable divisors on modular curves, bounding heights using Arakelov theory, bounding Arakelov invariants of certain modular curves, approximation of divisors using complex numbers, and using finite fields.

The main result on the computation of Galois representations is proved in Chapter 14, where one finds a detailed description of the algorithm and a rigorous proof of the complexity bound.

Chapter 15 contains the application of the main result to the computation of coefficients of modular forms.

The Epilogue announces some work on generalisations and applications that will be completed in the near future, as well as a direction of further research outside the context of modular forms.

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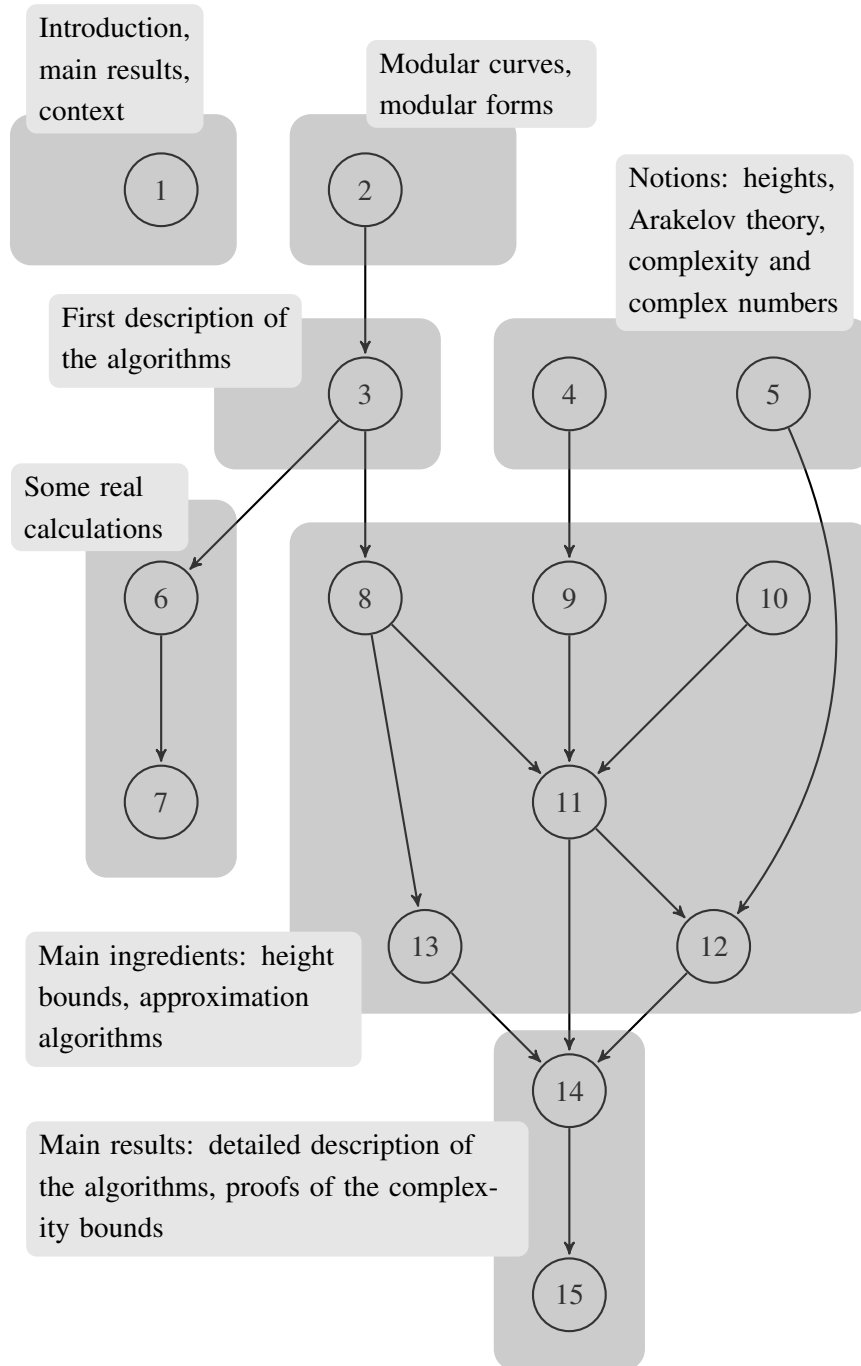
From 2005 until 2010, our project was supported financially in the Netherlands by NWO, the Dutch organisation for scientific research (Nederlandse organisatie voor Wetenschappelijk Onderzoek) in the form of a VICI-grant to Bas Edixhoven.

As the contract with the CELAR stipulated, the report was published on internet (arxiv, homepage), in May 2006, and our aim was to extract a research article from it. We thank Eyal Goren for his suggestion to expand it into this book instead.

Bas Edixhoven thanks René Schoof for asking the question, in 1995, that this book answers.

We thank Peter Bruin for carefully reading preliminary versions and pointing out some errors, and Hendrik Lenstra for some suggestions and references.

**DEPENDENCIES BETWEEN THE CHAPTERS**





# Chapter One

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## Introduction, main results, context

B. Edixhoven

### 1.1 STATEMENT OF THE MAIN RESULTS

As the final results in this book are about fast computation of coefficients of modular forms, I start by describing the state of the art in this subject.

A convenient way to view modular forms and their coefficients in this context is as follows, in terms of Hecke algebras. For  $N$  and  $k$  positive integers, let  $S_k(\Gamma_1(N))$  be the finite dimensional complex vector space of cuspforms of weight  $k$  on the congruence subgroup  $\Gamma_1(N)$  of  $\mathrm{SL}_2(\mathbb{Z})$ . Each  $f$  in  $S_k(\Gamma_1(N))$  has a power series expansion  $f = \sum_{n \geq 1} a_n(f)q^n$ , a complex power series converging on the unit disk. These  $a_n(f)$  are the coefficients of  $f$  that we want to compute, in particular for large  $n$ . For each positive integer  $n$  we have an endomorphism  $T_n$  of  $S_k(\Gamma_1(N))$ , and we let  $\mathbb{T}(N, k)$  denote the sub- $\mathbb{Z}$ -algebra of  $\mathrm{End}(S_k(\Gamma_1(N)))$  generated by them. The  $\mathbb{T}(N, k)$  are commutative, and free  $\mathbb{Z}$ -modules of rank the dimension of  $S_k(\Gamma_1(N))$ , which is of polynomially bounded growth in  $N$  and  $k$ . For each  $N, k$  and  $n$  one has the identity  $a_n(f) = a_1(T_n f)$ . The  $\mathbb{C}$ -valued pairing between  $S_k(\Gamma_1(N), \mathbb{C})$  and  $\mathbb{T}(N, k)$  given by  $(f, t) \mapsto a_1(tf)$  identifies  $S_k(\Gamma_1(N), \mathbb{C})$  with the space of  $\mathbb{Z}$ -linear maps from  $\mathbb{T}(N, k)$  to  $\mathbb{C}$ , and we can write  $f(T_n)$  for  $a_n(f)$ . All together this means that the key to the computation of coefficients of modular forms is the computation of the Hecke algebras  $\mathbb{T}(N, k)$  and their elements  $T_n$ . A modular form  $f$  in  $S_k(\Gamma_1(N))$  is determined by the  $f(T_i)$  with  $i \leq k \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)]/12$ , hence if  $T_n$  is known as a  $\mathbb{Z}$ -linear combination of these  $T_i$ , then  $f(T_n)$  can be computed as the same  $\mathbb{Z}$ -linear combination of the  $f(T_i)$ .

The state of the art in computing the algebras  $\mathbb{T}(N, k)$  can now be summarised as follows.

*There is a deterministic algorithm, that on input positive integers  $N$  and  $k \geq 2$ , computes  $\mathbb{T}(N, k)$ : it gives a  $\mathbb{Z}$ -basis and the multiplication table for this basis, in running time polynomial in  $N$  and  $k$ . Moreover, the Hecke operator  $T_n$  can be expressed in this  $\mathbb{Z}$ -basis in deterministic polynomial time in  $N$ ,  $k$  and  $n$ .*

We do not know a precise reference for this statement, but it is rather obvious from the literature on calculations with modular forms for which we refer to William Stein's book [Ste2], and in particular to Section 8.10.2 of it. The algorithms alluded to above use that  $S_k(\Gamma_1(N))$ , viewed as  $\mathbb{R}$ -vector space, is naturally isomorphic to the  $\mathbb{R}$ -vector space obtained from the so-called ‘‘cuspidal subspace’’  $H^1(\Gamma_1(N), \mathbb{Z}[x, y]_{k-2})_{\text{cusp}}$  of the  $\mathbb{Z}$ -module  $H^1(\Gamma_1(N), \mathbb{Z}[x, y]_{k-2})$  in group cohomology. Here,  $\mathbb{Z}[x, y]_{k-2}$  is the homogeneous part of degree  $k-2$  of the polynomial ring  $\mathbb{Z}[x, y]$  on which  $\text{SL}_2(\mathbb{Z})$  acts via its standard representation on  $\mathbb{Z}[x, y]_1$ . In this way,  $H^1(\Gamma_1(N), \mathbb{Z}[x, y]_{k-2})_{\text{cusp}}$ , modulo its torsion subgroup, is a free  $\mathbb{Z}$ -module of finite rank that is a faithful  $\mathbb{T}(N, k)$ -module, and the action of the  $T_n$  is described explicitly. Algorithms based on this typically use a presentation of  $H^1(\Gamma_1(N), \mathbb{Z}[x, y]_{k-2})_{\text{cusp}}$  in terms of so-called ‘‘modular symbols’’, and we call them therefore modular symbols algorithms. The theory of modular symbols was developed by Birch, Manin, Shokurov, Cremona, Merel, . . . It has led to many algorithms, implementations and calculations, which together form the point of departure for this book.

The computation of the element  $T_n$  of  $\mathbb{T}(N, k)$ , using modular symbols algorithms, involves sums of a number of terms that grows at least linearly in  $n$ . If one computes such sums by evaluating and adding the terms one by one, the computation of  $T_n$ , for  $N$  and  $k$  fixed, will take time at least linear in  $n$ , and hence exponential in  $\log n$ . The same is true for other methods for computing  $T_n$  that we know of: computations with  $q$ -expansions that involve multiplication of power series, using linear combinations of theta series, the ‘‘graph method’’ of Mestre and Oesterlé, and the Lefschetz trace formula for correspondences, holomorphic or not. Efforts to evaluate the en-

countered sums more quickly seem to lead, in each case, again to the problem of computing coefficients of modular forms. For example, the graph method leads to the problem of computing quickly representation numbers of integer quadratic forms in 4 variables. In the case of the trace formula, there are maybe only  $O(\sqrt{n})$  terms, but they contain class numbers of imaginary quadratic orders, these numbers being themselves directly related to coefficients of modular forms of half integral weight.

Here is one of the main results in this book, Theorem 15.2.1.

*Assume that the generalised Riemann hypothesis (GRH) holds. There exists a deterministic algorithm that on input positive integers  $n$  and  $k$ , together with the factorisation of  $n$  into prime factors, computes the element  $T_n$  of  $\mathbb{T}(1, k)$  in running time polynomial in  $k$  and  $\log n$ .*

The restriction to modular forms of level 1 in this result is there for a technical reason. The result will certainly be generalised to much more general levels; see the Epilogue at the end of this book. The condition that the factorisation of  $n$  into primes must be part of the input is necessary because we do not have a polynomial time algorithm for factoring integers. Vice versa, see Remark 2.2.4 for evidence that factoring is not harder than computing coefficients of modular forms.

I now describe how the computation of Galois representations is used for the computation of  $T_n$ . Standard identities express  $T_n$  in terms of the  $T_p$  for  $p$  dividing  $n$ . These  $T_p$  are computed, via the LLL basis reduction algorithm, from sufficiently many of their images under morphisms  $f$  from  $\mathbb{T}(1, k)$  to finite fields, analogously to Schoof's algorithm for counting points of an elliptic curve over a finite field. Indeed, for such an  $f: \mathbb{T}(1, k) \rightarrow \mathbb{F}$ , with  $p$  not the characteristic,  $l$ , say, of  $\mathbb{F}$ , the image  $f(T_p)$  is equal to the trace of  $\rho_f(\text{Frob}_p)$ , where  $\rho_f: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$  is the Galois representation attached to  $f$ , and  $\rho_f(\text{Frob}_p)$  a Frobenius element at  $p$ . The representation  $\rho_f$  is characterised by the following three conditions: it is semi-simple, it is unramified outside  $l$ , and for all prime numbers  $p \neq l$  one has:

$$\text{trace}(\rho_f(\text{Frob}_p)) = f(T_p) \quad \text{and} \quad \det(\rho_f(\text{Frob}_p)) = p^{k-1} \quad \text{in } \mathbb{F}.$$

It is *the* main result of this book, Theorem 14.1.1, plus some standard com-



putational number theory, that enables us to compute  $\rho_f(\text{Frob}_p)$  in time polynomial in  $k$ ,  $\#\mathbb{F}$  and  $\log p$  (note the log!). Under GRH, existence of sufficiently many maximal ideals of small enough index is guaranteed. Theorem 14.1.1 says:

*There is a deterministic algorithm that on input a positive integer  $k$ , a finite field  $\mathbb{F}$ , and a surjective ring morphism  $f$  from  $\mathbb{T}(1, k)$  to  $\mathbb{F}$  such that the associated Galois representation  $\rho_f: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$  is reducible or has image containing  $\text{SL}_2(\mathbb{F})$ , computes  $\rho_f$  in time polynomial in  $k$  and  $\#\mathbb{F}$ .*

By “computing  $\rho_f$ ” we mean the following. Let  $K_f \subset \overline{\mathbb{Q}}$  be the finite Galois extension such that  $\rho_f$  factors as the natural surjection from  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  to  $\text{Gal}(K_f/\mathbb{Q})$ , followed by an injection into  $\text{GL}_2(\mathbb{F})$ . Then to give  $\rho_f$  means to give  $K_f$  as  $\mathbb{Q}$ -algebra, in terms of a multiplication table with respect to a  $\mathbb{Q}$ -basis, together with a list of all elements of  $\text{Gal}(K_f/\mathbb{Q})$ , as matrices with coefficients in  $\mathbb{Q}$ , and, for each  $\sigma$  in  $\text{Gal}(K_f/\mathbb{Q})$ , to give the corresponding element  $\rho_f(\sigma)$  of  $\text{GL}_2(\mathbb{F})$ .

Before I describe in more detail, in the next sections, some history and context concerning our main results, I give one example and I make some brief remarks. Many of these remarks are treated with more detail further on.

The first non-trivial example is given by  $k = 12$ . The space of cuspidal modular forms of level one and weight 12 is one-dimensional, generated by the discriminant modular form  $\Delta$ , whose coefficients are given by Ramanujan’s  $\tau$ -function:

$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n = q - 24q^2 + 252q^3 + \cdots \quad \text{in } \mathbb{Z}[[q]].$$

In this case, the Hecke algebra  $\mathbb{T}(1, 12)$  is the ring  $\mathbb{Z}$ , and, for each  $n$  in  $\mathbb{Z}_{>0}$ , we have  $T_n = \tau(n)$ . The results above mean that:

*for  $p$  prime, Ramanujan’s  $\tau(p)$  can be computed in time polynomial in  $\log p$ .*

For  $l$  prime, let  $\rho_l$  denote the Galois representation to  $\text{GL}_2(\mathbb{F}_l)$  attached to  $\Delta$ . It was proved by Swinnerton-Dyer that for  $l$  not in  $\{2, 3, 5, 7, 23, 691\}$

the image of  $\rho_l$  contains  $\mathrm{SL}_2(\mathbb{F}_l)$ . This means that for all  $l$  not in this short list the representation  $\rho_l$  has non-solvable image, and so cannot be computed using computational class field theory. The classical congruences for Ramanujan's  $\tau$ -function correspond to the  $l$  in the list above. Our results provide a generalisation of these congruences in the sense that the number fields  $K_l$  that give the  $\rho_l$  "encode" the  $\tau(p) \bmod l$  in such a way that  $\tau(p) \bmod l$  can be computed in time polynomial in  $l$  and  $\log p$ , i.e., just the same complexity as in the case where one has explicit congruences.

More generally, we hope that non-solvable global field extensions whose existence and local properties are implied by the Langlands program can be made accessible to computation and so become even more useful members of the society of mathematical objects. Explicit descriptions of these fields make the study of global properties such as class groups and groups of units possible. Certainly, if we only knew the maximal abelian extension of  $\mathbb{Q}$  as described by general class field theory, then roots of unity would be very much welcomed.

The natural habitat for Galois representations such as the  $\rho_f$  above is that of higher degree étale cohomology with  $\mathbb{F}_\ell$ -coefficients of algebraic varieties over  $\overline{\mathbb{Q}}$ , together with the action of  $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . Our results provide some evidence that, also in interesting cases, such objects can be computed in reasonable time. Note that this question is not restricted to varieties related to modular forms or automorphic forms. In fact, thinking of elliptic curves, over  $\mathbb{Q}$ , say, knowing that these are modular does not help for computing their number of points over finite fields: Schoof's algorithm uses algebraic geometry, not modularity.

The problem of computing étale cohomology with Galois action is clearly related to the question of the existence of polynomial time algorithms for computing the number of solutions in  $\mathbb{F}_p$  of a fixed system of polynomial equations over  $\mathbb{Z}$ , when  $p$  varies. Our results treat this problem for the 11-dimensional variety that gives rise to  $\Delta$ ; see Section 1.5 for more details and also for an explicit variety of dimension 19 related to this.

The Epilogue at the end of this book describes a striking application of a generalisation of our results to the problem of computing representation numbers of the  $\mathbb{Z}^{2k}$  equipped with the standard inner product. This again is an example where only for small  $k$  there are explicit formulas, but where

in general there (surely) exists an algorithm that computes such numbers as quickly as if such formulas did exist. Hence, from a computational perspective, such algorithms form a natural generalisation of the finite series of formulas.

I very briefly describe the method by which we compute the  $\rho_f$ . Their duals occur in the higher degree étale cohomology of certain higher dimensional varieties, but no-one seems to know how to compute with this directly.

Via some standard methods in étale cohomology (the Leray spectral sequence, and passing to a finite cover to trivialise a locally constant sheaf of finite dimensional  $\mathbb{F}_l$ -vector spaces), or from the theory of congruences between modular forms, it is well known that the  $\rho_f$  are realised by subspaces  $V_f$  in the  $l$ -torsion  $J_l(\overline{\mathbb{Q}})[l]$  of the Jacobian variety  $J_l$  of some modular curve  $X_l$  defined over  $\mathbb{Q}$ . The field  $K_l$  is then the field generated by suitable “coordinates” of the points  $x \in V_l \subset J_l(\overline{\mathbb{Q}})[l]$ . We are now in the more familiar situation of torsion points on abelian varieties. But the price that we have paid for this is that the abelian variety  $J_l$  depends on  $l$ , and that its dimension, equal to the genus of  $X_l$ , i.e., equal to  $(l-5)(l-7)/24$ , grows quadratically with  $l$ . This makes it impossible to directly compute the  $x \in V_l$  using computer algebra: known algorithms for solving systems of non-linear polynomial equations take time exponential in the dimension.

Instead of using computer algebra directly, Jean-Marc Couveignes suggested that we use approximations and height bounds. In its simplest form, this works as follows. Suppose that  $x$  is a rational number,  $x = a/b$ , with  $a$  and  $b$  in  $\mathbb{Z}$  coprime. Suppose that we have an upper bound  $M$  for  $\max(|a|, |b|)$ . Then  $x$  is determined by any approximation  $y \in \mathbb{R}$  of  $x$  such that  $|y - x| < 1/2M^2$ , simply because for all  $x' \neq x$  with  $x' = a'/b'$ , where  $a'$  and  $b'$  in  $\mathbb{Z}$  satisfy  $\max(|a'|, |b'|) < M$ , we have  $|x' - x| = |(a'b - ab')/bb'| \geq 1/M^2$ .

For the computation of  $K_l$ , we consider the minimal polynomial  $P_l$  in  $\mathbb{Q}[T]$  of a carefully theoretically constructed generator  $\alpha$  of  $K_l$ . We use approximations of all Galois conjugates of  $\alpha$ , i.e., of all roots of  $P_l$ . Instead of working directly with torsion points of  $J_l$ , we work with divisors on the curve  $X_l$ . Using this strategy, the problem of showing that  $P_l$  can be computed in time polynomial in  $l$  is divided into two different tasks. Firstly, to show that the number of digits necessary for a good enough approxima-

tion of  $P_l$  is bounded by a fixed power of  $l$ . Secondly, to show that, given  $l$  and  $n$ , the coefficients of  $P_l$  can be approximated with a precision of  $n$  digits in time polynomial in  $n \cdot l$ . The first problem is dealt with in Chapters 9, 10, and 11, using Arakelov geometry. The second problem is solved in Chapters 12 and 13, in two ways: complex approximations (numerical analysis), and approximations in the sense of reductions modulo many small primes, using exact computations in Jacobians of modular curves over finite fields. These five chapters form the technical heart of this book. The preceding chapters are meant as an introduction to them, or motivation for them, and the two chapters following them give the main results as relatively straightforward applications.

Chapters 6 and 7 stand a bit apart, as they are concerned with some real computations of Galois representations attached to modular forms. They use the method by complex approximations, but do not use a rigorously proven bound for a sufficient accuracy. Instead, the approximations provide good *candidates* for polynomials  $P_l$ . The  $P_l$  that are found have the correct Galois group, and the right ramification properties. Recent modularity results by Khare, Wintenberger and Kisin, see [Kh-Wi1], [Kh-Wi2], and [Kis1] and [Kis2], are then applied to *prove* that the candidates do indeed give the right Galois representations.

## 1.2 HISTORICAL CONTEXT: SCHOOF'S ALGORITHM

The computation of Hecke operators from Galois representations and congruences can be viewed as a generalisation of Schoof's method to count points on elliptic curves over finite fields, see [Sch2] and [Sch3]. René Schoof gave an algorithm to compute, for  $E$  an elliptic curve over a finite field  $\mathbb{F}_q$ , the number  $\#E(\mathbb{F}_q)$  of  $\mathbb{F}_q$ -rational points in a time  $O((\log q)^{5+\varepsilon})$ . His algorithm works as follows.

The elliptic curve is embedded, as usual, in the projective plane  $\mathbb{P}_{\mathbb{F}_q}^2$  as the zero locus of a Weierstrass equation, which, in inhomogeneous coordinates, is of the form:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

with the  $a_i$  in  $\mathbb{F}_q$ . We let  $\mathbb{F}_q \rightarrow \overline{\mathbb{F}_q}$  be an algebraic closure. We let

$F_q: E \rightarrow E$  denote the so-called  $q$ -Frobenius. It is the endomorphism of  $E$  with the property that for all  $(a, b)$  in the affine part of  $E(\overline{\mathbb{F}}_q)$  given by the Weierstrass equation above we have  $F_q((a, b)) = (a^q, b^q)$ . The theory of elliptic curves over finite fields says:

1. there is a unique integer  $a$ , called the *trace* of  $F_q$ , such that in the endomorphism ring of  $E$  one has  $F_q^2 - aF_q + q = 0$ ;
2.  $\#E(\mathbb{F}_q) = 1 - a + q$ ;
3.  $|a| \leq 2q^{1/2}$ .

So, computing  $\#E(\mathbb{F}_q)$  is equivalent to computing this integer  $a$ . Schoof's idea is now to compute  $a$  modulo  $l$  for small prime numbers  $l$ . If the product of the prime numbers  $l$  exceeds  $4q^{1/2}$ , the length of the interval in which we know  $a$  to lie, then the congruences modulo these  $l$  determine  $a$  uniquely. Analytic number theory tells us that it will be sufficient to take all primes  $l$  up to approximately  $(\log q)/2$ .

Then the question is how one computes  $a$  modulo  $l$ . This should be done in time polynomial in  $\log q$  and  $l$ . The idea is to use the elements of order dividing  $l$  in  $E(\overline{\mathbb{F}}_q)$ . We assume now that  $l$  does not divide  $q$ , i.e., we avoid the characteristic of  $\mathbb{F}_q$ . For each  $l$ , the kernel  $E(\overline{\mathbb{F}}_q)[l]$  of multiplication by  $l$  on  $E(\overline{\mathbb{F}}_q)$  is a two-dimensional vector space over  $\mathbb{F}_l$ . The map  $F_q$  gives an endomorphism of  $E(\overline{\mathbb{F}}_q)[l]$ , and it follows that the image of  $a$  in  $\mathbb{F}_l$  is the unique element of  $\mathbb{F}_l$ , also denoted  $a$ , such that for each  $v$  in  $E(\overline{\mathbb{F}}_q)[l]$  we have  $aF_q(v) = F_q^2v + qv$ . We remark that the image of  $a$  in  $\mathbb{F}_l$  is the trace of the endomorphism of  $E(\overline{\mathbb{F}}_q)[l]$  given by  $F_q$ , but this is not really used at this point.

To find this element  $a$  of  $\mathbb{F}_l$ , one proceeds as follows. We suppose that  $l \neq 2$ . There is a unique monic element  $\psi_l$  of  $\mathbb{F}_q[x]$  of degree  $(l^2 - 1)/2$ , whose roots in  $\overline{\mathbb{F}}_q$  are precisely the  $x$ -coordinates of the  $l^2 - 1$  non-zero elements in  $E(\overline{\mathbb{F}}_q)[l]$  (the rational function  $x$  on  $E$  is a degree two map to  $\mathbb{P}_{\mathbb{F}_q}^1$ , which as such is the quotient for the multiplication by  $-1$  map on  $E$ ). One then lets  $A_l$  be the  $\mathbb{F}_q$ -algebra obtained as:

$$A_l := \mathbb{F}_q[x, y]/(y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6, \psi_l(x)).$$

The dimension of  $A_l$  as  $\mathbb{F}_q$ -vector space is  $l^2 - 1$ . An equivalent description

of  $A_l$  is to say that it is the affine coordinate ring of the subscheme of points of order  $l$  of  $E$ . By construction of  $A_l$ , there is a tautological  $A_l$ -valued point  $v$  in  $E(A_l)$  (its coordinates are the images of  $x$  and  $y$  in  $A_l$ ). Now to find the element  $a$  of  $\mathbb{F}_l$  that we are looking for one then tries one by one the elements  $i$  in  $0, \pm 1, \dots, \pm(l-1)/2$  until  $iF_q(v) = F_q^2 v + qv$ ; then  $i = a \pmod{l}$ .

It is easy to see that all required computations can be done in time  $O((\log q)^{5+\varepsilon})$  (using fast arithmetic for the elementary operations, e.g., a multiplication in  $A_l$  costs about  $(l^2(\log q))^{1+\varepsilon}$  time;  $l^2(\log q)$  is the number of bits needed to store one element of  $A_l$ ).

For the sake of completeness, I mention that shortly after the appearance of Schoof's algorithm, Atkin and Elkies have added some improvements to it, making it possible in certain cases to reduce the dimension of the  $\mathbb{F}_q$ -algebra from  $l^2 - 1$  to linear in  $l + 1$  or  $l - 1$ . This improvement, called the Schoof-Atkin-Elkies (SEA) algorithm, is important mainly for implementations. Its (average) complexity is  $O((\log q)^{4+\varepsilon})$ ; for details, the reader is referred to [Sch3].

### 1.3 SCHOOF'S ALGORITHM DESCRIBED IN TERMS OF ÉTALE COHOMOLOGY

In order to describe Schoof's algorithm in the previous section, we referred to the theory of elliptic curves over finite fields. But there is a more general framework for getting information on the number of rational points of algebraic varieties over finite fields: cohomology, and Lefschetz's trace formula. Cohomology exists in many versions. The version directly related to Schoof's algorithm is étale cohomology with coefficients in  $\mathbb{F}_l$ . Standard references for étale cohomology are [SGA4], [SGA4.5], [SGA5], [Mil1], [Fr-Ki]. The reader is referred to these references for the notions that we will use below. We also recommend Appendix C of [Hart].

For the sake of precision, let I say that we define the notion of *algebraic variety over a field  $k$*  to mean  *$k$ -scheme that is separated and of finite type*. Attached to an algebraic variety  $X$  over a field  $k$  there are *étale cohomology groups with compact supports*  $H_c^i(X_{\text{et}}, \mathbb{F}_l)$ , for all  $i \geq 0$  and for all prime numbers  $l$ . Actually, the coefficients  $\mathbb{F}_l$  can be replaced by more general

objects, sheaves of Abelian groups on the étale site  $X_{\text{et}}$  of  $X$ , but we do not need this now. If  $X$  is a proper  $k$ -scheme, then the  $H_c^i(X_{\text{et}}, \mathbb{F}_l)$  are equal to the étale cohomology groups  $H^i(X_{\text{et}}, \mathbb{F}_l)$  without condition on supports.

If  $k$  is separably closed then the  $H_c^i(X_{\text{et}}, \mathbb{F}_l)$  are finite dimensional  $\mathbb{F}_l$ -vector spaces, zero for  $i > 2 \dim(X)$ . In that case, they are the analog of the more easily defined cohomology groups  $H_c^i(X, \mathcal{F})$  for complex analytic varieties: the derived functors of the functor that associates to a sheaf  $\mathcal{F}$  of  $\mathbb{Z}$ -modules on  $X$  equipped with its Archimedean topology its  $\mathbb{Z}$ -module of global sections whose support is compact.

The construction of the  $H_c^i(X_{\text{et}}, \mathbb{F}_l)$  is functorial for proper morphisms: a proper morphism  $f: X \rightarrow Y$  of algebraic varieties over  $k$  induces a pull-back morphism  $f^*$  from  $H_c^i(Y_{\text{et}}, \mathbb{F}_l)$  to  $H_c^i(X_{\text{et}}, \mathbb{F}_l)$ .

Let now  $X$  be an algebraic variety over  $\mathbb{F}_q$ . Then we have the  $q$ -Frobenius morphism  $F_q$  from  $X$  to itself, and, by extending the base field from  $\mathbb{F}_q$  to  $\overline{\mathbb{F}}_q$ , from  $X_{\overline{\mathbb{F}}_q}$  to itself. This morphism  $F_q$  is proper, hence induces maps:

$$F_q^*: H_c^i(X_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{F}_l) \longrightarrow H_c^i(X_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{F}_l).$$

Hence, for each  $i$  in  $\mathbb{Z}$ , the trace  $\text{trace}(F_q^*, H_c^i(X_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{F}_l))$  of the map above is defined, and it is zero for  $i < 0$  and  $i > 2 \dim(X)$ . The set of fixed points of  $F_q$  on  $X(\overline{\mathbb{F}}_q)$  is precisely the subset  $X(\mathbb{F}_q)$ . The Lefschetz trace formula then gives the following identity in  $\mathbb{F}_l$ :

$$(1.3.1) \quad \#X(\mathbb{F}_q) = \sum_i (-1)^i \text{trace}(F_q^*, H_c^i(X_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{F}_l)).$$

We can now say how Schoof's algorithm is related to étale cohomology. We consider again an elliptic curve  $E$  over a finite field  $\mathbb{F}_q$ . We assume that  $l$  does not divide  $q$ . Then, as for any smooth proper geometrically connected curve,  $H^0(E_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{F}_l) = \mathbb{F}_l$  and  $F_q^*$  acts on it as the identity, and  $H^2(E_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{F}_l)$  is one-dimensional and  $F_q^*$  acts on it by multiplication by  $q$ , the degree of  $F_q$ . According to the trace formula (1.3.1), we have:

$$\#E(\mathbb{F}_q) = 1 - \text{trace}(F_q^*, H^1(E_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{F}_l)) + q.$$

It follows that for the integer  $a$  of the previous section, the trace of Frobenius, we have, for all  $l$  not dividing  $p$  the identity in  $\mathbb{F}_l$ :

$$a = \text{trace}(F_q^*, H^1(E_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{F}_l)).$$

This identity is explained by the fact that there is a natural isomorphism, compatible with the action of  $F_q$ :

$$H^1(E_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{F}_l) = E(\overline{\mathbb{F}}_q)[l].$$

Let I describe how one constructs this isomorphism. On  $E_{\text{et}}$  we have the short exact sequence of sheaves, called the Kummer sequence:

$$0 \longrightarrow \mu_l \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 0$$

where the map on  $\mathbb{G}_m$  is multiplication by  $l$  in the group law of  $\mathbb{G}_m$ , i.e., taking  $l$ th powers. This short exact sequence gives an exact sequence of cohomology groups after pullback to  $E_{\overline{\mathbb{F}}_q, \text{et}}$ :

$$\begin{aligned} \{1\} \longrightarrow \mu_l(\overline{\mathbb{F}}_q) \longrightarrow \overline{\mathbb{F}}_q^\times \longrightarrow \overline{\mathbb{F}}_q^\times \longrightarrow H^1(E_{\overline{\mathbb{F}}_q, \text{et}}, \mu_l) \longrightarrow \\ \longrightarrow H^1(E_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{G}_m) \longrightarrow H^1(E_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{G}_m) \longrightarrow \dots \end{aligned}$$

Just as for any scheme, one has:

$$H^1(E_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{G}_m) = \text{Pic}(E_{\overline{\mathbb{F}}_q})$$

It follows that

$$H^1(E_{\overline{\mathbb{F}}_q, \text{et}}, \mu_l) = \text{Pic}(E_{\overline{\mathbb{F}}_q})[l].$$

Finally, using the exact sequence:

$$0 \longrightarrow \text{Pic}^0(E_{\overline{\mathbb{F}}_q}) \longrightarrow \text{Pic}(E_{\overline{\mathbb{F}}_q}) \xrightarrow{\text{deg}} \mathbb{Z} \longrightarrow 0$$

and the fact that  $E$  is its own Jacobian variety, i.e.,  $\text{Pic}^0(E_{\overline{\mathbb{F}}_q}) = E(\overline{\mathbb{F}}_q)$ , we obtain:

$$H^1(E_{\overline{\mathbb{F}}_q, \text{et}}, \mu_l) = \text{Pic}(E_{\overline{\mathbb{F}}_q})[l] = \text{Pic}^0(E_{\overline{\mathbb{F}}_q})[l] = E(\overline{\mathbb{F}}_q)[l].$$

The choice of an isomorphism between  $\mu_l(\overline{\mathbb{F}}_q)$  and  $\mathbb{F}_l$  gives us the desired isomorphism between  $H^1(E_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{F}_l)$  and  $E(\overline{\mathbb{F}}_q)[l]$ . In fact, we note that by using the Weil pairing from  $E(\overline{\mathbb{F}}_q)[l] \times E(\overline{\mathbb{F}}_q)[l]$  to  $\mu_l(\overline{\mathbb{F}}_q)$ , we get an isomorphism:

$$H^1(E_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{F}_l) = E(\overline{\mathbb{F}}_q)[l]^\vee$$

that is more natural than the one used above; in particular, it does not depend on the choice of an isomorphism  $\mathbb{F}_l \rightarrow \mu_l(\overline{\mathbb{F}}_q)$ .



## 1.4 SOME NATURAL NEW DIRECTIONS

We have seen that the two-dimensional  $\mathbb{F}_l$ -vector spaces that are used in Schoof's algorithm for elliptic curves can also be seen as étale cohomology groups. A natural question that arises is then the following.

*Are there other interesting cases where étale cohomology groups can be used to construct polynomial time algorithms for counting rational points of varieties over finite fields?*

A more precise question is the following.

*Let  $n$  and  $m$  be in  $\mathbb{Z}_{\geq 0}$ , and let  $f_1, \dots, f_m$  be in  $\mathbb{Z}[x_1, \dots, x_n]$ . Is there an algorithm that on input a prime number  $p$  computes  $\#\{a \in \mathbb{F}_p^n \mid \forall i : f_i(a) = 0\}$  in time polynomial in  $\log p$ ?*

We believe that the answer to this question is yes, and that the way in which such an algorithm can work is to compute étale cohomology.

### 1.4.1 Curves of higher genus

The first step in the direction of this question was taken by Jonathan Pila. In [Pil] he considered principally polarised Abelian varieties of a fixed dimension, and curves of a fixed genus, and showed that in those cases polynomial time algorithms for computing the number of rational points over finite fields exist. In these cases, the only relevant cohomology groups are in degree one, i.e., they are of the form  $H^1(X_{\overline{\mathbb{F}}_q, \text{et}}, \mathbb{F}_l)$  with  $X$  a smooth proper curve, or an Abelian variety, over the field  $\mathbb{F}_q$ . As in Schoof's algorithm, the way to deal with these cohomology groups is to view them as  $J(\overline{\mathbb{F}}_q)[l]$ , the kernel of multiplication by  $l$  on the Abelian variety  $J$ . In the case where  $X$  is a curve, one lets  $J$  be the Jacobian variety of  $X$ .

As Pila makes use of explicit systems of equations for Abelian varieties, his algorithm has a running time that is at least exponential in the dimension of the Abelian variety, and hence, in the case of curves, as a function of the genus of the curve.

The current state of affairs concerning the question of counting the rational points of curves over finite fields seems still to be the same: algorithms have a running time that is exponential in the genus. As an illustration, let

I mention that in [Ad-Hu] Adleman and Huang give an algorithm that computes  $\#X(\mathbb{F}_q)$  in time  $(\log q)^{O(g^2 \log g)}$ , where  $X$  is a hyperelliptic curve over  $\mathbb{F}_q$ , and where  $g$  is the genus of  $X$ .

Recent progress in the case where the characteristic of the finite fields  $\mathbb{F}_q$  is fixed, using so-called  $p$ -adic methods, will be discussed in Section 1.6 below. In that case, there are algorithms whose running time is polynomial in  $g$  and  $\log q$ .

#### 1.4.2 Higher degree cohomology, modular forms

Another direction in which one can try to generalise Schoof's algorithm is to varieties of higher dimension, where non-trivial cohomology groups of degree higher than one are needed. In this context, we would call the degree 2 cohomology group of a curve trivial, because the trace of  $F_q$  on it is  $q$ .

More generally speaking, cohomology groups, but now with  $l$ -adic coefficients, that are of dimension one are expected to have the property that the trace of  $F_q$  can only be of the form  $q^n \zeta$ , with  $n$  an integer greater than or equal to zero, and  $\zeta$  a root of unity. This means that one-dimensional cohomology groups are not so challenging. Indeed, it is the fact that for elliptic curves over  $\mathbb{F}_p$  all integers in the Hasse interval  $[p+1-2p^{1/2}, p+1+2p^{1/2}]$  can occur that makes the problem of point counting very different from point counting on non-singular quadric surfaces in  $\mathbb{P}_{\mathbb{F}_q}^3$ , for example, where the outcome can only be  $q^2 + 2q + 1$  or  $q^2 + 1$ .

It follows that the simplest case to consider is cohomology groups of dimension two, in degree at least two, on which the action of  $F_q$  is not given by a simple rule as in the one-dimensional case. Such cohomology groups are provided by modular forms, as will be explained later in Section 2.2. Let me just say for the moment, that there is a direct relation with elliptic curves, via the concept of *modularity* of elliptic curves over  $\mathbb{Q}$ , that I will now sketch.

Let  $E$  be an elliptic curve over  $\mathbb{Q}$ , given by some Weierstrass equation. Such a Weierstrass equation can be chosen to have its coefficients in  $\mathbb{Z}$ . A Weierstrass equation for  $E$  with coefficients in  $\mathbb{Z}$  is called *minimal* if its *discriminant* is minimal among all Weierstrass equations for  $E$  with coefficients in  $\mathbb{Z}$ ; this discriminant then only depends on  $E$  and will be de-

noted  $\text{discr}(E)$ . In fact, two minimal Weierstrass equations define isomorphic curves in  $\mathbb{P}_{\mathbb{Z}}^2$ , the projective plane over  $\mathbb{Z}$ . In other words,  $E$  has a Weierstrass minimal model over  $\mathbb{Z}$ , that will be denoted by  $E_{\mathbb{Z}}$ . For each prime number  $p$ , let  $E_{\mathbb{F}_p}$  denote the curve over  $\mathbb{F}_p$  given by reducing a minimal Weierstrass equation modulo  $p$ ; it is the fibre of  $E_{\mathbb{Z}}$  over  $\mathbb{F}_p$ . The curve  $E_{\mathbb{F}_p}$  is smooth if and only if  $p$  does not divide  $\text{discr}(E)$ . The possible singular fibres have exactly one singular point: an ordinary double point with rational tangents, or with conjugate tangents, or an ordinary cusp. The three types of reduction are called split multiplicative, non-split multiplicative and additive, respectively, after the type of group law that one gets on the complement of the singular point. For each  $p$  we then get an integer  $a_p$  by requiring the following identity:

$$p + 1 - a_p = \#E(\mathbb{F}_p).$$

This means that for all  $p$ ,  $a_p$  is the trace of  $F_p$  on the degree one étale cohomology of  $E_{\mathbb{F}_p}$ , with coefficients in  $\mathbb{F}_l$ , or in  $\mathbb{Z}/l^n\mathbb{Z}$  or in the  $l$ -adic numbers  $\mathbb{Z}_l$ . For  $p$  not dividing  $\text{discr}(E)$  we know that  $|a_p| \leq 2p^{1/2}$ . If  $E_{\mathbb{F}_p}$  is multiplicative, then  $a_p = 1$  or  $-1$  in the split and non-split case. If  $E_{\mathbb{F}_p}$  is additive, then  $a_p = 0$ . We also define, for each  $p$  an element  $\varepsilon(p)$  in  $\{0, 1\}$  by setting  $\varepsilon(p) = 1$  for  $p$  not dividing  $\text{discr}(E)$  and setting  $\varepsilon(p) = 0$  for  $p$  dividing  $\text{discr}(E)$ . The *Hasse-Weil L-function* of  $E$  is then defined as:

$$L_E(s) = \prod_p L_{E,p}(s), \quad L_{E,p}(s) = (1 - a_p p^{-s} + \varepsilon(p) p p^{-2s})^{-1},$$

for  $s$  in  $\mathbb{C}$  with  $\Re(s) > 3/2$  (indeed, the fact that  $|a_p| \leq 2p^{1/2}$  implies that the product converges for such  $s$ ). To explain this function more conceptually, we note that for all  $p$  and for all  $l \neq p$  we have the identity:

$$1 - a_p t + \varepsilon(p) p t^2 = \det(1 - tF_p^*, H^1(E_{\mathbb{F}_p, \text{et}}, \mathbb{Q}_l))$$

The reader should notice that now we use étale cohomology with coefficients in  $\mathbb{Q}_l$ , the field of  $l$ -adic numbers, and not in  $\mathbb{F}_l$ . The reason for this is that we want the last identity above to be an identity between polynomials with integer coefficients, and not with coefficients in  $\mathbb{F}_l$ .

The function  $L_E$  was conjectured to have a holomorphic continuation over all of  $\mathbb{C}$ , and to satisfy a certain precisely given functional equation

relating the values at  $s$  and  $2 - s$ . In that functional equation appears a certain positive integer  $N_E$  called the *conductor* of  $E$ , composed of the primes  $p$  dividing  $\text{discr}(E)$  with exponents that depend on the behaviour of  $E$  at  $p$ , i.e., on  $E_{\mathbb{Z}_p}$ . This conjecture on continuation and functional equation was proved for semistable  $E$  (i.e.,  $E$  such that there is no  $p$  where  $E$  has additive reduction) by Wiles and Taylor-Wiles, and in the general case by Breuil, Conrad, Diamond and Taylor; see [Edi2] for an overview of this. In fact, the continuation and functional equation are direct consequences of the modularity of  $E$  that was proved by Wiles, Taylor-Wiles etc. (see below). The weak Birch and Swinnerton-Dyer conjecture says that the dimension of the  $\mathbb{Q}$ -vector space  $\mathbb{Q} \otimes E(\mathbb{Q})$  is equal to the order of vanishing of  $L_E$  at 1. Anyway, the function  $L_E$  gives us integers  $a_n$  for all  $n \geq 1$  as follows:

$$L_E(s) = \sum_{n \geq 1} a_n n^{-s}, \quad \text{for } \Re(s) > 3/2.$$

From these  $a_n$  one can then consider the following function:

$$f_E: \mathbb{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\} \rightarrow \mathbb{C}, \quad \tau \mapsto \sum_{n \geq 1} a_n e^{2\pi i n \tau}.$$

Equivalently, we have:

$$f_E = \sum_{n \geq 1} a_n q^n, \quad \text{with } q: \mathbb{H} \rightarrow \mathbb{C}, \quad \tau \mapsto e^{2\pi i \tau}.$$

A more conceptual way to state the relation between  $L_E$  and  $f_E$  is to say that  $L_E$  is obtained, up to elementary factors, as the *Mellin transform* of  $f_E$ :

$$\int_0^\infty f_E(it) t^s \frac{dt}{t} = (2\pi)^{-s} \Gamma(s) L_E(s), \quad \text{for } \Re(s) > 3/2.$$

After all these preparations, one can finally state what the modularity of  $E$  means:

*$f_E$  is a modular form of weight two for the congruence subgroup  $\Gamma_0(N_E)$  of  $\text{SL}_2(\mathbb{Z})$ .*

For some more details on the concept of modular forms see Section 2.2. At this moment, I just want to say that the last statement means that  $f_E$  has, as Mazur says in Singh's BBC documentary on Wiles's proof of Fermat's Last

Theorem, an enormous amount of symmetry. This symmetry is with respect to the action of  $GL_2(\mathbb{Q})^+$ , the group of invertible 2 by 2 matrices with coefficients in  $\mathbb{Q}$  whose determinant is positive, on the upper half plane  $\mathbb{H}$ . This symmetry gives, by Mellin transformation, the functional equation of  $L_E$ . Conversely, it had been proved in [Wei1] by Weil that if sufficiently many twists of  $L_E$  by Dirichlet characters satisfy the conjectured holomorphic continuation and functional equation, then  $f_E$  is a modular form of the type mentioned.

We now remark that Schoof's algorithm implies that, for  $p$  prime, the coefficient  $a_p$  in the  $q$ -expansion of  $f_E = \sum_{n \geq 1} a_n q^n$  can be computed in time polynomial in  $\log p$ . One of the aims of the research project described in this report is to generalise this last fact to certain modular forms of higher weight. Before giving precise definitions in Section 2.2, I will discuss a typical case in the next section.

### 1.5 MORE HISTORICAL CONTEXT: CONGRUENCES FOR RAMANUJAN'S $\tau$ -FUNCTION

References for this section are the articles [Ser2], [Swi] and [Del1] by Serre, Swinnerton-Dyer and Deligne.

A typical example of a modular form of weight higher than two is the *discriminant* modular form, usually denoted  $\Delta$ . One way to view  $\Delta$  is as the holomorphic function on the upper half plane  $\mathbb{H}$  given by:

$$(1.5.1) \quad \Delta = q \prod_{n \geq 1} (1 - q^n)^{24},$$

where  $q$  is the function from  $\mathbb{H}$  to  $\mathbb{C}$  given by  $z \mapsto \exp(2\pi iz)$ . The coefficients in the power series expansion:

$$(1.5.2) \quad \Delta = \sum_{n \geq 1} \tau(n) q^n$$

define the famous *Ramanujan  $\tau$ -function*.

To say that  $\Delta$  is a modular form of weight 12 for the group  $SL_2(\mathbb{Z})$  means that for all elements  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  of  $SL_2(\mathbb{Z})$  the following identity holds for all  $z$

in  $\mathbb{H}$ :

$$(1.5.3) \quad \Delta \left( \frac{az + b}{cz + d} \right) = (cz + d)^{12} \Delta(z),$$

which is equivalent to saying that the multi-differential form  $\Delta(z)(dz)^{\otimes 6}$  is invariant under the action of  $\mathrm{SL}_2(\mathbb{Z})$ . As  $\mathrm{SL}_2(\mathbb{Z})$  is generated by the elements  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , it suffices to check the identity in (1.5.3) for these two elements. The fact that  $\Delta$  is  $q$  times a power series in  $q$  means that  $\Delta$  is a *cuspidal form*: it vanishes at “ $q = 0$ ”. It is a fact that  $\Delta$  is the first example of a non-zero cuspidal form for  $\mathrm{SL}_2(\mathbb{Z})$ : there is no non-zero cuspidal form for  $\mathrm{SL}_2(\mathbb{Z})$  of weight smaller than 12, i.e., there are no non-zero holomorphic functions on  $\mathbb{H}$  satisfying (1.5.3) with the exponent 12 replaced by a smaller integer, whose Laurent series expansion in  $q$  is  $q$  times a power series. Moreover, the  $\mathbb{C}$ -vector space of such functions of weight 12 is one-dimensional, and hence  $\Delta$  is a basis of it.

The one-dimensionality of this space has as a consequence that  $\Delta$  is an eigenform for certain operators on this space, called *Hecke operators*, that arise from the action on  $\mathbb{H}$  of  $\mathrm{GL}_2(\mathbb{Q})^+$ , the subgroup of  $\mathrm{GL}_2(\mathbb{Q})$  of elements whose determinant is positive. This fact explains that the coefficients  $\tau(n)$  satisfy certain relations which are summarised by the following identity of Dirichlet series (converging for  $\Re(s) \gg 0$ , for the moment, or just formal series, if one prefers that):

$$(1.5.4) \quad L_{\Delta}(s) := \sum_{n \geq 1} \tau(n)n^{-s} = \prod_p (1 - \tau(p)p^{-s} + p^{11}p^{-2s})^{-1}.$$

These relations:

$$(1.5.5) \quad \begin{aligned} \tau(mn) &= \tau(m)\tau(n) && \text{if } \gcd(m, n) = 1 \\ \tau(p^n) &= \tau(p^{n-1})\tau(p) - p^{11}\tau(p^{n-2}) && \text{if } p \text{ is prime and } n \geq 2 \end{aligned}$$

were conjectured by Ramanujan, and proved by Mordell. Using these identities,  $\tau(n)$  can be expressed in terms of the  $\tau(p)$  for  $p$  dividing  $n$ .

As  $L_{\Delta}$  is the Mellin transform of  $\Delta$ ,  $L_{\Delta}$  is holomorphic on  $\mathbb{C}$ , and satisfies the functional equation (Hecke):

$$(2\pi)^{-(12-s)}\Gamma(12-s)L_{\Delta}(12-s) = (2\pi)^{-s}\Gamma(s)L_{\Delta}(s).$$

The famous *Ramanujan conjecture* states that for all primes  $p$  one has the inequality:

$$(1.5.6) \quad |\tau(p)| < 2p^{11/2},$$

or, equivalently, that the complex roots of the polynomial  $x^2 - \tau(p)x + p^{11}$  are complex conjugates of each other, and hence are of absolute value  $p^{11/2}$ . This conjecture was proved by Deligne as a consequence of his article [Del1] and his proof of the analog of the Riemann hypothesis in the Weil conjectures in [Del2].

Finally, Ramanujan conjectured congruences for the integers  $\tau(p)$  with  $p$  prime, modulo certain powers of certain small prime numbers. In order to state these congruences we set, for  $n \geq 1$  and  $r \geq 0$ :

$$\sigma_r(n) := \sum_{1 \leq d|n} d^r,$$

i.e.,  $\sigma_r(n)$  is the sum of the  $r$ th powers of the positive divisors of  $n$ . I now list the congruences that are given in the first pages of [Swi]:

$$\begin{array}{lll} \tau(n) \equiv \sigma_{11}(n) & \text{mod } 2^{11} & \text{if } n \equiv 1 \pmod{8} \\ \tau(n) \equiv 1217\sigma_{11}(n) & \text{mod } 2^{13} & \text{if } n \equiv 3 \pmod{8} \\ \tau(n) \equiv 1537\sigma_{11}(n) & \text{mod } 2^{12} & \text{if } n \equiv 5 \pmod{8} \\ \tau(n) \equiv 705\sigma_{11}(n) & \text{mod } 2^{14} & \text{if } n \equiv 7 \pmod{8} \end{array}$$

$$\begin{array}{lll} \tau(n) \equiv n^{-610}\sigma_{1231}(n) & \text{mod } 3^6 & \text{if } n \equiv 1 \pmod{3} \\ \tau(n) \equiv n^{-610}\sigma_{1231}(n) & \text{mod } 3^7 & \text{if } n \equiv 2 \pmod{3} \end{array}$$

$$\tau(n) \equiv n^{-30}\sigma_{71}(n) \pmod{5^3} \quad \text{if } n \text{ is prime to } 5$$

$$\begin{array}{lll} \tau(n) \equiv n\sigma_9(n) & \text{mod } 7 & \text{if } n \equiv 0, 1, 2 \text{ or } 4 \pmod{7} \\ \tau(n) \equiv n\sigma_9(n) & \text{mod } 7^2 & \text{if } n \equiv 3, 5 \text{ or } 6 \pmod{7} \end{array}$$

$$\begin{aligned} \tau(p) &\equiv 0 \pmod{23} && \text{if } p \text{ is prime and not a square mod } 23 \\ \tau(p) &\equiv 2 \pmod{23} && \text{if } p \neq 23 \text{ is a prime of the form } u^2 + 23v^2 \\ \tau(p) &\equiv -1 \pmod{23} && \text{for other primes } p \neq 23 \end{aligned}$$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}$$

The reader is referred to [Swi] for the origin and for proofs of these congruences. There, Swinnerton-Dyer remarks that the proofs do little explain why such congruences occur. Serre conjectured an explanation in [Ser2]. First of all, Serre conjectured the existence, for each prime number  $l$ , of a continuous representation:

$$(1.5.7) \quad \rho_l: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{Aut}(V_l),$$

with  $V_l$  a two-dimensional  $\mathbb{Q}_l$ -vector space, such that  $\rho_l$  is unramified at all primes  $p \neq l$ , and such that for all  $p \neq l$  the characteristic polynomial of  $\rho_l(\text{Frob}_p)$  is given by:

$$(1.5.8) \quad \det(1 - x\text{Frob}_p, V_l) = 1 - \tau(p)x + p^{11}x^2.$$

To help the reader, let I explain what unramified at  $p$  means, and what the Frobenius elements  $\text{Frob}_p$  are. For  $p$  prime, let  $\mathbb{Q}_p$  denote the topological field of  $p$ -adic numbers, and  $\overline{\mathbb{Q}_p}$  an algebraic closure. The action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on the set  $\text{Hom}(\overline{\mathbb{Q}}, \overline{\mathbb{Q}_p})$  of embeddings of  $\overline{\mathbb{Q}}$  into  $\overline{\mathbb{Q}_p}$  is transitive, and each embedding induces an injection from  $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$  into  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ , the image of which is called a decomposition group of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  at  $p$ . The injections from  $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$  into  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  and the corresponding decomposition groups at  $p$  obtained like this are all conjugated by the action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . In order to go further we need to say a bit about the structure of  $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ . Let  $\mathbb{Q}_p^{\text{unr}}$  be the maximal unramified extension of  $\mathbb{Q}_p$  in  $\overline{\mathbb{Q}_p}$ , i.e., the composite of all finite extensions  $K$  of  $\mathbb{Q}_p$  in  $\overline{\mathbb{Q}_p}$  such that  $p$  is a uniformiser for the integral closure  $O_K$  of  $\mathbb{Z}_p$  in  $K$ . Let  $\mathbb{Z}_p^{\text{unr}}$  be the integral closure of  $\mathbb{Z}_p$  in  $\mathbb{Q}_p^{\text{unr}}$ ; it is a local ring, and its residue



field is an algebraic closure  $\overline{\mathbb{F}}_p$  of  $\mathbb{F}_p$ . The sub-extension  $\mathbb{Q}_p^{\text{unr}}$  gives a short exact sequence:

$$(1.5.9) \quad I_p \hookrightarrow \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p).$$

The subgroup  $I_p$  of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  is called the inertia subgroup. The quotient  $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$  is canonically isomorphic to  $\hat{\mathbb{Z}}$ , the profinite completion of  $\mathbb{Z}$ , by demanding that the element 1 of  $\hat{\mathbb{Z}}$  corresponds to the Frobenius element  $\text{Frob}_p$  of  $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$  that sends  $x$  to  $x^p$  for each  $x$  in  $\overline{\mathbb{F}}_p$ .

Let now  $\rho_l$  be a continuous representation from  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  to  $\text{GL}(V_l)$  with  $V_l$  a finite dimensional  $\mathbb{Q}_l$ -vector space. Each embedding of  $\overline{\mathbb{Q}}$  into  $\overline{\mathbb{Q}}_p$  then gives a representation of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  on  $V_l$ . Different embeddings give isomorphic representations because they are conjugated by an element in the image of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  under  $\rho_l$ . We now choose one embedding, and call the representation  $\rho_{l,p}$  of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  on  $V_l$  obtained like this the local representation at  $p$  attached to  $\rho_l$ . This being defined,  $\rho_l$  is then said to be unramified at a prime  $p$  if  $\rho_{l,p}$  factors through the quotient  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ , i.e., if  $I_p$  acts trivially on  $V_l$ . If  $\rho_l$  is unramified at  $p$ , then we get an element  $\rho_l(\text{Frob}_p)$  in  $\text{GL}(V_l)$ . This element depends on our chosen embedding of  $\overline{\mathbb{Q}}$  into  $\overline{\mathbb{Q}}_p$ , but its conjugacy class under  $\rho_l(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$  does not. In particular, we get a well-defined conjugacy class in  $\text{GL}(V_l)$ , and so the characteristic polynomial of  $\rho_l(\text{Frob}_p)$  is now defined if  $\rho_l$  is unramified at  $p$ .

Continuous representations such as  $\rho_l$  can be reduced modulo powers of  $l$  as follows. The compactness of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  implies that with respect to a suitable basis of  $V_l$  the representation  $\rho_l$  lands in  $\text{GL}_2(\mathbb{Z}_l)$ , and hence gives representations to  $\text{GL}_2(\mathbb{Z}/l^n\mathbb{Z})$  for all  $n \geq 0$ . This reduction of  $\rho_l$  modulo powers of  $l$  is not unique, but the semi-simplification of the reduction modulo  $l$  is well-defined, i.e., two reductions lead to the same Jordan-Hölder constituents. According to Serre, the congruences above would then be explained by properties of the image of  $\rho_l$ .

For example, if the image of the reduction modulo  $l$  of  $\rho_l$  is reducible, say an extension of two characters  $\alpha$  and  $\beta$  from  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  to  $\mathbb{F}_l^\times$ , then one has the identity in  $\mathbb{F}_l$ , for all  $p \neq l$ :

$$(1.5.10) \quad \tau(p) \equiv \alpha(\text{Frob}_p) + \beta(\text{Frob}_p).$$

The characters  $\alpha$  and  $\beta$  are unramified outside  $l$ . By the Kronecker-Weber theorem, the maximal Abelian subextension of  $\mathbb{Q} \rightarrow \overline{\mathbb{Q}}$  that is unramified outside  $l$  is the cyclotomic extension generated by all  $l$ -power roots of unity, with Galois group  $\mathbb{Z}_l^\times$ . It then follows that  $\alpha = \chi_l^n$  and  $\beta = \chi_l^m$  for suitable  $n$  and  $m$ , where  $\chi_l$  is the character giving the action on the  $l$ th roots of unity in  $\overline{\mathbb{Q}}$ : for all  $\sigma$  in  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  and for all  $\zeta$  in  $\overline{\mathbb{Q}}^\times$  with  $\zeta^l = 1$  one has  $\sigma(\zeta) = \zeta^{\chi_l(\sigma)}$ . The identity (1.5.10) in  $\mathbb{F}_l$  above then takes the form:

$$(1.5.11) \quad \tau(p) = p^n + p^m \pmod{l}, \quad \text{for all } p \neq l,$$

which indeed is of the same form as the congruences mod  $l$  for  $\tau(p)$  listed above. For example, the congruence mod 691 corresponds to the statement that the reduction modulo  $l$  of  $\rho_l$  contains the two characters 1 and  $\chi_l^{11}$ .

Deligne, in [Del1], proved the existence of the  $\rho_l$ , as conjectured by Serre, by showing that they occur in the degree one  $l$ -adic étale cohomology of certain sheaves on certain curves, and in the degree 11 étale cohomology with  $\mathbb{Q}_l$ -coefficients of a variety of dimension 11. This last variety is, loosely speaking, the 10-fold fibred product of the universal elliptic curve. Deligne's constructions will be discussed in detail in Sections 2.2 and 2.4. It should be said that Shimura had already shown how to construct Galois representations in the case of modular forms of weight two; in that case one does not need étale cohomology, but torsion points of Jacobians of modular curves suffice, see [Shi1].

At this point I give the following precise statement, relating Ramanujan's  $\tau$ -function to point counting on an algebraic variety  $C_{10}$  (more precisely, a quasi-projective scheme over  $\mathbb{Z}$ ), for which one easily writes down a system of equations. Moreover, the statement relates the weight of  $\Delta$  to the classical question in geometry on cubic plane curves passing through a given set of points: up to 9 points the situation is easy and the count is given by a polynomial.

**1.5.12 Proposition** *For  $n \in \mathbb{Z}_{\geq 0}$ ,  $q$  a prime power and  $\mathbb{F}_q$  a finite field with  $q$  elements, let  $C_n(\mathbb{F}_q)$  be the set of  $(C, P_1, \dots, P_n)$ , where  $C$  is a smooth cubic in  $\mathbb{P}_{\mathbb{F}_q}^2$ , and  $P_i \in C(\mathbb{F}_q)$ . Then there are  $f_0, \dots, f_{10} \in \mathbb{Z}[x]$  such that for all  $\mathbb{F}_q$  and  $n \leq 9$  one has  $\#C_n(\mathbb{F}_q)/\#\text{PGL}_3(\mathbb{F}_q) = f_n(q)$ , and for all*

prime numbers  $p$ :

$$\#C_{10}(\mathbb{F}_p) / \#\mathrm{PGL}_3(\mathbb{F}_p) = -\tau(p) + f_{10}(p).$$

**Proof** For  $n$  in  $\mathbb{Z}_{\geq 0}$  and  $\mathbb{F}_q$  a field with  $q$  elements, let  $\mathcal{E}_n(\mathbb{F}_q)$  denote the category, and also its set of objects, of  $(E/\mathbb{F}_q, P_1, \dots, P_n)$ , where  $E/\mathbb{F}_q$  is an elliptic curve, and  $P_i \in E(\mathbb{F}_q)$ ; the morphisms are the isomorphisms  $\phi: E \rightarrow E'$  such that  $\phi(P_i) = P'_i$ . For each  $n$ , the category  $\mathcal{E}_n(\mathbb{F}_q)$  has only finitely many objects up to isomorphism, and one defines:

$$\#\mathcal{E}_n(\mathbb{F}_q) = \sum_{x \in \mathcal{E}_n(\mathbb{F}_q)} \frac{1}{\#\mathrm{Aut}(x)},$$

where, in the sum, one takes one  $x$  per isomorphism class. It is well known (see [Del1], [Beh]) that for  $n \leq 9$  the functions  $q \mapsto \#\mathcal{E}_n(\mathbb{F}_q)$  are given by certain elements  $f_n$  in  $\mathbb{Z}[x]$ , and that there is an  $f_{10}$  in  $\mathbb{Z}[x]$  such that for all prime numbers  $p$  one has  $\#\mathcal{E}_{10}(\mathbb{F}_p) = -\tau(p) + f_{10}(p)$ . In view of this, the claims in Proposition 1.5.12 are a consequence of the following equality, for all  $n \in \mathbb{Z}_{\geq 0}$  and all prime powers  $q$ :

(1.5.13)

$$\text{for all } n \in \mathbb{Z}_{\geq 0} \text{ and all } \mathbb{F}_q: \#C_n(\mathbb{F}_q) = \#\mathrm{PGL}_3(\mathbb{F}_q) \cdot \#\mathcal{E}_n(\mathbb{F}_q).$$

We prove (1.5.13) by comparing the subsets on both sides in which the underlying curves are fixed.

Let  $n \in \mathbb{Z}_{\geq 0}$  and  $q$  a prime power. Let  $F$  be a nonsingular projective geometrically irreducible curve of genus one over  $\mathbb{F}_q$ , and let  $E_0$  be its Jacobian. Then  $F$  is an  $E_0$ -torsor. By Lang's theorem, Theorem 2 of [Lan2],  $F(\mathbb{F}_q)$  is not empty.

Let  $C_n(\mathbb{F}_q)_F$  be the subset of  $C_n(\mathbb{F}_q)$  consisting of the  $(C, P_1, \dots, P_n)$  with  $C$  isomorphic to  $F$ . The number of  $C$  in  $\mathbb{P}_{\mathbb{F}_q}^2$  that are isomorphic to  $F$  is the number of embeddings  $i: F \rightarrow \mathbb{P}_{\mathbb{F}_q}^2$ , divided by  $\#\mathrm{Aut}(F)$ . Such embeddings are obtained from line bundles  $\mathcal{L}$  of degree 3 on  $F$ , together with a basis, up to  $\mathbb{F}_q^\times$ , of  $\mathcal{L}(F)$ . Hence the number of embeddings is  $\#\mathrm{PGL}_3(\mathbb{F}_q) \cdot \#E_0(\mathbb{F}_q)$ . The group  $\mathrm{Aut}(F)$  has the subgroup of translations,  $E_0(\mathbb{F}_q)$ , with quotient  $\mathrm{Aut}(E_0)$ . So we find:

$$\#C_n(\mathbb{F}_q)_F = \#\mathrm{PGL}_3(\mathbb{F}_q) \cdot (\#E_0(\mathbb{F}_q))^n / \#\mathrm{Aut}(E_0).$$

On the other hand, let  $\mathcal{E}_n(\mathbb{F}_q)_{E_0}$  be the full subcategory of  $\mathcal{E}_n(\mathbb{F}_q)$  with objects the  $(E_0, P_1, \dots, P_n)$ , with  $P_i$  in  $E_0(\mathbb{F}_q)$ . The group  $\text{Aut}(E_0)$  acts on the set of objects of  $\mathcal{E}_n(\mathbb{F}_q)_{E_0}$ , and this action is the set of morphisms in  $\mathcal{E}_n(\mathbb{F}_q)_{E_0}$ . This means that:

$$\#\mathcal{E}_n(\mathbb{F}_q)_{E_0} = (\#E_0(\mathbb{F}_q))^n / \#\text{Aut}(E_0).$$

Summing over the isomorphism classes of  $F$  gives (1.5.13).  $\square$

**1.5.14 Remark** The polynomials  $f_n$  mentioned in Proposition 1.5.12 have been computed by Carel Faber and Gerard van der Geer. Their result is:

$$\begin{aligned} f_0 &= x, \\ f_1 &= x^2 + x, \\ f_2 &= x^3 + 3x^2 + x - 1, \\ f_3 &= x^4 + 6x^3 + 6x^2 - 2x - 3, \\ f_4 &= x^5 + 10x^4 + 20x^3 + 4x^2 - 14x - 7, \\ f_5 &= x^6 + 15x^5 + 50x^4 + 40x^3 - 30x^2 - 49x - 15, \\ f_6 &= x^7 + 21x^6 + 105x^5 + 160x^4 - 183x^2 - 139x - 31, \\ f_7 &= x^8 + 28x^7 + 196x^6 + 469x^5 + 280x^4 - 427x^3 - 700x^2 \\ &\quad - 356x - 63, \\ f_8 &= x^9 + 36x^8 + 336x^7 + 1148x^6 + 1386x^5 - 406x^4 - 2436x^3 \\ &\quad - 2224x^2 - 860x - 127, \\ f_9 &= x^{10} + 45x^9 + 540x^8 + 2484x^7 + 4662x^6 + 1764x^5 - 6090x^4 \\ &\quad - 9804x^3 - 6372x^2 - 2003x - 255, \\ f_{10} &= x^{11} + 55x^{10} + 825x^9 + 4905x^8 + 12870x^7 + 12264x^6 \\ &\quad - 9240x^5 - 33210x^4 - 33495x^3 - 17095x^2 - 4553x - 511. \end{aligned}$$

We refer to Birch [Bir] for results on the distribution of the number of rational points on elliptic curves over finite fields, that also make  $\tau(p)$  appear.

In [Swi], Swinnerton-Dyer gives results, partly resulting from his correspondence with Serre, in which the consequences of the existence of the  $\rho_l$  for congruences of  $\tau(p)$  modulo  $l$  are explored. A natural question to ask is if there are primes  $l$  other than 2, 3, 5, 7, 23 and 691 modulo which there are similar congruences for  $\tau(p)$ .

For each  $p \neq l$ ,  $\tau(p)$  is the trace of  $\rho_l(\text{Frob}_p)$ , and the determinant of  $\rho_l(\text{Frob}_p)$  equals  $p^{11}$ . Hence, a polynomial relation between  $\tau(p)$  and  $p^{11}$ , valid modulo some  $l^n$  for all  $p \neq l$ , is a relation between the determinant and the trace of all  $\rho_l(\text{Frob}_p)$  in  $\text{GL}_2(\mathbb{Z}/l^n\mathbb{Z})$ . But Chebotarev's theorem (see [Lan6], or [Ca-Fr], for example) implies that every element of the image of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  in  $\text{GL}_2(\mathbb{Z}/l^n\mathbb{Z})$  is of the form  $\text{Frob}_p$  for infinitely many  $p$ . Hence, such a polynomial relation is then valid for all elements in the image of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  in  $\text{GL}_2(\mathbb{Z}/l^n\mathbb{Z})$ . For this reason, the existence of non-trivial congruences modulo  $l^n$  as above for  $\tau(p)$  depends on this image.

The image of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  in  $\mathbb{Z}_l^\times$  under  $\det \circ \rho_l$  is equal to the subgroup of 11th powers in  $\mathbb{Z}_l^\times$ . To explain this, I note that  $\det \circ \rho_l$  is a continuous character from  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  to  $\mathbb{Z}_l^\times$ , unramified outside  $l$ , and such that  $\text{Frob}_p$  is mapped to  $p^{11}$  for all  $p \neq l$ ; this implies that  $\det \circ \rho_l$  is the 11th power of the  $l$ -adic cyclotomic character  $\chi_l: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{Z}_l^\times$ , giving the action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on the  $l$ -power roots of unity.

In order to state the results in [Swi], one calls a prime number  $l$  *exceptional* (for  $\Delta$ ) if the image of  $\rho_l$ , taking values in  $\text{GL}_2(\mathbb{Z}_l)$ , does *not* contain  $\text{SL}_2(\mathbb{Z}_l)$ . For  $l$  not exceptional, i.e., such that the image of  $\rho_l$  contains  $\text{SL}_2(\mathbb{Z}_l)$ , the image of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  in  $\text{GL}_2(\mathbb{Z}_l) \times \mathbb{Z}_l^\times$ , under  $(\rho_l, \chi_l)$ , is the subgroup  $H$  of elements  $(g, t)$  such that  $\det(g) = t^{11}$ . This subgroup  $H$  maps surjectively to  $\mathbb{F}_l \times \mathbb{F}_l^\times$  under  $(g, t) \mapsto (\text{trace}(g), t)$ , and therefore there can be no congruence for  $\tau(p)$  modulo  $l$  as above.

The Corollary to Theorem 4 in [Swi] states, among others, that the list of primes that are exceptional for  $\Delta$  is  $\{2, 3, 5, 7, 23, 691\}$ . The main tool that is used and that I have not discussed is the theory of modular forms modulo  $l$ , or, equivalently, the theory of congruences modulo  $l$  between modular forms. As a consequence, there are no similar congruences for  $\tau(p)$  modulo primes other than the ones listed above. The special form of the congruences modulo 23 is explained by the fact that in that case the image of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  in  $\text{GL}_2(\mathbb{F}_{23})$  is dihedral; in the other cases the residual representation, i.e., the representation to  $\text{GL}_2(\mathbb{F}_l)$ , is reducible. In the case  $l = 2$ , Swinnerton-Dyer has determined the image of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  in  $\text{GL}_2(\mathbb{Z}_2)$  exactly: see the appendix in [Swi].

The direction in which we generalise Schoof's algorithm is to give an algorithm that computes for prime numbers  $l$  that are not exceptional for

$\Delta$  the field extension  $\mathbb{Q} \rightarrow K_l$  that corresponds to the representation of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  to  $\text{GL}_2(\mathbb{F}_l)$  that comes from  $\Delta$ . The field  $K_l$  is given in the form  $\mathbb{Q}[x]/(f_l)$ . The computation has a running time that is polynomial in  $l$ . It is fair to say that this algorithm makes the mod  $l$  Galois representations attached to  $\Delta$  accessible to computation, at least theoretically. As the field extensions that are involved are non-solvable, this should be seen as a step beyond computational class field theory, and beyond the case of elliptic curves, in the direction to make the results of Langlands's program accessible to computations.

As a consequence, one can compute  $\tau(p) \bmod l$  in time polynomial in  $\log p$  and  $l$ , by reducing  $f_l$  as above mod  $p$  and some more computations that will be described later (see Section 15.1). By doing this for sufficiently many  $l$ , just as in Schoof's algorithm, one then gets an algorithm that computes  $\tau(p)$  in time polynomial in  $\log p$ .

In Section 15.2 the method used here is generalised to the case of modular forms for  $\text{SL}_2(\mathbb{Z})$  of arbitrary weight. The main result there is Theorem 15.2.1.

## 1.6 COMPARISON WITH $p$ -ADIC METHODS

Before we start seriously with the theory of modular forms and the Galois representations attached to them in the next chapters, I make a comparison between our generalisation of Schoof's algorithm and the so-called  $p$ -adic methods that have been developed since 2000 by Satoh [Sat], Kedlaya [Ked] (see also [Edi3]), Hubrechts, [Hub]), Lauder and Wan [La-Wa1], [La-Wa2], [Lau1] and [Lau2], Fouquet, Gaudry, Gürel and Harley [Fo-Ga-Ha], [Ga-Gu], Denef and Vercauteren and Castryk [De-Ve], [Ca-De-Ve], Mestre, Lercier and Lubicz [Le-Lu], Carls, Kohel and Lubicz, [Ca-Ko-Lu], [Ca-Lu], and Gerkmann, [Ger1] and [Ger2]. Actually, I should notice that such a method was already introduced in [Ka-Lu] in 1982, but that this article seems to have been forgotten (we thank Fre Vercauteren for having drawn my attention to this article).

In all these methods, one works with fields of small characteristic  $p$ , hence of the form  $\mathbb{F}_q$  with  $q = p^m$  and  $p$  fixed. All articles cited in the previous paragraph have the common property that they give algorithms for comput-

ing the number of  $\mathbb{F}_q$ -rational points on certain varieties  $X$  over  $\mathbb{F}_q$ , using, sometimes indirectly, cohomology groups with  $p$ -adic coefficients, whence the terminology “ $p$ -adic methods”.

For example, Satoh [Sat] uses the canonical lift of ordinary elliptic curves and the action of the lifted Frobenius endomorphism on the tangent space, which can be interpreted in terms of the algebraic de Rham cohomology of the lifted curve. Kedlaya [Ked] uses Monsky-Washnitzer cohomology of certain affine pieces of hyperelliptic curves. In fact, all cohomology groups used here are de Rham type cohomology groups, given by complexes of differential forms on certain  $p$ -adic lifts of the varieties in question. Just as an example, let us mention that Kedlaya [Ked] gives an algorithm that for fixed  $p \neq 2$  computes the zeta functions of hyperelliptic curves given by equations:

$$y^2 = f(x),$$

where  $f$  has arbitrary degree, in time  $m^3 \deg(f)^4$ . The running times of the other algorithms are all similar, but all have in common that the running time grows at least linearly in  $p$  (or linear in  $O(p^{1/2})$ , in [Harv]), hence exponentially in  $\log p$ . The explanation for this is that somehow in each case non-sparse polynomials of degree at least linear in  $p$  have to be manipulated.

Summarising this recent progress, one can say that, at least from a theoretical point of view, the problem of counting the solutions of systems of polynomial equations over finite fields of a fixed characteristic  $p$  and in a fixed number of variables has been solved. If  $p$  is not bounded, then almost nothing is known about the existence of polynomial time algorithms.

A very important difference between the project described here, using étale cohomology with coefficients in  $\mathbb{F}_l$ , and the  $p$ -adic methods, is that the Galois representations on  $\mathbb{F}_l$ -vector spaces that we obtain are *global* in the sense that they are representations of the absolute Galois group of the global field  $\mathbb{Q}$ . The field extensions such as the  $K_l = \mathbb{Q}[x]/(f_l)$  arising from  $\Delta$  discussed in the previous section have the advantage that one can choose to do the required computations over the complex numbers, approximating  $f_l$ , or  $p$ -adically at some suitable prime  $p$ , or in  $\mathbb{F}_p$  for sufficiently many small  $p$ . Also, as I have said already, being able to compute such field extensions  $K_l$ , that give mod  $l$  information on the Frobenius elements at all primes  $p \neq l$ ,

is very interesting. On the other hand, the  $p$ -adic methods force one to compute with  $p$ -adic numbers, or, actually, modulo some sufficiently high power of  $p$ , and it gives information only on the Frobenius at  $p$ . The main drawback of the étale cohomology with  $\mathbb{F}_l$ -coefficients seems to be that the degree of the field extensions as  $K_l$  to be dealt with grows exponentially in the dimension of the cohomology groups; for that reason, we do not know how to use étale cohomology to compute  $\#X(\mathbb{F}_q)$  for  $X$  a curve of arbitrary genus in a time polynomial in  $\log q$  and the genus of  $X$ . Nevertheless, for modular curves, see the end of Section 15.3.





## Chapter Two

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### Modular curves, modular forms, lattices, Galois representations

B. Edixhoven

#### 2.1 MODULAR CURVES

As a good reference for getting an overview of the theory of modular curves and modular forms we recommend the article [Di-Im] by Fred Diamond and John Im. This reference is quite complete as results are concerned, and gives good references for the proofs of those results. Moreover, it is one of the few references that treats the various approaches to the theory of modular forms, from the classical analytic theory on the upper half plane to the more modern representation theory of adelic groups. Another good first introduction could be the book [Di-Sh]. Let us also mention that there is a forthcoming book [Conr] by Brian Conrad, and also the information in the *wikipedia* is getting more and more detailed.

In this section our aim is just to give the necessary definitions and results for what we need later (and we need at least to fix our notation). Readers who want more details, or more conceptual explanations are encouraged to consult [Di-Im].

**2.1.1 Definition** For  $n$  an integer greater than or equal to one we let  $\Gamma(n)$  be the kernel of the surjective morphism of groups  $\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$  given by reduction of the coefficients modulo  $n$ , and we let  $\Gamma_1(n)$  be the inverse image of the subgroup of  $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$  that fixes the element  $(1, 0)$  of  $(\mathbb{Z}/n\mathbb{Z})^2$ . Similarly, we let  $\Gamma_0(n)$  be the inverse image of the subgroup

of  $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$  that fixes the subgroup  $\mathbb{Z}/n\mathbb{Z} \cdot (1, 0)$  of  $(\mathbb{Z}/n\mathbb{Z})^2$ . Hence the elements of  $\Gamma_0(n)$  are the  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  of  $\mathrm{SL}_2(\mathbb{Z})$  such that  $c \equiv 0 \pmod{n}$ , those of  $\Gamma_1(n)$  are the ones that satisfy the extra conditions  $a \equiv 1 \pmod{n}$  and  $d \equiv 1 \pmod{n}$  and those of  $\Gamma(n)$  are the ones that satisfy the extra condition  $b \equiv 0 \pmod{n}$ .

The group  $\mathrm{SL}_2(\mathbb{R})$  acts on the upper half plane  $\mathbb{H}$  by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

The subgroup  $\mathrm{SL}_2(\mathbb{Z})$  of  $\mathrm{SL}_2(\mathbb{R})$  acts discontinuously in the sense that for each  $z$  in  $\mathbb{H}$  the stabiliser  $\mathrm{SL}_2(\mathbb{Z})_z$  is finite and there is an open neighbourhood  $U$  of  $z$  such that each translate  $\gamma U$  with  $\gamma$  in  $\mathrm{SL}_2(\mathbb{Z})$  contains exactly one element of the orbit  $\mathrm{SL}_2(\mathbb{Z}) \cdot z$  and any two translates  $\gamma U$  and  $\gamma' U$  with  $\gamma$  and  $\gamma'$  in  $\mathrm{SL}_2(\mathbb{Z})$  are either equal or disjoint. This property implies that the quotient  $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ , equipped with the quotient topology and with, on each open subset  $U$ , the  $\mathrm{SL}_2(\mathbb{Z})$ -invariant holomorphic functions on the inverse image of  $U$ , is a complex analytic manifold of dimension one, i.e., each point of the quotient has an open neighbourhood that is isomorphic to the complex unit disk. Globally, the well-known  $j$ -function from  $\mathbb{H}$  to  $\mathbb{C}$  is in fact the quotient map for this action. One way to see this is to associate to each  $z$  in  $\mathbb{H}$  the elliptic curve  $E_z := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}z)$ , and to note that for  $z$  and  $z'$  in  $\mathbb{H}$  the elliptic curves  $E_z$  and  $E_{z'}$  are isomorphic if and only if  $z$  and  $z'$  are in the same  $\mathrm{SL}_2(\mathbb{Z})$ -orbit, and to use the fact that two complex elliptic curves are isomorphic if and only if their  $j$ -invariants are equal.

The quotient set  $\Gamma(n) \backslash \mathbb{H}$  can be identified with the set of isomorphism classes of pairs  $(E, \phi)$ , where  $E$  is a complex elliptic curve and  $\phi: (\mathbb{Z}/n\mathbb{Z})^2 \rightarrow E[n]$  is an isomorphism of groups, compatible with the Weil pairing  $E[n] \times E[n] \rightarrow \mu_n(\mathbb{C})$  and the  $\mu_n(\mathbb{C})$ -valued pairing on  $(\mathbb{Z}/n\mathbb{Z})^2$  that sends  $((a_1, a_2), (b_1, b_2))$  to  $\zeta_n^{a_1 b_2 - a_2 b_1}$ , where  $\zeta_n = e^{2\pi i/n}$ .

The quotient  $\Gamma_0(n) \backslash \mathbb{H}$  is then identified with the set of pairs  $(E, G)$  where  $E$  is a complex elliptic curve, and  $G \subset E$  a subgroup that is isomorphic to  $\mathbb{Z}/n\mathbb{Z}$ . Equivalently, we may view  $\Gamma_0(n) \backslash \mathbb{H}$  as the set of isomorphism classes of  $E_1 \xrightarrow{\phi} E_2$ , where  $\phi$  is a morphism of complex elliptic curves, and  $\ker(\phi)$  is isomorphic to  $\mathbb{Z}/n\mathbb{Z}$ .

Finally, the quotient  $\Gamma_1(n)\backslash\mathbb{H}$  is then identified with the set of pairs  $(E, P)$  where  $E$  is a complex elliptic curve, and  $P$  is a point of order  $n$  of  $E$ . Explicitly: to each  $z$  in  $\mathbb{H}$  corresponds the pair  $(\mathbb{C}/(\mathbb{Z}z + \mathbb{Z}), [1/n])$ , where  $[1/n]$  denotes the image of  $1/n$  in  $\mathbb{C}/(\mathbb{Z}z + \mathbb{Z})$ .

In order to understand that the quotients considered above are in fact the complex analytic varieties associated with affine complex algebraic curves, it is necessary (and sufficient!) to show that these quotients can be compactified to compact Riemann surfaces by adding a finite number of points, called *the cusps*. As the quotient by  $SL_2(\mathbb{Z})$  is given by  $j: \mathbb{H} \rightarrow \mathbb{C}$ , it can be compactified easily by embedding  $\mathbb{C}$  into  $\mathbb{P}^1(\mathbb{C})$ ; the point  $\infty$  of  $\mathbb{P}^1(\mathbb{C})$  is called the cusp. Another way to view this is to note that the equivalence relation on  $\mathbb{H}$  given by the action of  $SL_2(\mathbb{Z})$  identifies two elements  $z$  and  $z'$  with  $\Im(z) > 1$  and  $\Im(z') > 1$  if and only if  $z' = z + n$  for some  $n$  in  $\mathbb{Z}$ ; this follows from the identity, for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in  $SL_2(\mathbb{R})$  and  $z$  in  $\mathbb{H}$ :

$$(2.1.2) \quad \Im\left(\frac{az + b}{cz + d}\right) = \frac{\Im(z)}{|cz + d|^2}.$$

Indeed, if moreover  $c \neq 0$ , then:

$$(2.1.3) \quad \frac{\Im(z)}{|cz + d|^2} \leq \frac{\Im(z)}{(\Im(cz))^2} = \frac{1}{c^2\Im(z)}.$$

Hence on the part “ $\Im(z) > 1$ ” of  $\mathbb{H}$  the equivalence relation given by  $SL_2(\mathbb{Z})$  is given by the action of  $\mathbb{Z}$  by translation. As the quotient for that action is given by the map  $q: \mathbb{H} \rightarrow D(0, e^{-2\pi})^*$ ,  $z \mapsto \exp(2\pi iz)$ , where  $D(0, e^{-2\pi})^*$  is the open disk of radius  $e^{-2\pi}$ , centred at 0, and with 0 removed, we get an open immersion of  $D(0, e^{-2\pi})^*$  into  $SL_2(\mathbb{Z})\backslash\mathbb{H}$ . The compactification is then obtained by replacing  $D(0, e^{-2\pi})^*$  with  $D(0, e^{-2\pi})$ , i.e., by adding the centre back into the punctured disk.

Let us now consider the problem of compactifying the other quotients above. Let  $\Gamma$  be one of the groups considered above, or, in fact, any subgroup of finite index in  $SL_2(\mathbb{Z})$ . We consider the morphism  $f: \Gamma\backslash\mathbb{H} \rightarrow SL_2(\mathbb{Z})\backslash\mathbb{H} = \mathbb{C}$ , and our compactification  $\mathbb{P}^1(\mathbb{C})$  of  $\mathbb{C}$ . By construction,  $f$  is proper (i.e., the inverse image of a compact subset of  $\mathbb{C}$  is compact). Also, we know that ramification can only occur at points with  $j$ -invariant 0 or 1728. Let  $D^*$  be the punctured disk described above. Then  $f: f^{-1}D^* \rightarrow D^*$  is an unramified covering of degree  $\#SL_2(\mathbb{Z})/\Gamma$  if  $\Gamma$  does

not contain  $-1$ , and of degree  $(\#\mathrm{SL}_2(\mathbb{Z})/\Gamma)/2$  if  $-1$  is in  $\Gamma$ . Up to isomorphism, the only connected unramified covering of degree  $n$ , with  $n \geq 1$ , of  $D^*$  is the map  $D_n^* \rightarrow D^*$ , with  $D_n^* = \{z \in \mathbb{C} \mid 0 < |z| < e^{-2\pi/n}\}$ , sending  $z \mapsto z^n$ . It follows that  $f^{-1}D^*$  is, as a covering of  $D^*$ , a disjoint union of copies of such  $D_n^* \rightarrow D^*$ . Each  $D_n^*$  has the natural compactification  $D_n := \{z \in \mathbb{C} \mid |z| < e^{-2\pi/n}\}$ . We compactify  $\Gamma \backslash \mathbb{H}$  by adding the origin to each punctured disk in  $f^{-1}D^*$ . The points that we have added are called the cusps. By construction, the morphism  $f: \Gamma \backslash \mathbb{H} \rightarrow \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$  extends to the compactifications. It is a fact that a compact Riemann surface can be embedded into some projective space, using the theorem of Riemann-Roch, and that the image of such an embedding is a complex algebraic curve. This means that our quotients are, canonically, the Riemann surfaces associated with smooth complex algebraic curves.

**2.1.4 Definition** For  $n \geq 1$  we define  $X(n)$ ,  $X_1(n)$  and  $X_0(n)$  to be the proper smooth complex algebraic curves obtained via the compactifications of  $\Gamma(n) \backslash \mathbb{H}$ ,  $\Gamma_1(n) \backslash \mathbb{H}$ , and  $\Gamma_0(n) \backslash \mathbb{H}$ , respectively. The affine parts obtained by removing the cusps are denoted  $Y(n)$ ,  $Y_1(n)$  and  $Y_0(n)$ .

The next step in the theory is to show that these complex algebraic curves are naturally defined over certain number fields. Let us start with the  $X_0(n)$  and  $X_1(n)$ , which are defined over  $\mathbb{Q}$ . A simple way to produce a model of  $X_0(n)$  over  $\mathbb{Q}$ , i.e., an algebraic curve  $X_0(n)_{\mathbb{Q}}$  over  $\mathbb{Q}$  that gives  $X_0(n)$  via extension of scalars via  $\mathbb{Q} \rightarrow \mathbb{C}$ , is to use the map:

$$(j, j'): \mathbb{H} \longrightarrow \mathbb{C} \times \mathbb{C}, \quad z \mapsto (j(z), j(nz)).$$

This map factors through the action of  $\Gamma_0(n)$ , and induces a map from  $X_0(n)$  to  $\mathbb{P}^1 \times \mathbb{P}^1$  that is birational to its image. This image is a curve in  $\mathbb{P}^1 \times \mathbb{P}^1$ , hence the zero locus of a bi-homogeneous polynomial often denoted  $\Phi_n$ , the minimal polynomial of  $j'$  over  $\mathbb{C}(j)$ . One can then check, using some properties of the  $j$ -function, that  $\Phi_n$  has integer coefficients. The normalisation of the curve in  $\mathbb{P}_{\mathbb{Q}}^1 \times \mathbb{P}_{\mathbb{Q}}^1$  defined by  $\Phi_n$  is then the desired curve  $X_0(n)_{\mathbb{Q}}$ . As  $\Phi_n$  has coefficients in  $\mathbb{Z}$ , it even defines a curve in  $\mathbb{P}_{\mathbb{Z}}^1 \times \mathbb{P}_{\mathbb{Z}}^1$  (here, one has to work with schemes), whose normalisation  $X_0(n)_{\mathbb{Z}}$  can be characterised as a so-called *coarse moduli space*. For this notion, and for the necessary proofs, the reader is referred to [Di-Im, II.8], to [De-Ra]

and to [Ka-Ma]. One consequence of this statement is that for any algebraically closed field  $k$  in which  $n$  is invertible, the  $k$ -points of  $Y_0(n)_{\mathbb{Z}}$  (the complement of the cusps) correspond bijectively to isomorphism classes of  $E_1 \xrightarrow{\phi} E_2$  where  $\phi$  is a morphism of elliptic curves over  $k$  of which the kernel is cyclic of order  $n$ .

The notion of moduli space also gives natural models over  $\mathbb{Z}[1/n]$  of  $X_1(n)$  and  $Y_1(n)$ . For  $n \geq 4$  the defining property of  $Y_1(n)_{\mathbb{Z}[1/n]}$  is not hard to state. There is an elliptic curve  $\mathbb{E}$  over  $Y_1(n)_{\mathbb{Z}[1/n]}$  with a point  $\mathbb{P}$  in  $E(Y_1(n)_{\mathbb{Z}[1/n]})$  that has order  $n$  in every fibre, such that *any* pair  $(E/S, P)$  with  $S$  a  $\mathbb{Z}[1/n]$ -scheme and  $P$  in  $E(S)$  of order  $n$  in all fibres arises by a *unique* base change:

$$\begin{array}{ccc} E & \longrightarrow & \mathbb{E} \\ \downarrow & & \downarrow \\ S & \longrightarrow & Y_1(n)_{\mathbb{Z}[1/n]} \end{array}$$

that is compatible with the sections  $P$  and  $\mathbb{P}$ . The pair  $(\mathbb{E}/Y_1(n)_{\mathbb{Z}[1/n]}, \mathbb{P})$  is therefore called *universal*.

The moduli interpretation of  $X(n)$  is a bit more complicated, because of the occurrence of the Weil pairing on  $E[n]$  that we have seen above. The curve  $X(n)$  has a natural model  $X(n)_{\mathbb{Z}[1/n, \zeta_n]}$  over  $\mathbb{Z}[1/n, \zeta_n]$ . The complement of the cusps  $Y(n)_{\mathbb{Z}[1/n, \zeta_n]}$  then has an elliptic curve  $\mathbb{E}$  over it, and an isomorphism  $\phi$  between the constant group scheme  $(\mathbb{Z}/n\mathbb{Z})^2$  and  $\mathbb{E}[n]$  that respects the pairings on each side. The pair  $(\mathbb{E}/Y(n)_{\mathbb{Z}[1/n, \zeta_n]}, \phi)$  is universal in the same sense as above. We warn the reader that the notation  $X(n)$  is also used sometimes for the moduli scheme for pairs  $(E, \phi)$  where  $\phi$  does not necessarily respect the pairings on the two sides.

For  $n$  and  $m$  in  $\mathbb{Z}_{>1}$  that are relatively prime we will sometimes view  $Y_1(nm)_{\mathbb{Z}[1/nm]}$  as the moduli space of triples  $(E/S, P_n, P_m)$ , where  $S$  is a scheme over  $\mathbb{Z}[1/nm]$ ,  $E/S$  an elliptic curve,  $P_n$  and  $P_m$  in  $E(S)$  that are everywhere (i.e., in every geometric fibre of  $E/S$ ) of orders  $n$  and  $m$ , respectively. Indeed, for such a triple,  $P_n + P_m$  is everywhere of order  $nm$ , and the inverse construction starting with a point  $P_{nm}$  that is everywhere of order  $nm$  is given by multiplying with the two idempotents of  $\mathbb{Z}/nm\mathbb{Z}$  corresponding to the isomorphism of rings  $\mathbb{Z}/nm\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ .

## 2.2 MODULAR FORMS

Let us now turn our attention to modular forms. It will be enough for us to work with modular forms for the congruence subgroups  $\Gamma_1(n)$ . Therefore, we restrict ourselves to that case.

**2.2.1 Definition** Let  $n \geq 1$  and  $k$  an integer. A (holomorphic) *modular form* for  $\Gamma_1(n)$  is a holomorphic function  $f: \mathbb{H} \rightarrow \mathbb{C}$  that satisfies the following properties:

1. for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(n)$  and for all  $z \in \mathbb{H}$ :

$$f((az + b)/(cz + d)) = (cz + d)^k f(z);$$

2.  $f$  is holomorphic at the cusps (see below for an explanation).

A modular form is called a *cuspsform* if it vanishes at the cusps.

We still need to explain the condition that  $f$  is holomorphic at the cusps. In order to do that, we first explain what this means at the cusp  $\infty$ . That cusp is the point that was added to the punctured disk obtained by taking the quotient of  $\mathbb{H}$  by the unipotent subgroup  $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ , which acts on  $\mathbb{H}$  by translations by integers. The coordinate of that disk is  $q$ , the map that sends  $z$  to  $\exp(2\pi iz)$ . Therefore,  $f$  admits a Laurent series expansion in  $q$ :

$$(2.2.2) \quad f = \sum_{n \in \mathbb{Z}} a_n(f) q^n, \quad \text{called the } q\text{-expansion at } \infty.$$

With this notation,  $f$  is called holomorphic at  $\infty$  if  $a_n(f)$  is zero for all  $n < 0$ , and  $f$  is said to vanish at  $\infty$  if  $a_n(f)$  is zero for all  $n \leq 0$ .

To state this condition at the other cusps, we need some description of the set of cusps. First, we note that  $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$  is the same as  $\mathbb{C} - \mathbb{R}$ , and therefore the disjoint union of  $\mathbb{H}$  and its complex conjugate (which explains, by the way, that  $\text{GL}_2(\mathbb{R})^+$  acts by fractional linear transformations on  $\mathbb{H}$ ). We can then consider  $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$  inside  $\mathbb{P}^1(\mathbb{C})$ , with the  $\text{SL}_2(\mathbb{Z})$  action on it. Then the subgroup  $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$  stabilises the point  $\infty = (1 : 0)$  of  $\mathbb{P}^1(\mathbb{Q})$ , and  $\infty$  can be naturally identified with the origin that we added to the disk  $D^*$  above, because  $\infty$  is the unique element of  $\mathbb{P}^1(\mathbb{Q})$  that lies in the closure of the inverse image “ $\Im(z) > 1$ ” of  $D^*$  in  $\mathbb{H}$ . Then, the images of

the region “ $\Im(z) > 1$ ” under the action of elements of  $\mathrm{SL}_2(\mathbb{Z})$  correspond bijectively to the elements of  $\mathbb{P}^1(\mathbb{Q})$  (note that  $\mathrm{SL}_2(\mathbb{Z})$  acts transitively on  $\mathbb{P}^1(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Z})$ ), and also to the maximal unipotent subgroups of  $\mathrm{SL}_2(\mathbb{Z})$  (i.e., the subgroups that consist of elements whose eigenvalues are 1). It follows that we can identify the set of cusps of  $X_1(n)$  with  $\Gamma_1(n) \backslash \mathbb{P}^1(\mathbb{Q})$ , and that the images of the region “ $\Im(z) > 1$ ” under  $\mathrm{SL}_2(\mathbb{Z})$  give us punctured disks around the other cusps. Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be an element of  $\mathrm{SL}_2(\mathbb{Z})$ . The conditions of holomorphy and vanishing at the cusp  $\gamma\infty = (a : c)$  are then given in terms of the  $q$ -expansion of  $z \mapsto (cz + d)^{-k} f(\gamma z)$  at  $\infty$ . The group  $\gamma^{-1}\Gamma_1(n)\gamma$  contains the group  $\begin{pmatrix} 1 & n^* \\ 0 & 1 \end{pmatrix}$  (indeed,  $\Gamma_1(n)$  contains  $\Gamma(n)$  and that one is normal in  $\mathrm{SL}_2(\mathbb{Z})$ ). Therefore, putting  $q_n: \mathbb{H} \rightarrow \mathbb{C}$ ,  $z \mapsto \exp(2\pi iz/n)$ , the function  $z \mapsto (cz + d)^{-k} f(\gamma z)$  then has a Laurent series expansion in  $q_n$ , and one asks that this Laurent series is a power series (for holomorphy) or a power series with constant term zero (for vanishing).

The space of modular forms of weight  $k$  on  $\Gamma_1(n)$  will be denoted  $M_k(\Gamma_1(n))$ , and the subspace of cuspforms by  $S_k(\Gamma_1(n))$ . We define  $M(\Gamma_1(n))$  to be the direct sum over the  $k$  in  $\mathbb{Z}$  of the  $M_k(\Gamma_1(n))$ ; it is a  $\mathbb{Z}$ -graded commutative  $\mathbb{C}$ -algebra under pointwise multiplication.

**2.2.3 Example** Some simple examples of modular forms for  $\mathrm{SL}_2(\mathbb{Z})$  are given by Eisenstein series. For each even  $k \geq 4$  one has the function  $E_k$ :

$$E_k: \mathbb{H} \rightarrow \mathbb{C}, \quad z \mapsto \frac{1}{2\zeta(k)} \sum_{\substack{(n,m) \in \mathbb{Z}^2 \\ (n,m) \neq (0,0)}} \frac{1}{(n + mz)^k}.$$

The  $q$ -expansions of these  $E_k$  are given by:

$$E_k = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where the  $B_k$  are the Bernoulli numbers defined by:

$$\frac{te^t}{e^t - 1} = \sum_{k \geq 0} B_k \frac{t^k}{k!},$$

and where, as before,  $\sigma_r(n)$  denotes the sum of the  $r$ th powers of the positive



divisors of  $n$ . In particular, one has the formulas:

$$E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n)q^n, \quad E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n)q^n,$$

$$\Delta = \frac{E_4^3 - E_6^2}{1728}.$$

**2.2.4 Remark** We note that, from a computational point of view, the coefficients of  $q^p$  with  $p$  prime of the  $E_k$  are very easy to compute, namely, up to a constant factor they are  $1 + p^{k-1}$ , but that computing the  $\sigma_{k-1}(n)$  for composite  $n$  is equivalent to factoring  $n$ . This is a strong indication that, for computing coefficients  $a_n(f)$  of a modular form  $f$ , there is a real difference between the case where  $n$  is prime and the case where  $n$  is composite.

Indeed, Denis Charles and Eric Bach have shown that for  $n = pq$  a product of two distinct primes such that  $\tau(n) \neq 0$ , one can compute  $p$  and  $q$  from  $n$ ,  $\tau(n)$  and  $\tau(n^2)$  in time polynomial in  $\log n$ ; see [Ba-Ch].

The argument is very simple: one uses the identities in (1.5.5) to compute the rational number  $\tau(p)^2/q^{11}$  and notes that the denominator is of the form  $q^r$  with  $r$  odd. According to a conjecture by Lehmer,  $\tau(n) \neq 0$  for all  $n \in \mathbb{Z}_{\geq 1}$ . See Corollary 7.1.4.

The Eisenstein series  $E_4$  and  $E_6$  generate the  $\mathbb{C}$ -algebra  $M(\mathrm{SL}_2(\mathbb{Z}))$ , and are algebraically independent:

$$M(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6].$$

In particular, we have:

$$\dim_{\mathbb{C}} M_k(\mathrm{SL}_2(\mathbb{Z})) = \#\{(a, b) \in \mathbb{Z}_{\geq 0}^2 \mid 4a + 6b = k\}.$$

The space  $S_k(\Gamma_1(n))$  can be interpreted as the space of sections of some holomorphic line bundle  $\underline{\omega}^{\otimes k}(-\mathrm{Cusps})$  on  $X_1(n)$ , if  $n \geq 5$  (for  $n < 4$  the action of  $\Gamma_1(n)$  on  $\mathbb{H}$  is not free, and for  $n = 4$  there is a cusp whose stabiliser is not unipotent):

$$(2.2.5) \quad S_k(\Gamma_1(n)) = H^0(X_1(n), \underline{\omega}^{\otimes k}(-\mathrm{Cusps})), \quad \text{if } n \geq 5.$$

This implies that the spaces  $S_k(\Gamma_1(n))$  are finite dimensional, and in fact zero if  $k \leq 0$  because the line bundle in question then has negative degree.

The restriction to  $Y_1(n)$  of the line bundle giving the weight  $k$  forms is given by dividing out the action of  $\Gamma_1(n)$  on  $\mathbb{C} \times \mathbb{H}$  given by:

$$(2.2.6) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}: (x, z) \mapsto \left( (cz + d)^k x, \frac{az + b}{cz + d} \right).$$

The extension of this line bundle over the cusps is then given by decreeing that, at the cusp  $\infty$ , the constant section 1 (which is indeed invariant under the translations  $z \mapsto z + n$ ) is a generator for the bundle of holomorphic forms, and  $q$  times 1 is a generator for the bundle of cusp forms.

The moduli interpretation for  $Y_1(n)$  can be extended to the holomorphic line bundles giving the modular forms as follows. Recall that a point on  $Y_1(n)$  is an isomorphism class of a pair  $(E, P)$  with  $E$  a complex elliptic curve and  $P$  a point of order  $n$  of  $E$ . The complex line at  $(E, P)$  of the bundle of forms of weight  $k$  is then  $\omega_E^{\otimes k}$ , the  $k$ th tensor power of the dual of the tangent space at 0 of  $E$ . In this way, a modular form  $f$  of weight  $k$  for  $\Gamma_1(n)$  can be described as follows: it is a function that assigns to each  $(E, P)$  an element  $f(E, P)$  of  $\omega_E^{\otimes k}$ , varying holomorphically with  $(E, P)$ , and such that it has the right property at the cusps (being holomorphic or vanishing). The function  $f$  has to be compatible with isomorphisms: if  $\phi: E \rightarrow E'$  is an isomorphism, and  $\phi(P) = P'$ , then  $f(E, P)$  has to be equal to  $(\phi^*)^{\otimes k} f(E', P')$ . In what follows we will simply write  $\phi^*$  for  $(\phi^*)^{\otimes k}$ .

The fact that  $f$  should be holomorphic can be stated by evaluating it on the family of elliptic curves that we have over  $\mathbb{H}$ . Recall that to  $z$  in  $\mathbb{H}$  we attached the pair  $(\mathbb{C}/(\mathbb{Z}z + \mathbb{Z}), [1/n])$ . Let us denote  $x$  the coordinate of  $\mathbb{C}$ , then  $dx$  is a generator of the cotangent space at 0 of this elliptic curve. Then for  $f$  a function as above, we can write:

$$(2.2.7) \quad f((\mathbb{C}/(\mathbb{Z}z + \mathbb{Z}), [1/n])) = F_f(z) \cdot (dx)^{\otimes k}, \quad F_f: \mathbb{H} \rightarrow \mathbb{C}.$$

The function  $F_f$  is then required to be holomorphic. The requirement that  $f$  is compatible with isomorphisms means precisely that  $F_f$  transforms under  $\Gamma_1(n)$  as in Definition 2.2.1 above. The requirement that  $f$  vanishes at the cusps is equivalent to the statement that the Laurent expansions in  $q^{1/n}: z \mapsto \exp(2\pi iz/n)$  obtained by evaluating  $f$  on all pairs  $(\mathbb{C}/(\mathbb{Z}z + \mathbb{Z}), (az + b)/n)$ , with  $a$  and  $b$  in  $\mathbb{Z}$  such that  $(az + b)/n$  is of order  $n$  are in fact power series with constant term zero.

The spaces  $S_k(\Gamma_1(n))$  are equipped with certain operators, called Hecke operators and diamond operators. These operators arise from the fact that for every element  $\gamma$  of  $\mathrm{GL}_2(\mathbb{Q})^+$  the subgroups  $\Gamma_1(n)$  and  $\gamma\Gamma_1(n)\gamma^{-1}$  are commensurable, i.e., their intersection has finite index in each of them. The diamond operators are then the simplest to describe. For each  $a$  in  $(\mathbb{Z}/n\mathbb{Z})^\times$ ,  $Y_1(n)$  has the automorphism  $\langle a \rangle$  given by the property that it sends  $(E, P)$  to  $(E, aP)$ . This action is then extended on modular forms by:

$$(2.2.8) \quad (\langle a \rangle f)(E, P) = f(E, aP).$$

Similarly, there are Hecke operators  $T_m$  on  $S_k(\Gamma_1(n))$  for all integers  $m \geq 1$ , defined by:

$$(2.2.9) \quad (T_m f)(E, P) = \frac{1}{m} \sum_{\phi} \phi^* f(E_{\phi}, \phi(P)),$$

where the sum runs over all quotients  $\phi: E \rightarrow E_{\phi}$  of degree  $m$  such that  $\phi(P)$  is of order  $n$ . Intuitively, the operator  $T_m$  is to be understood as a kind of averaging operator over all possible isogenies of degree  $m$ . However, the normalising factor  $1/m$  is not equal to the inverse of the number of such isogenies. Instead, this factor is there to make the Eichler-Shimura isomorphism (see (2.4.5))  $T_m$ -equivariant.

Of course, each element  $f$  of  $S_k(\Gamma_1(n))$  is determined by its  $q$ -expansion  $\sum_{m \geq 1} a_m(f)q^m$  at the cusp  $\infty$ . The action of the Hecke operators can be expressed in terms of these  $q$ -expansions (see [Di-Im, (12.4.1)]):

$$(2.2.10) \quad a_m(T_r f) = \sum_{\substack{0 < d | (r, m) \\ (d, n) = 1}} d^{k-1} a_{rm/d^2}(\langle d \rangle f),$$

for  $f$  in  $S_k(\Gamma_1(n))$ ,  $r$  and  $m$  positive integers.

From this formula, a lot can be deduced. It can be seen that the  $T_r$  commute with each other (but there are better ways to understand this). The  $\mathbb{Z}$ -algebra generated by the  $T_m$  for  $m \geq 1$  and the  $\langle a \rangle$  for  $a$  in  $(\mathbb{Z}/n\mathbb{Z})^\times$  is in fact generated by the  $T_m$  with  $m \geq 1$ , i.e., one does not need the diamond operators, and also by the  $T_p$  for  $p$  prime and the  $\langle a \rangle$  with  $a$  in  $(\mathbb{Z}/n\mathbb{Z})^\times$  (see [Di-Im, §3.5]). The multiplication rules for the  $T_m$  acting on  $S_k(\Gamma_1(n))$  can

be read off from the formal identity ([Di-Im, §3.4]):

$$(2.2.11) \quad \sum_{m \geq 1} T_m m^{-s} = \prod_p (1 - T_p p^{-s} + p^{k-1} \langle p \rangle p^{-2s})^{-1},$$

where  $\langle p \rangle$  is to be interpreted as zero when  $p$  divides  $n$ . The fact that the Hecke and diamond operators commute means that they have common eigenspaces. Taking  $m = 1$  in (2.2.10) gives:

$$(2.2.12) \quad a_1(T_r f) = a_r(f).$$

It follows that if  $f$  is a non-zero eigenvector for all  $T_r$ , then  $a_1(f) \neq 0$ , so that we can assume that  $a_1(f) = 1$ . Then, for all  $r \geq 1$ ,  $a_r(f)$  is the eigenvalue for  $T_r$ . In particular, this means that the common eigenspaces for the  $T_r$  are one-dimensional, and automatically eigenspaces for the diamond operators. Eigenforms with  $a_1(f) = 1$  are called *normalised eigenforms*.

From (2.2.11) above it follows that for a normalised eigenform  $f$  one has:

$$(2.2.13) \quad \begin{aligned} L_f(s) &:= \sum_{m \geq 1} a_m(f) m^{-s} \\ &= \prod_p (1 - a_p(f) p^{-s} + p^{k-1} \varepsilon_f(p) p^{-2s})^{-1}, \end{aligned}$$

where  $\varepsilon_f: (\mathbb{Z}/n\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$  is the character via which the diamond operators act on  $f$ , with the convention that  $\varepsilon_f(p) = 0$  if  $p$  divides  $n$ . In particular, the  $L$ -function of a modular form has such an Euler product expansion if and only if the modular form is an eigenform for all Hecke operators.

An element of  $S_k(\Gamma_1(n))$  that is a normalised eigenform for all Hecke operators is called a *newform* if the system of eigenvalues  $a_p(f)$ , with  $p$  not dividing  $n$ , does not occur in a level strictly smaller than  $n$ , i.e., in some  $S_k(\Gamma_1(m))$  with  $m < n$  (actually, we will see in a moment that one only needs to consider the  $m$ 's dividing  $n$ ). The set of newforms in  $S_k(\Gamma_1(n))$  will be denoted  $S_k(\Gamma_1(n))^{\text{new}}$ .

Now we want to recall briefly how one obtains a basis of  $S_k(\Gamma_1(n))$  in terms of the sets of newforms  $S_k(\Gamma_1(m))^{\text{new}}$  for  $m$  dividing  $n$ . For details and references to proofs, see [Di-Im, I.6]. First of all, for each  $n$ ,  $S_k(\Gamma_1(n))^{\text{new}}$  is a linearly independent subset of  $S_k(\Gamma_1(n))$ , hence finite. For  $m$  dividing  $n$  and for  $d$  dividing  $n/m$ , we have a map

$B_{n,m,d}: X_1(n) \rightarrow X_1(m)$ , whose moduli interpretation is that it maps  $(E, P)$  to  $(E/\langle(n/d)P\rangle, d'P)$ , where  $dd' = n/m$ . For example, this means:

$$(2.2.14) \quad B_{n,m,d}: (\mathbb{C}/(\mathbb{Z}z + \mathbb{Z}), 1/n) \mapsto (\mathbb{C}/(\mathbb{Z}zd + \mathbb{Z}), 1/m),$$

which means that the cusp  $\infty$  of  $X_1(n)$  is mapped to the cusp  $\infty$  of  $X_1(m)$ . Each such map  $B_{n,m,d}$  induces by pullback a map:

$$(2.2.15) \quad B_{n,m,d}^*: S_k(\Gamma_1(m)) \rightarrow S_k(\Gamma_1(n)).$$

In terms of  $q$ -expansions at the cusp  $\infty$  we have, for  $f$  in  $S_k(\Gamma_1(m))$ :

$$(2.2.16) \quad B_{n,m,d}^* f = \sum_{r \geq 1} a_r(f) q^{dr},$$

i.e., the effect is just substitution of  $q$  by  $q^d$ . With these definitions, we can describe a basis for  $S_k(\Gamma_1(n))$ :

$$(2.2.17) \quad \prod_{m|n} \prod_{d|(n/m)} B_{n,m,d}^* S_k(\Gamma_1(m))^{\text{new}} \quad \text{is a basis of } S_k(\Gamma_1(n)).$$

In the case where  $\Gamma_1(n)$  is replaced by  $\Gamma_0(n)$ , this kind of basis is due to Atkin and Lehner.

In the sequel, we will also make use of a (hermitian) inner product on the  $S_k(\Gamma_1(n))$ : the Petersson scalar product. It is defined as follows. For  $f$  and  $g$  in  $S_k(\Gamma_1(n))$ , viewed as functions on  $\mathbb{H}$  as in Definition 2.2.1 one has:

$$(2.2.18) \quad \langle f, g \rangle = \int_{\Gamma_1(n) \backslash \mathbb{H}} f(z) \overline{g(z)} y^k \frac{dx dy}{y^2},$$

where the integral over  $\Gamma_1(n) \backslash \mathbb{H}$  means that one can perform it over any fundamental domain. Indeed, formula 2.1.2 shows that the function  $z \mapsto f(z) \overline{g(z)} y^k$  is invariant under  $\Gamma_1(n)$ .

We also want to explain the definition of  $\langle f, g \rangle$  in terms of the moduli interpretation of  $S_k(\Gamma_1(n))$ , if  $k \geq 2$ . For simplicity, let us suppose  $n \geq 5$  now. Then  $S_k(\Gamma_1(n))$  is the space of global sections of  $\underline{\omega}^{\otimes k}(-\text{Cusps})$  on  $X_1(n)$ . Now we let  $\Omega^1 := \Omega_{X_1(n)}^1$  denote the line bundle of holomorphic differentials on  $X_1(n)$ . Then there is an isomorphism, named after Kodaira and Spencer:

$$(2.2.19) \quad \text{KS: } \underline{\omega}^{\otimes 2}(-\text{Cusps}) \xrightarrow{\sim} \Omega^1, \quad \text{Kodaira-Spencer isomorphism.}$$

Explicitly, for  $f$  in  $S_2(\Gamma_1(n))$ , viewed as a  $\Gamma_1(n)$ -invariant section of  $\omega^{\otimes 2}$  for the family of elliptic curves over  $\mathbb{H}$  whose fibre at  $z$  is  $\mathbb{C}/(\mathbb{Z}z + \mathbb{Z})$  we have:

$$(2.2.20) \quad \text{KS: } f(dx)^{\otimes 2} \mapsto (2\pi i)^{-2} f \frac{dq}{q}.$$

Equivalently, for this family of elliptic curves, the Kodaira-Spencer isomorphism sends  $(dx)^{\otimes 2}$  to  $(2\pi i)^{-2}(dq)/q$ . Note that indeed  $(dx)^{\otimes 2}$  and  $(dq)/q$  transform in the same way under the action of  $\text{SL}_2(\mathbb{R})$ . We note that without  $f$  being required to vanish at the cusps,  $\text{KS}(f)$  could have poles of order one at the cusps. The factor  $(2\pi i)^{-2}$  is to make the isomorphism compatible with the coordinates  $t = \exp(2\pi ix)$  on  $\mathbb{C}^\times / \langle \exp(2\pi iz) \rangle$  (which is another way to write  $\mathbb{C}/(\mathbb{Z}z + \mathbb{Z})$ ), and the coordinate  $q = \exp(2\pi iz)$  on the unit disk. In those coordinates, that have a meaning “over  $\mathbb{Z}$ ”, which means that formulas relating them are power series (or Laurent series) with integer coefficients, KS sends  $((dt)/t)^{\otimes 2}$  to  $(dq)/q$ .

For every complex elliptic curve, the one dimensional complex vector space  $\omega_E$  has the inner product given by:

$$(2.2.21) \quad \langle \alpha, \beta \rangle = \frac{i}{2} \int_E \alpha \bar{\beta},$$

where we interpret  $\alpha$  and  $\beta$  as translation invariant differential forms on  $E$ . The factor  $i/2$  is explained by the fact that, for  $z = x + iy$ , one has  $dx dy = (i/2) dz d\bar{z}$ . Applying this to the family of elliptic curves  $\mathbb{C}/(\mathbb{Z}z + \mathbb{Z})$  over  $\mathbb{H}$  gives an inner product on the line bundle  $\omega$  on  $\mathbb{H}$ , and also on the line bundle  $\omega$  on  $Y_1(n)$  (recall that we are supposing that  $n \geq 5$ ). Taking tensor powers and duals, this induces inner products on  $\omega^{\otimes k}$  for all  $k$ . The Kodaira-Spencer isomorphism (2.2.20) gives isomorphisms:

$$(2.2.22) \quad \text{KS: } \omega^{\otimes k}(-\text{Cusps}) \xrightarrow{\sim} \Omega^1 \otimes \omega^{\otimes(k-2)}.$$

For  $f$  and  $g$  in  $S_k(\Gamma_1(n))$ , now viewed as sections of  $\omega^{\otimes k}(-\text{Cusps})$  over  $X_1(n)$ , one has:

$$(2.2.23) \quad \langle f, g \rangle = \frac{i}{2} \int_{X_1(n)} \langle \text{KS}(f), \text{KS}(g) \rangle,$$

where the inner product on the left hand side is the Petersson scalar product (2.2.18), and where for two local sections  $\omega_1 \otimes \alpha_1^{\otimes(k-2)}$  and  $\omega_2 \otimes \alpha_2^{\otimes(k-2)}$

of  $\Omega^1 \otimes \underline{\omega}^{\otimes(k-2)}$  we have defined:

$$(2.2.24) \quad \langle \omega_1 \otimes \alpha_1^{\otimes(k-2)}, \omega_2 \otimes \alpha_2^{\otimes(k-2)} \rangle = \langle \alpha_1, \alpha_2 \rangle^{k-2} \omega_1 \overline{\omega_2}.$$

The operators  $T_m$  on  $S_k(\Gamma_1(n))$  with  $m$  relatively prime to  $n$  are normal: they commute with their adjoint. As a consequence, distinct newforms in  $S_k(\Gamma_1(n))$  are orthogonal to each other. On the other hand, the basis (2.2.17) above of  $S_k(\Gamma_1(n))$  is not orthogonal if it consists of more than only newforms.

### 2.3 LATTICES AND MODULAR FORMS

Before we move on to Galois representations attached to modular forms, we briefly discuss the relation between modular forms and lattices.

Let us consider a free  $\mathbb{Z}$ -module  $L$  of finite rank  $n$ , equipped with a positive definite symmetric bilinear form  $b: L \times L \rightarrow \mathbb{Z}$ . Then  $L_{\mathbb{R}} := \mathbb{R} \otimes L$  is an  $\mathbb{R}$ -vector space of dimension  $n$  on which  $b$  gives an inner product, and hence  $L$  is a lattice in the euclidean space  $L_{\mathbb{R}}$ . For  $m$  in  $\mathbb{Z}$  the *representation numbers* of  $(L, b)$  are defined as:

$$(2.3.1) \quad r_L(m) = r_{L,b}(m) := \#\{x \in L \mid b(x, x) = m\}.$$

In this situation, one considers the *theta-function* attached to  $(L, b)$ :

$$(2.3.2) \quad \theta_L = \theta_{L,b} = \sum_{x \in L} q^{b(x,x)/2} = \sum_{m \geq 0} r_L(m) q^{m/2}, \quad \mathbb{H} \rightarrow \mathbb{C},$$

where  $q^{1/2}: \mathbb{H} \rightarrow \mathbb{C}$  is the function  $q^{1/2}: z \mapsto \exp(\pi iz)$ .

If  $(L, b)$  is the orthogonal direct sum of  $(L_1, b_1)$  and  $(L_2, b_2)$  then we have:

$$(2.3.3) \quad \theta_L = \sum_{x \in L} q^{b(x,x)/2} = \sum_{(x_1, x_2) \in L_1 \times L_2} q^{b_1(x_1, x_1)/2 + b_2(x_2, x_2)/2} = \theta_{L_1} \cdot \theta_{L_2}.$$

The *discriminant* of  $b$  is  $\det(B)$ , where  $B$  is the matrix of  $b$  with respect to some basis of  $L$  (indeed, this determinant does not depend on the choice of basis); we denote it by  $\text{discr}(b)$ .

We define the positive integer  $N_L$  to be the exponent of the cokernel of the map  $\phi_b: L \rightarrow L^\vee$  given by  $b$ , or, equivalently, to be the denominator

of  $B^{-1}$ , where  $B$  is the matrix of  $b$  with respect to some basis  $e$  of  $L$ . The map  $N_L \phi_b^{-1}: L_{\mathbb{Q}}^{\vee} \rightarrow L_{\mathbb{Q}}$  restricts to a map  $N_L \phi_b^{-1}: L^{\vee} \rightarrow L$ . Viewing  $L$  as  $(L^{\vee})^{\vee}$  in the usual way, this gives a positive definite symmetric bilinear form  $b': L^{\vee} \times L^{\vee} \rightarrow \mathbb{Z}$ . The matrix of this form with respect to the basis  $e^{\vee}$  dual to  $e$  is  $N_L B^{-1}$ . Applying this same construction to  $b'$  gives a  $b'': L \times L \rightarrow \mathbb{Z}$  that is not necessarily equal to  $b$ : one has  $b = mb''$ , with  $m \in \mathbb{Z}_{>0}$  and  $b''$  primitive (i.e., the  $\mathbb{Z}$ -linear map  $L \otimes L \rightarrow \mathbb{Z}$  induced by  $b''$  is surjective). Poisson's summation formula gives the following functional equation; see [Ser5, VII, §6, Prop. 16].

**2.3.4 Theorem** *Let  $L$  be a free  $\mathbb{Z}$ -module, of finite rank  $n$ , equipped with a positive definite symmetric bilinear form  $b: L \times L \rightarrow \mathbb{Z}$ . We have, with the notation as above, for all  $z \in \mathbb{H}$ :*

$$\theta_{L,b}(-1/N_L z) = \frac{(-N_L iz)^{n/2}}{\text{discr}(b)^{1/2}} \theta_{L^{\vee},b'}(z),$$

where the square root of  $-N_L iz$  is holomorphic in  $z$  and positive for  $z \in \mathbb{R}i$ .

The form  $b$  is called *even* if  $b(x, x)$  is even for all  $x$  in  $L$ . Equivalently,  $b$  is even if and only if the matrix  $B$  of  $b$  with respect to some basis of  $L$  has only even numbers on the diagonal.

The form  $b$  is called *unimodular* if  $\phi_b: L \rightarrow L^{\vee}$  is an isomorphism, or, equivalently, if  $N_L = 1$ . In this case,  $\phi_b$  is an isomorphism from  $(L, b)$  to  $(L^{\vee}, b')$ .

With this terminology, one has the following result, see [Miy, Cor. 4.9.5], the proof of which has as main ingredient the functional equation of Theorem 2.3.4.

**2.3.5 Theorem** *Let  $L$  be a free  $\mathbb{Z}$ -module of finite rank  $n$ , equipped with a positive definite symmetric bilinear form  $b: L \times L \rightarrow \mathbb{Z}$ . Assume that  $n$  is even. Let  $N_L$  be as defined above, and let  $\chi_L$  be the character given by:*

$$\chi_L: (\mathbb{Z}/N_L \mathbb{Z})^{\times} \rightarrow \mathbb{C}^{\times}, \quad (a \bmod N_L) \mapsto \left( \frac{(-1)^{n/2} \text{discr}(b)}{a} \right),$$

where the fraction denotes the Kronecker symbol.

1. The function  $z \mapsto \theta_L(2z)$  is a (non-cuspidal) modular form on  $\Gamma_1(4N_L)$  of weight  $n/2$  and with character  $\chi_L$ .



2. If  $b$  is even then the function  $\theta_L$  is a modular form on  $\Gamma_1(2N_L)$  of weight  $n/2$  and with character  $\chi_L$ .
3. If both  $b$  and  $b'$  (see above for its definition) are even, then the function  $\theta_L$  is a modular form on  $\Gamma_1(N_L)$  of weight  $n/2$  and with character  $\chi_L$ .

This theorem says nothing about the case where  $n$  is odd. In that case,  $\theta_L$  is a modular form of half-integral weight  $n/2$ ; see [Miy, Cor. 4.9.7]. For even unimodular forms, we have the following corollary of Theorem 2.3.5.

**2.3.6 Corollary** *Let  $L$  be a free  $\mathbb{Z}$ -module of finite rank  $n$ , and let  $b: L \times L \rightarrow \mathbb{Z}$  be bilinear, symmetric, positive definite, even and unimodular. Then  $n$  is even, and  $\theta_L$  is a modular form on  $\mathrm{SL}_2(\mathbb{Z})$  of weight  $n/2$ .*

**Proof** As  $b$  is unimodular, we have  $N_L = 1$ . The fact that  $n$  is even follows from the fact that  $b$  induces a non-degenerate alternating bilinear form on  $\mathbb{F}_2 \otimes L$ . As  $\phi_b: L \rightarrow L^\vee$  is an isomorphism between  $(L, b)$  and  $(L^\vee, b')$ , we have that  $b'$  is even as well. Theorem 2.3.5 gives the conclusion.  $\square$

**2.3.7 Remark** In fact, the rank of an even unimodular lattice is a multiple of 8. This follows directly from Theorems 2.3.4 and 2.3.5 (see [Miy, Cor. 4.9.6], or [Ser5, VII, §6, Thm. 8]).

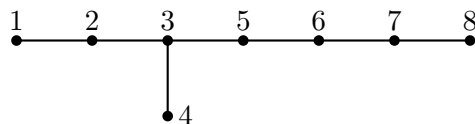
Let us consider some examples.

**2.3.8 Example** For  $n \in \mathbb{Z}_{\geq 0}$  we consider  $\mathbb{Z}^n$  with its standard inner product. For  $m$  in  $\mathbb{Z}$  we have:

$$r_{\mathbb{Z}^n}(m) = \#\{x \in \mathbb{Z}^n \mid x_1^2 + \cdots + x_n^2 = m\},$$

the number of ways in which  $m$  can be written as a sum of  $n$  squares of integers. Theorem 2.3.5 tells us that for even  $n$  the theta function  $z \mapsto \theta_{\mathbb{Z}^n}(2z)$  is a modular form on  $\Gamma_1(4)$  of weight  $n/2$ . According to (2.3.3) we have  $\theta_{\mathbb{Z}^n} = \theta_{\mathbb{Z}}^n$ , and so all the functions  $z \mapsto \theta_{\mathbb{Z}^n}(2z)$  are powers of the modular form  $\sum_{m \in \mathbb{Z}} q^{m^2}$  of weight  $1/2$  on  $\Gamma_1(4)$ .

**2.3.9 Example** We consider the  $E_8$ -lattice, i.e.,  $E_8 := \mathbb{Z}^8$  equipped with the inner product given by the Dynkin diagram  $E_8$  with numbered vertices:



that is, whose matrix with respect to the standard basis is:

$$\begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & -1 & & & \\ & & -1 & 2 & & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}.$$

The lattice  $E_8$  is unimodular and even, hence, by Corollary 2.3.6,  $\theta_{E_8}$  is a modular form on  $SL_2(\mathbb{Z})$  of weight 4, i.e.,  $\theta_{E_8}$  is an element of  $M_4(SL_2(\mathbb{Z}))$ . The dimension space of  $M_4(SL_2(\mathbb{Z}))$  is one, with the Eisenstein series  $E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n)q^n$  as basis. Therefore,  $\theta_{E_8}$  is a constant times  $E_4$ . Comparing constant terms, we get:

$$\theta_{E_8} = E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n)q^n.$$

**2.3.10 Example** Let  $L$  be the *Leech lattice*. This lattice, which is of rank 24, even and unimodular, is named after John Leech, see [Lee2] and [Lee1]. Apparently, it had already been discovered by Ernst Witt in 1940 (unpublished, see [Wit]). John Horton Conway showed in [Conw] that  $L$  is the only non-zero even unimodular lattice of rank less than 32 with  $r_L(2) = 0$ ; this also follows from Hans-Volker Niemeier’s classification of even unimodular lattices of rank 24 in [Nie]. According to Henry Cohn and Abhinav Kumar [Co-Ku], the Leech lattice gives the densest lattice sphere packing in dimension 24.

Theorem 2.3.5 above shows that the theta function  $\theta_L$  of the Leech lattice is a modular form of level 1 and weight 12. The space of such modular

forms is two-dimensional, with basis the Eisenstein series  $E_{12}$  and the discriminant form  $\Delta$ , where:

$$E_{12} = 1 + \frac{65520}{691} \sum_{m \geq 1} \sigma_{11}(m) q^m.$$

Hence  $\theta_L$  is a linear combination of  $E_{12}$  and  $\Delta$ . Comparing the coefficients of  $q^m$  for  $m = 0$  and  $m = 1$  gives:

$$\theta_L = E_{12} - \frac{65520}{691} \Delta.$$

## 2.4 GALOIS REPRESENTATIONS ATTACHED TO EIGENFORMS

The aim of this section is to describe the construction of the Galois representations attached to modular forms, that came up in the case of  $\Delta$  in Section 1.5. Before giving the construction, let us state the result, which is due, for  $k = 2$ , to Eichler and Shimura [Shi1], to Deligne [Del1] for  $k > 2$ , and to Deligne and Serre [De-Se] for  $k = 1$ . See Section 12.5 in [Di-Im]. A long account of the construction in the case  $k \geq 2$  will be given in the book [Conr].

**2.4.1 Theorem** *Let  $f$  be a normalised newform, let  $n$  be its level, let  $k$  be its weight, and let  $\varepsilon: (\mathbb{Z}/n\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$  be its character. Then the subfield  $K$  of  $\mathbb{C}$  generated over  $\mathbb{Q}$  by the  $a_n(f)$ ,  $n \geq 1$ , and the image of  $\varepsilon$  is finite over  $\mathbb{Q}$ . For every prime number  $l$  and for any embedding  $\lambda$  of  $K$  into  $\overline{\mathbb{Q}}_l$ , there is a continuous two-dimensional representation  $V_\lambda$  over  $\overline{\mathbb{Q}}_l$  of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  that is unramified outside  $nl$  and such that for each prime number  $p$  not dividing  $nl$  the characteristic polynomial of the Frobenius at  $p$  acting on  $V_\lambda$  equals:*

$$\det(1 - x\text{Frob}_p, V_\lambda) = 1 - a_p(f)x + \varepsilon(p)p^{k-1}x^2.$$

For  $k \geq 2$  the representations  $V_\lambda$  can be found in the  $l$ -adic étale cohomology in degree  $k - 1$  of some variety of dimension  $k - 1$ , or in the cohomology in degree one of some sheaf on a curve, as we will describe below. The determinant of the action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on  $V_\lambda$  is easily described. We let  $\chi_l: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{Z}_l^\times$  be the  $l$ -adic cyclotomic character defined by  $\sigma(z) = z^{\chi_l(\sigma)}$ , for all  $\sigma$  in  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  and all  $z$  in  $\overline{\mathbb{Q}}^\times$  of  $l$ -power order. We let  $\varepsilon: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow K^\times$  be the composition of

the character  $\varepsilon: (\mathbb{Z}/n\mathbb{Z})^\times \rightarrow K^\times$  with the mod  $n$  cyclotomic character  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow (\mathbb{Z}/n\mathbb{Z})^\times$  given by the action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on  $\mu_n(\overline{\mathbb{Q}})$ . With these definitions, the determinant of the action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on  $V_\lambda$  is given by the character  $\varepsilon\chi_l^{k-1}$ . As the image of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  under the determinant of  $V_\lambda$  is infinite, its image in  $\text{GL}(V_\lambda)$  is infinite.

On the other hand, for  $k = 1$ , the image of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  in  $\text{GL}(V_\lambda)$  is finite, and in fact all these representations when  $\lambda$  varies can be realised over some fixed finite extension of  $\mathbb{Q}$ . The proof of Theorem 2.4.1 by Deligne and Serre in the case  $k = 1$  is quite different from the case  $k \geq 2$ : the reductions to finite coefficient fields (see Section 2.5) can be constructed via congruences to forms of weight 2, and then it is shown that these representations can be lifted to characteristic zero. No direct construction of the characteristic zero Galois representations for forms of weight one is known. We remark that in the case  $k = 2$  the  $V_\lambda$  occur in the first degree étale cohomology with constant coefficients  $\mathbb{Q}_l$  of modular curves, hence can be constructed from  $l$ -power torsion points of Jacobians of modular curves (in fact, of the modular curve  $X_1(n)$ ).

The representation  $V_\lambda$  is irreducible by a theorem of Ribet, see Theorem 2.3 of [Rib2], and hence it is characterised by its trace. As the Frobenius conjugacy classes at the primes not dividing  $nl$  are dense by Chebotarev's theorem, the representation  $V_\lambda$  is unique up to isomorphism. Non-cuspidal eigenforms lead to Galois representations that are reducible; as our interest lies in going beyond class field theory, we do not discuss this case.

Let us now start the description of the construction, by Deligne, of the representation  $V_\lambda$  as in Theorem 2.4.1 above in the case where  $k \geq 2$ . First, if  $n < 5$ , we replace  $n$  by say  $5n$  and  $f$  by a normalised Hecke eigenform in the 2-dimensional  $\mathbb{C}$ -vector space generated by  $f(q)$  and  $f(q^5)$ . Then  $f$  is no longer a newform, but it is an eigenform, which will be good enough, and as  $n \geq 5$  we can view it as a section of the line bundle  $\omega^{\otimes k}(-\text{Cusps})$  on the smooth complex projective curve  $X_1(n)$ . The eigenvalues at primes other than 5 have not been changed by this operation. As one can compute from the formulas in the previous section, the two possible eigenvalues for  $T_5$  on the space generated by  $f(q)$  and  $f(q^5)$  are the two roots of the polynomial  $x^2 - a_5(f)x + \varepsilon(5)5^{k-1}$ , i.e., the two eigenvalues of the Frobenius element at 5 attached to  $f$  if  $\lambda$  does not divide 5. For a detailed computation for this,

see Section 4 of [Co-Ed]; that article also explains why one should expect the two eigenvalues always to be distinct, and that this is a theorem if  $k = 2$ .

On  $Y_1(n)$ , we have a universal family  $(\mathbb{E}/Y_1(n), \mathbb{P})$  of elliptic curves with a given point of order  $n$ . Taking fibre-wise the cohomology  $H^1(\mathbb{E}_s, \mathbb{Z})$  gives us a locally constant sheaf on  $Y_1(n)$ , denoted  $R^1p_*\mathbb{Z}_{\mathbb{E}}$  because it is the first higher direct image of the constant sheaf  $\mathbb{Z}_{\mathbb{E}}$  on  $\mathbb{E}$  via the morphism  $p: \mathbb{E} \rightarrow Y_1(n)$ . The stalks of the locally constant sheaf  $R^1p_*\mathbb{Z}_{\mathbb{E}}$  on  $Y_1(n)$  are free  $\mathbb{Z}$ -modules of rank 2. More concretely, the sheaf  $R^1p_*\mathbb{Z}_{\mathbb{E}}$  is obtained from the constant sheaf  $\mathbb{Z}^2$  on  $\mathbb{H}$  by dividing out the  $\Gamma_1(n)$ -action given by:

$$(2.4.2) \quad \gamma \cdot \left( \begin{pmatrix} n \\ m \end{pmatrix}, \tau \right) = \left( \gamma \cdot \begin{pmatrix} n \\ m \end{pmatrix}, \gamma \cdot \tau \right) = \left( \begin{pmatrix} an + bm \\ cn + dm \end{pmatrix}, \frac{a\tau + b}{c\tau + d} \right),$$

where  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .

We will also use other locally constant sheaves on  $Y_1(n)$  that are obtained from  $R^1p_*\mathbb{Z}_{\mathbb{E}}$  by tensor constructions. The classification of the irreducible representations of the algebraic group  $GL_2$  over  $\mathbb{Q}$  implies that these tensor constructions are finite direct sums in which each term is a symmetric power of  $R^1p_*\mathbb{Z}_{\mathbb{E}}$ , tensored with a power of the determinant of  $R^1p_*\mathbb{Z}_{\mathbb{E}}$ . We define:

$$(2.4.3) \quad \mathcal{F}_k := \text{Sym}^{k-2}(R^1p_*\mathbb{Z}_{\mathbb{E}}),$$

where  $\text{Sym}^{k-2}$  denotes the operation of taking the  $(k-2)$ th symmetric power. The sheaf  $\mathcal{F}_k$  is then obtained by dividing out the  $\Gamma_1(n)$ -action on the constant sheaf  $\text{Sym}^{k-2}(\mathbb{Z}^2)$  on  $\mathbb{H}$ . It is useful to view  $\mathbb{Z}^2$  as the  $\mathbb{Z}$ -submodule  $\mathbb{Z}x \oplus \mathbb{Z}y$  of the polynomial ring  $\mathbb{Z}[x, y]$ . The grading  $\mathbb{Z}[x, y] = \bigoplus_i \mathbb{Z}[x, y]_i$  by the degree then gives the symmetric powers of  $\mathbb{Z}x \oplus \mathbb{Z}y$ :

$$(2.4.4) \quad \text{Sym}^{k-2}(\mathbb{Z}^2) = \mathbb{Z}[x, y]_{k-2} = \bigoplus_{i+j=k-2} \mathbb{Z}x^i y^j.$$

We extend the sheaf  $\mathcal{F}_k$  to  $X_1(n)$  by taking the direct image via the open immersion  $j: Y_1(n) \rightarrow X_1(n)$ ; this gives us  $j_*\mathcal{F}_k$  on  $X_1(n)$ , again denoted  $\mathcal{F}_k$ . Outside the cusps,  $\mathcal{F}_k$  is locally constant, with stalks free of rank  $k-1$  as  $\mathbb{Z}$ -modules. At the cusps, the stalks of  $\mathcal{F}_k$  are free of rank one. At the cusp  $\infty$  this follows from the fact that the subring of invariants of

$\mathbb{Z}[x, y]$  for the action of  $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$  is  $\mathbb{Z}[x]$ . At the other cusps it then follows by conjugating with a suitable element of  $\mathrm{SL}_2(\mathbb{Z})$ . We note that for  $k = 2$  the sheaf  $\mathcal{F}_k$  is the constant sheaf  $\mathbb{Z}$  on  $X_1(n)$ .

The *Eichler-Shimura isomorphism* gives a relation between modular forms and the cohomology of  $\mathcal{F}_k$ . One way to view this, due to Deligne, is in terms of Hodge structures. More precisely, the  $\mathbb{C}$ -vector space  $\mathbb{C} \otimes H^1(X_1(n), \mathcal{F}_k)$  carries a Hodge decomposition:

$$(2.4.5) \quad \mathbb{C} \otimes H^1(X_1(n), \mathcal{F}_k) \xrightarrow{\sim} S_k(\Gamma_1(n)) \oplus \overline{S_k(\Gamma_1(n))},$$

where the two terms on the right are of type  $(k-1, 0)$  and  $(0, k-1)$ , respectively. The complex conjugation on the second term on the right comes from the complex conjugation on the tensor factor  $\mathbb{C}$  on the left. A good reference for this decomposition and its properties is [Ba-Ne]; we will not go into details here. For an account using group cohomology we refer to Section 12.2 of [Di-Im]. For  $k = 2$  all of this is quite easy. Via the Kodaira-Spencer isomorphism (2.2.20) it then is the decomposition:

$$(2.4.6) \quad H^1(X_1(n), \mathbb{C}) = H^0(X_1(n), \Omega^1) \oplus \overline{H^0(X_1(n), \Omega^1)}.$$

We should mention that instead of working with the sheaf  $\mathcal{F}_k$  on the curve  $X_1(n)$ , one can also work with a constant sheaf on a  $k-1$ -dimensional variety. As before, we let  $(\mathbb{E}, \mathbb{P})$  denote the universal object over  $Y_1(n)$ . Then we let  $\mathbb{E}^{k-2}$  denote the  $k-2$ -fold fibre power of  $\mathbb{E}$  over  $Y_1(n)$ ; these are the simplest cases of so-called Kuga-Sato varieties. The graded commutative algebra structure on cohomology gives, for  $s$  in  $Y_1(n)$ , a map, equivariant for the action of the symmetric group  $S_{k-2}$ :

$$\mathbb{Z}(\varepsilon) \otimes H^1(\mathbb{E}_s, \mathbb{Z}) \otimes \cdots \otimes H^1(\mathbb{E}_s, \mathbb{Z}) \longrightarrow H^{k-2}(\mathbb{E}_s^{k-2}, \mathbb{Z}),$$

where  $\mathbb{Z}(\varepsilon)$  denotes the sign representation. Twisting this map by  $\mathbb{Z}(\varepsilon)$  and taking co-invariants gives a map:

$$(2.4.7) \quad \mathcal{F}_{k,s} = \mathrm{Sym}^{k-2}(H^1(\mathbb{E}_s, \mathbb{Z})) \longrightarrow H^{k-2}(\mathbb{E}_s^{k-2}, \mathbb{Z})_{\varepsilon},$$

where the subscript  $\varepsilon$  means the largest quotient on which  $S_{k-2}$  acts via the sign representation. In view of the Leray spectral sequence for the cohomology  $H(\mathbb{E}^{k-2}, \mathbb{Z})$  of  $\mathbb{E}^{k-2}$  in terms of the cohomology of the higher derived direct images  $H(Y_1(n), R p_* \mathbb{Z}_{\mathbb{E}^{k-2}})$  it is then not so surprising that

$S_k(\Gamma_1(n))$  can be identified with a piece of  $H^{k-1}(\overline{\mathbb{E}^{k-2}}, \mathbb{C})$ , where  $\overline{\mathbb{E}^{k-2}}$  is a certain smooth projective model of  $\mathbb{E}^{k-2}$  over  $X_1(n)$ . Some details for this can be found in [Del1], and more of them in [Sch1], and probably still more in [Conr]. A very explicit way to describe this identification is the map:

$$(2.4.8) \quad \begin{aligned} S_k(\Gamma_1(n)) &\longrightarrow H^{k-1}(\overline{\mathbb{E}^{k-2}}, \mathbb{C}), \\ f &\mapsto (2\pi i)^{k-1} f d\tau dz_1 \cdots dz_{k-2}, \end{aligned}$$

where  $\tau$  is the coordinate on  $\mathbb{H}$ , and the  $z_j$  are the coordinates on the copies of  $\mathbb{C}$  using  $\mathbb{E}_\tau = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ . It is indeed easy to verify that the differential form on the right is invariant under the actions of  $\mathbb{Z}^{2(k-2)}$  and  $\mathrm{SL}_2(\mathbb{Z})$ , precisely because  $f$  is a modular form of weight  $k$  for  $\Gamma_1(n)$ . The claim (proved in the references above) is that it extends without poles over  $\overline{\mathbb{E}^{k-2}}$ . As it is a holomorphic form of top-degree, it is automatically closed, and hence defines a class in the de Rham cohomology of  $\overline{\mathbb{E}^{k-2}}$ , hence in  $H^{k-1}(\overline{\mathbb{E}^{k-2}}, \mathbb{C})$ .

There are natural Hecke correspondences on  $\mathbb{C} \otimes H^1(X_1(n), \mathcal{F}_k)$  and on  $H^{k-1}(\overline{\mathbb{E}^{k-2}}, \mathbb{C})$ , and the identification of  $S_k(\Gamma_1(n))$  as a piece of these cohomology groups is compatible with these correspondences. Let now  $f$  be our eigenform in  $S_k(\Gamma_1(n))$  as above. Then the Hecke eigenspace in  $\mathbb{C} \otimes H^1(X_1(n), \mathcal{F}_k)$  with the eigenvalues  $a_m(f)$  for  $T_m$  is two-dimensional: the sum of the one-dimensional subspace  $\mathbb{C}f$  in  $S_k(\Gamma_1(n))$  and the one-dimensional subspace  $\mathbb{C}\overline{f'}$  in  $\overline{S_k(\Gamma_1(n))}$ , where  $f' = \sum_{m \geq 1} \overline{a_m(f)} q^m$ , the Galois conjugate of  $f$  obtained by letting complex conjugation act on the coefficients of  $f$ . This element  $f'$  has eigenvalue  $\overline{a_m(f)}$  for  $T_m$ , hence  $\overline{f'}$  has eigenvalue  $a_m(f)$  again. The  $(k-1)$ -form corresponding to  $\overline{f'}$  is  $\overline{f'} d\overline{\tau} d\overline{z_1} \cdots d\overline{z_{k-2}}$ , indeed a form of type  $(0, k-1)$ .

We let  $\mathbb{T}(n, k)$  or just  $\mathbb{T}$  denote the  $\mathbb{Z}$ -algebra in  $\mathrm{End}_{\mathbb{C}}(S_k(\Gamma_1(n)))$  generated by the  $T_m$  ( $m \geq 1$ ) and the  $\langle a \rangle$  ( $a$  in  $(\mathbb{Z}/n\mathbb{Z})^\times$ ). The fact that the Eichler-Shimura isomorphism (2.4.5) is equivariant for the Hecke correspondences acting on both sides implies that the image of  $H^1(X_1(n), \mathcal{F}_k)$  in  $\mathbb{C} \otimes H^1(X_1(n), \mathcal{F}_k)$  is a faithful  $\mathbb{T}(n, k)$ -module. As this image is free of finite rank as  $\mathbb{Z}$ -module,  $\mathbb{T}(n, k)$  is free of finite rank as  $\mathbb{Z}$ -module.

Let us for a moment drop the assumption that  $n \geq 5$ . For  $A$  a subring of  $\mathbb{C}$  and for  $n \geq 1$ , we let  $M_k(\Gamma_1(n), A)$  be the sub- $A$ -module of  $M_k(\Gamma_1(n))$  consisting of elements  $g$  such that  $a_m(g) \in A$  for all  $m \geq 0$ .

In particular,  $M_k(\Gamma_1(n), \mathbb{Z})$  is the submodule of forms whose  $q$ -expansion has all its coefficients in  $\mathbb{Z}$ . Similarly, for  $A$  a subring of  $\mathbb{C}$  and for  $n \geq 1$ , we let  $S_k(\Gamma_1(n), A)$  be the sub- $A$ -module of  $S_k(\Gamma_1(n))$  consisting of elements  $g$  such that  $a_m(g) \in A$  for all  $m \geq 1$ . For example,  $\Delta$  belongs to  $S_{12}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$ . The  $S_k(\Gamma_1(n), A)$  are  $\mathbb{T}(n, k)$ -submodules of  $S_k(\Gamma_1(n))$ ; see Propositions 12.3.11 and 12.4.1 of [Di-Im].

We have the following pairing between  $\mathbb{T}(n, k)$  and  $S_k(\Gamma_1(n), \mathbb{Z})$ :

$$(2.4.9) \quad S_k(\Gamma_1(n), \mathbb{Z}) \times \mathbb{T}(n, k) \longrightarrow \mathbb{Z}, \quad (g, t) \mapsto a_1(tg).$$

This pairing is perfect, in the sense that it identifies each side with the  $\mathbb{Z}$ -linear dual of the other; this follows easily from the identity (2.2.12). It follows that the  $\mathbb{Z}$ -dual  $S_k(\Gamma_1(n), \mathbb{Z})^\vee$  of  $S_k(\Gamma_1(n), \mathbb{Z})$  is free of rank one as  $\mathbb{T}(n, k)$ -module. See [Di-Im, 12.4.13].

For any  $\mathbb{Z}$ -algebra  $A$  we let  $\mathbb{T}_A = \mathbb{T}(n, k)_A$  denote  $A \otimes \mathbb{T}(n, k)$ , and  $\mathbb{T}_A^\vee$  will denote the  $A$ -linear dual of  $\mathbb{T}_A$ . It can be proved that  $\mathbb{T}_\mathbb{Q}^\vee$  is free of rank one as  $\mathbb{T}_\mathbb{Q}$ -module, i.e., that  $\mathbb{T}_\mathbb{Q}$  is *Gorenstein*. One proof is by explicit computation, see Theorem 3.5 and Corollary 3.6 of [Par]. Another, more conceptual proof, uses the Petersson inner product, and a so-called Atkin-Lehner pseudo-involution  $w_{\zeta_n}$ , to show that  $S_k(\Gamma_1(n))^\vee$  is isomorphic as  $\mathbb{T}_\mathbb{C}$ -module to  $S_k(\Gamma_1(n))$  itself; see [Di-Im, 12.4.14]. It follows that  $S_k(\Gamma_1(n))$  is free of rank one as  $\mathbb{T}_\mathbb{C}$ -module, and, if  $n \geq 5$ , that  $\mathbb{Q} \otimes H^1(X_1(n), \mathcal{F}_k)$  and its dual  $\mathbb{Q} \otimes H^1(X_1(n), \mathcal{F}_k)^\vee$  are free of rank two as  $\mathbb{T}_\mathbb{Q}$ -module. It is this freeness result that will lead to the fact that the Galois representations we get are two-dimensional.

We assume again that  $n \geq 5$ . The step from the cohomological interpretation of modular forms, given, over the complex numbers, by the Eichler-Shimura isomorphism (2.4.5), to two-dimensional  $l$ -adic Galois representations is made by comparing the cohomology groups above to their  $l$ -adic counterparts for the étale topology, and noting that  $p: \mathbb{E} \rightarrow X_1(n)$  is naturally defined over  $\mathbb{Z}[1/n]$  as we have seen at the end of Section 2.1. From now on we will denote by  $X_1(n)$  this model over  $\mathbb{Z}[1/n]$ , and by  $X_1(n)(\mathbb{C})$  the Riemann surface given by  $X_1(n)$ . For any  $\mathbb{Z}[1/n]$ -algebra  $A$ ,  $X_1(n)_A$  will denote the  $A$ -scheme obtained from  $X_1(n)$  by extending scalars via  $\mathbb{Z}[1/n] \rightarrow A$ .



We let  $\mathcal{F}_{k,l}$  denote the sheaf of  $\mathbb{Q}_l$ -vector spaces  $\mathbb{Q}_l \otimes \mathcal{F}_k$  on  $X_1(n)$ . Then we have a canonical isomorphism:

$$(2.4.10) \quad H^1(X_1(n)(\mathbb{C}), \mathcal{F}_{k,l}) = \mathbb{Q}_l \otimes H^1(X_1(n)(\mathbb{C}), \mathcal{F}_k).$$

The sheaves  $\mathcal{F}_{k,l}$  can also be constructed on the étale site  $X_1(n)_{\text{et}}$ , by taking the first derived direct image of the constant sheaf  $\mathbb{Q}_l$  on  $\mathbb{E}_{\text{et}}$  under  $p: \mathbb{E} \rightarrow Y_1(n)$ , then the  $(k-2)$ th symmetric power of that and finally the pushforward from  $Y_1(n)$  to  $X_1(n)$ .

The usual comparison theorems (comparing cohomology for étale and Archimedean topology, and étale cohomology over various algebraically closed fields) give:

$$(2.4.11) \quad \begin{aligned} H^1(X_1(n)(\mathbb{C}), \mathcal{F}_{k,l}) &= H^1(X_1(n)_{\mathbb{C},\text{et}}, \mathcal{F}_{k,l}) \\ &= H^1(X_1(n)_{\overline{\mathbb{Q}},\text{et}}, \mathcal{F}_{k,l}). \end{aligned}$$

We put:

$$(2.4.12) \quad W_l := H^1(X_1(n)_{\overline{\mathbb{Q}},\text{et}}, \mathcal{F}_{k,l})^\vee.$$

By the results and the comparisons above,  $W_l$  is, as  $\mathbb{T}_{\mathbb{Q}_l}$ -module, free of rank 2, and  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  acts continuously on it. To be precise: an element  $\sigma$  of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  acts as  $((\text{id} \times \text{Spec}(\sigma^{-1}))^*)^\vee$ , which is indeed covariant in  $\sigma$ . The fact that the Hecke correspondences exist over  $\mathbb{Q}$  makes that the  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -action on  $W_l$  commutes with the Hecke operators. The choice of a  $\mathbb{T}_{\mathbb{Q}_l}$ -basis of  $W_l$  gives us a representation:

$$(2.4.13) \quad \rho_l: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GL}_2(\mathbb{T}_{\mathbb{Q}_l}).$$

Recall that we have fixed an eigenform  $f$  in  $S_k(\Gamma_1(n), \mathbb{C})$ . Sending a Hecke operator to its eigenvalue for  $f$  then gives us a morphism of rings:

$$(2.4.14) \quad \phi_f: \mathbb{T} \longrightarrow \mathbb{C}.$$

We let  $K(f)$  be the image of  $\mathbb{T}_{\mathbb{Q}}$  under  $\phi_f$ ; it is the finite extension of  $\mathbb{Q}$  obtained by adjoining all coefficients  $a_m(f)$  of the  $q$ -expansion of  $f$ . We now view  $\phi_f$  as a morphism from  $\mathbb{T}$  to  $K(f)$ . The tensor product  $\mathbb{Q}_l \otimes K(f)$  is the product of the completions  $K(f)_\lambda$ , with  $\lambda$  ranging through the finite

places of  $K(f)$  that divide  $l$ . For each such  $\lambda$  we then get a morphism  $\phi_{f,\lambda}: \mathbb{T}_{\mathbb{Q}_l} \rightarrow K(f)_\lambda$ , and a representation:

$$(2.4.15) \quad \rho_{f,\lambda}: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GL}_2(K(f)_\lambda).$$

These are the representations mentioned in Theorem 2.4.1. It may be useful to note that the space on which the representation is realised is:

$$(2.4.16) \quad V_{f,\lambda} := K(f)_\lambda \otimes_{\mathbb{T}_{\mathbb{Q}_l}} W_l.$$

The representations  $\rho_{f,\lambda}$  are continuous by construction. The sheaves  $\mathcal{F}_{k,l}$  on  $X_1(n)_{\mathbb{Z}[1/nl]}$  are “lisse” away from the cusps, and tamely ramified at the cusps, hence, by Proposition 2.1.9 of [SGA7, XIII, §2],  $\rho_{f,\lambda}$  is unramified at all  $p$  not dividing  $nl$ .

In the case where  $k = 2$  the construction of the  $\rho_{f,\lambda}$  is much simpler, because then the sheaf  $\mathcal{F}_k$  is the constant sheaf  $\mathbb{Z}$  on  $X_1(n)(\mathbb{C})$ . The use of étale cohomology can then be replaced by Tate modules of the Jacobian variety of  $X_1(n)$ . We let  $J := J_1(n)$  be this Jacobian variety, actually an Abelian scheme over  $\mathbb{Z}[1/n]$ . Then we have:

$$(2.4.17) \quad W_l = \mathbb{Q} \otimes \varprojlim_m J(\overline{\mathbb{Q}})[l^m].$$

The fact that for  $p$  a prime not dividing  $nl$  the characteristic polynomial of  $\rho_{f,\lambda}(\text{Frob}_p)$  is as stated in Theorem 2.4.1 is obtained by studying the reduction modulo  $p$  of the Hecke correspondence  $T_p$ , i.e., as a correspondence on  $X_1(nl)_{\mathbb{F}_p}$ , compatibly with the sheaf  $\mathcal{F}_{k,l}$ . For details we refer to Conrad’s book [Conr] and Deligne’s article [Del1]. In the case  $k = 2$  this result is known as the *Eichler-Shimura congruence relation*, expressing the endomorphism  $T_p$  of  $J_{\mathbb{F}_p}$  as  $F + \langle p \rangle V$ , where  $F$  denotes the Frobenius endomorphism, and  $V$  its dual, i.e, the endomorphism satisfying  $FV = p = VF$  in  $\text{End}(J_{\mathbb{F}_p})$ . For details in the case  $k = 2$  we refer to Section 12.5 of [Di-Im].

Now that we have sketched the construction of the  $l$ -adic Galois representations attached to modular forms, we mention some more of their properties, that are not mentioned in Theorem 2.4.1 and in the remarks directly following that theorem.

The fact that Deligne proved the Riemann hypothesis part of Weil’s conjectures in [Del2] implies very precise bounds on the coefficients of

modular forms. The reason for that is that the roots of the equation  $x^2 - a_p(f)x + \varepsilon_f(p)p^{k-1}$  are eigenvalues of the Frobenius at  $p$  on the space  $H^{k-1}(\overline{\mathbb{E}^{k-2}}_{\mathbb{F}_p, \text{et}}, \mathbb{Q}_l)$ . We state these bounds, called *Ramanujan bounds*, in a theorem, due to Deligne in the case  $k \geq 2$ , and to Deligne-Serre ([De-Se]) in the case  $k = 1$ .

**2.4.18 Theorem** *Let  $f$  be a normalised newform, let  $n$  be its level and  $k$  its weight. Then for  $p$  not dividing  $n$ , we have:*

$$(2.4.19) \quad |a_p(f)| \leq 2 \cdot p^{(k-1)/2}.$$

A slightly weaker result than in the theorem above, stating that, for a given  $f$  as above,  $|a_m(f)| = O(m^{k/2})$ , can be obtained in a very elementary way; see [Miy, Cor. 2.1.6] (the idea is to use that the function  $z \mapsto |f(z)|(\Im(z))^k$  is bounded on  $\mathbb{H}$  and to view  $a_m(f)$  as a residue).

Theorem 2.4.1 gives us information on the restriction  $\rho_{f,\lambda,p}$  of  $\rho_{f,\lambda}$  to decomposition groups  $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$  for  $p$  not dividing  $nl$ . Namely, the theorem says that such restrictions  $\rho_{f,\lambda,p}$  are unramified, and it gives the eigenvalues of  $\rho_{f,\lambda,p}(\text{Frob}_p)$ . Unfortunately, it is not known if  $\rho_{f,\lambda,p}(\text{Frob}_p)$  is semi-simple; see [Co-Ed] for information on this.

We should note that also in the case that  $p$  divides  $nl$  almost everything is known about  $\rho_{f,\lambda,p}$ . For  $p$  not dividing  $l$ , this is the very general statement that the ‘‘Frobenius semi-simplification’’ of  $\rho_{f,\lambda,p}$  corresponds, via a suitably normalised local Langlands correspondence, to a certain representation  $\pi_{f,p}$  of  $\text{GL}_2(\mathbb{Q}_p)$  attached to  $f$ . This result is due, in increasing order of generality, to Langlands, Deligne, and Carayol. For details on this the reader is referred to [Car], which gives this result in the more general context of Hilbert modular forms (i.e.,  $\mathbb{Q}$  is replaced by a totally real number field). The result for  $p = l$  uses Fontaine’s  $p$ -adic Hodge theory, and is due to Saito ([Sai1] for the case of modular forms, and [Sai2] for the case of Hilbert modular forms).

## 2.5 GALOIS REPRESENTATIONS OVER FINITE FIELDS, AND REDUCTION TO TORSION IN JACOBIANS

We start this section by explaining how to pass from  $l$ -adic Galois representations to Galois representations over finite fields.

Let  $f = \sum a_m q^m$  be a (complex) normalised cuspidal eigenform for all Hecke operators  $T_m$ ,  $m \geq 1$ , of some level  $n \geq 1$  and of some weight  $k \geq 2$ . As in Theorem 2.4.1 we have the Galois representations  $\rho_{f,\lambda}$ , from  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  to  $\text{GL}_2(\overline{\mathbb{Q}}_l)$ . It follows from the construction of those representations that there is a finite subextension  $\mathbb{Q}_l \rightarrow E$  of  $\mathbb{Q}_l \rightarrow \overline{\mathbb{Q}}_l$  such that  $\rho_{f,\lambda}$  takes its values in  $\text{GL}_2(E)$ . (Actually, this can also be deduced from the continuity alone; see the proof of Corollary 5 in [Dic1] for an argument.) The question as to what the smallest possible  $E$  is can be easily answered. Such an  $E$  must contain the traces  $a_p(f)$  of the  $\rho_{f,\lambda}(\text{Frob}_p)$  for all  $p$  not dividing  $nl$ . So let  $K$  be the extension of  $\mathbb{Q}$  generated by the  $a_p(f)$  with  $p$  not dividing  $n$ , i.e.,  $K$  is the field of definition of the newform corresponding to  $f$ . Then  $E$  can be taken to be  $K_\lambda$ , the  $l$ -adic completion of  $K$  specified by the embedding  $\lambda$  of  $K$  into  $\overline{\mathbb{Q}}_l$  (see Section 12.5 in [Di-Im]).

Let now  $\rho_{f,\lambda}: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(E)$  be a realisation of  $\rho_{f,\lambda}$  over  $E$  as above. As  $\rho_{f,\lambda}$  is semisimple (it is even irreducible), such a realisation is unique up to isomorphism (because it is determined by the traces). Let  $O_E$  be the ring of integers in  $E$ , i.e., the integral closure of  $\mathbb{Z}_l$  in  $E$ . As  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  is compact, it stabilises some  $O_E$  lattice in  $E^2$  (in the set of lattices, the orbits under  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  are finite, take the intersection, or the sum, of the lattices in one orbit). This means that, after suitable conjugation (choose such a lattice, and an  $O_E$ -basis of it),  $\rho_{f,\lambda}$  takes values in  $\text{GL}_2(O_E)$ . We let  $O_E \rightarrow \overline{\mathbb{F}}_l$  denote the morphism induced by the given embedding of  $E$  into  $\overline{\mathbb{Q}}_l$  (we view  $\overline{\mathbb{F}}_l$  as the residue field of the subring of integers  $\overline{\mathbb{Z}}_l$  of  $\overline{\mathbb{Q}}_l$ ). We can then define the *residual* Galois representation  $\overline{\rho}_{f,\lambda}$  to be the *semi-simplification* of the composed representation  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(O_E) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_l)$ . Another choice of  $E$  or of lattice or basis leads to an isomorphic  $\overline{\rho}_{f,\lambda}$ , but we note that without the operation of semi-simplification this would not be true (see Chapter III of [Ser7]).

Given  $f$ , all but finitely many of the  $\overline{\rho}_{f,\lambda}$  are irreducible. This was proved for  $f$  of level one and with coefficients in  $\mathbb{Z}$  in Theorem 4 of [Swi]. The

general case follows easily from Theorem 2.3 of [Fa-Jo], which says that if  $\bar{\rho}_{f,\lambda}$  is reducible with  $l > k$  not dividing  $n$ , then  $\bar{\rho}_{f,\lambda} = \alpha \oplus \beta \bar{\chi}_l^{k-1}$  with  $\alpha$  and  $\beta$  unramified outside  $n$ , and  $\bar{\chi}_l: \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{F}_l^\times$  the mod  $l$  cyclotomic character. Moreover, the proof shows that the set of  $l$  such that some  $\bar{\rho}_{f,\lambda}$  is reducible can be bounded explicitly.

The next question that we want to answer is the following: what is the smallest subfield of  $\bar{\mathbb{F}}_l$  over which  $\bar{\rho}_{f,\lambda}$  can be realised? Just as for  $\rho_{f,\lambda}$  itself, that subfield must contain the traces of the  $\bar{\rho}_{f,\lambda}(\text{Frob}_p)$  for all  $p$  not dividing  $nl$ . That condition turns out to be sufficient, as we will now show. So we let, in this paragraph,  $\mathbb{F}$  be the subfield of  $\bar{\mathbb{F}}_l$  that is generated by the images  $\overline{a_p(f)}$  in  $\bar{\mathbb{F}}_l$  of the  $a_p(f)$  in  $\bar{\mathbb{Z}}_l$ . Then for any  $\sigma$  in  $\text{Gal}(\bar{\mathbb{F}}_l/\mathbb{F})$  the conjugate  $\bar{\rho}_{f,\lambda}^\sigma$  of  $\bar{\rho}_{f,\lambda}$  and  $\bar{\rho}_{f,\lambda}$  itself are both semisimple and give the same characteristic polynomials as functions on  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ . Therefore, by a theorem of Brauer-Nesbitt (see Theorem 30.16 of [Cu-Re]),  $\bar{\rho}_{f,\lambda}$  is isomorphic to all its conjugates over  $\mathbb{F}$ . (A more general statement of this kind is given in Exercise 1 of Section 18.2 of [Ser7].) The fact that  $\text{Gal}(\bar{\mathbb{F}}_l/\mathbb{F})$  is equal to  $\hat{\mathbb{Z}}$  then implies that  $\bar{\rho}_{f,\lambda}$  can be realised over  $\mathbb{F}$ . Let us give an argument for that in terms of matrices, although a much more conceptual argument would be to say that a “gerbe over a finite field is trivial”. Let  $\sigma$  be the Frobenius element of  $\text{Gal}(\bar{\mathbb{F}}_l/\mathbb{F})$ , and let  $s$  be an element of  $\text{GL}_2(\bar{\mathbb{F}}_l)$  such that for all  $g$  in the image of  $\bar{\rho}_{f,\lambda}$  we have  $\sigma(g) = sgs^{-1}$ . Then take a  $t$  in  $\text{GL}_2(\bar{\mathbb{F}}_l)$  such that  $s = \sigma(t)^{-1}t$ . Then all  $tgt^{-1}$  are in  $\text{GL}_2(\mathbb{F})$ . By Brauer-Nesbitt, the realisation over  $\mathbb{F}$  is unique.

For a discussion on possible images of  $\bar{\rho}_{f,\lambda}$  we refer the reader to the introduction of [Ki-Ve] (we note however that for  $f$  a “CM-form”, i.e., a form for which all  $l$ -adic Galois representations have dihedral image) infinitely many of the  $\bar{\rho}_{f,\lambda}$  can have dihedral image in  $\text{PGL}_2(\bar{\mathbb{F}}_l)$ ). In particular, Theorem 2.1 of [Rib3] states that for  $f$  not a CM-form only finitely many of the images of the  $\bar{\rho}_{f,\lambda}$  are exceptional in the sense that they are of order prime to  $l$ . See also Theorem 2.5.18 for the case where  $n = 1$ . For  $l > 3$  such that  $\bar{\rho}_{f,\lambda}$  is irreducible and not exceptional, a result of Dickson, see Chapter XII of [Dic2], or the proof of Theorem 2.5 of [Rib4], says that the image of  $\bar{\rho}_{f,\lambda}$  in  $\text{PGL}_2(\bar{\mathbb{F}}_l)$  is, after suitable conjugation, equal to  $\text{PGL}_2(\mathbb{F})$  or  $\text{SL}_2(\mathbb{F})/\{1, -1\}$  for some finite extension  $\mathbb{F}$  of  $\mathbb{F}_l$ . We note that this field  $\mathbb{F}$  can be smaller than the extension  $\mathbb{F}_l$  generated by the traces of  $\bar{\rho}_{f,\lambda}$

(indeed, twisting does not change the projective image, but it can make the field generated by the traces bigger).

We will use later on the following lemma.

**2.5.1 Lemma** *Let  $l$  be a prime number, let  $V$  be a two-dimensional  $\mathbb{F}_l$ -vector space, and let  $G$  be a subgroup of  $\text{Aut}(V)$  of order a multiple of  $l$ , and such that  $V$  is irreducible as a representation of  $G$ . Then  $G$  contains  $\text{SL}(V)$ , and acts transitively on  $V - \{0\}$ .*

**Proof** Let  $g_1$  be an element of  $G$  of order  $l$ . Then the kernel  $L_1$  of  $g_1 - \text{id}_V$  is a line. As  $V$  is irreducible,  $L_1$  is not  $G$ -invariant, hence we can take an element  $h$  in  $G$  such that  $L_2 := hL_1$  is not  $L_1$ . Then  $g_2 := hg_1h^{-1}$  is of order  $l$  and fixes  $L_2$ . Let  $e_1$  and  $e_2$  be non-zero elements of  $L_1$  and  $L_2$ , respectively. Then with respect to the basis  $e = (e_1, e_2)$  of  $V$ ,  $g_1$  and  $g_2$  are given by elementary matrices of the form  $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ , respectively, with  $a$  and  $b$  non-zero, and hence generate  $\text{SL}(V)$ .  $\square$

It follows that if  $\bar{\rho}_{f,\lambda}$  takes values in  $\text{GL}_2(\mathbb{F}_l)$ , and is irreducible and not exceptional, then  $\text{im}\bar{\rho}_{f,\lambda}$  contains  $\text{SL}_2(\mathbb{F}_l)$ , and therefore is the subgroup of elements of  $\text{GL}_2(\mathbb{F}_l)$  whose determinant is in the image of the character  $\bar{\varepsilon}_f \cdot \chi_l^{k-1}$ . In that case,  $\text{im}\bar{\rho}_\lambda$  acts transitively on  $\mathbb{F}_l^2 - \{0\}$ .

The properties of residual Galois representations that we have seen above show that we do not need to define them via  $l$ -adic Galois representations, but that we can start from maximal ideals in Hecke algebras.

**2.5.2 Theorem** *Let  $n$  and  $k$  be positive integers. Let  $\mathbb{F}$  be a finite field, and  $f: \mathbb{T}(n, k) \rightarrow \mathbb{F}$  a surjective morphism of rings. Then there is a continuous semi-simple representation:*

$$\rho_f: \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GL}_2(\mathbb{F})$$

that is unramified outside  $nl$ , where  $l$  is the characteristic of  $\mathbb{F}$ , such that for all  $p$  not dividing  $nl$  we have, in  $\mathbb{F}$ :

$$\text{trace}(\rho(\text{Frob}_p)) = f(T_p) \quad \text{and} \quad \det(\rho(\text{Frob}_p)) = f(\langle p \rangle)p^{k-1}.$$

Such a  $\rho_f$  is unique up to isomorphism (i.e., up to conjugation).

**Proof** Let  $n, k, \mathbb{F}$  and  $f$  be given. As  $\mathbb{T} := \mathbb{T}(n, k)$  is free of finite rank as  $\mathbb{Z}$ -module,  $\text{Spec}(\mathbb{T})$  has only finitely many irreducible components, each of which is one-dimensional and finite over  $\text{Spec}(\mathbb{Z})$ . Therefore, the maximal ideal  $\ker(f)$  of  $\mathbb{T}_{\mathbb{F}_l}$  is the specialisation of a maximal ideal  $m$  of  $\mathbb{T}_{\mathbb{Q}}$ . Let  $K$  be the quotient  $\mathbb{T}_{\mathbb{Q}}/m$ . Then the quotient morphism  $\mathbb{T}_{\mathbb{Q}} \rightarrow K$  is a normalised eigenform  $\tilde{f}$  in  $S_k(\Gamma_1(n))_K$ , and  $\rho_f$  is the realisation over  $\mathbb{F}$  of the reduction of some  $\rho_{\tilde{f}, \lambda}$ .  $\square$

Let now  $f$  be as in Theorem 2.5.2, and let us suppose now that  $\rho_f$  is irreducible. The construction of  $l$ -adic Galois representations that we recalled in Section 2.4 implies that the dual of  $\rho_f$  occurs in  $H^{k-1}(\mathbb{E}^{k-2}_{\mathbb{Q}, \text{et}}, \mathbb{F}_l)$ , as well as in  $H^1(X_1(n)_{\overline{\mathbb{Q}}, \text{et}}, \overline{\mathcal{F}}_{k,l})$ , where  $\overline{\mathcal{F}}_{k,l}$  is defined as  $\mathcal{F}_{k,l}$  but with  $\mathbb{Q}_l$  replaced by  $\mathbb{F}_l$ . Let us now assume that  $k > 2$ . Then both these realisations are difficult to deal with computationally. In the first representation the difficulty arises from the degree  $k - 1$  étale cohomology; it seems to be unknown how to deal explicitly with elements of such cohomology groups. In the second representation, the elements of the cohomology group are isomorphism classes of  $\overline{\mathcal{F}}_{k,l}$ -torsors, on  $X_1(n)_{\overline{\mathbb{Q}}, \text{et}}$ . Such torsors can be described explicitly, as certain covers of  $X_1(n)_{\overline{\mathbb{Q}}}$  with certain extra data. The set of such torsors can probably be described by a system of polynomial equations that can be written down in time polynomial in  $nl$  (think of the variables as coefficients of certain equations for the torsors). But the problem is that, apparently, there are no good methods known to solve these systems of equations (the number of variables grows too fast with  $l$  and the equations are not linear). In fact, the *satisfiability problem* SAT, which is known to be NP-complete (Cook's theorem, see for example [Mor1], or wikipedia), is a special case of the problem of deciding whether or not a polynomial system of equations over  $\mathbb{F}_2$  has a solution over  $\mathbb{F}_2$ . We note that the description of the set of torsors by a system of polynomial equations should also work over suitable finite extensions of finite fields  $\mathbb{F}_p$ , in time polynomial in  $l \log p$ .

Another place where  $\rho_f$  occurs is in  $J_1(nl)(\overline{\mathbb{Q}})[l]$ , i.e., in the  $l$ -torsion of the Jacobian of the modular curve with level  $nl$ , if  $l + 1 \geq k$  and  $l \nmid n$ . This means that at the cost of increasing the level by a factor  $l$ , we are reduced to dealing with torsion points on Abelian varieties. Of course, the  $l$ -adic

representations  $\rho_{\tilde{f},\lambda}$  attached to lifts of  $f$  do not occur in the Jacobian of any curve, simply because the Frobenius eigenvalues are Weil numbers of the wrong weight. What happens here for  $\rho_f$  is a “mod  $l$  phenomenon” having to do with “congruences” between modular forms. Before we give a detailed statement, let us explain why this happens (such explanations date back at least to the 1960’s; Shimura, Igusa, Serre, . . .).

For simplicity, and only during this explanation, we assume that  $n \geq 5$ . Then we have a universal elliptic curve with a given point of order  $n$  over  $\mathbb{Z}[1/nl]$ -schemes:  $(\mathbb{E}/Y_1(n), \mathbb{P})$ . We let  $p: \mathbb{E} \rightarrow Y_1(n)$  denote the structure morphism. By definition, we have:

$$(2.5.3) \quad \overline{\mathcal{F}}_{k,l} = \mathrm{Sym}^{k-2} R^1 p_* \mathbb{F}_l.$$

As explained at the end of Section 1.3, we have a natural isomorphism:

$$(2.5.4) \quad R^1 p_* \mathbb{F}_l = \mathbb{E}[l]^\vee.$$

And by the definition of  $Y_1(nl)$ , and the Weil pairing, we have an exact sequence on  $Y_1(nl)_{\mathrm{et}}$ :

$$(2.5.5) \quad 0 \longrightarrow \mathbb{F}_l \longrightarrow \mathbb{E}[l] \longrightarrow \mu_l \longrightarrow 0,$$

where  $\mathbb{F}_l$  and  $\mu_l$  denote the corresponding constant sheaves. It follows that the pullback of  $R^1 p_* \mathbb{F}_l$  to  $Y_1(nl)_{\mathrm{et}}$  has a 2-step filtration with successive quotients  $\mathbb{F}_l$  and  $\mu_l^\vee$ . Therefore,  $\overline{\mathcal{F}}_{k,l}$  has a filtration in  $k - 1$  steps, with successive quotients  $\mathbb{F}_l^{\otimes i} \otimes (\mu_l^\vee)^{\otimes j} = \mu_l^{\otimes -j}$ , with  $i + j = k - 2$ ,  $i \geq 0$ ,  $j \geq 0$ . In particular, we get a map:

$$(2.5.6) \quad \begin{aligned} \mathrm{H}^1(X_1(n)_{\overline{\mathbb{Q}},\mathrm{et}}, \overline{\mathcal{F}}_{k,l}) &\longrightarrow \mathrm{H}^1(X_1(nl)_{\overline{\mathbb{Q}},\mathrm{et}}, \overline{\mathcal{F}}_{k,l}) \longrightarrow \\ &\longrightarrow \mathrm{H}^1(X_1(nl)_{\overline{\mathbb{Q}},\mathrm{et}}, \mathbb{F}_l) = J_1(nl)(\overline{\mathbb{Q}})[l]^\vee. \end{aligned}$$

This map explains that  $\rho_f$  is likely to occur in  $J_1(nl)(\overline{\mathbb{Q}})[l]$ . A better way to analyse this map is in fact by studying the direct image of the constant sheaf  $\mathbb{F}_l$  via the map  $X_1(nl) \rightarrow X_1(n)$ . A recent detailed treatment of this method, and precise results can be found in [Wie1].

Another way to show that  $\rho_f$  occurs in  $J_1(nl)(\overline{\mathbb{Q}})[l]$  is to study modular forms mod  $l$  of level  $nl$  and of weight 2. This is more complicated than the modular forms that we have seen before, as it uses the study of the reduction



mod  $l$  of the modular curve  $X_1(nl)$ , which is not smooth. The study of these reductions has its roots in Kronecker's congruence relation. The most complete modern accounts of such material are given in the article [De-Ra] by Deligne and Rapoport and in the book [Ka-Ma] by Katz and Mazur. A construction of  $\rho_f$  in  $J_1(nl)(\overline{\mathbb{Q}})[l]$ , following suggestions from Serre, was given by Gross in [Gro].

We are now in a position to state the following theorem, that, combining Gross's result with a so-called multiplicity one theorem, gives us a useful realisation of  $\rho_f$ . As it is nowadays customary to say, it is a result due to "many people" (mainly Mazur, Ribet, Gross (and Edixhoven for the multiplicity one part)).

**2.5.7 Theorem** *Let  $n$  and  $k$  be positive integers,  $\mathbb{F}$  a finite field and  $l$  its characteristic, and  $f: \mathbb{T}(n, k) \rightarrow \mathbb{F}$  a surjective ring morphism. Assume that  $2 < k \leq l+1$  and that the associated Galois representation  $\rho_f$  from  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  to  $\text{GL}_2(\mathbb{F})$  is absolutely irreducible. Then there is a unique ring morphism  $f_2: \mathbb{T}(nl, 2) \rightarrow \mathbb{F}$  such that for all  $i \geq 1$  one has  $f_2(T_i) = f(T_i)$ . The morphism  $f_2$  is surjective. Let  $m_f = \ker(f_2)$ , and let  $V_f \subset J_1(nl)(\overline{\mathbb{Q}})$  denote the kernel of  $m_f$ , i.e., the  $\mathbb{F}$ -vector space of elements  $x$  in  $J_1(nl)(\overline{\mathbb{Q}})$  such that  $tx = 0$  for all  $t$  in  $m_f$ . Then  $V_f$  is a finite, non-zero, direct sum of copies of  $\rho_f$ . If  $k < l$  then the multiplicity of  $\rho_f$  in  $V_f$  is one, i.e.,  $V_f$  realises  $\rho_f$ . For all  $a \in (\mathbb{Z}/nl\mathbb{Z})^\times$ , one has  $f_2(\langle a \rangle) = f(\langle a \rangle)a^{k-2}$ , where we still denote by  $a$  its images in  $\mathbb{Z}/n\mathbb{Z}$  and in  $\mathbb{F}_l$ .*

**Proof** The existence of  $f_2$  and the statement that  $V_f$  is a successive extension of copies of  $\rho_f$  are given in [Gro] (see his Proposition 11.8). In Section 6 of [Edi1] it is proved, applying results from [Bo-Le-Ri], that  $V_f$  is a direct sum of copies of  $\rho_f$ . Case 1 of Theorem 9.2 of [Edi1] gives the multiplicity one result.  $\square$

**2.5.8 Remark** See [Wie2], Corollary 4.5, for a complete result on the multiplicity one question for weights  $k$  with  $2 \leq k \leq l+1$ . In particular, if  $k = l$  and  $\rho_f$  is unramified at  $l$  and  $\rho_f(\text{Frob}_l)$  is scalar, then this multiplicity is not one.

As we want to describe  $V_f$  explicitly, we will need a bound on the amount

of Hecke operators needed to describe  $\mathbb{T}(nl, 2)$  and its ideal  $m_f$ . We start by quoting a result of Jacob Sturm (see [Stu]).

**2.5.9 Theorem (Sturm)** *Let  $N \geq 1$  be an integer,  $\Gamma$  a subgroup of  $\mathrm{SL}_2(\mathbb{Z})$  containing  $\Gamma(N)$ . Let  $N'$  be the “width” of the cusp  $\infty$  for  $\Gamma$ , i.e., the positive integer defined by  $\Gamma \cap (\frac{1}{0} \frac{\mathbb{Z}}{1}) = (\frac{1}{0} \frac{N'\mathbb{Z}}{1})$ . Let  $f$  be a modular form on  $\Gamma$  of weight  $k$ , with coefficients in a discrete valuation ring  $R$  contained in  $\mathbb{C}$ . Let  $F$  be the residue field of  $R$ , and suppose that the image  $\sum a_n q^{n/N'}$  in  $F[[q^{1/N'}]]$  of the  $q$ -expansion of  $f$  has  $a_n = 0$  for all  $n \leq k[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]/12$ . Then  $a_n = 0$  for all  $n$ , i.e.,  $f$  is congruent to 0 modulo the maximal ideal of  $R$ .*

This result of Sturm gives as a direct consequence a bound for up to where one has to take  $T_i$  so that one gets a system of generators of the Hecke algebra as  $\mathbb{Z}$ -module, for a given level and weight. See Section 9.4 of [Ste2] for a detailed proof of Sturm’s result, and of this consequence. For convenience we also state and prove this result in the precise context where we use it.

**2.5.10 Theorem** *Let  $N \geq 1$  and  $k \geq 1$  be integers, and let  $\mathbb{T}(N, k)$  be the Hecke algebra attached to  $S_k(\Gamma_1(N))$ , i.e.,  $\mathbb{T}(N, k)$  is the  $\mathbb{Z}$ -submodule of  $\mathrm{End}_{\mathbb{C}}(S_k(\Gamma_1(N)))$  generated by the  $T_n$ , for  $n \geq 1$ , and the  $\langle a \rangle$ , for  $a$  in  $(\mathbb{Z}/N\mathbb{Z})^\times$ . Then  $\mathbb{T}(N, k)$  is generated, as  $\mathbb{Z}$ -module, by the  $T_i$  with  $1 \leq i \leq k[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)]/12$ .*

**Proof** Let  $S$  be the  $\mathbb{Z}$ -module  $S_k(\Gamma_1(N), \mathbb{Z})$ . Then by (2.4.9) we have isomorphisms of  $\mathbb{T}(N, k)$ -modules:  $S = \mathbb{T}(N, k)^\vee$ , and  $\mathbb{T}(N, k) = S^\vee$ . Now the result of Sturm above says that for each prime number  $p$ , the elements  $T_i$ ,  $1 \leq i \leq k[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)]/12$ , generate the  $\mathbb{F}_p$ -vector space  $\mathbb{F}_p \otimes S^\vee$ , and hence they generate  $\mathbb{F}_p \otimes \mathbb{T}(N, k)$ . So, indeed, these  $T_i$  generate  $\mathbb{T}(N, k)$  as a  $\mathbb{Z}$ -module.  $\square$

We can now state a complement to Theorem 2.5.7.

**2.5.11 Proposition** *In the situation of Theorem 2.5.7, the Hecke algebra  $\mathbb{T}(nl, 2)$  is generated, as  $\mathbb{Z}$ -module, by the Hecke operators  $T_i$  with  $1 \leq i \leq [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(nl)]/6$ .*

We remark that, still in the same situation, giving generators of  $m_f$  is then a matter of simple linear algebra over  $\mathbb{F}_l$  in a vector space of suitably bounded dimension.

We consider the particular case of a mod  $l$  eigenform of level one and of weight  $k$ , viewed as a ring morphism  $f$  from  $\mathbb{T}(1, k)$  to a finite extension of  $\mathbb{F}_l$ . Then we have the following result, that states explicitly how the Galois representation attached to  $f$  is realised in the Jacobian  $J_1(l)(\overline{\mathbb{Q}})$ . Recall that  $\mathbb{T}(l, 2)$ , the Hecke algebra acting on weight two cusp forms on  $\Gamma_1(l)$ , is generated as  $\mathbb{Z}$ -module by the  $T_j$  with  $1 \leq j \leq (l^2 - 1)/6$ .

**2.5.12 Theorem** *Let  $l$  be a prime number, let  $k$  be an integer such that  $2 < k \leq l+1$ , and  $f: \mathbb{F}_l \otimes \mathbb{T}(1, k) \rightarrow \mathbb{F}$  a surjective ring morphism with  $\mathbb{F}$  a finite field of characteristic  $l$ , such that the associated Galois representation  $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$  is irreducible. Let  $f_2: \mathbb{F}_l \otimes \mathbb{T}(l, 2) \rightarrow \mathbb{F}$  be the morphism of rings such that for all  $m \in \mathbb{Z}_{\geq 1}$  we have  $f_2(T_m) = f(T_m)$  (see Theorem 2.5.7). Let  $(t_1, \dots, t_r)$  be a system of generators for  $\ker(f_2)$ . Let:*

$$V_f := \bigcap_{1 \leq i \leq r} \ker(t_i, J_1(l)(\overline{\mathbb{Q}})[l]).$$

*Then  $V_f$  is a 2-dimensional  $\mathbb{F}$ -vector space realising  $\rho$ . For  $p \neq l$  prime,  $\langle p \rangle$  acts on  $V_f$  as multiplication by  $p^{k-2}$ .*

*One can obtain a system of generators of  $\ker(f_2)$  as follows. For  $i$  in  $\{1, \dots, (l^2 - 1)/6\}$ , either  $f(T_i)$  is an  $\mathbb{F}_l$ -linear combination of the  $f(T_j)$  with  $j < i$ , or it is not. If it is not, then let  $t_i = 0$ . If it is, then pick one:  $f(T_i) = \sum_{j < i} a_{i,j} f(T_j)$ , and let  $t_i = T_i - \sum_{j < i} a_{i,j} T_j$ .*

**Proof** Just as in the proof of Theorem 2.5.7, we use Theorem 9.2 of [Edi1], but this time in the case of level one. Case 1 of that theorem deals with the  $k$  that satisfy  $2 < k < l$ . Case 3 deals with the case  $k = l$ , because  $\rho$ , being unramified outside  $\{l\}$ , and being irreducible of dimension two, is ramified at  $l$ . Case 4 deals with the case  $k = l+1$ , because there are no nonzero cusp forms of weight two and level one.  $\square$

We also state the following definition and theorem here, because the result, to be used later, is directly related to Theorem 2.5.7. The theorem is due, again, to “many people”, just as Theorem 2.5.7 itself.

**2.5.13 Definition** Let  $N \geq 1$ , and let  $\mathbb{Z}[\zeta_N]$  be the subring of  $\mathbb{C}$  generated by a root of unity of order  $N$ . To a pair  $(E/S/\mathbb{Z}[1/N, \zeta_N], P)$  consisting of an elliptic curve  $E$  over a  $\mathbb{Z}[1/N, \zeta_N]$ -scheme  $S$ , together with a point  $P$  in  $E(S)$  that is of order  $N$  everywhere on  $S$ , we associate another such pair  $(E'/S'/\mathbb{Z}[1/N, \zeta_N], P')$  as follows. Let  $\beta: E \rightarrow E'$  be the isogeny whose kernel is the subgroup of  $E$  generated by  $P$ . Let  $\beta^\vee: E' \rightarrow E$  be the dual of  $\beta$  (see Section 2.5 of [Ka-Ma]). Let  $P'$  be the unique element of  $\ker(\beta^\vee)(S)$  such that  $e_\beta(P, P') = \zeta_N$ , where  $e_\beta$  is the perfect  $\mu_N$ -valued pairing between  $\ker(\beta)$  and  $\ker(\beta^\vee)$  as described in Section 2.8 of [Ka-Ma]. This construction induces an automorphism  $w_{\zeta_N}$  of the modular curve  $X_1(N)_{\mathbb{Z}[1/N, \zeta_N]}$ , called an ‘‘Atkin-Lehner pseudo-involution’’.

**2.5.14 Theorem** *In the situation of Theorem 2.5.7 the completion  $\mathbb{T}_{m_f, \lambda}$  of  $\mathbb{T}$  at  $m_{f, \lambda}$  is Gorenstein, i.e., the  $\mathbb{Z}_l$ -linear dual of  $\mathbb{T}_{m_f, \lambda}$  is free of rank one as  $\mathbb{T}_{m_f, \lambda}$ -module. For all  $r \geq 1$ , the  $(\mathbb{Z}/l^r\mathbb{Z}) \otimes \mathbb{T}_{m_f, \lambda}$ -module  $J_1(nl)(\overline{\mathbb{Q}})[l^r]_{m_f, \lambda}$  is free of rank 2.*

*For any  $t$  in  $\mathbb{T}$  we have  $t^\vee = wtw^{-1}$ , where  $t^\vee$  is the dual of  $t$  as endomorphism of the self-dual Abelian variety  $J_1(nl)_{\mathbb{Q}(\zeta_{nl})}$ , and where  $w$  is the endomorphism of  $J_1(nl)_{\mathbb{Q}(\zeta_{nl})}$  induced via Picard functoriality by the automorphism  $w_{\zeta_{nl}}$  of  $X_1(nl)_{\mathbb{Q}(\zeta_{nl})}$ .*

*For  $r \geq 0$ , let  $\langle \cdot, \cdot \rangle_r$  denote the Weil pairing on  $J_1(nl)(\overline{\mathbb{Q}})[l^r]$ , and let  $\langle \cdot, \cdot \rangle_r$  denote the pairing defined by:*

$$\langle x, y \rangle_r = (x, w(y))_r.$$

*Then  $\langle \cdot, \cdot \rangle_r$  is a perfect pairing on  $J_1(nl)(\overline{\mathbb{Q}})[l^r]$  for which the action of  $\mathbb{T}$  is self-adjoint. As a consequence,  $\langle \cdot, \cdot \rangle_r$  induces a perfect pairing on  $J_1(nl)(\overline{\mathbb{Q}})[l^r]_{m_f, \lambda}$ .*

**Proof** See Sections 6.4 and 6.8 of [Edi1]. □

**2.5.15 Remark** See Corollary 4.2 of [Wie2] for a proof that  $\mathbb{T}_{m_f, \lambda}$  is *not* Gorenstein if the multiplicity of  $\rho_f$  in  $V_f$  is not one.

The next result gives an effective criterion whether two modular forms give isomorphic residual Galois representations.

**2.5.16 Proposition** *Let  $l$  be a prime number,  $\mathbb{F}$  a finite extension of  $\mathbb{F}_l$ ,  $k_1$  and  $k_2$  in  $\mathbb{Z}_{\geq 0}$ , and  $f_1: \mathbb{T}(1, k_1) \rightarrow \mathbb{F}$  and  $f_2: \mathbb{T}(1, k_2) \rightarrow \mathbb{F}$  two morphisms of rings, and  $i$  in  $\{0, \dots, l-2\}$ . Then  $\rho_{f_1}$  and  $\rho_{f_2} \otimes \chi_l^i$  are isomorphic if and only if  $k_1 = k_2 + 2i$  in  $\mathbb{Z}/(l-1)\mathbb{Z}$  and for all primes  $p \neq l$  with  $p \leq (l^2-1)/12$  we have  $f_1(T_p) = p^i f_2(T_p)$ .*

**Proof** Assume first that  $\rho_{f_1}$  and  $\rho_{f_2} \otimes \chi_l^i$  are isomorphic. Then we have  $\det \rho_{f_1} = \det(\rho_{f_2} \otimes \chi_l^i)$ , hence  $\chi_l^{k_1-1} = \chi_l^{k_2-1+2i}$ , hence  $k_1 = k_2 + 2i$  in  $\mathbb{Z}/(l-1)\mathbb{Z}$ . For all primes  $p \neq l$ , we have  $f_1(T_p) = p^i f_2(T_p)$  because they are the traces of the images under  $\rho_{f_1}$  and  $\rho_{f_2} \otimes \chi_l^i$  of the Frobenius at  $p$ .

Assume now that  $k_1 = k_2 + 2i$  in  $\mathbb{Z}/(l-1)\mathbb{Z}$  and that for all primes  $p \neq l$  with  $p \leq (l^2-1)/12$  we have  $f_1(T_p) = p^i f_2(T_p)$ . Then  $\det \rho_{f_1}$  and  $\det(\rho_{f_2} \otimes \chi_l^i)$  are equal, hence it suffices to prove that for all primes  $p \neq l$  we have:  $f_1(T_p) = p^i f_2(T_p)$ .

We will use some theory on ‘‘Katz modular forms’’; see Sections 2 and 3 of [Edi1] for a short account. For  $a$  in  $\mathbb{Z}$ , we denote by  $M_a(1, \mathbb{F}_l)$  the space of Katz modular forms of level one and weight  $a$  over  $\mathbb{F}_l$ , and by  $S_a(1, \mathbb{F}_l)$  its subspace of cuspidal forms. Our reason to use Katz modular forms over  $\mathbb{F}_l$  is that this gives us the Hasse invariant  $A$  in  $M_{l-1}(1, \mathbb{F}_l)$  and the operators  $\theta: M_a(1, \mathbb{F}_l) \rightarrow S_{a+l+1}(1, \mathbb{F}_l)$ , for all  $a \in \mathbb{Z}_{\geq 0}$ , that, on  $q$ -expansions, act as the differential operator  $q \cdot d/dq$ . See [Edi1, §3] for the properties of  $\theta$  that we will use.

The idea in what follows is to use  $\theta$  to pass to eigenforms that are annihilated by  $T_l$ , and to pass to eigenforms of weight at most  $l^2-1$  by dividing by  $A$  as many times as possible. Recall that the  $q$ -expansion of  $A$  is the constant 1.

We write  $\theta^{l-1} f_1 = A^{n_1} f'_1$ , with  $n_1$  maximal, and we let  $k'_1$  be the weight of  $f'_1$ . Then  $k'_1 \leq l^2-1$  by Theorem 3.4 of [Edi1] and the definition of  $\theta$ -cycles; note that  $l+1 + (l-2)(l+1) = l^2-1$ . Similarly, we write  $\theta^{l-1-i} f_2 = A^{n_2} f'_2$ , with  $n_2$  maximal, and we let  $k'_2$  be the weight of  $f'_2$ . Then  $k'_1 = k'_2$  in  $\mathbb{Z}/(l-1)\mathbb{Z}$ ,  $f'_1$  and  $f'_2$  are eigenforms, annihilated by  $T_l$ , and with the same eigenvalues for all  $T_p$  with  $p \leq (l^2-1)/12$  prime. This implies that for all  $m \leq (l^2-1)/12$  we have  $f'_1(T_m) = f'_2(T_m)$ . If  $k'_1 \geq k'_2$ , then  $f'_1 = A^{(k'_1-k'_2)/(l-1)} f'_2$  by Sturm’s bound in this case: if the difference were non-zero, then the order of vanishing at  $\infty$  contradicts

the degree of the line bundle of which it is a section. If  $k'_2 \geq k'_1$ , then  $f'_2 = A^{(k'_2 - k'_1)/(l-1)} f'_1$  for the same reason. We conclude that for all primes  $p \neq l$  we have  $f_1(T_p) = p^i f_2(T_p)$ .  $\square$

**2.5.17 Remark** Proposition 2.5.16 can be generalised to forms of higher level, by the similar trick of passing to a higher level  $n$  at which one has forms that gave the same Galois representation, but with eigenvalue 0 for all  $T_p$  with  $p$  dividing  $n$ .

The next result gives some conditions under which the Galois representation  $\rho$  attached to a surjective ring morphism  $f: \mathbb{T}(1, k) \rightarrow \mathbb{F}$  has large image in the sense that it contains  $\mathrm{SL}_2(\mathbb{F})$ . It is an effective version of Theorem 5.1 of [Rib1]. We will need such a result later on.

**2.5.18 Theorem** *Let  $k$  be a positive integer,  $l$  a prime number with  $l > 6(k - 1)$ ,  $\mathbb{F}$  a finite field and  $l$  its characteristic, and  $f: \mathbb{T}(1, k) \rightarrow \mathbb{F}$  a surjective morphism of rings such that the associated Galois representation  $\rho: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_2(\mathbb{F})$  is irreducible. Then the image of  $\rho$  contains  $\mathrm{SL}_2(\mathbb{F})$ , and is equal to the subgroup of  $\mathrm{GL}_2(\mathbb{F})$  of elements  $g$  whose determinant is in the subgroup of  $k-1$ th powers in  $\mathbb{F}_l^\times$ .*

**Proof** As  $\mathbb{T}(1, k) = 0$  for  $k < 12$ , we have  $k \geq 12$ , and hence  $l > 66$ . As  $l > 2$  and  $\rho$  is odd,  $\rho$  is absolutely irreducible. We also have  $k \leq l + 1$ .

We apply what is known about the restriction of  $\rho$  to an inertia subgroup  $I$  at  $l$ . We denote by  $\psi$  and  $\psi' := \psi^l$  the two fundamental characters from  $I$  to  $\mathbb{F}_{l^2}^\times$  of level 2 (the tame quotient of  $I$  is the projective limit of the  $\mathbb{F}_{l^n}^\times$  and the fundamental characters of level  $n$  to  $\overline{\mathbb{F}}_l^\times$  are those that are induced by ring morphisms  $\mathbb{F}_{l^n} \rightarrow \overline{\mathbb{F}}_l$ ). By Theorems 2.5 (due to Deligne) and 2.6 (due to Fontaine) in [Edi1], we have:

$$\begin{aligned} \rho|_I &= \begin{pmatrix} \chi_l^{k-1} & * \\ 0 & 1 \end{pmatrix} & \text{if } f(T_l) \neq 0, \text{ and} \\ \overline{\mathbb{F}}_l \otimes_{\mathbb{F}} \rho|_I &= \begin{pmatrix} \psi^{k-1} & 0 \\ 0 & \psi'^{k-1} \end{pmatrix} & \text{if } f(T_l) = 0. \end{aligned}$$

The classification of subgroups of  $\mathrm{GL}_2(\mathbb{F})$  of order prime to  $l$  (see for example [Ser3, §2.5, Prop. 16]) says that the image in  $\mathrm{PGL}_2(\mathbb{F})$  of such a subgroup is either cyclic, dihedral or isomorphic to  $A_4$ ,  $S_4$  or  $A_5$ .

As  $\rho$  is absolutely irreducible, its image in  $\mathrm{PGL}_2(\mathbb{F})$  cannot be cyclic (note that the kernel of  $\mathrm{GL}_2(\mathbb{F}) \rightarrow \mathrm{PGL}_2(\mathbb{F})$  is the center of  $\mathrm{GL}_2(\mathbb{F})$ ).

Let us show that the projective image of  $\rho$  cannot be  $A_4$ ,  $S_4$  or  $A_5$ . Assume that it is. Then the image of  $\rho(I)$  in  $\mathrm{PGL}_2(\mathbb{F})$  is cyclic and has order at least  $(l-1)/(k-1)$  (the order of  $\psi/\psi'$  is  $l+1$ ). As we assume that  $l-1$  is at least  $6(k-1)$ , this image has an element of order at least 6, a contradiction.

Let us show that the projective image of  $\rho$  is not dihedral. Assume that it is. Then the image of  $\rho$  is contained in the normaliser of a Cartan subgroup (i.e., the group of points of a split or non-split maximal torus), and there is a quadratic extension  $K$  of  $\mathbb{Q}$  such that  $\rho$  is the induction from  $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$  to  $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  of a character of  $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$  that is not equal to its conjugate under  $\mathrm{Gal}(K/\mathbb{Q})$ . As  $\rho$  is unramified outside  $l$ ,  $K$  must be the quadratic extension of  $\mathbb{Q}$  that is ramified precisely at  $l$ . As  $l-1 > 2(k-1)$  the description above of  $\rho|_I$  shows that there are precisely two lines in  $\overline{\mathbb{F}}_l^2$  whose orbit under  $\rho$  in  $\mathbb{P}^1(\overline{\mathbb{F}}_l)$  has order at most 2: these are the coordinate axes (in the first case, the extension must be split). But in the first case the characters on these two lines are not conjugate under  $\mathrm{Gal}(K/\mathbb{Q})$ , and in the second case the action of  $\rho$  on the set of these two lines is not ramified. These contradictions show that the image of  $\rho$  cannot be dihedral.

We conclude that the order of the image of  $\rho$  is divisible by  $l$ . As  $l > 3$  a result of Dickson, see [Dic2], Chapter XII, or rather the proof of Theorem 2.5 in [Rib4], says that the image of  $\rho$  in  $\mathrm{PGL}_2(\mathbb{F})$  is, after suitable conjugation, equal to  $\mathrm{PGL}_2(\mathbb{F}')$  or  $\mathrm{SL}_2(\mathbb{F}')/\{1, -1\}$  for some subfield  $\mathbb{F}'$  of  $\mathbb{F}$ .

We claim that  $\mathbb{F}' = \mathbb{F}$ . Assume that it is not. We let  $f'$  and  $\rho'$  be the conjugates of  $f$  and  $\rho$  by the Frobenius automorphism of  $\mathbb{F}$  over  $\mathbb{F}'$ . Then  $\rho$  and  $\rho'$  are not isomorphic because the traces of the image of  $\rho$  generate  $\mathbb{F}$  (use that  $\mathbb{T}(1, k)$  is generated as  $\mathbb{Z}$ -module by the  $T_i$  with  $i \leq k/12 < l$ ). But their projective representations to  $\mathrm{PGL}_2(\mathbb{F})$  are equal. Hence  $\rho'$  is a twist of  $\rho$  by some character  $\chi: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{F}^\times$ . As  $\rho$  and  $\rho'$  are unramified outside  $l$ ,  $\chi$  is unramified outside  $l$  and hence a power of  $\chi_l$ . But then we have  $\theta^{i+1}f = \theta f'$ , with  $\theta$  as in the proof of Proposition 2.5.16. A look at the theta cycles in Section 3 of [Edi1] or Section 7 of [Joc] shows that then  $k = (l+3)/2$  if  $f(T_l) = 0$ , and  $k = (l+1)/2$  if  $f(T_l) \neq 0$ . This contradicts

our assumption that  $l > 6(k - 1)$ .

So the image  $G$  of  $\rho$  in  $\mathrm{PGL}_2(\mathbb{F})$  contains the image of  $\mathrm{SL}_2(\mathbb{F})$ . Then, for each  $a \in \mathbb{F}$ ,  $G$  contains elements of the form  $\begin{pmatrix} t & a \\ 0 & t \end{pmatrix}$  and  $\begin{pmatrix} s & 0 \\ a & s \end{pmatrix}$ , for some  $t$  and  $s$  in  $\mathbb{F}^\times$ . Taking suitable powers, we conclude that  $G$  contains all  $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ , where  $a$  ranges through  $\mathbb{F}$ . These generate  $\mathrm{SL}_2(\mathbb{F})$ . As  $\det \rho = \chi_l^{k-1}$  the last claim in the theorem follows.  $\square$

**2.5.19 Remark** Eigenforms  $f: \mathbb{T}(1, k) \rightarrow \mathbb{F}$  such that the projective image of  $\rho$  is  $A_4$ ,  $S_4$  or  $A_5$  are related to complex modular forms of weight one and level  $l$  or  $l^2$ , see [Kh-Wi1], Theorem 10.1. There are tables of these. For example, in [Ba-Ki], page 110, one finds an  $A_5$ -example with  $l = 2083$ , and an  $S_4$ -example with  $l = 751$ . See also Section 4.3 of [Ki-Wi]. We note that for  $f = \Delta$  the prime 23 with  $23 - 1 = 2(12 - 1)$  nicely illustrates one of the arguments that is used in the proof above:  $\rho$  is then dihedral. More generally,  $f$  with  $\rho$  dihedral come from class groups of imaginary quadratic orders that are unramified outside  $l$ .





## Chapter Three

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### First description of the algorithms

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We put ourselves in the situation of Theorem 2.5.7, and we ask how we can compute the Galois representation. More explicitly, let  $n$  and  $k$  be positive integers,  $\mathbb{F}$  a finite field and  $l$  its characteristic, and  $f: \mathbb{T}(n, k) \rightarrow \mathbb{F}$  a surjective ring morphism. Assume that  $2 < k \leq l+1$ , and that the associated Galois representation  $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$  is absolutely irreducible. Let  $f_2: \mathbb{T}(nl, 2) \rightarrow \mathbb{F}$  be the weight two eigenform as in Theorem 2.5.7 and let  $m = \ker(f_2)$ . Assume that the multiplicity of  $\rho$  in  $V := J_1(nl)(\overline{\mathbb{Q}})[m]$  is one, i.e., that  $\rho$  is realised by  $V$ .

We let  $K \subset \overline{\mathbb{Q}}$  be the field “cut out by  $\rho$ ”, i.e., the finite Galois extension of  $\mathbb{Q}$  contained in  $\overline{\mathbb{Q}}$  consisting of the elements of  $\overline{\mathbb{Q}}$  that are fixed by all elements in  $\ker(\rho)$ . Then we have, by definition, the following factorisation of  $\rho$ :

$$\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \text{Gal}(K/\mathbb{Q}) \hookrightarrow \text{GL}_2(\mathbb{F}).$$

Our aim is then to compute such residual representations  $\rho$ , in time polynomial in  $n$ ,  $k$  and  $\#\mathbb{F}$ . By computing  $\rho$  we mean giving  $K$  as a  $\mathbb{Q}$ -algebra, in the form of a monic polynomial in  $\mathbb{Q}[T]$  that is the minimal polynomial of some generator  $t$  of  $K$ , and giving the elements  $\sigma$  of  $\text{Gal}(K/\mathbb{Q})$  by giving their matrices with respect to the  $\mathbb{Q}$ -basis of  $K$  consisting of the first so many powers of  $t$ , together with the element  $\rho(\sigma)$  of  $\text{GL}_2(\mathbb{F})$ . Once given such an explicit description of  $\rho$  it becomes possible to compute  $f(T_p) \in \mathbb{F}$  in deterministic polynomial time in  $\log p$ . Indeed this boils down to computing the Frobenius endomorphism at  $p$  for the algebra  $A$ . Chapter 15 explains how to do this.

It will be convenient for us to use the modern version of Galois theory that says that the functor  $A \mapsto \text{Hom}_{\mathbb{Q}}(A, \overline{\mathbb{Q}})$  is an anti-equivalence from the category of finite separable  $\mathbb{Q}$ -algebras to that of finite discrete (continuous)  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -sets. An inverse is given by the functor that sends  $X$  to  $\text{Hom}_{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(X, \overline{\mathbb{Q}})$ , the  $\mathbb{Q}$ -algebra of functions  $f$  from  $X$  to  $\overline{\mathbb{Q}}$  such that  $f(gx) = g(f(x))$  for all  $g$  in  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  and all  $x$  in  $X$ . Under this correspondence, fields correspond to transitive  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -sets.

As a first step towards the computation of  $\rho$  we let  $A$  be the  $\mathbb{Q}$ -algebra corresponding to the  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -set  $V$ . Before we explain our strategy to compute  $A$ , we sketch how one gets from  $A$  to  $K$  and  $\rho$ . The  $\mathbb{Q}$ -algebra corresponding to  $V \times V$  is  $A \otimes A$ . The addition map  $V \times V \rightarrow V$  corresponds to a morphism  $A \rightarrow A \otimes A$ , the co-addition. The  $\mathbb{F}^\times$ -action on  $V$  corresponds to an  $\mathbb{F}^\times$ -action on  $A$ . We will see later that the co-addition and the  $\mathbb{F}^\times$ -action on  $A$  can be computed by the same method by which  $A$  will be computed. Viewing  $V \times V$  as  $\text{Hom}_{\mathbb{F}}(\mathbb{F}^2, V)$  gives a right-action by  $\text{GL}_2(\mathbb{F})$  on  $V \times V$ , hence a left-action on  $A \otimes A$ . This action can be expressed in the co-addition and the  $\mathbb{F}^\times$ -action. Let  $B$  be the  $\mathbb{Q}$ -algebra corresponding to the subset  $\text{Isom}_{\mathbb{F}}(\mathbb{F}^2, V)$  of  $\text{Hom}_{\mathbb{F}}(\mathbb{F}^2, V)$ . This factor  $B$  of  $A \otimes A$  can be computed by linear algebra over  $\mathbb{Q}$ , using the  $\text{GL}_2(\mathbb{F})$ -action on  $A \otimes A$ . In terms of  $V \times V$ , one removes the subset of  $(v_1, v_2)$  that are linearly dependent, i.e., the point  $(0, 0)$  and the  $\text{GL}_2(\mathbb{F})$ -orbit of  $(V - \{0\}) \times \{0\}$ . The field  $K$  then corresponds to a  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -orbit in  $\text{Isom}(\mathbb{F}^2, V)$ , hence is obtained by factoring  $B$  as a product of fields, using factoring algorithms, and choosing one of the factors. See [Le-Le-Lo], [Len1] and [Lan1] for the fact that such factoring can be done in polynomial time. The equivalence between factoring algebras and polynomials is given in [Len2]. Let  $G \subset \text{GL}_2(\mathbb{F})$  be the stabiliser of the chosen factor  $K$ . Then  $G = \text{Gal}(K/\mathbb{Q})$  and the inclusion  $is$  is a representation from  $G$  to  $\text{GL}_2(\mathbb{F})$ . Let  $\phi$  be in the chosen  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -orbit in  $\text{Isom}_{\mathbb{F}}(\mathbb{F}^2, V)$ . As this orbit is a right  $G$ -torsor on which  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  acts, there is, for every  $\sigma$  in  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ , a unique  $g(\sigma)$  in  $G$  such that  $\rho(\sigma) \circ \phi = \phi \circ g(\sigma)$ . Note also that evaluation at  $\phi$  is an embedding of  $K$  in  $\overline{\mathbb{Q}}$ , such that  $\sigma$  in  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  induces  $g(\sigma)$  on  $K$ . It follows that  $\phi$  is an isomorphism between  $\rho$  and the representation  $g: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Gal}(K/\mathbb{Q}) = G \subset \text{GL}_2(\mathbb{F})$ .

We now turn to the question of how to compute the  $\mathbb{Q}$ -algebra  $A$  cor-

responding to  $V$ . We wish to produce a generator of  $A$ , and its minimal polynomial over  $\mathbb{Q}$ . This means that we must produce a  $\overline{\mathbb{Q}}$ -valued function  $a$  on  $V$  such that  $a(\sigma x) = \sigma(k(x))$  for all  $x$  in  $V$  and all  $\sigma$  in  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . Such a function is a generator of  $A$  if and only if it does not arise from a strictly smaller quotient of  $V$  as  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -set (such quotients correspond to subalgebras), hence, equivalently, if and only if  $a$  is injective. The minimal polynomial over  $\mathbb{Q}$  of such a generator  $a$  is given as follows:

$$(3.1) \quad P(T) = \prod_{x \in V} (T - a(x)).$$

The question is now how to produce such a generator? A direct way would be to compute the elements of  $V$  in  $J_1(nl)(\overline{\mathbb{Q}})$ , by writing down polynomial equations in a suitable coordinate system that is defined over  $\mathbb{Q}$ , and solving them, using computer algebra. This is essentially how Schoof's algorithm deals with elliptic curves. However, the dimension of  $J_1(nl)$  is quadratic in  $l$ . Writing down equations in polynomial time still seems possible. But we do not know of a way of solving the equations in a time that is not exponential in the dimension.

The decisive idea is to use numerical computations to approximate the coefficients of a minimal polynomial  $P$  as above, in combination with a bound on the *height* of those coefficients. We recall that the (standard, logarithmic) height of a rational number  $a/b$ , with  $a$  and  $b$  integers that are relatively prime, is  $\log \max\{|a|, |b|\}$  (a variant would be  $\log(a^2 + b^2)$ ). This rational number  $x = a/b$  is known if we know an upper bound  $h$  for its height, and an approximation  $y$  of it (in  $\mathbb{R}$ , say), with  $|x - y| < e^{-2h}/2$ . Indeed, if  $x' = a'/b'$  also has height at most  $h$ , and  $x' \neq x$ , then:

$$|x - x'| = \left| \frac{a}{b} - \frac{a'}{b'} \right| = \left| \frac{ab' - ba'}{bb'} \right| \geq \frac{1}{|bb'|} \geq e^{-2h}.$$

We also note that there are good algorithms to deduce  $x$  from such a pair of an approximation  $y$  and a bound  $h$ , for example by using continued fractions, as we will now explain.

In practice we will use rational approximations  $y$  of  $x$ . Every rational

number  $y$  can be written uniquely as:

$$[a_0, a_1, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}}$$

where  $n \in \mathbb{Z}_{\geq 0}$ ,  $a_0 \in \mathbb{Z}$ ,  $a_i \in \mathbb{Z}_{>0}$  for all  $i > 0$ , and  $a_n > 1$  if  $n > 0$ . To find these  $a_i$ , one defines  $a_0 := \lfloor y \rfloor$  and puts  $n = 0$  if  $y = a_0$ ; otherwise, one puts  $y_1 := 1/(y - a_0)$  and  $a_1 = \lfloor y_1 \rfloor$  and  $n = 1$  if  $y_1 = a_1$ , and so on. The rational numbers  $[a_0, a_1, \dots, a_i]$  with  $0 \leq i \leq n$  are called the *convergents* of the continued fraction of  $y$ . Then one has the following well known result, see Theorem 184 from [Ha-Wr].

**3.2 Proposition** *Let  $y$  be in  $\mathbb{Q}$ ,  $a$  and  $b$  in  $\mathbb{Z}$  with  $b \neq 0$  and:*

$$\left| \frac{a}{b} - y \right| < \frac{1}{2b^2}.$$

*Then  $a/b$  is a convergent of the continued fraction of  $y$ .*

The question is now: how we are going to implement this method? The basic idea in doing this is to not work on the Abelian variety  $J_1(nl)$  but rather on the product  $X_1(nl)^g$  of copies of  $X_1(nl)$ , where  $g$  is the genus of  $X_1(nl)$ . To compare the two, we first choose an effective divisor  $D_0 = P_1 + \dots + P_g$  on  $X_1(nl)_{\mathbb{Q}}$ , and we consider the well-known map:

$$(3.3) \quad \begin{aligned} X_1(nl)^g &\longrightarrow J_1(nl), \\ (Q_1, \dots, Q_g) &\mapsto [Q_1 + \dots + Q_g - D_0]. \end{aligned}$$

To understand the definition of this map rigorously, one must use the interpretation of  $X_1(nl)$  as its functor of points with values in  $\mathbb{Z}[1/nl]$ -schemes, and that of  $J_1(nl)$  as the degree zero part of the relative Picard functor  $\text{Pic}_{X_1(nl)/\mathbb{Z}[1/nl]}^0$ . For the necessary background on this, see Chapters 8 and 9 of [Bo-Lu-Ra]. The divisor  $D_0$  lives on  $X_1(nl)_{\mathbb{Q}}$ , and it extends uniquely over  $\mathbb{Z}[1/nl]$  to an effective relative Cartier divisor of degree  $g$  on  $X_1(nl)$ . The points  $P_i$  of which  $D_0$  is the sum need not be rational over  $\mathbb{Q}$ .

The inverse image of a point  $x$  in  $J_1(nl)(\overline{\mathbb{Q}})$  under the map (3.3) can be described as follows. Let  $\mathcal{L}_x$  denote a line bundle of degree zero on  $X_1(nl)_{\overline{\mathbb{Q}}}$  that corresponds to  $x$  ( $x$  is an isomorphism class of such line bundles). Then the inverse image of  $x$  is the set of  $(Q_1, \dots, Q_g)$  such that  $\mathcal{L}_x$  has a rational section whose divisor is  $Q_1 + \dots + Q_g - D_0$ , or, equivalently, the set of  $(Q_1, \dots, Q_g)$  such that there is a non-zero section of  $\mathcal{L}_x(D_0)$  with divisor  $Q_1 + \dots + Q_g$ .

When  $x$  ranges over  $J_1(nl)(\overline{\mathbb{Q}})$ , the class of the  $\mathcal{L}_x(D_0)$  ranges over the set  $\text{Pic}^g(X_1(nl)_{\overline{\mathbb{Q}}})$ . The function on  $J_1(nl)(\overline{\mathbb{Q}})$  that assigns to  $x$  the dimension  $h^0(\mathcal{L}_x(D_0))$  of the space of global sections of  $\mathcal{L}_x(D_0)$  is semi-continuous in the sense that for each  $i$  the locus of  $x$  where  $h^0(\mathcal{L}_x(D_0)) \geq i$  is closed (the condition  $h^0(\mathcal{L}_x(D_0)) \leq i$  need not be closed). On a non-empty open subset of  $J_1(nl)(\overline{\mathbb{Q}})$  this value is one, as can be seen using the theorem of Riemann-Roch, and Serre duality. This means that for  $x$  outside a proper closed subset of  $J_1(nl)(\overline{\mathbb{Q}})$ , the inverse image in  $X_1(nl)^g(\overline{\mathbb{Q}})$  of  $x$  consists of the  $g$ -tuples obtained by permutation of coordinates of a single  $(Q_1, \dots, Q_g)$ . Another way to express this is to say that the map (3.3) above factors through the symmetric product  $X_1(nl)^{(g)}$  and that the map from  $X_1(nl)^{(g)}$  to  $J_1(nl)$  is birational (i.e., an isomorphism on suitable non-empty open parts).

It is then reasonable to assume that we can take  $D_0$  such that for all  $x$  in  $V$  there is, up to permutation of the coordinates, a unique  $Q = (Q_1, \dots, Q_g)$  in  $X_1(nl)^g(\overline{\mathbb{Q}})$  that is mapped to  $x$  via the map (3.3). On the other hand, on a curve of high genus such as  $X_1(nl)$  it is not clear how to make a large supply of inequivalent effective divisors  $D_0$  on  $X_1(nl)_{\mathbb{Q}}$ . We will see later, in Theorem 8.1.7, that we can indeed find a suitable divisor, supported on the cusps, and defined over  $\mathbb{Q}(\zeta_l)$ , on the  $X_1(5l)$ , which will suffice for treating almost all modular forms of level one.

*Remark.* In situations where such a cuspidal divisor cannot be found, one could try at random  $P_1, \dots, P_g$  in  $X_1(nl)(L)$ , corresponding to elliptic curves lying in one isogeny class, with complex multiplications, for example by  $\mathbb{Q}(i)$ . Then  $L$  is a solvable Galois extension of  $\mathbb{Q}$ , so that  $K$  can be reconstructed from the compositum  $KL$ . If one chooses the  $P_i$  reasonably, the degree of  $L$  and the logarithm of the discriminant of  $L$  are polynomial

in  $l$ . Another possibility is to try to work with a divisor  $D_0$  of degree smaller than  $g$ , for example a multiple of a rational cusp.

Let us now assume that we have a divisor  $D_0$  as described above. Then we choose a non-constant function:

$$f: X_1(nl)_{\mathbb{Q}} \rightarrow \mathbb{P}_{\mathbb{Q}}^1,$$

that will have to satisfy some conditions that will be given in a moment.

With these two choices,  $D_0$  and  $f$ , and a choice of an integer  $m$ , we get an element  $a_{D_0, f, m}$  of the  $\mathbb{Q}$ -algebra  $A$  corresponding to  $V$  as follows. For  $x$  in  $V$  we let  $D_x = Q_{x,1} + \cdots + Q_{x,g}$  be the unique effective divisor of degree  $g$  such that:

$$(3.4) \quad x = [D_x - D_0].$$

Note that indeed for  $x = 0$  we have  $D_x = D_0$ . We assume that for all  $x \in V$  the divisor  $D_x$  is disjoint from the poles of  $f$ . Then, for each  $x$  in  $V$ , we define:

$$P_{D_0, f, x} = \prod_{i=1}^g (t - f(Q_{x,i})) \quad \text{in } \overline{\mathbb{Q}}[t].$$

We then get an element  $a_{D_0, f, m}$  of  $A$  by evaluating the  $P_{D_0, f, x}$  at  $m$ :

$$(3.5) \quad a_{D_0, f, m}: V \longrightarrow \overline{\mathbb{Q}}, \quad x \mapsto P_{D_0, f, x}(m) = \prod_{i=1}^g (m - f(Q_{x,i})).$$

The condition that all the  $D_x$  are disjoint from the locus of poles of  $f$  will not be guaranteed to hold later when we treat forms of level one, but then it will be possible to omit the  $Q_{x,i}$  at which  $f$  has a pole from the sum in (3.5) ( $f$  will have its poles at certain cusps). For the moment, let us just assume that this condition is satisfied. Then the  $f_*D_x$ , for  $x$  in  $V$ , are effective divisors of degree  $g$  on  $\mathbb{A}_{\mathbb{Q}}^1$ .

We will choose  $f$  in such a way that the  $f_*D_x$  are distinct; we assume now that this is so. Then there is an integer  $m \geq 0$  with  $m \leq g \cdot (\#\mathbb{F})^4$  such that  $a_{D_0, f, m}$  is injective, and hence a generator of  $A$ : the polynomials  $P_{D_0, f, x}$  are distinct when  $x$  varies, and  $m$  must not be a root of any difference of two of them.

Finally, we want to have control on the heights of the coefficients of the minimal polynomial of  $a_{D_0, f, m}$ , because these heights determine the required precision of the approximations of those coefficients that we must compute. The whole strategy depends on the possibility to choose a divisor  $D_0$  and a function  $f$ , such that, when  $n$ ,  $k$  and  $\mathbb{F}$  vary, those heights grow at most polynomially in  $n$ ,  $k$  and  $\#\mathbb{F}$ . Using a great deal of machinery from Arakelov theory, we will show (at least in the case  $n = 1$ ) that any reasonable choices of  $D_0$  and  $f$  will lead to an at most polynomial growth of those heights. Intuitively, and completely non-rigorously, one can believe that this should work, because of the following argument. Our  $x$  are torsion points, so that their Néron-Tate height is zero. As  $x$  and  $D_0$  determine  $D_x$ , the height of  $D_x$  should be not much bigger than the height of  $D_0$ . As we choose  $D_0$  ourselves, it should have small height. Finally, the height of  $a_{D_0, f, m}$  should be not much bigger than the sum of those of  $f$  and  $m$  and the  $D_x$ . Turning these optimistic arguments into rigorous statements implies a lot of work that will be done in Chapters 9–11. An important problem here is that in Arakelov theory many results are available that deal with a single curve over  $\mathbb{Q}$ , but in our situation we are dealing with the infinitely many curves  $X_1(nl)$  as  $l$  varies.

A few words about the numerical computations involved. What we need is that these can be done in a time that is polynomial in  $n$  and  $\#\mathbb{F}$  and the number of significant digits that one wants for the coefficients of the minimal polynomial  $P_{D_0, f, m}$  of  $a_{D_0, f, m}$ . It is not at all obvious that this can be done, as the genus of  $X_1(nl)$  and hence the dimension of  $J_1(nl)$  are quadratic in  $l$ .

One way to do the computations is to use the complex uniformisations of  $X_1(nl)(\mathbb{C})$  and  $J_1(nl)(\mathbb{C})$ . The Riemann surface  $X_1(nl)(\mathbb{C})$  can be obtained by adding finitely many cusps (the set  $\Gamma_1(nl) \backslash \mathbb{P}^1(\mathbb{Z})$ ) to the quotient  $\Gamma_1(nl) \backslash \mathbb{H}$  (see Section 2.1). This means that  $X_1(nl)(\mathbb{C})$  is covered by disks around the cusps, which are well suited for computations (functions have  $q$ -expansions, for example). In order to describe  $J_1(nl)(\mathbb{C})$  as  $\mathbb{C}^g$  modulo a lattice, we need a basis of the space of holomorphic differential forms  $H^0(X_1(nl)(\mathbb{C}), \Omega^1)$ . The basis that we work with is the one provided by Atkin-Lehner theory, as given in (2.2.17); we write it as  $\omega = (\omega_1, \dots, \omega_g)$ .



Then we have the following complex description of the map (3.3):

$$(3.6) \quad X_1(nl)(\mathbb{C})^g \longrightarrow \twoheadrightarrow J_1(nl)(\mathbb{C}) \xlongequal{\quad} \mathbb{C}^g/\Lambda$$

$$(Q_1, \dots, Q_g) \longmapsto \left[ \sum_{i=1}^g Q_i - \sum_{i=1}^g P_i \right] \xlongequal{\quad} \sum_{i=1}^g \int_{P_i}^{Q_i} \omega,$$

where  $\Lambda$  is the period lattice with respect to this basis, i.e., the image of  $H_1(X_1(nl)(\mathbb{C}), \mathbb{Z})$  under integration of the  $\omega_i$ . This map can be computed up to any desired precision by formal integration of power series on the disks mentioned above. The coefficients needed from the power series expansions of the  $\omega_i$  can be computed using the method of modular symbols, as has been implemented by William Stein in Magma (see his book [Ste2]). We note that modular symbols algorithms can be used very well to locate  $V$  inside  $l^{-1}\Lambda/\Lambda$ , hence in  $J_1(nl)(\mathbb{C})$ . A strategy to approximate a point  $Q_x = (Q_{x,1}, \dots, Q_{x,g})$  as above for a non-zero  $x$  in  $V$  is to lift the straight line that one can draw in  $\mathbb{C}^g/\Lambda$  from 0 to  $x$  (within a suitable fundamental domain for  $\Lambda$ ) to a path in  $X_1(nl)(\mathbb{C})^g$  starting at  $(P_1, \dots, P_g)$ . In practice this seems to work reasonably well, see Bosman's Chapters 6 and 7. A theoretical difficulty with this approach is that one needs to bound from below the distance to the ramification locus of  $X_1(nl)^{(g)} \rightarrow J_1(nl)$ . Chapter 12 gets around this difficulty and provides a proven algorithm for inverting the Jacobi map (3.3). The starting idea is to set  $y = x/N$  for  $N$  a large enough integer. This  $y$  is no longer an  $l$ -torsion point but it is close to the origin in the torus  $J_1(nl)(\mathbb{C})$ , and this helps finding a preimage  $Q_y$  of  $y$ , because the behaviour of the Jacobi map (3.3) is well understood at least in the neighborhood of the origin. The divisor  $Q_x$  we are looking for is such that  $Q_x - D_0$  and  $N(Q_y - D_0)$  are linearly equivalent. So  $Q_x$  can be computed from  $Q_y$  by repeated application of an explicit form of the Riemann-Roch theorem. The resulting algorithm reduces to computing approximations of the complex zeros of a great number of modular forms with level  $5l$  and weight 4. Chapter 5 explains how to approximate the complex zeros of entire series. It also contains a reminder of the necessary notions from computational complexity theory.

Another way to do the ‘‘approximation’’ is to compute the minimal poly-

nomial  $P_{D_0, f, m}$  of  $a_{D_0, f, m}$  modulo many small primes  $p$ . Indeed, the map (3.3) can be reduced mod  $p$ . In this case one has no analytic description of the curve and its Jacobian, but one can make random points in  $J_1(nl)(\mathbb{F}_q)$  for a suitable finite extension  $\mathbb{F}_p \rightarrow \mathbb{F}_q$ . Such random points can then be projected, using Hecke operators, into  $V$ . Elements of  $J_1(nl)(\mathbb{F}_q)$  can be represented by divisors on  $X_1(nl)_{\mathbb{F}_q}$ , and all necessary operations can be done in polynomial time. This approach is explained in detail in Chapter 13. In order to deduce a rational number  $x = a/b$  from the knowledge sufficiently many of its reductions modulo primes  $p$  not dividing  $b$  we have the following well-known result.

**3.7 Proposition** *Let  $x = a/b$  be in  $\mathbb{Q}$ , with  $a$  and  $b$  in  $\mathbb{Z}$ , relatively prime. Let  $M = \max\{|a|, |b|\}$ . Let  $S$  be a finite set of prime numbers  $p$  with  $p$  not dividing  $b$ , such that  $\prod_{p \in S} p > 2M^2$ . For each  $p$  in  $S$ , let  $x_p$  in  $\mathbb{F}_p$  be the reduction of  $x$ , and let  $L \subset \mathbb{Z}^2$  be the submodule of  $(n, m)$  with the property that for all  $p$  in  $S$ :  $n - x_p m = 0$  in  $\mathbb{F}_p$ . Then  $(a, b)$  and  $(-a, -b)$  are the shortest non-zero elements of  $L$  with respect to the standard inner product on  $\mathbb{R}^2$ , and the lattice reduction algorithm in dimension two, Algorithm 1.3.14 in [Coh], finds these in time polynomial in  $\log M$ .*

**Proof** The lattice reduction gives a shortest non-zero element, so it suffices to show that, under the assumptions in the Proposition, the two shortest non-zero elements of  $L$  are precisely  $\pm(a, b)$ . The volume of  $\mathbb{R}^2/L$  is the index of  $L$  in  $\mathbb{Z}^2$ , hence equals  $\prod_{p \in S} p$ . Let  $l_1$  be a shortest non-zero element of  $L$ . Then  $\|l_1\| \leq \|(a, b)\| \leq \sqrt{2}M$ . Let  $l_2$  in  $L$  be linearly independent of  $l_1$ . Then:

$$2M^2 < \text{Vol}(\mathbb{R}^2/L) \leq \text{Vol}(\mathbb{R}^2/(\mathbb{Z} \cdot l_1 + \mathbb{Z} \cdot l_2)) \leq \|l_1\| \cdot \|l_2\| \leq \sqrt{2}M \|l_2\|.$$

Hence  $\|l_2\| > \sqrt{2}M \geq \|(a, b)\|$ . It follows that  $(a, b)$  and  $l_1$  are linearly dependent, and hence  $l_1 = \pm(a, b)$ .  $\square$

*Remark.* In case one has a natural rigid analytic uniformisation at some prime  $p$ , one may want to use that. For the modular curves that we are dealing with this is not the case, but the closely related Shimura curves attached to quaternion algebras over  $\mathbb{Q}$  do admit such uniformisations at the

primes where the quaternion algebra is ramified (as was proved by Cerednik, Drinfeld, see [Bo-Ca]).

## Chapter Four

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### Short introduction to heights and Arakelov theory

B. Edixhoven and R. de Jong

In Chapter 3 it has been explained how the computation of the Galois representations  $V$  attached to modular forms over finite fields should proceed. The essential step is to approximate the minimal polynomial  $P$  of (3.1) with sufficient precision so that  $P$  itself can be obtained. The topic to be addressed now is to bound from above the precision that is needed for this. This means that we must bound the heights of the coefficients of  $P$ . As was hinted to in Chapter 3, we get such bounds using Arakelov theory, a tool that we discuss in this section. It is not at all excluded that a direct approach to bound the coefficients of  $P$  exists, thus avoiding the complicated theory that we use. On the other hand, it is clear that the use of Arakelov theory provides a way to split the work to be done in smaller steps, and that the quantities occurring in each step are intrinsic in the sense that they do not depend on coordinate systems or other choices that one could make. We also want to point out that our method does not depend on cancellations of terms in the estimates that we will do; all contributions encountered can be bounded appropriately.

A good reference for a more detailed introduction to heights is Chapter 6 of [Co-Si]. Good references for the Arakelov theory that we will use are [Fal1] and [Mor2]. A general reference for heights in the context of Diophantine geometry is [Bo-Gu].

#### 4.1 HEIGHTS ON $\mathbb{Q}$ AND $\overline{\mathbb{Q}}$

The definition of the height of an element of  $\mathbb{Q}$  has already been given in Chapter 3; for  $x = a/b$  with  $a$  and  $b \neq 0$  relatively prime integers, we

have  $h(x) = \log \max\{|a|, |b|\}$ . We will now give an equivalent definition in terms of absolute values  $|\cdot|_v$  on  $\mathbb{Q}$  attached to all places  $v$  of  $\mathbb{Q}$ , the finite places, indexed by the prime numbers, and the infinite place denoted  $\infty$ .

The absolute value  $|\cdot|_\infty$  is just the usual absolute value on  $\mathbb{R}$ , restricted to  $\mathbb{Q}$ . We note that  $\mathbb{R}$  is the completion of  $\mathbb{Q}$  for  $|\cdot|_\infty$ . For  $p$  prime, we let  $v_p$  be the  $p$ -adic valuation:

$$(4.1.1) \quad v_p: \mathbb{Z} \longrightarrow \mathbb{Z} \cup \{\infty\},$$

sending an integer to the maximal number of times that it can be divided by  $p$ . This valuation  $v_p$  extends uniquely to  $\mathbb{Q}$  subject to the condition that  $v_p(xy) = v_p(x) + v_p(y)$ ; we have  $v_p(a/b) = v_p(a) - v_p(b)$  for integers  $a$  and  $b \neq 0$ . We let  $|\cdot|_p$  denote the absolute value on  $\mathbb{Q}$  defined by:

$$(4.1.2) \quad |x|_p = p^{-v_p(x)}, \quad |0|_p = 0.$$

The completion of  $\mathbb{Q}$  with respect to  $|\cdot|_p$  is the locally compact topological field  $\mathbb{Q}_p$ . An important property of these absolute values is that all together they satisfy the product formula:

$$(4.1.3) \quad \prod_v |x|_v = 1, \quad \text{for all } x \text{ in } \mathbb{Q}^\times.$$

With these definitions, we have:

$$(4.1.4) \quad h(x) = \sum_v \log \max\{1, |x|_v\}, \quad \text{for all } x \in \mathbb{Q},$$

where  $v$  ranges over the set of all places of  $\mathbb{Q}$  (note that almost all terms in the sum are equal to 0).

The height function on  $\mathbb{Q}$  generalises as follows to number fields. First of all, for a local field  $F$  we define the natural absolute value  $|\cdot|$  on it by letting, for  $x$  in  $F^\times$ ,  $|x|_F$  be the factor by which all Haar measures on  $F$  are scaled by the homothety  $y \mapsto xy$  on  $F$ . For example, for  $F = \mathbb{C}$  we have  $|z|_{\mathbb{C}} = z\bar{z} = |z|^2$ , the square of the usual absolute value. Let now  $K$  be a number field. By a finite place of  $K$  we mean a maximal ideal of  $O_K$ . An infinite place of  $K$  is an embedding of  $K$  into  $\mathbb{C}$ , up to complex conjugation. For each place  $v$  of  $K$ , let  $K_v$  be its completion at  $v$ ; as  $K_v$  is a local field, we have the natural absolute value  $|\cdot|_v := |\cdot|_{K_v}$  on  $K_v$  and on  $K$ . In this case, the product formula is true (this can be shown easily by considering the

adèles of  $K$ , see Chapter IV, Section 4, Theorem 5 of [Wei2]). The height function on  $\mathbb{Q}$  also generalises to  $K$ . For all for all  $x$  in  $K$  we define:

$$(4.1.5) \quad \begin{aligned} h_K(x) &:= \sum_v \log \max\{1, |x|_v\} = \\ &= \sum_{v \text{ finite}} \log \max\{1, |x|_v\} + \sum_{\sigma: K \rightarrow \mathbb{C}} \log \max\{1, |\sigma(x)|\}. \end{aligned}$$

This function  $h_K$  is called the *height function of  $K$* . For  $K \rightarrow K'$  an extension of number fields, and for  $x$  in  $K$ , we have

$$h_{K'}(x) = (\dim_K K') \cdot h_K(x).$$

Therefore one has the *absolute* height function  $h$  on  $\overline{\mathbb{Q}}$  defined by:

$$(4.1.6) \quad h: \overline{\mathbb{Q}} \rightarrow \mathbb{R}, \quad h(x) = \frac{h_K(x)}{\dim_{\mathbb{Q}} K},$$

where  $K \subset \overline{\mathbb{Q}}$  is any number field that contains  $x$ .

## 4.2 HEIGHTS ON PROJECTIVE SPACES AND ON VARIETIES

For  $n \geq 0$  and for  $K$  a number field, we define a height function on the projective space  $\mathbb{P}^n(K)$  by:

$$(4.2.1) \quad \begin{aligned} h_K((x_0 : \cdots : x_n)) &:= \sum_v \log \max\{|x_0|, \dots, |x_n|\}, \\ h(x) &:= \frac{h_K(x)}{\dim_{\mathbb{Q}} K}, \end{aligned}$$

where  $v$  ranges through the set of all places of  $K$ . We note that it is because of the product formula that  $h_K(x)$  is well-defined, and that this definition is compatible with our earlier definition of the height and absolute height on  $K$  if we view  $K$  as the complement of  $\infty$  in  $\mathbb{P}^1(K)$ . The functions  $h$  on  $\mathbb{P}^n(K)$  for varying  $K$  naturally induce the absolute height function on  $\mathbb{P}^n(\overline{\mathbb{Q}})$ .

A fundamental result, not difficult to prove, but too important to omit here (even though we will not use it), is Northcott's finiteness theorem.

**4.2.2 Theorem (Northcott)** *Let  $n, d$  and  $C$  be integers. Then:*

$$\{x \in \mathbb{P}^n(\overline{\mathbb{Q}}) \mid h(x) \leq C \text{ and } \dim_{\mathbb{Q}}(\mathbb{Q}(x)) \leq d\}$$

*is a finite set.*

For a proof the reader is referred to Chapter 6 of [Co-Si], or to Section 2.4 of [Ser10].

For any algebraic variety  $X$  embedded in a projective space  $\mathbb{P}_K^n$  over some number field  $K$ , we get height functions  $h_K$  on  $X(K)$  and  $h$  on  $X(\overline{\mathbb{Q}})$  by restricting those from  $\mathbb{P}^n$  to  $X$ .

For later use, we include here some simple facts. The height functions on the projective spaces  $\mathbb{P}^n(\overline{\mathbb{Q}})$  are compatible with embeddings as coordinate planes, for example by sending  $(x_0 : \cdots : x_n)$  to  $(x_0 : \cdots : x_n : 0)$ , or to  $(0 : x_0 : \cdots : x_n)$ .

For all  $n \in \mathbb{N}$ , we view  $\mathbb{A}^n(\overline{\mathbb{Q}})$  as a subvariety of  $\mathbb{P}^n(\overline{\mathbb{Q}})$ , embedded in one of the  $n+1$  standard ways by sticking in a 1 at the extra coordinate. For example, by sending  $(x_1, \dots, x_n)$  to  $(1 : x_1 : \cdots : x_n)$ . This gives us, for each  $n$ , a height function  $h : \mathbb{A}^n(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$ . These height functions are also compatible with embeddings as coordinate planes. For  $n = 1$  the height function on  $\mathbb{A}^1(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}$  is the function in (4.1.6).

**4.2.3 Lemma** *Let  $n \in \mathbb{Z}_{\geq 1}$ , and  $x_1, \dots, x_n$  in  $\overline{\mathbb{Q}}$ . Then:*

$$\begin{aligned} h(x_1 \cdots x_n) &\leq \sum_{i=1}^n h(x_i), & h(x_1 + \cdots + x_n) &\leq \log n + \sum_{i=1}^n h(x_i) \\ h(x_i) &\leq h((x_1, \dots, x_n)) \leq h(x_1) + \cdots + h(x_n) && \text{for each } i. \end{aligned}$$

**Proof** Let  $n \in \mathbb{Z}_{\geq 1}$ , and  $x_1, \dots, x_n$  in  $\overline{\mathbb{Q}}$ . Let  $K \subset \overline{\mathbb{Q}}$  be a finite extension of  $\mathbb{Q}$  containing the  $x_i$ . For the first inequality, we have:

$$\begin{aligned} h_K(x_1 \cdots x_n) &= \sum_v \log \max\{1, |x_1 \cdots x_n|_v\} \\ &\leq \sum_v \log(\max\{1, |x_1|_v\} \cdots \max\{1, |x_n|_v\}) \\ &= h_K(x_1) + \cdots + h_K(x_n). \end{aligned}$$

For the second inequality, let  $x = x_1 + \cdots + x_n$ . Then we have:

$$\begin{aligned}
 h_K(x) &= \sum_{v \text{ finite}} \log \max\{1, |\sum_i x_i|_v\} + \sum_{\sigma: K \rightarrow \overline{\mathbb{Q}}} \log \max\{1, |\sigma \sum_i x_i|\} \\
 &\leq \sum_{v \text{ finite}} \log \max\{1, \max_i |x_i|_v\} + \sum_{\sigma} \log \max\{1, \sum_i |\sigma(x_i)|\} \\
 &\leq \sum_{v \text{ finite}} \sum_i \log \max\{1, |x_i|_v\} + \sum_{\sigma} \max\{0, \log(n \cdot \max_i |\sigma(x_i)|)\} \\
 &\leq \sum_{v \text{ finite}, i} \log \max\{1, |x_i|_v\} + \sum_{\sigma} \log n + \sum_{\sigma, i} \max\{0, \log |\sigma(x_i)|\} \\
 &= (\dim_{\mathbb{Q}} K) \cdot \log n + \sum_i h_K(x_i).
 \end{aligned}$$

For the third inequality, let  $i$  be in  $\{1, \dots, n\}$ . We have:

$$\begin{aligned}
 h_K(x_1, \dots, x_n) &= h_K(1 : x_1 : \cdots : x_n) \\
 &= \sum_v \log \max\{1, |x_1|_v, \dots, |x_n|_v\} \\
 &\geq \sum_v \log \max\{1, |x_i|_v\} = h_K(x_i).
 \end{aligned}$$

Finally, for the last inequality:

$$\begin{aligned}
 h_K(x_1, \dots, x_n) &= h_K(1 : x_1 : \cdots : x_n) \\
 &= \sum_v \log \max\{1, |x_1|_v, \dots, |x_n|_v\} \\
 &\leq \sum_v \sum_i \log \max\{1, |x_i|_v\} = h_K(x_1) + \cdots + h_K(x_n).
 \end{aligned}$$

□

**4.2.4 Lemma** *Let  $d \geq 1$  and  $n \geq d$  be integers. Let  $\Sigma_d$  denote the elementary symmetric polynomial of degree  $d$  in  $n$  variables. Let  $y_1, \dots, y_n$  be in  $\overline{\mathbb{Q}}$ . Then we have:*

$$h(\Sigma_d(y_1, \dots, y_n)) \leq n \log 2 + \sum_{1 \leq i \leq n} h(y_i).$$

**Proof** Let  $K$  be the compositum of the fields  $\mathbb{Q}(y_i)$  for  $i = 1, \dots, n$ . For each place  $v$  of  $K$ , we let  $|\cdot|_v$  be the natural absolute value on  $K_v$  and on  $K$



as at the end of Section 4.1. By the triangle inequality we obtain, for each place  $v$  of  $K$ :

$$\begin{aligned} |\Sigma_d(y_1, \dots, y_n)|_v &\leq c(v, n) \max_{1 \leq i_1 < \dots < i_d \leq n} |y_{i_1} \cdots y_{i_d}|_v \leq \\ &\leq c(v, n) \prod_{1 \leq i \leq n} \max\{1, |y_i|_v\}, \end{aligned}$$

where  $c(v, n) = 2^n$  if  $v$  is Archimedean, and  $c(v, n) = 1$  if  $v$  is non-Archimedean. It follows that:

$$\max\{1, |\Sigma_d(y_1, \dots, y_n)|_v\} \leq c(v, n) \prod_{i=1}^n \max\{1, |y_i|_v\}.$$

The proof of the lemma is finished by taking logarithms, summing over the places  $v$ , and dividing by  $\dim_{\mathbb{Q}} K$ .  $\square$

**4.2.5 Lemma** *Let  $x \neq 0$  be in  $\overline{\mathbb{Q}}$ , let  $K = \mathbb{Q}(x)$ , and let  $\sigma: K \rightarrow \mathbb{C}$ . Then:*

$$|\sigma(x)| \geq e^{-(\dim_{\mathbb{Q}} K) \cdot h(x)}.$$

**Proof** We have:

$$\begin{aligned} -\log |\sigma(x)| &= \log (|\sigma(x)|^{-1}) \leq \log \max\{1, |\sigma(x)|^{-1}\} \\ &\leq \sum_v \log \max\{1, |x|_v^{-1}\} = (\dim_{\mathbb{Q}} K) \cdot h(x^{-1}) \\ &= (\dim_{\mathbb{Q}} K) \cdot h(x), \end{aligned}$$

where the sum is over all places of  $K$ .  $\square$

**4.2.6 Lemma** *Let  $K$  be a number field, let  $n$  be in  $\mathbb{Z}_{\geq 1}$ , and let  $a$  be in  $M_n(K)$ . Then:*

$$h(\det(a)) \leq \sum_{i,j} h(a_{i,j}) + \frac{1}{2}n \log n.$$

**Proof** Let  $v$  be a finite place of  $K$ . Then we have:

$$\begin{aligned} |\det(a)|_v &= \left| \sum_{s \in S_n} a_{1,s(1)} \cdots a_{n,s(n)} \right|_v \leq \max_s |a_{1,s(1)}|_v \cdots |a_{n,s(n)}|_v \\ &\leq \prod_{i,j} \max\{1, |a_{i,j}|_v\}. \end{aligned}$$

For  $\sigma: K \rightarrow \mathbb{C}$  we have, by Hadamard's inequality and the comparison  $\|\cdot\| \leq n^{1/2} \|\cdot\|_{\max}$  in  $\mathbb{C}^n$  of the euclidean norm and the max-norm:

$$\begin{aligned} |\det \sigma(a)| &\leq \prod_j \|\sigma(a_j)\| = \prod_j \left( n^{1/2} \|\sigma(a_j)\|_{\max} \right) \\ &\leq n^{n/2} \prod_{i,j} \max\{1, |\sigma(a_{i,j})|\}, \end{aligned}$$

where  $a_j$  is the  $j$ th column of  $a$ . Then we have:

$$\begin{aligned} h_K(\det(a)) &= \sum_v \log \max\{1, |\det(a)|_v\} + \sum_\sigma \log \max\{1, |\det \sigma(a)|\} \\ &\leq \sum_v \sum_{i,j} \log \max\{1, |a_{i,j}|_v\} \\ &\quad + \sum_\sigma \left( \frac{n}{2} \log n + \sum_{i,j} \log \max\{1, |\sigma(a_{i,j})|\} \right) \\ &= \frac{1}{2} (\dim_{\mathbb{Q}} K) \cdot n \log n + \sum_{i,j} h_K(a_{i,j}). \end{aligned}$$

Dividing by  $\dim_{\mathbb{Q}} K$  gives the result.  $\square$

**4.2.7 Lemma** *Let  $K$  be a number field, let  $n$  be in  $\mathbb{Z}_{\geq 1}$ , let  $a$  be in  $\mathrm{GL}_n(K)$ , and  $y$  in  $K^n$ . Let  $x$  be the unique element in  $K^n$  such that  $ax = y$ . Let  $b$  be the maximum of all  $h(a_{i,j})$  and  $h(y_i)$ . Then we have, for all  $i$ :*

$$h(x_i) \leq 2n^2b + n \log n.$$

**Proof** We apply Cramer's rule:  $x_i = \det(a(i))/\det(a)$ , where  $a(i)$  in  $M_n(K)$  is obtained by replacing the  $i$ th column by  $y$ . Lemma 4.2.6 gives us:

$$h(\det(a)) \leq n^2b + \frac{1}{2}n \log n, \quad h(\det(a(i))) \leq n^2b + \frac{1}{2}n \log n.$$

Therefore:  $h(x_i) = h(\det(a(i))/\det(a)) \leq 2n^2b + n \log n.$   $\square$

### 4.3 THE ARAKELOV PERSPECTIVE ON HEIGHT FUNCTIONS

We have just defined height functions  $h_K$  and  $h$  on a variety  $X$  over a number field  $K$ , embedded into some projective space  $\mathbb{P}_K^n$ . Such an embedding

determines a line bundle  $\mathcal{L}$  on  $X$ : the restriction of the line bundle  $\mathcal{O}(1)$  of  $\mathbb{P}_K^n$  that corresponds to homogeneous forms of degree 1, in the variables  $x_0, \dots, x_n$ , say. The embedding of  $X$  into  $\mathbb{P}_K^n$  is given by the global sections  $s_0, \dots, s_n$  of  $\mathcal{L}$  obtained by restricting the global sections  $x_0, \dots, x_n$  to  $X$ . Now any finite set of generating global sections  $t_0, \dots, t_m$  of  $\mathcal{L}$  determines a morphism  $f: X \rightarrow \mathbb{P}_K^m$ , inducing height functions  $h_{K,f}$  and  $h_f$  via pull-back along  $f$ . For  $f$  and  $f'$  two such morphisms, the difference  $|h_f - h_{f'}|$  is bounded on  $X(\overline{\mathbb{Q}})$  (see Theorem 3.1 of Chapter 6 of [Co-Si]). For this reason, one usually associates to a line bundle  $\mathcal{L}$  on a variety  $X$  a class of height functions  $f_{\mathcal{L}}$ , i.e., an element in the set of functions  $X(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$  modulo bounded functions; this map is then a morphism of groups on  $\text{Pic}(X)$ :  $f_{\mathcal{L}_1 \otimes \mathcal{L}_2} \equiv f_{\mathcal{L}_1} + f_{\mathcal{L}_2}$ . However, in our situation, we cannot permit ourselves to work just modulo bounded functions on each variety, as we have infinitely many curves  $X_1(l)$  to deal with.

There is a geometric way to associate to a line bundle a specific height function, not just a class of functions modulo bounded functions. For this, the contributions from the finite as well as the infinite places must be provided. Those from the finite places come from a model of  $X$  over the ring of integers  $O_K$  of  $K$ , i.e., an  $O_K$ -scheme  $X_{O_K}$  whose fibre over  $K$  is  $X$ , together with a line bundle  $\mathcal{L}$  on  $X_{O_K}$  whose restriction to  $X$  is the line bundle that we had. The  $O_K$ -scheme  $X_{O_K}$  is required to be proper (e.g., projective). The contributions from the infinite places are provided by a hermitian metric (or inner product) on  $\mathcal{L}$ , a notion that we will briefly explain.

A hermitian metric on a locally free  $\mathcal{O}_X$ -module of finite rank  $\mathcal{E}$  consists of a hermitian metric  $\langle \cdot, \cdot \rangle_x$  on all  $\mathbb{C}$ -vector spaces  $x^* \mathcal{E}$ , where  $x$  runs through  $X(\mathbb{C})$ , the set of  $x: \text{Spec}(\mathbb{C}) \rightarrow X$ . Each  $x$  in  $X(\mathbb{C})$  induces a morphism  $\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(K)$ , i.e., an embedding of  $K$  into  $\mathbb{C}$ . Therefore,  $X(\mathbb{C})$  is the disjoint union of the complex analytic varieties  $X_\sigma$ , indexed by the  $\sigma: K \rightarrow \mathbb{C}$ . A hermitian metric on  $\mathcal{E}$  consists of hermitian metrics on all the holomorphic vector bundles  $\mathcal{E}_\sigma$  that  $\mathcal{E}$  induces on the  $X_\sigma$ . The metrics to be used are required to be continuous, i.e., for  $U$  open in  $X$  and  $s$  and  $t$  in  $\mathcal{E}(U)$ , the function  $x \mapsto \langle s(x), t(x) \rangle_x$  on  $U(\mathbb{C})$  must be continuous. Actually, the metrics that we will use will live on non-singular  $X$ , and will be required to be smooth (infinitely differentiable). Another condition that is usually imposed is a certain compatibility between the metrics

at a point  $x$  in  $X(\mathbb{C})$  and its complex conjugate  $\bar{x}$ . We do not give this condition in detail, but note that it will be fulfilled by the metrics that we will use. It is also customary to denote a hermitian metric  $\langle \cdot, \cdot \rangle$  by its norm  $\|\cdot\|$ , given by  $\|s\|^2 = \langle s, s \rangle$ . Indeed, a suitable polarisation identity expresses the hermitian metric in terms of its norm. A pair  $(\mathcal{E}, \|\cdot\|)$  of a locally free  $\mathcal{O}_{X_{O_K}}$ -module with a hermitian metric  $\|\cdot\|$  is called a metrised vector bundle on  $X_{O_K}$ . Metrised vector bundles can be pulled back via morphisms  $f: W_{O_K} \rightarrow X_{O_K}$  between  $O_K$ -schemes of the type considered.

An important example of the above is the case where  $X = \text{Spec}(K)$ , just a point, and  $X_{O_K} = \text{Spec}(O_K)$ . A metrised line bundle  $(\mathcal{L}, \|\cdot\|)$  then corresponds to an invertible  $O_K$ -module,  $L$ , say, with hermitian metrics on the  $L_\sigma := \mathbb{C} \otimes_{\sigma, O_K} L$ . The *Arakelov degree* of  $(\mathcal{L}, \|\cdot\|)$  is the real number defined by:

$$(4.3.1) \quad \deg(\mathcal{L}, \|\cdot\|) = \log \#(L/O_K s) - \sum_{\sigma: K \rightarrow \mathbb{C}} \log \|s\|_\sigma,$$

where  $s$  is any non-zero element of  $L$  (independence of the choice of  $s$  follows from the product formula). This definition should be compared to that of the degree of a line bundle on a smooth projective curve over a field: there one takes a rational section, and counts zeros and poles. The first term in (4.3.1) counts the zeros of  $s$  at the finite places. Interpreting this term in terms of valuations, and then norms, at the finite places, then leads to the second term which “counts” the “zeros” (or minus the “poles”, for that matter) at the infinite places. For a finite extension  $K \rightarrow K'$ , and  $(\mathcal{L}, \|\cdot\|)$  on  $\text{Spec}(O_K)$  as above, the pullback  $(\mathcal{L}', \|\cdot\|)$  to  $\text{Spec}(O_{K'})$  has degree  $\dim_K K'$  times that on  $\text{Spec}(O_K)$ .

We can now give the definition of the height given by a proper  $O_K$ -scheme  $X$  together with a hermitian line bundle  $(\mathcal{L}, \|\cdot\|)$ . Let  $x$  be in  $X(K)$ . Then, by the properness of  $X$  over  $O_K$ ,  $x$  extends uniquely to an  $O_K$ -valued point, also denoted  $x$ , and one defines:

$$(4.3.2) \quad h_K(x) := \deg x^*(\mathcal{L}, \|\cdot\|).$$

The same method as the one use above can be applied to get an absolute height  $h: X(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$ . For  $K \rightarrow K'$  a finite extension, each  $x$  in  $X(K')$

extends uniquely to an  $x$  in  $X(O_{K'})$ , and one defines:

$$(4.3.3) \quad h(x) := \frac{\deg x^*(\mathcal{L}, \|\cdot\|)}{\dim_{\mathbb{Q}} K'}.$$

It is not hard to verify that this height function  $h$  is in the class (modulo bounded functions) that is attached to  $X_K$  and  $\mathcal{L}_K$  (without metric); see Proposition 7.2 of Chapter 6 of [Co-Si], or Theorem 4.5 of Chapter V in [Ed-Ev]. In fact, for  $X = \mathbb{P}_{O_K}^n$ , and  $\mathcal{L} = \mathcal{O}(1)$  with a suitable metric, the height  $h$  just defined is *equal* to the one defined in (4.2.1).

#### 4.4 ARITHMETIC RIEMANN-ROCH AND INTERSECTION THEORY ON ARITHMETIC SURFACES

The context in which we are going to apply Arakelov theory is that of smooth projective curves  $X$  over number fields  $K$ . In [Ara] Arakelov defined an intersection theory on the *arithmetic surfaces* attached to such curves, with the aim of proving certain results, known in the case of function fields, in the case of number fields. The idea is to take a regular projective model  $\mathcal{X}$  over  $B := \text{Spec}(O_K)$  of  $X$ , and try to develop an intersection theory on the surface  $\mathcal{X}$ , analogous to the theory that one has when  $K$  is a function field. If  $K$  is a function field over a finite field  $k$ , say, one gets a projective surface  $\mathcal{X}$  over  $k$ , fibred over the nonsingular projective curve  $B$  over  $k$  that corresponds to  $K$ . On such a projective surface, intersecting with principal divisors gives zero, hence the intersection pairing factors through the Picard group of  $\mathcal{X}$ , the group of isomorphism classes of invertible  $\mathcal{O}_{\mathcal{X}}$ -modules. In the number field case one “compactifies”  $B$  by formally adding the infinite places of  $K$ ; the product formula then means that principal divisors have degree zero. Instead of the Picard group of  $\mathcal{X}$ , one considers the group of isomorphism classes of certain metrised line bundles on  $\mathcal{X}$ , as defined above. In [Fal1], see also Chapters II, III and I of [Szp], Faltings extended Arakelov’s work by establishing results such as a Grothendieck-Riemann-Roch theorem in this context. Since then, Arakelov theory has been generalised by Gillet and Soulé (see [Sou] and [Fal2]). Below, we will use the theory as given in [Fal1] and Chapter II of [Szp]. We start with some preparations concerning Riemann surfaces. The aim of this subsection is to

give the arithmetic Riemann-Roch theorem as stated and proved by Faltings.

Let  $X$  be a compact Riemann surface of genus  $g > 0$ . The space of holomorphic differentials  $H^0(X, \Omega_X^1)$  carries a natural hermitian inner product:

$$(4.4.1) \quad (\omega, \eta) \mapsto \frac{i}{2} \int_X \omega \wedge \bar{\eta}.$$

Let  $(\omega_1, \dots, \omega_g)$  be an orthonormal basis with respect to this inner product.

This leads to a positive  $(1, 1)$ -form  $\mu$  on  $X$  given by:

$$(4.4.2) \quad \mu = \frac{i}{2g} \sum_{k=1}^g \omega_k \wedge \bar{\omega}_k,$$

independent of the choice of orthonormal basis. Note that  $\int_X \mu = 1$ . We refer to [Ara] for a proof of the following proposition. Denote by  $C^\infty$  the sheaf of complex valued  $C^\infty$ -functions on  $X$ , and by  $\mathcal{A}^1$  the sheaf of complex  $C^\infty$  1-forms on  $X$ . Recall that we have a tautological differential operator  $d: C^\infty \rightarrow \mathcal{A}^1$ . It decomposes as  $d = \partial + \bar{\partial}$  where, for any local  $C^\infty$  function  $f$  and any holomorphic local coordinate  $z$ , with real and imaginary parts  $x$  and  $y$ , one has  $\partial f = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}) \cdot dz$  and  $\bar{\partial} f = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}) \cdot d\bar{z}$ .

**4.4.3 Proposition** *For each  $a$  in  $X$ , there exists a unique real-valued  $g_{a,\mu}$  in  $C^\infty(X - \{a\})$  such that the following properties hold:*

1. *we can write  $g_{a,\mu} = \log |z - z(a)| + h$  in an open neighbourhood of  $a$ , where  $z$  is a local holomorphic coordinate and where  $h$  is a  $C^\infty$ -function;*
2.  *$\partial\bar{\partial}g_{a,\mu} = \pi i\mu$  on  $X - \{a\}$ ;*
3.  *$\int_X g_{a,\mu}\mu = 0$ .*

We refer to  $\mu$  and the  $g_{a,\mu}$  as the Arakelov  $(1, 1)$ -form and the Arakelov-Green function, respectively. A fundamental property of the functions  $g_{a,\mu}$  is that they give an inverse to the map  $C^\infty \rightarrow \mathcal{A}^2$ ,  $f \mapsto (-1/\pi i)\partial\bar{\partial}f$ , with  $\mathcal{A}^2$  the sheaf of complex  $C^\infty$  2-forms on  $X$ , up to constants. For all  $f$  in  $C^\infty(X)$  we have:

$$(4.4.4) \quad f(x) = \int_{y \in X} g(x, y) \frac{-1}{\pi i} (\partial\bar{\partial}f)y + \int_X f \mu_X.$$

For a proof of this see [Fal1, pp. 393–394], or [Elk, Lemme 4].

We note that Stokes' theorem implies  $g_{a,\mu}(b) = g_{b,\mu}(a)$  for all  $a$  and  $b$  in  $X$ . The Arakelov-Green functions determine certain metrics, called admissible metrics, on all line bundles  $\mathcal{O}_X(D)$ , where  $D$  is a divisor on  $X$ , as well as on the holomorphic cotangent bundle  $\Omega_X^1$ . To start, consider line bundles of the form  $\mathcal{O}_X(a)$  with  $a$  a point in  $X$  (the general case with  $D$  follows by taking tensor products). Let  $s$  be the tautological section of  $\mathcal{O}_X(a)$ , i.e. the constant function 1. We define a smooth hermitian metric  $\|\cdot\|_{\mathcal{O}_X(a)}$  on  $\mathcal{O}_X(a)$  by putting  $\log \|s\|_{\mathcal{O}_X(a)}(b) = g_{a,\mu}(b)$  for any  $b$  in  $X$ . By property 2 of the Arakelov-Green function, the curvature form  $(2\pi i)^{-1} \partial \bar{\partial} \log(\|s\|^2)$  of  $\mathcal{O}_X(a)$  is equal to  $\mu$ . To continue, it is clear that the functions  $g_{a,\mu}$  can be used to put a hermitian metric on the line bundle  $\mathcal{O}_{X \times X}(\Delta_X)$ , where  $\Delta_X$  is the diagonal on  $X \times X$ , by putting  $\log \|s\|(a, b) = g_{a,\mu}(b)$  for the tautological section  $s$  of  $\mathcal{O}_{X \times X}(\Delta_X)$ . Restricting to the diagonal we have a canonical adjunction isomorphism  $\mathcal{O}_{X \times X}(-\Delta_X)|_{\Delta_X} \xrightarrow{\sim} \Omega_X^1$ . We define a hermitian metric  $\|\cdot\|_{\text{Ar}}$  on  $\Omega_X^1$  by insisting that this adjunction isomorphism be an isometry. It is proved in [Ara] that this gives a smooth hermitian metric on  $\Omega_X^1$ , and that its curvature form is a multiple of  $\mu$ . From now on we will work with these metrics on  $\mathcal{O}_X(P)$  and  $\Omega_X^1$  (as well as on tensor product combinations of them) and refer to them as Arakelov metrics. Explicitly: for  $D = \sum_P n_P P$  a divisor on  $X$ , we define  $g_{D,\mu} := \sum_P n_P g_{P,\mu}$ , and equip  $\mathcal{O}_X(D)$  with the metric  $\|\cdot\|$  for which  $\log \|1\|(Q) = g_{D,\mu}(Q)$ , for all  $Q$  away from the support of  $D$ . A metrised line bundle  $\mathcal{L}$  in general is called admissible if, up to a constant scaling factor, it is isomorphic to one of the admissible bundles  $\mathcal{O}_X(D)$ , or, equivalently, if its curvature form  $\text{curv}(\mathcal{L})$  is a multiple of  $\mu$ . Note that then necessarily we have  $\text{curv}(\mathcal{L}) = (\deg \mathcal{L}) \cdot \mu$  by Stokes' theorem.

For any admissible line bundle  $\mathcal{L}$ , Faltings defines a certain metric on the determinant of cohomology  $\lambda(\mathcal{L}) = \det H^0(X, \mathcal{L}) \otimes \det H^1(X, \mathcal{L})^\vee$  of the underlying line bundle. This metric is the unique metric satisfying a set of axioms. We recall these axioms (cf. [Fal1], Theorem 1): (i) any isometric isomorphism  $\mathcal{L}_1 \xrightarrow{\sim} \mathcal{L}_2$  of admissible line bundles induces an isometric isomorphism  $\lambda(\mathcal{L}_1) \xrightarrow{\sim} \lambda(\mathcal{L}_2)$ ; (ii) if we scale the metric on  $\mathcal{L}$  by a factor  $\alpha$ , the metric on  $\lambda(\mathcal{L})$  is scaled by a factor  $\alpha^{\chi(\mathcal{L})}$ , where  $\chi(\mathcal{L}) = \deg \mathcal{L} - g + 1$  is the Euler-Poincaré character-

istic of  $\mathcal{L}$ ; (iii) for any divisor  $D$  and any point  $P$  on  $X$ , the exact sequence  $0 \rightarrow \mathcal{O}_X(D - P) \rightarrow \mathcal{O}_X(D) \rightarrow P_*P^*\mathcal{O}_X(D) \rightarrow 0$  induces an isometry  $\lambda(\mathcal{O}_X(D)) \xrightarrow{\sim} \lambda(\mathcal{O}_X(D - P)) \otimes P^*\mathcal{O}_X(D)$ ; (iv) for  $\mathcal{L} = \Omega_X^1$ , the metric on  $\lambda(\mathcal{L}) \cong \det H^0(X, \Omega_X^1)$  is defined by the hermitian inner product  $(\omega, \eta) \mapsto (i/2) \int_X \omega \wedge \bar{\eta}$  on  $H^0(X, \Omega_X^1)$ . In particular, for an admissible line bundle  $\mathcal{L}$  of degree  $g - 1$ , the metric on the determinant of cohomology  $\lambda(\mathcal{L})$  is independent of scaling.

It was proved by Faltings that we can relate the metric on the determinant of cohomology to theta functions on the Jacobian of  $X$ . Let  $\mathbb{H}_g$  be the Siegel upper half space of complex symmetric  $g$ -by- $g$ -matrices with positive definite imaginary part. Let  $\tau$  in  $\mathbb{H}_g$  be the period matrix attached to a symplectic basis of  $H_1(X, \mathbb{Z})$  and consider the analytic Jacobian  $J_\tau(X) = \mathbb{C}^g / (\mathbb{Z}^g + \tau\mathbb{Z}^g)$  attached to  $\tau$ . On  $\mathbb{C}^g$  one has a theta function  $\vartheta(z; \tau) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i {}^t n \tau n + 2\pi i {}^t n z)$ , giving rise to a reduced effective divisor  $\Theta_0$  and a line bundle  $\mathcal{O}(\Theta_0)$  on  $J_\tau(X)$ . Now consider on the other hand the set  $\text{Pic}_{g-1}(X)$  of divisor classes of degree  $g - 1$  on  $X$ . It comes with a canonical subset  $\Theta$  given by the classes of effective divisors. A fundamental theorem of Abel-Jacobi-Riemann says that there is a canonical bijection  $\text{Pic}_{g-1}(X) \xrightarrow{\sim} J_\tau(X)$  mapping  $\Theta$  onto  $\Theta_0$ . As a result, we can equip  $\text{Pic}_{g-1}(X)$  with the structure of a compact complex manifold, together with a divisor  $\Theta$  and a line bundle  $\mathcal{O}(\Theta)$ .

The function  $\vartheta$  is not well-defined on  $\text{Pic}_{g-1}(X)$  or  $J_\tau(X)$ . We can remedy this by putting

$$(4.4.5) \quad \|\vartheta\|(z; \tau) = (\det \Im(\tau))^{1/4} \exp(-\pi {}^t y (\Im(\tau))^{-1} y) |\vartheta(z; \tau)|,$$

with  $y = \Im(z)$ . One can check that  $\|\vartheta\|$  descends to a function on  $J_\tau(X)$ . By our identification  $\text{Pic}_{g-1}(X) \xrightarrow{\sim} J_\tau(X)$  we obtain  $\|\vartheta\|$  as a function on  $\text{Pic}_{g-1}(X)$ . It can be checked that this function is independent of the choice of  $\tau$ . Note that  $\|\vartheta\|$  gives a canonical way to put a metric on the line bundle  $\mathcal{O}(\Theta)$  on  $\text{Pic}_{g-1}(X)$ . For any line bundle  $\mathcal{L}$  of degree  $g - 1$  there is a canonical isomorphism from  $\lambda(\mathcal{L})$  to  $\mathcal{O}(-\Theta)[\mathcal{L}]$ , the fibre of  $\mathcal{O}(-\Theta)$  at the point  $[\mathcal{L}]$  in  $\text{Pic}_{g-1}(X)$  determined by  $\mathcal{L}$ . Faltings proves that when we give both sides the metrics discussed above, the norm of this isomorphism is a constant independent of  $\mathcal{L}$ ; he writes it as  $\exp(\delta(X)/8)$ . In more explicit terms, this means that for any line bundle  $\mathcal{L}$  of degree  $g - 1$  on  $X$  with



$h^0(\mathcal{L}) = 0$  (and hence  $h^1(\mathcal{L}) = 0$ ) we have:

$$(4.4.6) \quad \lambda(\mathcal{L}) = \mathbb{C}, \quad \|1\|_{\lambda(\mathcal{L})}^{-1} = \exp(\delta(X)/8) \cdot \|\theta\|([\mathcal{L}]).$$

The invariant  $\delta(X)$  of  $X$  appears in the Noether formula, see below.

We will now turn to intersections on an arithmetic surface. For us, an arithmetic surface is a proper, flat morphism  $p: \mathcal{X} \rightarrow B$  with  $\mathcal{X}$  a regular scheme, with  $B$  the spectrum of the ring of integers  $O_K$  in a number field  $K$ , and with generic fibre a geometrically connected and smooth curve  $X/K$ . We say that  $\mathcal{X}$  is of genus  $g$  if the generic fibre is of genus  $g$ . We will always assume that  $p$  is a semi-stable curve, unless explicitly stated otherwise. After extending the base field if necessary, any geometrically connected, smooth proper curve  $X/K$  of positive genus with  $K$  a number field is the generic fibre of a unique semi-stable arithmetic surface.

An Arakelov divisor on  $\mathcal{X}$  is a finite formal integral linear combination of integral closed subschemes of codimension 1 of  $\mathcal{X}$  plus a contribution  $\sum_{\sigma} \alpha_{\sigma} \cdot F_{\sigma}$  running over the complex embeddings of  $K$ . Here  $\alpha_{\sigma}$  is a real number, and the symbols  $F_{\sigma}$  correspond to the compact Riemann surfaces  $X_{\sigma}$  obtained by base changing  $X/K$  to  $\mathbb{C}$  via  $\sigma$ . We have an  $\mathbb{R}$ -valued intersection product  $(\cdot, \cdot)$  for such divisors, respecting linear equivalence. When we want to indicate which model  $\mathcal{X}$  is used for this intersection product, we will use the notation  $(\cdot, \cdot)_{\mathcal{X}}$ . The notion of principal divisor is given as follows: let  $f$  be a non-zero rational function in  $K(X)$ , then  $(f) = (f)_{\text{fin}} + (f)_{\text{inf}}$  with  $(f)_{\text{fin}}$  the usual Weil divisor of  $f$  on  $\mathcal{X}$ , and with  $(f)_{\text{inf}} = \sum_{\sigma} v_{\sigma}(f) \cdot F_{\sigma}$  with  $v_{\sigma}(f) = -\int_{X_{\sigma}} \log |f|_{\sigma} \mu_{\sigma}$ . For a list of properties of this intersection product we refer to [Ara], [Fal1] or Chapter II of [Szp].

It is proved in [Ara] that the group of linear equivalence classes of Arakelov divisors is canonically isomorphic to the group  $\widehat{\text{Pic}}(\mathcal{X})$  of isometry classes of admissible line bundles on  $\mathcal{X}$ . By an admissible line bundle on  $\mathcal{X}$  we mean the datum of a line bundle  $\mathcal{L}$  on  $\mathcal{X}$ , together with admissible metrics on the restrictions  $\mathcal{L}_{\sigma}$  of  $\mathcal{L}$  to the  $X_{\sigma}$ . In particular we have a canonical admissible line bundle  $\omega_{\mathcal{X}/B}$  whose underlying line bundle is the relative dualising sheaf of  $p$ . In many situations it is convenient to treat intersection numbers from the point of view of admissible line bundles.

For example, if  $P: B \rightarrow \mathcal{X}$  is a section of  $p$ , and  $D$  is an Arakelov divisor

on  $\mathcal{X}$ , the pull-back  $P^*\mathcal{O}_{\mathcal{X}}(D)$  is a metrised line bundle on  $B$ , and we have:

$$(4.4.7) \quad (D, P) = \deg P^*\mathcal{O}_{\mathcal{X}}(D),$$

where the degree  $\deg$  of a metrised line bundle is as defined in (4.3.1). As a second example, we mention that by definition of the metric on  $\omega_{\mathcal{X}/B}$ , we have for each section  $P: B \rightarrow \mathcal{X}$  of  $p$  an adjunction formula:

$$(4.4.8) \quad (P, P + \omega_{\mathcal{X}/B}) = 0.$$

For an admissible line bundle  $\mathcal{L}$  on  $\mathcal{X}$ , we have the notion of determinant of cohomology on  $B$ , in this context denoted by  $\det \mathrm{R}p_*\mathcal{L}$  (see Chapter II of [Szp]). By using the description above for its metrisation over the complex numbers, we obtain the determinant of cohomology on  $B$  as a metrised line bundle. One of its most important features is a metrised Riemann-Roch formula (cf. [Fal1], Theorem 3), also called *arithmetic Riemann-Roch formula*:

$$(4.4.9) \quad \deg \det \mathrm{R}p_*\mathcal{L} = \frac{1}{2}(\mathcal{L}, \mathcal{L} \otimes \omega_{\mathcal{X}/B}^{-1}) + \deg \det p_*\omega_{\mathcal{X}/B}$$

for any admissible line bundle  $\mathcal{L}$  on  $\mathcal{X}$ .

The term  $\deg \det p_*\omega_{\mathcal{X}/B}$  is also known as the *Faltings height* of  $X$ , the definition of which we will now recall. We let  $J_K$  be the Jacobian variety of  $X$ , and  $J$  its Néron model over  $B$ . Then we have the locally free  $O_K$ -module  $\mathrm{Cot}_0(J) := 0^*\Omega_{J/O_K}^1$  of rank  $g$ , and hence the invertible  $O_K$ -module of rank one:

$$\omega_J := \bigwedge^g 0^*\mathrm{Cot}_0(J).$$

For each  $\sigma: K \rightarrow \mathbb{C}$  we have the scalar product on  $\mathbb{C} \otimes_{O_K} \omega_J$  given by:

$$\langle \omega | \eta \rangle_{\sigma} = (i/2)^g (-1)^{g(g-1)/2} \int_{J_{\sigma}(\mathbb{C})} \omega \wedge \bar{\eta}.$$

The Faltings height  $h_K(X)$  is then defined to be the Arakelov degree of this metrised line bundle:

$$(4.4.10) \quad h_K(X) = \deg(\omega_J),$$

and the absolute Faltings height (also called stable Faltings height)  $h_{\mathrm{abs}}(X)$  of  $X$  is defined as:

$$(4.4.11) \quad h_{\mathrm{abs}}(X) = [K : \mathbb{Q}]^{-1} \deg(\omega_J).$$

We remark that the stable Faltings height of  $X$  does not change after base change to larger number fields; that is why it is called stable. Therefore,  $h_{\text{abs}}(X)$  can be computed from any model of  $X$  over a number field as long as that model has stable reduction over the ring of integers of that number field.

As  $\mathcal{X} \rightarrow B$  is semi-stable, a result of Raynaud gives that the connected component of 0 of  $J$  is the Picard scheme  $\text{Pic}_{\mathcal{X}/B}^0$ , whose tangent space at 0 is  $R^1 p_* \mathcal{O}_{\mathcal{X}}$ . Therefore,  $\text{Cot}_0(J)$  is the same as  $p_* \omega$ , as locally free  $\mathcal{O}_K$ -modules. A simple calculation (see lemme 3.2.1 in Chapter I of [Szp]) shows that, with these scalar products,  $\omega_J$  and  $\det p_* \omega$  are the same as metrised  $\mathcal{O}_K$ -modules. Therefore we have:

$$(4.4.12) \quad h_K(X) = \deg(\omega_J) = \deg \det p_* \omega.$$

One may derive from (4.4.9) the following projection formula: let  $E$  be a metrised line bundle on  $B$ , and  $\mathcal{L}$  an admissible line bundle on  $\mathcal{X}$ . Then the formula:

$$(4.4.13) \quad \deg \det R p_*(\mathcal{L} \otimes p^* E) = \deg \det R p_* \mathcal{L} + \chi(\mathcal{L}) \cdot \deg E$$

holds. Here again  $\chi(\mathcal{L})$  is the Euler-Poincaré characteristic of  $\mathcal{L}$  on the fibres of  $p$ .

## Chapter Five

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### Computing complex zeros of polynomials and series

J.-M. Couveignes

The purpose of this chapter is twofold. We first want to prove Theorems 5.3.1 and 5.4.3 below about the complexity of computing complex roots of polynomials and zeros of power series. The existence of a deterministic polynomial time algorithm for these purposes plays an important role in this book. More importantly, we want to explain what it means for us to compute with real or complex data in polynomial time. All the necessary concepts and algorithms already exist and are provided partly by numerical analysis and partly by algorithmic complexity theory. However, the computational model of numerical analysis is not quite a Turing machine, but rather a real computer with floating point arithmetic. Such a computer makes rounding errors at almost every step in the computation. In this context, it is good enough to estimate the conditioning of the problem and the stability of the used algorithm. Statements about conditioning and stability tend to be local and qualitative. And this suffices to identify and overcome most difficulties and design optimal methods.

Our situation however is quite different. We don't really care about efficiency. Being polynomial time is enough to us. On the other hand, we want a rigorous, unconditional and fully general proof that the algorithms we use are polynomial time and return a result that is correct up to a small error that must be bounded rigorously in any case. For this reason, we shall not use floating point registers: we don't want to worry about the accumulation of rounding errors. We rather decompose the computation in big blocks. Inside every block we only allow exact computations (e.g. using integers or

rational numbers). We also check that the function computed by every such block is well conditioned and we make a precise statement for that. Finally, we need to control the accumulation of errors in a chain of big blocks. But this shall not be too difficult because, since the blocks are big and efficient enough, the general organization of the algorithm is simple and involves few blocks.

In Section 5.1 we recall basic definitions in computational complexity theory. Section 5.2 deals with the problem of computing square roots. We illustrate on this simple example what is expected from an algorithm in our context. The more general problem of computing complex roots of polynomials is treated in Section 5.3. Finally, we study in Section 5.4 the problem of finding zeros of a converging power series.

**Notation:** The symbol  $\Theta$  in this chapter stands for a positive effective absolute constant. So any statement containing this symbol becomes true if the symbol is replaced in every occurrence by some large enough real number.

## 5.1 POLYNOMIAL TIME COMPLEXITY CLASSES

In this section we briefly recall classical definitions from computational complexity theory. Since we only need to define the polynomial time complexity classes, we shall not go into the details. We refer the reader to Papadimitriou's book [Pap] for a complete treatment of these matters.

*Turing machines* are a theoretical model for computers. They are finite automata (they have finitely many inner states) but they can write or read on an infinite tape with a tape head. A Turing machine can be defined by a transition table. For a given inner state and current character read by the head, the transition table provides the next inner state, which character to write on the tape in place of the current one, and how the head should move on the tape (one step left, one step right, or no move at all). See [Pap, Chapter 2] for a formal definition.

A *decision problem* is a question that must be answered by yes or no. For example deciding if an integer is prime. The answer of a *functional problem* is a more general function of the question. For example factoring

an integer is a functional problem. If we want to solve a problem with a Turing machine, we write the *input* on the tape, we run the Turing machine, and we wait until it stops. We then read the *output* on the tape. If the machine always stops and returns the correct answer, we say that it solves the problem in question. The time *complexity* is the number of steps before the Turing machine has solved a given problem. Such a Turing machine is said to be *deterministic* because its behaviour only depends on the input. The *size* of the input is the number of bits required to encode it. This is the space used on the tape to write this input. For example, the size of an integer is the number of bits in its binary expansion. A problem is said to be *deterministic polynomial time* if there exists a deterministic Turing machine that solves it in time polynomial in the size of the input. The *class* of all functional problems that can be solved in deterministic polynomial time is denoted **FP** or **FPTIME**. The class of deterministic polynomial time decision problems is denoted **P** or **PTIME**.

There exist other models for complexity theory. For example one may define *multitape Turing machines*. There also exist *random access machines*. All these models lead to equivalent definitions of the polynomial complexity classes. An *algorithm* is a sequence of elementary operations and instructions. Any algorithm can be turned into a Turing machine, but this is fastidious and rather useless since conceptual description of the algorithm suffices to decide if the number of elementary operations performed by the algorithm is polynomial in the size of the input. If this is the case, we say that the algorithm is deterministic polynomial time and we know that the corresponding problem is in **PTIME** or **FPTIME**.

For example, if we want to multiply two positive integers  $N_1$  and  $N_2$ , then the size of the input  $(N_1, N_2)$  is the number of digits in  $N_1$  and  $N_2$  and this is  $\lceil \log_{10}(N_1 + 1) \rceil + \lceil \log_{10}(N_2 + 1) \rceil$ . The number of elementary operations required by the elementary school algorithm for multiplication is  $\Theta \times \lceil \log_{10}(N_1 + 1) \rceil \times \lceil \log_{10}(N_2 + 1) \rceil$ . The constant here depends on the (reasonable) definition we have chosen for what an elementary operation is. We don't care about constants anyway. We say that the elementary school algorithm is deterministic polynomial time. There also exists a deterministic polynomial time algorithm for Euclidean division (e.g. the elementary school one). The extended Euclidean algorithm computes coefficients in

Bézout's identity in deterministic polynomial time also. So addition, subtraction, multiplication and inversion in the ring  $\mathbb{Z}/N\mathbb{Z}$  can be performed in time polynomial in  $\log N$ . The class  $a \bmod N$  in  $\mathbb{Z}/N\mathbb{Z}$  is represented by its smallest non-negative element. We denote it  $a \% N$ . This is the remainder in the Euclidean division of  $a$  by  $N$ .

A very important problem is exponentiation: given  $a \bmod N$  with  $0 \leq a \leq N - 1$  and an integer  $e \geq 1$ , compute  $a^e \bmod N$ .

Computing  $a^e$  then reducing modulo  $N$  is not a good idea because  $a^e$  might be very large. Another option would be to set  $a_1 = a$  and compute  $a_k = (a_{k-1} \times a) \% N$  for  $2 \leq k \leq e$ . This requires  $e - 1$  multiplications and  $e - 1$  Euclidean divisions. And we never deal with integers bigger than  $N^2$ . The complexity of this method is thus  $\Theta \times e \times (\log N)^2$  using elementary school algorithms. It is well known however that we can do much better. We write the expansion of  $e$  in base 2

$$e = \sum_{0 \leq k \leq K} \epsilon_k 2^k$$

and we set  $b_0 = a$  and  $b_k = b_{k-1}^2 \% N$  for  $1 \leq k \leq K$ . We then notice that

$$a^e \equiv \prod_{0 \leq k \leq K} b_k^{\epsilon_k} \bmod N.$$

So we can compute  $(a^e) \% N$  at the expense of  $\Theta \times \log e$  multiplications and Euclidean divisions between integers  $\leq N^2$ . The total number of elementary operations is thus  $\Theta \times \log e \times (\log N)^2$  with this method. So exponentiation in  $\mathbb{Z}/N\mathbb{Z}$  lies in **FPTIME**. This is an elementary but decisive result in algorithmic number theory. The algorithm above is called *fast exponentiation* and it makes sense in any group. We shall use it many times and in many different contexts.

A first interesting consequence is that for  $p$  an odd prime and  $a$  an integer such that  $1 \leq a \leq p - 1$ , we can compute the Legendre symbol

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \bmod p$$

at the expense of  $\Theta(\log p)^3$  elementary operations. So testing quadratic residues is achieved in polynomial deterministic time. Assume now that we are interested in the following problem

Given an odd prime integer  $p$ , find an integer  $a$  such that  
 $1 \leq a \leq p - 1$  and  $a$  is not a square modulo  $p$ . (★)

This looks like a very easy problem because half of the non-zero residues modulo  $p$  are not squares. So we may just pick a random integer  $a$  between 1 and  $p - 1$  and compute the Legendre symbol  $\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}}$ . If the symbol is  $-1$  we output  $a$ . Otherwise we output FAIL. The probability of success is  $1/2$  and failing is not such a big problem because we can rerun the algorithm: we just pick another random integer  $a$ .

This is a typical example of a *randomized Las Vegas* algorithm. The behavior of the algorithm depends on the input of course, but also on the result of some random choices. One has to flip coins. A nice model for such an algorithm would be a Turing machine that receives besides the input, a long enough (say infinite) one-dimensional array  $\mathcal{R}$  consisting of 0's and 1's. Whenever the machine needs to flip a coin, she looks at the next entry in the array  $\mathcal{R}$ . So the Turing machine does not need to flip coins: we provide her with enough random data at the beginning. We assume that the running time of the algorithm is bounded from above in terms of the size of the input only (this upper bound should not depend on the random data  $\mathcal{R}$ ). For each input, we ask that the probability (on  $\mathcal{R}$ ) that the Turing machine provides the correct answer is  $\geq 1/2$ . The random data  $\mathcal{R}$  takes values in  $\{0, 1\}^{\mathbb{N}}$ . The measure on this latter set is the limit of the uniform measures on  $\{0, 1\}^k$  when  $k$  tends to infinity. If the Turing machine fails to return the correct answer, she should return FAIL instead.

We just proved that finding a non-quadratic residue modulo  $p$  can be done in Las Vegas probabilistic polynomial time. There is no known algorithm that can be proven to solve this problem in deterministic polynomial time. The class of Las Vegas probabilistic polynomial time decision problems is denoted **ZPP**.

We now consider another slightly more difficult problem

Given an odd prime integer  $p$ , find a generating set  $(g_i)_{1 \leq i \leq I}$  for the cyclic group  $(\mathbb{Z}/p\mathbb{Z})^*$ . (★★)

We have a simple probabilistic algorithm for this problem. We compute an integer  $I$  such that

$$\log_2(3 \log_2(p - 1)) \leq I \leq \log_2(3 \log_2(p - 1)) + 2$$



and we pick  $I$  random integers  $(a_i)_{1 \leq i \leq I}$  in the interval  $[1, p-1]$ . The  $a_i$  are uniformly distributed and pairwise independent. We set  $g_i = a_i \bmod p$  and we show that the  $(g_i)_{1 \leq i \leq I}$  generate the group  $(\mathbb{Z}/p\mathbb{Z})^*$  with probability  $\geq 2/3$ . Indeed, if they don't, they must all lie in a maximal subgroup of  $(\mathbb{Z}/p\mathbb{Z})^*$ . The maximal subgroups of  $(\mathbb{Z}/p\mathbb{Z})^*$  correspond to prime divisors of  $p-1$ . Let  $q$  be such a prime divisor. The probability that the  $(g_i)_{1 \leq i \leq I}$  all lie in the subgroup of index  $q$  is bounded from above by

$$\frac{1}{q^I} \leq \frac{1}{2^I}$$

so the probability that the  $(g_i)_{1 \leq i \leq I}$  don't generate  $(\mathbb{Z}/p\mathbb{Z})^*$  is bounded from above by  $2^{-I}$  times the number of prime divisors of  $p-1$ . Since the latter is  $\leq \log_2(p-1)$ , the probability of failure is

$$\leq \frac{\log_2(p-1)}{2^I}$$

and this  $\leq 1/3$  by definition of  $I$ .

Note that here we have a new kind of probabilistic algorithm: the answer is correct with probability  $\geq 2/3$  but when the algorithm fails, he may return a false answer. Such an algorithm (a Turing machine) is called *Monte Carlo* probabilistic. This is weaker than a Las Vegas algorithm. We just proved that problem  $(\star\star)$  can be solved in Monte Carlo probabilistic polynomial time. We don't know of any Las Vegas probabilistic polynomial time algorithm for this problem. The class of Las Vegas probabilistic polynomial time decision problems is denoted **BPP**.

In general, a Monte Carlo algorithm can be turned into a Las Vegas one provided the answer can be checked efficiently, because we then can force the algorithm to admit that he has failed. Note also that if we set

$$I = f + \lceil \log_2(\log_2(p-1)) \rceil$$

where  $f$  is a positive integer; then the probability of failure in the algorithm above is bounded from above by  $2^{-f}$ . So we can make this probability arbitrarily small at almost no cost.

The main purpose of this book is to prove statements about the complexity of computing coefficients of modular forms. For example, Theorem 15.1.1 states that on input a prime integer  $p$ , computing the Ramanujan function

$\tau(p)$  can be done in deterministic polynomial time in  $\log p$ . An important intermediate result is to prove that one can compute some Galois representations modulo  $l$  in time polynomial in  $l$ . We shall present two methods for computing such representations. Both methods rely on computing approximations. The first method computes complex approximations and leads to a deterministic algorithm. This is explained in Chapter 12 using the main results in this Chapter 5. We also present in Chapter 13 a probabilistic method that relies on computations modulo small auxiliary primes. The main reason why the latter methods are probabilistic is that they require to find generating sets for the Picard group of curves over finite fields. This is a generalization of problem (\*\*\*) and solving it in deterministic polynomial time is out of reach at the moment.

## 5.2 COMPUTING THE SQUARE ROOT OF A POSITIVE REAL NUMBER

In this section, we consider the following problem:

*Given a positive real number  $a$ ,  
compute the positive square root  $b = \sqrt{a}$  of  $a$ .* (\*\*\*)

We need an algorithm that runs in deterministic polynomial time. This raises a few simple minded but important questions about what should be called an algorithm in this context. In Section 5.2.1 we try to formulate problem (\*\*\*) in a more precise way. We explain what is meant by an algorithm in this context, and which properties one would expect from such an algorithm. In Section 5.2.2 we present the classical dichotomy algorithm and check that it has polynomial time complexity. The notions presented in this section are classical and elementary and come from computational complexity theory [Pap] and numerical analysis [Hig, Hen]. The algorithms and methods we present are not original either, and they are far from optimal. We stress that our unique goal here is to prove that a polynomial time algorithm (in a sense that can be made rigorous) exists for some classical computational problem regarding real numbers.

### 5.2.1 Turing machines and real numbers

We shall use deterministic Turing machines; as defined in [Pap, Chapter 2] for example. There is something annoying with problem (\*\*\*) however: both the input and the output are real numbers. Both existing computing devices and Turing machines only deal with discrete data. So they can't deal with real numbers. We may imagine a Turing machine or a computer handling registers with real numbers as in [Bl-Sh-Sm]. However, this would not be of great use to us, because we plan to perform computations on real numbers as an intermediate step in the computation of a discrete quantity: our basic idea is to compute an integer (having some arithmetic significance) from a good enough real approximation of it. In the end, we want a rigorous proof that the discrete information we are interested in can be computed by a deterministic Turing machine. We need to prove that a standard deterministic Turing machine can efficiently and safely compute with real and complex numbers, or at least with approximations of them. One possible approach to this classical problem is interval arithmetic as presented in [Hay]. We shall follow a slightly different track, which is better adapted to our situation. Our goal is to prove that a certain number of more or less elementary calculations on complex numbers can be safely and efficiently performed (in a way that will be made more precise soon) by an ordinary Turing machine. These calculations include root finding of polynomials and power series, computation with divisors on modular curves, direct and inverse Jacobi problems on these curves. In this section, the problem (\*\*\*) will be used to illustrate a few simple ideas that will be applied more systematically in the sequel. The first question to be addressed concerns the input.

*What is the input of problem (\*\*\*) ?*

Well, if a classical Turing machine is supposed to solve problem (\*\*\*) it cannot be given the real number  $a$  all at a time. That would be too big for her. Instead of that, we assume that the Turing machine is given a black box  $\text{BOX}_a$ . On input a positive integer  $m$ , the black box  $\text{BOX}_a$  returns a decimal fraction  $N_a \times 10^{-m_a}$  such that  $|a - N_a \times 10^{-m_a}| \leq 10^{-m_a}$ . If the black box answers immediately, we will call it an *oracle* for  $a$ . A more realistic situation is that the black box answers in polynomial time. This means that on input a positive integer  $m_a$ , the black box outputs the expected numerator

$N_a$  in time  $\leq A_a m_a^{d_a}$  where  $A_a$  and  $d_a$  are positive integers depending on  $a$  but not on  $m_a$ . We assume that a Turing machine calling to a black box (or an oracle) must take the time to read and copy the integrality of the oracle's answer. For example, a Turing machine with an oracle for  $\pi$  cannot access the  $10^{100}$ -th digit without reading the previous ones.

In all the situations we shall be facing, there will be a Turing machine in the black box. However, not every real number can be associated with such a Turing machine: the set of Turing machines is countable and the set of real numbers is not. This is the theoretical reason for introducing black boxes there.

*What should be the output of a Turing machine solving problem (★★) ?*

Again, we don't expect the Turing machine to provide us with the real  $b = \sqrt{a}$  all at a time. We would be a bit embarrassed with it anyway. We rather expect the Turing machine, on input a positive integer  $m_b$  and a black box for  $a$ , will return a decimal fraction  $N_b \times 10^{-m_b}$  such that  $|b - N_b \times 10^{-m_b}| \leq 10^{-m_b}$ . The square root Turing machine may call the black box for  $a$  once or several times.

Altogether, the input of the square root Turing machine should consist of a black box  $\text{BOX}_a$  for  $a$  and a positive integer  $m_b$  telling her the desired absolute accuracy of the expected result. And the output will be a decimal fraction approximating  $b$ .

*How do we define the complexity of a square root Turing machine ? What does it mean for such a Turing machine to be polynomial time ?*

Assume that we have a Turing machine  $\text{SQRT}$  that computes square roots. Assume that the input of the square root machine  $\text{SQRT}$  consists of a black box  $\text{BOX}_a$  for  $a$  and a positive integer  $m_b$  (the required absolute accuracy of the result). We look for an upper bound for the number of elementary operations performed by  $\text{SQRT}$ , as a function of  $\log \max(a, 1)$  and  $m_b$ . Such a bound will be called a *complexity* estimate for  $\text{SQRT}$ . Notice that a call to the black box  $\text{BOX}_a$  will be counted as a single operation.

We assume that there exist two positive integers  $A_{\text{SQRT}}$  and  $d_{\text{SQRT}}$  such that the complexity of  $\text{SQRT}$  is bounded above by a polynomial  $A_{\text{SQRT}}(m_b + \log \max(a, 1))^{d_{\text{SQRT}}}$ . Then the number of calls to  $\text{BOX}_a$  is certainly bounded by this number. And the absolute accuracy required

from  $\text{BOX}_a$  cannot exceed this number either, otherwise the machine SQR T would not even find the time to read the digits provided to her by  $\text{BOX}_a$ . So by combining a black box  $\text{BOX}_a$  for the input and the Turing machine SQR T, we obtain a black box  $\text{BOX}_b$  for the output  $b$ . And if both SQR T and  $\text{BOX}_a$  have polynomial time complexity, so is the resulting black box  $\text{BOX}_b$ .

*Is problem (\*\*\*) well conditioned ?*

We have seen that the Turing machine SQR T cannot always access the exact value of the input  $a$ . Instead of that SQR T is provided with a black box that sends to her approximations of  $a$ . We want to make sure that a good approximation of  $b = \sqrt{a}$  can be deduced from a good approximation of  $a$ . Since the function  $a \mapsto \sqrt{a}$  is  $\frac{1}{2}$ -Lipschitz on the interval  $[1, \infty]$ , we have

$$|\Delta b| = |\sqrt{a + \Delta a} - \sqrt{a}| \leq \frac{1}{2} |\Delta a| \quad (5.1)$$

as soon as  $a \geq 2$  and  $|\Delta a| \leq 1$ . So a small perturbation of the input results in a small perturbation of the expected output in that case. One says that the problem is *well conditioned*.

Not every computational problem is well conditioned. For example, computing the rounding function  $a \mapsto \lceil a \rceil$  is not well conditioned if one gets close to  $\frac{1}{2}$  because the function is not even continuous there.

We shall not need to formalize a definition of conditioning, but we shall check in several occasions that the function we want to evaluate is  $A$ -Lipschitz for a reasonable constant  $A$ . A weaker condition may suffice in some cases: for example, assume that we want to compute a function  $a \mapsto b$  and assume that  $-\log \min(1, |\Delta b|)$  is lower bounded by  $(-\log \min(1, |\Delta a|))^{\frac{1}{e}}$  for some fixed positive integer  $e$ . Then the loss of accuracy is polynomial in some sense: one can obtain  $m$  digits of  $b$  from  $\Theta m^e$  digits of  $a$ .

For example, if we consider the problem of computing the square root of a positive real number  $a$ , we notice that the function  $a \mapsto \sqrt{a}$  is not Lipschitz on  $[0, +\infty[$  but we have

$$\begin{aligned}
|\Delta b| &= |\sqrt{a + \Delta a} - \sqrt{a}| = \sqrt{a} \times \left| \sqrt{1 + \frac{\Delta a}{a}} - 1 \right| \\
&\leq \sqrt{a} \times \sqrt{|\Delta a|}
\end{aligned} \tag{5.2}$$

whenever  $|\Delta a| \leq \min(a, a^2)$ .

So for small values of  $a$ , we loose (no more than) half the absolute accuracy when taking the square root. This is enough for us to say that the problem is well conditioned.

### 5.2.2 The dichotomy algorithm

Given a real interval  $[M_1, M_2]$  and a continuous function

$$f : [M_1, M_2] \rightarrow \mathbb{R}$$

such that  $f(M_1)f(M_2) < 0$ , the dichotomy algorithm finds an approximation of a real zero of  $f$  in  $[M_1, M_2]$ . We use the dichotomy algorithm to compute the positive square root  $b = \sqrt{a}$  of a positive decimal number  $a = N_a \times 10^{-m_a}$  where  $N_a \geq 1$  and  $m_a \geq 0$  are integers. So we call  $f : [0, \infty[ \rightarrow [0, \infty[$  the map  $x \mapsto x^2 - a$ . The algorithm below only handles integers and decimal fractions. Let  $h$  be the smallest integer such that  $10^h \geq N_a$ . If  $h - m_a$  is even we set  $M_2 = 10^{\frac{h-m_a}{2}}$ , otherwise we take  $M_2 = 10^{\frac{h-m_a+1}{2}}$ . We set  $M_1 = M_2/10$ . We assume that we are given also a positive integer  $m_b$  (the required absolute accuracy of the result).

We use two registers  $R_1$  and  $R_2$  containing decimal fractions. The initial value of  $R_1$  is  $M_1$  and the initial value of  $R_2$  is  $M_2$ .

The algorithm then goes as follows:

1. If  $f(\frac{R_1+R_2}{2})$  is zero or if  $|R_1 - R_2| \leq 10^{-m_b}$ , output  $\frac{R_1+R_2}{2}$  and stop.
2. If  $f(R_1)f(\frac{R_1+R_2}{2}) > 0$ , set  $R_1 := \frac{R_1+R_2}{2}$  and go to step 1.
3. If  $f(R_1)f(\frac{R_1+R_2}{2}) < 0$ , set  $R_2 := \frac{R_1+R_2}{2}$  and go to step 1.

The algorithm outputs a decimal fraction  $\tilde{b}$  such that  $|b - \tilde{b}| \leq 10^{-m_b}$ . The loop is not executed more than  $\Theta(m_b + h - m_a + 1)$  times. So the denominator of  $\tilde{b}$  is bounded above by  $10^{\Theta(m_a + m_b + h + 1)}$  and the same holds

for any intermediate result in the course of the algorithm. So the *complexity* of the algorithm is polynomial in  $m_a$ ,  $\log N$  and  $m_b$ . And so is the size of the output.

We now consider a few questions raised by this algorithm.

*What to do if the input  $a$  is a real number rather than a decimal one ?*

In that case, we assume that we are given a black box  $\text{BOX}_a$  for  $a$  and a positive integer  $m_b$  (the required absolute accuracy of the result). Our first task is to obtain from  $\text{BOX}_a$  a positive lower bound for  $a$ . So we first ask her for a decimal approximation of  $a$  within  $10^{-10}$ . If the answer is zero, we ask her for a decimal approximation of  $a$  within  $10^{-20}$ . If the answer is zero again, we ask her for a decimal approximation of  $a$  within  $10^{-40}$ . After a few steps, either we obtain a positive lower bound for  $a$  or we prove that  $a$  is smaller than  $10^{-2m_b}$ . In the later case, we output  $\hat{b} = 0$  which is a good enough approximation for  $b = \sqrt{a}$ .

The above shows that we can assume that we know the smallest positive integer  $s$  such that  $a$  is bigger than  $10^{-s}$ .

Inequalities (5.2) and (5.1) show that if  $l \geq 2s$  and  $|\hat{a} - a| \leq 10^{-\Theta l}$  then  $|\sqrt{\hat{a}} - \sqrt{a}| \leq 10^{-l}$  for some absolute constant  $\Theta$ . So we ask  $\text{BOX}_a$  for a decimal approximation  $\hat{a}$  of  $a$  within  $10^{-\Theta(m_b+2s+1)}$ . We set  $b = \sqrt{a}$  and  $\hat{b} = \sqrt{\hat{a}}$  and we check that  $|\hat{b} - b| \leq 10^{-m_b-2s-1}$ . We send the decimal number  $\hat{a}$  to the dichotomy algorithm above and ask it for an approximation of  $\hat{b} = \sqrt{\hat{a}}$  within  $10^{-m_b-1}$ .

We obtain a decimal number  $\tilde{b}$  such that  $|\tilde{b} - \hat{b}| \leq 10^{-m_b-1}$ . We output  $\tilde{b}$  and check that

$$|\tilde{b} - b| \leq |\tilde{b} - \hat{b}| + |\hat{b} - b| \leq 2 \times 10^{-m_b-1} \leq 10^{-m_b}. \quad (5.3)$$

So we have designed an algorithm (a Turing machine) that computes the positive square root of a positive number  $a$  in time polynomial in  $\log(a+1)$  and the required accuracy.

*Is the above algorithm stable ?*

People in numerical analysis say that an algorithm is *stable* when the value output by the algorithm is not too far from the exact value. As we

just proved, the dichotomy algorithm can compute  $b = \sqrt{a}$  within  $10^{-m_b}$  in time polynomial in  $\log(a + 1)$  and  $m_b \geq 1$ . So it can be said to be stable. We shall allow ourselves to use this terminology; but we prefer to state and prove clear and accurate complexity estimates like the one above.

It is important to make a distinction between stability and conditioning. A *problem* can be said to be well conditioned. An *algorithm* can be said to be stable. In the above proof that the dichotomy algorithm for computing square roots is stable, we have used the fact that the problem itself is well conditioned. This is illustrated by inequality (5.3).

### 5.3 COMPUTING THE COMPLEX ROOTS OF A POLYNOMIAL

In this section, we consider the following problem:

*Given a degree  $d \geq 2$  unitary polynomial with complex coefficients*

$$P(x) = x^d + \sum_{0 \leq k \leq d-1} a_k x^k,$$

*compute the complex roots of  $P(x)$ .* (★★★)

The input of problem (★★★) consists of an integer  $d \geq 2$  and a black box  $\text{BOX}_P$  for the coefficients of  $P(x)$ . On input a positive integer  $m$  and an index  $k$  such that  $0 \leq k \leq d - 1$ , the black box  $\text{BOX}_P$  returns a decimal fraction

$$(N_1 + N_2 i) \times 10^{-m-1} \text{ such that } |a_k - (N_1 + N_2 i) \times 10^{-m-1}| \leq 10^{-m}$$

where  $i = \sqrt{-1} \in \mathbb{C}$ .

A Turing machine **ROOTS** solving problem (★★★) should be given also a positive integer  $m_Z$  telling her the required accuracy of the result. Let  $Z = [z_1] + [z_2] + \cdots + [z_d]$  be the divisor of  $P$ . This is the formal sum of complex roots, counted with multiplicities. This is an effective divisor of degree  $d$ . On input a positive integer  $m_Z$  and a black box for the coefficients of  $P(x)$ , the machine **ROOTS** is expected to return an approximation  $\hat{Z} = [\hat{z}_1] + [\hat{z}_2] + \cdots + [\hat{z}_d]$  of  $Z$  within  $10^{-m_Z}$ . This means that there should



exist a permutation of the indices  $\tau \in \mathcal{S}_d$  such that  $|z_{\tau(k)} - \hat{z}_k| \leq 10^{-m_Z}$  for every  $1 \leq k \leq d$ .

The rest of this section is devoted to proving the following theorem.

**5.3.1 Theorem (Computing roots of polynomials)** *There exists a deterministic algorithm that on input a degree  $d$  unitary polynomial*

$$P(x) = x^d + \sum_{0 \leq k \leq d-1} a_k x^k$$

*in  $\mathbb{C}[x]$  and a positive integer  $m_Z$ , computes an approximation*

$$\hat{Z} = [\hat{z}_1] + [\hat{z}_2] + \cdots + [\hat{z}_d]$$

*of the divisor*

$$Z = [z_1] + [z_2] + \cdots + [z_d]$$

*of  $P$ , within  $10^{-m_Z}$ . This means that there exists a permutation of the indices  $\tau \in \mathcal{S}_d$  such that  $|z_{\tau(k)} - \hat{z}_k| \leq 10^{-m_Z}$  for every  $1 \leq k \leq d$ . The running time is polynomial in  $d$ ,  $\log H$  and the required accuracy  $m_Z \geq 1$ . Here  $H$  is the smallest integer bigger than the absolute value of all coefficients in  $P(x)$ .*

In Section 5.3.2 we prove that the problem is well conditioned in some sense. In Section 5.3.4 we recall the principles of Weyl's Quadtree algorithm. We recall in Section 5.3.5 that there exists an exclusion function that is sharp enough and easy to compute. This finishes the proof of Theorem 5.3.1.

### 5.3.2 Conditioning

Our first concern is to check that the problem of finding roots is well conditioned in some sense. We first need to define *clusterings* of roots. Let  $\epsilon$  be a positive real number. An  $\epsilon$ -clustering for  $P(x)$  consists of a positive integer  $K$  and a pair  $(c_k, n_k)$  for every  $1 \leq k \leq K$  such that the following conditions are satisfied:

- $c_k$  is a complex number and  $n_k$  is a positive integer for every  $1 \leq k \leq K$ .

- if  $k_1 \neq k_2$  then  $|c_{k_1} - c_{k_2}| > 2\epsilon$ .
- There are  $n_k$  roots of  $P(x)$ , counting multiplicities, in the open disk  $D(c_k, \epsilon/2)$  of center  $c_k$  and radius  $\epsilon/2$ .
- There are  $n_k$  roots of  $P(x)$  in the open disk  $D(c_k, \epsilon)$ , counting multiplicities.
- $\sum_{1 \leq k \leq K} n_k = d$ .

So we want to squeeze the complex roots of  $P(x)$  into small disks that are distant enough from each other. Note that there may not exist an  $\epsilon$ -clustering for every  $\epsilon$ . Problems may occur when the distance between two roots of  $P$  is close to  $\epsilon$ . However, for every positive  $\epsilon$ , there exists an  $\epsilon'$  such that

$$2^{-d^2} \epsilon \leq \epsilon' \leq \epsilon \tag{5.4}$$

and an  $\epsilon'$ -clustering for  $P(x)$ . Indeed, we consider the interval

$$S = [-d^2 \log 2 + \log \epsilon, \log \epsilon]$$

and for every pair  $(z, z')$  of distinct roots of  $P(x)$  we remove the interval

$$[\log |z - z'| - \log 2, \log |z - z'| + \log 2]$$

to  $S$ . The resulting set  $T$  is not empty because there are at most  $d(d-1)/2$  pairs of distinct roots. Any  $\epsilon'$  such that  $\log \epsilon'$  belongs to  $T$  is fine.

Now, call  $Z = [z_1] + [z_2] + \dots + [z_d]$  the divisor of  $P$ . We call  $A$  the smallest integer bigger than the absolute values of the coefficients of  $P(x)$ . Let  $\Delta(x)$  be a polynomial of degree  $\leq d-1$ . Let  $\delta$  be the maximum of the absolute values of the coefficients of  $\Delta(x)$ . We want to compare the roots of  $P$  and the roots of  $P + \Delta$ . The absolute values of the roots of  $P$  are  $\leq dA$ . We fix an  $\epsilon \leq 1$ . We know that there exists an  $\epsilon'$ -clustering  $(c_k, n_k)_{1 \leq k \leq K}$  of the roots of  $P$  for some  $2^{-d^2} \epsilon \leq \epsilon' \leq \epsilon$ . Let  $z$  be a complex number such that  $|z - c_k| = \epsilon'$  for some  $1 \leq k \leq K$ . The absolute value of  $P(z)$  is lower bounded by  $(\epsilon')^d 2^{-d}$ . On the other hand, the absolute value of  $\Delta(z)$  is upper bounded by  $d\delta(dA + 2\epsilon')^{d-1}$ . So if

$$\delta < \left( \frac{\epsilon'}{2(dA + 2)} \right)^d \quad (5.5)$$

we deduce from Rouché's theorem that  $P + \Delta$  has  $n_k$  roots inside  $D(c_k, \epsilon')$ . As a consequence, the roots  $\hat{z}_1, \hat{z}_2, \dots, \hat{z}_d$  of  $P + \Delta$  can be indexed in such a way that  $|z_j - \hat{z}_j| \leq 2\epsilon' \leq 2\epsilon$  for every  $1 \leq j \leq d$ . Roughly speaking, the meaning of inequality (5.5) is that when passing from coefficients to roots, the accuracy is divided by (no significantly more than) the degree  $d$  of the polynomial  $P$ . We thus have proven the following lemma.

**5.3.3 Lemma (Conditioning of the roots)** *There exists a positive constant  $\Theta$  such that the following is true. Let  $d \geq 1$  be an integer and let  $P(x)$  be a degree  $d$  unitary polynomial with complex coefficients. Let  $A$  be the smallest integer bigger than the absolute values of the coefficients of  $P$ . Let  $\Delta(x)$  be a degree  $d - 1$  polynomial with coefficients bounded above by  $\delta$  in absolute value. Let  $\epsilon$  be the unique positive real such that*

$$\log \delta = d \log \epsilon - d \log A - \Theta d^3. \quad (5.6)$$

*Assume that  $\epsilon \leq 1$ . Let  $Z = [z_1] + [z_2] + \dots + [z_d]$  be the divisor of  $P$  and let  $\hat{Z} = [\hat{z}_1] + [\hat{z}_2] + \dots + [\hat{z}_d]$  be the divisor of  $P(x) + \Delta(x)$ . There exists a permutation of the indices  $\tau \in \mathcal{S}_d$  such that  $|z_{\tau(k)} - \hat{z}_k| \leq \epsilon$  for every  $1 \leq k \leq d$ .*

This lemma tells us that if we are looking for an approximation within  $10^{-m_Z}$  of the divisor  $Z = [z_1] + [z_2] + \dots + [z_d]$  of a unitary polynomial  $P(x) \in \mathbb{C}[x]$  given by a blackbox  $\text{BOX}_P$ , we may replace  $P$  by a good enough approximation of it having e.g. decimal coefficients. Indeed, we first compute the smallest integer  $A$  bigger than the absolute values of all coefficients of  $P(x)$ . We set  $\epsilon = 10^{-m_Z - 1}$  and let  $\delta$  be the real number given by Equation (5.6). Let  $m_P$  be the smallest integer such that  $10^{-m_P} \leq \delta/d$ . We call the black box  $\text{BOX}_P$  and obtain for every  $0 \leq k \leq d - 1$  a decimal fraction  $\hat{a}_k$  with denominator  $10^{m_P + 1}$  such that  $|\hat{a}_k - a_k| \leq 10^{-m_P} \leq \delta/d$ . Let  $\hat{P} = x^d + \sum_{0 \leq k \leq d-1} \hat{a}_k x^k$ . We compute the discriminant of  $\hat{P}$ . If it is zero, we replace  $\hat{P}$  by  $\hat{P} + 10^{-m_P}$  (we increase the constant term by  $10^{-m_P}$ .) If the discriminant is zero again, we add  $10^{-m_P}$  to the constant term again. We

go on like that until the discriminant of  $\hat{P}$  is non-zero. This process stops after  $d$  steps at most (seen as a polynomial in the indeterminate  $a_0$  the discriminant has degree  $d-1$ , so it cannot cancel  $d$  times.) In the end we obtain a unitary polynomial  $\hat{P}$  with decimal coefficients  $\hat{a}_k$  such that  $|\hat{a}_k - a_k| \leq \delta$  and the discriminant of  $\hat{P}$  is non zero. If  $\hat{Z} = [\hat{z}_1] + [\hat{z}_2] + \dots + [\hat{z}_d]$  is the divisor of  $\hat{P}$ , there exists a permutation of the indices  $\tau \in \mathcal{S}_d$  such that  $|z_{\tau(k)} - \hat{z}_k| \leq \epsilon$  for every  $1 \leq k \leq d$ . The coefficients of  $\hat{P}$  are decimal fractions with denominator  $10^{m_P+1}$  where  $m_P$  is an integer such that

$$m_P \leq \Theta d(m_Z + d^2 + \log A). \tag{5.7}$$

In order to approximate the roots of  $P$  within  $10^{-m_Z}$  it suffices to approximate the roots of  $\hat{P}$  within  $10^{-m_Z-1}$ . So in the sequel we shall assume that we are given a unitary polynomial with coefficients in  $\mathbb{Z}[i, \frac{1}{10}]$  having no multiple root.

### 5.3.4 Weyl's Quadtree algorithm

We now describe a simple-minded variant of the celebrated Weyl's Quadtree algorithm to compute complex roots of polynomials. Let

$$P(x) = x^d + \sum_{0 \leq k \leq d-1} a_k x^k$$

be a degree  $d \geq 1$  unitary polynomial with coefficients in  $\mathbb{Z}[i, \frac{1}{10}]$ . So every coefficient  $a_k$  is a fraction  $N_k \times 10^{-m_P-1}$  with  $N_k = N_{1,k} + N_{2,k}i$  and  $N_{1,k}, N_{2,k}$  are in  $\mathbb{Z}$ . We assume that  $a_0$  is not zero. Let  $g$  be the smallest positive integer such that the distance between any two distinct roots of  $P$  is at least  $10^{-g}$ . For every complex number  $z$  we denote by  $r(z)$  the distance between  $z$  and the closest root of  $P$ . We denote by  $R(z)$  the distance between  $z$  and the furthest root of  $P$ . Computing  $r(z)$  seems difficult unless one already knows the roots of  $P$ . However, we assume that we can compute for every  $z$  in  $\mathbb{Z}[i, \frac{1}{10}]$  a decimal  $\rho(z)$  such that  $\rho(z) \leq r(z) \leq 1.01 \times \rho(z)$ . Such a  $\rho$  is called an *exclusion function*. We shall give in Section 5.3.5 an example of such an exclusion function that can be computed efficiently.

We first construct a square  $Q$  in the complex plane  $\mathbb{C}$  that is large enough to contain all roots of  $P$ . We take for  $Q$  a square with center the origin and

side length

$$s = 2dA$$

where  $A$  is the smallest integer bigger than the absolute values of all coefficients in  $P$ .

Now we divide  $\mathcal{Q}$  into four squares of side length  $s/2$ : the top left square  $\mathcal{Q}_1$ , the top right square  $\mathcal{Q}_2$ , the bottom left square  $\mathcal{Q}_3$  and the bottom right square  $\mathcal{Q}_4$ . For each  $1 \leq k \leq 4$  we evaluate the exclusion function  $\rho$  at the center  $c_k$  of  $\mathcal{Q}_k$ . If  $\rho(c_k)$  is bigger than the half diagonal  $s/(2\sqrt{2})$  of  $\mathcal{Q}_k$  then we know that there is no root of  $P$  in  $\mathcal{Q}_k$ . So we erase this square.

Next we consider all those squares that have not been erased and we divide each of them into four smaller squares with side length  $s/4$ . We evaluate the exclusion function at the center of every such square. If the value of the exclusion function is bigger than the half diagonal  $s/(4\sqrt{2})$  of the square in question, we erase it.

We go on like that, dividing all remaining squares into four smaller ones at each step. The number of remaining squares is never bigger than  $4d$ . The reason is that a given root of  $P$  cannot compromise more than 4 squares at a time (such a situation would occur if the root in question were very close to the intersection of four contiguous squares).

After  $n$  steps, the side length of the remaining squares is  $s/2^n$ . If  $s/2^n$  is much smaller than the minimum distance between two roots of  $P$ , then there remains exactly  $d$  groups of contiguous squares, and they each contain a single root of  $P$ . So the number of steps is bounded by a constant times

$$\log(dA) + m_Z + g \tag{5.8}$$

where  $A$  is the smallest integer bigger than the absolute values of all coefficients in  $P$ , and  $m_Z$  is the required accuracy of the result, and  $10^{-g}$  is a lower bound for the distance between any two roots of  $P$ .

We illustrate this process on Figure 5.1. The two roots are represented by two bullets.

Note that if the discriminant of  $P$  is non-zero, its absolute value is at least  $10^{-(m_P+1)(2d-1)}$  so the distance between any two distinct roots is at least  $10^{-(m_P+1)(2d-1)}(2dA)^{-d(d-1)}$ . We deduce

$$g \leq \Theta d(m_P + d^2 + d \log A).$$

Combining this with the estimates in Equations (5.8) and (5.7) we deduce that the number of steps in Weyl's Quadtree algorithm is

$$\leq \Theta d^2(\log(A) + d^2 + m_Z) \tag{5.9}$$

where  $d$  is the degree of the polynomial,  $A$  the smallest integer bigger than the coefficients and  $m_Z$  the required accuracy for the roots of  $P(x)$ .

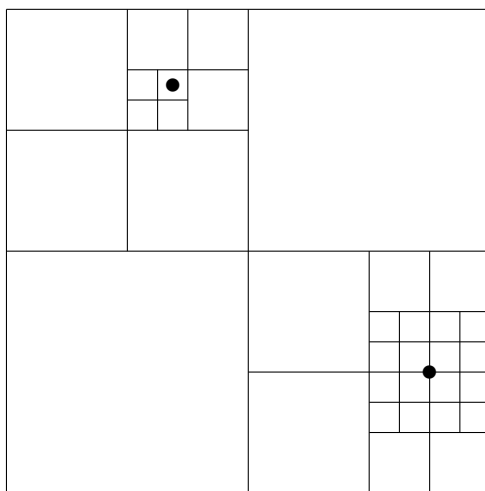


Figure 5.1 Weyl's Quadtree method

To finish the proof of Theorem 5.3.1 it remains to prove that there exists an exclusion function  $\rho$  that can be evaluated quickly enough. This is done in the next Section 5.3.5.

### 5.3.5 Buckholtz inequalities

In this section we recall useful inequalities due to Buckholtz [Buc1, Buc2] and we explain how to deduce a nice exclusion function following Pan [Pan]. Let  $P(x) = x^d + \sum_{0 \leq k \leq d-1} a_k x^k$  be a degree  $d \geq 2$  unitary polynomial with complex coefficients. Assume that  $a_0$  is not zero. Let

$Z = [z_1] + [z_2] + \cdots + [z_d]$  be the divisor of  $P$ . For every integer  $v$  (positive or negative) we call

$$\nu_v = z_1^v + \cdots + z_d^v$$

the  $v$ -th power sum. Buckholtz has shown the following inequality

$$\frac{1}{5} \max_{1 \leq v \leq d} |z_v| \leq \max_{1 \leq v \leq d} \left( \frac{|\nu_v|}{d} \right)^{\frac{1}{v}} \leq \max_{1 \leq v \leq d} |z_v|. \quad (5.10)$$

One can easily deduce a more general and sharper statement. If  $M \geq 1$  is an integer then

$$5^{-\frac{1}{M}} \max_{1 \leq v \leq d} |z_v| \leq \max_{1 \leq v \leq d} \left( \frac{|\nu_{Mv}|}{d} \right)^{\frac{1}{Mv}} \leq \max_{1 \leq v \leq d} |z_v|. \quad (5.11)$$

Recall that for any  $z \in \mathbb{C}$  we call  $r(z)$  the distance between  $z$  and the closest root of  $P$  and  $R(z)$  the distance between  $z$  and the furthest root of  $P$ . From Equation (5.11) we deduce an estimate for  $R(0)$ .

$$\max_{1 \leq v \leq d} \left( \frac{|\nu_{Mv}|}{d} \right)^{\frac{1}{Mv}} \leq R(0) \leq 5^{\frac{1}{M}} \max_{1 \leq v \leq d} \left( \frac{|\nu_{Mv}|}{d} \right)^{\frac{1}{Mv}}. \quad (5.12)$$

If we apply inequality (5.12) to the reciprocal polynomial of  $P(x)$  we obtain

$$\frac{5^{-\frac{1}{M}}}{\max_{1 \leq v \leq d} \left( \frac{|\nu_{-Mv}|}{d} \right)^{\frac{1}{Mv}}} \leq r(0) \leq \frac{1}{\max_{1 \leq v \leq d} \left( \frac{|\nu_{-Mv}|}{d} \right)^{\frac{1}{Mv}}}. \quad (5.13)$$

The power sum  $\nu_v$  can be computed using Newton formulae. Assume that  $P(x)$  has coefficients  $a_k = N_k \times 10^{-m_P}$  with  $N_k = N_{k,1} + N_{k,2}i$  and  $N_{k,1}, N_{k,2}$  are in  $\mathbb{Z}$ . Let  $A$  be the smallest integer bigger than the absolute values of the coefficients  $a_k$ . Then computing  $\nu_v$  takes time polynomial in  $v, m_P$  and  $\log A$ .

Extracting the  $v$ -th power of a decimal fraction  $N \times 10^{-m}$  can be done in time polynomial in  $v$ ,  $\log N$ ,  $m$  and the required accuracy, using a dichotomy algorithm as in Section 5.2. If  $M \geq 200$  then the approximation factor  $5^{\frac{1}{N}}$  is smaller than 1.01. So we obtain a very sharp estimate for  $r(0)$ .

For any  $z \in \mathbb{C}$  we can apply inequality (5.13) to the polynomial  $P(x+z)$  and obtain a good approximation of  $r(z)$ . Assume that  $P(x)$  has coefficients  $a_i = N_i \times 10^{-m_P}$  with  $N_i = N_{i,1} + N_{i,2}i$ . Assume that  $z = (z_1 + z_2i) \times 10^{-m_z}$  and  $z_1, z_2$  are in  $\mathbb{Z}$ . Let  $A$  be smallest integer bigger than the absolute values  $|a_i|$ . Then, using Equation (5.13) for  $N = 200$  and the change of variable  $x \mapsto x + z$ , we can compute a decimal number  $\rho(z)$  such that

$$\rho(z) \leq r(z) \leq 1.01 \times \rho(z)$$

and this takes time polynomial in  $d$ ,  $m_P$ ,  $\log A$ ,  $m_z$ , and

$$\log \max(|z_1|, |z_2|, 1).$$

In the course Weyl's Quadtree algorithm,  $m_z$  is the number of steps which is bounded in (5.9). Both  $|z_1|$  and  $|z_2|$  are bounded by  $dA10^{m_z}$ . An upper bound for  $m_P$  is given in (5.7).

So the calculation of any value of the exclusion function that is required in the course of Weyl's algorithm takes polynomial time in  $d$ ,  $\log A$  and the required accuracy  $m_Z$ . This finishes the proof of Theorem 5.3.1.

#### 5.4 COMPUTING THE ZEROS OF A POWER SERIES

Given a power series  $f(x) = f_0 + \sum_{k \geq 1} f_k x^k \in \mathbb{C}[[x]]$  with positive radius of convergence  $R$ , we may want to compute the zeros of  $f$  inside the open disk  $D(0, R)$ . However, the number of such zeros may very well be infinite. So we should restrict to a smaller disc  $D(0, r)$  with  $0 < r < R$ . Then it makes sense to wonder how many zeros there are in the disk  $D(0, r)$  and try to compute approximations of these zeros. However, the problem of counting zeros in such a disk is not a well conditioned problem, because if  $f$  has a zero  $z$  of absolute value exactly  $r$ , then an infinitesimal perturbation of  $f$



may push this zero inside or outside  $D(0, r)$ . So we should allow the algorithm to choose a  $r'$  that is very close to  $r$  and compute the number of zeros inside  $D(0, r')$ . Then it makes sense to ask for approximations of these zeros. The input of a Turing machine ZERO computing zeros of power series would consist of a black box  $\text{BOX}_f$  for the coefficients of  $f(x)$ . On input an integer  $K \geq 0$  and an integer  $m \geq 1$ , the black box  $\text{BOX}_f$  returns for every  $0 \leq k \leq K$  a decimal fraction  $(N_{1,k} + N_{2,k}i) \times 10^{-m-1}$  such that  $|f_k - (N_{1,k} + N_{2,k}i) \times 10^{-m-1}| \leq 10^{-m}$ . Note that the Turing machine ZERO is not allowed to ask about one coefficient individually. In particular, she needs at least time  $\Theta k$  to receive any information about coefficient  $f_k$  from the black box  $\text{BOX}_f$ . Unfortunately this is not enough for her to compute an approximation of the zeros, because there may appear a huge coefficient  $f_k$  very far away in the development of  $f$ . If we want ZERO to be able to compute zeros without knowing all coefficients of the power series, we should at least provide her with an upper bound for the coefficients in  $f(x)$ . We introduce the following definition.

**5.4.1 Definition (Type of a power series)** Let  $A \geq 1$  be a real and  $n \geq 1$  an integer. A power series  $f(x) = f_0 + \sum_{k \geq 1} f_k x^k$  is said to be of type  $(A, n)$  if for every  $k \geq 0$  we have

$$|f_k| \leq A(k+1)^n.$$

The radius of convergence of a power series of type  $(A, n)$  is at least 1. Elementary results about series of type  $(A, n)$  are collected in Section 5.4.17. The reader is advised to read quickly the results in this section before going further. We shall assume that the Turing machine ZERO is given two integers  $a \geq 1$  and  $n \geq 1$  such that the power series  $f(x)$  is of type  $(\exp(a), n)$ .

There is still one difficulty to overcome. For any finite set  $F \subset \mathbb{C}$  we may consider the polynomial  $P_F(x) = \prod_{z \in F} (x - z)$ . If we divide  $P_F(x)$  by its  $L^\infty$  norm we obtain a polynomial  $Q_F(x)$  having all its coefficient bounded by 1 in absolute value. We may regard  $Q_F$  as a power series of type  $(1, 1)$ . Since  $F$  can be arbitrarily large, we deduce that we cannot bound the complexity of finding zeros of a power series in a given disk  $D(0, r)$  just in terms of the type  $(A, n)$  of the series. However, we guess that

a power series of type  $(1, 1)$  having too many zeros inside a small compact set contained in its disk of convergence, must be small everywhere on this compact set. Also we shall assume that the Turing machine ZERO is given a lower bound for the maximum of  $f$  on  $D(0, r)$ . More precisely, we assume that  $1/2 \leq r < 1$  and provide the Turing machine with a positive integer  $\mu$  such that  $|f(z)| > \exp(-\mu)$  for *at least one*  $z$  in  $D(0, 1/2)$ .

In this section, we shall prove the following theorem.

**5.4.2 Theorem (Counting zeros of a power series)** *Let*

$$f(x) = f_0 + \sum_{k \geq 1} f_k x^k \in \mathbb{C}[[x]]$$

*be a power series. Let  $a \geq 1$  be an integer and set  $A = \exp(a)$ . Let  $n \geq 1$  be an integer such that  $f$  is of type  $(A, n)$ . Let  $r$  be a real number such that  $1/2 \leq r < 1$ . Let  $\mu$  be a positive integer such that there exists at least one  $z$  in  $D(0, 1/2)$  such that  $|f(z)| > \exp(-\mu)$ .*

*The number of zeros of  $f$  in the closed disk  $\bar{D}(0, r)$  is bounded by a polynomial in  $n, a, \mu$  and  $(1-r)^{-1}$ . More precisely, there exists an absolute constant  $\Theta$  such that the number of zeros of  $f$  in the closed disk  $\bar{D}(0, r)$  is bounded above by*

$$\frac{\Theta(n^2 + \mu + a)^2}{(1-r)^{13}}.$$

We shall prove also the existence of an algorithm that computes approximations of these zeros.

**5.4.3 Theorem (Approximating the zeros of a power series)** *There is a deterministic algorithm that on input an integer  $m \geq 1$ , an integer  $a \geq 1$ , an integer  $n \geq 1$ , an integer  $\mu \geq 1$ , a rational number  $r = 1 - 1/o$  where  $o \geq 2$  is an integer, and a power series*

$$f(x) = f_0 + \sum_{k \geq 1} f_k x^k \in \mathbb{C}[[x]]$$

*of type  $(\exp(a), n)$ , such that  $|f(z)| > \exp(-\mu)$  for at least one  $z$  in  $D(0, 1/2)$  returns*

- A rational number  $r'$  such that  $|r - r'| \leq 10^{-m}$ ,
- The number  $J$  of the zeros of  $f(x)$  in the closed disk  $\bar{D}(0, r')$ , or equivalently the degree  $J$  of the divisor  $Z = [z_1] + [z_2] + \cdots + [z_J]$  of  $f$  restricted to the closed disk  $\bar{D}(0, r')$ ,
- Assuming  $J \geq 1$ , for every  $1 \leq j \leq J$ , a decimal

$$\hat{z}_j = (M_{1,j} + M_{2,j}i) \times 10^{-m-1}$$

where  $M_{1,j}$  and  $M_{2,j}$  are integers, such that

$$\hat{Z} = [\hat{z}_1] + [\hat{z}_2] + \cdots + [\hat{z}_J]$$

approximates the divisor  $Z = [z_1] + [z_2] + \cdots + [z_J]$  within  $10^{-m}$ . More precisely, there exists a permutation of the indices  $\tau \in \mathcal{S}_J$  such that  $|z_{\tau(j)} - \hat{z}_j| \leq 10^{-m}$  for every  $1 \leq j \leq J$ .

The algorithm runs in time polynomial in  $a, n, \mu, o = (1 - r)^{-1}$ , and  $m$ .

Here is the main idea in the proof of these theorems. For every positive integer  $u$  we write  $f(x) = P_u(x) + R_u(x)$  where  $P_u(x) = \sum_{0 \leq k \leq u-1} f_k x^k$  is the principal part and  $R_u(x) = \sum_{k \geq u} f_k x^k$  is the remainder term of order  $u$ . We expect that if  $u$  is large enough, the roots of  $P_u(x)$  in  $D(0, r)$  sharply approximate the zeros of  $f(x)$  in  $D(0, r)$ . It would then suffice to apply the algorithm and theorem in Section 5.3.

Our first concern will be to prove that the problem of finding zeros of a power series is well conditioned in some sense: a small perturbation does not affect too much the zeros. We cannot use Lemma 5.3.3 about the conditioning of roots of polynomials, and we cannot adapt its proof either, because both the statement and the proof involve the degree of the polynomial. Instead of that, we shall first study the conditioning of every zero independently. In Section 5.4.4 we introduce the Newton polynomial of a power series and we show how it can help locating the zeros in the neighborhood of the origin. In Section 5.4.6 we deduce that if  $f(0)$  is very small, then  $f(x)$  has a zero in a small neighborhood of 0. This proves that a zero at the origin is well conditioned. In Section 5.4.8 we state and prove a simple effective version of the analytic continuation theorem. In the next Section 5.4.11, we

deduce that any zero of  $f(x)$  is well conditioned in some sense. A global statement about the conditioning of all zeros is stated and proved in Section 5.4.13. In the next Section 5.4.15 we use this result to finish the proof of Theorem 5.4.2. We give the proof of Theorem 5.4.3 and the corresponding algorithm in Section 5.4.16. Section 5.4.17 is devoted to the proof of a few elementary results about power series of type  $(A, n)$ .

#### 5.4.4 The Newton polygon of a power series

Let  $f(x) = 1 + \sum_{k \geq 1} f_k x^k \in \mathbb{C}[[x]]$  be a power series such that  $f(0) = 1$ . We assume that the radius  $R$  of convergence is positive (it might be infinite). Let  $r$  be a real such that  $0 < r < R$ . We try to mimic non-archimedean analysis and relate the modulus of zeros of  $f(x)$  and the slopes of its Newton polygon. Let  $d$  be the degree of  $f(x)$  in the indeterminate  $x$ . So  $0 \leq d \leq \infty$  and most of the time  $d = \infty$ . The *Newton cloud* of  $f(x)$  is the subset of  $\mathbb{R} \times \mathbb{R}$  consisting of couples  $(k, -\log |f_k|)$  for all  $k \geq 0$  such that  $f_k \neq 0$ . Let  $\Phi$  be the set of all affine functions  $\phi : [0, d] \rightarrow \mathbb{R}$  whose graph stays below the Newton cloud: for every  $k \geq 0$  one has  $\phi(k) \leq -\log |f_k|$ . The *Newton function* is a function  $\mathcal{N} : [0, d] \rightarrow \mathbb{R}$  defined as the sup of all  $\phi$  in  $\Phi$ . For every  $t \in [0, d]$

$$\mathcal{N}(t) = \sup_{\phi \in \Phi} \phi(t).$$

This is a convex function on  $[0, d]$ . It is continuous and piecewise affine. Indeed, it is affine on the interval  $[k, k+1]$  for every  $0 \leq k \leq d-1$ . The *Newton polygon* is the set  $\mathcal{P} = \{(x, y) \in [0, d] \times \mathbb{R} \mid \mathcal{N}(x) \leq y\}$ . The *vertices* of the Newton polygon are the points  $(k, \mathcal{N}(k))$  for all  $k$  where  $\mathcal{N}$  is not differentiable (such a  $k$  must be an integer). This includes the point  $(0, 0)$ , and the point  $(d, \mathcal{N}(d))$  also when  $d$  is finite. The Newton polygon has a vertical *edge*  $] (0, +\infty), (0, 0) ]$ . If  $d$  is finite and non-zero, there is another vertical edge  $[(d, \mathcal{N}(d)), (d, +\infty)[$ . The remaining edges connect successive vertices of  $\mathcal{P}$ . A *slope* of the Newton polygon is a slope of one of its edges. A *supporting line* is a line  $L$  such that  $\mathcal{P}$  is entirely contained in one of the two closed half planes determined by  $L$ , and  $L$  contains at least one point of  $\mathcal{P}$ . For every element  $\alpha$  in  $[-\infty, \log R[$  there is a single

supporting line  $L_\alpha$  with slope  $\alpha$ . If the Newton polygon has finitely many edges, then there also exists a supporting line with slope  $\log R$ . If  $\alpha$  is a slope of the Newton polygon, then  $L_\alpha$  contains the corresponding edge of  $\mathcal{P}$ .

Assume that  $\alpha \in \mathbb{R}$  is not a slope of the Newton polygon. The supporting line  $L_\alpha$  contains a single vertex  $v = (k, \mathcal{N}(k))$  of  $\mathcal{P}$ . Let  $\alpha^- \in [-\infty, +\infty[$  be the slope of the edge just before  $v$ . Let  $\alpha^+ \in ]-\infty, +\infty]$  be the slope of the edge just after  $v$ . The vector  $(-\alpha, 1)$  is orthogonal to  $L_\alpha$ . The Newton polygon  $\mathcal{P}$  is contained in the closed half plane  $H = \{(x, y) | (-\alpha, 1) \cdot (x, y) \geq (-\alpha, 1) \cdot (k, \mathcal{N}(k))\}$ . We deduce that for  $z$  a complex number such that  $\log |z| = \alpha$ , the term  $f_k z^k$  dominates the power series in some sense. Indeed, for every  $j \geq 0$  we have

$$\begin{aligned} -\log |f_j z^j| &= -\log |f_j| - j\alpha = (-\alpha, 1) \cdot (j, -\log |f_j|) \\ &\geq (-\alpha, 1) \cdot (k, -\log |f_k|) \end{aligned}$$

so  $f_k z^k$  has a bigger absolute value than any other term in the power series.

Assume now that  $j > k$ . The point  $(j, -\log |f_j|)$  is above the supporting line  $L_{\alpha^+}$  with slope  $\alpha^+$ . So  $|f_j| \leq |f_k| \exp(-(j-k)\alpha^+)$ . Since  $\log |z| = \alpha$ , the term  $f_j z^j$  has absolute value bounded above by  $|f_k| |z|^k$  times  $\exp(-(j-k)(\alpha^+ - \alpha))$ . The sum  $\sum_{j>k} |f_j| |z|^j$  is thus bounded above by  $|f_k| |z|^k$  times  $\frac{a}{1-a}$  where  $a = \exp(-(\alpha^+ - \alpha))$ .

If  $j < k$ , then the point  $(j, -\log |f_j|)$  is above the supporting line  $L_{\alpha^-}$  with slope  $\alpha^-$ . So  $|f_j| \leq |f_k| \exp(-(k-j)\alpha^-)$ . Since  $\log |z| = \alpha$ , the term  $f_j z^j$  has absolute value bounded above by  $|f_k| |z|^k$  times  $\exp(-(k-j)(\alpha - \alpha^-))$ . The sum  $\sum_{j<k} |f_j| |z|^j$  is thus bounded above by  $|f_k| |z|^k$  times  $\frac{b}{1-b}$  where  $b = \exp(-(\alpha - \alpha^-))$ .

If  $\frac{a}{1-a} + \frac{b}{1-b} < 1$  then the power series  $f(x)$  is dominated by the term  $f_k z^k$  for  $|z| = \exp(\alpha)$ . We deduce from Rouché's theorem that  $f(x)$  has exactly  $k$  zeros inside  $D(0, \exp(\alpha))$  counting multiplicities. This is the case in particular if both  $a$  and  $b$  are smaller than  $\frac{1}{3}$ . We deduce the following lemma.

**5.4.5 Lemma (Newton polygon)** *Let  $f(x) = 1 + \sum_{k \geq 1} f_k x^k$  be a power series. Let  $R > 0$  be its radius of convergence. Let  $z \in D(0, R)$  be*

a zero of  $f(x)$ . There exists a slope  $\sigma$  of the Newton polygon such that  $|\log |z| - \sigma| \leq \log 3$ .

Let  $\mathcal{P}_3$  be the set obtained by removing to  $] - \infty, \log R[$  all the intervals  $[\sigma - \log 3, \sigma + \log 3]$  where  $\sigma$  is any slope of the Newton polygon.

If a real  $\alpha$  belongs to  $\mathcal{P}_3$ , then there is a unique vertex  $(k, \mathcal{N}(k))$  lying on the supporting line with slope  $\alpha$ . The power series  $f(x)$  has exactly  $k$  zeros, counting multiplicities, inside the open disk  $D(0, \exp(\alpha))$ ; and none on its boundary.

Let  $\mathcal{P}_4$  be the set obtained by removing to  $] - \infty, \log R[$  all the intervals  $[\sigma - \log 4, \sigma + \log 4]$  where  $\sigma$  is any slope of the Newton polygon.

If a real  $\alpha$  belongs to  $\mathcal{P}_4$ , then there is a unique vertex  $(k, \mathcal{N}(k))$  lying on the supporting line with slope  $\alpha$ . For any complex number  $z$  such that  $|z| = \exp(\alpha)$ , one has  $|f(z)| \geq \frac{1}{3} \times |f_k| \times |z|^k$ .

#### 5.4.6 Conditioning of a zero at the origin

In this section, we prove that a power series taking a very small value at the origin must have a very small zero. More precisely we prove the following lemma.

**5.4.7 Lemma (The smaller zero of a power series)** Consider a power series  $F(x) = F_0 + \sum_{k \geq 1} F_k x^k \in \mathbb{C}[[x]]$  of type  $(A, n)$  where  $A \geq 1$  is a real and  $n \geq 1$  is an integer. Assume that  $|F_0| < 1$  and

$$\sqrt{-\log |F_0|} \geq \Theta(n^2 + \log A).$$

Then at least one of the following two statements holds true

- $F(x)$  has a zero  $\xi$  such that  $-\log |\xi| \geq \sqrt{-\log |F_0|}$ ,
- for every  $z$  in  $D(0, 1/2)$  the absolute value of  $F(z)$  is upper bounded by  $\exp(-0.031 \sqrt{-\log |F_0|})$ .

Before giving the proof of this lemma, let us sketch the main idea in this proof: if  $F$  is not small everywhere, there must be a coefficient in this series that is not too small. Since  $F_0$  is very small, the first finite slope in the Newton polygon must be very negative. But the lack of roots in

the neighborhood of 0 and Lemma (5.4.5) force the Newton polygon to be smooth at the beginning: the many first slopes must keep very small. So the series has many huge coefficients, and it cannot be of type  $(A, n)$ . A contradiction.

So let  $F(x)$  be a power series of type  $(A, n)$  where  $A \geq 1$  is a real and  $n \geq 1$  is an integer. If  $F_0 = 0$  then the first condition in the conclusion of the lemma is met and we are done. So we assume that  $F_0 \neq 0$  and we set  $f(x) = F(x)/F_0$ . If  $F(x)$  is constant, then the second condition in the conclusion of the lemma is met and we are done. So we assume that  $f(x)$  is not constant. The Newton polygon of  $f(x)$  has at least one finite slope. We call  $\sigma_0 = -\infty, \sigma_1, \sigma_2, \dots$  the successive slopes of the Newton polygon. If  $\sigma_1 \geq 0$  then all coefficients  $f_i$  are upper bounded by 1 in absolute value and the second condition in the conclusion of the lemma is met. So we may assume that  $\sigma_1$  is negative. Recall the definition of  $\mathcal{P}_3$  in Lemma 5.4.5 and consider the intersection  $\mathcal{P}_3 \cap ]\sigma_1, 0[$ . If this intersection is not empty, we call  $\rho$  its infimum and set  $r = \exp(\rho)$ . If the intersection is empty, we set  $\rho = 0$  and  $r = 1$ .

We assume that the first condition in the conclusion of Lemma 5.4.7 is not met, and we show that in that case the second condition holds true.

Since the first condition in the conclusion of Lemma 5.4.7 is not met, we know that all zeros of  $f(x)$  have absolute value bigger than  $\exp(-\sqrt{-\log |F_0|})$ . Using Lemma 5.4.5 we deduce that

$$-\rho = -\log r \leq \sqrt{-\log |F_0|}. \quad (5.14)$$

The interval  $]\sigma_1, \log r[$  does not meet  $\mathcal{P}_3$ . So it is contained in the union of the intervals  $[\sigma_1, \sigma_1 + \log 3], [\sigma_2 - \log 3, \sigma_2 + \log 3], [\sigma_3 - \log 3, \sigma_3 + \log 3], [\sigma_4 - \log 3, \sigma_4 + \log 3], \dots$

We deduce that all these intervals should adjust to tile all of  $[\sigma_1, \log r]$ . So  $\sigma_2 \leq \sigma_1 + 2 \log 3, \sigma_3 \leq \sigma_2 + 2 \log 3, \dots, \sigma_k \leq \sigma_{k-1} + 2 \log 3$  as long as  $\sigma_{k-1} < \log r - \log 3$ . So  $\sigma_k \leq \sigma_1 + 2(k-1) \log 3$  for every  $k$  from 1 to  $\ell$  where

$$\ell = \left\lfloor \frac{\log r - \sigma_1}{2 \log 3} \right\rfloor. \quad (5.15)$$

We deduce by summation that

$$\mathcal{N}(\ell) \leq \sigma_1 + \sigma_2 + \cdots + \sigma_\ell \leq \ell\sigma_1 + \ell(\ell - 1) \log 3. \quad (5.16)$$

where  $\mathcal{N}$  is the Newton function of the power series  $f(x)$ .

From (5.15) we deduce that  $\ell \leq \frac{\log r - \sigma_1}{2 \log 3}$  and  $\sigma_1 \leq \log r - 2\ell \log 3$ . Reporting in (5.16) we obtain

$$\mathcal{N}(\ell) \leq \ell \log r - \ell^2 \log 3. \quad (5.17)$$

On the other hand, the power series  $F(x)$  is of type  $(A, n)$ , so  $f(x)$  is of type  $(A/|F_0|, n)$ . So the Newton function  $k \mapsto \mathcal{N}(k)$  is lower bounded by the convex function  $k \mapsto -\log A + \log |F_0| - n \log(k + 1)$ . We deduce

$$\mathcal{N}(\ell) \geq -\log A + \log |F_0| - n \log(\ell + 1) \geq -\log A + \log |F_0| - n\ell. \quad (5.18)$$

From (5.17) and (5.18) we deduce

$$\ell^2 \log 3 - (n + \log r)\ell + \log |F_0| - \log A \leq 0.$$

In case  $\ell \geq n + \log r$  we deduce from the inequality above that  $\ell^2(\log 3 - 1) \leq \log A - \log |F_0|$ . So

$$\begin{aligned} \ell &\leq \max(n + \log r, \sqrt{\frac{\log A - \log |F_0|}{\log 3 - 1}}) \\ &\leq \max(n, \sqrt{\frac{\log A - \log |F_0|}{\log 3 - 1}}). \end{aligned} \quad (5.19)$$

We have made the assumption that

$$\sqrt{-\log |F_0|} \geq \Theta(\log A + n^2). \quad (5.20)$$

From (5.19) and (5.20) we deduce



$$\ell \leq 3.2\sqrt{-\log |F_0|}. \quad (5.21)$$

From (5.15) we know that  $\ell \geq \frac{\log r - \sigma_1}{2 \log 3} - 1$  so

$$\sigma_1 \geq -2\ell \log 3 - 2 \log 3 + \log r. \quad (5.22)$$

From (5.21) and (5.22) and (5.14) we deduce

$$\sigma_1 \geq -9\sqrt{-\log |F_0|}. \quad (5.23)$$

So for every  $k \geq 0$  we have

$$|F_k| \leq |F_0| \exp(-k\sigma_1) \leq |F_0| \exp(9k\sqrt{-\log |F_0|}). \quad (5.24)$$

This estimate will show very useful for small values of  $k$ . For bigger values of  $k$ , we use the fact that the power series  $F(x)$  is of type  $(A, n)$ .

Now we take some  $z \in D(0, 1/2)$  and we try to bound  $|F(z)|$ . We set

$$u = \left\lceil \frac{\sqrt{-\log |F_0|}}{10} \right\rceil \quad (5.25)$$

and we check that  $u \geq \frac{4n^2}{(\log 2)^2}$  because of (5.20).

We write  $F(x) = P_u(x) + R_u(x)$  where  $P_u(x) = \sum_{0 \leq k \leq u-1} F_k x^k$  is the principal part and  $R_u(x) = \sum_{k \geq u} F_k x^k$  is the remainder term of order  $u$ . We now bound  $|P_u(z)|$  and  $|R_u(z)|$  separately.

On the one hand, using Equation (5.24)

$$\begin{aligned} |P_u(z)| &\leq \sum_{0 \leq k \leq u-1} 2^{-k} \exp(\log |F_0| + 9(u-1)\sqrt{-\log |F_0|}) \\ &\leq \exp\left(\frac{\log |F_0|}{10}\right). \end{aligned} \quad (5.26)$$

On the other hand, using Lemma 5.4.18

$$|R_u(z)| \leq n!A2^{-\frac{u}{2}+n+1} \leq 2^{-\frac{\sqrt{-\log |F_0|}}{21}} \quad (5.27)$$

because of (5.20).

Altogether we have

$$|F(z)| \leq |P_u(z)| + |R_u(z)| \leq \exp\left(\frac{\log |F_0|}{10}\right) + 2^{-\frac{\sqrt{-\log |F_0|}}{21}} \leq 2^{-\frac{\sqrt{-\log |F_0|}}{22}}$$

using (5.26) and (5.27), then (5.20). This proves that the second condition in the conclusion of Lemma 5.4.7 is met. So the proof of this lemma is finished.  $\square$

#### 5.4.8 Analytic continuation of a power series of type $(A, n)$

In this section we consider a power series  $f(x) = \sum_{k \geq 0} f_k x^k$  of type  $(A, n)$  with  $A \geq 1$  a real and  $n \geq 1$  an integer. Let  $c$  be a complex number with absolute value smaller than 1 and let  $r$  be a real such that  $0 < r < 1 - |c|$ . The disk  $D(c, r)$  is contained in  $D(0, 1)$ . We want to prove that if  $|f(z)|$  is very small for every  $z \in D(c, r)$  then  $|f(z)|$  is small for every  $z \in D(0, 1/2)$ . We shall need the following definition.

**5.4.9 Definition (Balanced disk)** Let  $D(c, r)$  be a disk contained in the unit disk  $D(0, 1)$ . We say that  $D(c, r)$  is *balanced* if  $r = \frac{1-|c|}{2}$ . The meaning of this condition is that the distance between  $D(c, r)$  and the unit circle is equal to the radius of  $D(c, r)$ . If this condition is met, the circle  $D(c, r)$  is denoted  $D_c$ .

Assume further that  $|c| > \frac{1}{5}$  and set  $c' = (|c| - \frac{r}{2}) \times \frac{c}{|c|}$ . Let  $D_{c'}$  be the balanced disk with center  $c'$ . The radius of  $D_{c'}$  is  $r' = \frac{5}{4}r$  and  $1 - |c'| = \frac{5}{4}(1 - |c|)$ . We say that  $D_{c'}$  is the *son* of  $D_c$ .

If  $|c| \leq \frac{1}{5}$ , then the son of  $D_c$  is defined to be  $D(0, 1/2)$ .

The following lemma states that a power series of type  $(A, n)$  that is very small on a balanced disk, must be small also on the son of this disk.

**5.4.10 Lemma (Analytic continuation)** Let  $f(x) = \sum_{k \geq 0} f_k x^k$  be a power series of type  $(A, n)$  where  $A \geq 1$  is a real and  $n \geq 1$  is an integer. Let  $c$  be a complex number with absolute value smaller than 1 and

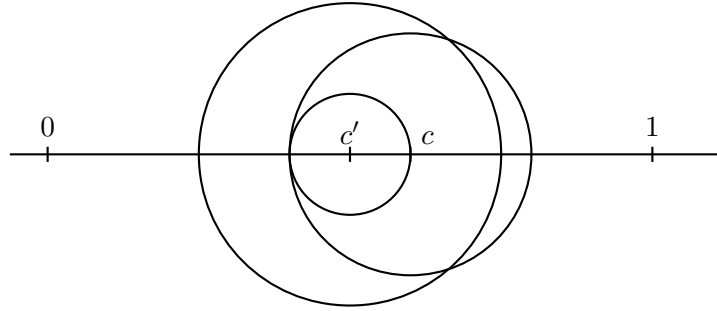


Figure 5.2 Son of a balanced disk

let  $D_c$  be the balanced disk with center  $c$ . Call  $r = \frac{1-|c|}{2}$  the radius of  $D_c$ . Let  $D_{c'}$  be the son of  $D_c$ . Let  $\epsilon$  be a real number in  $]0, 1[$  such that  $-\log \epsilon \geq \Theta(\log A + n(n + |\log r|))$ . Assume that  $|f(z)| \leq \epsilon$  for every  $z \in D_c$ . Then  $|f(z)| \leq \epsilon^{\frac{1}{3}}$  for every  $z \in D_{c'}$ .

In order to prove this lemma we observe that the disk  $D(c', r/2)$  is contained in  $D_c$ . So the absolute value of  $f$  is bounded by  $\epsilon$  in  $D(c', r/2)$ . Cauchy's integral formula then gives an upper bound for the successive derivatives of  $f$  at  $c'$ .

$$|f^{(k)}(c')| = \left| \frac{k!}{(2i\pi)} \int_{|\zeta|=r/2} \frac{f(c' + \zeta)}{\zeta^{k+1}} d\zeta \right| \leq \epsilon \frac{2^k k!}{r^k}. \quad (5.28)$$

In order to bound  $|f(z)|$  for  $z \in D_{c'}$  we call  $T(x) = \sum_{k \geq 0} \frac{f^{(k)}(c')}{k!} x^k$  the Taylor expansion of  $f$  at  $c'$ . We choose an integer  $u \geq 0$  and write  $T(x) = P_{c',u}(x) + R_{c',u}(x)$  where  $P_{c',u}(x) = \sum_{0 \leq k \leq u-1} \frac{f^{(k)}(c')}{k!} x^k$  is the principal part and  $R_{c',u}(x) = \sum_{k \geq u} \frac{f^{(k)}(c')}{k!} x^k$  is the remainder term of order  $u$ . Any  $z$  in the son  $D_{c'}$  can be written  $z = c' + t$  which  $|t| < r' = \frac{5r}{4}$ . We now bound  $|P_{c',u}(t)|$  and  $|R_{c',u}(t)|$  separately.

On the one hand, using Equation (5.28) we obtain

$$\begin{aligned}
 |P_{c',u}(t)| &\leq \sum_{0 \leq k \leq u-1} \left| \frac{f^{(k)}(c')}{k!} t^k \right| \\
 &\leq \sum_{0 \leq k \leq u-1} \left( \frac{5r}{4} \right)^k \frac{2^k \epsilon}{r^k} \leq \epsilon \left( \frac{5}{2} \right)^u. \tag{5.29}
 \end{aligned}$$

On the other hand, we set  $t = (1 - |c'|)y$  and  $z = c' + t = c' + (1 - |c'|)y$ . The remainder  $R_{c',u}((1 - |c'|)y)$  is nothing but the remainder of order  $u$  of the refocused<sup>1</sup> power series  $y \mapsto f_{c'}(y) = f(c' + (1 - |c'|)y)$ . Since  $z = c' + y(1 - |c'|)$  belongs to the balanced disk  $D_{c'}$  we know that  $y$  belongs to  $D(0, 1/2)$ , the balanced disk with center 0. So  $|y| < 1/2$ . We now apply the refocusing Lemma 5.4.19 together with Lemma 5.4.18.

The refocusing Lemma 5.4.19 tells us that  $f_{c'}$  has type  $(A', n + 1)$  where  $A' \leq n!A \left( \frac{2e}{1 - |c'|} \right)^{n+2} \leq n!A \left( \frac{e}{r} \right)^{n+2}$ . Lemma 5.4.18 applied to the refocused series  $f_{c'}$  then says that

$$|R_{c',u}(t)| \leq A''(u + 1)^{n+1} 2^{-u} \tag{5.30}$$

where

$$A'' \leq (n + 1)!n!A \left( \frac{2e}{r} \right)^{n+2} \leq A \exp(Kn(n + |\log r|)) \tag{5.31}$$

for some positive constant  $K$ .

We set  $u = \left\lceil \frac{|\log \epsilon|}{2 \log \frac{5}{2}} \right\rceil$ . From (5.29) we deduce

$$\begin{aligned}
 \log |P_{c',u}(t)| &\leq \log \epsilon + u \log \frac{5}{2} \leq \log \epsilon + \frac{|\log \epsilon|}{2} + \log \frac{5}{2} \\
 &\leq 0.49 \log \epsilon \tag{5.32}
 \end{aligned}$$

using the hypothesis

$$-\log \epsilon \geq \Theta(\log A + n(n + |\log r|)). \tag{5.33}$$

---

<sup>1</sup>The refocused series is defined in section 5.4.17

As far as the remainder is concerned we obtain from (5.30) and (5.31) that  $\log |R_{c',u}(t)|$  is

$$\leq \log A + Kn(n + |\log r|) + (n + 1) \log(u + 1) - u \log 2. \quad (5.34)$$

We show that the negative term  $-u \log 2$  dominates this sum. First of all, from the definition of  $u$  and the hypothesis (5.33) we deduce that

$$\log A + Kn(n + |\log r|) \leq 0.01 \times u \log 2.$$

The same hypothesis (5.33) implies that

$$u \geq \left( \frac{n + 1}{0.01 \times \log 2} \right)^2$$

so  $0.01 \times u \log 2 \geq (n + 1)\sqrt{u} \geq (n + 1) \log(u + 1)$ . Inequality (5.34) then implies

$$\log |R_{c',u}(t)| \leq -u \times 0.98 \log 2 \leq \frac{0.98 \log 2}{2 \log \frac{5}{2}} \log \epsilon \leq 0.37 \log \epsilon. \quad (5.35)$$

From (5.32) and (5.35) we deduce that if  $z \in D_{c'}$  and  $z = c' + t$  then

$$\begin{aligned} \log |f(z)| &= \log |P_{c',u}(t) + R_{c',u}(t)| \\ &\leq \log (2 \max(|P_{c',u}(t)|, |R_{c',u}(t)|)) \\ &\leq \log 2 + 0.37 \log \epsilon \leq \frac{\log \epsilon}{3}. \end{aligned} \quad (5.36)$$

This finishes the proof of Lemma 5.4.10. We notice that the exponent  $\frac{1}{3}$  in the conclusion of this lemma could be replaced by any real smaller than  $\frac{\log 2}{\log \frac{5}{2}}$ .  $\square$

#### 5.4.11 Conditioning of any zero

In this section we combine Lemma 5.4.7 and Lemma 5.4.10 to prove that any zero of a power series is well conditioned in some sense. In other words, we prove that a power series taking a very small value at some point  $c$  inside its disk of convergence, must have a zero that is very close to  $c$ . More precisely we prove the following lemma.

**5.4.12 Lemma (Conditioning of a zero)** *Let*

$$f(x) = f_0 + \sum_{k \geq 1} f_k x^k \in \mathbb{C}[[x]]$$

*be a power series of type  $(A, n)$  where  $A \geq 1$  is a real and  $n \geq 1$  is an integer. Let  $c \in D(0, 1)$  be a complex number with absolute value smaller than 1. Assume that  $|f(c)| < 1$  and*

$$(1 - |c|)^6 \times \sqrt{-\log |f(c)|} \geq \Theta(n^2 + \log A). \quad (5.37)$$

*Then at least one of the following two statements holds true*

- *$f(x)$  has a zero  $\xi$  such that  $-\log |\xi - c| \geq \sqrt{-\log |f(c)|}$ ,*
- *for every  $z$  in  $D(0, 1/2)$  the absolute value of  $f(z)$  is upper bounded by*

$$\exp\left(-\frac{(1 - |c|)^5 \times \sqrt{-\log |f(c)|}}{100}\right).$$

Indeed, let  $f_c(y) = f(c + y(1 - |c|))$  be the refocused series of  $f$  at  $c$ . According to Lemma 5.4.19, it has type  $(A', n + 1)$  where

$$\begin{aligned} \log(A') &\leq \log\left(n!A \left(\frac{2e}{1 - |c|}\right)^{n+2}\right) \leq K(n^2 + \log A + n|\log(1 - |c|)|) \\ &\leq K(n^2 + \log A + \frac{n}{1 - |c|}) \end{aligned} \quad (5.38)$$

where  $K$  is a positive constant.

We now apply Lemma 5.4.7 to  $f_c$ . From inequality (5.38) and the hypothesis (5.37) we deduce that  $f_c$  satisfies the hypothesis in Lemma 5.4.7. We deduce that one at least of the two following conditions is met:

- *Either  $f_c$  has a zero  $y_0$  such that  $\log |y_0| \leq -\sqrt{-\log |f(c)|}$ , in which case we call  $\xi = c + y_0(1 - |c|)$  the corresponding zero of  $f$ . And we are done, because  $|\xi - c| \leq |y_0|$ .*

• Otherwise,  $|f_c(y)|$  is bounded by  $\exp(-0.031 \times \sqrt{-\log |f(c)|})$  for any  $y$  in  $D(0, 1/2)$ . Equivalently,  $f(z)$  is bounded by

$$\exp(-0.031 \times \sqrt{-\log |f(c)|})$$

for any  $z$  in the balanced disk  $D_c$  with center  $c$  and radius  $(1 - |c|)/2$ . In that case, we apply Lemma 5.4.10 several times. Indeed, we set

$$w = \left\lceil \frac{-\log(1 - |c|)}{\log \frac{5}{4}} \right\rceil$$

and apply  $w$  times Lemma 5.4.10.

We conclude that for every  $z \in D(0, 1/2)$ , the absolute value of  $f(z)$  is bounded by

$$\exp(-0.031 \times 3^{-w} \times \sqrt{-\log |f(c)|}).$$

Since

$$3^w \leq 3(1 - |c|)^{-\frac{\log 3}{\log \frac{5}{4}}} \leq 3(1 - |c|)^{-5}$$

we find that  $f$  is bounded in absolute value by

$$\exp(-0.01 \times (1 - |c|)^5 \times \sqrt{-\log |f(c)|})$$

on  $D(0, 1/2)$ .

Of course, we must check that the hypothesis of Lemma 5.4.10 are satisfied every time we apply it. This is the case if

$$(1 - |c|)^5 \times \sqrt{-\log |f(c)|} \geq \Theta(\log A + n(n + |\log(1 - |c|)|)).$$

We conclude using the hypothesis (5.37) and the fact that

$$|\log(1 - |c|)| \leq (1 - |c|)^{-1}.$$

□

**5.4.13 Global conditioning of zeros**

In this section, we apply Lemma 5.4.12 to several zeros at a time and evaluate the conditioning of the divisor of zeros of a power series. The following lemma states that the zeros of a power series can be approximated by the zeros of its principal part.

**5.4.14 Lemma (Global conditioning of the zeros)** *Let*

$$f(x) = f_0 + \sum_{k \geq 1} f_k x^k \in \mathbb{C}[[x]]$$

*be a power series of type  $(A, n)$  where  $A \geq 1$  is a real and  $n \geq 1$  is an integer. Let  $r$  and  $\rho$  be two real numbers such that  $1/2 \leq r < 1$  and  $0 < \rho < 1$ . Assume*

$$(1 - r)^6 \times |\log \rho| \geq \Theta(n^2 + \log A). \quad (5.39)$$

*Let  $u$  be an integer such that*

$$u \geq \frac{16(\log \rho)^2}{|\log r|}$$

*and  $u \leq \frac{16(\log \rho)^2}{1-r}$ . Let  $v$  be an integer such that the two following conditions hold true*

- $1 \leq v \leq u$ ,
- *For any integer  $w$  such that  $v \leq w$  and  $w \leq u - 1$ , the coefficient  $f_w$  is bounded in absolute value by  $\exp(-6(\log \rho)^2)$ .*

*Note in particular that  $v = u$  is fine.*

*Then for every  $z$  in  $D(0, r)$ , the remainder  $R_v(z)$  of order  $v$  is bounded in absolute value by  $\exp(-4(\log \rho)^2)$*

$$|R_v(z)| \leq \exp(-4(\log \rho)^2), \quad (5.40)$$

*and at least one of the following two statements holds true*



- for every  $z$  in  $D(0, 1/2)$  the absolute value of  $f(z)$  is upper bounded by

$$\exp(0.01 \times (1 - r)^5 \times \log \rho),$$

- there exists a positive real  $R$  such that  $r - \rho \leq R \leq r$  and such that, inside the disk  $D(0, R)$ , the zeros of  $f(x)$  are approximated within  $\rho$  by the zeros of its principal part  $P_v(x) = \sum_{0 \leq k \leq v-1} f_k x^k$ . In particular, the number  $J$  of these zeros (counting multiplicities) is at most  $v-1$ . More precisely, let  $Z = [z_1] + [z_2] + \cdots + [z_J]$  be the divisor of  $f(x)$  restricted to  $D(0, R)$ , and let  $\hat{Z} = [\hat{z}_1] + [\hat{z}_2] + \cdots + [\hat{z}_J]$  be the divisor of  $P_v(x)$  restricted to  $D(0, R)$ . Then, either  $J = 0$  or there exists a permutation of the indices  $\tau \in \mathcal{S}_J$  such that  $|z_{\tau(j)} - \hat{z}_j| \leq \rho$  for every  $1 \leq j \leq J$ .

We can be a bit more explicit about the real  $R$  above. Any  $R$  such that  $R \in [r - \rho, r]$  and  $|R - |z|| > 2\rho^2$  for every root  $z$  of  $P_v(x)$  is fine.

The meaning of this lemma is that the zeros of  $f(x)$  are well approximated by the zeros of a well chosen truncation of  $f(x)$ . The lemma tells us where the series should be truncated, depending on the required absolute accuracy: if we want  $m$  digits of accuracy, we need to consider the  $m^2$  first terms in the series. One may wonder why we have introduced the integer  $v$ . The point is that if for the given integer  $u$ , the coefficient  $f_u$  is very small, then we can ignore it and truncate a bit earlier. This freedom will be used to ensure that the leading coefficient in  $P_v(z)$  is not too small. Then, we can normalize  $P_v(z)$  and apply Theorem 5.3.1 of Section 5.3. It is important to normalize because Theorem 5.3.1 only applies to unitary polynomials.

Now let us prove Lemma 5.4.14.

Let  $\epsilon = \exp(-4(\log \rho)^2)$ . Using Lemma 5.4.18 we prove that  $R_u(z)$  is bounded in absolute value by  $\epsilon/2$  for any  $z$  in  $D(0, r)$ . Indeed, the hypothesis  $u \geq \frac{4n^2}{(\log |z|)^2}$  in Lemma 5.4.18 results from Equation (5.39) and the definition of  $u$ . Equation (5.46) then gives

$$\begin{aligned} \log |R_u(z)| &\leq n^2 + \log A + \frac{u \log r}{2} - (n+1) \log(1-r) \\ &\leq n^2 + \log A - 8(\log \rho)^2 - (n+1) \log(1-r) \end{aligned}$$

from the definition of  $u$ .

Using (5.39) we deduce

$$\log |R_u(z)| \leq -7(\log \rho)^2 \leq -\log 2 + \log \epsilon.$$

We check that

$$\begin{aligned} u &\leq \frac{16(\log \rho)^2}{1-r} \\ &\leq \Theta(\log \rho)^{2+\frac{1}{6}} \text{ because of (5.39).} \end{aligned} \tag{5.41}$$

Now, we prove that  $R_v(z)$  is bounded in absolute value by  $\epsilon$ . Indeed

$$\begin{aligned} |R_v(z)| &\leq |R_u(z)| + (u-v) \exp(-6(\log \rho)^2) \\ &\leq \epsilon/2 + \Theta(\log \rho)^{2+\frac{1}{6}} \exp(-6(\log \rho)^2) \text{ using equation (5.41),} \\ &\leq \epsilon/2 + \exp(-5(\log \rho)^2) \text{ because of (5.39),} \\ &\leq \epsilon. \end{aligned}$$

The principal part  $P_v(x)$  is a degree  $v-1$  polynomial. It has no more than  $u-1$  zeros. From Equation (5.41) we deduce that  $u\rho^2 \leq \Theta\rho^2(\log \rho)^{2+\frac{1}{6}}$ . So  $4u\rho^2 \leq \rho$  using (5.39). In particular  $4u\rho^2$  is smaller than  $r$  and smaller than  $\rho$ . The interval  $[r-\rho, r]$  then consists of positive reals and must contain at least one real number  $R$  such that  $|R-|z|| > 2\rho^2$  for every root  $z$  of  $P_v(x)$ . We fix any such  $R$ .

Let  $\mathcal{U}$  be the union of all open disks  $D(\xi, 2\rho^2)$  where  $\xi$  is any root of  $P_v(x)$  lying in  $\bar{D}(0, R)$ . We check that  $\mathcal{U}$  is contained in  $D(0, R)$ . Let  $\mathcal{D} = \bar{D}(0, R) - \mathcal{U}$ . It is clear that  $P_v(x)$  has no root in  $\mathcal{D}$ .

- We first assume that  $|P_v(z)| > \epsilon$  for every  $z$  on the boundary of  $\mathcal{D}$ . We apply Rouché's theorem to  $\bar{D}(0, R)$  and deduce that  $f(x)$  and  $P_v(x)$  have the same number of zeros with absolute value  $\leq R$ .

Now let  $\mathcal{V}$  be any connected component of  $\mathcal{U}$ . The boundary of  $\mathcal{V}$  is contained in the boundary of  $\mathcal{D}$ . Applying Rouché's theorem to  $\bar{\mathcal{V}}$  we deduce that  $f(x)$  and  $P_v(x)$  have the same number of zeros in  $\bar{\mathcal{V}}$ . Since  $\bar{\mathcal{V}}$  is the union of a number  $< v$  of closed disks of radius  $2\rho^2$ , we know that its diameter is  $\leq 4u\rho^2$ . We deduce that the zeros of  $P_v(x)$  inside  $\bar{D}(0, R)$  approximate the zeros of  $f(x)$  inside  $\bar{D}(0, R)$  within  $4u\rho^2 \leq \rho$ . More precisely, let  $Z = [z_1] + [z_2] + \cdots + [z_J]$  be the divisor of  $f(x)$  restricted to  $D(0, R)$ . Then the divisor  $\hat{Z}$  of  $P_v(x)$  restricted to  $D(0, R)$  has degree  $J$  also. Assume that  $J \geq 1$  and let  $\hat{Z} = [\hat{z}_1] + [\hat{z}_2] + \cdots + [\hat{z}_J]$  be the divisor of  $P_v(x)$  restricted to  $D(0, R)$ . Then there exists a permutation of the indices  $\tau \in \mathcal{S}_J$  such that  $|z_{\tau(j)} - \hat{z}_j| \leq \rho$  for every  $1 \leq j \leq J$ . So the lemma is proven in that case.

- We now assume that  $|P_v(\zeta)| \leq \epsilon$  for some  $\zeta$  on the boundary of  $\mathcal{D}$ . Since  $|\zeta| \leq r$  and  $P_v(x)$  has no zero in  $\bar{D}(\zeta, \rho^2)$ , we deduce, using Lemma 5.4.12, that  $P_v(z)$  is bounded by

$$\exp(-0.01 \times (1 - r)^5 \times |\log \rho^2|) = \exp(0.02 \times (1 - r)^5 \times \log \rho)$$

for every  $z \in D(0, 1/2)$ . So for every such  $z$  we have

$$\begin{aligned} |f(z)| &\leq |P_v(z)| + |R_v(z)| \leq \exp(0.02 \times (1 - r)^5 \times \log \rho) + \epsilon \\ &\leq \exp(0.01 \times (1 - r)^5 \times \log \rho) \end{aligned}$$

and the lemma is proven in that case. □

In the next two Sections 5.4.15 and 5.4.16 we use Lemma 5.4.14 to prove Theorem 5.4.2 and Theorem 5.4.3.

#### 5.4.15 Counting zeros of power series

We now can give a proof of Theorem 5.4.2.

Let  $K$  be a large enough positive constant and let  $0 < \rho < 1$  be the unique real number such that

$$|\log \rho| \times (1 - \rho)^6 = K(\mu + n^2 + \log A). \quad (5.42)$$

We set  $r' = (1 + r)/2$  and check that  $1 - r' = (1 - r)/2$ . We apply Lemma 5.4.14 to the series  $f(x)$  over the disk  $D(0, r')$ .

It is clear that condition (5.39) is satisfied provided  $K$  is large enough. We note also that

$$u \leq \frac{16(\log \rho)^2}{1 - r'} = \frac{16K^2(\mu + n^2 + \log A)^2}{(1 - r')^{13}}.$$

We set  $v = u$  and observe that, provided  $K$  is large enough, the first conclusion in Lemma 5.4.14 is not compatible with the definition of  $\rho$  given by Equation (5.42). Therefore, the second conclusion in Lemma 5.4.14 must hold true: there exists an  $R$  such that  $r' - \rho \leq R \leq r'$  and the number of zeros of  $f(x)$  inside  $D(0, R)$  is bounded by  $u$ . If  $K$  is large enough then  $\rho$  is smaller than  $r' - r = (1 - r)/2$ . So  $D(0, r) \subset D(0, R)$  and the number of zeros in  $D(0, r)$  is bounded by  $u$ .  $\square$

#### 5.4.16 An algorithm for finding zeros of power series

In this section we describe the algorithm announced by Theorem 5.4.3 for computing the zeros of a power series  $f(x)$ . The input of the algorithm is described in the statement of Theorem 5.4.3 and at the beginning of Section 5.4. We are given a black box  $\text{BOX}_f$  for the coefficients of  $f(x)$ . We are given two integers  $a \geq 1$  and  $n \geq 1$  such that  $f(x)$  is of type  $(A, n)$  where  $A = \exp(a)$ . We are given also an integer  $\mu \geq 1$  such that there exists at least one  $z$  in  $D(0, 1/2)$  such that  $|f(z)| > \exp(-\mu)$ . We don't need the value of this  $z$ . Knowing its existence suffices. Finally, we are given two integers  $m \geq 1$  and  $o \geq 2$  and we are supposed to compute approximations within  $10^{-m}$  for the zeros of  $f(x)$  in some disk  $D(0, r')$  where  $|r' - r| \leq 10^{-m}$  and  $r = 1 - 1/o$ .

Let  $K$  be a large enough constant. We assume that  $K$  is an integer. We set

$$m' = K(m + (\mu + n^2 + a)o^6) \text{ and } \rho' = \exp(-m'). \quad (5.43)$$

We apply Lemma 5.4.14 to the series  $f(x)$  over the disk  $D(0, r)$  with the accuracy  $\rho'$ . It is clear that condition (5.39) is satisfied provided  $K$  is

large enough. It is clear also that the first conclusion in this lemma is not compatible with the definitions of  $\mu$  and  $m'$ . So the second conclusion must hold true. We take for  $u$  the value

$$u = \frac{16(\log \rho')^2}{1-r} = 16oK^2(m + (\mu + n^2 + a)o^6)^2.$$

We first show that there exists at least one integer  $w$  such that  $1 \leq w \leq u$  and  $|f_{w-1}| \geq 10^{-6(m')^2}$ . Otherwise, for any  $z \in D(0, 1/2)$  we would have

$$\begin{aligned} |f(z)| &\leq |P_u(z)| + |R_u(z)| \\ &\leq u10^{-6(m')^2} + \exp(-4(\log \rho')^2) \text{ using equation (5.40)} \\ &\leq \exp(-\mu) \end{aligned}$$

using the definition of  $\rho'$  in (5.43) and assuming  $K$  is large enough. But the later inequality contradicts the hypothesis on  $\mu$  in Theorem 5.4.3.

So let  $v$  be an integer such that  $1 \leq v \leq u$  and  $|f_{v-1}| \geq 0.5 \times 10^{-6(m')^2}$  and  $|f_w| \leq 10^{-6(m')^2}$  for all  $w$  such that  $w \geq v$  and  $w \leq u - 1$ . Let  $R$  be any real such that  $R \in [r - \rho', r]$  and  $|R - |z|| > 2(\rho')^2$  for every root  $z$  of  $P_v(x)$ . Then, inside the disk  $D(0, R)$ , the zeros of  $P_v(x)$  approximate the zeros of  $f(x)$  within  $\rho'$ . If  $K$  is large enough then  $\rho' < 0.5 \times 10^{-m}$ .

We call  $P(x) = P_v(x)/f_{v-1}$  the unitary polynomial associated with  $P_v(x)$ . Its coefficients are bounded above by  $2Av^n 10^{6(m')^2}$ . So we can compute approximations  $(\alpha_j)_{1 \leq j \leq v-1}$  of the roots of  $P(x)$  within  $(\rho')^2$  using the algorithm provided by Theorem 5.3.1 of Section 5.3. We assume that the  $\alpha_j$  are sorted by increasing absolute value. We consider the interval  $[r - \rho', r]$  and we remove to it all intervals  $[|\alpha_j| - 3(\rho')^2, |\alpha_j| + 3(\rho')^2]$ . Let  $r'$  be a rational number in this set. We count the  $\alpha_j$  that belong to the disk  $D(0, r')$ . Assume that there are  $J$  of them. We output the divisor  $[\alpha_1] + [\alpha_2] + \dots + [\alpha_J]$ .

This leads to the following algorithm:

1. Set

$m' = K(m + o^6(\mu + n^2 + a))$  and  $u = 16K^2o(m + o^6(\mu + n^2 + a))^2$   
and look for an integer  $v$  such that

$1 \leq v \leq u$  and  $|f_{v-1}| \geq 0.5 \times 10^{-6(m')^2}$  and  $|f_w| \leq 10^{-6(m')^2}$  for all  $w$  such that  $w \geq v$  and  $w \leq u - 1$ .

2. Using the algorithm provided by Theorem 5.3.1, compute approximations  $(\alpha_j)_{1 \leq j \leq v-1}$  of the roots of  $P(x) = P_v(x)/f_{v-1}$  within  $\exp(-2m')$ .
3. Pick a rational number  $r'$  in the set  $[r - \exp(-m'), r] - \bigcup_j [|\alpha_j| - 3 \exp(-2m'), |\alpha_j| + 3 \exp(-2m')]$ .
4. Output  $r'$  and those  $\alpha_j$  that have absolute value  $\leq r'$ .

#### 5.4.17 Power series of type $(A, n)$

In this section, we review simple elementary results about power series in one variable. Recall Definition 5.4.1 of the *type* of a power series in one variable and let  $f(x) = \sum_{k \geq 0} f_k x^k$  be a power series of type  $(A, n)$  with  $A \geq 1$  a real number and  $n \geq 1$  an integer. Let  $u \geq 0$  be an integer. We write  $f(x) = P_u(x) + R_u(x)$  where  $P_u(x) = \sum_{0 \leq k \leq u-1} f_k x^k$  is the principal part and  $R_u(x) = \sum_{k \geq u} f_k x^k$  is the remainder term of order  $u$ . We first want to bound  $|R_u(z)|$  for  $z \in D(0, 1)$ .

$$\begin{aligned}
 |R_u(z)| &= \left| \sum_{k \geq u} f_k z^k \right| \leq \sum_{k \geq u} |f_k| |z|^k \\
 &\leq A \sum_{k \geq u} (k+1)^n |z|^k \\
 &\leq A |z|^u \sum_{k \geq 0} (k+u+1)^n |z|^k \\
 &\leq A |z|^u (u+1)^n \sum_{k \geq 0} (k+1)^n |z|^k \\
 &\leq A |z|^u (u+1)^n \frac{n!}{(1-|z|)^{n+1}} \tag{5.44}
 \end{aligned}$$

We set

$$\kappa = \frac{n! A |z|^{\frac{u}{2}}}{(1-|z|)^{n+1}} \tag{5.45}$$

and we show that  $|R_u(z)| \leq \kappa$  provided  $u \geq \frac{4n^2}{(\log|z|)^2}$ . Indeed, if  $u \geq \frac{4n^2}{(\log|z|)^2}$  then  $n \log(u+1) \leq n\sqrt{u} \leq \frac{u \log|z|}{2}$  so

$$u \log|z| + n \log(u+1) \leq \frac{u \log|z|}{2}.$$

Using (5.44) we deduce

$$\log|R_u(z)| \leq \log A + \frac{u \log|z|}{2} + \log \frac{n!}{(1-|z|)^{n+1}} = \log \kappa.$$

We thus have proven the following lemma.

**5.4.18 Lemma (Bounding the remainder of a power series)** *Let  $f(x)$  be a power series of type  $(A, n)$  where  $A \geq 1$  is a real number and  $n \geq 1$  is an integer. Let  $z$  be a complex number such that  $|z| < 1$ . Let  $u \geq 0$  be an integer and let  $R_u(x) = \sum_{k \geq u} f_k x^k$  be the remainder of order  $u$  of  $f(x)$ . We have*

$$|R_u(z)| \leq A|z|^u (u+1)^n \frac{n!}{(1-|z|)^{n+1}}$$

and if  $u \geq \frac{4n^2}{(\log|z|)^2}$  then

$$|R_u(z)| \leq \frac{n!A|z|^{\frac{u}{2}}}{(1-|z|)^{n+1}} \quad (5.46)$$

Note that if we set  $u = 0$  in (5.44) we obtain

$$|f(z)| \leq \frac{n!A}{(1-|z|)^{n+1}} \quad (5.47)$$

Now let  $f(x)$  be a series of type  $(A, n)$  and let  $c$  be a complex number such that  $|c| < 1$ . We set  $f_c(x) = f(c+x(1-|c|))$  and call  $f_c$  the *refocused* series of  $f(x)$  at  $c$ . We want to bound the coefficients of the power series  $f_c(x)$ . These coefficients are related to the successive derivatives of  $f$  at  $c$ . Let  $k \geq 0$  be an integer. We set

$$\rho = \frac{(k+1) + (n+1)|c|}{k+n+2}.$$

From Cauchy's integral formula

$$\begin{aligned} |f^{(k)}(c)| &= \left| \frac{k!}{2i\pi} \int_{|\zeta|=\rho} \frac{f(\zeta)}{(\zeta-c)^{k+1}} d\zeta \right| \\ &\leq n!k!A \frac{(k+n+2)^{k+n+2}}{(k+1)^{k+1}(n+1)^{n+1}} \frac{1}{(1-|c|)^{n+k+2}} \end{aligned} \quad (5.48)$$

using (5.47).

We notice that

$$\begin{aligned} \frac{(k+n+2)^{k+n+2}}{(k+1)^{k+1}(n+1)^{n+1}} &\leq \left(1 + \frac{n+1}{k+1}\right)^{k+1} \left(1 + \frac{k+1}{n+1}\right)^{n+1} \\ &\leq \exp(n+1)2^{n+1}(k+1)^{n+1}. \end{aligned} \quad (5.49)$$

From (5.47) and (5.49) we deduce that the refocused power series  $f_c(x)$  has type

$$(n!A(1-|c|)^{-n-2} \exp(n+1)2^{n+1}, n+1).$$

We thus have proven the following lemma.

**5.4.19 Lemma (Refocusing a power series in one variable)** *Let*

$$f(x) \in \mathbb{C}[[x]]$$

*be a power series of type  $(A, n)$  where  $A \geq 1$  is a real and  $n \geq 1$  is an integer. Let  $c$  be a complex number with absolute value smaller than 1. The refocused series  $f_c(x)$  is defined as  $f_c(x) = f(c+x(1-|c|))$ . It is a power series of type  $(n!A(1-|c|)^{-n-2} \exp(n+1)2^{n+1}, n+1)$ .*





## Chapter Six

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### Computations with modular forms and Galois representations

J. Bosman

In this chapter we will discuss several aspects of the practical side of computing with modular forms and Galois representations. We start by discussing computations with modular forms and from there on work towards the computation of polynomials associated with modular Galois representations. Throughout this chapter, we will denote the space of cusp forms of weight  $k$ , group  $\Gamma_1(N)$  and character  $\varepsilon$  by  $S_k(N, \varepsilon)$ .

#### 6.1 MODULAR SYMBOLS

Modular symbols provide a way of doing symbolic calculations with modular forms, as well as the homology of modular curves. In this section our intention is to give the reader an idea of what is going on rather than a complete and detailed account of the material. For more details and further reading on the subject of modular symbols, the reader could take a look at [Man1], [Sho] and [Mer]. A computational approach to the material can be found in [Ste1] and [Ste2].

##### 6.1.1 Definitions

Let  $A$  be the free abelian group on the symbols  $\{\alpha, \beta\}$  with  $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ . Consider the subgroup  $I \subset A$  generated by all elements of the forms:

$$\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\}, \quad \{\alpha, \beta\} + \{\beta, \alpha\}, \quad \text{and} \quad \{\alpha, \alpha\}.$$

We define the group:

$$\mathbb{M}_2 := (A/I)/\text{torsion}$$

as the quotient of  $A/I$  by its torsion subgroup. By a slight abuse of notation, we will denote the class of  $\{\alpha, \beta\}$  in this quotient also by  $\{\alpha, \beta\}$ . We have an action  $\text{GL}_2^+(\mathbb{Q})$  on  $\mathbb{M}_2$  by:

$$\gamma\{\alpha, \beta\} := \{\gamma\alpha, \gamma\beta\},$$

where  $\gamma$  acts on  $\mathbb{P}^1(\mathbb{Q})$  by fractional linear transformations.

For  $k \geq 2$ , we consider also the abelian group  $\mathbb{Z}[x, y]_{k-2} \subset \mathbb{Z}[x, y]$  of homogeneous polynomials of degree  $k - 2$  and we let matrices in  $\text{GL}_2^+(\mathbb{Q})$  with integer coefficients act on it on the left by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P(x, y) := P(dx - by, -cx + ay).$$

We define:

$$\mathbb{M}_k := \mathbb{Z}[x, y]_{k-2} \otimes \mathbb{M}_2,$$

and we equip  $\mathbb{M}_k$  with the component-wise action of integral matrices in  $\text{GL}_2^+(\mathbb{Q})$  (that is  $\gamma(P \otimes \alpha) = \gamma(P) \otimes \gamma(\alpha)$ ).

**6.1.2 Definition** Let  $k \geq 2$  be an integer. Let  $\Gamma \subset \text{SL}_2(\mathbb{Z})$  be a subgroup of finite index and let  $I \subset \mathbb{M}_k$  be the subgroup generated by all elements of the form  $\gamma x - x$  with  $\gamma \in \Gamma$  and  $x \in \mathbb{M}_k$ . Then we define the space of *modular symbols* of weight  $k$  for  $\Gamma$  to be the quotient of  $\mathbb{M}_k/I$  by its torsion subgroup and we denote this space by  $\mathbb{M}_k(\Gamma)$ :

$$\mathbb{M}_k(\Gamma) := (\mathbb{M}_k/I)/\text{torsion}.$$

In the special case  $\Gamma = \Gamma_1(N)$ , which we will mostly be interested in,  $\mathbb{M}_k(\Gamma)$  is called the space of modular symbols of weight  $k$  and level  $N$ . The class of  $\{\alpha, \beta\}$  in  $\mathbb{M}_k(\Gamma)$  will be denoted by  $\{\alpha, \beta\}_\Gamma$  or, if no confusion exists, by  $\{\alpha, \beta\}$ .

The group  $\Gamma_0(N)$  acts naturally on  $\mathbb{M}_k(\Gamma_1(N))$  and induces an action of  $(\mathbb{Z}/N\mathbb{Z})^\times$  on  $\mathbb{M}_k(\Gamma_1(N))$ . We denote this action by the diamond symbol  $\langle d \rangle$ . The operator  $\langle d \rangle$  on  $\mathbb{M}_k(\Gamma_1(N))$  is called a *diamond operator*. This leads to the notion of modular symbols with character.

**6.1.3 Definition** Let  $\varepsilon: (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$  be a Dirichlet character. Denote by  $\mathbb{Z}[\varepsilon] \subset \mathbb{C}$  the subring generated by all values of  $\varepsilon$ . Let  $I$  be the  $\mathbb{Z}[\varepsilon]$ -submodule of  $\mathbb{M}_k(\Gamma_1(N)) \otimes \mathbb{Z}[\varepsilon]$  generated by all elements of the form  $\langle d \rangle x - \varepsilon(d)x$  with  $d \in (\mathbb{Z}/N\mathbb{Z})^\times$  and  $x \in \mathbb{M}_k(\Gamma_1(N))$ . Then we define the space  $\mathbb{M}_k(N, \varepsilon)$  of modular symbols of weight  $k$ , level  $N$  and character  $\varepsilon$  as the  $\mathbb{Z}[\varepsilon]$ -module:

$$\mathbb{M}_k(N, \varepsilon) := (\mathbb{M}_k(\Gamma_1(N)) \otimes \mathbb{Z}[\varepsilon]/I)/\text{torsion}.$$

We denote the elements of  $\mathbb{M}_k(N, \varepsilon)$  by  $\{\alpha, \beta\}_{N, \varepsilon}$  or simply by  $\{\alpha, \beta\}$ . If  $\varepsilon$  is trivial, then we have  $\mathbb{M}_k(N, \varepsilon) \cong \mathbb{M}_k(\Gamma_0(N))$ .

Let  $\mathbb{B}_2$  be the free abelian group on the symbols  $\{\alpha\}$  with  $\alpha \in \mathbb{P}^1(\mathbb{Q})$ , equipped with action of  $\text{SL}_2(\mathbb{Z})$  by  $\gamma\{\alpha\} = \{\gamma\alpha\}$  and define  $\mathbb{B}_k$  as  $\mathbb{Z}[x, y]_{k-2} \otimes \mathbb{B}_2$  with component-wise  $\text{SL}_2(\mathbb{Z})$ -action. Elements of  $\mathbb{B}_k$  are called *boundary modular symbols*. For a subgroup  $\Gamma < \text{SL}_2(\mathbb{Z})$  of finite index, we define  $\mathbb{B}_k(\Gamma)$  as:

$$\mathbb{B}_k(\Gamma) := (\mathbb{B}_k/I)/\text{torsion}$$

where  $I$  is the subgroup of  $\mathbb{B}_k$  generated by all elements  $\gamma x - x$  with  $\gamma \in \Gamma$  and  $x \in \mathbb{B}_k$ . We define:

$$\mathbb{B}_k(N, \varepsilon) := ((\mathbb{B}_k(\Gamma_1(N)) \otimes \mathbb{Z}[\varepsilon])/I)/\text{torsion},$$

where  $I$  is the  $\mathbb{Z}[\varepsilon]$ -submodule of  $\mathbb{B}_k(\Gamma_1(N)) \otimes \mathbb{Z}[\varepsilon]$  generated by the elements  $\gamma x - \varepsilon(\gamma)x$  with  $\gamma \in \Gamma_0(N)$ .

We have *boundary homomorphisms*:

$$\delta: \mathbb{M}_k(\Gamma) \rightarrow \mathbb{B}_k(\Gamma) \quad \text{and} \quad \delta: \mathbb{M}_k(N, \varepsilon) \rightarrow \mathbb{B}_k(N, \varepsilon)$$

defined by:

$$\delta(P \otimes \{\alpha, \beta\}) = P \otimes \{\beta\} - P \otimes \{\alpha\}.$$

The spaces of *cuspidal modular symbols*, denoted by  $\mathbb{S}_k(\Gamma)$  and  $\mathbb{S}_k(N, \varepsilon)$  respectively are defined as the kernel of  $\delta$ .

### 6.1.4 Properties

One can interpret the symbol  $\{\alpha, \beta\}$  as a smooth path in  $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$  from the cusp  $\alpha$  to the cusp  $\beta$ , lying in  $\mathbb{H}$  except for the endpoints  $\alpha$  and  $\beta$ . It can

be shown that this interpretation induces an isomorphism:

$$\mathbb{M}_2(\Gamma) \cong H_1(X_\Gamma, \text{cusps}, \mathbb{Z}).$$

Here the homology is taken of the topological pair  $(X_1(N), \text{cusps})$ . We also get an isomorphism:

$$\mathbb{S}_2(\Gamma) \cong H_1(X_\Gamma, \mathbb{Z}).$$

So we immediately see that there is a perfect pairing:

$$(\mathbb{S}_2(\Gamma(N)) \otimes \mathbb{C}) \times (S_2(\Gamma(N)) \oplus \overline{S}_2(\Gamma(N))) \rightarrow \mathbb{C}$$

defined by:

$$(\{\alpha, \beta\}, f \oplus g) \mapsto \int_\alpha^\beta \left( f \frac{dq}{q} + g \frac{d\bar{q}}{\bar{q}} \right).$$

More generally, there is a pairing

$$(6.1.5) \quad \mathbb{M}_k(\Gamma_1(N)) \times (S_k(\Gamma_1(N)) \oplus \overline{S}_k(\Gamma_1(N))) \rightarrow \mathbb{C}$$

defined by:

$$(P \otimes \{\alpha, \beta\}, f \oplus g) \mapsto 2\pi i \int_\alpha^\beta (f(z)P(z, 1)dz - g(z)P(\bar{z}, 1)d\bar{z}),$$

which becomes perfect if we restrict and then tensor the left factor to  $\mathbb{S}_k(\Gamma(N)) \otimes \mathbb{C}$ . This pairing induces a pairing:

$$(\mathbb{M}_k(N, \varepsilon)) \times (S_k(N, \varepsilon) \oplus \overline{S}_k(N, \varepsilon)) \rightarrow \mathbb{C}$$

which becomes perfect when the left factor is restricted and then tensored to  $\mathbb{S}_k(N, \varepsilon) \otimes_{\mathbb{Z}[\varepsilon]} \mathbb{C}$ . From now on we will denote all these pairings with the notation:

$$(x, f) \mapsto \langle x, f \rangle.$$

### 6.1.6 The star involution

On the spaces  $\mathbb{M}_k(\Gamma_1(N))$  and  $\mathbb{M}_k(N, \varepsilon)$  we have an involution  $\iota^*$ :

$$\iota^*(P(x, y) \otimes \{\alpha, \beta\}) := -P(x, -y) \otimes \{-\alpha, -\beta\},$$

which is called the *star involution*. It preserves cuspidal subspaces. We define  $\mathbb{S}_k(\Gamma_1(N))^+$  and  $\mathbb{S}_k(\Gamma_1(N))^-$  subspaces of  $\mathbb{S}_k(\Gamma_1(N))$  where  $\iota^*$  acts as  $+1$  and  $-1$  respectively and we use similar definitions for  $\mathbb{S}_k(N, \varepsilon)^\pm$ . It can be shown that the pairing (6.1.5) induces perfect pairings:

$$(\mathbb{S}_k(\Gamma_1(N))^+ \otimes \mathbb{C}) \times S_k(\Gamma_1(N)) \rightarrow \mathbb{C}$$

and:

$$(\mathbb{S}_k(\Gamma_1(N))^- \otimes \mathbb{C}) \times \overline{S}_k(\Gamma_1(N)) \rightarrow \mathbb{C}$$

and similarly for the spaces with character. This allows us to work sometimes in modular symbols spaces of half the dimension of the full cuspidal space.

### 6.1.7 Hecke operators

Let  $k \geq 2$  and  $N \geq 1$  be given. Then for  $\gamma \in \mathrm{GL}_2^+(\mathbb{Q}) \cap \mathrm{M}_2(\mathbb{Z})$  we define an operator  $T_\gamma$  on  $\mathbb{M}_k(\Gamma_1(N))$  by letting  $\gamma_1, \dots, \gamma_r$  be double coset representatives for  $\Gamma_1(N) \backslash \Gamma_1(N)\gamma\Gamma_1(N)$  and putting

$$(6.1.8) \quad T_\gamma(x) := \sum_{i=1}^r \gamma_i x \quad \text{for } x \in \mathbb{M}_k(\Gamma_1(N)).$$

It follows from [Sho, Theorem 4.3] that this operator is well-defined. For a prime number  $p$  we put  $T_p = T_\gamma$  for  $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$  and for positive integers  $n$  we define  $T_n$  by means of the formal identity (2.2.11). The operators  $T_n$  are called Hecke operators.

The Hecke operators preserve the subspace  $\mathbb{S}_k(\Gamma_1(N))$  and induce an action on the spaces  $\mathbb{M}_k(N, \varepsilon)$  and  $\mathbb{S}_k(N, \varepsilon)$ . Furthermore, from [Sho, Theorem 4.3] one can conclude that the diamond and Hecke operators are self-adjoint with respect to the pairings defined in the previous subsection: one has

$$(6.1.9) \quad \langle Tx, f \rangle = \langle x, Tf \rangle$$

for any modular symbol  $x$ , cusp form  $f$  and diamond or Hecke operator  $T$  for which this relation is well-defined. Here, for anti-holomorphic cusp forms we define the Hecke action by  $T\overline{f} = \overline{Tf}$ . Also, the Hecke operators commute with the star involution  $\iota^*$ .

In conclusion, we have seen how we can write cusp forms spaces as the dual of modular symbols spaces. The computation of Hecke operators on these modular symbols spaces would enable us to compute  $q$ -expansions of cusp forms:  $q$ -coefficients of newforms can be computed once we can compute the eigenvalues of Hecke operators. But because of (6.1.9) this reduces to the computation of the eigenvalues of Hecke operators on modular symbols spaces. In computations one often works with the spaces  $\mathbb{S}_k(N, \varepsilon)^+ \otimes_{\mathbb{Z}[\varepsilon]} \mathbb{Q}(\varepsilon)$  because these have smaller dimension than  $\mathbb{S}_k(\Gamma_1(N)) \otimes \mathbb{Q}$ . Since we also know how all cusp forms arise from newforms of possibly lower level (see (2.2.17)), this allows us to compute the  $q$ -expansions of a basis for the spaces  $S_k(\Gamma_1(N))$  and  $S_k(N, \varepsilon)$ . For precise details on how these computations work, please read [Ste2, Chapter 9].

### 6.1.10 Manin symbols

If we want to do symbolic calculations with modular symbols, then the above definitions are not quite applicable since the groups of which we take quotients are not finitely generated. The *Manin symbols* enable us to give finite presentations for the spaces of modular symbols.

First we need some definitions and lemmas. For a positive integer  $N$  we define a set:

$$E_N := \{(c, d) \in (\mathbb{Z}/N\mathbb{Z})^2 : \gcd(N, c, d) = 1\}.$$

Define the following equivalence relation on  $E_N$ :

$$(c, d) \sim (c', d') \stackrel{\text{def}}{\iff} \text{there is an } a \in (\mathbb{Z}/N\mathbb{Z})^\times \text{ with } (c, d) = (ac', ad')$$

and denote the quotient by  $P_N$ :

$$(6.1.11) \quad P_N := E_N / \sim.$$

The following lemma is easily verified:

**6.1.12 Lemma** *Let  $N$  be a positive integer. Then the maps*

$$\begin{aligned} \Gamma_1(N) \backslash \text{SL}_2(\mathbb{Z}) &\rightarrow E_N : \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \mapsto (\bar{c}, \bar{d}) \quad \text{and} \\ \Gamma_0(N) \backslash \text{SL}_2(\mathbb{Z}) &\rightarrow P_N : \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \mapsto \overline{(c, d)} \end{aligned}$$

are well-defined and bijective.

This lemma enables us to write down an explicit set of coset representatives for the orbit spaces  $\Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})$  and  $\Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z})$ . The following lemma provides us a first step in reducing the set of generators for the spaces of modular symbols:

**6.1.13 Lemma** *Each space  $\mathbb{M}_2(\Gamma_1(N))$  or  $\mathbb{M}_2(N, \varepsilon)$  is generated by the symbols  $\{a/c, b/d\}$  with  $a, b, c, d \in \mathbb{Z}$  and  $ad - bc = 1$ , where in this notation a fraction with denominator equal to zero denotes the cusp at infinity.*

Calculating the continued fraction expansion at each cusp in  $\mathbb{Q}$  gives us immediately an algorithm to write a given element of  $\mathbb{M}_2$  in terms of the generators in the lemma. Furthermore, note that:

$$\left\{ \frac{a}{c}, \frac{b}{d} \right\} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \{ \infty, 0 \},$$

so that we can write each element of  $\mathbb{M}_2$  as a sum of  $\gamma\{\infty, 0\}$  with  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ .

Let's consider the space  $\mathbb{M}_2(\Gamma_1(N))$ . As we saw, it is generated by the elements  $\gamma\{\infty, 0\}$  where  $\gamma$  runs through  $\mathrm{SL}_2(\mathbb{Z})$ . Now, two matrices  $\gamma$  define the same element this way if they are in the same coset of the quotient  $\Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})$ . According to Lemma 6.1.12 such a coset can be uniquely identified with a pair  $(c, d) \in (\mathbb{Z}/N\mathbb{Z})^2$ . The corresponding element in  $\mathbb{M}_2(\Gamma_1(N))$  is also denoted by  $(c, d)$ . This element  $(c, d)$  is called a *Manin symbol*. Clearly, there are only a finite number of Manin symbols so we now know a finite set of generators for  $\mathbb{M}_2(\Gamma_1(N))$ .

For arbitrary  $k$  we define the Manin symbols in  $\mathbb{M}_k(\Gamma_1(N))$  as the symbols of the form  $P \otimes (c, d)$  where  $P$  is a monomial in  $\mathbb{Z}[x, y]_{k-2}$  and  $(c, d)$  a Manin symbol in  $\mathbb{M}_2(\Gamma_1(N))$ . In this case as well there are finitely many Manin symbols and they generate the whole space.

In the modular symbols spaces with a given character  $\varepsilon$  we have, for all  $\gamma \in \Gamma_0(N)$ , that  $\gamma(\alpha) = \varepsilon(\alpha)$ . Now for each element of  $P_N$  we choose according to Lemma 6.1.12 a corresponding element  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  and hence an element in  $\mathbb{M}_2(N, \varepsilon)$ , which we call again a Manin symbol. Note that this Manin symbol depends on the choice of  $\gamma$ , but because of the relation



$\gamma(x) = \varepsilon(x)$  these chosen Manin symbols always form a finite set of generators for  $\mathbb{M}_2(N, \varepsilon)$  as a  $\mathbb{Z}[\varepsilon]$ -module. Likewise,  $\mathbb{M}_k(N, \varepsilon)$  is generated by elements  $P \otimes (c, d)$  with  $P$  a monomial in  $\mathbb{Z}[x, y]_{k-2}$  and  $(c, d)$  a Manin symbol in  $\mathbb{M}_2(N, \varepsilon)$ .

If we want to do symbolic calculations, then besides generators we also need to know the relations between the Manin symbols. For  $\mathbb{M}_k(\Gamma_1(N))$  one can do the following.

**6.1.14 Proposition** *Let  $N$  be a positive integer and let  $A$  be the free abelian group on the Manin symbols of the space  $\mathbb{M}_k(\Gamma_1(N))$ . Let  $I \subset A$  be the subgroup generated by the following elements:*

$$\begin{aligned} &P(x, y) \otimes (c, d) + P(-y, x) \otimes (-d, -c), \\ &P(x, y) \otimes (c, d) + P(-y, x - y) \otimes (-d, -c - d) \\ &\quad + P(-x + y, -x) \otimes (-c - d, -c), \\ &P(x, y) \otimes (c, d) - P(-x, -y) \otimes (c, d), \end{aligned}$$

where  $P(x, y) \otimes (c, d)$  runs through all Manin symbols. Then  $\mathbb{M}_k(\Gamma_1(N))$  is naturally isomorphic to the quotient of  $A/I$  by its torsion subgroup.

For the modular symbols spaces  $\mathbb{M}_k(N, \varepsilon)$  we have a similar proposition.

**6.1.15 Proposition** *Let  $N$  and  $\varepsilon$  be given. Let  $A$  be the free  $\mathbb{Z}[\varepsilon]$ -module on the Manin symbols of  $\mathbb{M}_k(N, \varepsilon)$ . Let  $I \subset A$  be the submodule generated by the elements given in Proposition 6.1.14 plus for each  $n \in (\mathbb{Z}/N\mathbb{Z})^\times$  the elements:*

$$P(x, y) \otimes \overline{(nc, nd)} - \varepsilon(n)P(x, y) \otimes \overline{(c, d)}.$$

Then  $\mathbb{M}_k(N, \varepsilon)$  is naturally isomorphic to the quotient of  $A/I$  by its torsion submodule.

These presentations enable us to perform symbolic calculations very efficiently.

A remark on the computation of Hecke operators is in order here. The formula (6.1.8) does not express the Hecke action on Manin symbols in terms of Manin symbols. However, one can use other formulas to compute

Hecke operators. The following theorem, due to Merel, allows us to express Hecke operators more directly in terms of Manin symbols:

**6.1.16 Theorem (see [Mer, Theorem 2])** *On the spaces  $\mathbb{M}_k(\Gamma_1(N))$  and  $\mathbb{M}_k(N, \varepsilon)$  the Hecke operator  $T_n$  satisfies the following relation:*

$$T_n(P(x, y) \otimes (u, v)) = \sum'_{\substack{a > b \geq 0 \\ d > c \geq 0 \\ ad - bc = n}} P(ax + by, cx + dy) \otimes (au + cv, bu + dv),$$

where the prime in the notation for the sum means that terms with  $\gcd(N, au + cv, bu + dv) \neq 1$  have to be omitted.

One would also like to express  $\mathbb{S}_k(\Gamma_1(N))$  and  $\mathbb{S}_k(N, \varepsilon)$  in terms of the Manin symbols. The following proposition will help us.

**6.1.17 Proposition (see [Mer, Proposition 4])** *Let integers  $N \geq 1$  and  $k \geq 2$  be given. Define an equivalence relation on the vector space  $\mathbb{Q}[\Gamma_1(N) \setminus \mathbb{Q}^2]$  by:*

$$[\lambda x] \sim \text{sign}(\lambda)^k [\bar{x}] \quad \text{for } \lambda \in \mathbb{Q}^\times \text{ and } x \in \mathbb{Q}^2.$$

Then the map:

$$\mu: \mathbb{B}_k(\Gamma_1(N)) \rightarrow \mathbb{Q}[\Gamma_1(N) \setminus \mathbb{Q}^2] / \sim$$

given by:

$$\mu: P \otimes \left\{ \frac{a}{b} \right\} \mapsto P(a, b) \left[ \begin{pmatrix} a \\ b \end{pmatrix} \right] \quad (a, b \text{ coprime integers})$$

is well-defined and injective.

The vector space  $\mathbb{Q}[\Gamma_1(N) \setminus \mathbb{Q}^2] / \sim$  is finite dimensional. The above proposition shows that  $\mathbb{S}_k(\Gamma_1(N))$  is the kernel of  $\mu\delta$ , which is a map that can be computed in terms of Manin symbols. The computation of  $\mathbb{S}_k(N, \varepsilon)$  can be done in a similar way, see [Ste2, Section 8.4].

## 6.2 INTERMEZZO: ATKIN-LEHNER OPERATORS

In the rest of this chapter, we will be using the Atkin-Lehner operators on  $\mathbb{S}_k(\Gamma_1(N))$  from time to time. This section provides a brief treatment of the properties that we need. The main reference for this material is [At-Li].

Let  $Q$  be a positive divisor of  $N$  such that  $\gcd(Q, N/Q) = 1$ . Let  $w_Q \in \mathrm{GL}_2^+(\mathbb{Q})$  be any matrix of the form:

$$(6.2.1) \quad w_Q = \begin{pmatrix} Qa & b \\ Nc & Qd \end{pmatrix}$$

with  $a, b, c, d \in \mathbb{Z}$  and  $\det(w_Q) = Q$ . The assumption  $\gcd(Q, N/Q) = 1$  ensures that such a  $w_Q$  exists. A straightforward verification shows  $f|_k w_Q \in S_k(\Gamma_1(N))$ . Now, given  $Q$ , this  $f|_k w_Q$  still depends on the choice of  $a, b, c, d$ . However, we can use a normalisation in our choice of  $a, b, c, d$  which will ensure that  $f|_k w_Q$  only depends on  $Q$ . Be aware of the fact that different authors use different normalisations here. The one we will be using is:

$$(6.2.2) \quad a \equiv 1 \pmod{N/Q}, \quad b \equiv 1 \pmod{Q},$$

which is the normalisation used in [At-Li]. We define:

$$(6.2.3) \quad \begin{aligned} W_Q(f) &:= Q^{1-k/2} f|_k w_Q \\ &= \frac{Q^{k/2}}{(Ncz + Qd)^k} f\left(\frac{Qaz + b}{Ncz + Qd}\right), \end{aligned}$$

which is now independent of the choice of  $w_Q$  and call  $W_Q$  an *Atkin-Lehner operator*. In particular we have:

$$W_N(f) = \frac{1}{N^{k/2} z^k} f\left(\frac{-1}{Nz}\right).$$

An unfortunate thing about these Atkin-Lehner operators is that they do not preserve the spaces  $S_k(N, \varepsilon)$ . But we can say something about it. Let  $\varepsilon: (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$  be a character and suppose that  $f$  is in  $S_k(N, \varepsilon)$ . By the Chinese Remainder Theorem, one can write  $\varepsilon$  in a unique way as  $\varepsilon = \varepsilon_Q \varepsilon_{N/Q}$  such that  $\varepsilon_Q$  is a character on  $(\mathbb{Z}/Q\mathbb{Z})^\times$  and  $\varepsilon_{N/Q}$  is a character on  $(\mathbb{Z}/(N/Q)\mathbb{Z})^\times$ . It is a fact that:

$$W_Q(f) \in S_k(N, \bar{\varepsilon}_Q \varepsilon_{N/Q}).$$

Also, there is a relation between the  $q$ -expansions of  $f$  and  $W_Q(f)$ :

**6.2.4 Theorem** *Let  $f$  be a newform in  $S_k(N, \varepsilon)$ . Take  $Q$  dividing  $N$  with  $\gcd(Q, N/Q) = 1$ . Then:*

$$W_Q(f) = \lambda_Q(f)g$$

with  $\lambda_Q(f)$  an algebraic number in  $\mathbb{C}$  of absolute value 1 and  $g$  a newform in  $S_k(N, \bar{\varepsilon}_Q \varepsilon_{N/Q})$ . Suppose now that  $n$  is a positive integer and write  $n = n_1 n_2$  where  $n_1$  consists only of prime factors dividing  $Q$  and  $n_2$  consists only of prime factors not dividing  $Q$ . Then we have:

$$a_n(g) = \varepsilon_{N/Q}(n_1) \bar{\varepsilon}_Q(n_2) \overline{a_{n_1}(f)} a_{n_2}(f).$$

The number  $\lambda_Q(f)$  in the above theorem is called a *pseudo-eigenvalue* for the Atkin-Lehner operator. In some cases there exists a closed expression for it. In the notation of the following theorem,  $g(\chi)$  denotes the Gauss sum of a Dirichlet character  $\chi$  of conductor  $N(\chi)$ :

$$(6.2.5) \quad g(\chi) := \sum_{\nu \in (\mathbb{Z}/N(\chi)\mathbb{Z})^\times} \chi(\nu) \exp\left(\frac{2\pi i \nu}{N(\chi)}\right).$$

**6.2.6 Theorem** *Let  $f \in S_k(N, \varepsilon)$  be a newform and suppose  $q$  is a prime that divides  $N$  exactly once. Then we have:*

$$\lambda_q(f) = \begin{cases} g(\varepsilon_q) q^{-k/2} \overline{a_q(f)} & \text{if } \varepsilon_q \text{ is non-trivial,} \\ -q^{1-k/2} a_q(f) & \text{if } \varepsilon_q \text{ is trivial.} \end{cases}$$

**6.2.7 Theorem (see [Asa, Theorem 2])** *Let  $f \in S_k(N, \varepsilon)$  be a newform with  $N$  square-free. For  $Q \mid N$  we have:*

$$\lambda_Q(f) = \varepsilon(Qd - \frac{N}{Q}a) \prod_{q \mid Q} \varepsilon(Q/q) \lambda_q(f).$$

Here,  $a$  and  $d$  are defined by (6.2.1). Moreover, this identity holds without any normalisation assumptions on the entries of  $w_Q$ , as long as we define  $\lambda_q(f)$  by the formula given in Theorem 6.2.6.

### 6.3 BASIC NUMERICAL EVALUATIONS

In this section we will describe how to perform basic numerical evaluations, such as the evaluation of a cusp form at a point in  $\mathbb{H}$  and the evaluation of an integral of a cusp form between two points in  $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ . Again, the focus will be on performing actual computations.

### 6.3.1 Period integrals: the direct method

In this subsection we will stick to the case  $k = 2$ , referring to [Ste2, Chapter 10] for a more general approach (see also [Cre, Section 2.10] for a treatment of  $\Gamma_0(N)$ ). So fix a positive integer  $N$  and an  $f \in S_2(\Gamma_1(N))$ . Our goal is to efficiently evaluate the integral pairing  $\langle x, f \rangle$  for  $x \in \mathbb{S}_2(\Gamma_1(N))$ .

Let us indicate why it suffices to look at newforms  $f$ . Because of (2.2.17), it suffices to look at  $f = \alpha_d(f')$  with  $f' \in S_k(\Gamma_1(M))$  a newform for some  $M \mid N$  and  $d \mid N/M$ . By [Sho, Theorem 4.3] we have:

$$\langle x, f \rangle = \langle x, \alpha_d(f') \rangle = d^{1-k} \left\langle \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} x, f' \right\rangle$$

so that computing period integrals for  $f$  reduces to computing period integrals of the newform  $f'$ .

Let us now make the important remark that for each  $z \in \mathbb{H}$  we can numerically compute  $\int_{\infty}^z f dq/q$  by formally integrating the  $q$ -expansion of  $f$ :

$$(6.3.2) \quad \int_{\infty}^z f \frac{dq}{q} = \sum_{n \geq 1} \frac{a_n(f)}{n} q^n \quad \text{where } q = \exp(2\pi iz).$$

The radius of convergence of this series is 1 and the coefficients are small (that is, estimated by  $\tilde{O}(n^{(k-3)/2})$ ). So if  $\Im z \gg 0$  then we have  $|q| \ll 1$  and the series converges rapidly. To be more concrete, for  $\Im z > M$  we have  $|q^n| < \exp(-2\pi Mn)$  so if we want to compute  $\int_{\infty}^z f dq/q$  to a precision of  $p$  decimals, we need to compute about  $\frac{p \log 10}{2\pi M} \approx 0.37 \frac{p}{M}$  terms of the series.

To compute a period integral we remark that for any  $\gamma \in \Gamma_1(N)$  and any  $z \in \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$  any continuous, piecewise smooth path  $\delta$  in  $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$  from  $z$  to  $\gamma z$ , the homology class of  $\delta$  pushed forward to  $X_1(N)(\mathbb{C})$  depends only on  $\gamma$  [Man1, Proposition 1.4]. Let us denote this homology class by:

$$\{\infty, \gamma\infty\} \in \mathbb{S}_2(\Gamma_1(N)) \cong H_1(X_1(N)(\mathbb{C}), \mathbb{Z})$$

and remark that all elements of  $H_1(X_1(N)(\mathbb{C}), \mathbb{Z})$  can be written in this way. As we also have  $S_2(\Gamma_1(N)) \cong H^0(X_1(N)_{\mathbb{C}}, \Omega^1)$ , this means we can calculate  $\int_{\{\infty, \gamma\infty\}} f \frac{dq}{q}$  by choosing a smart path in  $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ :

$$\int_{\infty}^{\gamma\infty} f \frac{dq}{q} = \int_z^{\gamma z} f \frac{dq}{q} = \int_{\infty}^{\gamma z} f \frac{dq}{q} - \int_{\infty}^z f \frac{dq}{q}.$$

If we write  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  then a good choice for  $z$  is:

$$z = -\frac{d}{c} + \frac{i}{|c|}.$$

In this case we have  $\Im z = \Im \gamma z = 1/|c|$  so in view of (6.3.2), to compute the integral to a precision of  $p$  decimals we need about  $\frac{pc \log 10}{2\pi} \approx 0.37pc$  terms of the series.

Another thing we can use is the Hecke compatibility from (6.1.9). Put:

$$W_f := (\mathbb{S}_2(\Gamma_1(N))/I_f \mathbb{S}_2(\Gamma_1(N))) \otimes \mathbb{Q},$$

where  $I_f$  is the Hecke ideal belonging to  $f$ , i.e. the kernel of the map  $\mathbb{T} \rightarrow \mathbb{C}$  that sends  $T_n$  to  $a_n$  for all  $n$  (here, as usual,  $\mathbb{T}$  denotes the Hecke algebra attached to  $S_2(\Gamma_1(N))$ ). The space  $W_f$  has the structure of a vector space over  $(\mathbb{T}/I_f) \otimes \mathbb{Q} \cong K_f$  of dimension 2. This means that computing any period integral of  $f$ , we only need to precompute 2 period integrals. So one tries to find a  $K_f$ -basis of  $W_f$  consisting of elements  $\{\infty, \gamma\infty\}$  where  $\gamma \in \Gamma_1(N)$  has a very small  $c$ -entry. In practice it turns out that we do not need to search very far.

### 6.3.3 Period integrals: the twisted method

In this subsection we have the same set-up as in the previous subsection. There is another way of computing period integrals for  $f \in S_2(\Gamma_1(N))$  which sometimes beats the method described in the previous subsection. The method described in this subsection is similar to [Cre, Section 2.11] and makes use of winding elements and twists.

The *winding element* of  $\mathbb{M}_2(\Gamma_1(N))$  is simply defined as the element  $\{\infty, 0\}$  (some authors define it as  $\{0, \infty\}$ , this is just a matter of sign convention). Integration over this element is easy to perform because we can break up the path in a very neat way:

$$\begin{aligned} \int_{\infty}^0 f \frac{dq}{q} &= \int_{\infty}^{i/\sqrt{N}} f \frac{dq}{q} + \int_{i/\sqrt{N}}^0 f \frac{dq}{q} = \\ &= \int_{\infty}^{i/\sqrt{N}} f \frac{dq}{q} + \int_{i/\sqrt{N}}^{\infty} W_N(f) \frac{dq}{q} = \int_{\infty}^{i/\sqrt{N}} (f - W_N(f)) \frac{dq}{q}. \end{aligned}$$

Now, choose an odd prime  $\ell$  not dividing  $N$  and a primitive Dirichlet character  $\chi: \mathbb{Z} \rightarrow \mathbb{C}$  of conductor  $\ell$ . If  $f \in S_k(\Gamma_1(N))$  is a newform then  $f \otimes \chi$  is a newform in  $S_k(\Gamma_1(N\ell^2))$ , where:

$$f \otimes \chi = \sum_{n \geq 1} a_n(f) \chi(n) q^n.$$

The following formula to express  $\chi$  as a linear combination of additive characters is well-known:

$$\chi(n) = \frac{g(\chi)}{\ell} \sum_{\nu=1}^{\ell-1} \bar{\chi}(-\nu) \exp\left(\frac{2\pi i \nu n}{\ell}\right),$$

where  $g(\chi)$  is the Gauss sum of  $\chi$  (see (6.2.5)). It follows now immediately that:

$$f \otimes \chi = \frac{g(\chi)}{\ell} \sum_{\nu=1}^{\ell-1} \chi(-\nu) f\left(z + \frac{\nu}{\ell}\right) = \frac{g(\chi)}{\ell} \sum_{\nu=1}^{\ell-1} \chi(-\nu) f \left| \begin{pmatrix} \ell & \nu \\ 0 & \ell \end{pmatrix} \right.$$

For  $f \in S_2(\Gamma_1(N))$  we now get the following useful formula for free:

$$(6.3.4) \quad \langle \{\infty, 0\}, f \otimes \chi \rangle = \frac{g(\chi)}{\ell} \left\langle \sum_{\nu=0}^{\ell-1} \chi(-\nu) \left\{ \infty, \frac{\nu}{\ell} \right\}, f \right\rangle.$$

The element  $\sum_{\nu=0}^{\ell-1} \chi(-\nu) \left\{ \infty, \frac{\nu}{\ell} \right\}$  of  $\mathbb{M}_k(\Gamma_1(N)) \otimes \mathbb{Z}[\chi]$  or of some other modular symbols space where it is well-defined is called a *twisted winding element* or, more precisely the  $\chi$ -*twisted winding element*. Because of formula (6.3.4), we can calculate the pairings of newforms in  $S_2(\Gamma_1(N))$  with twisted winding elements quite efficiently as well.

We can describe the action of the Atkin-Lehner operator  $W_{N\ell^2}$  on  $f \otimes \chi$ :

$$W_{N\ell^2}(f \otimes \chi) = \frac{g(\chi)}{g(\bar{\chi})} \varepsilon(\ell) \chi(-N) \lambda_N(f) \tilde{f} \otimes \bar{\chi},$$

where  $\tilde{f} = \sum_{n \geq 1} \overline{a_n(f)} q^n$  (see for example [At-Li, Section 3]). So in particular we have the following integral formula for a newform  $f \in S_2(N, \varepsilon)$ :

$$(6.3.5) \quad \begin{aligned} \int_{\infty}^0 f \otimes \chi \frac{dq}{q} &= \int_{\infty}^{i/(\ell\sqrt{N})} (f \otimes \chi - W_{N\ell^2}(f \otimes \chi)) \frac{dq}{q} = \\ &= \int_{\infty}^{i/(\ell\sqrt{N})} \left( f \otimes \chi - \frac{g(\chi)}{g(\bar{\chi})} \chi(-N) \varepsilon(\ell) \lambda_N(f) \tilde{f} \otimes \bar{\chi} \right) \frac{dq}{q}. \end{aligned}$$

So to calculate:

$$\left\langle \sum_{\nu=0}^{\ell-1} \chi(-\nu) \left\{ \infty, \frac{\nu}{\ell} \right\}, f \right\rangle$$

we need to evaluate the series (6.3.2) at  $z$  in  $\mathbb{H}$  with  $\Im z = 1/(\ell\sqrt{N})$  which means that for a precision of  $p$  decimals we need to sum about  $\frac{p\ell\sqrt{N}\log 10}{2\pi} \approx 0.37p\ell\sqrt{N}$  terms of the series. In the spirit of the previous subsection, we try several  $\ell$  and  $\chi$ , as well as the untwisted winding element  $\{\infty, 0\}$ , until we can make a  $K_f$ -basis for  $W_f$ . It follows from [Shi2, Theorems 1 and 3] that we can always find such a basis. Also here, it turns out that in practice we do not need to search very far. The method that requires the least amount of  $q$ -expansion terms is preferred.

### 6.3.6 Computation of $q$ -expansions at various cusps

The upper half plane  $\mathbb{H}$  is covered by neighbourhoods of the cusps. If we want to evaluate a cusp form  $f \in S_k(\Gamma_1(N))$  or an integral of a cusp form at a point in such a neighbourhood then it is useful to be able to calculate the  $q$ -expansion of  $f$  at the corresponding cusp. We shall mean by this the following: A cusp  $a/c$  can be written as  $\gamma\infty$  with  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ . Then a  $q$ -expansion of  $f$  at  $a/c$  is simply the  $q$ -expansion of  $f|_k\gamma$ . This notation is abusive, since it depends on the choice of  $\gamma$ . The  $q$ -expansion will be an element of the power series ring  $\mathbb{C}[[q^{1/w}]]$  where  $w$  is the width of the cusp  $a/c$  and  $q^{1/w} = \exp(2\pi iz/w)$ .

If the level  $N$  is square-free this can be done symbolically. However, for general  $N$  it is not known how to do this, but we shall give some attempts that do at least give numerical computations of  $q$ -expansions. We use that we can compute the  $q$ -expansions of newforms in  $S_k(\Gamma_1(N))$  at  $\infty$  using modular symbols methods.

#### *The case of square-free $N$*

The method we present here is due to Asai [Asa]. Let  $N$  be square-free and let  $f \in S_k(\Gamma_1(N))$  be a newform of character  $\varepsilon$ . The main reason for being able to compute  $q$ -expansions at all cusps in this case is because the group generated by  $\Gamma_0(N)$  and all  $w_Q$  (see (6.2.1)) acts transitively on the cusps,



something that is not true when  $N$  is not square-free.

So let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  be given. Put:

$$c' = \frac{c}{\gcd(N, c)}, \quad \text{and} \quad Q = \frac{N}{\gcd(N, c)}.$$

Let  $r \in \mathbb{Z}$  be such that  $d \equiv cr \pmod{Q}$  and define  $b', d' \in \mathbb{Z}$  by:

$$Qd' = d - cr \quad \text{and} \quad b' = b - ar.$$

Then we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} Qa & b' \\ Nc' & Qd' \end{pmatrix} \begin{pmatrix} Q^{-1} & rQ^{-1} \\ 0 & 1 \end{pmatrix}.$$

Theorems 6.2.4 and 6.2.7 tell us how  $\begin{pmatrix} Qa & b' \\ Nc' & Qd' \end{pmatrix}$  acts on  $q$ -expansions. The action of  $\begin{pmatrix} Q^{-1} & rQ^{-1} \\ 0 & 1 \end{pmatrix}$  on  $q$ -expansions is simply:

$$\sum_{n \geq 1} a_n q^n \mapsto Q^{1-k} \sum_{n \geq 1} a_n \zeta_Q^{rn} q^{n/Q} \quad \text{with } \zeta_Q = \exp\left(\frac{2\pi i}{Q}\right).$$

This shows how the  $q$ -expansion of  $f|_k \gamma$  can be derived from the  $q$ -expansion of  $f$ .

Let us now explain how to do it for oldforms as well. By induction and (2.2.17) we may suppose  $f = \alpha_p(f')$  with  $p \mid N$  prime,  $f' \in S_k(\Gamma_1(N/p))$  and that we know how to compute the  $q$ -expansions of  $f'$  at all the cusps. Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be given. Then we have:

$$f|_k \gamma = p^{1-k} f'|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \gamma = p^{1-k} f'|_k \begin{pmatrix} pa & pb \\ c & d \end{pmatrix}.$$

We will now distinguish on two cases:  $p \mid c$  and  $p \nmid c$ . If  $p \mid c$  then we have a decomposition:

$$\begin{pmatrix} pa & pb \\ c & d \end{pmatrix} = \begin{pmatrix} a & pb \\ c/p & d \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

and we know how both matrices on the right hand side act on  $q$ -expansions.

If  $p \nmid c$ , choose  $b', d'$  with  $pad' - b'c = 1$ . Then we have:

$$\begin{pmatrix} pa & pb \\ c & d \end{pmatrix} = \begin{pmatrix} pa & b' \\ c & d' \end{pmatrix} \beta$$

with  $\beta \in \mathrm{GL}_2^+(\mathbb{Q})$  upper triangular, so also in this case we know how both matrices on the right hand side act on  $q$ -expansions.

*The general case*

In a discussion with Peter Bruin, the author figured out an attempt to drop the assumption that  $N$  be square-free and compute  $q$ -expansions of cusp forms numerically in this case. The idea is to generalise the  $W_Q$  operators from Section 6.2.

So let  $N$  be given. Let  $Q$  be a divisor of  $N$  and put  $R = \gcd(Q, N/Q)$ . Let  $w_Q$  be any matrix of the form:

$$w_Q = \begin{pmatrix} RQa & b \\ RNc & Qd \end{pmatrix} \quad \text{with } a, b, c, d \in \mathbb{Z}$$

such that  $\det w_Q = QR^2$  (the conditions guarantee us that such matrices do exist). One can then verify:

$$\Gamma_1(NR^2) < w_Q^{-1}\Gamma_1(N)w_Q,$$

so that slashing with  $w_Q$  defines a linear map:

$$S_k(\Gamma_1(N)) \oplus \bar{S}_k(\Gamma_1(N)) \xrightarrow{|w_Q} S_k(\Gamma_1(NR^2)) \oplus \bar{S}_k(\Gamma_1(NR^2))$$

which is injective since the slash operator defines a group action on the space of all functions  $\mathbb{H} \rightarrow \mathbb{C}$ .

On the other hand,  $w_Q$  defines an operation on  $\mathbb{M}_k$  which can be shown to induce a linear map:

$$w_Q: \mathbb{S}_k(\Gamma_1(NR^2)) \otimes \mathbb{Q} \rightarrow \mathbb{S}_k(\Gamma_1(N)) \otimes \mathbb{Q}$$

that satisfies the following compatibility with respect to the integration pairing between modular symbols and cusp forms (see [Sho, Theorem 4.3]):

$$(6.3.7) \quad \langle w_Q x, f \rangle = \langle x, f|_k w_Q \rangle.$$

Let  $(x_1, \dots, x_r)$  and  $(y_1, \dots, y_s)$  be bases of  $\mathbb{S}_k(\Gamma_1(N)) \otimes \mathbb{Q}$  and of  $\mathbb{S}_k(\Gamma_1(NR^2)) \otimes \mathbb{Q}$  respectively. Then one can write down a matrix  $A$  in terms of these basis that describes the map  $w_Q$  since we can express any symbol  $P \otimes \{\alpha, \beta\}$  in terms of Manin symbols. The matrix  $A^t$  then defines the action of  $w_Q$  in terms of the bases of the cusp forms spaces that are dual to  $(x_1, \dots, x_r)$  and  $(y_1, \dots, y_s)$ .

Now, let  $(f_1, \dots, f_r)$  be a basis of  $S_k(\Gamma_1(N)) \oplus \bar{S}_k(\Gamma_1(N))$  and let  $(g_1, \dots, g_s)$  be a basis of  $S_k(\Gamma_1(NR^2)) \oplus \bar{S}_k(\Gamma_1(NR^2))$  (for instance we

could take bases consisting of eigenforms for the Hecke operators away from  $N$ ). Define matrices:

$$B := (\langle x_i, f_j \rangle)_{i,j} \quad \text{and} \quad C := (\langle y_i, g_j \rangle)_{i,j}.$$

These can be computed numerically as the entries are period integrals. Then the matrix  $C^{-1}A^tB$  describes the map  $\cdot|_k w_Q$  in terms of the bases  $(f_1, \dots, f_r)$  and  $(g_1, \dots, g_s)$ . Hence if we can invert  $C$  efficiently, then we can numerically compute the  $q$ -expansion of  $f|_k w_Q$  with  $f \in S_k(\Gamma_1(N))$ .

Let now a matrix  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  be given. Put:

$$c' := \gcd(N, c) \quad \text{and} \quad Q := N/c'.$$

Because of  $\gcd(c/c', Q) = 1$  we can find  $\alpha \in (\mathbb{Z}/Q\mathbb{Z})^\times$  with  $\alpha c/c' \equiv 1 \pmod{Q}$ . If we lift  $\alpha$  to  $(\mathbb{Z}/N\mathbb{Z})^\times$  then we have  $\alpha c \equiv c' \pmod{N}$ . Let now  $d' \in \mathbb{Z}$  be a lift of  $\alpha d \in (\mathbb{Z}/N\mathbb{Z})^\times$ . Because  $\alpha c$  and  $\alpha d$  together generate  $\mathbb{Z}/N\mathbb{Z}$  we have  $\gcd(c', d') = 1$  and so we can find  $a', b' \in \mathbb{Z}$  that satisfy  $a'd' - b'c' = 1$ . According to Lemma 6.1.12, we have:

$$\gamma = \gamma_0 \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \quad \text{with } \gamma_0 \in \Gamma_0(N).$$

Put  $R = \gcd(c', Q)$ . Then we have  $\gcd(NR, Q^2 Ra') = QR \gcd(c', Qa')$  and hence  $\gcd(NR, Q^2 Ra') = QR^2$ , so there exist  $b'', d'' \in \mathbb{Z}$  with:

$$w_Q := \begin{pmatrix} QRa' & b'' \\ NR & Qd'' \end{pmatrix}$$

having determinant  $QR^2$ . One can now verify that we have  $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = w_Q \beta$  with  $\beta \in \mathrm{GL}_2^+(\mathbb{Q})$  upper triangular. So in the decomposition:

$$\gamma = \gamma_0 w_Q \beta$$

we can compute the slash action of all three matrices on the right hand side in terms of  $q$ -expansions, hence also of  $\gamma$ .

In conclusion we see that in this method we have to increase the level and go to  $S_k(\Gamma_1(NR^2))$  for the square divisors  $R^2$  of  $N$  to compute  $q$ -expansions of cusp forms in  $S_k(\Gamma_1(N))$  at arbitrary cusps.

### 6.3.8 Numerical evaluation of cusp forms

For  $f \in S_k(\Gamma_1(N))$  and a point  $P \in \mathbb{H}$  we wish to compute  $f(P)$  to a high numerical precision. Before we do this let us say some words on how  $P$  should be represented. The transformation property of modular forms implies that representing  $P$  as  $x + iy$  with  $x, y \in \mathbb{R}$  is not a good idea, as this would be numerically very unstable when  $P$  is close to the real line. Instead, we represent  $P$  as:

$$(6.3.9) \quad P = \gamma z \quad \text{with } \gamma \in \text{SL}_2(\mathbb{Z}), z = x + iy, x \ll \infty \text{ and } y \gg 0.$$

For instance, one could demand that  $z$  be in the standard fundamental domain:

$$\mathcal{F} := \{z \in \mathbb{H} : |\Re z| \leq 1/2 \text{ and } |z| \geq 1\}$$

for  $\text{SL}_2(\mathbb{Z})$  acting on  $\mathbb{H}$ , although this is not strictly necessary.

So let  $P = \gamma z$  be given, with  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$  and  $\Im z > M$ , say. Let  $w = w(\gamma)$  be the width of the cusp  $\gamma\infty$  with respect to  $\Gamma_1(N)$ . To compute  $f(P)$  we make use of a  $q$ -expansion of  $f$  at  $\gamma\infty$ :

$$f(P) = (cz + d)^k (f|_k \gamma)(z) = (cz + d)^k \sum_{n \geq 1} a_n q^{n/w}$$

The radius of convergence is 1 and the coefficients are small (estimated by  $\tilde{O}(n^{(k-1)/2})$ ). So to compute  $f(P)$  to a precision of  $p$  decimals we need about  $\frac{pw \log 10}{2\pi M} \approx 0.37 \frac{pw}{M}$  terms of the  $q$ -expansion of  $f|_k \gamma$ .

Of course, we have some freedom in choosing  $\gamma$  and  $z$  to write down  $P$ . We want to find  $\gamma$  such that  $P = \gamma z$  with  $\Im z/w(\gamma)$  as large as possible. In general, one can always write  $P = \gamma z$  with  $z \in \mathcal{F}$  so one obtains:

$$(6.3.10) \quad \max_{\gamma \in \text{SL}_2(\mathbb{Z})} \frac{\Im \gamma^{-1} P}{w(\gamma)} \geq \frac{\sqrt{3}}{2N}.$$

We see that in order to calculate  $f(P)$  to a precision of  $p$  decimals it suffices to use about  $\frac{pN \log 10}{\sqrt{3}\pi} \approx 0.42pN$  terms of the  $q$ -expansions at each cusp. Although for many points  $P$  there is a better way of writing it as  $\gamma z$  in this respect than taking  $z \in \mathcal{F}$ , it seems hard to improve the bound  $\frac{\sqrt{3}}{2N}$  in general.

We wish to adjust the representation sometimes from  $P = \gamma z$  to  $P = \gamma' z'$  where  $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$  is another matrix, for instance because during our calculations  $\Re z$  has become too large or  $\Im z$  has become too small (but still within reasonable bounds). We can make  $\Re z$  smaller by putting  $z' := z - n$  for appropriate  $n \in \mathbb{Z}$  and putting  $\gamma' := \gamma \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ . Making  $\Im z$  larger is rather easy as well. We want to find  $\gamma'' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  such that:

$$\Im \gamma'' z = \frac{\Im z}{|cz + d|^2}$$

is large. This simply means that we have to find a small vector  $cz + d$  in the lattice  $\mathbb{Z}z + \mathbb{Z}$ , something which can be done easily if  $\Re z \ll \infty$  and  $\Im z \gg 0$ . If  $c$  and  $d$  are not coprime we can divide both by their greatest common divisor to obtain a smaller vector. The matrix  $\gamma''$  can now be completed and we put  $z' := \gamma'' z$  and  $\gamma' := (\gamma'')^{-1}$ .

### 6.3.11 Numerical evaluation of integrals of cusp forms

In this subsection we will describe for  $f \in S_2(\Gamma_1(N))$  and  $P \in \mathbb{H}$  how to evaluate the integral  $\int_{\infty}^P f dq/q$ . As in the previous subsection, we assume  $P$  to be given by means of (6.3.9). The path of integration will be broken into two parts: first we go from  $\infty$  to a cusp  $\alpha$  near  $P$  and then we go from  $\alpha$  to  $P$ .

*Integrals over paths between cusps*

The pairing (6.1.5) gives a map:

$$\Theta: \mathbb{M}_2(\Gamma_1(N)) \rightarrow \mathrm{Hom}_{\mathbb{C}}(S_2(\Gamma_1(N)), \mathbb{C}),$$

which is injective when restricted to  $S_2(\Gamma_1(N))$ . The image of  $\Theta$  is a lattice of full rank, hence the induced map:

$$S_2(\Gamma_1(N)) \otimes \mathbb{R} \rightarrow \mathrm{Hom}_{\mathbb{C}}(S_2(\Gamma_1(N)), \mathbb{C})$$

is an isomorphism. In particular we obtain a map:

$$\Phi: \mathbb{M}_2(\Gamma_1(N)) \rightarrow S_2(\Gamma_1(N)) \otimes \mathbb{R},$$

which is an interesting map to compute if we want to calculate integrals of cusp forms along paths between cusps. The map  $\Phi$  is called a *period mapping*.

The Manin-Drinfel'd theorem (see [Man1, Corollary 3.6] and [Dri, Theorem 1]) tells us that  $\text{im}(\Phi) \subset \mathbb{S}_2(\Gamma_1(N)) \otimes \mathbb{Q}$ . This is equivalent to saying that each degree 0 divisor of  $X_1(N)$  which is supported on cusps defines a torsion point of  $J_1(N)$ . The proof given in [Dri] already indicates how to compute  $\Phi$  with symbolic methods: let  $p$  be a prime that is 1 mod  $N$ . Then the operator  $p + 1 - T_p$  on  $\mathbb{M}_2(\Gamma_1(N))$  has its image in  $\mathbb{S}_2(\Gamma_1(N))$ . The same operator is invertible on  $\mathbb{S}_2(\Gamma_1(N)) \otimes \mathbb{Q}$ . So we simply have:

$$\Phi = (p + 1 - T_p)^{-1}(p + 1 - T_p),$$

where the rightmost  $p+1-T_p$  denotes the map from  $\mathbb{M}_2(\Gamma_1(N))$  to  $\mathbb{S}_2(\Gamma_1(N))$  and the leftmost  $p+1-T_p$  denotes the invertible operator on  $\mathbb{S}_2(\Gamma_1(N)) \otimes \mathbb{Q}$ . For other methods to compute  $\Phi$ , see [Ste2, Section 10.6]. So we can express the integral of  $f dq/q$  between any two cusps  $\alpha$  and  $\beta$  in terms of period integrals, which we have already seen how to compute:

$$\int_{\alpha}^{\beta} f \frac{dq}{q} = \langle \Phi(\{\alpha, \beta\}), f \rangle.$$

*Integrals over general paths*

We can imitate the previous subsection pretty much. Write  $P \in \mathbb{H}$  as  $P = \gamma z$  with  $\gamma \in \text{SL}_2(\mathbb{Z})$  such that  $\Im z/w(\gamma\infty)$  is as large as possible. Then we have

$$\begin{aligned} \int_{\infty}^P f \frac{dq}{q} &= \int_{\infty}^{\gamma\infty} f \frac{dq}{q} + \int_{\gamma\infty}^{\gamma z} f \frac{dq}{q} \\ (6.3.12) \quad &= \int_{\infty}^{\gamma\infty} f \frac{dq}{q} + \int_{\infty}^z (f|_{2\gamma}) \frac{dq}{q}. \end{aligned}$$

The integral  $\int_{\infty}^{\gamma\infty} f \frac{dq}{q}$  is over a path between two cusps so we can compute it by the above discussion and the integral  $\int_{\infty}^z (f|_{2\gamma}) \frac{dq}{q}$  can be computed using the  $q$ -expansion of  $f|_{2\gamma}$ :

$$\int_{\infty}^z (f|_{2\gamma}) \frac{dq}{q} = w \sum_{n \geq 1} \frac{a_n}{n} q^{n/w},$$

where  $w = w(\gamma)$ ,  $q^{1/w} = \exp(2\pi iz/w)$  and  $f|_2\gamma = \sum a_n q^{n/w}$ . Because of (6.3.10), computing about  $\frac{p^N \log 10}{\sqrt{3\pi}} \approx 0.42pN$  terms of the series should suffice to compute  $\int_{\infty}^P f \frac{dq}{q}$  for any  $P \in \mathbb{H}$ .

Note also that we can use formula (6.3.12) to compute the pseudo-eigenvalue  $\lambda_Q(f)$  by plugging in  $\gamma = w_Q$  and a  $z$  for which both  $\Im z$  and  $\Im w_Q z$  are high and for which  $\int_{\infty}^z W_q(f) dq/q$  is not too close to zero.

### 6.4 APPLYING NUMERICAL CALCULATIONS TO GALOIS REPRESENTATIONS

Let  $f$  be a newform (of some level and weight) and let  $\lambda \mid \ell$  be a prime of its coefficient field. From Section 2.5 we know that a residual Galois representation  $\bar{\rho} = \bar{\rho}_{f,\lambda}$  is attached to the pair  $(f, \lambda)$ . The fixed field  $K_\lambda$  of  $\ker(\bar{\rho})$  in  $\bar{\mathbb{Q}}$  is a number field. The results from Chapter 14 point out that we know that computing  $\bar{\rho}$  essentially boils down to computing a polynomial that has  $K_\lambda$  as splitting field. In this section we describe how numerical calculations can be used to compute such a polynomial. We will follow ideas from Chapter 3.

Theorem 2.5.7 shows that we can reduce this problem to the case of a form of weight 2 in most interesting cases. Hence we will assume that  $f$  is a newform in  $S_2(\Gamma_1(N))$ . Assume that the representation  $\bar{\rho}_{f,\lambda}$  is absolutely irreducible and let  $\mathbb{T}$  be the Hecke algebra acting on  $J_1(N)$ . There is a subspace  $V_\lambda$  of  $J_1(N)(\bar{\mathbb{Q}})[\ell]$  on which both  $\mathbb{T}$  and  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  act, such that the action of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  defines  $\bar{\rho}_{f,\lambda}$ .

#### 6.4.1 Approximation of torsion points

The Jacobian  $J_1(N)_{\mathbb{C}}$  can be described as follows. Pick a basis  $f_1, \dots, f_g$  of  $S_2(\Gamma_1(N))$ . Put:

$$\Lambda := \left\{ \int_{\gamma} (f_1, \dots, f_g) \frac{dq}{q} : [\gamma] \in H_1(X_1(N)(\mathbb{C}), \mathbb{Z}) \right\} \subset \mathbb{C}^g.$$

This is a lattice in  $\mathbb{C}^g$  of full rank. By the Abel-Jacobi theorem we have an isomorphism:

$$J_1(N)(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^g/\Lambda,$$

$$\left[ \sum_i ([Q_i] - [R_i]) \right] \mapsto \sum_i \int_{R_i}^{Q_i} (f_1, \dots, f_g) \frac{dq}{q}.$$

Let now a divisor  $\sum_{i=1}^g [R_i]$  on  $X_1(N)$  be given. Identifying  $J_1(N)(\mathbb{C})$  with  $\mathbb{C}^g/\Lambda$  in this way, we get a birational morphism

$$\phi: \text{Sym}^g X_1(N)(\mathbb{C}) \rightarrow \mathbb{C}^g/\Lambda,$$

$$(Q_1, \dots, Q_g) \mapsto \sum_{i=1}^g \int_{R_i}^{Q_i} (f_1, \dots, f_g) \frac{dq}{q}.$$

The homology group  $H_1(X_1(N)(\mathbb{C}), \mathbb{Z})$  is canonically isomorphic to the modular symbols space  $\mathbb{S}_2(\Gamma_1(N))$ . The period lattice  $\Lambda$  can thus be computed numerically using the methods from Subsections 6.3.1 and 6.3.3. Since we can compute the action of  $\mathbb{T}$  on  $\mathbb{S}_2(\Gamma_1(N)) \cong \Lambda$ , we can write down the points in  $\frac{1}{\ell}\Lambda/\Lambda \subset \mathbb{C}^g/\Lambda$  that correspond to the points of  $V_\lambda$ . The aim is now to compute the divisors on  $X_1(N)_\mathbb{C}$  that map to these points along  $\phi$ . In our computations, we assume without proof that  $V_\lambda$  lies beneath the good locus of  $\phi$ , i.e. the map  $X_1(N)^g \rightarrow \mathbb{C}^g/\Lambda$  induced by  $\phi$  is étale above  $V_\lambda$ .

We start calculating with a small precision. Let a non-zero  $P$  in  $V_\lambda(\mathbb{C}) \subset \mathbb{C}^g/\Lambda$  be given. First we try out a lot of random points  $Q=(Q_1, \dots, Q_g)$  in  $X_1(N)(\mathbb{C})^g$ . Here, each  $Q_i$  will be written as  $Q_i = \gamma_i w_i$ , with  $\gamma_i$  in a set of representatives for  $\Gamma_1(N) \setminus \text{SL}_2(\mathbb{Z})$  and  $w_i \in \mathcal{F}$ . We can compute  $\phi(Q)$  using methods from Subsection 6.3.11. We work with the point  $Q$  for which  $\phi(Q)$  is closest to  $P$ . If we in fact already know some points  $Q$  with  $\phi(Q)$  approximately equal to a point in  $V_\lambda(\mathbb{C})$ , then we could also take one of those points as a starting point  $Q$  to work with.

The next thing to do is adjust  $Q$  so that  $\phi(Q)$  comes closer to  $P$ . We'll make use of the Newton-Raphson approximation method. Let  $\phi': \mathbb{H}^g \rightarrow \mathbb{C}^g/\Lambda$  be the function defined by:

$$\phi'(z_1, \dots, z_g) = \phi(\gamma_1 z_1, \dots, \gamma_g z_g).$$



We observe that for a small vector  $h = (h_1, \dots, h_g) \in \mathbb{C}^g$  we have:

$$\phi'(w_1 + h_1, \dots, w_g + h_g) = \phi(Q) + hD + O(\|h\|^2)$$

with:

$$D = \left( \begin{array}{ccc} \frac{\partial \phi'_1}{\partial z_1} & \dots & \frac{\partial \phi'_g}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi'_1}{\partial z_g} & \dots & \frac{\partial \phi'_g}{\partial z_g} \end{array} \right) \Big|_{(w_1, \dots, w_g)} .$$

From the definition of  $\phi$  we can immediately deduce:

$$\frac{\partial \phi'_i}{\partial z_j}(w_1, \dots, w_g) = 2\pi i \cdot (f_i|_2 \gamma_j)(w_j),$$

where we apologise for the ambiguous  $i$ . We can thus compute the matrix  $D$  using the methods of Subsection 6.3.8. Now choose a small vector  $v = (v_1, \dots, v_g) \in \mathbb{C}^g$  such that  $\phi(Q) + v$  is closer to  $P$  than  $\phi(Q)$  is. For example,  $v$  can be chosen among all vectors of a bounded length so that  $\phi(Q) + v$  is closest to  $P$ . If we write:

$$h = vD^{-1},$$

then we expect  $\phi'(w_1+h_1, \dots, w_g+h_g)$  to be approximately equal to  $\phi(Q)+v$ . If this is not the case, then we try the same thing with a smaller  $v$ . It could be that this still fails, for instance because we are too close to the bad locus of the map  $\phi$ . In that case, we start with a new random point  $Q$ .

We repeat the above adjustments until we are (almost) as close as we can get considering our calculation precision. It might happen that the  $w_i$  become too wild, i.e.  $|\Re w_i|$  becomes too large or  $\Im w_i$  becomes too small. If this is the case we adjust the way we write  $Q_i$  as  $\gamma_i w_i$  using the method described in Subsection 6.3.8. We can always replace the  $\gamma_i$  then by a small matrix in the same coset of  $\Gamma_1(N) \setminus \mathrm{SL}_2(\mathbb{Z})$ .

Once we have for each  $P \in V_\lambda - \{0\}$  a point  $Q$  such that  $\phi(Q)$  is approximately equal to  $P$ , we can start increasing the precision. We double our calculation precision and repeat the above adjustments ( $\phi(Q)+v$  will in this case be equal to  $P$ ). We repeat this a few times until we have very good approximations.

### 6.4.2 Computation of polynomials

Now, we will choose a function in  $h \in \mathbb{Q}(X_1(N))$  and evaluate it at the components of the points in  $\phi^{-1}(V_\lambda)$ . With the discussion of Chapter 3 in mind, we want  $h$  to take values of small height. Since  $h$  multiplies heights of points roughly by  $\deg(h)$ , we want to find a function of small degree. Take any  $k$  and a basis  $h_1, \dots, h_n$  of  $S_k(\Gamma_1(N))$  such that the  $q$ -expansions of the  $h_i$  lie in  $\mathbb{Z}[[q]]$  and such that the exponents of the first non-zero terms of these  $q$ -expansions form a strictly increasing sequence. We propose to use  $h = W_N(h_{n-1})/W_N(h_n)$  as a function to use (assuming  $n \geq 2$ ). Remember from Section 2.2 that  $S_k(\Gamma_1(N))$  is the space of global sections of the line bundle  $\mathcal{L} = \omega^{\otimes k}(-\text{cusps})$  on  $X_1(N)$ , base changed to  $\mathbb{C}$ . Be aware of the fact that the cusp  $\infty$  of  $X_1(N)$  is not defined over  $\mathbb{Q}$ , but the cusp 0 is. Since we demand the  $q$ -expansions to have rational coefficients, the sections  $W_N(h_1), \dots, W_N(h_n)$ , with  $W_N$  an Atkin-Lehner operator, are defined over  $\mathbb{Q}$  and they have increasing order at 0. One can now verify that for  $h = W_N(h_{n-1})/W_N(h_n)$  we have:

$$\deg(h) \leq \deg(\mathcal{L}) - v_\infty(h_{n-1}) \leq \deg(\mathcal{L}) - \dim H^0(\mathcal{L}) + 2 \leq g + 1.$$

For  $k = 2$  and  $g \geq 2$  we have  $\mathcal{L} \cong \Omega^1(X_1(N))$  and we get  $g$  as an upper bound for  $\deg(h)$ . Using methods from Subsection 6.3.8, we can evaluate  $h$  numerically. The author is not aware of a sophisticated method for finding a function  $h \in \mathbb{Q}(X_1(N))$  of minimal degree in general; this minimal degree is called the *gonality* of the curve  $X_1(N)$ . Published results on these matters seem to either be limited to  $X_0(N)$  or to concern only *lower* bounds for the gonality of modular curves, see for example [Abr], [Bak, Chapter 3] or [Poo].

Now put, for  $P \in V_\lambda(\mathbb{C}) - \{0\}$ :

$$\alpha_P = \sum_{i=1}^g h(Q_i), \quad \text{where } \phi(Q_1, \dots, Q_g) = P.$$

We work out the product in:

$$P_\lambda(x) := \prod_{P \in V_\lambda(\mathbb{C}) - \{0\}} (x - \alpha_P) = \sum_{k=0}^n a_k x^k, \quad \text{where } n = \deg P_\lambda.$$

The coefficients  $a_k$  are rational numbers that we have computed numerically. Since the height of  $P_\lambda$  is expected to be not too large, the denominators of the  $a_k$  should have a relative small common denominator. The LLL algorithm can be used to compute integers  $p_0, \dots, p_{n-1}, q$  such that  $|p_k - a_k q|$  is small for all  $k$ , see [Le-Le-Lo, Proposition 1.39]. If the sequence  $(a_k)$  is arbitrary, then we'll be able to find  $p_k$  and  $q$  such that  $|p_k - a_k q|$  is roughly of order  $q^{-1/n}$  for each  $k$ , but not much better than that. So if it happens that we find  $p_k$  and  $q$  with  $|p_k - a_k q|$  much smaller than  $q^{-1/n}$  for all  $k$ , then we guess that  $a_k$  is equal to  $p_k/q$ . If we cannot find such  $p_k$  and  $q$  then we will double the precision and repeat all the calculations described above.

Heuristically, the calculation precision that is needed to find the true value of  $a_k$  is about  $(1 + 1/n) \cdot \text{height}(P_\lambda) / \log(10)$  decimals. Another way of finding rational approximations of the  $a_k$  is by approximating them using continued fractions. For this method, the precision needed to find the true value of  $a_k$  would be about  $2 \cdot \text{height}(P_\lambda) / \log(10)$  decimals.

Since the degree of  $P_\lambda$  will be quite large, we won't be able to do many further calculations with it. In particular it may be hard to verify whether all the guesses we made were indeed correct. Instead, we will look at the following variant. If  $\mathfrak{m}$  is the Hecke ideal of  $f \bmod \lambda$ , then  $V_\lambda$  is a vector space over  $\mathbb{T}/\mathfrak{m}$ . The representation  $\bar{\rho}_{f,\lambda}$  induces an action  $\tilde{\rho}_\lambda$  of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  on the set  $\mathbb{P}(V_\lambda)$  of lines in  $V_\lambda$ . We can attach a polynomial  $\tilde{P}_\lambda$  to this projectivised representation  $\tilde{\rho}_\lambda$ , analogously to the way this was done for  $\bar{\rho}$ . This polynomial will have a smaller degree than  $P_\lambda$ . We put:

$$\tilde{P}_\lambda(x) = \prod_{L \in \mathbb{P}(V_\ell)} \left( x - \sum_{P \in L - \{0\}} \alpha_P \right) = \sum_{k=0}^m b_k x^k, \quad \text{where } m = \deg \tilde{P}_\lambda.$$

As above, if the calculation precision is sufficient we can use lattice reduction algorithms to compute the exact values of the  $b_k$ .

### 6.4.3 Reduction of polynomials

Although the polynomial  $\tilde{P}_\lambda$  will not have a very huge height, its height is still too large to do any useful computations with it. The first step in making a polynomial of smaller height defining the same number field is computing

the maximal order of that number field. Let  $q$  be the common denominator of the coefficients and put  $p_k = b_k q$ . Consider the polynomial:

$$Q(x) = q \cdot \tilde{P}_\lambda(x) = qx^m + p_{m-1}x^{m-1} + \cdots + p_0.$$

We make ourselves confident that we correctly computed  $Q(x)$  (although we won't prove anything at this point yet). For instance, we verify that  $Q(x)$  is irreducible and that its discriminant has the prime factors of  $N\ell$  in it. We can also compute for several primes  $p$  not dividing  $\text{Disc}(Q(x))$  the decomposition type of  $Q(x)$  modulo  $p$  and verify that it could be equal to the cycle type of  $\tilde{\rho}(\text{Frob}_p)$ . If not, we again double the precision and repeat the above calculations.

Let now  $\alpha$  be a root of  $\tilde{P}_\lambda(x)$  and write down the order:

$$\mathcal{O} := \mathbb{Z} + \sum_{k=1}^{m-1} \left( \mathbb{Z} \cdot \sum_{j=0}^{k-1} a_{m-j} \alpha^{k-j} \right),$$

which is an order that is closer to the maximal order than  $\mathbb{Z}[q\alpha]$  (see [Len2, Subsection 2.10]). Being confident in the correctness of  $Q(x)$ , we know where the number field  $K$  defined by it ramifies and thus we can compute its maximal order (see [Bu-Le, Section 6 and Theorems 1.1 and 1.4]). Having done this, we embed  $\mathcal{O}_K$  as a lattice into  $\mathbb{C}^m$  in the usual way and we use the LLL algorithm to compute a basis of small vectors in  $\mathcal{O}_K$ . We can then search for an element of small length in  $\mathcal{O}_K$  that generates  $K$  over  $\mathbb{Q}$ . Its defining polynomial  $\tilde{P}'_\lambda$  will have small coefficients. See also [Co-Di].

In the computation of the polynomials  $P_\lambda$  and  $\tilde{P}'_\lambda$  we made several guesses and assumptions that we cannot prove to be correct. In Chapter 7, we work out in special cases how we can use established parts of Serre's conjecture to prove afterwards for polynomials of the style  $\tilde{P}'_\lambda$  that they indeed belong to the modular Galois representations that we claim they belong to. See [Bos1] for another example of this. In the unlikely case that such tests may fail we can of course make adjustments like choosing another function  $h$  or another divisor to construct  $\phi$ .

#### 6.4.4 Further refinements

The Jacobian  $J_1(N)$  has large dimension (for  $N$  prime this dimension is  $(N-5)(N-7)/24$ ). It could be that our newform  $f$  is an element of  $S_2(\Gamma)$  with  $\Gamma_1(N) \subsetneq \Gamma < \Gamma_0(N)$ . In that case we work with the curve  $X_\Gamma$ , which is given its  $\mathbb{Q}$ -structure by defining it as a quotient of  $X_1(N)$ . The Jacobian  $J_\Gamma$  of  $X_\Gamma$  is isogenous to an abelian subvariety of  $J_1(N)$  that contains  $V_\lambda$ , so this works perfectly well.

In the case  $\Gamma = \Gamma_0(N)$  we can sometimes go a step further. The operator  $W_N$  on  $X_0(N)$ , sending  $z$  to  $-1/Nz$ , is defined over  $\mathbb{Q}$ . If  $f$  is invariant under  $W_N$ , one can work with the curve  $X_0^+(N) := X_0(N)/\langle W_N \rangle$ . Its Jacobian  $J_0^+(N)$  is isogenous to an abelian subvariety of  $J_1(N)$  that contains  $V_\lambda$ , so also here it works. Some words on the computation of the homology of  $X_0^+(N)$  are in order. The action of  $W_N$  on  $X_0(N)$  induces an action on  $H_1(X_0(N)(\mathbb{C}), \mathbb{Z})$  and on  $H_1(X_0(N)(\mathbb{C}), \text{cusps}, \mathbb{Z})$ . Since paths between cusps on  $X_0^+(N)(\mathbb{C})$  lift to paths between cusps on  $X_0(N)(\mathbb{C})$  we have a surjection:

$$H_1(X_0(N), \text{cusps}, \mathbb{Z}) \twoheadrightarrow H_1(X_0^+(N)(\mathbb{C}), \text{cusps}, \mathbb{Z}).$$

The kernel of this map consists of the elements  $[\gamma]$  in  $H_1(X_0(N), \text{cusps}, \mathbb{Z})$  satisfying  $W_N([\gamma]) = -[\gamma]$ . So modular symbols methods allow us to compute  $H_1(X_0^+(N)(\mathbb{C}), \text{cusps}, \mathbb{Z})$  as a quotient of  $\mathbb{M}_2(\Gamma_0(N))$ . Let  $\mathbb{B}_2^+(\Gamma_0(N))$  be the free abelian group on the cusps of  $X_0^+(N)(\mathbb{C})$  and define:

$$\delta: H_1(X_0^+(N)(\mathbb{C}), \text{cusps}, \mathbb{Z}) \rightarrow \mathbb{B}_2^+(\Gamma_0(N)), \quad \{\alpha, \beta\} \mapsto \{\beta\} - \{\alpha\}.$$

Then  $H_1(X_0^+(N)(\mathbb{C})) = \ker(\delta)$ .

## Chapter Seven

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### Polynomials for projective representations of level one forms

J. Bosman

#### 7.1 INTRODUCTION

In this chapter we explicitly compute mod- $\ell$  Galois representations attached to modular forms. To be precise, we look at cases with  $\ell \leq 23$  and the modular forms considered will be cusp forms of level 1 and weight up to 22. We present the result in terms of polynomials associated with the projectivised representations. As an application, we will improve a known result on Lehmer's non-vanishing conjecture for Ramanujan's tau function (see [Leh, p. 429]).

To fix a notation, for any  $k \in \mathbb{Z}$  satisfying  $\dim S_k(\mathrm{SL}_2(\mathbb{Z})) = 1$  we will denote the unique normalised cusp form in  $S_k(\mathrm{SL}_2(\mathbb{Z}))$  by  $\Delta_k$ . We will denote the coefficients of the  $q$ -expansion of  $\Delta_k$  by  $\tau_k(n)$ :

$$\Delta_k(z) = \sum_{n \geq 1} \tau_k(n) q^n \in S_k(\mathrm{SL}_2(\mathbb{Z})).$$

From  $\dim S_k(\mathrm{SL}_2(\mathbb{Z})) = 1$  it follows that the numbers  $\tau_k(n)$  are integers. For every  $\Delta_k$  and every prime  $\ell$  there is a continuous representation:

$$\bar{\rho}_{\Delta_k, \ell}: \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_2(\mathbb{F}_\ell)$$

such that for every prime  $p \neq \ell$  we have that the characteristic polynomial of  $\bar{\rho}_{\Delta_k, \ell}(\mathrm{Frob}_p)$  is congruent to  $X^2 - \tau_k(p)X + p^{k-1} \pmod{\ell}$ . For a summary on the exceptional representations  $\bar{\rho}_{\Delta_k, \ell}$  and the corresponding congruences for  $\tau_k(n)$ , see [Swi].

### 7.1.1 Notational conventions

Throughout this chapter, for every field  $K$  we will fix an algebraic closure  $\overline{K}$  and all algebraic extension fields of  $K$  will be regarded as subfields of  $\overline{K}$ . Furthermore, for each prime number  $p$  we will fix an embedding  $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$  and hence an embedding  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ , whose image we call  $D_p$ . We will use  $I_p$  to denote the inertia subgroup of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ .

All representations (either linear or projective) in this chapter will be *continuous*. For any field  $K$ , a linear representation  $\rho: G \rightarrow \text{GL}_n(K)$  defines a projective representation  $\tilde{\rho}: G \rightarrow \text{PGL}_n(K)$  via the canonical map  $\text{GL}_n(K) \rightarrow \text{PGL}_n(K)$ . We say that  $\tilde{\rho}: G \rightarrow \text{PGL}_n(K)$  is *irreducible* if the induced action of  $G$  on  $\mathbb{P}^{n-1}(K)$  fixes no proper subspace. So for  $n = 2$  this means that every point of  $\mathbb{P}^1(K)$  has its stabiliser subgroup not equal to  $G$ .

### 7.1.2 Statement of results

**7.1.3 Theorem** *For every pair  $(k, \ell)$  occurring the table in Section 7.5, let the polynomial  $P_{k,\ell}$  be defined as in that same table. Then the splitting field of each  $P_{k,\ell}$  is the fixed field of  $\text{Ker}(\tilde{\rho}_{\Delta_{k,\ell}})$  and has Galois group  $\text{PGL}_2(\mathbb{F}_\ell)$ . Furthermore, if  $\alpha \in \overline{\mathbb{Q}}$  is a root of  $P_{k,\ell}$  then the subgroup of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  fixing  $\alpha$  corresponds via  $\tilde{\rho}_{\Delta_{k,\ell}}$  to a subgroup of  $\text{PGL}_2(\mathbb{F}_\ell)$  fixing a point of  $\mathbb{P}^1(\mathbb{F}_\ell)$ .*

For completeness we also included the pairs  $(k, \ell)$  for which  $\rho_{k,\ell}$  is isomorphic to the action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on the  $\ell$ -torsion of an elliptic curve. These are the pairs in the table in Section 7.5 with  $\ell = k - 1$ , as there the representation is the  $\ell$ -torsion of  $J_0(\ell)$ , which happens to be an elliptic curve for  $\ell \in \{11, 17, 19\}$ . A simple calculation with division polynomials [Lan5, Chapter II] can be used to treat these cases. In the general case, one has to work in the more complicated Jacobian variety  $J_1(\ell)$ , which has dimension 12 for  $\ell = 23$  for instance.

We can apply Theorem 7.1.3 to verify the following result.

**7.1.4 Corollary** *The non-vanishing of  $\tau(n)$  holds for all  $n$  such that:*

$$n < 22798241520242687999 \approx 2 \cdot 10^{19}.$$

In [Jo-Ke], the non-vanishing of  $\tau(n)$  was verified for all  $n$  such that:

$$n < 22689242781695999 \approx 2 \cdot 10^{16}.$$

To compute the polynomials, the author used the approach described in Section 6.4. After the initial computations some of the polynomials had coefficients of almost 2000 digits, so reduction techniques were absolutely necessary. The used algorithms do not give a proven output, so we have to concentrate on the verification. We will show how to verify the correctness of the polynomials in Section 7.3 after setting up some preliminaries about Galois representations in Section 7.2. In Section 7.4 we will point out how to use Theorem 7.1.3 in a calculation that verifies Corollary 7.1.4. All the calculations were performed using MAGMA (see [Magma]).

## 7.2 GALOIS REPRESENTATIONS

This section will be used to state some results on Galois representations that we will need in the proof of Theorem 7.1.3.

### 7.2.1 Liftings of projective representations

Let  $G$  be a topological group, let  $K$  be a topological field and let  $\tilde{\rho}: G \rightarrow \mathrm{PGL}_n(K)$  be a projective representation. Let  $L$  be an extension field of  $K$ . By a *lifting* of  $\tilde{\rho}$  over  $L$  we shall mean a representation  $\rho: G \rightarrow \mathrm{GL}_n(L)$  that makes the following diagram commute:

$$\begin{array}{ccc} G & \xrightarrow{\tilde{\rho}} & \mathrm{PGL}_n(K) \\ \rho \downarrow & & \downarrow \\ \mathrm{GL}_n(L) & \twoheadrightarrow & \mathrm{PGL}_n(L) \end{array}$$

where the maps on the bottom and the right are the canonical ones. If the field  $L$  is not specified then by a lifting of  $\tilde{\rho}$  we shall mean a lifting over  $\bar{K}$ .

An important theorem of Tate arises in the context of liftings. For the proof we refer to [Ser6, Section 6]. Note that in the reference representations over  $\mathbb{C}$  are considered, but the proof works for representations over arbitrary algebraically closed fields.



**7.2.2 Theorem (Tate)** *Let  $\tilde{\rho}: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{PGL}_n(K)$  be a projective representation of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  over a field  $K$ . Then for each prime number  $p$ , there exists a lifting  $\rho'_p: D_p \rightarrow \text{GL}_n(\overline{K})$  of  $\tilde{\rho}|_{D_p}$ . If these liftings  $\rho'_p$  have been chosen so that all but finitely many of them are unramified, then there is a unique lifting  $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_n(\overline{K})$  such that for all primes  $p$  we have:*

$$\rho|_{I_p} = \rho'_p|_{I_p}.$$

**7.2.3 Lemma** *Let  $p$  be a prime number and let  $K$  be a field. Suppose that we are given a projective representation  $\tilde{\rho}_p: \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \rightarrow \text{PGL}_n(K)$  that is unramified. Then there exists a lifting  $\rho_p: \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \rightarrow \text{GL}_n(\overline{K})$  of  $\tilde{\rho}_p$  that is unramified as well.*

**Proof** Since  $\tilde{\rho}$  is an unramified representation of  $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ , it factors through  $\text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$  and is determined whenever we know the image of  $\text{Frob}_p \in \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ . By continuity, this image is an element of  $\text{PGL}_n(K)$  of finite order, say of order  $m$ . If we take any lift  $F$  of  $\tilde{\rho}(\text{Frob}_p)$  to  $\text{GL}_n(K)$  then we have  $F^m = a$  for some  $a \in K^\times$ . So  $F' := \alpha^{-1}F$ , where  $\alpha \in \overline{K}$  is any  $m$ -th root of  $a$ , has order  $m$  in  $\text{GL}_n(\overline{K})$ . Hence the homomorphism  $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \rightarrow \text{GL}_n(\overline{K})$  obtained by the composition:

$$\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \twoheadrightarrow \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \xrightarrow{\sim} \hat{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/m\mathbb{Z} \xrightarrow{1 \mapsto F'} \text{GL}_n(\overline{K})$$

lifts  $\tilde{\rho}$  and is continuous as well as unramified. □

#### 7.2.4 Serre invariants and Serre's conjecture

Let  $\ell$  be a prime. A Galois representation  $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{F}_\ell})$  has a *level*  $N(\rho)$  and a *weight*  $k(\rho)$ . The definitions were introduced by Serre (see [Ser9, Sections 1.2 & 2]). Later on, Edixhoven found an improved definition for the weight, see [Edi1, Section 4]. The definitions agree in the cases of our interest, but in the general formulation of Theorem 7.2.7 later on, Edixhoven's definition applies. The level  $N(\rho)$  is defined as the prime-to- $\ell$  part of the Artin conductor of  $\rho$  and equals 1 if  $\rho$  is unramified outside  $\ell$ . The weight is defined in terms of the local representation  $\rho|_{D_\ell}$ ; its definition is rather lengthy so we will not write

it out here. When we need results about the weight we will just state them. Let us for now mention that one can consider the weights of the twists  $\rho \otimes \chi$  of a representation  $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_\ell)$  by a character  $\chi: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \overline{\mathbb{F}}_\ell^\times$ . If one chooses  $\chi$  so that  $k(\rho \otimes \chi)$  is minimal, then we always have  $1 \leq k(\rho \otimes \chi) \leq \ell + 1$  and we can in fact choose our  $\chi$  to be a power of the mod  $\ell$  cyclotomic character.

Serre conjectured [Ser9, Conjecture 3.2.4] that if  $\rho$  is irreducible and odd, then  $\rho$  belongs to a modular form of level  $N(\rho)$  and weight  $k(\rho)$ . Oddness here means that the image of a complex conjugation has determinant  $-1$ . A proof of this conjecture in the case  $N(\rho) = 1$  has been published by Khare, building on ideas of himself and Wintenberger:

**7.2.5 Theorem (Khare & Wintenberger, [Kha, Theorem 1.1])** *Let  $\ell$  be a prime number and let  $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_\ell)$  be an odd irreducible representation of level  $N(\rho) = 1$ . Then there exists a modular form  $f$  of level 1 and weight  $k(\rho)$  which is a normalised eigenform and a prime  $\lambda \mid \ell$  of  $K_f$  such that  $\rho$  and  $\bar{\rho}_{f,\lambda}$  become isomorphic after a suitable embedding of  $\mathbb{F}_\lambda$  into  $\overline{\mathbb{F}}_\ell$ .*

### 7.2.6 Weights and discriminants

If a representation  $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_\ell)$  is wildly ramified at  $\ell$  it is possible to relate the weight to discriminants of certain number fields. In this subsection we will present a theorem of Moon and Taguchi, [Mo-Ta, Theorem 3], on this matter and derive some results from it that are of use to us.

**7.2.7 Theorem (Moon & Taguchi)** *Consider a wildly ramified representation from  $\rho: \text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_\ell)$ . Let  $\alpha \in \mathbb{Z}$  be such that  $k(\rho \otimes \chi_\ell^{-\alpha})$  is minimal where  $\chi_\ell: \text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \rightarrow \overline{\mathbb{F}}_\ell^\times$  is the mod  $\ell$  cyclotomic character. Put  $\tilde{k} = k(\rho \otimes \chi_\ell^{-\alpha})$ , put  $d = \gcd(\alpha, \tilde{k} - 1, \ell - 1)$  and define  $m$  in  $\mathbb{Z}$  by letting  $\ell^m$  be the wild ramification degree of  $K := \overline{\mathbb{Q}}_\ell^{\text{Ker}(\rho)}$  over  $\mathbb{Q}_\ell$ . Then we have:*

$$v_\ell(\mathcal{D}_{K/\mathbb{Q}_\ell}) = \begin{cases} 1 + \frac{\tilde{k}-1}{\ell-1} - \frac{\tilde{k}-1+d}{(\ell-1)\ell^m} & \text{if } 2 \leq \tilde{k} \leq \ell, \\ 2 + \frac{1}{(\ell-1)\ell} - \frac{2}{(\ell-1)\ell^m} & \text{if } \tilde{k} = \ell + 1, \end{cases}$$

where  $\mathcal{D}_{K/\mathbb{Q}_\ell}$  denotes the different of  $K$  over  $\mathbb{Q}_\ell$  and  $v_\ell$  is normalised by  $v_\ell(\ell) = 1$ .

We can simplify this formula to one which is useful in our case. In the following corollaries  $v_\ell$  denotes a valuation at a prime above  $\ell$  that is normalised by  $v_\ell(\ell) = 1$ .

**7.2.8 Corollary** *Let  $\tilde{\rho}: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{PGL}_2(\mathbb{F}_\ell)$  be an irreducible projective representation that is wildly ramified at  $\ell$ . Take a point in  $\mathbb{P}^1(\mathbb{F}_\ell)$ , let  $H \subset \text{PGL}_2(\mathbb{F}_\ell)$  be its stabiliser subgroup and let  $K$  be the number field defined as:*

$$K = \overline{\mathbb{Q}}^{\tilde{\rho}^{-1}(H)}.$$

Then the  $\ell$ -primary part of  $\text{Disc}(K/\mathbb{Q})$  is related to the minimal weight  $k$  of the liftings of  $\tilde{\rho}$  by the following formula:

$$v_\ell(\text{Disc}(K/\mathbb{Q})) = k + \ell - 2.$$

**Proof** Let  $\rho$  be a lifting of  $\tilde{\rho}$  of minimal weight. Since  $\rho$  is wildly ramified, after a suitable conjugation in  $\text{GL}_2(\overline{\mathbb{F}}_\ell)$  we may assume:

$$(7.2.9) \quad \rho|_{I_\ell} = \begin{pmatrix} \chi_\ell^{k-1} & * \\ 0 & 1 \end{pmatrix},$$

where  $\chi_\ell: I_\ell \rightarrow \mathbb{F}_\ell^\times$  denotes the mod  $\ell$  cyclotomic character; this follows from the definition of weight. The canonical map  $\text{GL}_2(\overline{\mathbb{F}}_\ell) \rightarrow \text{PGL}_2(\overline{\mathbb{F}}_\ell)$  is injective on the subgroup  $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ , so the subfields of  $\overline{\mathbb{Q}}_\ell$  cut out by  $\rho|_{I_\ell}$  and  $\tilde{\rho}|_{I_\ell}$  are equal, call them  $K_2$ . Also, let  $K_1 \subset K_2$  be the fixed field of the diagonal matrices in  $\text{Im } \rho|_{I_\ell}$ . We see from (7.2.9) that in the notation of Theorem 7.2.7 we can put  $\alpha = 0$ ,  $m = 1$  and  $d = \gcd(\ell - 1, k - 1)$ . So we have the following diagram of field extensions:

$$\begin{array}{ccc} & & K_2 \\ & \nearrow \chi_\ell^{k-1} & \downarrow \begin{pmatrix} \chi_\ell^{k-1} & * \\ 0 & 1 \end{pmatrix} \\ K_1 & & \\ & \searrow \text{deg} = \ell & \\ & & \mathbb{Q}_\ell^{\text{un}} \end{array}$$

The extension  $K_2/K_1$  is tamely ramified of degree  $(\ell-1)/d$  hence we have:

$$v_\ell(\mathcal{D}_{K_2/K_1}) = \frac{(\ell-1)/d-1}{(\ell-1)\ell/d} = \frac{\ell-1-d}{(\ell-1)\ell}.$$

Consulting Theorem 7.2.7 for the case  $2 \leq k \leq \ell$  now yields

$$\begin{aligned} v_\ell(\mathcal{D}_{K_1/\mathbb{Q}_\ell^{\text{un}}}) &= v_\ell(\mathcal{D}_{K_2/\mathbb{Q}_\ell^{\text{un}}}) - v_\ell(\mathcal{D}_{K_2/K_1}) \\ &= 1 + \frac{k-1}{\ell-1} - \frac{k-1+d}{(\ell-1)\ell} - \frac{\ell-1-d}{(\ell-1)\ell} = \frac{k+\ell-2}{\ell} \end{aligned}$$

and also in the case  $k = \ell + 1$  we get:

$$v_\ell(\mathcal{D}_{K_1/\mathbb{Q}_\ell^{\text{un}}}) = 2 + \frac{1}{(\ell-1)\ell} - \frac{2}{(\ell-1)\ell} - \frac{\ell-2}{(\ell-1)\ell} = \frac{k+\ell-2}{\ell}.$$

Let  $L$  be the number field  $\overline{\mathbb{Q}}^{\text{Ker}(\tilde{\rho})}$ . From the irreducibility of  $\tilde{\rho}$  and the fact that  $\text{Im } \tilde{\rho}$  has an element of order  $\ell$  it follows that the induced action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on  $\mathbb{P}^1(\mathbb{F}_\ell)$  is transitive and hence that  $L$  is the normal closure of  $K$  in  $\overline{\mathbb{Q}}$ . This in particular implies that  $K/\mathbb{Q}$  is wildly ramified. Now from  $[K : \mathbb{Q}] = \ell + 1$  it follows that there are two primes in  $K$  above  $\ell$ : one is unramified and the other has inertia degree 1 and ramification degree  $\ell$ . From the considerations above it now follows that any ramification subgroup of  $\text{Gal}(L/\mathbb{Q})$  at  $\ell$  is isomorphic to a subgroup of  $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \subset \text{GL}_2(\overline{\mathbb{F}}_\ell)$  of order  $(\ell-1)\ell/d$  with  $d \mid \ell-1$ . Up to conjugacy, the only subgroup of index  $\ell$  is the subgroup of diagonal matrices. Hence  $K_1$  and  $K_{\lambda_2}^{\text{un}}$  are isomorphic field extensions of  $\mathbb{Q}_\ell^{\text{un}}$ , from which:

$$v_\ell(\text{Disc}(K/\mathbb{Q})) = v_\ell(\text{Disc}(K_1/\mathbb{Q}_\ell^{\text{un}})) = \ell \cdot v_\ell(\mathcal{D}_{K_1/\mathbb{Q}_\ell^{\text{un}}}) = k + \ell - 2.$$

follows. □

**7.2.10 Corollary** *Let  $\tilde{\rho}: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{PGL}_2(\mathbb{F}_\ell)$  be an irreducible projective representation and let  $\rho$  be a lifting of  $\tilde{\rho}$  of minimal weight. Let  $K$  be the number field belonging to a point of  $\mathbb{P}^1(\mathbb{F}_\ell)$ , as in the notation of Corollary 7.2.8. If  $k \geq 3$  is such that:*

$$v_\ell(\text{Disc}(K/\mathbb{Q})) = k + \ell - 2$$

*holds, then we have  $k(\rho) = k$ .*

**Proof** From  $v_\ell(\text{Disc}(K/\mathbb{Q})) = k + \ell - 2 \geq \ell + 1$  it follows that  $\tilde{\rho}$  is wildly ramified at  $\ell$  so we can apply Corollary 7.2.8. □

### 7.3 PROOF OF THE THEOREM

To prove Theorem 7.1.3 we need to do several verifications. We will derive representations from the polynomials  $P_{k,\ell}$  and verify that they satisfy the conditions of Theorem 7.2.5. Then we know there are modular forms attached to them that have the right level and weight and uniqueness follows then easily.

First we will verify that the polynomials  $P_{k,\ell}$  from the table in Section 7.5 have the right Galois group. The algorithm described in [Ge-KI, Algorithm 6.1] can be used perfectly to do this verification; proving  $A_{\ell+1} \not\leq \text{Gal}(P_{k,\ell})$  is the most time-consuming part of the calculation here. It turns out that in all cases we have:

$$(7.3.1) \quad \text{Gal}(P_{k,\ell}) \cong \text{PGL}_2(\mathbb{F}_\ell).$$

That the action of  $\text{Gal}(P_{k,\ell})$  on the roots of  $P_{k,\ell}$  is compatible with the action of  $\text{PGL}_2(\mathbb{F}_\ell)$  follows from the following lemma.

**7.3.2 Lemma** *Let  $\ell$  be a prime and let  $G$  be a subgroup of  $\text{PGL}_2(\mathbb{F}_\ell)$  of index  $\ell+1$ . Then  $G$  is the stabiliser subgroup of a point in  $\mathbb{P}^1(\mathbb{F}_\ell)$ . In particular, any transitive permutation representation of  $\text{PGL}_2(\mathbb{F}_\ell)$  of degree  $\ell+1$  is isomorphic to the standard action on  $\mathbb{P}^1(\mathbb{F}_\ell)$ .*

**Proof** This follows from [Suz, Proof of Theorem 6.25]. □

So now we have shown that the second assertion in Theorem 7.1.3 follows from the first one.

Next we will verify that we can obtain representations from this that have the right Serre invariants. Let us first note that every automorphism of the group  $\text{PGL}_2(\mathbb{F}_\ell)$  is an inner automorphism. This implies that for every  $P_{k,\ell}$ , two isomorphisms (7.3.1) define isomorphic representations  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{PGL}_2(\mathbb{F}_\ell)$  via composition with the canonical map  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \text{Gal}(P_{k,\ell})$ . In other words, every  $P_{k,\ell}$  gives a projective representation  $\tilde{\rho}: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{PGL}_2(\mathbb{F}_\ell)$  that is well-defined up to isomorphism.

Now, for each  $(k, \ell)$  in the table in Section 7.5, the polynomial  $P_{k,\ell}$  is

irreducible and hence defines a number field:

$$K_{k,\ell} := \mathbb{Q}[x]/(P_{k,\ell}),$$

whose ring of integers we will denote by  $\mathcal{O}_{k,\ell}$ . It is possible to compute  $\mathcal{O}_{k,\ell}$  using the algorithm from [Bu-Le, Section 6] (see also [Bu-Le, Theorems 1.1 & 1.4]), since we know what kind of ramification behaviour to expect. In all cases it turns out that we have:

$$\text{Disc}(K_{k,\ell}/\mathbb{Q}) = (-1)^{(\ell-1)/2} \ell^{k+\ell-2}.$$

We see that for each  $(k, \ell)$  the representation  $\tilde{\rho}_{k,\ell}$  is unramified outside  $\ell$ . From Lemma 7.2.3 it follows that for each  $p \neq \ell$ , the representation  $\tilde{\rho}_{k,\ell}|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$  has an unramified lifting. Above we saw that via  $\tilde{\rho}_{k,\ell}$  the action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on the set of roots of  $P_{k,\ell}$  is compatible with the action of  $\text{PGL}_2(\mathbb{F}_\ell)$  on  $\mathbb{P}^1(\mathbb{F}_\ell)$ , hence we can apply Corollary 7.2.10 to show that the minimal weight of a lifting of  $\tilde{\rho}_{k,\ell}$  equals  $k$ . Theorem 7.2.2 now shows that every  $\tilde{\rho}_{k,\ell}$  has a lifting  $\rho_{k,\ell}$  that has level 1 and weight  $k$ . From  $\text{Im } \tilde{\rho}_{k,\ell} = \text{PGL}_2(\mathbb{F}_\ell)$  it follows that each  $\rho_{k,\ell}$  is absolutely irreducible.

To apply Theorem 7.2.5 we should still verify that  $\rho_{k,\ell}$  is odd in each case. I thank Robin de Jong for pointing out that this is immediate: Since the weight of  $\rho_{k,\ell}$  is  $k$ , we have  $\det \rho_{k,\ell}|_{I_\ell} = \chi_\ell^{k-1}|_{I_\ell}$  where  $\chi_\ell$  is the mod  $\ell$  cyclotomic character. Now,  $\det \rho_{k,\ell}|_{D_p}$  is unramified for  $p \neq \ell$  and hence  $\det \rho_{k,\ell}$  must equal  $\chi_\ell^{k-1}$  on all of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  (apply Theorem 7.2.2 with  $n = 1$  for instance). But then, since  $k$  is even, we have that  $\det \rho_{k,\ell}$  evaluated at a complex conjugation equals  $-1$  hence  $\rho_{k,\ell}$  is odd.

So now that we have verified all the conditions of Theorem 7.2.5 we remark as a final step that all spaces of modular forms  $S_k(\text{SL}_2(\mathbb{Z}))$  involved here are 1-dimensional. So the modularity of each  $\rho_{k,\ell}$  implies immediately the isomorphism  $\rho_{k,\ell} \cong \bar{\rho}_{\Delta_k,\ell}$ , hence also  $\tilde{\rho}_{k,\ell} \cong \tilde{\rho}_{\Delta_k,\ell}$ , which completes the proof of Theorem 7.1.3.

#### 7.4 PROOF OF THE COROLLARY

If  $\tau$  vanishes somewhere, then the smallest positive integer  $n$  for which  $\tau(n)$  is zero is a prime. This was observed by Lehmer [Leh, Theorem 2] and can also be seen using the following argument: Suppose  $n$  is the smallest

positive integer with  $\tau(n) = 0$ . From the multiplicative property of  $\tau$  it follows that  $n$  is a power of a prime  $p$ . If  $\tau(p) \neq 0$  then from  $|\tau(p)| < p^6$  and the recursion for  $\tau(p^r)$  it follows that  $v_p(\tau(p^r)) = r \cdot v_p(\tau(p))$  for all  $r$ , so  $\tau(p^r)$  would never be zero.

Using results on the exceptional representations for  $\tau(p)$ , Serre pointed out [Ser8, Section 3.3] that if  $p$  is a prime number with  $\tau(p) = 0$  then  $p$  can be written as:

$$p = hM - 1$$

with

$$M = 2^{14}3^75^3691 = 3094972416000,$$

$$\left(\frac{h+1}{23}\right) = 1 \quad \text{and} \quad h \equiv 0, 30 \text{ or } 48 \pmod{49}.$$

In fact  $p$  is of this form if and only if  $\tau(p) \equiv 0 \pmod{23 \cdot 49 \cdot M}$  holds. Knowing this, we will do a computer search on these primes  $p$  and verify whether  $\tau(p) \equiv 0 \pmod{\ell}$  for  $\ell \in \{11, 13, 17, 19\}$ . To do that we will use the following lemma.

**7.4.1 Lemma** *Let  $K$  be a field of characteristic not equal to 2. Then the following conditions on  $M \in \text{GL}_2(K)$  are equivalent:*

- (1)  $\text{tr } M = 0$ .
- (2) *For the action of  $M$  on  $\mathbb{P}^1(K)$ , there are 0 or 2 orbits of length 1 and all other orbits have length 2.*
- (3) *The action of  $M$  on  $\mathbb{P}^1(K)$  has an orbit of length 2.*

**Proof** We begin with verifying (1)  $\Rightarrow$  (2). Suppose  $\text{tr } M = 0$ . Matrices of trace 0 in  $\text{GL}_2(K)$  have distinct eigenvalues in  $\overline{K}$  because of  $\text{char}(K) \neq 2$ . It follows that two such matrices are conjugate if and only if their characteristic polynomials coincide. Hence  $M$  and  $M' := \begin{pmatrix} 0 & 1 \\ -\det M & 0 \end{pmatrix}$  are conjugate so without loss of generality we assume  $M = M'$ . Since  $M^2$  is a scalar matrix, all the orbits of  $M$  on  $\mathbb{P}^1(K)$  have length 1 or 2. If there are at least 3 orbits of length 1 then  $K^2$  itself is an eigenspace of  $M$  hence  $M$  is scalar, which is not the case. If there is exactly one orbit of length 1 then  $M$  has a

non-scalar Jordan block in its Jordan decomposition, which contradicts the fact that the eigenvalues are distinct.

The implication (2)  $\Rightarrow$  (3) is trivial so that leaves proving (3)  $\Rightarrow$  (1). Suppose that  $M$  has an orbit of length 2 in  $\mathbb{P}^1(K)$ . After a suitable conjugation, we may assume that this orbit is  $\{[(\frac{1}{0})], [(\frac{0}{1})]\}$ . But this means that  $M \sim \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$  for certain  $a, b \in K$  hence  $\text{tr } M = 0$ .  $\square$

In view of the above lemma it follows from Theorem 7.1.3 that for  $\ell \in \{11, 13, 17, 19\}$  and  $p \neq \ell$  we have  $\tau(p) \equiv 0 \pmod{\ell}$  if and only if the prime  $p$  decomposes in the number field  $\mathbb{Q}[x]/(P_{12,\ell})$  as a product of primes of degree 1 and 2, with degree 2 occurring at least once. For  $p \nmid \text{Disc}(P_{12,\ell})$ , which is a property that all primes  $p$  satisfying Serre's criteria possess, we can verify this condition by checking whether  $P_{12,\ell}$  has an irreducible factor of degree 2 over  $\mathbb{F}_p$ . This can be easily checked by verifying:

$$\bar{x}^{p^2} = \bar{x} \quad \text{and} \quad \bar{x}^p \neq \bar{x} \quad \text{in} \quad \mathbb{F}_p[x]/(\bar{P}_{12,\ell}).$$

Having done a computer search, it turns out that the first few primes satisfying Serre's criteria as well as  $\tau(p) \equiv 0 \pmod{11 \cdot 13 \cdot 17 \cdot 19}$  are

$$\begin{aligned} &22798241520242687999, 60707199950936063999, \\ &93433753964906495999, 102797608484376575999. \end{aligned}$$

*Remark.* The unpublished paper [Jo-Ke] in which Jordan and Kelly obtained the previous bound for the verification of Lehmer's conjecture seems to be unfindable. Kevin Buzzard asked me the question what method they could have used. If we weaken the above search to using only the prime  $\ell = 11$  we obtain the same bound as Jordan and Kelly did. So our speculation is that they searched for primes  $p$  satisfying Serre's criteria as well as  $\tau(p) \equiv 0 \pmod{11}$ . This congruence can be verified using an elliptic curve computation, as was already remarked in Subsection 7.1.2.

## 7.5 THE TABLE OF POLYNOMIALS

In this section we present the table of polynomials that is referred to throughout this chapter.



## Polynomials belonging to projective modular representations

$(k, \ell)$	$P_{k,\ell}$
(12, 11)	$x^{12} - 4x^{11} + 55x^9 - 165x^8 + 264x^7 - 341x^6 + 330x^5 - 165x^4 - 55x^3 + 99x^2 - 41x - 111$
(12, 13)	$x^{14} + 7x^{13} + 26x^{12} + 78x^{11} + 169x^{10} + 52x^9 - 702x^8 - 1248x^7 + 494x^6 + 2561x^5 + 312x^4 - 2223x^3 + 169x^2 + 506x - 215$
(12, 17)	$x^{18} - 9x^{17} + 51x^{16} - 170x^{15} + 374x^{14} - 578x^{13} + 493x^{12} - 901x^{11} + 578x^{10} - 51x^9 + 986x^8 + 1105x^7 + 476x^6 + 510x^5 + 119x^4 + 68x^3 + 306x^2 + 273x + 76$
(12, 19)	$x^{20} - 7x^{19} + 76x^{17} - 38x^{16} - 380x^{15} + 114x^{14} + 1121x^{13} - 798x^{12} - 1425x^{11} + 6517x^{10} + 152x^9 - 19266x^8 - 11096x^7 + 16340x^6 + 37240x^5 + 30020x^4 - 17841x^3 - 47443x^2 - 31323x - 8055$
(16, 17)	$x^{18} - 2x^{17} - 17x^{15} + 204x^{14} - 1904x^{13} + 3655x^{12} + 5950x^{11} - 3672x^{10} - 38794x^9 + 19465x^8 + 95982x^7 - 280041x^6 - 206074x^5 + 455804x^4 + 946288x^3 - 1315239x^2 + 606768x - 378241$
(16, 19)	$x^{20} + x^{19} + 57x^{18} + 38x^{17} + 950x^{16} + 4389x^{15} + 20444x^{14} + 84018x^{13} + 130359x^{12} - 4902x^{11} - 93252x^{10} + 75848x^9 - 1041219x^8 - 1219781x^7 + 3225611x^6 + 1074203x^5 - 3129300x^4 - 2826364x^3 + 2406692x^2 + 6555150x - 5271039$
(16, 23)	$x^{24} + 9x^{23} + 46x^{22} + 115x^{21} - 138x^{20} - 1886x^{19} + 1058x^{18} + 59639x^{17} + 255599x^{16} + 308798x^{15} - 1208328x^{14} - 6156732x^{13} - 10740931x^{12} + 2669403x^{11} + 52203054x^{10} + 106722024x^9 + 60172945x^8 - 158103380x^7 - 397878081x^6 - 357303183x^5 + 41851168x^4 + 438371490x^3 + 484510019x^2 + 252536071x + 55431347$

Continued on next page

Table continued from previous page

$(k, \ell)$	$P_{k,\ell}$
(18, 17)	$x^{18} - 7x^{17} + 17x^{16} + 17x^{15} - 935x^{14} + 799x^{13}$ $+ 9231x^{12} - 41463x^{11} + 192780x^{10} + 291686x^9$ $- 390014x^8 + 6132223x^7 - 3955645x^6 + 2916112x^5$ $+ 45030739x^4 - 94452714x^3 + 184016925x^2$ $- 141466230x + 113422599$
(18, 19)	$x^{20} + 10x^{19} + 57x^{18} + 228x^{17} - 361x^{16} - 3420x^{15}$ $+ 23446x^{14} + 88749x^{13} - 333526x^{12} - 1138233x^{11}$ $+ 1629212x^{10} + 13416014x^9 + 7667184x^8$ $- 208954438x^7 + 95548948x^6 + 593881632x^5$ $- 1508120801x^4 - 1823516526x^3 + 2205335301x^2$ $+ 1251488657x - 8632629109$
(18, 23)	$x^{24} + 23x^{22} - 69x^{21} - 345x^{20} - 483x^{19} - 6739x^{18}$ $+ 18262x^{17} + 96715x^{16} - 349853x^{15} + 2196684x^{14}$ $- 7507476x^{13} + 59547x^{12} + 57434887x^{11}$ $- 194471417x^{10} + 545807411x^9 + 596464566x^8$ $- 9923877597x^7 + 33911401963x^6 - 92316759105x^5$ $+ 157585411007x^4 - 171471034142x^3$ $+ 237109280887x^2 - 93742087853x + 97228856961$
(20, 19)	$x^{20} - 5x^{19} + 76x^{18} - 247x^{17} + 1197x^{16} - 8474x^{15}$ $+ 15561x^{14} - 112347x^{13} + 325793x^{12} - 787322x^{11}$ $+ 3851661x^{10} - 5756183x^9 + 20865344x^8$ $- 48001353x^7 + 45895165x^6 - 245996344x^5$ $+ 8889264x^4 - 588303992x^3 - 54940704x^2$ $- 538817408x + 31141888$

Continued on next page

Table continued from previous page

$(k, \ell)$	$P_{k,\ell}$
(20, 23)	$ \begin{aligned} & x^{24} - x^{23} - 23x^{22} - 184x^{21} - 667x^{20} - 5543x^{19} \\ & - 22448x^{18} + 96508x^{17} + 1855180x^{16} \\ & + 13281488x^{15} + 66851616x^{14} + 282546237x^{13} \\ & + 1087723107x^{12} + 3479009049x^{11} + 8319918708x^{10} \\ & + 8576048755x^9 - 19169464149x^8 - 111605931055x^7 \\ & - 227855922888x^6 - 193255204370x^5 \\ & + 176888550627x^4 + 1139040818642x^3 \\ & + 1055509532423x^2 + 1500432519809x \\ & + 314072259618 \end{aligned} $
(22, 23)	$ \begin{aligned} & x^{24} - 2x^{23} + 115x^{22} + 23x^{21} + 1909x^{20} + 22218x^{19} \\ & + 9223x^{18} + 121141x^{17} + 1837654x^{16} - 800032x^{15} \\ & + 9856374x^{14} + 52362168x^{13} - 32040725x^{12} \\ & + 279370098x^{11} + 1464085056x^{10} + 1129229689x^9 \\ & + 3299556862x^8 + 14586202192x^7 + 29414918270x^6 \\ & + 45332850431x^5 - 6437110763x^4 - 111429920358x^3 \\ & - 12449542097x^2 + 93960798341x - 31890957224 \end{aligned} $

## Chapter Eight

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### Description of $X_1(5l)$

B. Edixhoven

#### 8.1 CONSTRUCTION OF A SUITABLE CUSPIDAL DIVISOR ON $X_1(5l)$

In this section we put ourselves in the situation of Theorem 2.5.12:  $l$  is a prime number,  $k$  is an integer such that  $2 < k \leq l+1$ , and  $f$  is a surjective ring morphism  $\mathbb{T}(1, k) \rightarrow \mathbb{F}$  with  $\mathbb{F}$  a finite field of characteristic  $l$ , such that the associated Galois representation  $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$  is irreducible. We let  $V$  denote the two-dimensional  $\mathbb{F}$ -vector space in  $J_1(l)(\overline{\mathbb{Q}})[l]$  that realises  $\rho$ .

As explained in Chapter 3, we would like to have an effective divisor  $D_0$  on  $X_1(l)_{\mathbb{Q}}$  of degree the genus of  $X_1(l)$  such that for all non-zero  $x$  in the submodule  $V$  of  $J_1(l)(\overline{\mathbb{Q}})$  we have  $h^0(\mathcal{L}_x(D_0)) = 1$ . It would be nice to have a cuspidal divisor (i.e., a divisor supported on the cusps) with this property. The first complication is that the cusps are not all rational over  $\mathbb{Q}$ : half of them have the maximal real subfield of  $\mathbb{Q}(\zeta_l)$  as field of definition. Moreover, even working with all the cusps, we have not succeeded to find a cuspidal divisor  $D_0$  with the desired properties. On the other hand, below we will give explicitly a cuspidal divisor  $D_0$  on the curve  $X_1(5l)_{\mathbb{Q}(\zeta_l)}$  that has the property that  $h^0(\mathcal{L}_x(D_0)) = 1$  for each  $x$  in  $J_1(5l)(\overline{\mathbb{Q}})$  that specialises to 0 at some place of  $\overline{\mathbb{Q}}$  over  $l$ . In particular,  $D_0$  has the required property for  $V$  embedded in  $J_1(5l)(\overline{\mathbb{Q}})$  in an arbitrary way, provided that the image of  $\rho$  contains  $\text{SL}_2(\mathbb{F})$  (this will be shown in Section 8.2). We have chosen to work with  $X_1(5l)$ , but the same method will work for modular curves corresponding to some level structure if the prime to  $l$  part of the level structure is fine, and of genus zero.

For the rest of this section, our assumptions are the following:  $l$  is a prime number, not equal to 5. We let  $X := X_1(5l)_{\mathbb{Q}(\zeta_l)}$  over  $\mathbb{Q}(\zeta_l)$ . The genus of  $X$  is  $(l - 2)^2$ . References for facts about  $X$  that we use can be found in [Gro], and also in [Edi1]; they are derived from results in [De-Ra] and in [Ka-Ma].

The curve  $X_0(5)_{\mathbb{Q}}$  has 2 cusps, both  $\mathbb{Q}$ -rational, called 0 and  $\infty$  (after the points of  $\mathbb{P}^1(\mathbb{Q})$  of which they come). The cusp  $\infty$  has as moduli interpretation the degenerate elliptic curve (generalised elliptic curve in the terminology of [De-Ra]): the 1-gon, equipped with the unique subgroup of order 5 of  $\mathbb{G}_m$ . The cusp 0 corresponds to the 5-gon, equipped with a subgroup of order 5 that meets all 5 components. The group  $\mathbb{F}_5^\times$  acts (as diamond operators) on  $X_1(5)$ , with quotient  $X_0(5)$ ; the subgroup  $\{\langle \pm 1 \rangle\}$  acts trivially, and the quotient  $\mathbb{F}_5^\times / \{\langle \pm 1 \rangle\}$  by this subgroup acts faithfully. The inverse images of 0 and  $\infty$  both consist of two cusps. Those over 0 are  $\mathbb{Q}$ -rational (the subgroup of order 5 of the 5-gon is the constant group-scheme  $\mathbb{Z}/5\mathbb{Z}$ ), whereas those over  $\infty$  are conjugated over  $\mathbb{Q}(\sqrt{5})$ . We fix one  $\mathbb{Q}$ -rational cusp  $c$  of  $X_1(5)$ .

The group  $\mathbb{F}_l^\times$  acts faithfully, and in fact, even freely, on  $X$ . The set of cusps of  $X$  over the cusp  $c$  of  $X_1(5)$  form two  $\mathbb{F}_l^\times$ -orbits, corresponding to the type of degenerate elliptic curve that they correspond to: 5-gon or  $5l$ -gon. The orbit corresponding to the 5-gon consists of points rational over  $\mathbb{Q}(\zeta_l)$ , all conjugates of each other. The orbit corresponding to the  $5l$ -gon consists of  $\mathbb{Q}$ -points.

We let  $J$  denote the Jacobian of  $X$ . What we want is an effective divisor  $D_0$  of degree  $g$  on  $X$  (with  $g$  the genus of  $X$ ), supported on the cusps over  $c$ , such that for all  $x$  in  $J(\overline{\mathbb{Q}})$  that specialise to 0 at some place of  $\overline{\mathbb{Q}}$  over  $l$  we have  $h^0(X_{\overline{\mathbb{Q}}}, \mathcal{L}_x(D_0)) = 1$ . For the notion of specialisation we use Néron models; the reader is referred to [Bo-Lu-Ra] for this notion. For  $x$  in  $J(K)$  with  $K \subset \overline{\mathbb{Q}}$  a finite extension of  $\mathbb{Q}(\zeta_l)$ , and  $\lambda$  a place of  $\overline{\mathbb{Q}}$  over  $l$ , we say that  $x$  specialises to 0 at  $\lambda$  if  $x$ , viewed as element of  $J_{O_K}(O_K)$ , with  $J_{O_K}$  the Néron model of  $J$  over  $O_K$ , specialises to 0 at the place of  $K$  given by  $\lambda$ . For  $K \subset K'$  a finite extension we have  $J(K) \subset J(K')$ , hence we can also view  $x$  as element of  $J(K')$ . The notion of  $x$  specialising to zero is the same for  $K$  and  $K'$ , because  $J_{\mathbb{Z}[\zeta_l]}$  is semi-stable at  $l$ .

The moduli interpretation of  $X$  gives a semi-stable model  $X_{\mathbb{Z}[\zeta_l, 1/5]}$  over

$\mathbb{Z}[\zeta_l, 1/5]$ , described in [Gro] for example. A result of Raynaud identifies the connected component of the Néron model  $J_{O_K}$  with the connected component of the Picard scheme of  $X_{O_K}$  (see Section 9.5 of [Bo-Lu-Ra]). This means that for  $x$  in  $J(K)$  specialising to 0 at  $\lambda$  the line bundle  $\mathcal{L}_x$  on  $X_K$  associated with  $x$  can be extended uniquely over the local ring  $O_{K,\lambda}$  to a line bundle  $\mathcal{L}_x$  on  $X_{O_{K,\lambda}}$  such that the restriction  $\overline{\mathcal{L}}_x$  of  $\mathcal{L}_x$  to the special fibre  $X_{\mathbb{F}_\lambda}$  is trivial. The divisor  $D_0$  on  $X_K$  extends, by taking the Zariski closure, to an effective Cartier divisor on  $X_{O_{K,\lambda}}$ .

We now note that  $h^0(X_K, \mathcal{L}_x(D_0))$  is at least one, by Riemann-Roch, and that  $h^0(X_K, \mathcal{L}_x(D_0))$  is at most  $h^0(X_{\mathbb{F}_\lambda}, \overline{\mathcal{L}}_x(\overline{D_0}))$  by semi-continuity of cohomology of coherent sheaves. As  $\overline{\mathcal{L}}_x(\overline{D_0}) = \mathcal{O}(\overline{D_0})$ , it now suffices to take  $D_0$  such that  $h^0(X_{\mathbb{F}_l}, \mathcal{O}(\overline{D_0})) = 1$ . We do this by looking at the geometry of  $X_{\mathbb{F}_l}$ . As  $\mathbb{Z}[\zeta_l]$  has a unique morphism to  $\overline{\mathbb{F}_l}$ , the curve  $X_{\mathbb{F}_l}$  does not depend on  $K$ . The scheme of cusps of  $X_{\mathbb{Z}_l[\zeta_l]}$  is finite étale over  $\mathbb{Z}_l[\zeta_l]$ , hence the cusps lying over  $c$  specialise injectively to  $X_{\mathbb{F}_l}$ .

The curve  $X_{\overline{\mathbb{F}_l}}$  is the union of two irreducible components,  $X_1$  and  $X_2$ , say, both isomorphic to the Igusa curve of level  $l$  over  $X_1(5)$  over  $\overline{\mathbb{F}_l}$ , that meet transversally in the set  $\Sigma$  of supersingular points. We will take  $\overline{D_0} = D_1 + D_2$ , with  $D_1$  on  $X_1$  and  $D_2$  on  $X_2$ ; note that the cusps are disjoint from  $\Sigma$ , so  $\overline{D_0}$  lies in the smooth locus of  $X_{\overline{\mathbb{F}_l}}$ .

In order to simplify the notation, we let  $X$  and  $D_0$  denote  $X_{\overline{\mathbb{F}_l}}$  and  $\overline{D_0}$ , from now on, in this section. We let  $\Omega_X$  be the dualising sheaf on  $X$  (see Section 8 of [Gro], or [Ma-Ri]): it is the invertible  $\mathcal{O}_X$ -module obtained by gluing  $\Omega_{X_1}^1(\Sigma)$  and  $\Omega_{X_2}^1(\Sigma)$  along  $\Sigma$  via the residue maps at the points of  $\Sigma$  on  $X_1$  and minus the residues maps at the points of  $\Sigma$  on  $X_2$ . By Riemann-Roch, what we want is that  $h^1(X, \mathcal{O}(D_0)) = 0$ , and hence, by Serre duality, that  $h^0(X, \Omega_X(-D_0)) = 0$ . In other words, an element of  $H^0(X, \Omega_X)$  that vanishes on  $D_0$  must be zero. Restriction to  $X_1$  gives a short exact sequence:

$$\begin{aligned} 0 \rightarrow H^0(X_2, \Omega_{X_2/\overline{\mathbb{F}_l}}^1(-D_2)) &\rightarrow H^0(X, \Omega_X(-D_0)) \rightarrow \\ &\rightarrow H^0(X_1, \Omega_{X_1/\overline{\mathbb{F}_l}}^1(\Sigma - D_1)) \rightarrow 0. \end{aligned}$$

Hence, it suffices to take  $D_1$  such that  $H^0(X_1, \Omega_{X_1/\overline{\mathbb{F}_l}}^1(\Sigma - D_1)) = 0$  and  $D_2$  such that  $H^0(X_2, \Omega_{X_2/\overline{\mathbb{F}_l}}^1(-D_2)) = 0$ . Let us now first see of which degrees

$d_1$  and  $d_2$  we want to take  $D_1$  and  $D_2$ . Let  $g_1$  and  $g_2$  be the genera of  $X_1$  and  $X_2$  (note: they are equal). Then we have that  $g = g_1 + g_2 + \#\Sigma - 1$ , and  $g = \deg D_0 = d_1 + d_2$ . It can be shown in several ways that the degree of the sheaf  $\underline{\omega}$  on  $X_1(5)_{\mathbb{F}_l}$  is one. Either by explicit computation, using the equations of Proposition 8.2.8, or by the following argument. The curve  $X_1(5)_{\mathbb{F}_l}$  over  $\mathbb{F}_l$  has genus zero. The Kodaira-Spencer isomorphism on  $X_1(5)_{\mathbb{F}_l}$ , from  $\underline{\omega}^{\otimes 2} \rightarrow \Omega^1(\text{Cusps})$ , plus the fact that the divisor of cusps has degree 4, give that the degree of  $\underline{\omega}^{\otimes 2}$  is 2. Therefore, the Hasse invariant, being a global section of  $\underline{\omega}^{\otimes l-1}$ , has exactly  $l-1$  zeros on  $X_1(5)$  over  $\overline{\mathbb{F}_l}$  and therefore we have:

$$(8.1.1) \quad \#\Sigma = l - 1.$$

Applying Hurwitz's formula to the covering  $X_1 \rightarrow X_1(5)_{\overline{\mathbb{F}_l}}$ , which is totally ramified over  $\Sigma$  and unramified outside it, gives:

$$2g_1 - 2 = -2(l - 1) + (l - 1)(l - 2),$$

and hence:

$$(8.1.2) \quad g_1 = \frac{1}{2}(l - 2)(l - 3).$$

This implies that we want to take:

$$(8.1.3) \quad \begin{aligned} d_1 &= g_1 + \#\Sigma - 1 = \frac{1}{2}(l - 1)(l - 2), \\ d_2 &= g_2 = \frac{1}{2}(l - 2)(l - 3). \end{aligned}$$

Now we use equations to compute with. We choose a coordinate  $z$  on  $X_1(5)_{\overline{\mathbb{F}_l}}$ , i.e., an isomorphism from  $X_1(5)_{\overline{\mathbb{F}_l}}$  to  $\mathbb{P}_{\overline{\mathbb{F}_l}}^1$ , such that  $z(\Sigma)$  does not contain 0 or  $\infty$  and such that  $z^{-1}0$  is the (rational) cusp 0 of  $X_1(5)_{\overline{\mathbb{F}_l}}$ . Let  $f$  be the monic polynomial in  $z$  whose zeros are the elements of  $\Sigma$ , each with multiplicity one. Then  $X_1$  and  $X_2$  are both isomorphic to the cover of  $X_1(5)_{\overline{\mathbb{F}_l}}$  given by the equation  $y^{l-1} = f$ , by the following argument. The complete local rings of  $X_1$  at the points of  $\Sigma$ , with their  $\mathbb{F}_l^\times$ -actions, are all isomorphic to each other because, by a theorem of Serre and Tate, these can all be described in terms of the deformation theory of one  $l$ -divisible group over  $\overline{\mathbb{F}_l}$ ; see [Ka-Ma, §5.2–5.3]. There is a general theory of cyclic

possibly ramified covers such as  $X_1 \rightarrow X_1(5)_{\overline{\mathbb{F}}_l}$ , based on the decomposition of  $\mathcal{O}_{X_1}$  as  $\mathcal{O}$ -module on  $X_1(5)_{\overline{\mathbb{F}}_l}$  for the  $\mathbb{F}_l^\times$ -action. It shows that the cover  $X_1 \rightarrow X_1(5)_{\overline{\mathbb{F}}_l}$  is the cover of  $l-1$ th roots of the global section 1 of the invertible sheaf  $\mathcal{O}(\Sigma)$  on  $X_1(5)_{\overline{\mathbb{F}}_l}$ , in an invertible  $\mathcal{O}$ -module  $\mathcal{L}$  with a given isomorphism  $\mathcal{O}(\Sigma) \rightarrow \mathcal{L}^{\otimes(l-1)}$ , where the  $\mathbb{F}_l^\times$ -action may have been changed by an automorphism of  $\mathbb{F}_l^\times$ . As  $X_1(5)_{\overline{\mathbb{F}}_l}$  has genus zero, we can take  $\mathcal{L}$  to be  $\mathcal{O}_{X_1}(z^{-1}\infty)$ . In fact, Section 12.8 of [Ka-Ma] shows that  $X_1$  is obtained from  $X_1(5)_{\overline{\mathbb{F}}_l}$  by extracting the  $l-1$ th root of the Hasse invariant, in  $\omega$ .

We compute a basis of  $H^0(X_1, \Omega_{X_1/\overline{\mathbb{F}}_l}^1(\Sigma))$ . On  $X_1$  we have:

$$(8.1.4) \quad -y^{l-2}dy = f'dz.$$

Hence  $(dz)/y^{l-2} = -(dy)/f'$  is a generating section of  $\Omega_{X_1/\overline{\mathbb{F}}_l}^1$  on the affine part given by our equation. Hence  $(dz)/y^{l-1}$  is generating section of  $\Omega_{X_1/\overline{\mathbb{F}}_l}^1(\Sigma)$  on the affine part, and it is  $\mathbb{F}_l^\times$ -invariant. At each point of  $X_1$  over the point where  $z$  has its pole, both  $z$  and  $y$  have a simple pole, and  $(dz)/y^{l-1}$  has order  $-2 + l - 1 = l - 3$ . So we have a basis:

$$(8.1.5) \quad H^0(X_1, \Omega_{X_1/\overline{\mathbb{F}}_l}^1(\Sigma)) = \bigoplus_{i+j \leq l-3} \overline{\mathbb{F}}_l z^i y^j \cdot (dz)/y^{l-1}.$$

Note that this agrees with the fact that  $d_1 = \frac{1}{2}(l-1)(l-2)$ .

We can now say how to choose  $D_1$ . At each of the  $l-1$  points where  $z$  has a zero we must give a multiplicity. In the coordinate system given by  $z$  and  $y$ , these points are the ones of the form  $(0, b)$  with  $b \in \overline{\mathbb{F}}_l^\times$  satisfying  $b^{l-1} = f(0)$ . Here is how we choose  $D_1$ : just distribute the multiplicities  $(0, 1, \dots, l-2)$  over these points. Then one sees that any linear combination of our basis elements that vanishes on  $D_1$  is zero, as follows. At all points of  $D_1$ ,  $z$  has a simple zero. Let  $\omega$  be an element of  $H^0(X_1, \Omega_{X_1/\overline{\mathbb{F}}_l}^1(\Sigma))$ , with coordinates  $\lambda_{i,j}$  in the basis (8.1.5). Assume that  $\omega$  vanishes on  $D_1$  (taking multiplicities into account). As there are  $l-2$  points in  $D_1$  with multiplicity  $> 0$ , the polynomial  $\sum_j \lambda_{0,j} y^j$ , being of degree  $\leq l-3$ , must be zero. As there are  $l-3$  points in  $D_1$  with multiplicity  $> 1$ , the polynomial  $\sum_j \lambda_{1,j} z y^j$ , being of degree  $\leq l-4$ , must be zero. And so on.



Now we do  $D_2$ . A basis is the following:

$$(8.1.6) \quad H^0(X_2, \Omega_{X_2/\mathbb{F}_l}^1) = \bigoplus_{i+j \leq l-4} \mathbb{F}_l z^i y^j \cdot (dz)/y^{l-2}.$$

We note that this agrees with  $g_2 = (l-3)(l-2)/2$ . So, for  $D_2$ , just distribute the multiplicities  $(0, 0, 1, \dots, l-3)$  over the points where  $z$  has a zero. The same argument as the one we used for  $D_1$  shows that any  $\omega$  in  $H^0(X_2, \Omega_{X_2/\mathbb{F}_l}^1)$  that vanishes on  $D_2$  is zero.

We summarise our results. As the action of  $\mathbb{F}_5^\times$  permutes the two  $\mathbb{Q}$ -rational cusps of  $X_1(5)$ , our arguments above work for both of them.

**8.1.7 Theorem** *Let  $l$  be a prime number not equal to 5. Let  $c$  be one of the two  $\mathbb{Q}$ -rational cusps of  $X_1(5)$ . Then the cusps of  $X_1(5l)$  over  $c$  are  $\mathbb{Q}(\zeta_l)$ -rational, and consist of two  $\mathbb{F}_l^\times$ -orbits, on which  $\mathbb{F}_l^\times$  acts freely. Let  $D_1$  be a divisor on  $X_1(5l)_{\mathbb{Q}(\zeta_l)}$  obtained by distributing the multiplicities  $(0, 1, \dots, l-2)$  over one of these two orbits. Let  $D_2$  be the divisor obtained by distributing the multiplicities  $(0, 0, 1, \dots, l-3)$  over the other orbit. Then  $D_0 := D_1 + D_2$  has degree equal to the genus of  $X_1(5l)$  and has the property that for any  $\overline{\mathbb{Q}}$ -point  $x$  of the Jacobian of  $X_1(5l)$  that specialises to 0 at some place over  $l$  we have  $h^0(X_1(5l)_{\overline{\mathbb{Q}}}, \mathcal{L}_x(D_0)) = 1$ .*

## 8.2 THE EXACT SETUP FOR THE LEVEL ONE CASE

In Chapter 3 we described our strategy for computing the residual Galois representations attached to a fixed newform. That strategy depends on properties of divisors  $D_0$  and functions  $f$  to be chosen, on modular curves of varying level. These  $D_0$  and  $f$  must satisfy a number of conditions. In general we do not know how to choose divisors  $D_0$  of which we can prove, without a computer computation, that they have the required property. This is the main reason for which we will now restrict ourselves to just the case of modular forms of level one.

The aim of this section is to describe exactly our strategy for computing the residual representations  $V$  in the situation of Theorem 2.5.12:  $l$  is a prime number,  $k$  is an integer such that  $2 < k \leq l+1$ , and  $f$  is a surjective ring morphism  $\mathbb{T}(1, k) \rightarrow \mathbb{F}$  with  $\mathbb{F}$  a finite field of characteristic  $l$ , such

that the associated Galois representation  $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$  is irreducible, under the extra hypothesis that the image of  $\rho$  contains  $\text{SL}_2(\mathbb{F})$ . By Theorem 2.5.18, this hypothesis holds when  $\rho$  is irreducible and  $l \geq 6k - 5$ . We let  $V$  denote the two-dimensional  $\mathbb{F}$ -vector space in  $J_1(l)(\overline{\mathbb{Q}})[l]$  that realises  $\rho$ .

Theorem 8.1.7 gives us a divisor  $D_0$  on  $X_1(5l)_{\mathbb{Q}(\zeta_l)}$  that we want to use. Therefore, we want to embed  $V$  into  $J_1(5l)(\overline{\mathbb{Q}})[l]$ .

Let  $\pi: X_1(5l) \rightarrow X_1(l)$  be the standard map (i.e., the one that forgets the 5-part of the level structure, the one denoted  $B_{5l,l,1}$  in Section 2.2). Then the degree of  $\pi$  is  $5^2 - 1 = 24$ , which is prime to  $l$ . This implies that  $\pi_*\pi^*$  is multiplication by 24 on  $J_1(l)$ , and that  $\pi^*$  is injective on  $J_1(l)(\overline{\mathbb{Q}})[l]$ . We have a projector:

$$(8.2.1) \quad \frac{1}{24}\pi^*\pi_*: J_1(5l)(\overline{\mathbb{Q}})[l] \twoheadrightarrow \pi^*J_1(l)(\overline{\mathbb{Q}})[l] \subset J_1(5l)(\overline{\mathbb{Q}})[l].$$

We will consider  $V$  embedded in  $J_1(5l)(\overline{\mathbb{Q}})[l]$  via its embedding into  $J_1(l)(\overline{\mathbb{Q}})[l]$ , followed by  $\pi^*$ .

**8.2.2 Proposition** *Let  $l$  be a prime number, let  $k$  be an integer such that  $2 < k \leq l+1$ , and  $f$  a surjective ring morphism  $\mathbb{T}(1, k) \rightarrow \mathbb{F}$  with  $\mathbb{F}$  a finite field of characteristic  $l$ , such that the image of the associated Galois representation  $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$  contains  $\text{SL}_2(\mathbb{F})$ . We let  $V$  denote the pullback as above of the two-dimensional  $\mathbb{F}$ -vector space in  $J_1(l)(\overline{\mathbb{Q}})[l]$  that realises  $\rho$ . Let  $D_0$  be a divisor on  $X_1(5l)_{\mathbb{Q}(\zeta_l)}$  as given in Theorem 8.1.7. Then, for every  $x$  in  $V_l$ , we have  $h^0(X_1(5l)_{\overline{\mathbb{Q}}}, \mathcal{L}_x(D_0)) = 1$ .*

**Proof** In view of Theorem 8.1.7, it suffices to show that for each  $x$  in  $V$  there is a place of  $\overline{\mathbb{Q}}$  over  $l$  at which  $x$  specialises to 0. The notion of specialisation is explained in Section 8.1. Let  $J_{\mathbb{Z}[\zeta_l]}$  denote the Néron model of  $J := J_1(5l)$  over  $\mathbb{Z}[\zeta_l]$ . Then  $V$  is the group of  $\overline{\mathbb{Q}}$ -points of an  $\mathbb{F}$ -vector space scheme  $\mathcal{V}_{\mathbb{Q}(\zeta_l)}$  in  $J_{\mathbb{Q}(\zeta_l)}$ . Let  $\mathcal{V}$  be the Zariski closure of  $\mathcal{V}_{\mathbb{Q}(\zeta_l)}$  in  $J_{\mathbb{Z}[\zeta_l]}$ . Then it is shown in Section 12 of [Gro] and in Section 6 of [Edi1] that  $\mathcal{V}_{\mathbb{Z}_l[\zeta_l]}$  is finite locally free over  $\mathbb{Z}_l[\zeta_l]$ , and that the dimension as  $\mathbb{F}$ -vector space scheme of the local part of  $\mathcal{V}_{\mathbb{Z}_l[\zeta_l]}$  is 1 if  $f(T_l) \neq 0$  and 2 if  $f(T_l) = 0$ . We note that it does not matter if we take Zariski closure in  $J_1(l)$  or in  $J_1(5l)$ , as  $\pi^*$  gives a closed immersion of the  $l$ -torsion of  $J_1(l)$  over  $\mathbb{Z}_l$  into that of  $J_1(5l)$ .

This means that at each place of  $\overline{\mathbb{Q}}$  over  $l$  there is a non-zero  $x$  in  $V$  that specialises to 0. Under our assumptions, the image of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l))$  acting on  $V$  is  $\text{SL}(V)$ . Hence  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l))$  acts transitively on  $V - \{0\}$ . Hence for each  $x$  in  $V - \{0\}$  there is at least one place of  $\overline{\mathbb{Q}}$  over  $l$  where  $x$  specialises to 0.  $\square$

The fact that our divisor  $D_0$  lives on  $X_1(5l)_{\mathbb{Q}(\zeta_l)}$ , and not on  $X_1(5l)_{\mathbb{Q}}$ , forces us to work over  $\mathbb{Q}(\zeta_l)$ , and not over  $\mathbb{Q}$ , as in Chapter 3.

We let  $X_l$  denote  $X_1(5l)_{\mathbb{Q}}$  and  $g_l$  its genus, and we let  $A_{l, \mathbb{Q}(\zeta_l)}$  denote the  $\mathbb{Q}(\zeta_l)$ -algebra that corresponds to the  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l))$ -set  $V$ . In order to explain the notation  $A_{l, \mathbb{Q}(\zeta_l)}$ , we note that this  $\mathbb{Q}(\zeta_l)$ -algebra is obtained from the  $\mathbb{Q}$ -algebra  $A_l$  (that corresponds to the  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -set  $V$ ) by extension of scalars.

Proposition 8.2.2 gives that for each  $x$  in  $V$  there is a unique effective divisor  $D_x = \sum_{i=1}^{g_l} Q_{x,i}$  of degree  $g_l$  on  $X_{l, \overline{\mathbb{Q}}}$  such that  $x = [D_x - D_0]$  in  $J_l(\overline{\mathbb{Q}})$ . Note that, for  $x = 0$ , it is indeed true that  $D_x = D_0$ , hence the two notations are consistent. The following observation will make the exposition in Chapter 12 somewhat easier. As for each  $x$  in  $V$  specialises to 0 at some place of  $\overline{\mathbb{Q}}$  above  $l$ , the divisor  $D_x$  specialises to the cuspidal divisor  $D_0$  at such a place, and hence none of all  $Q_{x,i}$  can be a CM-point, in particular:

$$(8.2.3) \quad \text{for all } x \text{ and } i: \quad j(Q_{x,i}) \notin \{0, 1728\}.$$

The uniqueness of  $D_x$  implies that:

$$(8.2.4) \quad D_{gx} = gD_x, \quad \text{for all } x \text{ in } V \text{ and } g \text{ in } \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l)).$$

We write each  $D_x$  as:

$$(8.2.5) \quad D_x = D_x^{\text{fin}} + D_x^{\text{cusp}},$$

where  $D_x^{\text{cusp}}$  is supported on the cusps of  $X_{l, \overline{\mathbb{Q}}}$  and where  $D_x^{\text{fin}}$  is disjoint from the cusps. The next lemma shows that  $D_x^{\text{fin}}$  determines  $x$ , and its proof uses only that  $\rho$  is absolutely irreducible, not that its image contains  $\text{SL}_2(\mathbb{F})$ .

**8.2.6 Lemma** *In this situation, the map from  $V$  to the set of effective divisors on  $X_{l, \overline{\mathbb{Q}}}$  that sends  $x$  to  $D_x^{\text{fin}}$  is injective.*

**Proof** Suppose that it is not. We take  $x_1$  and  $x_2$  in  $V$ , distinct, such that  $D_{x_1}^{\text{fin}} = D_{x_2}^{\text{fin}}$ . Then the element  $x_1 - x_2$  in  $V$  is nonzero and is represented by the cuspidal divisor  $D_{x_1} - D_{x_2}$ . The cusps of  $X_l$  are rational over  $\mathbb{Q}(\zeta_{5l})$ . Hence  $x_1 - x_2$  gives an injection  $\mathbb{F} \rightarrow V$  of representations of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{5l}))$ , where  $\mathbb{F}$  has trivial action. But that gives, by adjunction of induction and restriction, a nonzero map from the regular representation of  $\text{Gal}(\mathbb{Q}(\zeta_{5l})/\mathbb{Q})$  over  $\mathbb{F}$  to  $V$ , necessarily surjective because  $V$  is irreducible. But then the image of  $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$  is abelian. As  $\rho$  is odd, the two eigenspaces in  $V$  of any complex conjugation then decompose  $V$ , in contradiction with the irreducibility of  $V$ .  $\square$

As the cusps of  $X_{l,\overline{\mathbb{Q}}}$  form a  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -stable subset of  $X_{l,\overline{\mathbb{Q}}}$  we have:

$$(8.2.7) \quad \text{for all } g \text{ in } \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l)): \quad D_{gx}^{\text{fin}} = gD_x^{\text{fin}}, \quad D_{gx}^{\text{cusp}} = gD_x^{\text{cusp}}.$$

Hence the map that sends  $x$  in  $V$  to  $D_x^{\text{fin}}$  is  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l))$ -equivariant.

We will now produce a suitable function  $f_l: X_l \rightarrow \mathbb{P}_{\mathbb{Q}}^1$ , in order to push the set of  $\{D_x^{\text{fin}} \mid x \in V\}$  injectively and  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l))$ -equivariantly to the set  $\{f_{l,*}D_x^{\text{fin}} \mid x \in V\}$  of divisors on  $\mathbb{A}_{\mathbb{Q}}^1$ .

We start by giving an explicit description of the curve  $Y_1(5)$  over  $\mathbb{Z}[1/5]$ . In order to do that, we determine a universal triple  $(E/S, P)$  where  $E/S$  is an elliptic curve over an arbitrary scheme, and  $P$  in  $E(S)$  is everywhere of order 5, i.e., for every  $\text{Spec}(A) \rightarrow S$  with  $A$  non-zero, the image of  $P$  in  $E(A)$  has order 5. The base of this triple is the open part  $Y_1(5)'$  of the model  $Y_1(5)$  over  $\mathbb{Z}$  (constructed in Chapter 8 of [Ka-Ma]) where the  $P$  has order 5 (i.e.,  $Y_1(5)'$  is the complement of the irreducible component of  $Y_1(5)_{\mathbb{F}_5}$  where the point  $P$  generates the kernel of Frobenius). The equation of this universal triple can also be found on page 7 of Tom Fisher's thesis, see [Fis].

**8.2.8 Proposition** *Let  $E/S$  be an elliptic curve, and  $P \in E(S)$  a point that is everywhere of order 5. Then  $(E/S, P)$  arises via a unique base change from the following triple:*

$$\begin{cases} E : y^2 + (b+1)xy + by = x^3 + bx^2 \\ Y_1(5)' = \text{Spec}(\mathbb{Z}[b, 1/\text{discr}(E)]), \quad \text{discr}(E) = -b^5(b^2 + 11b - 1) \\ P = (0, 0). \end{cases}$$

The  $j$ -invariant of  $E$  is given by:

$$j(E) = -(b^4 + 12b^3 + 14b^2 - 12b + 1)^3 / b^5 (b^2 + 11b - 1).$$

**Proof** Our proof is modelled on Section 2.2 of [Ka-Ma]; basic properties of Weierstrass equations for elliptic curves are used without being mentioned.

Let  $(E/S, P)$  be given, with  $P$  everywhere of order 5. Choose a parameter  $t$  at 0, up to order 2, i.e., a trivialisation of  $\omega_{E/S}$ . Note: we are working locally on  $S$ , here; in the end, as we will succeed in making things unique, our construction will be global. Note:  $t$  is unique up to  $t' = ut$ , with  $u \in R^\times$  ( $S = \text{Spec}(R)$  now).

Choose  $x$  a global function on  $E - 0(S)$  such that  $x = t^{-2} + \dots$ . Then  $x$  is unique up to  $x' = x + a$ ,  $a \in R$ . Make  $x$  unique by demanding that  $x(P) = 0$  (this is alright because 0 and  $P$  are disjoint).

Choose  $y = t^{-3} + \dots$  regular on  $E - 0(S)$ . Then  $y$  is unique up to  $y' = y + ax + b$ . Make  $y$  unique by demanding that  $y(P) = 0$  and that the tangent of  $E$  at  $P$  is the line given by the equation  $y = 0$ . (Indeed, use  $b$  (uniquely) to get  $y(P) = 0$ , then note that the tangent at  $P$  is nowhere the line given by  $x = 0$  because  $P$  is nowhere annihilated by 2).

The equation for  $E$  is of the form:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2,$$

because the coefficients usually called  $a_4$  and  $a_6$  are zero. We also see that  $a_3$  is a unit because  $E$  is smooth at  $(0, 0)$ . The coefficient  $a_2$  is a unit, because  $P$  is nowhere annihilated by 3.

Now we try to get rid of  $u$  (the ambiguity in the choice of  $t$ ). If  $t' = u^{-1}t$ , then  $a'_i = t^i a_i$ , hence we can make  $t$  unique by demanding that  $a_2 = a_3$ . We do that, and then we have the following equations.

The elliptic curve  $E$  and the point  $P$  are given by:

$$y^2 + axy + by = x^3 + bx^2, \quad P = (0, 0).$$

We have  $5 \cdot P \equiv 5 \cdot 0$ , hence there is a unique  $f$  on  $E - 0(S)$  of the form:

$$f = xy + \alpha y + \beta x^2 + \gamma x + \delta$$

such that the divisor of  $f$  is  $5 \cdot P - 5 \cdot 0$ . As  $x$  and  $y$  have order one and two, respectively, at  $P$ , we have  $\gamma = \delta = 0$ . The function  $f$  with divisor

$5 \cdot P - 5 \cdot 0$  is given by:

$$f = xy + \alpha y + \beta x^2.$$

Here we know that  $b$ ,  $\alpha$  and  $\beta$  are in  $R^\times$ , because  $v_P(x) = 1$  and  $v_P(y) = 2$  everywhere on  $S$ . Now we have to compute what it means that  $v_P(f) = 5$ . This means that the intersection multiplicity of the two curves  $E$  and  $V(f)$  at  $(0, 0)$  is 5. A systematic way to compute that is to do successive blow-ups; that works nicely, but we will not do that here. A much faster way to do the computation is to take suitable linear combinations of the equations for  $E$  and  $f$  directly. One finds the equations:

$$\beta = -\alpha, \quad a = \alpha^{-1}b + 1, \quad \alpha = 1.$$

□

The reason we give such a detailed description of  $Y_1(5)$  is that it gives us functions on all the  $Y_1(5l)$ , at least over  $\mathbb{Z}[1/5l]$ , as stated in the following proposition.

**8.2.9 Proposition** *Let  $l \neq 5$  be prime. Let  $E/Y_1(5)'$  be the elliptic curve given in Proposition 8.2.8. Then  $Y_1(5l)$  and  $E[l] - \{0\}$  agree over  $\mathbb{Z}[1/5l]$ : for  $S$  a  $\mathbb{Z}[1/5l]$ -scheme and  $Q$  in  $(E[l] - \{0\})(S)$  we get, by pullback from  $Y_1(5)'$ , an elliptic curve over  $S$  with an  $S$ -valued point  $P_5$  that is everywhere of order 5, and an  $S$ -valued point  $P_l$  that is everywhere of order  $l$ . In particular, the functions  $b$ ,  $x$  and  $y$  on  $E[l] - \{0\}$  give functions  $b_l$ ,  $x_l$  and  $y_l$  on  $Y_1(5l)$  over  $\mathbb{Z}[1/5l]$  that generate its coordinate ring.*

**Proof** This is standard. The construction above gives a morphism, over  $\mathbb{Z}[1/5l]$ , from  $E[l] - \{0\}$  to  $Y_1(5l)$ . Conversely, an elliptic curve over  $S$  with such points  $P_5$  and  $P_l$  gives a point of  $E[l] - \{0\}$  by the universality of  $Y_1(5)'$  and the fact that  $E[l]$  is finite étale over  $Y_1(5)'$  away from characteristic  $l$ . The second statement follows from the fact that  $E[l] - \{0\}$  is a closed subscheme of the affine scheme  $E - \{0\}$ . □

The functions  $b_l$ ,  $x_l$  and  $y_l$  on  $Y_1(5l)$  over  $\mathbb{Z}[1/5l]$  have the following moduli interpretations. For any  $\mathbb{Z}[1/5l]$ -algebra  $A$ , and any  $Q$  in  $Y_1(5l)(A)$ , a point corresponding to a triple  $(E, P_5, P_l)$  with  $E$  an elliptic curve over  $A$ ,  $P_5$  in

$E(A)$  a point that is everywhere of order 5 and  $P_l$  in  $E(A)$  a point that is everywhere of order  $l$ , there are unique elements  $b_l(Q)$ ,  $x_l(Q)$  and  $y_l(Q)$  in  $A$  such that  $(E/A, P_5)$  is uniquely isomorphic to the pair given by:

$$y^2 + (b_l(Q) + 1)xy + b_l(Q)y = x^3 + b_l(Q)x^2, \quad P_5 = (0, 0).$$

Then, in these coordinates, we have:

$$P_l = (x_l(Q), y_l(Q)).$$

Similarly, we define regular functions  $x'_l$  and  $y'_l$  on  $Y_1(5l)_{\mathbb{Z}[1/5l]}$  by the condition that, in the coordinates above, we have:

$$l^{-1}P_5 + P_l = (x'_l(Q), y'_l(Q)),$$

where  $l^{-1}P_5$  is the unique point  $Q$  in  $E[5](A)$  with  $lQ = P_5$ .

We note that the pair of functions  $(b_l, x'_l)$  embeds  $Y_1(5l)_{\mathbb{Z}[1/5l]}$  in the affine plane  $\mathbb{A}_{\mathbb{Z}[1/5l]}^2$ . Indeed, assume that  $Q$  and  $Q'$  are in  $Y_1(5l)(A)$ , corresponding to  $(E, P_5, P_l)$  and  $(E', P'_5, P'_l)$ , with  $b_l(Q) = b_l(Q')$  and  $x'_l(Q) = x'_l(Q')$ . Then, by Proposition 8.2.8,  $(E, P_5)$  is uniquely isomorphic to  $(E', P'_5)$ , and so we simply consider them to be equal. Then,  $l^{-1}P_5 + P_l$  and  $l^{-1}P_5 + P'_l$  have the same  $x$ -coordinate. Hence, locally on  $\text{Spec}(A)$ ,  $l^{-1}P_5 + P_l = \pm(l^{-1}P_5 + P'_l)$ . Multiplying by  $l$  we see that the sign cannot be a minus.

Using the functions  $b_l$  and  $x'_l$ , we can now say how we will choose the function  $f_l$ . We return to the situation right after Lemma 8.2.6.

For  $x \in V$ , let  $d_x$  be the degree of  $D_x^{\text{fin}}$ , and let us write  $D_x$  as a sum of points in  $X_l(\overline{\mathbb{Q}})$  as follows:

$$D_x = \sum_{i=1}^g Q_{x,i}, \quad \text{with} \quad D_x^{\text{fin}} = \sum_{i=1}^{d_x} Q_{x,i}, \quad D_x^{\text{cusp}} = \sum_{i=d_x+1}^g Q_{x,i}.$$

We note that  $d_0 = 0$ , as  $D_0$  is a cuspidal divisor, and that for all non-zero  $x$  in  $V$  the  $d_x$  are equal, as they are permuted transitively by  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l))$ .

The set  $S$  of points in  $\mathbb{A}^2(\overline{\mathbb{Q}})$  consisting of the  $(b_l(Q_{x,i}), x'_l(Q_{x,i}))$ , with  $x$  in  $V$  and  $i$  in  $\{1, \dots, d_x\}$  has at most  $g \cdot (\#\mathbb{F})^2$  elements. We want to project  $S$  injectively into  $\mathbb{A}^1(\overline{\mathbb{Q}})$  with a map of the form  $(a, b) \mapsto a + nb$  for a suitable integer  $n$ . As there are at most  $g^2 \cdot (\#\mathbb{F})^4$  pairs of distinct elements

in  $S$ , at most that number of integers  $n$  is excluded. Hence there exists an integer  $n$  with  $0 \leq n \leq g^2 \cdot (\#\mathbb{F})^4$  such that the function  $f_l := b_l + nx'_l$  has the required property that the  $f_{l,*}D_x^{\text{fin}}$ , for  $x \in V$ , are all distinct.

Let  $f_l$  be such a function. For each  $x$  in  $V$ ,  $f_{l,*}D_x^{\text{fin}}$  gives us a polynomial  $P_{D_0, f_l, x}$  with coefficients in  $\overline{\mathbb{Q}}$  given by:

$$P_{D_0, f_l, x}(t) = \prod_{i=1}^{d_x} (t - f_l(Q_{x,i})) \quad \text{in } \overline{\mathbb{Q}}[t].$$

Vice versa, each  $P_{D_0, f_l, x}$  gives us the divisor  $f_{l,*}D_x^{\text{fin}}$  by taking the roots, with multiplicity. Therefore, the map that sends  $x$  to  $P_{D_0, f_l, x}$  is injective, and  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l))$ -equivariant.

The next step is to “encode” each  $P_{D_0, f_l, x}$  in a single element of  $\overline{\mathbb{Q}}$ , respecting the action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l))$ . We do this by evaluating at a suitable integer  $m$ , i.e., by sending  $P_{D_0, f_l, x}$  to  $P_{D_0, f_l, x}(m)$ . For a given  $m$ , this map is injective if and only if for any distinct  $x_1$  and  $x_2$  in  $V$ ,  $m$  is not a root of the difference of  $P_{D_0, f_l, x_1}$  and  $P_{D_0, f_l, x_2}$ . Each of these differences has at most  $g$  roots, and as there are less than  $(\#\mathbb{F})^4$  such differences, there are at most  $g \cdot (\#\mathbb{F})^4$  integers to avoid. So there is a suitable  $m$  with  $0 \leq m \leq g \cdot (\#\mathbb{F})^4$ . Composing our maps, we obtain a generator for the  $\mathbb{Q}(\zeta_l)$ -algebra  $A_{l, \mathbb{Q}(\zeta_l)}$  associated with  $V$ :

$$a_{D_0, f_l, m}: V \rightarrow \overline{\mathbb{Q}}, \quad x \mapsto P_{D_0, f_l, x}(m).$$

We let:

$$(8.2.10) \quad P_{D_0, f_l, m} := \prod_{x \in V} (T - a_{D_0, f_l, m}(x)) \quad \text{in } \mathbb{Q}(\zeta_l)[T]$$

be the minimal polynomial over  $\mathbb{Q}(\zeta_l)$  of  $a_{D_0, f_l, m}$ .





## Chapter Nine

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### Applying Arakelov theory

B. Edixhoven and R. de Jong

In this chapter we start applying Arakelov theory in order to derive a bound for the height of the coefficients of the polynomials  $P_{D_0, f_l, m}$  as in (8.2.10). We proceed in a few steps. The first step, taken in Section 9.1, is to relate the height of the  $b_l(Q_{x,i})$  as in Section 8.2 to intersection numbers on  $X_l$ . The second step, taken in Section 9.2, is to get some control on the difference of the divisors  $D_0$  and  $D_x$  as in (3.4). Certain intersection numbers concerning this difference are bounded in Theorem 9.2.5, in terms of a number of invariants in the Arakelov theory on modular curves  $X_l$ . These invariants will then be bounded in terms of  $l$  in Sections 11.1, 11.2, and 11.3. Finally, in Section 11.7, the height of the coefficients of the  $P_{D_0, f_l, m}$  will be bounded. In this chapter, we do our best to formulate the most important results, Theorem 9.1.3, Theorem 9.2.1, and Theorem 9.2.5 in the context of curves over number fields, i.e., outside the context of modular curves.

#### 9.1 RELATING HEIGHTS TO INTERSECTION NUMBERS

We pick up the notation as at the end of Section 8.2, so we have a modular curve  $X_l = X_1(5l)_{\mathbb{Q}}$  with  $l > 5$ , non-constant morphisms  $b_l$  and  $x'_l: X_l \rightarrow \mathbb{P}_{\mathbb{Q}}^1$ , and certain divisors  $D_x^{\text{fin}} = \sum_{i=1}^{d_x} Q_{x,i}$  on  $X_{l, \overline{\mathbb{Q}}}$  that have support outside the cusps. It is our objective in this subsection to link the absolute height  $h(b_l(Q_{x,i}))$  of the algebraic numbers  $b_l(Q_{x,i})$  to certain quantities coming from Arakelov intersection theory. The height of  $x'_l(Q_{x,i})$  will be bounded in terms of  $h(b_l(Q_{x,i}))$  in Section 11.5. The final estimates for these heights, depending only on  $l$ , will be done in Section 11.7.

**9.1.1 Theorem** *Let  $x$  be in  $V$ , and let  $i$  be in  $\{1, \dots, d_x\}$ . Let  $K$  be a number field containing  $\mathbb{Q}(\zeta_{5l})$  and such that  $Q_{x,i}$  is defined over  $K$ . Let  $\mathcal{X}$  be the minimal regular model of  $X_l$  over  $K$ . Then we have:*

$$h(b_l(Q_{x,i})) \leq \frac{1}{[K : \mathbb{Q}]} \left( (Q_{x,i}, b_l^* \infty)_{\mathcal{X}} + l^2 \sum_{\sigma} \sup_{X_{\sigma}} g_{\sigma} + \frac{1}{2} \sum_{\sigma} \int_{X_{\sigma}} \log(|b_l|^2 + 1) \mu_{X_{\sigma}} \right) + \frac{1}{2} \log 2.$$

Here  $\sigma$  runs through the embeddings of  $K$  into  $\mathbb{C}$  and  $g_{\sigma}$  is the Arakelov-Green function on  $X_{l,\sigma}$ .

In the next chapters we shall derive bounds that are polynomial in  $l$  for all terms in the above estimate.

Theorem 9.1.1 will be derived from Theorem 9.1.3 below, which states a fairly general result. We start with a definition. Let  $K$  be a number field and consider  $\mathbb{P}_{O_K}^1$ . Let  $\infty$  denote the  $O_K$ -point  $(1 : 0)$  of  $\mathbb{P}_{O_K}^1$ . For any section  $P$  in  $\mathbb{P}_{O_K}^1(O_K)$  we define by  $(P, \infty)_{\mathbb{P}^1}$  the degree (see (4.3.1)) of  $P^*O_{\mathbb{P}^1}(\infty)$ , where  $O_{\mathbb{P}^1}(\infty)$  has the Fubini-Study metric, i.e. the metric  $\|\cdot\|_{\mathbb{P}^1}$  given by:

$$(9.1.2) \quad \|1\|_{\mathbb{P}^1}(x_0 : x_1) := \frac{|x_1|}{(|x_0|^2 + |x_1|^2)^{1/2}}$$

over  $\mathbb{P}_{\mathbb{C}}^1$ . Here 1 is the tautological section of  $O_{\mathbb{P}^1}(\infty)$ .

**9.1.3 Theorem** *Let  $X$  be a geometrically irreducible, smooth and complete curve of positive genus over a number field  $K$  and let  $\mathcal{X}$  be a proper semi-stable model of  $X$  over the ring of integers  $O_K$  of  $K$ . Suppose that we have a non-constant morphism  $f : X \rightarrow \mathbb{P}_K^1$  and a  $K$ -rational point  $Q$  of  $X$  with  $f(Q) \neq \infty$ . Assume the following: the Zariski closure of  $\text{Supp}(\text{div}(f)_+) \cup \text{Supp}(\text{div}(f)_-)$  in  $\mathcal{X}$  is étale over  $O_K$ . For any closed point  $s$  of  $\text{Spec}(O_K)$ , denote by  $m_s(f)$  the supremum of the multiplicities of  $\text{div}(f)_-$  on  $\mathcal{X}$  along the irreducible components of the fibre at  $s$  of  $\mathcal{X}$ .*

Then we have the inequality:

$$(f(Q), \infty)_{\mathbb{P}^1} \leq (Q, f^* \infty)_{\mathcal{X}} + \deg f \sum_{\sigma} \sup_{X_{\sigma}} g_{\sigma} \\ + (1/2) \sum_{\sigma} \int_{X_{\sigma}} \log(|f|^2 + 1) \mu_{X_{\sigma}} + \sum_s m_s(f) \log \#k(s).$$

Here the first sum runs over the embeddings of  $K$  into  $\mathbb{C}$ , and the last sum runs over the closed points of  $\text{Spec}(O_K)$ .

**Proof** Note that the locus of indeterminacy of  $f$  on  $\mathcal{X}$  consists of finitely many closed points. This implies that there exists a blow-up  $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$  of  $\mathcal{X}$  such that  $f$  extends to a regular map  $f: \tilde{\mathcal{X}} \rightarrow \mathbb{P}_{O_K}^1$ . For any such  $\tilde{\mathcal{X}}$  we have by construction:

$$(f(Q), \infty)_{\mathbb{P}^1} = \deg f(Q)^*(O_{\mathbb{P}^1}(\infty), \|\cdot\|_{\mathbb{P}^1}) \\ = \deg Q^* f^*(O_{\mathbb{P}^1}(\infty), \|\cdot\|_{\mathbb{P}^1}) \\ = \deg Q^*(O_{\tilde{\mathcal{X}}}(f^* \overline{\infty}), \|\cdot\|_{\mathbb{P}^1}),$$

where we write  $\overline{\infty}$  to emphasise that  $f^* \overline{\infty}$  is the inverse image under  $f$  of  $\infty(\text{Spec}(O_K))$ , and not the Zariski closure  $\overline{f^* \infty}$  of the inverse image under  $f$  of  $\infty(\text{Spec}(K))$ . If we let  $\|\cdot\|_X$  denote the canonical Arakelov metric on  $O_X(f^* \infty)$  then we can write:

$$\deg Q^*(O_{\tilde{\mathcal{X}}}(f^* \overline{\infty}), \|\cdot\|_{\mathbb{P}^1}) = \deg Q^*(O_{\tilde{\mathcal{X}}}(f^* \overline{\infty}), \|\cdot\|_X \cdot \frac{\|\cdot\|_{\mathbb{P}^1}}{\|\cdot\|_X}) \\ = \deg Q^*(O_{\tilde{\mathcal{X}}}(f^* \overline{\infty}), \|\cdot\|_X) \\ - \sum_{\sigma} \log\left(\frac{\|\cdot\|_{\mathbb{P}^1}}{\|\cdot\|_X}(Q_{\sigma})\right) \\ = (Q, f^* \overline{\infty})_{\tilde{\mathcal{X}}} + \sum_{\sigma} \log\left(\frac{\|\cdot\|_X}{\|\cdot\|_{\mathbb{P}^1}}(Q_{\sigma})\right).$$

A bound for  $\log((\|\cdot\|_X / \|\cdot\|_{\mathbb{P}^1})(Q_{\sigma}))$  follows by testing on the tautological section 1, giving:

$$\log \|1\|_X(Q_{\sigma}) - \log \|1\|_{\mathbb{P}^1}(Q_{\sigma}) = g_{\sigma}(f^* \infty, Q_{\sigma}) + \frac{1}{2} \log(|f(Q_{\sigma})|^2 + 1).$$

Applying Proposition 9.1.4 below at this point shows that:

$$g_\sigma(f^*\infty, Q_\sigma) + \frac{1}{2} \log(|f(Q_\sigma)|^2 + 1) \leq (\deg f) \sup_{X_\sigma} g_\sigma + \frac{1}{2} \int_{X_\sigma} \log(|f|^2 + 1) \mu_{X_\sigma}.$$

This accounts for the second and third terms in the bound of the theorem. We are finished once we prove that  $(Q, f^*\infty)_{\tilde{\mathcal{X}}} - (Q, \overline{f^*\infty})_{\mathcal{X}}$  is bounded by  $\sum_s m_s(f) \log \#k(s)$  for a particular choice of  $\tilde{\mathcal{X}}$ . (The usual projection formula shows that in fact  $(Q, f^*\infty)_{\tilde{\mathcal{X}}}$  is independent of the choice of  $\tilde{\mathcal{X}}$ .) On any  $\tilde{\mathcal{X}}$  we write  $f^*\infty$  as a sum  $f^*\infty = (f^*\infty)_{\text{hor}} + (f^*\infty)_{\text{vert}}$  of a horizontal and a vertical part. Note that  $(f^*\infty)_{\text{hor}} = \overline{f^*\infty}$ , with the Zariski closure now taken in  $\tilde{\mathcal{X}}$ . Since the local intersection multiplicities of  $Q$  and  $\overline{f^*\infty}$  do not go up when passing from  $\mathcal{X}$  to  $\tilde{\mathcal{X}}$ , we have  $(Q, (f^*\infty)_{\text{hor}})_{\tilde{\mathcal{X}}} = (Q, \overline{f^*\infty})_{\tilde{\mathcal{X}}} \leq (Q, \overline{f^*\infty})_{\mathcal{X}}$  and thus we are reduced to proving that  $(Q, (f^*\infty)_{\text{vert}})_{\tilde{\mathcal{X}}}$  is bounded from above by  $\sum_s m_s(f) \log \#k(s)$  for a particular choice of  $\tilde{\mathcal{X}}$ .

We exhibit a specific blow-up, and we calculate which multiplicities  $f$  acquires along the irreducible components of the vertical fibres of this blow-up. Note that the locus of indeterminacy of  $f$  on  $\mathcal{X}$  consists precisely of the closed points of  $\mathcal{X}$  where an irreducible component  $C$  from the zero divisor  $\text{div}(f)_+$  of  $f$  on  $\mathcal{X}$  and an irreducible component  $C'$  from its polar divisor  $\text{div}(f)_-$  meet. Now since by assumption the Zariski closure of  $\text{Supp}(\text{div}(f)_+) \cup \text{Supp}(\text{div}(f)_-)$  in  $\mathcal{X}$  is étale over  $O_K$ , this can only happen when at least one of  $C, C'$  is vertical. In such points where this happens we have to perform a sequence of successive blowings-up until a component arises with multiplicity 0 for  $f$ , so that the components with positive multiplicities and the components with negative multiplicities are separated from each other.

We begin by observing that it will cause no harm if we pass to a finite extension  $K \rightarrow K'$ . Indeed, both the left hand side and the right hand side of the inequality that we wish to prove get multiplied by  $[K' : K]$  if we do this. Here is why: for the terms  $(f(Q), \infty)_{\mathbb{P}^1}$  and  $(Q, f^*\infty)$  the scaling by a factor  $[K' : K]$  follows from general properties of the Arakelov intersection product, cf. [Fal1], p. 404 for example. (Note that it is understood that

over  $K'$ , intersection products are taken on the minimal resolution of the pullback of the model  $\mathcal{X}$ .) That the term  $\sum_{\sigma} \sup_{X_{\sigma}} g_{\sigma}$  scales by a factor  $[K' : K]$  is obvious. Finally, fix a closed point  $s$  of  $\text{Spec}(O_K)$  and let  $s'$  be any closed point of  $\text{Spec}(O_{K'})$  above it. Denoting by  $e_{s'}$  the ramification index of  $s'$  over  $s$  and by  $f_{s'}$  the degree of the residue field extension of  $s'$  over  $s$ , we see that for any  $s'$  above  $s$ , the integer  $m_s(f)$  gets multiplied by  $e_{s'}$ , and the number  $\log \#k(s)$  gets multiplied by  $f_{s'}$ . Using that  $\sum_{s'} e_{s'} f_{s'} = [K' : K]$ , the sum running over the closed points  $s'$  above  $s$ , we see finally that also the term  $\sum_s m_s(f) \log \#k(s)$  gets multiplied by  $[K' : K]$ .

Starting with  $\mathcal{X}$  over  $O_K$ , we first do the following. Let  $x$  be a closed point on  $\mathcal{X}$  that is the intersection of a vertical component  $C$  and a horizontal component  $C'$  having non-zero multiplicities  $m$  and  $m'$  for  $f$  that have different sign. After blowing up in  $x$ , we obtain an exceptional divisor  $E$  whose multiplicity for  $f$  is  $m + m'$ . We have two distinguished points on  $E$ , one lying on the strict transform of  $C$ , and one lying on the strict transform of  $C'$ . At exactly one of them there is a sign change for the multiplicities, or  $m + m' = 0$ . If a sign change happens at the double point lying on the strict transform of  $C'$ , then we repeat the process. If a sign change happens at the double point lying on the strict transform of  $C$  or if  $m + m' = 0$ , we stop, and continue with a new point  $x'$ , if available.

We end up with a blow-up  $\mathcal{X}' \rightarrow \mathcal{X}$  such that an intersection of two irreducible components  $C, C'$  that have different sign in  $\text{div}(f)$  on  $\mathcal{X}'$  only occurs for  $C, C'$  both vertical. For this, we did not yet need to extend the ground field  $K$ . In order to continue, we note the following. Suppose that we have a closed point  $x$  on the model  $\mathcal{X}'$  of  $X$  over  $O_K$  which is a double point of a vertical fibre, and two irreducible components  $C, C'$  of that vertical fibre pass through  $x$ , having non-zero multiplicities  $m$  and  $m'$  for  $f$  that differ in sign. Assume that  $K \rightarrow K'$  is a Galois extension that ramifies over the image of  $x$  in  $\text{Spec}(O_K)$ , with a ramification index  $e$  that is a multiple of  $m - m'$ . In passing to the minimal resolution  $\tilde{\mathcal{X}}$  of  $\mathcal{X}'_{O_{K'}}$ , the point  $x$  gets replaced by a chain of  $e - 1$  projective lines of self-intersection  $-2$ . The multiplicities of  $f$  along these components change in  $e$  steps from  $em$  to  $em'$ , so that the steps are  $m - m'$  and a multiplicity 0 will appear somewhere, because  $m - m'$  is a divisor of  $em$ .

Thus we see how we can reach our goal: take a Galois extension  $K \rightarrow K'$  that ramifies as specified above over the images in  $\text{Spec}(O_K)$  of the double points where components meet with a different sign for  $f$ . (This is always possible.) By our remarks above, it suffices to prove the inequality over  $K'$ . By construction, the morphism  $f$  extends over the model  $\tilde{\mathcal{X}}$  that arises in this way. Moreover, it follows from the construction that for  $s'$  a closed point of  $\text{Spec}(O_{K'})$  and  $s$  its image in  $\text{Spec}(O_K)$  we have  $m_{s'}(f) \leq e_{s'} m_s(f)$ . Hence the sum of the local intersection numbers  $(Q, (f^*\infty)_{\text{vert}})_{s'}$  for all  $s'$  over  $s$  is bounded from above by  $[K' : K] m_s(f) \log \#k(s)$ . This is what we needed to prove.  $\square$

**9.1.4 Proposition** *Let  $f: X \rightarrow \mathbb{P}^1$  be a finite morphism of Riemann surfaces with  $X$  connected and of positive genus. Consider on  $X - f^{-1}\infty$  the function:*

$$h(x) = g(f^*\infty, x) + \frac{1}{2} \log(|f(x)|^2 + 1).$$

*Then  $h$  extends uniquely to an element of  $C^\infty(X)$ , also denoted  $h$ . For all  $x \in X$  we have:*

$$h(x) \leq (\sup_X g) \deg f + \frac{1}{2} \int_X \log(|f|^2 + 1) \mu_X.$$

**Proof** Let us first show that  $h$  extends to a  $C^\infty$  function on  $X$ . In fact, as the beginning of the proof of Theorem 9.1.3 indicates,  $h$  is the logarithm of the function  $x \mapsto \|\cdot\|_X(x)/\|\cdot\|_{\mathbb{P}^1}(x)$  that gives the quotient of two metrics on  $O_X(f^*\infty)$ : the Arakelov metric  $\|\cdot\|_X$  and the pullback of the Fubini-Study metric  $\|\cdot\|_{\mathbb{P}^1}$  (see 9.1.2). But then  $h$  is in  $C^\infty(X)$ . We have, by (4.4.4):

$$h(x) = \int_{y \in X} -g(x, y) \frac{1}{\pi i} (\partial \bar{\partial} h) y + \int_X h \mu_X.$$

For  $s$  any local holomorphic generator of  $O_X(f^*\infty)$  we have:

$$\frac{1}{\pi i} \partial \bar{\partial} h = \frac{1}{2\pi i} \partial \bar{\partial} \log(\|s\|_X^2) - \frac{1}{2\pi i} \partial \bar{\partial} \log(\|s\|_{\mathbb{P}^1}^2) = (\deg f) \mu_X - f^* \mu_{\mathbb{P}^1},$$

where  $\mu_{\mathbb{P}^1}$  is the curvature form of the Fubini-Study metric on  $O_{\mathbb{P}^1}(\infty)$ . Substituting this in the previous equality, and using that for all  $x$  in  $X$  we have  $\int_{y \in X} g(x, y) \mu_X(y) = 0$ , we get:

$$h(x) = \int_{y \in X} g(x, y) (f^* \mu_{\mathbb{P}^1}) y + \int_X \frac{1}{2} \log(|f|^2 + 1) \mu_X.$$

As  $\mu_{\mathbb{P}^1}$  defined as the curvature form of the Fubini-Study metric on  $O_{\mathbb{P}^1}(\infty)$ , we have  $\int_{\mathbb{P}^1} \mu_{\mathbb{P}^1} = \deg(O_{\mathbb{P}^1}(\infty)) = 1$ . As the metric is invariant under the transitive action of  $SU_2$ ,  $\mu_{\mathbb{P}^1}$  is everywhere positive. In fact, one can compute that  $\mu_{\mathbb{P}^1} = (i/2\pi) dz d\bar{z}/(1 + |z|^2)^2$ . We end up with:

$$h(x) \leq (\sup_X g) \deg f + \frac{1}{2} \int_X \log(|f|^2 + 1) \mu_X$$

as required. □

**Proof** [Proof of Theorem 9.1.1] In order to simplify our notation, we drop the subscript  $l$  in  $b_l$ . For  $P$  in  $\mathbb{P}^1(O_K)$  we put:

$$h'(P) = (P, \infty)/[K : \mathbb{Q}].$$

Then we have  $h(P) \leq h'(P) + (1/2) \log 2$ , as for all  $x \in \mathbb{C}^2$  we have  $|x_1|^2 + |x_2|^2 \leq 2 \max\{|x_1|, |x_2|\}^2$ . In order to bound  $h'(b(Q_{x,i}))$  from above we want to apply Theorem 9.1.3. It follows from the definition of the morphism  $b$  that both the zero divisor  $\text{div}(b)_+$  and the polar divisor  $\text{div}(b)_-$  of  $b$  on  $X_l$  have as their support only  $K$ -rational closed points, namely, cusps. In particular, we never have  $b(Q_{x,i}) = \infty$ , by construction of  $D''_x$ . We have also seen that the Zariski closure in  $\mathcal{X}$  of  $\text{Supp}(\text{Cusps})$  is étale over  $O_K$  (as  $O_K$ -valued points the cusps are disjoint), and hence the same holds for the Zariski closure in  $\mathcal{X}$  of  $\text{Supp}(\text{div}(b)_+) \cup \text{Supp}(\text{div}(b)_-)$ . Theorem 9.1.3 now gives us that:

$$\begin{aligned} [K : \mathbb{Q}] \cdot h'(b(Q_{x,i})) &\leq (Q_{x,i}, b^* \infty)_{\mathcal{X}} + \deg b \sum_{\sigma} \sup_{X_{\sigma}} g_{\sigma} \\ &+ (1/2) \sum_{\sigma} \int_{X_{\sigma}} \log(|b|^2 + 1) \mu_{X_{\sigma}} + \sum_s m_s(b) \log \#k(s) \end{aligned}$$

with  $m_s(b)$  the supremum of the multiplicities of  $\text{div}(b)_-$  on  $\mathcal{X}$  along the irreducible components of the fibres of  $\mathcal{X}$  at  $s$ . We are done if we can prove that  $\deg b$  is at most  $l^2$ , and that  $m_s(b) = 0$  for all  $s$ . The definition of  $b$  shows directly that its degree is  $l^2 - 1$  (it is the degree of the natural morphism from  $X_1(5l)$  to  $X_1(5)$ ). Let us now show that  $m_s(b) = 0$  for all  $s$ . For this we evidently need information on the divisor  $\text{div}(b)$  on  $\mathcal{X}$ .

We start with working over  $\mathbb{Q}(\zeta_{5l})$ . From the discussion in Section 8.1 we recall that there is a fine moduli scheme  $Y_1(5l)_{\mathbb{Z}[\zeta_{5l}]}$  over  $\mathbb{Z}[\zeta_{5l}]$  of elliptic curves with balanced level structure (terminology from [Ka-Ma]).



Let  $E(5l) \rightarrow Y_1(5l)_{\mathbb{Z}[\zeta_{5l}]}$  be the universal elliptic curve and let  $P_5, P_l$  be the tautological points of order 5 and  $l$ . From Proposition 8.2.8 we recall the elliptic curve  $E \rightarrow Y_1(5)'$  with  $Y_1(5)' = \text{Spec}(\mathbb{Z}[b, 1/\text{discr}(E)])$ . The elliptic curve  $E(5l) \rightarrow Y_1(5l)_{\mathbb{Z}[\zeta_{5l}, 1/5]}$  arises from  $E \rightarrow Y_1(5)'$  by a unique base change  $Y_1(5l)_{\mathbb{Z}[\zeta_{5l}, 1/5]} \rightarrow Y_1(5)'$ . This gives the regular function  $b$  on  $Y_1(5l)_{\mathbb{Z}[\zeta_{5l}, 1/5]}$ . As  $b$  is invertible on  $Y_1(5)'$  it is invertible on  $Y_1(5l)_{\mathbb{Z}[\zeta_{5l}, 1/5]}$ . We conclude that  $\text{div}(b)$  on  $X_1(5l)_{\mathbb{Z}[\zeta_{5l}]}$  is a certain linear combination of the irreducible components of the closed subschemes  $\text{Cusps}$  and  $X_1(5l)_{\mathbb{F}_5[\zeta_l]}$ . In order to find this linear combination, we examine the multiplicities of  $b$  along the irreducible components that we have isolated.

We start with the multiplicities along the irreducible components of the divisor  $\text{Cusps}$ . It is sufficient to study the situation over  $\mathbb{C}$ , and here we can make a beginning by looking at  $\text{div}(b)$  on  $X_1(5)_{\mathbb{C}}$ . From the equations in Proposition 8.2.8 we obtain that over the cusp 0 of  $X_0(5)_{\mathbb{C}}$  lie two cusps, say  $c_1$  and  $c_2$ , with  $c_1$ , say, corresponding to the 5-gon with the tautological point of order 5 being on a component adjacent to the connected component of 0, and the other,  $c_2$ , corresponding to the 5-gon with the tautological point of order 5 being on a component that is not adjacent to the connected component of 0. We have  $\text{div}(b) = \pm(c_1 - c_2)$  on  $X_1(5)_{\mathbb{C}}$ ; we could compute the exact sign but that is not important for us. The divisor of  $b$  on  $X_1(5l)_{\mathbb{C}}$  is obtained by pulling back its divisor on  $X_1(5)_{\mathbb{C}}$  via the forgetful map  $X_1(5l)_{\mathbb{C}} \rightarrow X_1(5)_{\mathbb{C}}$ . Hence, pulling back the divisor  $c_1 - c_2$  we get plus or minus the divisor of  $b$  on  $X_1(5l)_{\mathbb{C}}$ ; the multiplicities are just the ramification indices above the cusps  $c_1$  and  $c_2$ . Since these are in  $\{1, l\}$ , we obtain that the multiplicities of  $b$  along the irreducible components of  $\text{Cusps}$  are just 1 or  $l$  in absolute value.

Next we calculate the multiplicities of  $b$  along the irreducible components of  $X_1(5l)_{\mathbb{F}_5[\zeta_l]}$ . The structure of a connected component of the scheme  $X_1(5l)_{\mathbb{F}_5[\zeta_l]}$  is as follows: it consists of two irreducible components, one on which  $P_5$  has order 1, and one having an open part where  $P_5$  has order 5. These two irreducible components intersect (transversally) in the supersingular points.

We denote by  $\Gamma$  the union of the irreducible components over  $\mathbb{F}_5$  on which  $P_5$  has order 1. The construction of the scheme  $Y_1(5)'$  immediately gives

us a forgetful map  $X_1(5l)_{\mathbb{Z}[\zeta_{5l}]} - \Gamma - \text{Supp}(\text{Cusps}) \rightarrow Y_1(5)'$ . Since  $b$  is invertible on  $Y_1(5)'$ , the same holds for  $b$  along the irreducible components of  $X_1(5l)_{\mathbb{F}_5[\zeta_l]}$ , except possibly for the irreducible components in  $\Gamma$ . But the multiplicity of  $b$  along such an irreducible component is then also zero, as can be seen by the following argument. Let  $C \cup C'$  be a connected component of  $X_1(5l)_{\mathbb{F}_5[\zeta_l]}$ , with the irreducible component  $C$  corresponding to  $P_5$  having order 1. All the horizontal components of  $\text{div}(b)$  on  $X_1(5l)_{\mathbb{Z}[\zeta_{5l}]}$  specialise to  $C'$ . We know that  $b$  has multiplicity 0 along  $C'$  and hence it restricts to a non-trivial rational function, also denoted  $b$ , on  $C'$ . The degree of  $b$  on  $C'$  is zero, or equivalently  $m(C', C) + (C', \text{div}(b)_{\text{hor}}) = 0$ , where  $m$  is the multiplicity of  $b$  along  $C$ . Now, since  $(C', \text{div}(b)_{\text{hor}})$  is zero and  $(C', C)$  isn't, we get  $m = 0$ .

All in all we conclude that the absolute values of the multiplicities of the irreducible components in  $\text{div}(b)$  on  $X_1(5l)_{\mathbb{Z}[\zeta_{5l}]}$  are bounded by a constant times  $l$ , and that all multiplicities along irreducible components of fibers over closed points  $s$  of  $\text{Spec}(\mathbb{Z}[\zeta_{5l}])$  are zero. In particular, for all closed points  $s$  we have  $m_s(b) = 0$ . This implies in fact that the rational function  $b$  on  $X_1(5l)_{\mathbb{Z}[\zeta_{5l}]}$  extends to a morphism to  $\mathbb{P}_{\mathbb{Z}}^1$ . As this is a useful fact, we record it in a Proposition. This completes the proof of Theorem 9.1.1.  $\square$

**9.1.5 Proposition** *Let  $l > 5$  be prime a prime number. The rational function  $b_l$  on  $X_1(5l)_{\mathbb{Z}[\zeta_{5l}]}$  from Proposition 8.2.9 extends to a morphism to  $\mathbb{P}_{\mathbb{Z}}^1$ .*

## 9.2 CONTROLLING $D_x - D_0$

In this subsection, the hypotheses are as follows (unless stated otherwise). We let  $K$  be a number field,  $O_K$  its ring of integers,  $B := \text{Spec}(O_K)$ ,  $p: \mathcal{X} \rightarrow B$  a regular, split semi-stable curve over  $B$  whose generic fibre  $X \rightarrow \text{Spec } K$  is geometrically irreducible and of genus  $g \geq 1$ . We let  $D$  be the closure in  $\mathcal{X}$  of an effective divisor of degree  $g$  (also denoted  $D$ ) on  $X$ . We let  $x$  be a  $K$ -rational torsion point of the Jacobian of  $X$ , i.e., a torsion element of  $\text{Pic}(X)$ , which has the property that there is a unique effective divisor  $D_x$  on  $X$  such that  $x = [D_x - D]$ . Finally, we let  $P: B \rightarrow \mathcal{X}$  be a section of  $p$ , i.e., an element of  $\mathcal{X}(B)$ .

We denote by  $\Phi_{x,P}$  the unique finite vertical *fractional* divisor  $\Phi$  (i.e.,

with rational coefficients that are not necessarily integral) on  $\mathcal{X}$  such that  $(D_x - D - \Phi, C) = 0$  for all irreducible components  $C$  of fibres of  $p$ , and such that  $P(B)$  is disjoint from the support of  $\Phi$ . It is not difficult to see that a  $\Phi$  satisfying the first condition exists and that it is unique up to adding multiples of fibres of  $p$  (the intersection pairing restricted to the divisors with support in a fibre is negative semi-definite); see Lemme 6.14.1 of [Mor2]. The second condition removes the ambiguity of adding multiples of fibres.

We denote by  $\delta_s$  the number of singular points in the geometric fibre at a closed point  $s$  of  $B$ .

**9.2.1 Theorem** *The  $O_B$ -module  $R^1 p_* O_{\mathcal{X}}(D_x)$  is a torsion module on  $B$ , and we have:*

$$\begin{aligned} & (D_x, P) + \log \# R^1 p_* O_{\mathcal{X}}(D_x) + \frac{1}{8} (\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) + \frac{1}{8} \sum_s \delta_s \log \# k(s) \\ &= (D, P) - \frac{1}{2} (D + \Phi_{x,P}, D + \Phi_{x,P} - \omega_{\mathcal{X}/B}) + \frac{1}{2} \deg \det p_* \omega_{\mathcal{X}/B} \\ & \quad + \sum_{\sigma} \int_{X_{\sigma}} \log \|\vartheta\| (D_x^{\sigma} - Q) \cdot \mu_{\sigma}(Q) + \frac{g}{2} [K : \mathbb{Q}] \log(2\pi). \end{aligned}$$

Here  $s$  runs over the closed points of  $B$ , and  $\sigma$  runs through the complex embeddings of  $K$ .

We derive Theorem 9.2.1 from three lemmas. For the moment we work in  $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\text{Pic}}(\mathcal{X})$ .

**9.2.2 Lemma** *The admissible line bundles:*

$$O_{\mathcal{X}}(D_x - D) \otimes p^* P^* O_{\mathcal{X}}(D_x - D)^{\vee} \quad \text{and} \quad O_{\mathcal{X}}(\Phi_{x,P})$$

are numerically equivalent. That is, for any admissible line bundle  $F$  on  $\mathcal{X}$  we have:

$$(O_{\mathcal{X}}(D_x - D) \otimes p^* P^* O_{\mathcal{X}}(D_x - D)^{\vee}, F) = (O_{\mathcal{X}}(\Phi_{x,P}), F).$$

**Proof** In this proof we just write  $\Phi$  for  $\Phi_{x,P}$ . We denote the first line bundle in the lemma by  $\Psi$ . Since  $D_x - D$  is torsion, there is a positive integer  $N$  such that  $\Psi^{\otimes N}$  is trivial on the generic fibre as a classical line bundle (that is, without taking the metrics into account). We have a canonical isomorphism  $P^* \Psi \xrightarrow{\sim} O_B$  on  $B$ . Combining, we find that  $\Psi^{\otimes N}$  has a rational

section  $s$  with  $\text{div}_{\mathcal{X}}(s)$  vertical and with  $P^*s \mapsto 1$ . The latter condition implies that  $P$  intersects to zero with  $\text{div}_{\mathcal{X}}(s)$  for the Arakelov intersection product. On the other hand, as  $p^*P^*O_{\mathcal{X}}(D_x - D)^{\vee}$  is trivial on the fibres of  $p$  over finite places of  $B$ , we have  $(N(D_x - D) - \text{div}_{\mathcal{X}}(s), C) = 0$  for all irreducible components  $C$  of fibres of  $p$ . Hence in fact  $\Phi = \frac{1}{N} \text{div}_{\mathcal{X}}(s)$ . To prove the lemma, it suffices now to prove that  $\Psi^{\otimes N} \xrightarrow{\sim} O_{\mathcal{X}}(\text{div}_{\mathcal{X}}(s))$  given by  $s \mapsto 1$ , with 1 the tautological section, is an isometry. Because of admissibility, it suffices to check that this is so when restricted to  $P$ ; but here we get the canonical isomorphism  $P^*\Psi \xrightarrow{\sim} O_B$ . This is indeed an isometry by the definition of  $\Psi$ .  $\square$

**9.2.3 Lemma** *Let  $X$  be a compact Riemann surface of genus  $g \geq 1$ . Let  $D$  be an effective divisor on  $X$  of degree  $g$  satisfying  $h^0(D) = 1$ . Then the determinant of cohomology  $\lambda(O_X(D))$  of  $D$  is identified with  $H^0(X, O_X(D))$ . Further, the formula:*

$$\log \|1\| + \frac{\delta(X)}{8} + \int_X \log \|\vartheta\|(D - Q) \cdot \mu_X(Q) = 0$$

*holds for the length (with respect to Faltings' metrisation of the determinant of cohomology) of the tautological section 1 of  $H^0(X, O_X(D))$ .*

**Proof** Since  $h^0(D) = 1$ ,  $H^0(X, O_X(D)) = \mathbb{C}$ . Therefore, the set of points  $Q$  on  $X$  such that  $h^0(D - Q) > 0$  is the support of  $D$ . Let  $Q$  be a point outside the support of  $D$ . Then  $h^0(D - Q) = 0$ . According to the axioms for the metrisation of the determinant of cohomology, the exact sequence:

$$0 \rightarrow O_X(D - Q) \rightarrow O_X(D) \rightarrow Q_*Q^*O_X(D) \rightarrow 0$$

gives rise to an isometry:

$$\begin{aligned} \lambda(O_X(D)) &\xrightarrow{\sim} \lambda(O_X(D - Q)) \otimes Q^*O_X(D) \cong \\ &\cong O(-\Theta)[O_X(D - Q)] \otimes Q^*O_X(D). \end{aligned}$$

Taking the norm on left and right of a tautological section we obtain, using (4.4.6):

$$\|1\| = \exp(-\delta(X)/8) \cdot \|\vartheta\|(D - Q)^{-1} \cdot G(D, Q),$$

where  $\log G(D, Q) = g_{D, \mu_X}(Q)$ . Taking logarithms and then integrating against  $\mu_X(Q)$  gives the result.  $\square$

**9.2.4 Lemma** (Noether formula) *We have:*

$$12 \deg \det p_* \omega_{\mathcal{X}/B} = (\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) + \sum_s \delta_s \log \#k(s) \\ + \sum_{\sigma} \delta(X_{\sigma}) - 4g[K : \mathbb{Q}] \log(2\pi),$$

*the first sum running over the closed points of  $B$ , the second sum running over the complex embeddings of  $K$ .*

**Proof** See [Fal1] and [Mor3].  $\square$

**Proof** [Proof of Theorem 9.2.1] We first show that  $R^1 p_* O_{\mathcal{X}}(D_x)$  is a torsion module. As it is a coherent  $O_B$ -module, it suffices to show that it is zero on the generic point of  $B$ , i.e., that  $H^1(X, O_X(D_x))$  is zero. By Riemann-Roch, we have  $h^0(D_x) - h^1(D_x) = 1$ . By definition of  $D_x$ , we have  $O_X(D_x) \cong \mathcal{L}_x(D)$ . And by construction of  $D$ , we have  $h^0(\mathcal{L}_x(D)) = 1$ . This shows that  $h^1(D_x) = 0$ .

Let us now prove the identity in Theorem 9.2.1. We start by noting that, by (4.4.7):

$$(D_x - D, P) = \deg P^* O_{\mathcal{X}}(D_x - D).$$

By Lemma 9.2.2,  $O_{\mathcal{X}}(D_x) \otimes p^* P^* O_{\mathcal{X}}(D_x - D)^{\vee}$  and  $O_{\mathcal{X}}(D + \Phi_{x,P})$  are numerically equivalent. The Riemann-Roch theorem then gives:

$$\deg \det R p_*(O_{\mathcal{X}}(D_x) \otimes p^* P^* O_{\mathcal{X}}(D_x - D)^{\vee}) = \\ = \frac{1}{2}(D + \Phi_{x,P}, D + \Phi_{x,P} - \omega_{\mathcal{X}/B}) + \deg \det p_* \omega_{\mathcal{X}/B}.$$

By the projection formula for the determinant of cohomology we can write the left-hand side as:

$$\deg \det R p_*(O_{\mathcal{X}}(D_x) \otimes p^* P^* O_{\mathcal{X}}(D_x - D)^{\vee}) = \\ = \deg \det R p_* O_{\mathcal{X}}(D_x) - \deg P^* O_{\mathcal{X}}(D_x - D)$$

Since  $p_*O_{\mathcal{X}}(D_x)$  is canonically trivialised by the function 1, the term  $\deg \det R p_* O_{\mathcal{X}}(D_x)$  can be computed as:

$$\deg \det R p_* O_{\mathcal{X}}(D_x) = - \sum_{\sigma} \log \|1\|_{\sigma} - \log \#R^1 p_* O_{\mathcal{X}}(D_x),$$

where for each complex embedding  $\sigma$ , the norm  $\|1\|_{\sigma}$  is the length of the tautological section 1 of  $\lambda(O_{X_{\sigma}}(D_x)) = H^0(O_{X_{\sigma}}(D_x))$ . By Lemma 9.2.3 we can then write:

$$\begin{aligned} \deg \det R p_* O_{\mathcal{X}}(D_x) &= \sum_{\sigma} \int_{X_{\sigma}} \log \|\vartheta\|_{\sigma}(D_x^{\sigma} - Q) \cdot \mu_{\sigma}(Q) \\ &\quad + \sum_{\sigma} \delta(X_{\sigma})/8 - \log \#R^1 p_* O_{\mathcal{X}}(D_x). \end{aligned}$$

Combining everything gives:

$$\begin{aligned} (D_x - D, P) &= -\frac{1}{2}(D + \Phi_{x,P}, D + \Phi_{x,P} - \omega_{\mathcal{X}/B}) - \deg \det p_* \omega_{\mathcal{X}/B} \\ &\quad + \sum_{\sigma} \int_{X_{\sigma}} \log \|\vartheta\|_{\sigma}(D_x^{\sigma} - Q) \cdot \mu_{\sigma}(Q) \\ &\quad + \sum_{\sigma} \delta(X_{\sigma})/8 - \log \#R^1 p_* O_{\mathcal{X}}(D_x). \end{aligned}$$

We obtain the required formula upon eliminating  $\sum_{\sigma} \delta(X_{\sigma})/8$  with the Noether formula, Lemma 9.2.4.  $\square$

**9.2.5 Theorem** *We have an upper bound:*

$$\begin{aligned} (D_x, P) + \log \#R^1 p_* O_{\mathcal{X}}(D_x) &\leq -\frac{1}{2}(D, D - \omega_{\mathcal{X}/B}) \\ &\quad + 2g^2 \sum_{s \in B} \delta_s \log \#k(s) \\ &\quad + \sum_{\sigma} \log \|\vartheta\|_{\sigma, \text{sup}} + \frac{g}{2}[K : \mathbb{Q}] \log(2\pi) \\ &\quad + \frac{1}{2} \deg \det p_* \omega_{\mathcal{X}/B} + (D, P), \end{aligned}$$

where  $s$  runs through the closed points of  $B$ , and where the supnorm  $\|\vartheta\|_{\sigma, \text{sup}}$  is taken over  $\text{Pic}^{g-1}(X_{\sigma})$ .

The required upper bound follows directly from Theorem 9.2.1 by using Lemma 9.2.6 below and the fact that  $(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) \geq 0$  (cf. Thm. 5 of [Fal1]).

**9.2.6 Lemma** *We have an upper bound:*

$$\begin{aligned} -\frac{1}{2}(D + \Phi_{x,P}, D + \Phi_{x,P} - \omega_{\mathcal{X}/B}) &\leq \\ &\leq -\frac{1}{2}(D, D - \omega_{\mathcal{X}/B}) + 2g^2 \sum_{s \in B} \delta_s \log \#k(s), \end{aligned}$$

with  $s$  running through the closed points of  $B$ .

**Proof** In this proof we just write  $\Phi$  for  $\Phi_{x,P}$ . By the definition of  $\Phi$ , we have  $(D_x - D - \Phi, \Phi) = 0$ , or in other words,  $(\Phi, \Phi) = (D_x - D, \Phi)$ . Using this we can write:

$$-\frac{1}{2}(D + \Phi, D + \Phi - \omega_{\mathcal{X}/B}) = -\frac{1}{2}(D, D - \omega_{\mathcal{X}/B}) + \frac{1}{2}(\Phi, \omega_{\mathcal{X}/B} - D - D_x).$$

We write  $\Phi = \sum_C \Phi(C) \cdot C$ , and for any finite fibre  $F_s$  of  $p$  we put  $A_s := \sup_C |\Phi(C)|$  with  $C$  running through the irreducible components of  $F_s$ . Since  $\omega_{\mathcal{X}/B}, D$  and  $D_x$  intersect any irreducible component  $C$  with non-negative multiplicity, we find

$$\begin{aligned} \frac{1}{2}(\Phi, \omega_{\mathcal{X}/B} - D - D_x) &\leq \frac{1}{2} \left( \sum_s A_s F_s, \omega_{\mathcal{X}/B} + D + D_x \right) \\ &\leq 2g \sum_s A_s \log \#k(s). \end{aligned}$$

We are going to prove that  $A_s \leq g\delta_s$ , and then we are done. So let  $s$  be a finite place of  $B$ . Let  $S_0$  be the set of irreducible components of  $F_s$ , and let  $S_1$  be the set of double points on  $F_s$ . Let  $\Gamma_s$  be the dual graph of  $F_s$  (thus, the set of vertices of  $\Gamma_s$  corresponds to  $S_0$ , the set of edges corresponds to  $S_1$ , and the graph is defined by the incidence relations). Choose an orientation on  $\Gamma_s$ . This gives rise to the usual source and target maps  $s$  and  $t: S_1 \rightarrow S_0$ . Consider the boundary and coboundary maps  $d_* = t_* - s_*: \mathbb{Q}^{S_1} \rightarrow \mathbb{Q}^{S_0}$  and  $d^* = t^* - s^*: \mathbb{Q}^{S_0} \rightarrow \mathbb{Q}^{S_1}$ . Then  $d_* d^*: \mathbb{Q}^{S_0} \rightarrow \mathbb{Q}^{S_0}$  is given by minus the intersection matrix of  $F_s$ . In particular, the map  $d_* d^*$  sends  $\Phi$  to the map  $u: C \mapsto -(\Phi, C) = (D - D_x, C)$ . The kernel of  $d_* d^*$  consists exactly of the constant functions, and the image consists of the orthogonal complement of the constant functions. Now consider the graph  $\Gamma_s$  as an electric circuit, where each edge has a resistance of 1 Ohm. By Ohm's law

and by spelling out the maps  $d_*$  and  $d^*$  we see that if we let at each vertex  $C$  a current of  $u(C)$  Ampère enter the circuit, subject to the condition that  $\sum_C u(C) = 0$ , the potentials  $\varphi(C)$  at each vertex  $C$  will be given, up to addition of a constant function, by a solution of the equation  $d_*d^*\varphi = u$ . Hence  $\Phi$  is the potential corresponding to the current  $C \mapsto (D - D_x, C)$ , normalised by the condition that  $\Phi(C_P) = 0$  with  $C_P$  the component that  $P$  specialises to. We must bound the  $|\Phi(C)|$  for  $C$  varying over  $S_0$ . The worst case that may happen is that  $\Gamma_s$  is a chain, with  $D'$  and  $D$  specialising entirely to the beginning and end point, respectively. In this case, the biggest potential difference is  $g \cdot (\#S_0 - 1)$  in absolute value, so that we arrive at  $|\Phi(C)| \leq g \cdot (\#S_0 - 1)$ . Now note that  $\Gamma_s$  is connected and that  $X/K$  has split semi-stable reduction. This gives  $\#S_0 - 1 \leq \delta_s$  and hence  $|\Phi(C)| \leq g\delta_s$ , as required.  $\square$





## Chapter Ten

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### An upper bound for Green functions on Riemann surfaces

F. Merkl

We begin with explaining the setup and the results of this subsection. Let  $X$  be a compact Riemann surface, endowed with a 2-form  $\mu \geq 0$  that fulfils  $\int_X \mu = 1$ . Let  $*$  denote rotation by  $90^\circ$  in the cotangential spaces (with respect to the holomorphic structure); in a coordinate  $z = x + iy$  this means  $*dx = dy$ ,  $*dy = -dx$  and, equivalently,  $*dz = -idz$ ,  $*d\bar{z} = id\bar{z}$ . In particular, the Laplace operator on real  $C^\infty$  functions on  $X$  can be written as  $d*d = 2i\partial\bar{\partial}$ .

For  $a, b \in X$ , let  $g_{a,b}: X - \{a, b\} \rightarrow \mathbb{R}$  be the (unique) solution on  $X$  (in the sense of distributions) of the following differential equation:

$$d*dg_{a,b} = \delta_a - \delta_b \quad \text{on } X$$

with the normalising condition:

$$\int_{X - \{a, b\}} g_{a,b} \mu = 0.$$

Note that  $g_{a,a} = 0$ . The distributional differential equation for  $g_{a,b}$  is equivalent to the following two more elementary conditions. Firstly,  $g_{a,b}$  is a real-valued harmonic function on  $X - \{a, b\}$ . Secondly, it has logarithmic singularities near  $a$  and near  $b$  of the following type: for any local coordinate  $z$  near  $a$ , the function  $P \mapsto g_{a,b}(P) - (2\pi)^{-1} \log |z(P) - z(a)|$  extends to a harmonic function in a neighbourhood of  $a$ , and for any local coordinate  $w$  near  $b$ , the function  $P \mapsto g_{a,b}(P) + (2\pi)^{-1} \log |w(P) - w(b)|$  extends to a harmonic function in a neighbourhood of  $b$ . The existence of such a function  $g_{a,b}$  is shown in the theorem in paragraph II.4.3 on page 49 in [Fa-Kr].

However, for  $a$  close to  $b$ , the proof of Lemma 10.4 below also shows the existence of  $g_{a,b}$  as a by-product. The difference of any two solutions of the differential equation for  $g_{a,b}$  extends to a global harmonic function on  $X$  and thus is a constant. Hence, the normalising condition for  $g_{a,b}$  determines the function  $g_{a,b}$  uniquely.

Now, for  $x \in X - \{a\}$ , let

$$g_{a,\mu}(x) := \int_{b \in X - \{x\}} g_{a,b}(x) \mu(b).$$

Then we have:

$$d*dg_{a,\mu}(x) = \delta_a - \mu$$

in the sense of distributions, and:

$$\int_{X - \{a\}} g_{a,\mu} \mu = 0.$$

We consider an atlas of  $X$  consisting of  $n$  local coordinates:

$$z^{(j)}: U^{(j)} \rightarrow \mathbb{C}, \quad j = 1, \dots, n,$$

such that each range  $z^{(j)}[U^{(j)}]$  contains the closed unit disk. For any radius  $0 < r \leq 1$  and  $j \in \{1, \dots, n\}$ , we define the disk:

$$U_r^{(j)} = \{P \in U^{(j)} \mid |z^{(j)}(P)| < r\}.$$

We fix a radius  $0 < r_1 < 1$  once and for all. Our aim is to prove the following result.

**10.1 Theorem** *Assume that the open sets  $U_{r_1}^{(j)}$  with  $j$  in  $\{1, \dots, n\}$  cover  $X$ . Next, assume that  $c_1$  is a positive real number such that for all  $j$  in  $\{1, \dots, n\}$  we have:*

$$\mu \leq c_1 |dz^{(j)} \wedge d\bar{z}^{(j)}| \quad \text{on } U_1^{(j)}.$$

*Finally, assume that for all  $j$  and  $k$  in  $\{1, \dots, n\}$ :*

$$\sup_{U_1^{(j)} \cap U_1^{(k)}} \left| \frac{dz^{(j)}}{dz^{(k)}} \right| \leq M$$

holds with some constant  $M \geq 1$ . Then for some positive constants  $c_7, c_9, c_{10}$  and  $c_{11}$ , depending only on  $r_1$ , we have, for all  $a$  in  $X$ :

$$(10.2) \quad g_{a,\mu} \leq n(c_{10} + c_1 c_{11} + c_7 \log M) + \frac{\log 2}{2\pi}$$

and, for all  $j$  such that  $a \in U_{r_1}^{(j)}$ :

$$(10.3) \quad \lim_{x \rightarrow a} \left| g_{a,\mu}(x) - \frac{1}{2\pi} \log |z^{(j)}(x) - z^{(j)}(a)| \right| \leq n(c_{10} + c_1 c_{11} + c_7 \log M) + \frac{\log M}{2\pi} + c_9.$$

We start by considering just one coordinate  $z = z^{(j)}$  for a fixed  $j$ . To simplify the notation in this section, we drop the superscript  $(j)$  in  $U = U^{(j)}$ ,  $z = z^{(j)}$ , and so on. We fix three radii  $0 < r_1 < r_2 < r_3 < 1$  once and for all. The radii  $r_2$  and  $r_3$  should depend only on  $r_1$ ; e.g.  $r_2 = (2r_1 + 1)/3$ ,  $r_3 = (r_1 + 2)/3$  is an admissible choice. Furthermore, we fix a partition of unity: let  $\chi: X \rightarrow [0, 1]$  be a  $C^\infty$  function which is compactly supported in the interior of  $U_1$  with  $\chi = 1$  on  $\overline{U_{r_2}}$ , and set  $\chi^c = 1 - \chi$ . More specifically, we take  $\chi = \tilde{\chi}(|z|)$  on  $U_1$  with a smooth function  $\tilde{\chi}: \mathbb{R} \rightarrow [0, 1]$  such that  $\tilde{\chi}(r) = 0$  for  $r \geq 1 - \epsilon$  with some  $\epsilon > 0$ , and  $\tilde{\chi}(r) = 1$  for  $r \leq r_2$ . The shape function  $\tilde{\chi}$  may be taken independently of  $X$  and the choice of the coordinate  $z$ , only depending on  $r_2$ .

We shall use the 2-norm of a (real valued) 1-form  $\omega$  over a measurable set  $Y \subseteq X$  defined by:

$$\|\omega\|_Y := \left( \int_Y \omega \wedge * \omega \right)^{1/2} = \left( 2i \int_Y \omega_{1,0} \wedge \omega_{0,1} \right)^{1/2},$$

where  $\omega = \omega_{1,0} + \omega_{0,1}$  is the decomposition of  $\omega$  in its components in  $T_{(1,0)}X$  and  $T_{(0,1)}X$ . In the case  $Y = X$ , we just write  $\|\omega\|$  for  $\|\omega\|_X$ .

Given  $a$  and  $b$  in  $U_{r_1}$ , we define the following function, having logarithmic singularities in  $a$  and  $b$ :

$$f_{a,b} := \frac{1}{2\pi} \log \left| \frac{(z - z(a))(\overline{z(a)}z - 1)}{(z - z(b))(\overline{z(b)}z - 1)} \right| \quad \text{on } U_1 - \{a, b\}.$$

Note that the singularities at  $1/\overline{z(a)}$  and  $1/\overline{z(b)}$  do not lie within the unit disk. We note that:

$$d*df_{a,b} = \delta_a - \delta_b$$

holds on  $U_1$  in the sense of distributions, and that  $f_{a,b}$  fulfils Neumann boundary conditions on  $\partial U_1$ . One can see this as follows. The meromorphic function on  $U_1$  given by:

$$P \mapsto \frac{(z(P) - z(a))(\overline{z(a)}z(P) - 1)}{(z(P) - z(b))(\overline{z(b)}z(P) - 1)} = \frac{(z(P) - z(a))(\frac{1}{z(P)} - \overline{z(a)})}{(z(P) - z(b))(\frac{1}{z(P)} - \overline{z(b)})}$$

takes positive real values on  $\partial U_1$ . Let  $\text{Log}$  denote the principal branch of the logarithm. The function

$$q_{a,b}(P) := \frac{1}{2\pi} \text{Log} \frac{(z(P) - z(a))(\overline{z(a)}z(P) - 1)}{(z(P) - z(b))(\overline{z(b)}z(P) - 1)}$$

is defined and holomorphic for  $P$  in a neighbourhood of  $\partial U_1$ , and it takes real values for  $P \in \partial U_1$ . As a consequence, the directional derivative of the imaginary part  $\Im q_{a,b}$  tangential to  $\partial U_1$  vanishes on  $\partial U_1$ . Using holomorphy, this implies that the directional derivative of the real part  $\Re q_{a,b}$  in normal direction to  $\partial U_1$  vanishes also on  $\partial U_1$ . Using  $\Re q_{a,b}(P) = f_{a,b}(P)$  for  $P$  in a neighbourhood of  $U_1$ , this proves the claimed Neumann boundary conditions for  $f_{a,b}$ .

Finally, for  $a \in U_{r_1}$  and  $P \in U_1 - \{a\}$ , we set:

$$l_a(P) := \frac{1}{2\pi} \chi(P) \log |z(P) - z(a)|,$$

extended by 0 to  $X - \{a\}$ .

Our first step in the proof of Theorem 10.1 is the following key lemma.

**10.4 Lemma** *For  $a$  and  $b$  in  $U_{r_1}$ , the supremum  $\sup_X |g_{a,b} - l_a + l_b|$  is bounded by a constant  $c_2 = c_4 + c_1 c_5$ , with  $c_4, c_5$  depending only on  $r_1$ .*

**10.5 Remark** Note that  $g_{a,b} - l_a + l_b$  has removable singularities at  $a$  and  $b$ , since the logarithmic singularities cancel. The constant  $c_2$  is uniform in the choice of the Riemann surface  $X$ , and uniform in the choice of  $a, b \in U_{r_1}$ . The choice of the coordinate  $z$  influences  $c_2$  only via the dependence of  $c_1$  on the choice of  $z$ . The radii  $r_2, r_3$  and the shape function  $\tilde{\chi}$  are viewed as  $r_1$ -dependent parameters; this is why we need not emphasise in the lemma that  $c_2$  also depends on these quantities.

**Proof** (of Lemma 10.4) We define the 2-form:

$$u_{a,b} := d*d(\chi^c f_{a,b}) \quad \text{on } U_1 - \overline{U_{r_1}}$$

and extend it by 0 to the whole surface  $X$ . Note that  $u_{a,b}$  is supported in  $\overline{U_1} - U_{r_2}$ , since  $\chi^c$  varies only there, and since  $f_{a,b}$  is harmonic. Consider the following variational principle on square integrable 1-forms  $\omega$ . We want to minimise  $\|\omega\|^2$  with the constraint:

$$d*\omega = u_{a,b}$$

in the sense of distributions. Writing the constraint with test functions, we see that the minimisation problem is taken over the following closed affine linear subspace of  $L^2(X, T^*X)$ :

$$V = \{\omega \in L^2(X, T^*X) : - \int_X dg \wedge *\omega = \int_X g u_{a,b} \text{ for all } g \in C^\infty(X)\}.$$

The space  $V$  is nonempty, since  $\tilde{\omega}_{a,b} \in V$  holds for the following 1-form:

$$\tilde{\omega}_{a,b} = \begin{cases} d(\chi^c f_{a,b}) & \text{on } U_1 - U_{r_1}, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, using Stokes' theorem, we have:

$$\begin{aligned} - \int_X dg \wedge *\tilde{\omega}_{a,b} &= - \int_{U_1} dg \wedge *\tilde{\omega}_{a,b} \\ &= - \int_{\partial U_1} g *\tilde{\omega}_{a,b} + \int_{U_1} g d*\tilde{\omega}_{a,b}. \end{aligned}$$

The first summand in the last expression vanishes by the Neumann boundary conditions of  $\chi^c f_{a,b} = f_{a,b}$  on  $\partial U_1$ , and the second summand equals:

$$\int_{U_1} g d*\tilde{\omega}_{a,b} = \int_X g u_{a,b}$$

by the definition of  $u_{a,b}$ .

Our minimisation problem has a unique solution  $\omega_{a,b} \in V$ . It fulfils:

$$(10.6) \quad \int_X \omega_{a,b} \wedge \sigma = 0 \quad \text{for all closed } C^\infty \text{ 1-forms } \sigma.$$

Indeed: if  $d\sigma = 0$ , then  $\omega_{a,b} + t * \sigma \in V$  holds for all  $t \in \mathbb{R}$ , since  $\omega_{a,b} \in V$  and  $d(*\sigma) = -d\sigma = 0$ . Thus:

$$0 = \left. \frac{d}{dt} \|\omega_{a,b} + t * \sigma\|^2 \right|_{t=0} = -2 \int_X \omega_{a,b} \wedge \sigma.$$

In particular:

$$\int_X \omega_{a,b} \wedge dg = 0$$

for all  $g \in C^\infty(X)$ , i.e.  $d\omega_{a,b} = 0$  in the sense of distributions. Since  $d*\omega_{a,b} = u_{a,b}$  and  $d\omega_{a,b} = 0$ , we get that  $\omega_{a,b}$  is smooth. This follows from (hypo-)elliptic regularity, as treated in Corollary 4.1.2 on page 101 in [Hor]. Precisely speaking, this corollary treats only the case of a single partial differential equation. However, as is mentioned at the end of Section 4.0 on page 97 of the reference, the extension of the result to systems of partial differential equations with as many equations as unknowns, as needed here, follows trivially. Then equation (10.6) implies that  $\omega_{a,b}$  is exact:

$$\omega_{a,b} = d\tilde{g}_{a,b}$$

for some  $\tilde{g}_{a,b} \in C^\infty(X)$ ; see for example [For], Corollary 19.13. We normalise  $\tilde{g}_{a,b}$  such that:

$$(10.7) \quad \int_X \tilde{g}_{a,b} \mu = 0,$$

to make it uniquely determined.

We set

$$h_{a,b} := \tilde{g}_{a,b} + \chi f_{a,b} - \int_X \chi f_{a,b} \mu.$$

We are now going to prove that  $d*dh_{a,b} = \delta_a - \delta_b$ . We claim that  $d*d(\chi f_{a,b}) = -u_{a,b} + \delta_a - \delta_b$  holds. We prove this equality separately on the three sets  $X - \text{Supp}\chi$ ,  $X - \text{Supp}\chi^c$ , and  $U_1 - \overline{U_{r_1}}$ , which cover  $X$ . The claimed equality holds on  $X - \text{Supp}\chi$ , because both sides vanish there. It holds also on  $X - \text{Supp}\chi^c$ , because there  $u_{a,b} = 0$  and  $d*d(\chi f_{a,b}) = d*df_{a,b} = \delta_a - \delta_b$  are valid. Finally, on  $U_1 - \overline{U_{r_1}}$ , the function  $f_{a,b}$  is harmonic, which implies that

$d*d(\chi f_{a,b}) = -d*d(\chi^c f_{a,b}) = -u_{a,b}$  on this annulus, which neither contains  $a$  nor  $b$ . Thus the claim  $d*d(\chi f_{a,b}) = -u_{a,b} + \delta_a - \delta_b$  holds in all cases.

Combining this with the fact  $d*d\tilde{g}_{a,b} = u_{a,b}$ , we conclude

$$d*dh_{a,b} = d*d\tilde{g}_{a,b} + d*d(\chi f_{a,b}) = u_{a,b} - u_{a,b} + \delta_a - \delta_b$$

and thus

$$d*dh_{a,b} = \delta_a - \delta_b.$$

Furthermore, using the normalisation  $\int_X \mu = 1$  and  $\int_X \tilde{g}_{a,b} \mu = 0$ , we observe

$$\int_X h_{a,b} \mu = 0.$$

Because  $g_{a,b}$  is uniquely characterised by its properties  $d*dg_{a,b} = \delta_a - \delta_b$  and  $\int_X g_{a,b} \mu = 0$ , we conclude  $g_{a,b} = h_{a,b}$ . Thus, we have shown

$$g_{a,b} = \tilde{g}_{a,b} + \chi f_{a,b} - \int_X \chi f_{a,b} \mu.$$

The function:

$$g_{a,b}^{(1)} = \tilde{g}_{a,b} + \chi f_{a,b}$$

is harmonic on  $X - \{a, b\}$ , and:

$$g_{a,b}^{(2)} = \tilde{g}_{a,b} - \chi^c f_{a,b}$$

is harmonic on  $U_1$ ; in particular both functions are harmonic on the annulus  $A := U_1 - \overline{U_{r_2}}$ . Now for every harmonic function  $g$  on  $A$ , we have a bound:

$$\max_{\partial U_{r_3}} g - \min_{\partial U_{r_3}} g \leq c_3 \|dg\|_A$$

with some positive constant  $c_3$  depending only on  $r_2$  and  $r_3$ ; note that the circle  $\partial U_{r_3}$  is relatively compact in the annulus  $A$ . We bound  $\|dg_{a,b}^{(2)}\|_A$  from above:

$$\|dg_{a,b}^{(2)}\|_A \leq \|d\tilde{g}_{a,b}\|_A + \|d(\chi^c f_{a,b})\|_A.$$



We estimate the first summand as follows, using that  $\omega_{a,b} = d\tilde{g}_{a,b}$  solves the above variational problem:

$$\|d\tilde{g}_{a,b}\|_A \leq \|d\tilde{g}_{a,b}\| = \|\omega_{a,b}\| \leq \|\tilde{\omega}_{a,b}\| = \|\tilde{\omega}_{a,b}\|_A = \|d(\chi^c f_{a,b})\|_A;$$

we used that  $\tilde{\omega}_{a,b}$  is supported in  $A$ . Thus we have:

$$\|dg_{a,b}^{(2)}\| \leq 2\|d(\chi^c f_{a,b})\|_A,$$

which is bounded by a constant, uniformly in  $a$  and  $b$  in  $U_{r_1}$ .

This also allows us to estimate  $g_{a,b}^{(1)}$ : on  $A$ , we know  $g_{a,b}^{(1)} = g_{a,b}^{(2)} + f_{a,b}$ , hence,

$$\|dg_{a,b}^{(1)}\|_A \leq \|dg_{a,b}^{(2)}\|_A + \|df_{a,b}\|_A \leq 2\|d(\chi^c f_{a,b})\|_A + \|df_{a,b}\|_A.$$

Both summands on the right hand side are bounded by constants, only depending on  $r_1$  and  $r_2$ , but uniformly in  $a$  and  $b$  in  $U_{r_1}$ . To summarise, we have shown that:

$$\max_{\partial U_{r_3}} g_{a,b}^{(j)} - \min_{\partial U_{r_3}} g_{a,b}^{(j)}$$

( $j = 1, 2$ ) are uniformly bounded by a constant depending only on  $r_1$ ,  $r_2$ , and  $r_3$ . However,  $g_{a,b}^{(1)}$  is harmonic on  $X - U_{r_3}$ , and  $g_{a,b}^{(2)}$  is harmonic on  $\overline{U_{r_3}}$ , which both have the same boundary  $\partial U_{r_3}$ . Thus, by the maximum principle:

$$\max_{X - U_{r_3}} g_{a,b}^{(1)} - \min_{X - U_{r_3}} g_{a,b}^{(1)} = \max_{\partial U_{r_3}} g_{a,b}^{(1)} - \min_{\partial U_{r_3}} g_{a,b}^{(1)}$$

and:

$$\max_{\overline{U_{r_3}}} g_{a,b}^{(2)} - \min_{\overline{U_{r_3}}} g_{a,b}^{(2)} = \max_{\partial U_{r_3}} g_{a,b}^{(2)} - \min_{\partial U_{r_3}} g_{a,b}^{(2)}.$$

Furthermore,  $\max_{X - U_{r_3}} |\chi f_{a,b}|$  and  $\max_{\overline{U_{r_3}}} |\chi^c f_{a,b}|$  are bounded, uniformly in  $a$  and  $b$  in  $U_{r_1}$ , by a constant only depending on  $r_1$  and  $r_3$ . Using  $\tilde{g}_{a,b} = g_{a,b}^{(1)} - \chi f_{a,b}$  on  $X - U_{r_3}$  and  $\tilde{g}_{a,b} = g_{a,b}^{(2)} + \chi^c f_{a,b}$  on  $\overline{U_{r_3}}$ , we conclude that  $\max_X \tilde{g}_{a,b} - \min_X \tilde{g}_{a,b}$  is bounded on  $X = (X - U_{r_3}) \cup \overline{U_{r_3}}$  by a constant  $c_6$  only depending on the radii  $r_1$ ,  $r_2$  and  $r_3$ . Using the normalisation condition (10.7), we know that:

$$\max_X \tilde{g}_{a,b} \geq 0 \geq \min_X \tilde{g}_{a,b}$$

holds; thus:

$$\max_X |\tilde{g}_{a,b}| \leq \max_X \tilde{g}_{a,b} - \min_X \tilde{g}_{a,b}$$

is also bounded by the same constant.

From this we get a bound for:

$$g_{a,b} - \chi f_{a,b} = \tilde{g}_{a,b} - \int_X \chi f_{a,b} \mu.$$

Indeed, we estimate:

$$\left| \int_X \chi f_{a,b} \mu \right| \leq \int_{U_1} |f_{a,b}| \mu \leq c_1 \int_{U_1} |f_{a,b}| dz \wedge d\bar{z},$$

which is uniformly bounded for  $a, b \in U_{r_1}$  by a constant  $c_1 c_5$  with  $c_5$  depending only on  $r_1$ ; note that the logarithmic singularities are integrable. Combining the bounds for  $\max_X |\tilde{g}_{a,b}|$  and  $\left| \int_X \chi f_{a,b} \mu \right|$ , we conclude that  $\sup_X |g_{a,b} - \chi f_{a,b}|$  is bounded by a constant  $c_6 + c_1 c_5$  with  $c_6, c_5$  depending on  $r_1$ . Since:

$$\sup_X |\chi f_{a,b} - l_a + l_b| = \frac{1}{2\pi} \sup_{U_1} \left| \chi \log \left| \frac{\overline{z(a)}z - 1}{z(b)z - 1} \right| \right|$$

is bounded, uniformly in  $a, b \in U_{r_1}$  and  $X$ , the key lemma follows (with  $c_4$  being the sum of  $c_6$  and the uniform upper bound last mentioned).  $\square$

**Proof** (of Theorem 10.1) Since we now work with varying coordinates, we include again the superscript coordinate index ( $j$ ) in the coordinate  $z^{(j)}$ , its domain  $U^{(j)}$ , but also in  $U_r^{(j)}$ ,  $\chi^{(j)}$ , and  $l_a^{(j)}$ .

**10.8 Lemma** Consider two coordinates  $z^{(j)}$  and  $z^{(k)}$ , with  $k$  and  $j$  in  $\{1, \dots, n\}$ . Assume that  $x$  is in  $U_{r_1}^{(j)} \cap U_{r_1}^{(k)}$  and that  $y$  is in  $U_{r_2}^{(j)}$  with  $|z^{(j)}(y) - z^{(j)}(x)| < (r_2 - r_1)/M$ . Then  $y$  is in  $U_{r_2}^{(k)}$ .

**Proof** The intersection  $U_{r_1}^{(j)} \cap U_{r_1}^{(k)}$  is an open neighbourhood of  $x$ . Assume that there exists  $y \in \overline{U_{r_2}^{(j)}}$  with  $|z^{(j)}(y) - z^{(j)}(x)| < (r_2 - r_1)/M$  and  $y \notin U_{r_2}^{(k)}$ . Then there is also such a point  $y$  with minimal distance  $|z^{(j)}(y) - z^{(j)}(x)|$  from  $x$ , since  $\overline{U_{r_2}^{(j)}} - U_{r_2}^{(k)}$  is compact. For this point  $y$ , we conclude  $y \in \partial U_{r_2}^{(k)} \subseteq \overline{U_{r_2}^{(k)}}$ , and the straight line from  $x$  to  $y$  in the

$z^{(j)}$ -coordinate is contained in  $\overline{U}_{r_2}^{(j)} \cap \overline{U}_{r_2}^{(k)}$ . By the mean value theorem, we conclude  $|z^{(k)}(y) - z^{(k)}(x)| \leq M|z^{(j)}(y) - z^{(j)}(x)| < r_2 - r_1$ , hence  $|z^{(k)}(y)| < r_2$ , since  $|z^{(k)}(x)| \leq r_1$ . This contradicts  $y \in \partial U_{r_2}^{(k)}$ .  $\square$

We choose a smooth partition of unity  $\phi^{(j)} : X \rightarrow [0, 1]$ ,  $j = 1, \dots, n$ , such that  $\phi^{(j)}$  is supported in  $U_{r_1}^{(j)}$ . For  $a \in X$ , we set:

$$h_a := \sum_j \phi^{(j)}(a) l_a^{(j)}.$$

**10.9 Lemma** *Let  $a \in U_{r_1}^{(k)}$ ,  $y \in X$ ,  $y \neq a$ . Then we have:*

$$l_a^{(k)}(y) \leq \frac{\log 2}{2\pi}.$$

**Proof** This follows immediately from the definition of the function  $l_a^{(k)}$ , since  $|z^{(k)}(y) - z^{(k)}(a)| \leq 2$  whenever  $y \in \text{Supp}(\chi^{(k)})$ .  $\square$

**10.10 Lemma** *For all  $a, b \in X$  we have the inequality:*

$$\sup_X |g_{a,b} - h_a + h_b| \leq n(c_{10} + c_1 c_5 + c_7 \log M)$$

with constants  $c_{10}$ ,  $c_5$ , and  $c_7$  depending only on  $r_1$ .

**Proof** We first show for  $a \in U_{r_1}^{(k)} \cap U_{r_1}^{(j)}$  that:

$$\sup_X |l_a^{(k)} - l_a^{(j)}| \leq \frac{1}{2\pi} [\log M + |\log(r_2 - r_1)| + \log 2].$$

To prove this, let  $y \in X$ . We distinguish 3 cases in order to prove that  $l_a^{(k)}(y) - l_a^{(j)}(y)$  is bounded from above by the right hand side.

**case 1:**

$y \in U_1^{(j)}$  with  $|z^{(j)}(y) - z^{(j)}(a)| < (r_2 - r_1)/M$ . In particular, we have  $|z^{(j)}(y)| < |z^{(j)}(a)| + (r_2 - r_1)/M \leq r_2$  (recall that  $M \geq 1$ ), hence  $a, y \in U_{r_2}^{(j)}$ . Consequently, the straight line  $[a, y]^{(j)}$  from  $a$  to  $y$  in the  $z^{(j)}$ -coordinate is contained in  $U_{r_2}^{(j)}$ . Then Lemma 10.8 implies that  $[a, y]^{(j)} \subseteq U_{r_2}^{(k)}$ . Using  $\chi^{(j)}(y) = \chi^{(k)}(y) = 1$ , we conclude by the mean value theorem that:

$$l_a^{(k)}(y) - l_a^{(j)}(y) = \frac{1}{2\pi} \log \left| \frac{z^{(k)}(y) - z^{(k)}(a)}{z^{(j)}(y) - z^{(j)}(a)} \right| \leq \frac{\log M}{2\pi},$$

which is bounded by the right hand side.

**case 2:**

$y \notin U_1^{(j)}$ . Then  $l_a^{(j)}(y) = 0$ , and we conclude, using Lemma 10.9, that:

$$l_a^{(k)}(y) - l_a^{(j)}(y) = l_a^{(k)}(y) \leq \frac{\log 2}{2\pi}.$$

**case 3:**

$y \in U_1^{(j)}$  and  $|z^{(j)}(y) - z^{(j)}(a)| \geq (r_2 - r_1)/M$ ; thus:

$$l_a^{(k)}(y) - l_a^{(j)}(y) \leq \frac{\log 2}{2\pi} - l_a^{(j)}(y) \leq \frac{1}{2\pi}(\log 2 - \chi^{(j)}(y) \log[(r_2 - r_1)/M]),$$

which is also bounded by the right hand side.

The upper bound for  $l_a^{(j)}(y) - l_a^{(k)}(y)$  in our claim is obtained by exchanging  $j$  and  $k$ . Thus the claim is proven.

We conclude:

$$\begin{aligned} |h_a - l_a^{(j)}| &\leq \sum_k \phi^{(k)}(a) |l_a^{(k)} - l_a^{(j)}| \leq \\ (10.11) \quad &\leq \frac{1}{2\pi} (\log M + |\log(r_2 - r_1)| + \log 2). \end{aligned}$$

Combining this with Lemma 10.4, we conclude for  $a, b \in U_{r_1}^{(j)}$ :

$$\begin{aligned} |g_{a,b} - h_a + h_b| &\leq |g_{a,b} - l_a^{(j)} + l_b^{(j)}| + |h_a - l_a^{(j)}| + |h_b - l_b^{(j)}| \\ &\leq c_{10} + c_1 c_5 + c_7 \log M \end{aligned}$$

with some constants  $c_{10}, c_5, c_7$  depending only on  $r_1$  (a possible choice is  $c_7 = (\log M)/\pi$  and  $c_{10} = (|\log(r_2 - r_1)| + \log 2)/\pi + c_4$ ).

Finally, for general  $a, b \in X$ , we choose a finite sequence of points  $a = a_0, a_1, \dots, a_m = b$  in  $X$  and indices  $j_1, \dots, j_m$  with  $m \leq n$  and  $a_{i-1}, a_i \in U_{r_1}^{(j_i)}$  for all  $i = 1, \dots, m$ . Using:

$$g_{a,b} = \sum_{i=1}^m g_{a_{i-1}, a_i},$$

we get by estimating:

$$|g_{a,b} - h_a + h_b| \leq \sum_{i=1}^m |g_{a_{i-1}, a_i} - h_{a_{i-1}} + h_{a_i}| \leq n(c_{10} + c_1 c_5 + c_7 \log M)$$

the claim of the lemma.  $\square$

We define:

$$h_\mu(x) := \int_{b \in X} h_b(x) \mu(b), \quad (x \in X).$$

**10.12 Lemma** *We have:*

$$\sup_X |h_\mu| \leq nc_1c_8,$$

*with some universal constant  $c_8$ . Furthermore, we have:*

$$\sup_{\substack{b, x \in X \\ b \neq x}} h_b(x) \leq \frac{\log 2}{2\pi}.$$

**Proof** We observe first that for all  $w \in \mathbb{C}$  with  $|w| \leq 1$  the integral:

$$\frac{1}{2\pi} \int_{|z| \leq 1} |\log |z - w|| |dz \wedge d\bar{z}|$$

is bounded from above by a universal constant  $c_8$ . We conclude that for all  $x \in X$  we have:

$$\begin{aligned} \int_{b \in U_{r_1}^{(j)}} |l_b^{(j)}(x)| \phi^{(j)}(b) \mu(b) &\leq c_1 \int_{b \in U_{r_1}^{(j)}} |l_b^{(j)}(x)| |dz^{(j)} \wedge d\bar{z}^{(j)}| \\ &\leq c_1c_8. \end{aligned}$$

Let  $x \in X$ . We get the first estimate:

$$|h_\mu(x)| \leq \sum_{j=1}^n \int_{U_1^{(j)}} |l_b^{(j)}(x)| \phi^{(j)}(b) \mu(b) \leq nc_1c_8.$$

Finally, the second estimate follows from Lemma 10.9:

$$h_b = \sum_{j=1}^n \phi^{(j)} l_b^{(j)} \leq \frac{\log 2}{2\pi},$$

as required. □

**10.13 Proposition** *For some positive constants  $c_{10}$ ,  $c_7$ , and  $c_{11}$  that depend only on  $r_1$  we have, uniformly in  $a$  and  $x \neq a$  on  $X$ :*

$$|g_{a,\mu}(x) - h_a(x)| \leq n(c_{10} + c_1c_{11} + c_7 \log M).$$

**Proof** Indeed, averaging Lemma 10.10 over  $b$  with respect to  $\mu$ , we obtain:

$$\sup_X |g_{a,\mu} - h_a + h_\mu| \leq n(c_{10} + c_1 c_5 + c_7 \log M).$$

By Lemma 10.12, one has  $|h_\mu| \leq n c_1 c_8$ . Combining gives what we want (we can take  $c_{11} = c_5 + c_8$ ).  $\square$

**10.14 Proposition** *Let  $c_{10}$ ,  $c_7$ , and  $c_{11}$  be as in Proposition 10.13, and let  $a$  be in  $X$ . Then  $\lim_{x \rightarrow a} |g_{a,\mu}(x) - h_a(x)|$  exists, and we have:*

$$\lim_{x \rightarrow a} |g_{a,\mu}(x) - h_a(x)| \leq n(c_{10} + c_1 c_{11} + c_7 \log M).$$

**Proof** The functions  $g_{a,\mu}$  and  $h_a$  have the same logarithmic singularity at  $a$ ; hence the limit exists. The estimate then follows from Proposition 10.13.  $\square$

We can now finish the proof of Theorem 10.1. We have seen in (10.11) that:

$$|h_a - l_a^{(j)}| \leq \frac{1}{2\pi} (\log M + |\log(r_2 - r_1)| + \log 2).$$

Combining this with Proposition 10.14 and using the definition of  $l_a^{(j)}$  gives the second estimate of the theorem. As to the first estimate, using:

$$g_{a,\mu} \leq h_a + |g_{a,\mu} - h_a|$$

we obtain it by applying the upper bound for  $h_a$  in Lemma 10.12 and the upper bound for  $|g_{a,\mu} - h_a|$  in Proposition 10.13. This ends the proof of Theorem 10.1.  $\square$



## Chapter Eleven

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### Bounds for Arakelov invariants of modular curves

B. Edixhoven and R. de Jong

In this chapter, we give bounds for all quantities on the right hand side in the inequality in Theorems 9.1.1 and 9.2.5, in the context of the modular curves  $X_1(5l)$  with  $l > 5$  prime, using the upper bounds for Green functions from the previous chapter. The final estimates are given in the last section.

#### 11.1 BOUNDING THE HEIGHT OF $X_1(pl)$

As before, for  $l > 5$  prime, we let  $X_l$  be the modular curve  $X_1(5l)$ , over a suitable base that will be clear from the notation. We let  $g_l$  denote the genus of  $X_l$ ; we have  $g_l > 1$ . A model  $X_{l,\mathbb{Z}}$  is given by [Ka-Ma], as well as a model  $X_{l,\mathbb{Z}[\zeta_{5l}]}$  that is semi-stable; see Chapter 8. The aim of this section is to prove a suitable bound for the stable Faltings height of  $X_l$  (see 4.4.10). We will in fact give such a bound for the modular curves  $X_1(pl)$  with  $p$  and  $l$  distinct primes. Before we get to that, we prove some intermediate results, that will also be important in the next section.

**11.1.1 Lemma** *Let  $N \geq 1$  be an integer, and let:*

$$\mathcal{B}_2(N) := \coprod_{M|N} \coprod_{d|(N/M)} B_{N,M,d}^* S_2(\Gamma_1(M))^{\text{new}}$$

*be the basis of  $S_2(\Gamma_1(N))$  obtained from newforms of levels dividing  $N$  as explained in (2.2.17). Let  $f = \sum_{n \geq 1} a_n(f)q^n$  be an element of  $\mathcal{B}_2(N)$ . Then we have for all  $n \geq 1$ :*

$$|a_n(f)| \leq 2n.$$



**Proof** As  $a_n(B_{N,M,d}^* f) = a_{n/d}(f)$  (see 2.2.16), it suffices to treat the case that  $f$  is a newform of some level  $M$  dividing  $N$ . We use the Weil bounds on the  $a_p(f)$  for all primes  $p$ . We recall from Section 1.8 of [De-Se] that we have an equality of formal Dirichlet series:

$$\sum_{n \geq 1} a_n(f) n^{-s} = \prod_{p|M} (1 - a_p(f) p^{-s})^{-1} \prod_{p \nmid M} (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}$$

with the following properties. For  $p \nmid M$  we have  $|\alpha_p| = |\beta_p| = \sqrt{p}$ . For  $p|M$  we have:

$$\begin{cases} a_p(f) = 0 & \text{if } p^2|M, \\ a_p(f) = 0 & \text{if } \varepsilon_f \text{ factors through } (\mathbb{Z}/(M/p)\mathbb{Z})^\times, \\ |a_p(f)| = p^{1/2} & \text{if } \varepsilon_f \text{ does not factor through } (\mathbb{Z}/(M/p)\mathbb{Z})^\times, \\ |a_p(f)| = 1 & \text{if } p^2|M \text{ and } \varepsilon_f \text{ factors through } (\mathbb{Z}/(M/p)\mathbb{Z})^\times. \end{cases}$$

Using that:

$$(1 - a_p(f) p^{-s})^{-1} = \sum_{k \geq 0} a_p(f)^k p^{-sk},$$

and that:

$$(1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1} = \left( \sum_{k \geq 0} \alpha_p^k p^{-sk} \right) \left( \sum_{k \geq 0} \beta_p^k p^{-sk} \right)$$

we find that for arbitrary  $n$  we have  $|a_n(f)| \leq \sigma_{0,M}(n) \sqrt{n}$ , where  $\sigma_{0,M}(n)$  is the number of positive divisors of  $n$  that are prime to  $M$ , and a simple estimate leads to  $|a_n(f)| \leq 2n$ .  $\square$

The following lemma states a very well known lower bound for the Petersson norm of a normalised cuspform.

**11.1.2 Lemma** *Let  $N \geq 1$  and let  $\omega = fdq/q$  be the holomorphic 1-form on  $X_1(N)(\mathbb{C})$  attached to a cusp form  $f = \sum_n a_n(f) q^n$  in  $S_2(\Gamma_1(N))$  with  $a_1(f) = 1$ . Then we have:*

$$\|\omega\|^2 = \frac{i}{2} \int_{X_1(N)} \omega \wedge \bar{\omega} \geq \pi e^{-4\pi}.$$

**Proof** We have  $\omega = \sum_{n \geq 1} a_n(f)q^n dq/q$  in the coordinate  $q = e^{2\pi iz}$ , where  $z$  is the standard coordinate on the upper half plane  $\mathbb{H}$ . If we let  $x$  and  $y$  be the real and imaginary parts of  $z$  we have:

$$\frac{i}{2}\omega \wedge \bar{\omega} = 4\pi^2 |f|^2 dx dy$$

Let  $F$  be the region in  $\mathbb{H}$  given by the conditions  $|x| < 1/2$  and  $y > 1$ . Then:

$$\begin{aligned} \|\omega\|^2 &\geq \int_F 4\pi^2 |f(z)|^2 dx dy \\ &= 4\pi^2 \sum_{m,n \geq 1} a_m(f) \overline{a_n(f)} \int_{-1/2}^{1/2} e^{2\pi i(m-n)x} \int_1^\infty e^{-2\pi(m+n)y} dy \\ &= 4\pi^2 \sum_{n \geq 1} |a_n(f)|^2 e^{-4\pi n} / 4\pi n. \end{aligned}$$

From the first term (note that  $a_1(f) = 1$ ) we obtain  $\|\omega\|^2 \geq \pi e^{-4\pi}$ . □

We now specialise to a slightly less special case than our curves  $X_l$ : the curves  $X_1(pl)$  with  $p$  and  $l$  two distinct prime numbers. We call an Atkin-Lehner basis for  $\Omega^1(X_1(pl))$  any basis of  $\Omega^1(X_1(pl))$  given by an ordering of the set  $\mathcal{B}_2(pl)$ . We start by describing, in a notation that is slightly different from the one used in (2.2.14), the degeneracy maps that are used for the definition of  $\mathcal{B}_2(pl)$ . This time, we call them source and target maps:

$$\begin{cases} s_l: X_1(pl) \rightarrow X_1(p), & (E, P, L) \mapsto (E, P) \\ t_l: X_1(pl) \rightarrow X_1(p), & (E, P, L) \mapsto (E/\langle L \rangle, P) \\ s_p: X_1(pl) \rightarrow X_1(l), & (E, P, L) \mapsto (E, L) \\ t_p: X_1(pl) \rightarrow X_1(l), & (E, P, L) \mapsto (E/\langle P \rangle, L) \end{cases}$$

where  $(E, P, L)$  denotes an elliptic curve  $E$  with a point  $P$  of order  $p$  and a point  $L$  of order  $l$ . Note that  $s_l$  and  $t_l$  have degree  $l^2 - 1$ , and that  $s_p$  and  $t_p$  have degree  $p^2 - 1$ . For any integer  $M \geq 1$  we denote by  $\Omega^1(X_1(M))^{\text{new}}$  the set of holomorphic 1-forms in  $\Omega^1(X_1(M))$  of the form  $f dq/q$  with  $f$  in  $S_2(X_1(M))^{\text{new}}$ . Our next goal is to get information on the Gram matrix of an Atkin-Lehner basis of  $\Omega^1(X_1(pl))$ . As described above, the contribution to  $\Omega^1(X_1(pl))$  of each  $f$  in  $S_2(\Gamma_1(pl))^{\text{new}}$  is the subspace  $\mathbb{C}f dq/q$ .

The contribution of an  $f$  in  $S_2(\Gamma_1(p))^{\text{new}}$  is the 2-dimensional space generated by  $s_l^*fdq/q$  and  $t_l^*fdq/q$ , and, of course, each  $f$  in  $S_2(\Gamma_1(l))^{\text{new}}$  contributes the 2-dimensional space generated by  $s_p^*fdq/q$  and  $t_p^*fdq/q$ .

**11.1.3 Lemma** *For  $f$  in  $S_2(\Gamma_1(l))^{\text{new}}$  and  $\omega = fdq/q$  we have:*

$$\begin{aligned}\langle s_p^*\omega, s_p^*\omega \rangle &= (p^2 - 1)\|\omega\|^2, \\ \langle t_p^*\omega, t_p^*\omega \rangle &= (p^2 - 1)\|\omega\|^2, \\ \langle s_p^*\omega, t_p^*\omega \rangle &= (p - 1)\overline{a_p(f)}\|\omega\|^2.\end{aligned}$$

We have similar equalities with  $p$  and  $l$  switched.

**Proof** The first two equalities are clear. As to the latter, note first that:

$$\begin{aligned}\langle s_p^*\omega, t_p^*\omega \rangle &= \frac{i}{2} \int_{X_1(pl)} s_p^*\omega \wedge \overline{t_p^*\omega} = \frac{i}{2} \int_{X_1(l)} s_{p,*}(s_p^*\omega \wedge \overline{t_p^*\omega}) = \\ &= \frac{i}{2} \int_{X_1(l)} \omega \wedge \overline{s_{p,*}t_p^*\omega}.\end{aligned}$$

Next note that  $s_p: X_1(pl) \rightarrow X_1(l)$  and  $t_p: X_1(pl) \rightarrow X_1(l)$  factor through the forget map  $X_1(pl) \rightarrow X_1(l; p)$  where the latter curve corresponds to the moduli problem  $(E, P, G)$  with  $P$  of order  $l$  and  $G$  a subgroup of order  $p$ . This forget map has degree  $p - 1$ , and the correspondence on  $X_1(l)$  induced by  $X_1(l; p)$  is the standard Hecke correspondence  $T_p$ . We find that  $s_{p,*}t_p^*\omega = (p - 1)T_p^*\omega$ . By the standard relation between eigenvalues and  $q$ -coefficients we have  $T_p^*\omega = a_p(f)\omega$ , so finally:

$$\langle s_p^*\omega, t_p^*\omega \rangle = \frac{i}{2} \int_{X_1(l)} \omega \wedge \overline{(p - 1)T_p^*\omega} = (p - 1)\overline{a_p(f)}\|\omega\|^2$$

as required. □

**11.1.4 Corollary** *Let  $p$  and  $l$  be two distinct primes. The structure of the Gram matrix  $(\langle \omega_i, \omega_j \rangle)_{i,j}$  of holomorphic 1-forms attached to an Atkin-Lehner basis for  $\Omega^1(X_1(pl))$  is as follows. Two subspaces attached to distinct elements of the union of  $S_2(\Gamma_1(pl))^{\text{new}}$ ,  $S_2(\Gamma_1(l))^{\text{new}}$  and  $S_2(\Gamma_1(p))^{\text{new}}$  are orthogonal to each other, hence the Gram matrix decomposes into blocks corresponding to these subspaces. The contribution of an*

element  $f$  in  $S_2(\Gamma_1(pl))^{\text{new}}$  is the 1-by-1 block  $\|fdq/q\|^2$ . The contribution of an element  $f$  in  $S_2(\Gamma_1(l))^{\text{new}}$  is the 2-by-2 block:

$$(p-1)\|fdq/q\|^2 \begin{pmatrix} p+1 & \overline{a_p(f)} \\ a_p(f) & p+1 \end{pmatrix},$$

where the norm  $\|fdq/q\|^2$  is taken on  $X_1(l)$ . The contribution of an element  $f$  in  $S_2(\Gamma_1(p))^{\text{new}}$  is the 2-by-2 block:

$$(l-1)\|fdq/q\|^2 \begin{pmatrix} l+1 & \overline{a_l(f)} \\ a_l(f) & l+1 \end{pmatrix},$$

where the norm  $\|fdq/q\|^2$  is taken on  $X_1(l)$ .

**11.1.5 Corollary** *The determinant of the Gram matrix of the holomorphic 1-forms attached to an Atkin-Lehner basis for  $\Omega^1(X_1(pl))$  is bounded below by  $(\pi e^{-4\pi})^g$ .*

**Proof** By the Weil-Ramanujan-Deligne bounds (or, in this case, the Weil bounds, as the weight of the modular forms here is two), the determinant of a 2-by-2 block as in Corollary 11.1.4 is bounded below by  $\|fdq/q\|^4$ . We obtain our corollary by invoking Lemma 11.1.2.  $\square$

**11.1.6 Corollary** *The Arakelov (1,1)-form  $\mu$  on  $X_1(pl)$  is given by:*

$$\begin{aligned} \mu = & \frac{i}{2g} \sum_{\omega} \frac{\omega \wedge \bar{\omega}}{\|\omega\|^2} + \frac{i}{2g} \sum_{\omega} \left( \frac{(p+1)s_p^*(\omega \wedge \bar{\omega}) + (p+1)t_p^*(\omega \wedge \bar{\omega})}{(p-1)\|\omega\|^2((p+1)^2 - |a_p(f_{\omega})|^2)} \right. \\ & \left. - \frac{a_p(f_{\omega})s_p^*\omega \wedge \overline{t_p^*\omega} + \overline{a_p(f_{\omega})}t_p^*\omega \wedge \overline{s_p^*\omega}}{(p-1)\|\omega\|^2((p+1)^2 - |a_p(f_{\omega})|^2)} \right) \\ & + \frac{i}{2g} \sum_{\omega} \left( \frac{(l+1)s_l^*\omega \wedge \overline{s_l^*\omega} + (l+1)t_l^*\omega \wedge \overline{t_l^*\omega}}{(l-1)\|\omega\|^2((l+1)^2 - |a_l(f_{\omega})|^2)} \right. \\ & \left. - \frac{a_l(f_{\omega})s_l^*\omega \wedge \overline{t_l^*\omega} + \overline{a_l(f_{\omega})}t_l^*\omega \wedge \overline{s_l^*\omega}}{(l-1)\|\omega\|^2((l+1)^2 - |a_l(f_{\omega})|^2)} \right), \end{aligned}$$

with the first sum running over  $\Omega^1(X_1(pl))^{\text{new}}$ , the second sum running over  $\Omega^1(X_1(l))^{\text{new}}$ , the third sum running over  $\Omega^1(X_1(p))^{\text{new}}$ , and where  $f_{\omega}$  is defined by  $\omega = f_{\omega}dq/q$ .

**Proof** Consider first an arbitrary compact Riemann surface  $X$  and let  $\omega = (\omega_1, \dots, \omega_g)$  be an arbitrary basis of  $\Omega^1(X)$ . Let  $a := \langle \omega, \omega \rangle$  be the  $g$ -by- $g$  matrix given by  $a_{i,j} = \langle \omega_i, \omega_j \rangle$ . Note that  $\bar{a} = a^t$ . Let  $b = a^{t,-1}$ , the inverse of the transpose of  $a$ . Then we claim that the Arakelov  $(1, 1)$ -form on  $X$  can be written as:

$$\mu = \frac{i}{2g} \sum_{i,j} b_{i,j} \omega_i \wedge \bar{\omega}_j.$$

To see this, note that for  $\omega$  an orthonormal basis this is the correct expression, and that changing to  $\omega' = \omega \cdot g$  with any invertible  $g$  does not change  $\mu$ , as one may directly calculate.

In our case, the basis  $(\omega_1, \dots, \omega_g)$  that we take is an Atkin-Lehner basis. Using Corollary 11.1.4 one obtains the expression that we gave.  $\square$

We remark that Abbes and Ullmo have determined the Arakelov  $(1, 1)$ -form on  $X_0(n)$  for all square free  $n \geq 1$  such that  $X_0(n)$  has genus at least one in [Ab-UI]. It should not be hard to generalise their result to  $X_1(n)$  for square free  $n$ .

Now we arrive at the main result of this section. We recall that the Faltings height of a curve, and its stable or absolute version, have been briefly described in (4.4.11).

**11.1.7 Theorem** *For the stable Faltings height of  $X_1(pl)$ , for distinct prime numbers  $p$  and  $l$ , one has:*

$$h_{\text{abs}}(X_1(pl)) = O((pl)^2 \log(pl)).$$

**Proof** This proof is an adaptation of an argument in Section 5 of [Co-Ed] where the case  $X_0(p)$  with  $p$  prime was treated. We may and do assume that  $X_1(pl)$  has genus at least one.

We start with a general observation. For  $X_K$  a curve over a number field, and  $K \rightarrow L$  a finite extension, we claim that:

$$[L : \mathbb{Q}]^{-1} h_L(X_L) \leq [K : \mathbb{Q}]^{-1} h_K(X_K).$$

This inequality simply results from the fact that for the Néron models of the Jacobians the identity morphism on the generic fibres extends to a mor-

phism:

$$(J_{O_K})_{O_L} \longrightarrow J_{O_L}.$$

For  $n$  a positive integer, we let  $X_\mu(n)_\mathbb{Q}$  denote the modular curve corresponding to elliptic curves with an embedding of  $\mu_n$ . The reason for considering this variant of  $X_1(n)$  is that the cusp  $\infty$  of  $X_\mu(n)$  is  $\mathbb{Q}$ -rational. Of course,  $X_1(n)_\mathbb{Q}$  and  $X_\mu(n)_\mathbb{Q}$  become isomorphic over  $\mathbb{Q}(\zeta_n)$ , and therefore we have, for all  $n$ :

$$h_{\text{abs}}(X_1(n)_\mathbb{Q}) = h_{\text{abs}}(X_\mu(n)_\mathbb{Q}).$$

For more details about these  $X_\mu(n)$  we refer to sections 9.3 and 12.3 of [Di-Im].

The general observation above gives:

$$h_{\text{abs}}(X_\mu(n)_\mathbb{Q}) \leq h_{\mathbb{Q}}(X_\mu(n)_\mathbb{Q}).$$

Because of this, it suffices to establish the bound of the theorem for the  $h_{\mathbb{Q}}(X_\mu(pl)_\mathbb{Q})$ .

Let  $p$  and  $l$  be given, and let  $X$  be the model over  $\mathbb{Z}$  of  $X_\mu(pl)_\mathbb{Q}$  obtained by normalisation of the  $j$ -line  $\mathbb{P}_{\mathbb{Z}}^1$  in the function field of  $X_\mu(pl)_\mathbb{Q}$ . As  $X$  is proper over  $\mathbb{Z}$ , the  $\mathbb{Q}$ -rational point  $\infty$  extends to an element  $\infty$  in  $X(\mathbb{Z})$ , which is known to lie in the open part  $X^{\text{sm}}$  of  $X$  where the structure morphism to  $\text{Spec}(\mathbb{Z})$  is smooth, see [Di-Im]. In terms of the Tate curve over  $\mathbb{Z}((q)) = \mathbb{Z}((1/j))$ , the cusp  $\infty$  is the immersion of  $\mu_n$ , over  $\mathbb{Z}$ , in the  $n$ -torsion of the Tate curve (see Sections 8.6–8.11 of [Ka-Ma]).

We let  $J$  be the Néron model over  $\mathbb{Z}$  of the Jacobian of the curve  $X_\mathbb{Q}$ . Then, by the defining property, the embedding of  $X_\mathbb{Q}$  into  $J_\mathbb{Q}$  that sends  $\infty$  to 0 extends to a morphism from  $X^{\text{sm}}$  to  $J$ . This morphism induces via pullback of differential forms a morphism from  $\text{Cot}_0(J)$  to  $S(\mathbb{Z})$ , the sub- $\mathbb{Z}$ -module of  $\Omega^1(X_\mathbb{Q})$  of forms whose  $q$ -expansion at  $\infty$  has coefficients in  $\mathbb{Z}$  (see around (2.4.9)). As  $\text{Cot}_0(J)$  and  $S(\mathbb{Z})$  are both  $\mathbb{Z}$ -structures on  $\Omega^1(X(\mathbb{C}))$ , we have (see around (4.4.10)):

$$\begin{aligned} h_{\mathbb{Q}}(X_\mathbb{Q}) &= \deg\left(\bigwedge^g 0^* \text{Cot}_0(J)\right) = -\log \text{Vol}((\mathbb{R} \otimes \text{Cot}_0(J))/\text{Cot}_0(J)) \\ &\leq -\log \text{Vol}(\mathbb{R} \otimes S(\mathbb{Z})/S(\mathbb{Z})), \end{aligned}$$

where the volume form on  $\mathbb{R} \otimes \text{Cot}_0(J)$  comes from integration over  $J(\mathbb{C})$ , and that on  $\mathbb{R} \otimes S(\mathbb{Z})$  from integration over  $X(\mathbb{C})$ .

Let  $\mathbb{T} \subset \text{End}(J)$  be the Hecke algebra, generated by all  $T_i$ ,  $i \geq 1$ , and the  $\langle a \rangle$ ,  $a$  in  $(\mathbb{Z}/pl\mathbb{Z})^\times$ . We have a perfect pairing (see (2.4.9)):

$$\mathbb{T} \times S(\mathbb{Z}) \rightarrow \mathbb{Z}, \quad (t, \omega) \mapsto a_1(t\omega).$$

Using the duality we can write:

$$-\log \text{Vol}(\mathbb{R} \otimes S(\mathbb{Z})/S(\mathbb{Z})) = \log \text{Vol}(\mathbb{R} \otimes \mathbb{T}/\mathbb{T})$$

where the volume form on  $\mathbb{R} \otimes \mathbb{T}$  is dual to the one on  $\mathbb{R} \otimes S(\mathbb{Z})$ . Now consider an Atkin-Lehner basis  $(\omega_1, \dots, \omega_g)$  of  $\Omega^1(X)_{\mathbb{C}}$ . Let  $\text{Vol}'$  denote the volume with respect to the volume form on  $\mathbb{R} \otimes \mathbb{T}$  induced by the one on  $\mathbb{C} \otimes S(\mathbb{Z})$  for which the basis  $(\omega_1, \dots, \omega_g)$  is an orthonormal basis. Then we have:

$$\log \text{Vol}(\mathbb{R} \otimes \mathbb{T}/\mathbb{T}) = \log \text{Vol}'(\mathbb{R} \otimes \mathbb{T}/\mathbb{T}) - \frac{1}{2} \log \det(\langle \omega, \omega \rangle),$$

where  $\langle \omega, \omega \rangle$  is the matrix whose  $(i, j)$ -coefficient is  $\langle \omega_i, \omega_j \rangle$ . By Corollary 11.1.5 we have:

$$-\log \det(\langle \omega, \omega \rangle) \leq g(4\pi - \log \pi) = O((pl)^2).$$

It remains to bound  $\log \text{Vol}'(\mathbb{R} \otimes \mathbb{T}/\mathbb{T})$ . Let  $\Gamma$  be the set of integers  $i \geq 1$  such that there exists an  $\omega$  in  $\Omega^1(X(\mathbb{C}))$  with a zero of exact order  $i - 1$  at  $\infty$ . Then  $\Gamma$  is the set of integers  $i \geq 1$  such that  $h^0(X(\mathbb{C}), \Omega^1(-i\infty))$  is strictly less (and hence exactly one less) than  $h^0(X(\mathbb{C}), \Omega^1((-i + 1)\infty))$ . As  $h^0(X(\mathbb{C}), \Omega^1) = g$ , and  $h^0(X(\mathbb{C}), \Omega^1(-(2g - 1)\infty)) = 0$ , there are exactly  $g$  such integers, and we can write  $\Gamma = \{i_1, \dots, i_g\}$  with:

$$1 = i_1 < i_2 < \dots < i_g \leq 2g - 1.$$

Under the pairing between  $\mathbb{T}$  and  $S(\mathbb{Z})$ , each Hecke operator  $T_i$  is sent to the element  $\omega \mapsto a_i(\omega)$  of the dual of  $S(\mathbb{Z})$ , where the  $a_i(\omega)$  are given by the  $q$ -expansion:

$$\omega = \sum_{i \geq 1} a_i(\omega) q^i (dq/q) = \sum_{i \geq 1} a_i(\omega) q^{i-1} dq.$$

It follows that the elements  $T_{i_1}, \dots, T_{i_g}$  of the free  $\mathbb{Z}$ -submodule  $\mathbb{T}$  are linearly independent. Hence  $\mathbb{T}'$ , the submodule of  $\mathbb{T}$  generated by these  $T_{i_j}$  has finite index. We thus find:

$$\log \text{Vol}(\mathbb{R} \otimes \mathbb{T}/\mathbb{T}) \leq \log \text{Vol}'(\mathbb{R} \otimes \mathbb{T}'/\mathbb{T}') - \frac{1}{2} \log \det(\langle \omega, \omega \rangle).$$

Now we have  $g = r_1 + 2r_2$ , where  $r_1$  is the number of elements of our basis  $(\omega_1, \dots, \omega_g)$  of  $\mathbb{C} \otimes \mathbb{T}^\vee$  that are fixed by the complex conjugation. We let:

$$\phi: \mathbb{R} \otimes \mathbb{T} \longrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \times \mathbb{C}^{r_2} \longrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$$

be the map obtained from our basis (each  $\omega_i$  gives  $t \mapsto a_1(t\omega_i)$ ), composed with the projection. We view  $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$  as  $\mathbb{R}^g$  by decomposing each factor  $\mathbb{C}$  as  $\mathbb{R} \oplus \mathbb{R}i$ . Then we have:

$$\text{Vol}'(\mathbb{R} \otimes \mathbb{T}/\mathbb{T}') = 2^{r_2} |\det(\phi(T_{i_1}), \dots, \phi(T_{i_g}))|.$$

By construction, each  $\phi(T_{i_j})_k$  is the real or imaginary part of some  $a_{i_j}(\omega_l)$ . Hence by Lemma 11.1.1 we have:

$$|\phi(T_{i_j})_k| \leq 2i_j.$$

We obtain:

$$|\det(\phi(T_{i_1}), \dots, \phi(T_{i_g}))| \leq \prod_{j=1}^g (2i_j \sqrt{g}) \leq (4g^2)^g.$$

Hence, finally:

$$\log \text{Vol}'(\mathbb{R} \otimes \mathbb{T}/\mathbb{T}) \leq r_2 \log 2 + g(\log 4 + 2 \log g).$$

Noting that  $r_2 \leq g/2$  and that  $g = O((pl)^2)$  completes our proof. □

## 11.2 BOUNDING THE THETA FUNCTION ON $\text{Pic}^{g-1}(X_1(pl))$

The aim of this section is to give a bound for the supnorm of the theta function that occurs in Theorem 9.2.5.

**11.2.1 Theorem** *For  $X = X_1(pl)$ , with  $p$  and  $l$  distinct primes for which the genus of  $X_1(pl)$  is at least one, we have  $\log \|\vartheta\|_{\text{sup}} = O((pl)^6)$ .*



We start with two lemmas, which are possibly of independent interest.

**11.2.2 Lemma** *Let  $X = V/\Lambda$  be a principally polarised complex Abelian variety and let  $H : V \times V \rightarrow \mathbb{C}$  be its Riemann form. Let  $\lambda_1, \dots, \lambda_{2g}$  be the successive minima of the lattice  $\Lambda$ , with norm defined by  $\|x\|^2 = H(x, x)$ . Let  $(e_1, \dots, e_{2g})$  be a symplectic basis of  $\Lambda$ , i.e., a basis with respect to which the matrix of the symplectic form  $\mathfrak{S}(H)$  is, in  $g$  by  $g$  block form, equal to  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . The one has:*

$$\sqrt{\det \mathfrak{S}(\tau)} \leq \frac{(2g)! V_{2g}}{2^g V_g} \lambda_{g+1} \cdots \lambda_{2g},$$

where  $\tau$  is the period matrix in  $\mathbb{H}_g$  corresponding to  $(e_1, \dots, e_{2g})$ . Here  $V_n$  denotes the volume of the unit ball in  $\mathbb{R}^n$  with its standard euclidean inner product.

**Proof** We consider the lattice  $M = \mathbb{Z} \cdot e_1 \oplus \cdots \oplus \mathbb{Z} \cdot e_g$  in the real subvector space  $W = \mathbb{R} \cdot e_1 \oplus \cdots \oplus \mathbb{R} \cdot e_g$  of  $V$ . Denote by  $\mu_1, \dots, \mu_g$  the successive minima of  $M$ , where the norm is given by restricting  $H$  to  $W$  (note that  $M$  is isotropic for the symplectic form, so that  $H$  takes real values on  $W$ ). We have  $(H(e_i, e_j))_{i,j} = (\mathfrak{S}(\tau))^{-1}$ , so that the volume (with respect to the inner product on  $W$  given by  $H$ ) of  $W/M$  is equal to  $(\det \mathfrak{S}(\tau))^{-1/2}$ , and hence by Minkowski's second fundamental inequality:

$$\mu_1 \cdots \mu_g \leq 2^g \frac{(\det \mathfrak{S}(\tau))^{-1/2}}{V_g}.$$

On the other hand we have:

$$\mu_1 \cdots \mu_g \geq \lambda_1 \cdots \lambda_g = (\lambda_1 \cdots \lambda_{2g}) \cdot (\lambda_{g+1} \cdots \lambda_{2g})^{-1}$$

and since the volume of  $V/\Lambda$  is 1 we obtain by Minkowski's first fundamental inequality:

$$\lambda_1 \cdots \lambda_{2g} \geq \frac{2^{2g}}{(2g)!} \frac{1}{V_{2g}}.$$

Combining we find a lower bound:

$$\mu_1 \cdots \mu_g \geq \frac{2^{2g}}{(2g)!} \frac{1}{V_{2g}} \cdot (\lambda_{g+1} \cdots \lambda_{2g})^{-1}.$$

Combining this with the upper bound for  $\mu_1 \cdots \mu_g$  we obtain the required formula.  $\square$

**11.2.3 Lemma** *Let  $N \geq 3$  be an integer. The group  $\Gamma_1(N)$  is generated by its elements whose entries are bounded from above in absolute value by  $N^6/4$ .*

**Proof** We first note the following: let  $G$  be a group, and let  $S \subset G$  be a set of generators. Let  $X$  be a transitive  $G$ -set and let  $x$  be in  $X$ . For each  $y$  in  $X$ , let  $g_y$  be an element of  $G$  such that  $g_y x = y$ ; we demand that  $g_x = 1$ . Then the  $g_{sy}^{-1} s g_y$ , for  $s$  in  $S$  and  $y$  in  $X$ , form a system of generators for the stabiliser  $G_x$  of  $x$ . To see this, first replace  $S$  by  $S \cup S^{-1}$ . Let  $g$  be in  $G_x$ . Write  $g = s_n \cdots s_1$  with  $s_i$  in  $S$ . Then we can write:

$$g = s_n \cdots s_1 = g_{s_n y_n}^{-1} s_n g_{y_n} \cdots g_{s_1 y_1}^{-1} s_1 g_{y_1} \quad \text{with } y_i = (s_{i-1} \cdots s_1)x.$$

The equality holds because  $s_i y_i = y_{i+1}$ , and  $g_{y_1} = 1$  and  $g_{s_n y_n} = 1$ . Now we apply this to our case. We take  $G = \text{SL}_2(\mathbb{Z})$ , and we take  $X$  to be the subset of  $(\mathbb{Z}/N\mathbb{Z})^2$  consisting of the elements of order  $N$ . This is a transitive  $G$ -set. We let  $x = (1, 0)$ ; then  $G_x$  is identified with  $\Gamma_1(N)$ . Let  $S$  be the set consisting of  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  and their inverses. Then  $S$  generates  $G$ . We now apply the previous argument to find generators of  $\Gamma_1(N)$ . So let  $y = (\bar{a}, \bar{b})$  be in  $X$ . Then, thinking of  $\mathbb{Z}/N\mathbb{Z}$  as the product of its local rings, we see that there is a  $u$  in  $\mathbb{Z}$  with  $|u| \leq N/2$  and  $\overline{a + bu}$  in  $(\mathbb{Z}/N\mathbb{Z})^\times$ . Put  $a_1 := a + bu$  and  $b_1 := b$ . Then:

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Next, there is a  $v$  in  $\mathbb{Z}$  with  $|v| \leq N/2$  and  $b_1 + a_1 v = 1 \pmod N$ , i.e.:

$$\begin{pmatrix} a_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \pmod N.$$

Finally, let  $w$  be in  $\mathbb{Z}$  with  $|w| \leq N/2$  and with image  $\overline{a_1}$  in  $\mathbb{Z}/N\mathbb{Z}$ . Then one has:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pmod N.$$

Writing out, we have:

$$\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} w - u(1 - vw) & -1 - uv \\ 1 - vw & v \end{pmatrix}$$

so that we can put:

$$g_y = g_{(a,b)} = \begin{pmatrix} w - u(1 - vw) & -1 - uv \\ 1 - vw & v \end{pmatrix}.$$

The absolute values of the coefficients of  $g_{(a,b)}$  are smaller than  $N^3/4$ , if  $N \geq 3$ , and the lemma follows.  $\square$

**Proof** [Proof of Theorem 11.2.1] Recall that  $\|\vartheta\|(z; \tau)$  is given by:

$$\|\vartheta\|(z; \tau) = (\det \Im(\tau))^{1/4} \exp(-\pi {}^t y (\Im(\tau))^{-1} y) |\vartheta(z; \tau)|,$$

where  $y = \Im(z)$  and where  $\tau$  is a period matrix in the Siegel upper half plane  $\mathbb{H}_g$  corresponding to  $X$ . We first deal with the factor  $\det \Im(\tau)$  and for this we invoke Lemma 11.2.2. Choose once more an Atkin-Lehner basis  $(\omega_1, \dots, \omega_g)$  for  $\Omega^1(X)$ . Using the dual basis in  $\Omega^1(X)^\vee$  we write:

$$J(X) = \mathbb{C}^g / \Lambda,$$

where:

$$\Lambda = \text{Image} \left( H_1(X, \mathbb{Z}) \rightarrow \mathbb{C}^g : \gamma \mapsto \int_\gamma (\omega_1, \dots, \omega_g) \right).$$

The polarisation form for  $J(X)$  is given by:

$$(z, w) \mapsto {}^t z \cdot (\langle \omega, \omega \rangle)_{i,j}^{-1} \cdot \bar{w}.$$

Denote by  $\|\cdot\|_P$  the corresponding norm on  $\mathbb{C}^g$ . We also consider the standard hermitian inner product on  $\mathbb{C}^g$ , which is just  $(z, w) \mapsto {}^t z \cdot \bar{w}$ . Here we denote the corresponding norm by  $\|\cdot\|_E$ . From the next two lemmas we obtain:

$$(\lambda_{g+1} \cdots \lambda_{2g})^2 \leq (ge^{4\pi} (pl)^{46} / \pi)^g$$

and hence, by Lemma 11.2.2, the estimate:

$$\log(\det \Im(\tau)) = O((pl)^2 \log(pl)).$$

**11.2.4 Lemma** *The lattice  $\Lambda$  is generated by the subset of its elements  $x$  that satisfy  $\|x\|_E^2 \leq g \cdot (pl)^{46}$ .*

**Proof** For the moment put  $N = pl$ . Following the natural surjections:

$$\Gamma_1(N) \twoheadrightarrow \Gamma_1(N)^{\text{ab}} = H_1(Y_1(N), \mathbb{Z}) \twoheadrightarrow H_1(X_1(N), \mathbb{Z})$$

we see that any generating set for  $\Gamma_1(N)$  gives a generating set for  $H_1(X_1(N), \mathbb{Z})$ . We are going to take generators for  $\Gamma_1(N)$  as given by Lemma 11.2.3. In particular, the absolute values of their coefficients are bounded by  $N^6/4$ . We have to see now what this implies for  $\|x\|_E$  for corresponding elements  $x$  of  $\Lambda$ . Concretely, choose a  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in  $\Gamma_1(N)$ . The image in  $\Lambda$  can be given as follows: choose any  $z$  in  $\mathbb{H}$  and any path in  $\mathbb{H}$  from  $z$  to  $gz$ . This gives us a loop in  $X_1(N)$ , and the class of that loop is the image of  $g$  in  $H_1(X_1(N), \mathbb{Z})$ . In order to get to  $\Lambda$  we compute the periods of  $(\omega_1, \dots, \omega_g)$  around this loop. We want to get bounds for these periods. In order to do this, note first that we can assume that  $c \neq 0$ . Indeed, the  $g$  with  $c = 0$  are unipotent, hence have trivial image in  $H_1(X_1(N), \mathbb{Z})$ . Now we make the following choices. First, we want to take a  $z$  in  $\mathbb{H}$  with  $\Im(z) = \Im(gz)$ . Since, for all  $z$  in  $\mathbb{H}$ ,  $\Im(gz) = |cz + d|^{-2}\Im(z)$ , the condition that  $\Im(z) = \Im(gz)$  is equivalent to  $|cz + d| = 1$ . We choose  $z = -d/c + i/|c|$ , i.e., such that  $cz + d = \pm i$ , depending on the sign of  $c$ . Second, the path that we take is the straight line from  $z$  to  $gz$ . We have  $|c| \geq N$  because  $g$  is in  $\Gamma_1(N)$ . Using furthermore that the absolute values of the coefficients of  $g$  are bounded by  $N^6/4$  we get  $|gz - z| \leq |az + b| + |z| \leq N^{11}/10$  (where we have used that  $N \geq 2$ ). For the period of an element  $\omega = fdq/q$  of the Atkin-Lehner basis we obtain from this that:

$$\left| \int_z^{gz} \omega \right| = \left| \int_z^{gz} f \cdot (dq)/q \right| = \left| \int_z^{gz} f \cdot 2\pi i dw \right| \leq \frac{2\pi N^{11}}{10} \|f\|,$$

where  $w$  denotes the standard coordinate of  $\mathbb{H}$  (i.e., the inclusion map into  $\mathbb{C}$ ), and  $\|f\|$  the supnorm of  $f$  on the straight line from  $z$  to  $gz$ ; recall that  $q = \exp(2\pi iw)$ . When writing  $f = \sum_{n \geq 1} a_n(f)q^n$  we have  $|a_n(f)| \leq 2n$  by Lemma 11.1.1. Noting furthermore that that we have

$\Im z = |c|^{-1} \geq 4/N^6$  it follows that:

$$\|f\| \leq 2 \sum_{n \geq 1} n e^{-8\pi N^{-6}n} = \frac{2r}{(1-r)^2}, \quad \text{where } r = e^{-8\pi/N^6}.$$

Hence:

$$\|f\| \leq (N^6/4\pi)^2, \quad \text{and} \quad \left| \int_z^{gz} f(dq)/q \right| \leq \frac{2\pi}{10} N^{11} \left( \frac{N^6}{4\pi} \right)^2 \leq N^{23}.$$

This means that all  $g$  coordinates of our element  $x$  of  $\mathbb{C}^g$  are, in absolute value, at most  $N^{23}$ . Hence  $\|x\|_E^2$ , being the sum of the squares of these coordinates, is at most  $g \cdot N^{46} = g \cdot (pl)^{46}$ .  $\square$

**11.2.5 Lemma** *For any  $x$  in  $\mathbb{C}^g$  we have the estimate:*

$$\|x\|_P^2 \leq (e^{4\pi}/\pi) \|x\|_E^2.$$

**Proof** By Lemma 11.1.4 the matrix  $(\langle \omega, \omega \rangle)^{-1}$  is almost diagonal, having in fact diagonal elements  $1/\|\omega\|^2$  corresponding to newforms  $\omega$  on  $X_1(pl)$ , and 2-by-2 blocks corresponding to newforms  $\omega$  on  $X_1(l)$  and  $X_1(p)$ . The 2-by-2 block corresponding to a newform  $\omega$  on  $X_1(l)$  is:

$$\frac{1}{(p-1)\|\omega\|^2((p+1)^2 - |a_p(\omega)|^2)} \begin{pmatrix} p+1 & -\overline{a_p(\omega)} \\ -a_p(\omega) & p+1 \end{pmatrix},$$

where the norm  $\|\omega\|^2$  is taken on  $X_1(l)$ .

The 2-by-2 block corresponding to a newform  $\omega$  on  $X_1(p)$  is:

$$\frac{1}{(l-1)\|\omega\|^2((l+1)^2 - |a_l(\omega)|^2)} \begin{pmatrix} l+1 & -\overline{a_l(\omega)} \\ -a_l(\omega) & l+1 \end{pmatrix},$$

where the norm  $\|\omega\|^2$  is taken on  $X_1(p)$ .

A short calculation shows that for any  $(z_1, z_2)$  in  $\mathbb{C}^2$  one has:

$$\frac{\begin{pmatrix} z_1 & z_2 \end{pmatrix} \begin{pmatrix} p+1 & -\overline{a_p(\omega)} \\ -a_p(\omega) & p+1 \end{pmatrix} \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix}}{(p-1)((p+1)^2 - |a_p(\omega)|^2)} \leq (|z_1|^2 + |z_2|^2),$$

and similarly for  $X_1(p)$ , so that all in all one gets:

$$\|(z_1, \dots, z_g)\|_P^2 \leq \frac{|z_1|^2}{\|\omega_1\|^2} + \dots + \frac{|z_g|^2}{\|\omega_g\|^2}.$$

The lemma follows by the lower bound from Lemma 11.1.2.  $\square$

Next we consider the factor  $\exp(-\pi {}^t y (\Im(\tau))^{-1} y) |\vartheta(z; \tau)|$ . Since in our previous estimates the choice of  $\tau$  was irrelevant, it will cause no loss of generality here if we restrict to  $\tau$  lying in the so-called Siegel fundamental domain  $F_g$ , which is the set of matrices  $\tau = x + iy$  satisfying the conditions:

1. for each entry  $x_{ij}$  of  $x$  one has  $|x_{ij}| \leq \frac{1}{2}$ ,
2. for all  $\gamma$  in  $\text{Sp}(2g, \mathbb{Z})$  one has  $\det \Im(\gamma \cdot \tau) \leq \det \Im(\tau)$ ,
3.  $y$  is Minkowski reduced, i.e., for each  $\xi = (\xi_1, \dots, \xi_g)$  in  $\mathbb{Z}^g$  and each  $i$  such that  $\xi_i, \dots, \xi_g$  are non-zero, one has  $\xi {}^t y \xi \geq y_{ii}$  and moreover, for each  $1 \leq i \leq g-1$  one has  $y_{i,i+1} \geq 0$ .

It is well known that  $F_g$  contains at least one representative from each  $\text{Sp}(2g, \mathbb{Z})$ -orbit on  $\mathbb{H}_g$ . We claim that for  $\tau$  in  $F_g$  the estimate:

$$\exp(-\pi {}^t y (\Im(\tau))^{-1} y) |\vartheta(z; \tau)| \leq 2^{3g^3+5g}$$

holds, for all  $z$  in  $\mathbb{C}^g$ . Thus, this factor gives us a contribution  $O((pl)^6)$ . In order to prove the estimate, write  $y = \Im(z) = (\Im(\tau)) \cdot b$  with  $b$  in  $\mathbb{R}^g$ . Then it is easy to see that:

$$\exp(-\pi {}^t y (\Im(\tau))^{-1} y) |\vartheta(z; \tau)| \leq \sum_{n \in \mathbb{Z}^g} \exp(-\pi {}^t (n+b) (\Im(\tau)) (n+b)).$$

Since the  $\Im(\tau)$  are Minkowski reduced we have, for any  $m$  in  $\mathbb{R}^g$  (cf. [Igu], V §4):

$${}^t m \Im(\tau) m \geq c(g) \sum_{i=1}^g m_i^2 (\Im(\tau))_{ii}, \quad c(g) = \left(\frac{4}{g^3}\right)^{g-1} \left(\frac{3}{4}\right)^{g(g-1)/2}.$$

Moreover, we have  $(\Im(\tau))_{i,i} \geq \sqrt{3}/2$  for  $i = 1, \dots, g$ . From this we derive:

$$\begin{aligned} \sum_{n \in \mathbb{Z}^g} \exp(-\pi {}^t(n+b)(\Im(\tau))(n+b)) &\leq \\ &\leq \sum_{n \in \mathbb{Z}^g} \exp\left(-\sum_{i=1}^g \pi c(g)(n_i + b_i)^2 (\Im(\tau))_{i,i}\right) \leq \\ &\leq \prod_{i=1}^g \sum_{n_i \in \mathbb{Z}} \exp(-\pi c(g)(n_i + b_i)^2 (\Im(\tau))_{i,i}) \leq \\ &\leq \prod_{i=1}^g \frac{2}{1 - \exp(-\pi c(g)(\Im(\tau))_{ii})} \leq \\ &\leq 2^g \left(1 + \frac{2}{\sqrt{3}\pi c(g)}\right)^g. \end{aligned}$$

From this and the formula for  $c(g)$  the required estimate follows and the proof of Theorem 11.2.1 is finished.  $\square$

### 11.3 UPPER BOUNDS FOR ARAKELOV GREEN FUNCTIONS ON THE CURVES $X_1(pl)$

The aim of this section is to give an upper bound for the Arakelov Green functions on the curves  $X_1(pl)$  that will enable us to bound from above the contributions of the intersection numbers in the right hand side of the inequality in Theorem 9.2.5. As the  $X_l(\mathbb{C})$  are compact, it is clear that for each  $l$  such an upper bound exists, but we need such upper bounds that grow as most as a power of  $l$ .

In order to establish such upper bounds we will use Franz Merkl's result on Green functions on arbitrary Riemann surfaces given in Chapter 10.

Instead of using the result of Merkl for our work we could certainly also have used recent work by Jorgenson and Kramer in [Jo-Kr]. The results of Jorgenson and Kramer date back to the same time as those of Merkl (early Spring 2004). We chose to use Merkl's results because his approach is more elementary, and we had the details earlier than those of Jorgenson and Kramer.

The following theorem gives a suitable upper bound for the Arakelov-Green functions  $g_{a,\mu}$  (see (4.4.2) and Proposition 4.4.3) on the modular

curves  $X_1(pl)$  with  $p$  and  $l$  distinct primes.

**11.3.1 Theorem** *There is a real number  $c$  such that for all pairs of distinct prime numbers  $p$  and  $l$  for which the genus of  $X_1(pl)$  is at least one and for all distinct  $a$  and  $b$  on  $X_1(pl)(\mathbb{C})$  we have:*

$$g_{a,\mu}(b) \leq c \cdot (pl)^6.$$

Let  $\infty$  denote the cusp  $\infty$  on  $X_1(pl)$ , and let  $q$  be the standard local coordinate around  $\infty$  given by the map  $\tau \mapsto \exp(2\pi i\tau)$  from the region  $\Im\tau > 1$  in  $\mathbb{H}$  to  $\mathbb{C}$ . Then we have:

$$|\log \|dq\|_{\text{Ar}}(\infty)| = O((pl)^6),$$

where  $\|\cdot\|_{\text{Ar}}$  denotes the Arakelov metric on  $\Omega^1$  (see Section 4.4).

**Proof** We write for the moment  $N$  for  $pl$ . We will apply Theorem 10.1, but we will carry out the estimates on the more symmetrical modular curve  $X(N)$  which for us is  $\Gamma(N)\backslash(\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}))$ . Let  $h: X(N) \rightarrow X_1(N)$  be the canonical map; it has degree  $N$ . We let  $\mu$  denote the Arakelov  $(1, 1)$ -form on  $X_1(N)$ , and we define  $\mu' = h^*\mu/N$ . The characterising properties of Green functions directly imply that:

$$(11.3.2) \quad h^*g_{a,\mu} = \sum_{h(b)=a} g_{b,\mu'},$$

where the  $b$  are counted with multiplicity.

As in Section 10 we fix a constant  $r_1$  with  $0 < r_1 < 1$ ; we take  $r_1 := 3/4$ . We need to construct an atlas with charts  $z^{(j)}: U^{(j)} \rightarrow \mathbb{C}$  for  $X(N)$  with all  $z^{(j)}(U^{(j)})$  containing the closed unit disk and with the  $U_{r_1}^{(j)}$  covering  $X(N)$ .

We start with a construction of a local coordinate  $z: U \rightarrow \mathbb{C}$  in a neighbourhood of the standard cusp  $\infty$ . Since  $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$  acts transitively on the set of cusps of  $X(N)$ , this construction will suffice to give the full atlas. Our initial coordinate is induced by the map  $z$  from  $\mathbb{H}$  to  $\mathbb{C}$  that sends  $\tau$  to  $e^{2\pi i\tau/N}$ . As the following lemma is valid for all integers  $n \geq 1$ , we state it in that generality, and will apply it with  $n := N$ .

**11.3.3 Lemma** *Let  $n$  be in  $\mathbb{Z}_{\geq 1}$ . The subset in  $\mathbb{H}$  given by the conditions  $-1/2 \leq \Re\tau < n - 1/2$  and  $\Im\tau > 1/n$  is mapped injectively to  $X(n)(\mathbb{C})$ .*



**Proof** First we note that for  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in  $\mathrm{SL}_2(\mathbb{Z})$  and for  $\tau$  in  $\mathbb{H}$  we have:

$$\Im \left( \frac{a\tau + b}{c\tau + d} \right) = \frac{\Im(\tau)}{|c\tau + d|^2}.$$

Let us call  $D$  the set of  $\tau \in \mathbb{H}$  that satisfy the two conditions of the lemma:

$$D = \{ \tau \in \mathbb{H} \mid -1/2 \leq \Re(\tau) < n - 1/2 \text{ and } \Im(\tau) > 1/n \}.$$

Let  $\tau$  be in  $D$ , and let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be in  $\Gamma(n)$ , such that  $\gamma\tau \neq \tau$ . If  $c = 0$  then, as  $ad = 1$ , we have  $a = d$  and  $\tau' = \tau \pm b$  with  $b$  a non-zero multiple of  $n$ , and so  $\tau'$  is not in  $D$ . If  $c \neq 0$  then we have  $|c| \geq n$  because  $n|c|$ , and:

$$\begin{aligned} \Im \left( \frac{a\tau + b}{c\tau + d} \right) &= \frac{\Im(\tau)}{|c\tau + d|^2} \leq \frac{\Im(\tau)}{(\Im(c\tau))^2} \leq \frac{\Im(\tau)}{n^2(\Im(\tau))^2} = \\ &= \frac{1}{n^2\Im(\tau)} < \frac{1}{n^2 \cdot (1/n)} = \frac{1}{n}, \end{aligned}$$

hence  $\gamma\tau$  is not in  $D$ . □

In particular, the region of  $\tau$  with  $-1/2 \leq \Re\tau < N - 1/2$  and  $\Im\tau > 1/2$  is mapped injectively into  $X(N)$  to give an open neighbourhood  $U$  of  $\infty$ . We could replace the condition “ $\Im\tau > 1/2$ ” by “ $\Im\tau > 1/N$ ”, but that would not make the work to be done significantly easier. The map  $\tau \mapsto e^{2\pi i\tau/N}$  gives an isomorphism:

$$(11.3.4) \quad z: U \longrightarrow D(0, e^{-\pi/N}) \subset \mathbb{C}.$$

The region of  $\tau$  with  $-1/2 \leq \Re\tau < N - 1/2$  and  $\Im\tau > 3/4$  gives an open neighbourhood  $V$  of  $\infty$ , contained in  $U$ , such that the translates of  $V$  under  $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$  cover  $X(N)$  (note that  $3/4 < \sqrt{3}/2$ ). The image of  $V$  under  $z$  is the disk  $D(0, e^{-3\pi/2N})$ . However, the quotient of the radii  $e^{-\pi/N}$  and  $e^{-3\pi/2N}$  tends to 1 as  $N$  tends to infinity, hence we cannot work with these disks directly.

What we do instead is the following. We define a new coordinate  $z' := e^{3\pi/2N}z$  to get  $z'V = D(0, 1)$ . Then  $z'U = D(0, e^{\pi/2N})$ . Let  $\varepsilon := \varepsilon(N) := e^{\pi/2N} - 1$  be the difference between the two new radii. Then  $\varepsilon > \pi/2N$ . We can choose  $O(\varepsilon^{-2})$  open disks  $D(a, \varepsilon)$  with centre  $a$  in  $D(0, 1)$ , such that the union of the  $D(a, \varepsilon/2)$  contains  $D(0, 1)$ ; we let  $A$  denote the set of these  $a$ . The  $D(a, \varepsilon)$  are contained in  $z'U = D(0, 1 + \varepsilon)$ .

The group  $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$  acts transitively on the set of cusps of  $X(N)$ . For each cusp  $c$ , we choose a  $g_c$  in  $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$  such that  $c = g_c\infty$ . The open sets of our atlas for  $X(N)$  are then the  $U^{(a,c)}$  with  $a$  in  $A$  and  $c$  a cusp, defined by:

$$U^{(a,c)} := g_c \cdot (z')^{-1} D(a, \varepsilon).$$

The required coordinates  $z^{(a,c)}$  on the  $U^{(a,c)}$  are defined by the composition of isomorphisms:

$$z^{(a,c)} : U^{(a,c)} \xrightarrow{g_c^{-1}} U^{(a,\infty)} \xrightarrow{z'} D(a, \varepsilon) \xrightarrow{-a} D(0, \varepsilon) \xrightarrow{\cdot 3/2\varepsilon} D(0, \frac{3}{2}).$$

Indeed, the images  $z^{(a,c)}U^{(a,c)}$  contain the unit disk, and  $U_{r_1}^{(a,c)}$  corresponds via  $z' \circ g_c^{-1}$  to the subdisk  $D(a, \varepsilon/2)$  of  $D(a, \varepsilon)$ , hence the  $U_{r_1}^{(a,c)}$  cover  $X(N)$ . The exact number  $n = n(N)$  of  $U^{(a,c)}$  is the cardinality of  $A$  times the number of cusps, hence  $n = O(N^4)$ . We choose a numbering of  $A \times \{\text{cusps}\}$  with the integers  $\{1, \dots, n\}$ , and we will denote our charts as:

$$(11.3.5) \quad z^{(j)} : U^{(j)} \rightarrow D(0, 3/2) \subset \mathbb{C}.$$

**11.3.6 Lemma** *For the local coordinates  $z^{(j)} : U^{(j)} \rightarrow \mathbb{C}$  on  $X(N)$  that we have just defined, the following holds. For all  $j$  and  $k$  in  $\{1, \dots, n\}$  we have:*

$$\sup_{U_1^{(j)} \cap U_1^{(k)}} \left| \frac{dz^{(j)}}{dz^{(k)}} \right| \leq M,$$

with  $M = 6$ .

**Proof** Let  $j$  and  $k$  be in  $\{1, \dots, n\}$ . If  $j$  and  $k$  arise from the same cusp, then  $z^{(j)}$  and  $z^{(k)}$  differ by a translation, hence  $dz^{(j)}/dz^{(k)} = 1$ . Now suppose that  $j$  and  $k$  arise from two distinct cusps. We may suppose then, by acting with an element of  $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ , that  $k$  arises from the standard cusp  $\infty$ . Let  $x$  denote the cusp that  $j$  arises from. The coordinate  $z^{(j)}$  is then obtained as above from an element  $g_x$  of  $\mathrm{SL}_2(\mathbb{Z})$  that sends  $\infty$  to  $x$ . Let us write  $g_x^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Note that  $c \neq 0$ , hence  $|c| \geq 1$ . Let  $z$  be a point in  $\mathbb{H}$  with  $-1/2 \leq \Re z < N - 1/2$  that maps to an element in  $U_1^{(j)} \cap U_1^{(k)}$ .

Then we know that  $1/2 < \Im z < 1$  because disks given by  $\Im z > 1$  around different cusps do not meet at all. Likewise, we then know that:

$$\frac{1}{2} < \Im \left( \frac{az + b}{cz + d} \right) = \frac{\Im(z)}{|cz + d|^2} < 1.$$

Hence, as  $(\Im z)/|cz + d|^2 \leq (\Im z)/c^2(\Im z)^2$ , we have  $\Im z < 2/c^2$  which gives in fact  $|c| = 1$ . Under these conditions, we estimate:

$$\begin{aligned} \left| \log \left| \frac{d \exp(2\pi i \frac{az+b}{cz+d}/N)}{d \exp(2\pi iz/N)} \right| \right| &= \left| \log \left| \frac{\exp(2\pi i \frac{az+b}{cz+d}/N) d \frac{az+b}{cz+d}}{\exp(2\pi iz/N) dz} \right| \right| = \\ &= \left| \log \left| \exp(2\pi i \frac{az+b}{cz+d}/N) \right| - \log |\exp(2\pi iz/N)| - \log |cz + d|^2 \right| \leq \\ &\leq 4\pi/N + 4\pi/N + \log 4. \end{aligned}$$

So indeed, for  $N \geq 6$ , we can take  $M = 6$ . Some explanations are perhaps in order here: as  $\Im z$  and  $\Im(az + b)/(cz + d)$  are between  $1/2$  and  $2$ ,  $|\exp(2\pi iz/N)|$  and  $|\exp(2\pi i \frac{az+b}{cz+d}/N)|$  are between  $\exp(-4\pi/N)$  and  $\exp(-\pi/N)$ . As  $\Im(az + b)/(cz + d) = (\Im z)/|cz + d|^2$ , we see that  $|cz + d|^2$  is between  $1/4$  and  $4$ .  $\square$

Our next task is to produce a suitable bound, as in Theorem 10.1, of the type  $\mu \leq c_1 |dz^{(j)} \wedge d\bar{z}^{(j)}|$ . We start with a bound for  $\mu$  on disks around  $\infty$  on  $X_1(pl)$ .

**11.3.7 Lemma** *Let  $r$  be a real number such that  $0 < r < 1$ . We map  $D(0, r)$  to  $X_1(pl)$  by sending  $q \neq 0$  to  $(\mathbb{C}^\times/q^{\mathbb{Z}}, \zeta_{pl})$ . The image of this map is the image in  $X_1(pl)$  of the region in  $\mathbb{H}$  defined by the condition “ $\Im \tau > -(\log r)/2\pi$ ”, plus the cusp  $\infty$ . We still denote by  $\mu$  the  $(1, 1)$ -form on  $D(0, r)$  induced by  $\mu$ . Then we have, on  $D(0, r)$ :*

$$\mu \leq \frac{28e^{4\pi}}{\pi} \frac{1}{(1-r)^4} \cdot \frac{i}{2} dqd\bar{q}.$$

**Proof** We first bound, on the disk  $D(0, r)$ , and for a newform  $f$ , the functions  $\sum_{n \geq 1} a_n(f)q^{n-1}$  and  $\sum_{n \geq 1} a_n(f)q^{n-1}$ . We have, for  $|q| < r$  by Lemma 11.1.1:

$$\left| \sum_{n \geq 1} a_n(f)q^{n-1} \right| \leq \sum_{n \geq 1} |a_n(f)|r^{n-1} \leq 2 \sum_{n \geq 1} nr^{n-1} = \frac{2}{(1-r)^2},$$

and next:

$$\left| \sum_{n \geq 1} a_n(f) q^{nl-1} \right| \leq 2 \sum_{n \geq 1} nr^{nl-1} = \frac{2r^{l-1}}{(1-r^l)^2} \quad \text{for } |q| < r.$$

Now recall from Corollary 11.1.6 that for  $\mu$  we have the expression:

$$\begin{aligned} \mu = & \frac{i}{2g} \sum_{\omega} \frac{\omega \wedge \bar{\omega}}{\|\omega\|^2} + \frac{i}{2g} \sum_{\omega} \left( \frac{(p+1)s_p^*(\omega \wedge \bar{\omega}) + (p+1)t_p^*(\omega \wedge \bar{\omega})}{(p-1)\|\omega\|^2((p+1)^2 - |a_p(f_\omega)|^2)} \right. \\ & \left. - \frac{a_p(f_\omega)s_p^*\omega \wedge \overline{t_p^*\omega} + \overline{a_p(f_\omega)t_p^*\omega} \wedge s_p^*\omega}{(p-1)\|\omega\|^2((p+1)^2 - |a_p(f_\omega)|^2)} \right) \\ & + \frac{i}{2g} \sum_{\omega} \left( \frac{(l+1)s_l^*\omega \wedge \overline{s_l^*\omega} + (l+1)t_l^*\omega \wedge \overline{t_l^*\omega}}{(l-1)\|\omega\|^2((l+1)^2 - |a_l(f_\omega)|^2)} \right. \\ & \left. - \frac{a_l(f_\omega)s_l^*\omega \wedge \overline{t_l^*\omega} + \overline{a_l(f_\omega)t_l^*\omega} \wedge s_l^*\omega}{(l-1)\|\omega\|^2((l+1)^2 - |a_l(f_\omega)|^2)} \right), \end{aligned}$$

the first sum running over  $\Omega^1(X_1(pl))^{\text{new}}$ , the second sum running over  $\Omega^1(X_1(l))^{\text{new}}$ , and the third sum running over  $\Omega^1(X_1(p))^{\text{new}}$ . We bound the different terms from the above expression for  $\mu$ . The contribution of an  $\omega$  in  $\Omega^1(X_1(pl))^{\text{new}}$  gives, for  $|q| < r$ :

$$\begin{aligned} \frac{i}{2g\|\omega\|^2} \omega \wedge \bar{\omega} &= \frac{1}{g\|\omega\|^2} \left| \sum_{n \geq 1} a_n(f_\omega) q^{n-1} \right|^2 \cdot \frac{i}{2} dqd\bar{q} \\ &\leq \frac{1}{g} \frac{e^{4\pi}}{\pi} \left( \frac{2}{(1-r)^2} \right)^2 \cdot \frac{i}{2} dqd\bar{q} \\ &= \frac{1}{g} \frac{e^{4\pi}}{\pi} \frac{4}{(1-r)^4} \cdot \frac{i}{2} dqd\bar{q} \end{aligned}$$

The contribution of an element  $\omega$  of  $\Omega^1(p)^{\text{new}}$  is:

$$\begin{aligned} & \frac{i}{2g} \sum_{\omega} \left( \frac{(l+1)s_l^* \omega \wedge \overline{s_l^* \omega} + (l+1)t_l^* \omega \wedge \overline{t_l^* \omega}}{(l-1)\|\omega\|^2((l+1)^2 - |a_l(f_\omega)|^2)} \right. \\ & \quad \left. - \frac{a_l(f_\omega)s_l^* \omega \wedge \overline{t_l^* \omega} + \overline{a_l(f_\omega)}t_l^* \omega \wedge \overline{s_l^* \omega}}{(l-1)\|\omega\|^2((l+1)^2 - |a_l(f_\omega)|^2)} \right) \leq \\ & \leq \frac{1}{g} \frac{e^{4\pi}}{\pi(l-1)^3} \left( \frac{4(l+1)}{(1-r)^4} + \frac{4(l+1)r^{2(l-1)}}{(1-r^l)^4} + \right. \\ & \quad \left. \frac{16\sqrt{l}r^{l-1}}{(1-r)^2(1-r^l)^2} \right) \cdot \frac{i}{2} dqd\bar{q}. \end{aligned}$$

Here one uses the Weil bounds on  $a_l(f_\omega)$ . Symmetrically (in  $p$  and  $l$ ), the contribution to  $\mu$  of an element  $\omega$  of  $\Omega^1(l)^{\text{new}}$  is:

$$\begin{aligned} & \frac{i}{2g} \sum_{\omega} \left( \frac{(p+1)s_p^* \omega \wedge \overline{s_p^* \omega} + (p+1)t_p^* \omega \wedge \overline{t_p^* \omega}}{(p-1)\|\omega\|^2((p+1)^2 - |a_p(f_\omega)|^2)} \right. \\ & \quad \left. - \frac{a_p(f_\omega)s_p^* \omega \wedge \overline{t_p^* \omega} + \overline{a_p(f_\omega)}t_p^* \omega \wedge \overline{s_p^* \omega}}{(p-1)\|\omega\|^2((p+1)^2 - |a_p(f_\omega)|^2)} \right) \leq \\ & \leq \frac{1}{g} \frac{e^{4\pi}}{\pi(p-1)^3} \left( \frac{4(p+1)}{(1-r)^4} + \frac{4(p+1)r^{2(p-1)}}{(1-r^p)^4} + \right. \\ & \quad \left. \frac{16\sqrt{p}r^{p-1}}{(1-r)^2(1-r^p)^2} \right) \cdot \frac{i}{2} dqd\bar{q}. \end{aligned}$$

Now we sum all contributions up, over the elements of  $\Omega^1(X_1(pl))^{\text{new}}$ ,  $\Omega^1(X_1(p))^{\text{new}}$ , and  $\Omega^1(X_1(l))^{\text{new}}$ . We get for  $|q| < r$ :

$$\begin{aligned} \mu \leq & \frac{4e^{4\pi}}{\pi} \left( \frac{1}{(1-r)^4} + \frac{1}{(1-r)^4} + \frac{r^{2(l-1)}}{(1-r^l)^4} + \frac{r^{l-1}}{(1-r)^2(1-r^l)^2} + \right. \\ & \left. \frac{1}{(1-r)^4} + \frac{r^{2(p-1)}}{(1-r^p)^4} + \frac{r^{p-1}}{(1-r)^2(1-r^p)^2} \right) \cdot \frac{i}{2} dqd\bar{q}, \end{aligned}$$

and finally:

$$\mu \leq \frac{28e^{4\pi}}{\pi} \frac{1}{(1-r)^4} \cdot \frac{i}{2} dqd\bar{q}, \quad \text{for } |q| < r.$$

□

Our next step is to consider the disks  $gU$  in  $X(pl)$ , where  $g$  is in  $SL_2(\mathbb{Z}/pl\mathbb{Z})$  and where  $U$  is as in (11.3.4).

**11.3.8 Lemma** *Let  $g$  be in  $SL_2(\mathbb{Z}/pl\mathbb{Z})$  and let  $z: U \rightarrow D(0, e^{-\pi/pl})$  be as in (11.3.4). Let  $z_g := z \circ g^{-1}: gU \rightarrow D(0, e^{-\pi/pl})$ . Then we have, for the restriction to  $gU$  of the pullback  $h^*\mu$  of  $\mu$  along  $h: X(N) \rightarrow X_1(N)$ :*

$$(h^*\mu)|_{gU} \leq c(pl)^4 |dz_g d\bar{z}_g|,$$

with  $c$  independent of  $p$  and  $l$ .

**Proof** To prove this, we consider the map  $h \circ g \circ z^{-1}$  from  $D(0, e^{-\pi/pl})$  to  $X_1(pl)$  and the pullback of  $\mu$  to  $D(0, e^{-\pi/pl})$ . We observe that  $\mu$  is invariant under all automorphisms of  $X_1(pl)$ . This applies in particular to the diamond operators and the Atkin-Lehner pseudo-involutions (defined in (2.5.13)). As the group generated by these automorphisms permutes the cusps of  $X_1(pl)$  transitively, we can take such an automorphism  $\alpha$  such that  $\alpha \circ h \circ g \circ z^{-1}$  sends  $D(0, e^{-\pi/pl})$  to a disk around the cusp  $\infty$ , where we can then apply Lemma 11.3.7. The pullbacks of  $\mu$  via  $h \circ g \circ z^{-1}$  and  $\alpha \circ h \circ g \circ z^{-1}$  are the same. We are also free to replace the coordinate  $z$  by  $\zeta z$  with  $\zeta \in \mathbb{C}$  such that  $|\zeta| = 1$ .

The map  $h \circ g \circ z^{-1}$  sends a point  $0 \neq q \in D(0, e^{-\pi/pl})$  to the point of  $X_1(pl)$  corresponding to  $(\mathbb{C}^\times/q^{pl\mathbb{Z}}, \zeta_p^a(q^l)^b, \zeta_l^c(q^p)^d)$  for certain  $a$  and  $b$  in  $\mathbb{F}_p$  and  $c$  and  $d$  in  $\mathbb{F}_l$  depending on  $g$ . After replacing  $h$  with  $h$  composed with a suitable diamond operator, and  $z$  by  $\zeta z$  with  $\zeta$  a suitable element of  $\mu_{pl}(\mathbb{C})$ , we are in one of four cases, that we will treat one by one.

In the first case,  $q$  is mapped to  $(\mathbb{C}^\times/q^{pl\mathbb{Z}}, \zeta_{pl})$ . Then the map  $h \circ g \circ z^{-1}$  factors as the cover  $D(0, e^{-\pi/pl}) \rightarrow D(0, e^{-\pi})$  of degree  $pl$  sending  $q$  to  $q^{pl}$ , followed by the map of Lemma 11.3.7 that sends  $q^{pl}$  to  $(\mathbb{C}^\times/q^{pl\mathbb{Z}}, \zeta_{pl})$ . Then we have, on  $D(0, e^{-\pi/pl})$ :

$$\begin{aligned} h^*\mu &\leq \frac{28e^{4\pi}}{\pi} \frac{1}{(1 - e^{-\pi})^4} \cdot \frac{i}{2} d(q^{pl}) d(\overline{q^{pl}}) \\ &\leq \frac{28e^{4\pi}}{\pi} \frac{1}{(1 - e^{-\pi})^4} (pl)^2 \cdot \frac{i}{2} dq d\bar{q}. \end{aligned}$$

In the second case,  $q$  is mapped to  $(\mathbb{C}^\times/q^{pl\mathbb{Z}}, q^l, \zeta_l)$ . In this case, we compose it with the pseudo-involution  $w_{\zeta_p}$ , which brings us to the

point  $(\mathbb{C}^\times/q^{l\mathbb{Z}}, \zeta_p, \zeta_l)$ . The map then factors as the  $l$ th power map from  $D(0, e^{-\pi/pl})$  to  $D(0, e^{-\pi/p})$ , followed by the map of Lemma 11.3.7 composed with a suitable diamond operator. We find:

$$\begin{aligned} h^* \mu &\leq \frac{28e^{4\pi}}{\pi} \frac{1}{(1 - e^{-\pi/p})^4} \cdot \frac{i}{2} d(q^l) d(\bar{q}^l) \\ &\leq \frac{28e^{4\pi}}{\pi} \frac{1}{(1 - e^{-\pi/p})^4} l^2 \cdot \frac{i}{2} dq d\bar{q}. \end{aligned}$$

The third case is obtained by interchanging the roles of  $p$  and  $l$ , so we will not make it explicit.

In the fourth case,  $q$  is mapped to  $(\mathbb{C}^\times/q^{pl\mathbb{Z}}, q)$ . We compose with the pseudo-involution  $w_{\zeta_{pl}}$ , which brings us to  $(\mathbb{C}^\times/q^{\mathbb{Z}}, \zeta_{pl})$ . This is the map of Lemma 11.3.7. We find:

$$h^* \mu \leq \frac{28e^{4\pi}}{\pi} \frac{1}{(1 - e^{-\pi/pl})^4} \cdot \frac{i}{2} dq d\bar{q}.$$

In these four cases, we see that the factor in front of  $(i/2)dq d\bar{q}$  in the upper bound for  $h^* \mu$  on  $D(0, e^{-\pi/pl})$  is  $O((pl)^4)$ . This gives the required estimate on  $gU$ , as  $dq$  on  $D(0, e^{-\pi/pl})$  corresponds to  $dz_g$  on  $gU$ .  $\square$

**11.3.9 Lemma** *For the local coordinates  $z^{(j)}$  and the real  $(1, 1)$ -form  $\mu'$  on  $X(pl)$  as defined in (11.3.5) we have:*

$$\mu' \leq c_1 |dz^{(j)} \wedge d\bar{z}^{(j)}|$$

on  $U_1^{(j)}$  with  $c_1 = c_1(pl) = O(pl)$ .

**Proof** First of all, we have, by definition:  $\mu' = (1/pl)h^* \mu$ . The definition of the  $z^{(j)}$  (see 11.3.5) plus the definitions  $z' = e^{3\pi/2pl}z$  and  $\varepsilon = e^{\pi/2pl} - 1$  give:

$$dz = \frac{2\varepsilon}{3} e^{-3\pi/2pl} \cdot dz^{(j)}.$$

If  $pl$  gets large, then the factor  $e^{3\pi/2pl}$  tends to 1 and for  $\varepsilon$  we have  $\varepsilon = (\pi/2pl)(1 + O(1/pl))$ . Combining all this with Lemma 11.3.8 finishes the proof.  $\square$

We can now finish the proof of Theorem 11.3.1. We apply Theorem 10.1 on  $X(pl)$  with the  $(1, 1)$ -form  $\mu'$ . Then we have  $n = O((pl)^4)$ ,  $M = 6$

(Lemma 11.3.6) and  $c_1 = O(pl)$  (Lemma 11.3.9). We obtain from (10.2) that there exists a constant  $c$  such that  $g_{b,\mu'}(b') \leq c \cdot (pl)^5$  for all distinct primes  $p$  and  $l$ , and all distinct  $b$  and  $b'$  on  $X(pl)$ . For distinct  $a$  and  $a'$  on  $X_1(pl)$  we then have (see (11.3.2)):

$$g_{a,\mu}(a') = \sum_{h(b)=a} g_{b,\mu'}(h(a')) \leq c \cdot (pl)^6.$$

The statement that  $\log \|dz^{(j)}\|_{\text{Ar}}(P) = O((pl)^6)$  follows from the inequality (10.3) in Theorem 10.1. Indeed, if locally we write  $g_{a,\mu}$  as  $g_{a,\mu} = \log |z - z(a)| + f$  then  $f(a) = -\log \|dz\|_{\text{Ar}}(a)$ .  $\square$

#### 11.4 BOUNDS FOR INTERSECTION NUMBERS ON $X_1(pl)$

In this section, we will bound the intersection numbers occurring in the right hand side of the inequality in Theorem 9.2.5, in the situation described in Section 8.2.

**11.4.1 Theorem** *Let  $p$  and  $l$  be two distinct prime numbers, both at least 5, and let  $\mathcal{X}$  be the semistable model over  $B := \text{Spec } \mathbb{Z}[\zeta_{pl}]$  provided by [Ka-Ma]. For two cusps  $P$  and  $Q$  (possibly equal) in  $\mathcal{X}(B)$  we have:*

$$(P, P) \leq 0, \quad \text{and} \quad |(P, Q)| = O((pl)^7).$$

*For a cuspidal effective divisor  $D$  of degree  $g$  on  $\mathcal{X}$  we have:*

$$|(D, D - \omega_{\mathcal{X}/B})| = O((pl)^{11}).$$

**Proof** As  $p$  and  $l$  are at least 5, the genus of  $X_1(pl)$  is at least two. By the adjunction formula (see (4.4.8)). we have  $-(P, P) = (P, \omega_{\mathcal{X}/B})$ , and by [Fal1], Theorem 5, we have  $(P, \omega_{\mathcal{X}/B}) \geq 0$ , hence  $(P, P) \leq 0$ .

Let us now derive an upper bound for  $(P, \omega_{\mathcal{X}/B})$ . As the automorphism group of  $\mathcal{X}$  over  $B$  preserves the Arakelov intersection product on  $\mathcal{X}$  and acts transitively on the cusps, it suffices to do this for the standard cusp  $\infty$ . The Fourier expansion at  $\infty$  of the rational function  $j$  on  $\mathcal{X}$  is of the form  $j = 1/q + f$  with  $f \in \mathbb{Z}[[q]]$ . Therefore,  $1/j$  is regular in a neighbourhood of  $\infty$ , and has a zero of order one along  $\infty$ . It follows that  $d(1/j)$  generates



$\infty^* \omega_{\mathcal{X}/B}$ , and  $d(1/j) = dq$  in  $\infty^* \omega_{\mathcal{X}/B}$ . By definition of the Arakelov intersection product and Theorem 11.3.1 we then have:

$$(\infty, \omega_{\mathcal{X}/B}) = -[\mathbb{Q}(\zeta_{pl})] : \mathbb{Q} \log \|dq\|_{\text{Ar}}(\infty) = O((pl)^7).$$

We now know  $|(P, P)| = O((pl)^7)$  for all cusps  $P$  in  $\mathcal{X}(B)$ . We will now show that  $|(P, Q)| = O((pl)^7)$ . By the Theorem of Manin-Drinfeld, see [Dri], the image of the divisor  $P - Q$  in  $J_1(pl)(\mathbb{Q}(\zeta_{pl}))$  is of finite order. Let  $\Phi$  be a vertical fractional divisor such that for any irreducible component  $C$  of a fibre of  $\mathcal{X}$  over  $B$  we have  $(P - Q - \Phi, C) = 0$ . By [Hri] or Theorem 4 of [Fal1] we have  $(P - Q - \Phi, P - Q - \Phi) = 0$ . Equivalently, we have:

$$2(P, Q) = (P, P) + (Q, Q) - (P - Q, \Phi).$$

The term  $(P - Q, \Phi)$  can be dealt with by the method used in the proof of Lemma 9.2.6. We work this out in this special situation. We make  $\Phi$  unique by demanding that its support is disjoint from  $P$ . Of course, this does not change the number  $(P - Q, \Phi)$ , but it makes it easier to talk about  $\Phi$ . The support of  $\Phi$  is contained in the reducible fibers. These are exactly the fibers in the characteristics  $p$  and  $l$ . Let us estimate the contribution at the prime  $p$ . We have to sum over the maximal ideals of the  $\mathbb{F}_p$ -algebra  $\mathbb{F}_p[x]/(x^{l-1} + \dots + 1)$ . All residue fields of this algebra are isomorphic to a finite extension  $\mathbb{F}$  of  $\mathbb{F}_p$ , and the number of them is  $(l-1)/\dim_{\mathbb{F}_p} \mathbb{F}$ . Let  $\mathcal{X}_{\mathbb{F}}$  be the fibre at one of these residue fields, and let  $\Phi_{\mathbb{F}}$  be the part of  $\Phi$  that has support in  $\mathcal{X}_{\mathbb{F}}$ . Then  $\mathcal{X}_{\mathbb{F}}$  is the union of two irreducible components  $I$  and  $I'$ , with transversal intersection at the supersingular points. The number of  $\overline{\mathbb{F}_p}$ -valued supersingular points is given by:

$$s := \#\mathcal{X}_{\mathbb{F}}(\overline{\mathbb{F}_p})^{\text{s.s.}} = \#(X_1(l)(\overline{\mathbb{F}_p})^{\text{s.s.}}) = \frac{(p-1)(l^2-1)}{24}.$$

The degree on  $I$  of the restriction to it of  $\mathcal{O}_{\mathcal{X}}(I')$  is  $s$ , and, symmetrically,  $\deg_{I'} \mathcal{O}_{\mathcal{X}}(I) = s$ . As  $I + I' = \mathcal{X}_{\mathbb{F}}$  is a principal divisor in a neighbourhood of  $\mathcal{X}_{\mathbb{F}}$ , the restrictions of  $\mathcal{O}_{\mathcal{X}}(I + I')$  to  $I$  and  $I'$  are trivial, we have  $\deg_I \mathcal{O}_{\mathcal{X}}(I) = -s$  and  $\deg_{I'} \mathcal{O}_{\mathcal{X}}(I') = -s$ . It follows that  $\Phi_{\mathbb{F}}$  is one of the following fractional divisors:  $\Phi_{\mathbb{F}} = 0$  if  $P$  and  $Q$  specialise to points on the same irreducible component of  $\mathcal{X}_{\mathbb{F}}$ ;  $\Phi_{\mathbb{F}} = (1/s) \cdot I$  if  $P$  specialises to a

point on  $I'$  and  $Q$  to a point on  $I$ ;  $\Phi_{\mathbb{F}} = (1/s) \cdot I'$  if  $P$  specialises to a point on  $I$  and  $Q$  to a point on  $I'$ . If we denote by  $(P - Q, \Phi)_{\mathbb{F}}$  the contribution to  $(P - Q, \Phi)$  at the fibre  $\mathcal{X}_{\mathbb{F}}$ , we have:

$$|(P - Q, \Phi)_{\mathbb{F}}| \leq (2/s) \cdot \log \#\mathbb{F}.$$

Summing this over the residue fields of  $\mathbb{F}_p[x]/(x^{l-1} + \dots + 1)$  gives, for the contribution  $(P - Q, \Phi)_p$  to  $(P - Q, \Phi)$  at  $p$ :

$$|(P - Q, \Phi)_p| \leq \frac{l-1}{\dim_{\mathbb{F}_p}(\mathbb{F})} \cdot \frac{2}{s} \cdot \log \#\mathbb{F} = \frac{48 \log p}{(p-1)(l+1)}.$$

Likewise, we have, for the contribution at  $l$  to  $(P - Q, \Phi)$ :

$$|(P - Q, \Phi)_l| \leq \frac{48 \log l}{(l-1)(p+1)}.$$

So, finally:

$$|(P - Q, \Phi)| \leq \frac{48 \log l}{(l-1)(p+1)} + \frac{48 \log p}{(p-1)(l+1)}.$$

The estimate  $|(P, Q)| = O((pl)^7)$  now follows.

To get to the second statement of the theorem, note that

$$\begin{aligned} (D, D - \omega_{\mathcal{X}/B}) &= (D, D + \omega_{\mathcal{X}/B}) - 2(D, \omega_{\mathcal{X}/B}) \\ &= \sum_{k \neq l} (P_k, P_l) - 2(D, \omega_{\mathcal{X}/B}), \end{aligned}$$

where  $D = P_1 + \dots + P_g$ , with repetitions allowed. By our previous estimates, we get  $|(D, D - \omega_{\mathcal{X}/B})| = O(g^2(pl)^7) = O((pl)^{11})$ .  $\square$

We will also need a lower bound for the intersection number of two distinct points on  $X_1(pl)$ .

**11.4.2 Theorem** *There is an integer  $c$  such that for all pairs of distinct primes  $p$  and  $l$  such that  $X_1(pl)$  has genus at least one, for any extension  $K$  of  $\mathbb{Q}(\zeta_{5l})$  and for  $P$  and  $Q$  distinct points in  $X_1(pl)(K)$  we have:*

$$\frac{1}{[K : \mathbb{Q}]}(P, Q) \geq c(pl)^6,$$

where  $(P, Q)$  is the Arakelov intersection number of  $P$  and  $Q$  on the minimal regular model of  $X_1(pl)$  over  $O_K$ .

**Proof** We have:

$$(P, Q) = (P, Q)_{\text{fin}} + (P, Q)_{\infty},$$

with  $(P, Q)_{\text{fin}}$  the contribution from the finite places of  $K$ , and  $(P, Q)_{\infty}$  the contribution from the infinite places. As  $P \neq Q$ , we have  $(P, Q)_{\text{fin}} \geq 0$ . On the other hand, we have:

$$(P, Q)_{\infty} = \sum_{\sigma} -g_{\sigma}(P_{\sigma}, Q_{\sigma}).$$

By Theorem 11.3.1 we have:

$$g_{\sigma}(P_{\sigma}, Q_{\sigma}) \leq c \cdot (pl)^6$$

for some absolute constant  $c$ . This finishes the proof.  $\square$

### 11.5 A BOUND FOR $h(x'_i(Q))$ IN TERMS OF $h(b_l(Q))$

In this section we do what was promised at the beginning of Section 9.1, by stating and proving the following proposition and a corollary.

**11.5.1 Proposition** *There is a real number  $c$  such that the following holds. Let  $b$  be in  $\overline{\mathbb{Q}}$ , such that  $b^5(b^2 + 11b - 1) \neq 0$ . Let  $(u, v)$  in  $\overline{\mathbb{Q}}^2$  be a torsion point on the elliptic curve  $E_b$  over  $\overline{\mathbb{Q}}$  given by the equation:*

$$(11.5.2) \quad y^2 + (b + 1)xy + by = x^3 + bx^2,$$

*i.e., on the fibre at  $b$  of the universal elliptic curve with a point of order 5 given in Proposition 8.2.8. Then the absolute heights  $h(u)$  and  $h(v)$  are bounded from above by  $c + 14h(b)$ .*

**Proof** Let  $b, u$  and  $v$  in  $\overline{\mathbb{Q}}$  be as in the statement of the proposition. We will now invoke known bounds for the difference between the Weil height and the Néron-Tate height on elliptic curves over number fields. Such bounds are given, for example, in [Dem], [Zim], and [Sil]. In [Sil] bounds are given for elliptic curves given by general Weierstrass equations, but under the assumption that the coefficients are algebraic integers. Therefore, for us it seems better to use the following bound in Zimmer's theorem on page 40 of [Zim].

**11.5.3 Theorem (Zimmer)** *Let  $E$  be an elliptic curve over  $\overline{\mathbb{Q}}$  given by a Weierstrass equation  $y^2 = x^3 + Ax + B$ , and let  $P \in E(\overline{\mathbb{Q}})$ . Let  $h$  denote the absolute Weil height on  $\mathbb{P}^2(\overline{\mathbb{Q}})$ , and  $\hat{h}$  the absolute Néron-Tate height on  $E$  attached to  $h$ . Then one has:*

$$\begin{aligned} -2^{-1}(2^{-1}h(1 : A^3 : B^2) + 7 \log 2) &\leq h(P) - \hat{h}(P) \leq \\ &\leq 2^{-1}h(1 : A^3 : B^2) + 6 \log 2. \end{aligned}$$

So, we must compare our plane elliptic curve  $E_b$  with one given by a standard Weierstrass equation. We put:

$$v_1 := v + ((b+1)u + b)/2, \quad u_1 := u + (b + (b+1)^2/4)/3.$$

Then  $(u_1, v_1)$  is a point on the elliptic curve  $E'_b$  given by a Weierstrass equation  $y^2 = x^3 + Ax + B$ , with  $A$  and  $B$  polynomials in  $b$ , with coefficients in  $\mathbb{Q}$ , of degrees at most 4 and 6, respectively. We note that  $A$  and  $B$  depend only on  $b$ , not on  $(u, v)$ . Using Lemma 4.2.3 and writing  $A$  as  $A_0 + b(A_1 + b(A_2 + b(A_3 + bA_4)))$ , we see that there is a real number  $c_1$ , such that for all  $b$  we have  $h(A) \leq c_1 + 4h(b)$ . Similarly, there is a  $c_2$  such that  $h(B) \leq c_2 + 6h(b)$ . Therefore, there is a  $c_3$  such that  $h(1 : A^3 : B^2) \leq c_3 + 24h(b)$ . Zimmer's theorem 11.5.3, plus the fact that the Néron-Tate height of torsion points is zero, tells us that there is a  $c_4$  such that for all  $b$  and for all torsion points  $(u, v)$  on  $E_b$ , we have:

$$h(u_1, v_1) \leq c_4 + 12h(b).$$

Expressing  $u$  and  $v$  in  $u_1$  and  $v_1$ , and using again Lemma 4.2.3, we get a real number  $c_5$  such that for all  $b$  and all  $(u, v)$  as in the proposition we are proving, we have:

$$h(u) \leq c_5 + 14h(b), \quad h(v) \leq c_5 + 14h(b).$$

This ends the proof of Proposition 11.5.1. □

**11.5.4 Corollary** *There is a real number  $c$  such that for each  $l$  and each  $Q_{x,i}$  as in the beginning of Section 9.1 we have:*

$$h(x'_l(Q_{x,i})) \leq c + 14h(b_l(Q_{x,i})).$$

**Proof** By definition (Section 8.2),  $x'_l(Q_{x,i})$  is the  $x$ -coordinate of a point of order  $5l$  on the fibre at  $b_l(Q_{x,i})$  of the elliptic curve  $E$  in Proposition 8.2.8. Proposition 11.5.1 gives the result.  $\square$

## 11.6 AN INTEGRAL OVER $X_1(5l)$

In this section we will give an upper bound for the integral appearing in the estimate in Theorem 9.1.1. We recall the situation:  $l$  is a prime number with  $l > 5$ , and  $b$  is the regular function on  $Y_1(5)_{\mathbb{Q}}$  given by Proposition 8.2.8. We also view  $b$  as a regular function  $b_l$  on  $Y_1(5l)_{\mathbb{Q}}$  via pullback along the map  $Y_1(5l)_{\mathbb{Q}} \rightarrow Y_1(5)_{\mathbb{Q}}$  that sends  $(E/S, P_5, P_l)$  to  $(E/S, P_5)$ , where  $S$  is any  $\mathbb{Q}$ -scheme,  $E/S$  an elliptic curve,  $P_5$  in  $E(S)$  everywhere of order 5, and  $P_l$  in  $E(S)$  everywhere of order  $l$ .

**11.6.1 Proposition** *There exist real numbers  $A$  and  $B$  such that for all primes  $l > 5$ :*

$$\int_{X_1(5l)(\mathbb{C})} \log(|b_l|^2 + 1) \mu \leq A + B \cdot l^6,$$

where  $\mu$  is the Arakelov  $(1, 1)$ -form.

**Proof** In order to simplify the notation in the proof, we will let  $X_1(5l)$  denote the Riemann surface  $X_1(5l)_{\mathbb{Q}}(\mathbb{C})$  of complex points of the curve  $X_1(5l)_{\mathbb{Q}}$  over  $\mathbb{Q}$ , and we will drop the subscript  $l$  in  $b_l$ . We will denote points of  $Y_1(5l)$  as triples  $(E, P_5, P_l)$  with  $E$  a complex elliptic curve, with points  $P_5$  of order 5 and  $P_l$  of order  $l$ . Similarly, we will denote points of  $Y_1(5)$  by pairs  $(E, P)$ , and points on the  $j$ -line just by elliptic curves.

The equations in Proposition 8.2.8 show that the rational function  $b$  on  $X_1(5)$  has exactly one pole, that it is of order one, and that at that point, the function  $j: X_1(5) \rightarrow \mathbb{P}^1$  has a pole of order 5. The region in  $\mathbb{H}$  consisting of the  $\tau$  with  $\Im(\tau) > 1$  gives an embedding of the disk  $D(0, e^{-2\pi})$  into the  $j$ -line, sending  $q \neq 0$  to  $\mathbb{C}^{\times}/q^{\mathbb{Z}}$ . The inverse image of this disk under  $j: X_1(5) \rightarrow \mathbb{P}^1$  consists of 4 disks, one around each cusp. The two disks around the cusps where  $j$  is ramified are given by the embeddings  $D(0, e^{-2\pi/5}) \rightarrow X_1(5)$ , sending  $q \neq 0$  to  $(\mathbb{C}^{\times}/q^{5\mathbb{Z}}, q)$  and to  $(\mathbb{C}^{\times}/q^{5\mathbb{Z}}, q^2)$ . As the integral in the proposition that we are proving does not change if we

replace  $b$  by its image under a diamond operator  $\langle a \rangle$  with  $a \in \mathbb{F}_5^\times$ , we may and do suppose that  $b$  has its pole at the center of the punctured disk

$$U = \{(\mathbb{C}^\times/q^{5\mathbb{Z}}, q) : q \in D(0, e^{-2\pi/5})^*\} \subset X_1(5).$$

The integral of  $\log(|b|^2 + 1)\mu$  over the complement of the inverse image of  $U$  in  $X_1(5l)$  is bounded by the supremum of  $\log(|b|^2 + 1)$  on  $X_1(5) - U$ . This upper bound does not depend on  $l$ . Hence it is enough to prove that the integral of  $\log(|b|^2 + 1)\mu$  over the inverse image of  $U$  in  $X_1(5l)$  is bounded by  $B \cdot l^6$ , for a suitable  $B$ . Now this inverse image of  $U$  is a union of  $2(l-1)$  punctured disks  $U_c$  and  $V_d$  with  $c$  and  $d$  running through  $\mathbb{F}_l^\times$  given by:

$$U_c = \{(\mathbb{C}^\times/q^{5\mathbb{Z}}, q, \zeta_l^c) : q \in D(0, e^{-2\pi/5})^*\}$$

and:

$$V_d = \{(\mathbb{C}^\times/q^{5l\mathbb{Z}}, q^l, q^{5d}) : q \in D(0, e^{-2\pi/5l})^*\}.$$

For each  $c$  in  $\mathbb{F}_l^\times$  the map  $X_1(5l) \rightarrow X_1(5)$  restricts to the isomorphism  $U_c \rightarrow U$  given by  $q \mapsto q$ . For each  $d$  in  $\mathbb{F}_l^\times$ , the restriction  $V_d \rightarrow U$  is given by  $q \mapsto q^l$ .

Around the standard unramified cusp of  $X_1(5l)$  we have the punctured disk:

$$W_l = \{(\mathbb{C}^\times/q^{\mathbb{Z}}, \zeta_5, \zeta_l) : q \in D(0, e^{-2\pi/5})^*\}.$$

By applying a suitable element from the group of automorphisms of  $X_1(5l)$  generated by the Atkin-Lehner pseudo-involutions and the diamond operators we can establish isomorphisms of the  $U_c$  with  $W_l$ . The point  $(\mathbb{C}^\times/q^{5\mathbb{Z}}, q, \zeta_l^c)$  in  $U_c$  is then first sent to  $(\mathbb{C}^\times/q^{5\mathbb{Z}}, q, \zeta_l)$  by  $\langle c^{-1} \rangle$ , and then to  $(\mathbb{C}^\times/q^{\mathbb{Z}}, \zeta_5, \zeta_l)$  by the Atkin-Lehner pseudo-involution that divides out by the group generated by the point of order 5 (see Definition 2.5.13). The coordinate  $q$  on  $W_l$  is then identified in this way with the coordinate  $q$  on  $U_c$ . We observe that  $\mu$  is invariant under each automorphism of  $X_1(5l)$ . Lemma 11.3.7 gives a real number  $C_1$  such that for all  $l$  and  $c$ , the positive real  $(1, 1)$ -form  $\mu$  on  $U_c$  can be estimated from above by:

$$\mu|_{U_c} \leq C_1 \cdot \frac{i}{2} dqd\bar{q}.$$

Similarly, for each  $d$  in  $\mathbb{F}_l^\times$  a suitable automorphism of  $X_1(5l)$  maps  $V_d$  to the punctured disk  $W'_l$ :

$$W'_l = \{(\mathbb{C}^\times/q^{\mathbb{Z}}, \zeta_5, \zeta_l) : q \in D(0, e^{-2\pi/5l})^*\}.$$

Lemma 11.3.7 gives a real number  $C_2$  such that for all  $l$  and  $d$ , the positive real  $(1, 1)$ -form  $\mu$  on  $V_d$  can be estimated from above by:

$$\mu|_{V_d} \leq C_2 \cdot \frac{1}{(1 - e^{-2\pi/5l})^4} \cdot \frac{i}{2} dqd\bar{q}.$$

Now the function  $qb$  on  $U$  extends to a holomorphic function on a disk containing  $U$ , hence  $|qb|$  is bounded on  $U$ . Hence there is a real number  $C_3 > 1$  such that  $|b|^2 + 1 \leq C_3 \cdot |q^{-2}|$  on  $U$ . It follows that on all  $U_c$  we have  $|b|^2 + 1 \leq C_3 \cdot |q^{-2}|$ , and that on all  $V_d$  we have  $|b|^2 + 1 \leq C_3 \cdot |q^{-2l}|$  (note that under  $V_d \rightarrow U$ ,  $(\mathbb{C}^\times/q^{5l\mathbb{Z}}, q^l, q^{5d})$  is sent to  $(\mathbb{C}^\times/q^{5l\mathbb{Z}}, q^l)$ ). We remark that  $1/(1 - e^{-x}) = x^{-1}(1 + O(x))$  as  $x$  tends to 0 from above. Hence there is a  $C_4 \in \mathbb{R}$  such that for all  $l$ :

$$\frac{1}{(1 - e^{-2\pi/5l})^4} \leq C_4 \cdot l^4.$$

We get, for all  $l$ :

$$\begin{aligned} \int_{b^{-1}U} \log(|b|^2 + 1) \mu &\leq (l-1)C_1 \int_{D(0, e^{-2\pi/5})} \log(C_3 \cdot |q^{-2}|) \cdot \frac{i}{2} dqd\bar{q} \\ &+ (l-1)C_2 \frac{1}{(1 - e^{-2\pi/5l})^4} \int_{D(0, e^{-2\pi/5l})} \log(C_3 \cdot |q^{-2l}|) \cdot \frac{i}{2} dqd\bar{q} \leq \\ &\leq \int_{|z|<1} \left( lC_1 \log(C_3 \cdot |z|^{-2}) + C_2 C_4 l^5 \log(C_3 \cdot |z|^{-2l}) \right) \frac{i}{2} dzd\bar{z} \\ &\leq \int_{|z|<1} \left( C_1 l \log(C_3 \cdot |z|^{-2}) + C_2 C_4 l^6 \log(C_3 \cdot |z|^{-2}) \right) \frac{i}{2} dzd\bar{z} \\ &= (C_1 l + C_2 C_4 l^6) \cdot (\pi \log C_3 + \pi). \end{aligned}$$

This finishes the proof of Proposition 11.6.1 □

### 11.7 FINAL ESTIMATES OF THE ARAKELOV CONTRIBUTION

We will now put the estimates of the preceding sections together, in the situation of Section 8.2. We briefly recall this situation. We have a prime

number  $l > 5$ , and  $X_l$  denotes the modular curve  $X_1(5l)$ , over  $\mathbb{Q}$ , and  $g_l$  its genus. The Jacobian variety of  $X_l$  is denoted by  $J_l$ . In  $J_l(\overline{\mathbb{Q}})[l]$  we have the  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -module  $V$  that realises the representation  $\rho$  from  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  to  $\text{GL}_2(\mathbb{F})$  attached to a surjective ring morphism  $f: \mathbb{T}(1, k) \twoheadrightarrow \mathbb{F}$  such that the image of  $\rho$  contains  $\text{SL}_2(\mathbb{F})$ . We have an effective divisor  $D_0$  on  $X_{l, \mathbb{Q}(\zeta_l)}$ , of degree  $g_l$ , supported on the cusps. For every  $x$  in  $V$  there is a unique effective divisor  $D_x = Q_{x,1} + \dots + Q_{x,g_l}$  of degree  $g_l$  such that  $x$  is the class of  $D_x - D_0$ . We have written  $D_x = D_x^{\text{fin}} + D_x^{\text{cusp}}$ , where  $D_x^{\text{cusp}}$  is the part of  $D_x$  supported on the cusps. The numbering of the  $Q_{x,i}$  is such that  $D_x^{\text{fin}} = Q_{x,1} + \dots + Q_{x,d_x}$ . We have morphisms  $b_l$  and  $x'_l$  from  $X_{l, \mathbb{Q}}$  to  $\mathbb{P}_{\mathbb{Q}}^1$  that, seen as rational functions, have their poles contained in the set of cusps of  $X_l$ .

The following proposition gives upper bounds for the absolute heights of the algebraic numbers  $b_l(Q_{x,i})$  and  $x'_l(Q_{x,i})$ , polynomial in  $l$ . The height function  $h$  used here is as defined in (4.1.6). The proof of the proposition combines the involved arguments of the previous sections.

**11.7.1 Proposition** *There is an integer  $c$ , such that for all  $x$  in  $V$  as above, and for all  $i \in \{1, \dots, d_x\}$ , the absolute heights of  $b_l(Q_{x,i})$  and  $x'_l(Q_{x,i})$  are bounded from above by  $c \cdot l^{12}$ .*

**Proof** We will just write  $b$  for  $b_l$ . Let  $V$  be as in the proposition. Theorem 9.1.1 shows that for all  $x \in V$ , all  $i \in \{1, \dots, d_x\}$  and all number fields  $K$  containing  $\mathbb{Q}(\zeta_{5l})$  over which  $Q_{x,i}$  is rational, we have:

$$(11.7.2) \quad h(b(Q_{x,i})) \leq \frac{1}{[K : \mathbb{Q}]} \left( (Q_{x,i}, b^* \infty)_{\mathcal{X}} + l^2 \sum_{\sigma} \sup_{X_{\sigma}} g_{\sigma} + \frac{1}{2} \sum_{\sigma} \int_{X_{\sigma}} \log(|b|^2 + 1) \mu_{X_{\sigma}} \right) + \frac{1}{2} \log 2.$$

Here  $\mathcal{X}$  is the minimal regular model of  $X_l$  over  $B := \text{Spec } O_K$ .

Let us first concentrate on the second and third terms of the right hand side of (11.7.2). Theorem 11.3.1 gives an integer  $c_1$  such that for all  $\sigma$  we have  $\sup_{X_{\sigma}} g_{\sigma} \leq c_1 \cdot l^6$ , uniformly in all  $l$ . Proposition 11.6.1 gives an integer  $c_2$  such that for all  $\sigma$  we have  $(1/2) \int_{X_{\sigma}} \log(|b|^2 + 1) \mu_{X_{\sigma}} \leq c_2 \cdot l^6$ , uniformly in all  $l$ . Hence, for all  $x \in V$ , all  $i \in \{1, \dots, d_x\}$  and all number



fields  $K$  containing  $\mathbb{Q}(\zeta_{5l})$  over which  $Q_{x,i}$  is rational, we have:

$$(11.7.3) \quad h(b(Q_{x,i})) \leq \frac{1}{[K : \mathbb{Q}]} (Q_{x,i}, b^* \infty)_{\mathcal{X}} + c_1 \cdot l^8 + c_2 \cdot l^6,$$

uniformly in all  $l$  and  $V$ .

We now concentrate on the first term. We recall that  $b^* \infty$  is an effective cuspidal divisor on  $X_l$ . Let  $x$  be in  $V$ . We apply Theorem 9.2.5, where (in the notation of that theorem) we assume that all  $Q_{x,i}$  ( $i \in \{1, \dots, g_l\}$ ) are  $K$ -rational and that  $K$  contains  $\mathbb{Q}(\zeta_{5l})$ , and that  $P$  is a cusp. We also use the obvious fact that  $\log \#R^1 p_* O_{\mathcal{X}}(D_x)$  is nonnegative. That gives:

$$\begin{aligned} \frac{(D_x, P)}{[K : \mathbb{Q}]} \leq \frac{1}{[K : \mathbb{Q}]} & \left( -\frac{1}{2} (D_0, D_0 - \omega_{\mathcal{X}/B}) + 2g_l^2 \sum_{s \in B} \delta_s \log \#k(s) \right. \\ & + \sum_{\sigma} \log \|\vartheta\|_{\sigma, \text{sup}} + \frac{g_l}{2} [K : \mathbb{Q}] \log(2\pi) \\ & \left. + \frac{1}{2} \deg \det p_* \omega_{\mathcal{X}/B} + (D_0, P) \right). \end{aligned}$$

Theorem 11.4.1, applied with  $B = \text{Spec}(\mathbb{Z}[\zeta_{5l}])$ , gives that:

$$\frac{1}{[K : \mathbb{Q}]} |(D_0, D_0 - \omega_{\mathcal{X}/B})| = O(l^{10}),$$

and that:

$$\frac{1}{[K : \mathbb{Q}]} |(D_0, P)| = O(l^8),$$

as  $D_0$  is an effective cuspidal divisor of degree  $g_l$  and  $g_l = O(l^2)$ . By Theorem 11.2.1 we have:

$$\frac{1}{[K : \mathbb{Q}]} \sum_{\sigma} \log \|\vartheta\|_{\sigma, \text{sup}} = O(l^6).$$

By Theorem 11.1.7 we have:

$$\frac{1}{[K : \mathbb{Q}]} \deg \det p_* \omega_{\mathcal{X}/B} = O(l^2 \log l).$$

Finally, we have:

$$\frac{g_l^2}{[K : \mathbb{Q}]} \sum_{s \in B} \delta_s \log \#k(s) = O(l^6),$$

by the following argument. The only non-trivial contributions come from  $s$  over 5 and over  $l$ . The total contribution at 5 is independent of which extension  $K$  of  $\mathbb{Q}(\zeta_5)$  we use, and for  $\mathbb{Q}(\zeta_5)$  there is only one  $s$  over 5,  $k(s) = \mathbb{F}_5$ , and  $\delta_s$  equals the number of supersingular points in  $X_1(l)(\overline{\mathbb{F}}_5)$ , which is  $O(l^2)$ . The contribution from  $l$  can be computed over  $\mathbb{Q}(\zeta_l)$ . Then there is one  $s$ , and  $k(s) = \mathbb{F}_l$ , and  $\delta_s$  is the number of supersingular points in  $X_1(5)(\overline{\mathbb{F}}_l)$ , which is  $O(l)$  (see Section 8.1).

Putting these last estimates together, we get that there is an integer  $c_3$  such that for all  $l$  and  $x$  we have:

$$(11.7.4) \quad \frac{1}{[K : \mathbb{Q}]}(D_x, P) \leq c_3 \cdot l^{10}.$$

Now  $D_x$  is the sum of  $g_l$  points  $Q_{x,i}$ . In order to get upper bounds for the individual  $(Q_{x,i}, P)$  we need a lower bound for these. Theorem 11.4.2 gives us a lower bound if  $Q_{x,i} \neq P$  and Theorem 11.4.1 gives us one if  $P = Q_{x,i}$ . Putting these together, we get an integer  $c_4$  such that for all  $l, x$  and  $i$ :

$$\frac{1}{[K : \mathbb{Q}]}(Q_{x,i}, P) \geq c_4 \cdot l^6.$$

The last two estimates together imply that there is an integer  $c_5$  such that for all  $l, x$  and  $i$  we have:

$$\frac{1}{[K : \mathbb{Q}]}(Q_{x,i}, P) \leq c_5 \cdot l^{10}.$$

As  $b^*\infty$  is an effective cuspidal divisor on  $X_l$ , of degree  $O(l^2)$ , the previous inequality implies that there is an integer  $c_6$  such that for all  $l, x$  and  $i \in \{1, \dots, d_x\}$  we have:

$$\frac{1}{[K : \mathbb{Q}]}(Q_{x,i}, b^*\infty)_X \leq c_6 \cdot l^{12}.$$

This finishes the proof concerning the height of  $b_l(Q_{x,i})$ . Corollary 11.5.4 then finishes the proof.  $\square$

We recall, from the end of Section 8.2, that, in the situation as described in the beginning of this section, we take a linear combination  $f_l := b_l + nx'_l$  with  $0 \leq n \leq g_l^2(\#\mathbb{F})^4$ , such that under the map  $f_l: X_{l,\overline{\mathbb{Q}}} \rightarrow \mathbb{P}_{\overline{\mathbb{Q}}}^1$  the divisors  $D_x^{\text{fin}} = Q_{x,1} + \dots + Q_{x,d_x}$ , for  $x \in V$ , have distinct images  $f_{l,*}D_x^{\text{fin}}$ .

Suppose that  $f_l$  is any such linear combination. The  $f_{l,*}D_x^{\text{fin}}$  are then distinct effective divisors of degree  $d_x$  on  $\mathbb{A}_{\overline{\mathbb{Q}}}^1$ .

For each  $x$  in  $V$ , we get a polynomial  $P_{D_0, f_l, x}$  with coefficients in  $\overline{\mathbb{Q}}$  given by:

$$P_{D_0, f_l, x}(t) = \prod_{i=1}^{d_x} (t - f_l(Q_{x,i})) \quad \text{in } \overline{\mathbb{Q}}[t],$$

and the map that sends  $x$  to  $P_{D_0, f_l, x}$  is injective, and  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l))$ -equivariant.

We have seen that there is an integer  $m$  with  $0 \leq m \leq g \cdot (\#\mathbb{F})^4$  such that the map:

$$a_{D_0, f_l, m}: V \rightarrow \overline{\mathbb{Q}}, \quad x \mapsto P_{D_0, f_l, x}(m) = \prod_{i=1}^{d_x} (m - f_l(Q_{x,i}))$$

is injective and hence a generator of the  $\mathbb{Q}(\zeta_l)$ -algebra  $A_{l, \mathbb{Q}(\zeta_l)}$  associated with  $V$ . Assume that  $m$  is such an integer. The following theorem gives our final upper bound for the absolute height of the coefficients of the minimal polynomial:

$$(11.7.5) \quad P_{D_0, f_l, m} = \prod_{x \in V} (T - a_{D_0, f_l, m}(x)) = \sum_j P_j T^j \quad \text{in } \mathbb{Q}(\zeta_l)[T]$$

of  $a_{D_0, f_l, m}$  over  $\mathbb{Q}(\zeta_l)$ .

**11.7.6 Theorem** *There exists an integer  $c$  such that for all  $l, V, D_0, f_l$  and  $m$  as above we have, for all  $P_j$  as in (11.7.5):*

$$h(P_j) \leq c \cdot l^{14} \cdot (\#\mathbb{F})^2.$$

**Proof** Let  $c_1$  be an integer as given by Proposition 11.7.1. Let  $l, V, D_0, f_l$  and  $m$  be as in the theorem. For each  $x \in V$  and each  $i$  in  $\{1, \dots, d_x\}$  we

have, by Proposition 11.7.1, the definition of  $f_l$ , and Lemma 4.2.3:

$$\begin{aligned}
 (11.7.7) \quad h(f_l(Q_{x,i})) &= h(b_l(Q_{x,i}) + nx'_l(Q_{x,i})) \\
 &\leq \log 2 + h(b_l(Q_{x,i})) + h(nx'_l(Q_{x,i})) \\
 &\leq \log 2 + h(b_l(Q_{x,i})) + h(n) + h(x'_l(Q_{x,i})) \\
 &\leq \log 2 + c_1 \cdot l^{12} + \log(l^2 \cdot (\#\mathbb{F})^4) + c_1 \cdot l^{12} \\
 &\leq 2c_1 \cdot l^{12} + 2 \log l + 4 \log(\#\mathbb{F}) + \log 2 \\
 &\leq c_2 \cdot l^{12} + 4 \log(\#\mathbb{F}),
 \end{aligned}$$

for  $c_2 = c_1 + 1$ .

In order to simplify the notation during the rest of this proof, we write  $a$  for  $a_{D_0, f_l, m}$  and  $P$  for  $P_{D_0, f_l, m}$ . Then we have, for each  $x \in V$  (using  $d_x \leq l^2$  and (11.7.7)):

$$\begin{aligned}
 (11.7.8) \quad h(a(x)) &= h\left(\prod_{i=1}^{d_x} (m - f_l(Q_{x,i}))\right) \leq \sum_{i=1}^{d_x} h(m - f_l(Q_{x,i})) \\
 &\leq \sum_{i=1}^{d_x} (\log 2 + h(m) + h(f_l(Q_{x,i}))) \\
 &\leq d_x \cdot (\log 2 + \log(l^2 (\#\mathbb{F})^4) + c_2 \cdot l^{12} + 4 \log(\#\mathbb{F})) \\
 &\leq c_3 \cdot l^{14} + 8l^2 \log(\#\mathbb{F}),
 \end{aligned}$$

where  $c_3 = c_2 + 1$ .

Let  $j$  be in  $\{0, \dots, \#V\}$ . Then  $P_j$  is, up to a sign, the value of the elementary symmetric polynomial of degree  $\#V - j$  evaluated in the  $a(x)$ , where  $x$  ranges through  $V$ . Lemma 4.2.4, together with (11.7.8), gives us:

$$\begin{aligned}
 h(P_j) &\leq \#V \cdot \log 2 + \sum_{x \in V} h(a(x)) \\
 &\leq \#V \cdot (\log 2 + c_3 \cdot l^{14} + 8l^2 \log(\#\mathbb{F})) \\
 &\leq c \cdot (\#\mathbb{F})^2 \cdot l^{14},
 \end{aligned}$$

where  $c = c_3 + 1$ , and where we have used that  $\log \#\mathbb{F} \leq l \log l$  since  $\mathbb{F}$  is a quotient of  $\mathbb{T}(1, k)$  and  $k \leq l + 1$ .  $\square$

A last consequence of all Arakelovian estimates is the following upper bound for the term  $\log \#R^1 p_* O_{\mathcal{X}}(D_x)$  in Theorem 9.2.5.

**11.7.9 Theorem** *There is an integer  $c$  such that for all  $l > 5$  prime and all  $x$  in  $V$  we have:*

$$\frac{1}{[K : \mathbb{Q}]} \log \#R^1 p_* O_{\mathcal{X}}(D_x) \leq c \cdot l^{10}.$$

**Proof** We have already seen, in the proof of Proposition 11.7.1, that the right hand side of the inequality in Theorem 9.2.5, divided by  $[K : \mathbb{Q}]$ , is bounded from above by a constant times  $l^{10}$ . We have also seen in Theorem 11.4.2 that the term  $(D_x, P)/[K : \mathbb{Q}]$  on the left hand side is bounded from below by a constant times  $l^8$  (recall that  $D_x$  is of degree  $O(l^2)$ ). This proves the inequality.  $\square$

This upper bound will be very useful for us, as the next interpretation shows. Recall that  $X_l$  has good reduction over  $\mathbb{Z}[1/5l]$ .

**11.7.10 Theorem** *There is an integer  $c$  with the following property. Let  $l, X_l, V$  and  $D_0$  as in the beginning of this section. A prime number  $p \nmid 5l$  is said to be  $V$ -good if for all  $x$  in  $V - \{0\}$  the following two conditions are satisfied:*

1. *at all places  $v$  of  $\overline{\mathbb{Q}}$  over  $p$  the specialisation  $D_{x, \overline{\mathbb{F}}_p}$  at  $v$  is the unique effective divisor on the reduction  $X_{l, \overline{\mathbb{F}}_p}$  such that the difference with  $D_{0, \overline{\mathbb{F}}_p}$  represents the specialisation of  $x$ ;*
2. *the specialisations of the non-cuspidal part  $D_x^{\text{fin}}$  of  $D_x$  at all  $v$  above  $p$  are disjoint from the cusps.*

Then we have:

$$\sum_{p \text{ not } V\text{-good}} \log p \leq c l^{12} \cdot (\#\mathbb{F})^2.$$

**Proof** First of all, a prime number  $p$  satisfies conditions (1) and (2) for all  $x$  in  $V - \{0\}$  if and only if it satisfies them one of them, as  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l))$  acts transitively on  $V - \{0\}$  by assumption.

We take  $K$  to be the extension of  $\mathbb{Q}(\zeta_l)$  that corresponds to the transitive  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l))$ -set  $V - \{0\}$ ; this is the field of definition of one  $x$

in  $V - \{0\}$ . Then  $[K : \mathbb{Q}] = O(l \cdot (\#\mathbb{F})^2)$ . We define  $S(V)$  to be the image in  $\text{Spec}(\mathbb{Z}[1/5l])$  of the support in  $\text{Spec}(O_K)$  of the finite  $O_K$ -module  $R^1 p_* O_{\mathcal{X}}(D_x)$ . By Theorem 11.7.9 we have:

$$\log \#(\mathbb{Z}[1/5l] \otimes R^1 p_* O_{\mathcal{X}}(D'_x)) = O(l^{11} \cdot (\#\mathbb{F})^2),$$

and hence:

$$\sum_{p \in S(V)} \log p = O(l^{11} \cdot (\#\mathbb{F})^2).$$

We claim that the  $p$  in  $S(V)$  are precisely the primes  $p \notin \{5, l\}$  such that condition (1) is not satisfied for  $x$ . To see this, we first note that for a morphism  $O_K \rightarrow \overline{\mathbb{F}}_p$  the canonical map from  $\overline{\mathbb{F}}_p \otimes_{O_K} R^1 p_* O_{\mathcal{X}}(D_x)$  to  $H^1(X_{l, \overline{\mathbb{F}}_p}, O_{\mathcal{X}_{\overline{\mathbb{F}}_p}}(D_{x, \overline{\mathbb{F}}_p}))$  is an isomorphism (see Theorem III.12.1 of [Hart]; base change and cohomology in top dimension commute). The divisor  $D_{x, \overline{\mathbb{F}}_p}$  is the unique effective divisor in its linear equivalence class if and only if  $h^0(X_{l, \overline{\mathbb{F}}_p}, O_{\mathcal{X}_{\overline{\mathbb{F}}_p}}(D_{x, \overline{\mathbb{F}}_p})) = 1$ , which, by Riemann-Roch, is equivalent to  $h^1(X_{l, \overline{\mathbb{F}}_p}, O_{\mathcal{X}_{\overline{\mathbb{F}}_p}}(D_{x, \overline{\mathbb{F}}_p})) = 0$ .

Now we let  $T(V)$  denote the set of primes  $p \notin \{5, l\}$  such that at least one specialisation of  $D_x^{\text{fin}}$  at a place of  $K$  above  $p$  is not disjoint from the cusps. Taking into account that  $[K : \mathbb{Q}] = O(l \cdot (\#\mathbb{F})^2)$ , equation (11.7.4) gives us an upper bound:

$$(D_x, P) = O(l^{11} \cdot (\#\mathbb{F})^2)$$

As the degree of  $D_x^{\text{cusp}}$  is at most  $O(l^2)$ , Theorem 11.4.1 gives us:

$$|(D_x^{\text{cusp}}, P)| = O(l^7 \cdot (\#\mathbb{F})^2), \quad \text{hence} \quad (D_x^{\text{fin}}, P) = O(l^{11} \cdot (\#\mathbb{F})^2).$$

As the divisor Cusps has degree  $O(l)$ , we have:

$$(D_x^{\text{fin}}, \text{Cusps}) = O(l^{12} \cdot (\#\mathbb{F})^2).$$

The intersection number  $(D_x^{\text{fin}}, \text{Cusps})$  is the sum of  $(D_x^{\text{fin}}, \text{Cusps})_{\text{fin}}$ , the contribution of the finite places, and  $(D_x^{\text{fin}}, \text{Cusps})_{\infty}$ , the contribution of the infinite places. We have:

$$(D_x^{\text{fin}}, \text{Cusps})_{\infty} = \sum_{i, P, \sigma} -g_{\sigma}(Q_{x, i}, P),$$

where the sum is taken over the  $i$  with  $1 \leq i \leq d_x$ , over the cusps  $P$  and the  $\sigma: K \rightarrow \mathbb{C}$ . Then Theorem 11.3.1 gives the upper bound:

$$(D_x^{\text{fin}}, \text{Cusps})_{\text{fin}} = (D_x^{\text{fin}}, \text{Cusps}) - (D_x^{\text{fin}}, \text{Cusps})_{\infty} = O(l^{12} \cdot (\#\mathbb{F})^2).$$

By the definition of our set  $T(V)$  and the definition of  $(D_x^{\text{fin}}, \text{Cusps})_{\text{fin}}$  we get:

$$\sum_{p \in T(V)} \log p \leq (D_x^{\text{fin}}, \text{Cusps})_{\text{fin}} = O(l^{12} \cdot (\#\mathbb{F})^2).$$

The proof of the theorem is then finished by noticing that the set of primes  $p \notin \{5, l\}$  that are not  $V$ -good is precisely the union of  $S(V)$  and  $T(V)$ .  $\square$

## Chapter Twelve

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### Approximating $V_f$ over the complex numbers

J.-M. Couveignes

In this chapter, we address the problem of computing torsion divisors on modular curves with an application to the explicit calculation of modular representations. We assume we are given an even integer  $k > 2$ , a prime integer  $l > 6(k - 1)$ , a finite field  $\mathbb{F}$  with characteristic  $l$ , and a ring epimorphism  $f : \mathbb{T}(1, k) \rightarrow \mathbb{F}$ . We want to compute the associated Galois representation  $\rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$ . This representation lies in the jacobian of the modular curve  $X_1(l)$ . Indeed, let  $f_2 : \mathbb{T}(l, 2) \rightarrow \mathbb{F}$  be the unique ring homomorphism such that  $f_2(T_m) = f(T_m)$  for every positive integer  $m$ . Let  $V_f$  be the subgroup of  $J_1(l)(\overline{\mathbb{Q}})[l]$  cut out by the kernel of  $f_2$ . This is a dimension 2 vector space over  $\mathbb{F}$ . Given a finite generating set  $(t_1, \dots, t_r)$  with  $r = (l^2 - 1)/6$  as in Theorem 2.5.12 we can rewrite  $V_f$  as a finite intersection

$$V_f = \bigcap_{1 \leq i \leq r} \ker(t_i, J_1(l)(\overline{\mathbb{Q}})[l]). \quad (12.1)$$

Then  $V_f$  realizes  $\rho_f$  and we may write  $\rho_f$  as a morphism

$$\rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(V_f).$$

We will assume that the image of  $\rho_f$  contains  $\text{SL}(V_f)$ . Otherwise  $\rho_f$  would be reducible according to Theorem 2.5.18. And the reducible case is treated in Section 14.2.

We want to compute the splitting field  $K_f$  of  $\rho_f$  as an extension of  $\mathbb{Q}$ . Computing  $V_f$  using general algorithms from computer algebra, like Buchberger's algorithm, seems difficult in this situation because  $V_f$  is defined as



a subset of the  $l$ -torsion subgroup inside  $J_1(l)$ . A naive algebraic description of  $V_f$  would lead us to write down an equation for  $J_1(l)[l]$ ; something similar to an  $l$ -division polynomial for  $J_1(l)$ . The degree of such a polynomial would be the cardinality of the group of torsion points, that is  $l^{2g(X_1(l))}$  where the genus  $g(X_1(l))$  of  $X_1(l)$  grows quadratically in  $l$ . Such a degree is far too large for us: we are looking for an algorithm with polynomial time complexity in  $l$ .

We describe below an algorithm for computing elements in  $V_f$ . This is a deterministic algorithm and the running time is polynomial in  $l$ . We shall work with the jacobian  $J_1(5l)$  rather than with  $J_1(l)$ . We set

$$X = X_1(5l)$$

and we denote by  $g$  the genus of the latter curve. We note that the conditions above imply  $k \geq 4$ ,  $l \geq 19$  and  $g \geq 289$ . Using the map  $B_{5l,l,1} : X_1(5l) \rightarrow X_1(l)$  defined in Section 2.2, and the associated morphism  $B_{5l,l,1}^* : J_1(l) \rightarrow J_1(5l)$  between Jacobian varieties, we can see the subspace  $V_f$  as an  $\mathbb{F}_l$ -subspace inside the  $l$ -torsion of the jacobian

$$J = J_1(5l).$$

To avoid confusions we shall call  $W_f \subset J_1(5l)$  the image of  $V_f \subset J_1(l)$  by  $B_{5l,l,1}^*$ . We call  $W_f$  the *Ramanujan subspace* associated with  $f$ . Elements in  $J$  are represented by divisors on the curve  $X$ . For every class in  $W_f$ , we compute a sharp enough approximation for some divisor in this class. It will be explained in Chapter 14 how one can compute the splitting field of this divisor from such a complex approximation, using the upper bounds for the naive height of torsion divisors on modular curves proven in Section 11.7.

This chapter is organized as follows. In Section 12.1 we recall how points on  $X$  are represented using standard coordinates taking values in the complex unit disk. Section 12.2 recalls state of the art algorithms for computing the lattice of periods  $\Lambda$  of the jacobian  $J$  of the modular curve  $X$ . In Section 12.3 we describe an algebraic model for  $X_1(5l)$  and we relate it to the analytic model  $\Gamma_1(5l) \backslash \mathbb{H}^*$ .

The four next sections collect useful intermediate results. Section 12.4 provides explicit inequalities relating coefficients and values of converging power series. In Section 12.5 we prove formal identities relating Jacobians

and Wronskians, that are necessary for the local study of the Jacobi integration map. Section 12.6 collects simple quantitative facts about the Jacobi integration map. Section 12.7 relates several natural norms on the space of parabolic modular forms of weight 4.

A point on  $J$  can be represented in two different ways. We may consider it as a class  $x + \Lambda$  in  $\mathbb{C}^g/\Lambda = J(\mathbb{C})$ . We may also fix a degree  $g$  divisor  $\Omega$  on  $X$  and represent an element in  $J(\mathbb{C})$  by a divisor  $Q - \Omega$  in the corresponding linear equivalence class, where  $Q$  is an effective degree  $g$  divisor on  $X$ . In Sections 12.8 and 12.9 we adopt this latter point of view and we show that it is very convenient for computational purposes and leads to polynomial time algorithms. Unfortunately, the points we are interested in (the  $x$  belonging to the Ramanujan subspace  $W_f$ ) are rather difficult to characterize and compute in this form. However, assuming the divisor  $\Omega$  has been chosen correctly (e.g. we take for  $\Omega$  the divisor  $D_0$  constructed in Section 8.1), to every  $x$  in  $W_f$  there corresponds a unique divisor  $Q_x$  such that  $Q_x - \Omega$  lies in the class represented by  $x + \Lambda \in J(\mathbb{C})$ . Such a  $Q_x$  will be called a *Ramanujan divisor*. Computing  $x \in \mathbb{C}^g$  is not too difficult because the defining equations of  $V_f$  given in Equation (12.1) become linear in the analytic model  $\mathbb{C}^g/\Lambda$ . The difficulty then is to compute  $Q_x$  once given  $x$ . This is a typical example of the inverse Jacobi problem. Section 12.10 provides a partial general solution for this inverse Jacobi problem: it explains how, given  $\Omega$  and some  $x + \Lambda$ , one can find a divisor  $Q$  such that  $Q - \Omega$  lies in the corresponding linear equivalent class. Since we are only working with approximations, we must control the error made in computing  $Q$  from  $x$ . The output divisor  $Q'$  is hopefully close to  $Q$  but most likely not equal to it. We call  $x'$  the image of  $Q'$  by the Jacobi integration map. Statements in Section 12.10 control the difference between  $x$  and  $x'$ . We can't hope a much better result in full generality since, in general, the divisor  $Q$  is not even unique, because  $x$  could lie in the singular locus of the Jacobi map. To relate the distance between  $Q'$  and  $Q$  and the distance between  $x'$  and  $x$ , we need some information about the local behavior of the Jacobi map at  $Q - \Omega$ . Using results from Arakelov theory proven in Section 11.7, we show in Sections 12.11, 12.12 and 12.13 that when  $x$  lies in  $W_f$  then the error on  $x$  and the error on  $Q_x$  are nicely related. This finishes the proof of the main Theorem 12.10.2 in this chapter. The last Section 12.14 provides a more

algebraic variant of this theorem.

In this chapter, we shall use several times the main statements in Chapter 5 and in particular Lemma 5.4.12 and Theorem 5.4.3. These statements basically say that it is possible to compute efficiently sharp approximations of zeros of power series, provided we don't prospect near the boundary of the disk of convergence. In particular these zeros are well conditioned: they are not dramatically affected by a small perturbation of the series.

We suggest that the reader look at the first pages and main statements in Chapter 5 before going further in this chapter.

*Remark.* The symbol  $\Theta$  in this chapter stands for a positive effective absolute constant. So any statement containing this symbol becomes true if the symbol is replaced in every occurrence by some large enough real number.

*Remark.* In this chapter the letter  $i$  stands for the square root of  $-1$  in  $\mathbb{C}$  having positive imaginary part.

## 12.1 POINTS, DIVISORS AND COORDINATES ON $X$

In this section we recall how points, functions, forms and divisors are represented on a modular curve. We denote by  $\mathcal{F}$  the classical fundamental domain for the action of  $\mathrm{SL}_2(\mathbb{Z})$  on the Poincaré upper half plane  $\mathbb{H}$ . We set  $D = \bar{\mathcal{F}} \cup \infty$ . We set

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For every positive integer  $w$  we set

$$D_w = \bigcup_{0 \leq k \leq w-1} T^k(D).$$

We denote by  $F_w$  the image of  $D_w$  by the map  $z \mapsto \exp(2iw^{-1}\pi z)$ . This is a compact subset of the open disk  $D(0, 1) \in \mathbb{C}$ . It is even contained

in  $D(0, \exp(-\pi/w))$ . Any cusp on  $X(\mathbb{C})$  can be written  $\gamma(\infty)$  for some  $\gamma \in \text{SL}_2(\mathbb{Z})$ . These  $\gamma$  can be chosen once for all with entries

$$\leq l^\Theta$$

in absolute value. We denote by  $\Xi$  the set of all these chosen  $\gamma$ . The set  $\Xi$  parametrizes the cusps of  $X$ . We assume that the identity belongs to  $\Xi$ . It parametrizes the cusp  $\infty$  itself. We write the topological space  $X(\mathbb{C})$  as a union

$$X(\mathbb{C}) = \bigcup_{\gamma \in \Xi} \gamma(D_{w_\gamma}) = \bigcup_{\gamma \in \Xi} \bigcup_{0 \leq k \leq w_\gamma - 1} \gamma(T^k(D))$$

where  $w_\gamma$  is the width of the cusp  $\gamma(\infty)$ . We say that the  $\gamma T^k$  arising in this union form a *standard system of right cosets representatives* for  $\Gamma_1(5l)$  in  $\text{SL}_2(\mathbb{Z})$ .

Every point on  $X$  is represented by a complex number  $z$  in  $\gamma(D_{w_\gamma}) \subset \mathbb{H}$  for some  $\gamma$  in  $\Xi$ . But  $\gamma^{-1}(z) \in D_{w_\gamma}$  will often be more convenient. And

$$q_\gamma = \exp(2iw_\gamma^{-1}\pi\gamma^{-1}(z)) \tag{12.2}$$

is even more convenient. So most of the time, a complex point on  $X$  will be given as a pair  $(\gamma, q)$  where  $\gamma \in \Xi$  and  $q \in F_{w_\gamma} \subset D(0, \exp(-\pi/w_\gamma))$  is the value of  $q_\gamma$  at this point. We set

$$D_\gamma = \gamma(D_{w_\gamma}) \subset \Gamma_1(5l) \backslash \mathbb{H}^* = X_1(5l).$$

When  $\gamma$  is the identity  $\text{Id}$ , we sometimes write  $q$  instead of  $q_{\text{Id}}$ . The parameter  $q$  traditionally plays a more important role. Functions and forms on  $X_1(5l)$  are often identified with their  $q$ -expansions. The function field  $\mathbb{C}(X_1(5l))$  can thus be identified with a subfield of the field of Puiseux series  $\mathbb{C}\{\{q\}\}$ . In particular, we have an action of  $\text{Aut}(\mathbb{C})$  on modular functions and we can define the field  $\mathbb{Q}(X_1(5l))$  of  $\mathbb{Q}$ -rational functions on  $X_1(5l)$  to be the field of functions having  $q$ -expansion with  $\mathbb{Q}$ -rational coefficients.

We define the distance

$$d_\gamma(P_1, P_2)$$

between two points  $P_1 = (\gamma, q_1)$  and  $P_2 = (\gamma, q_2)$  in  $D_\gamma$  to be the modulus  $|q_2 - q_1|$  of the difference of their  $q_\gamma$  coordinates. Of course  $d_\gamma$  extends to  $D(0, 1) \supset F_{w_\gamma}$ .

Given a complex number  $q$  in the open disk  $D(0, 1)$ , let  $z \in \mathbb{H} \cup \infty$  be such that

$$\exp(2iw_\gamma^{-1}\pi\gamma^{-1}(z)) = q.$$

In the special case  $q = 0$  we set  $z = \gamma(\infty)$ . Such a  $z$  may not be unique. But two such  $z$ 's are mapped onto each other by some power of  $\gamma \times T^{w_\gamma} \times \gamma^{-1}$ . Since the latter lies in  $\Gamma_1(5l)$  we have defined a map

$$\mu_\gamma : D(0, 1) \rightarrow X_1(5l).$$

This is the parameterization associated with  $\gamma$ . It sends  $F_{w_\gamma}$  onto  $D_\gamma$ . Any form (resp. function) on  $X_1(5l)$  can be lifted to  $D(0, 1)$  along the map  $\mu_\gamma$ . For example, Klein's modular function  $j(z)$  is usually given as a function of  $q = q_{\text{Id}}$ . There exists a Laurent series  $\mathbf{J}(x)$  in the indeterminate  $x$  such that  $j(z) = \mathbf{J}(q)$ . Further

$$\mathbf{J}(x) = \frac{1}{x} + 744 + \sum_{k \geq 1} c(k)x^k \quad (12.3)$$

where the coefficients  $c(k)$  are rational integers. It can be checked easily that the expansion of  $j$  at the cusp  $\gamma(\infty)$  is given by

$$j(z) = \mathbf{J}(q_\gamma^{w_\gamma}) \quad (12.4)$$

where  $w_\gamma$  is the width of the cusp  $\gamma(\infty)$ . It is a consequence of a famous theorem by Petersson and Rademacher that the coefficient  $c(k)$  is bounded from above by

$$\Theta^{\sqrt{k}}. \quad (12.5)$$

Now let  $d$  be a positive integer and let  $Q = Q_1 + Q_2 + \cdots + Q_d$  be a degree  $d$  effective divisor on  $X$ . Let  $\epsilon$  be a non-negative real number. We say that  $Q$  is  $\epsilon$ -simple if the following conditions hold true.

1. For every integer  $k$  such that  $1 \leq k \leq d$ , the point  $Q_k$  belongs to  $D_{\gamma_k}$  for a unique  $\gamma_k$  in  $\Xi$ . So  $Q_k = (\gamma_k, q_k)$ . We ask that  $q_k$  lies in the interior of  $F_{w_k}$  where  $w_k$  is the width of the cusp  $\gamma_k(\infty)$ .
2. The distance between  $q_k$  and the boundary of  $F_{w_k}$  is  $> \epsilon$ .
3. If  $1 \leq k_1 < k_2 \leq d$  and  $\gamma_{k_1} = \gamma_{k_2} = \gamma$ , we write  $Q_{k_1} = (\gamma, q_1)$  and  $Q_{k_2} = (\gamma, q_2)$  and we ask that  $|q_2 - q_1| > \epsilon$ .

Not every divisor  $Q$  is  $\epsilon$ -simple but if  $\epsilon < 1/(d\Theta)$  there exists an  $\epsilon$ -simple divisor  $Q' = Q'_1 + Q'_2 + \cdots + Q'_d$  such that for every  $1 \leq k \leq d$  we have  $Q'_k = (\gamma_k, q'_k)$  and  $|q'_k - q_k| \leq \Theta d \epsilon$ .

## 12.2 THE LATTICE OF PERIODS

This section is devoted to the explicit calculation of the lattice of periods of  $X$ . All the algorithms in this section are detailed in the two books by Cremona [Cre] and Stein [Ste2] and in Bosman's thesis [Bos2]. See also Chapter 6.

We need a complex analytic description of the torus  $J(\mathbb{C})$  as  $\mathbb{C}^g/\Lambda$  where  $\Lambda$  is the lattice of periods. We first compute an explicit description of the first group in singular cohomology

$$H_1^{\text{sing}}(X_1(5l), \mathbb{Z}).$$

Using Manin-Shokurov theory we find a basis  $\mathcal{B}_1^{\text{sing}}$  for this  $\mathbb{Z}$ -module. Every element in this basis is an integer linear combination of Manin symbols

$$\sum_{\gamma} c_{\gamma} \{\gamma(0), \gamma(\infty)\}. \tag{12.6}$$

The  $\gamma$  in the sum (12.6) runs over the standard system of right cosets representatives for  $\Gamma_1(5l)$  in  $\text{SL}_2(\mathbb{Z})$ . The integer coefficients  $c_{\gamma}$  can be chosen to be

$$\leq \exp(l^\Theta)$$

in absolute value.

We also need a basis  $\mathcal{B}_{\text{DR}}^1$  of the space of holomorphic differentials

$$\mathcal{H}^1 = H_{\text{DR}}^1(X_1(5l))$$

or equivalently a basis of the space  $S_2(\Gamma_1(5l))$  of weight two cusp forms. We shall use the standard basis made of normalized newforms of level  $5l$  together with normalized newforms of level  $l$  lifted to level  $5l$  by the two degeneracy maps.

Let  $f = \sum_{k \geq v} f_k q^k$  be a form in this basis. The  $q$ -valuation  $v$  of  $f$  is 1 or 5. The first non-zero coefficient  $f_v$  in the  $q$ -expansion of  $f$  is 1. The coefficients  $f_k$  in the  $q$ -expansion of  $f$  are algebraic integers. The modulus of  $f_k$  is  $k^\Theta$ . One can compute an approximation of  $f_k$  within  $\exp(-m)$  in deterministic polynomial time  $(klm)^\Theta$ .

The action of Atkin-Lehner involutions  $w_5$  and  $w_l$  is expressed in the basis  $\mathcal{B}_{\text{DR}}^1$  by theorem 2 of [Asa]. The action of the diamond operators is known also because every form in  $\mathcal{B}_{\text{DR}}^1$  is an eigenform for the Hecke algebra  $\mathbb{T}^{(5l)}$  generated by the operators  $T_n$  for  $n$  prime to  $5l$ .

We also need the expansion of every form  $f(q)$  in  $\mathcal{B}_{\text{DR}}^1$  at every cusp  $\gamma(\infty)$ . More precisely,  $f(q)q^{-1}dq$  should be rewritten as  $h(q_\gamma)q_\gamma^{-1}dq_\gamma$  for every  $\gamma$  in  $\Xi$ . Since the level  $5l$  is squarefree, the group generated by the Atkin-Lehner involutions and the diamond operators acts simply transitively on the cusps. So there is an automorphism in this group that sends  $\infty$  to  $\gamma(\infty)$ . This automorphism can be represented by a matrix  $W_\gamma$  in  $\text{GL}_2(\mathbb{Q})$  having integer entries as explained in Section 6.2 of Chapter 6. If

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then the width  $w_\gamma$  of the cusp  $\gamma(\infty)$  is

$$w_\gamma = \frac{5l}{\gcd(5l, c)}.$$

Let  $r$  be the unique integer in  $[0, w_\gamma[$  such that  $d \equiv cr \pmod{w_\gamma}$ . Set  $b' = b - ar$  and  $d' = (d - cr)/w_\gamma$  and  $c' = c/\gcd(5l, c)$ . Then

$$W_\gamma = \begin{pmatrix} aw_\gamma & b' \\ 5lc' & w_\gamma d' \end{pmatrix} \tag{12.7}$$

and the product

$$W_\gamma^{-1} \times \gamma = \begin{pmatrix} w_\gamma^{-1} & rw_\gamma^{-1} \\ 0 & 1 \end{pmatrix} \tag{12.8}$$

fixes  $\infty$  and it acts on Fourier expansions like the substitution  $q \mapsto \zeta q^{1/w_\gamma}$  for some root of unity  $\zeta = \exp(\frac{2ri\pi}{w_\gamma})$ . Since the action of  $W_\gamma$  on forms is known, we can compute the expansion of all forms in  $\mathcal{B}_{\text{DR}}^1$  at all cusps in deterministic polynomial time  $(klm)^\Theta$  where  $k$  is the  $q$ -adic accuracy and  $m$  the complex absolute accuracy of coefficients.

Once we have computed a basis for both the singular homology  $H_1^{\text{sing}}(X_1(5l), \mathbb{Z})$  and the de Rham cohomology  $H_{\text{DR}}^1(X_1(5l))$  we can compute the lattice  $\Lambda$  of periods. Since we are given a basis  $\mathcal{B}_{\text{DR}}^1$  of holomorphic differentials, the lattice  $\Lambda$  is well defined inside  $\mathbb{C}^{\mathcal{B}_{\text{DR}}^1}$  as the image of the integration map  $H_1^{\text{sing}}(X_1(5l), \mathbb{Z}) \rightarrow \mathbb{C}^{\mathcal{B}_{\text{DR}}^1}$  sending a cycle  $c$  onto the vector  $(\int_c \omega)_{\omega \in \mathcal{B}_{\text{DR}}^1}$ . The image of the basis  $\mathcal{B}_1^{\text{sing}}$  by the integration map is a basis  $\mathcal{B}_{\text{per}}$  of the lattice  $\Lambda$  of periods. The so-called matrix of periods has entries  $\int_c \omega$  where  $c$  is a cycle in the basis  $\mathcal{B}_1^{\text{sing}}$  of  $H_1^{\text{sing}}(X_1(5l), \mathbb{Z})$  and  $\omega = f(q)q^{-1}dq$  is a form in the basis  $\mathcal{B}_{\text{DR}}^1$  of  $H_{\text{DR}}^1(X_1(5l))$ . Computing these periods reduces to evaluating integrals of the form  $\int_\alpha^\beta f(q)q^{-1}dq$  where  $\alpha$  and  $\beta$  are two cusps. We first cut this integral in two pieces  $\int_\alpha^\beta f(q)q^{-1}dq = \int_i^\beta f(q)q^{-1}dq - \int_i^\alpha f(q)q^{-1}dq$ . Since the group generated by Atkin-Lehner involutions and diamond operators acts transitively on the cusps, we can reduce to the computation of integrals of the form  $\int_\alpha^\infty f(q)q^{-1}dq$  where  $\alpha = (a + bi)/c$  and  $a, b$  and  $c$  are integers bounded by  $l^\Theta$  in absolute value. Since the coefficient  $f_k$  in the  $q$ -expansion of  $f$  are bounded by  $k^\Theta$  we can compute approximations of the entries in the period matrix within  $\exp(-m)$  in deterministic polynomial time  $(lm)^\Theta$ .

We note that the  $L^\infty$  norm on  $\mathbb{C}^{\mathcal{B}_{\text{DR}}^1}$  induces a distance  $d_J$  on the quotient  $\mathbb{C}^{\mathcal{B}_{\text{DR}}^1}/\Lambda = J(\mathbb{C})$



$$d_J(x + \Lambda, y + \Lambda) = \min_{z \in \Lambda} |x - y - z|_\infty. \quad (12.9)$$

This distance will be useful when evaluating rounding errors in the course of numerical computations. We denote by

$$\phi : X \rightarrow J$$

the Jacobi integration map. This map is well defined once we have chosen a degree 1 divisor on  $X$  as origin. For any  $\gamma$  in  $\Xi$ , the restriction of  $\phi$  to  $D_\gamma$  is Lipschitz with constant  $l^\Theta$  according to Equation (12.57). More precisely, if  $P_1 = (\gamma, q_1)$  and  $P_2 = (\gamma, q_2)$  are two points in  $D_\gamma$  then

$$d_J(\phi(P_2), \phi(P_1)) \leq l^\Theta \times |q_2 - q_1|.$$

For every positive integer  $k$ , we also denote by  $\phi$  the integration map  $\phi : X^k \rightarrow J$ . We denote by

$$\phi' : \text{Div}(X) \rightarrow J$$

the map induced by  $\phi$  on the group of divisors on  $X$ . The restriction of  $\phi'$  to the subgroup  $\text{Div}^0(X)$  of degree 0 divisors is independent of the origin we have chosen.

## 12.3 MODULAR FUNCTIONS

Since we plan to compute the splitting field of some very special divisors on the modular curve  $X = X_1(5l)$  we must be able to evaluate some well chosen modular functions of weight 0 and level 5 or  $5l$  at a given point  $z \in \mathbb{H}$ . In this section we describe algebraic models for  $X_1(5)$  and  $X_1(5l)$  and we explain how to compute the expansions of the involved modular functions at every cusp.

### 12.3.1 The modular curve $X_1(5)$

In this section we recall the definition of several classical level 5 modular functions and we show how to compute their expansions at each of the four

cusps of  $X_1(5)$ . Let  $b$  be an indeterminate and consider the elliptic curve  $E_b$  in Tate normal form with equation

$$y^2 + (1+b)xy + by = x^3 + bx^2. \quad (12.10)$$

The point  $P = (0, 0)$  has order 5 and its multiples are  $2P = (-b, b^2)$ ,  $3P = (-b, 0)$ ,  $4P = (0, -b)$ . Call  $\mathbb{P}_b^1$  the projective line with parameter  $b$ . The modular invariant of  $E_b$  is

$$j = j(b) = -\frac{(b^4 + 12b^3 + 14b^2 - 12b + 1)^3}{b^5(b^2 + 11b - 1)} \quad (12.11)$$

Let

$$s = -\frac{11 + 5\sqrt{5}}{2}$$

and  $\bar{s}$  be the two complex roots of  $b^2 + 11b - 1$ . We call  $A_\infty, A_0, A_s, A_{\bar{s}}$  the points on  $\mathbb{P}_b^1$  corresponding to the values  $\infty, 0, s$  and  $\bar{s}$  of the parameter  $b$ . The elliptic curve  $E_b \rightarrow \mathbb{P}_b^1 - \{A_\infty, A_0, A_s, A_{\bar{s}}\}$  is the universal elliptic curve with one point of order 5. So there exists a unique isomorphism between the modular curve  $X_1(5) = \Gamma_1(5) \backslash \mathbb{H}^*$  and  $\mathbb{P}_b^1$  that is compatible with the moduli structure on either side. We want to compute this isomorphism. More precisely we compute the expansions of  $b$  at every cusp of  $X_1(5)$ . To this end we compare the curve in Equation (12.10) and the Tate curve [Tat2] with equation

$$y'^2 + x'y' = x'^3 + a_4x' + a_6 \quad (12.12)$$

where

$$a_4 = -5 \sum_{n \geq 1} \sigma_3(n) q^n$$

$$a_6 = - \sum_{n \geq 1} \frac{5\sigma_3(n) + 7\sigma_5(n)}{12} q^n$$

and  $q$  is a formal parameter. We note that the coefficients in the expansions above are integers and we have

$$a_4 = \frac{1 - E_4(q)}{48}$$

$$a_6 = \frac{1 - 3E_4(q) + 2E_6(q)}{1728}$$

where

$$E_4(q) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n$$

$$E_6(q) = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n.$$

The modular invariant of the Tate curve in Equation (12.12) is

$$j(q) = \frac{1728E_4^3(q)}{E_4^3(q) - E_6^2(q)} = \frac{1}{q} + 744 + 196884q + \dots \quad (12.13)$$

Any isomorphism between the two Weierstrass curves in Equations (12.10) and (12.12) must take the form

$$x = u^2 x' + r \quad (12.14)$$

$$y = u^3 y' + su^2 x' + t.$$

Straightforward calculation gives the following necessary and sufficient conditions for the affine transform in (12.14) to induce an isomorphism of Weierstrass curves:

$$\frac{1728E_4^3(q)}{E_4^3(q) - E_6^2(q)} = -\frac{(b^4 + 12b^3 + 14b^2 - 12b + 1)^3}{b^5(b^2 + 11b - 1)} \quad (12.15)$$

$$u^2 = -\frac{E_4(q)}{E_6(q)} \times \frac{(b^2 + 1)(b^4 + 18b^3 + 74b^2 - 18b + 1)}{(b^4 + 12b^3 + 14b^2 - 12b + 1)}$$

$$r = \frac{u^2 - b^2 - 6b - 1}{12}$$

$$s = \frac{u - b - 1}{2}$$

$$t = \frac{b^3 + 7b^2 - (5 + u^2)b + 1 - u^2}{24}$$

We can simplify a bit these expressions. The first one just means

$$j(q) = j(b).$$

From the classical [Sch3, Proposition 7.1] identities

$$\left(\frac{q dj}{dq}\right)^2 = j(j - 1728)E_4$$

and

$$\left(\frac{q dj}{dq}\right)^3 = -j^2(j - 1728)E_6$$

we deduce

$$\left(\frac{q db}{dq}\right) = \frac{E_6(q)}{E_4(q)} \times \frac{b(b^2 + 11b - 1)(b^4 + 12b^3 + 14b^2 - 12b + 1)}{5(b^2 + 1)(b^4 + 18b^3 + 74b^2 - 18b + 1)}.$$

So the expression for  $u^2$  can be written

$$u^2 = -\frac{b(b^2 + 11b - 1)}{5\left(\frac{q db}{dq}\right)} \tag{12.16}$$

The expansion of  $j$  as a series in  $q$  has integer coefficients and can be computed using Equation (12.13). We deduce the expansion of  $q$  as a series in  $j^{-1}$

$$q = j^{-1} + 744j^{-2} + 750420j^{-3} + \dots \tag{12.17}$$

It has integer coefficients and one can compute it from the expansion of  $j$  as a series in  $q$  using any reasonable algorithm for the reversion of a power series: brute force linear algebra or the more efficient algorithms in [Br-Ku] or the quasi-optimal algorithms in [Ke-Um].

We first study the situation locally at  $A_\infty$ . A local parameter for  $\mathbb{P}_b^1$  at  $A_\infty$  is  $b^{-1}$ . The expansion

$$j^{-1} = -b^{-5} + 25b^{-6} + \dots \tag{12.18}$$

of  $j^{-1}$  as a series in  $b^{-1}$  has integer coefficients and can be computed using Equation (12.11) and standard algorithms for polynomial arithmetic. We substitute (12.18) in (12.17) and find

$$q = -b^{-5} + 25b^{-6} + \dots$$

One more reversion gives the expansion of  $b^{-1}$  as a series in  $q^{\frac{1}{5}}$ .

$$b^{-1} = -q^{\frac{1}{5}} + 5q^{\frac{2}{5}} + \dots \tag{12.19}$$

This expansion defines an embedding of the local field at  $A_\infty$  inside the field of Puiseux series  $\mathbb{C}\{\{q\}\}$  in the formal parameter  $q$ . We deduce the

expansion of  $u$  from Equations (12.16) and (12.19)

$$u^2 = q^{-\frac{2}{5}} + 4q^{-\frac{1}{5}} + 4 - 10q^{\frac{1}{5}} - 30q^{\frac{2}{5}} + \dots \quad (12.20)$$

We also find

$$r = -1 - 5q^{\frac{1}{5}} - 10q^{\frac{2}{5}} + 35q^{\frac{4}{5}} + 45q + \dots \quad (12.21)$$

The coordinates of the 5-torsion  $P$  on  $E_b$  are  $x_P = 0$  and  $y_P = 0$ . The image of  $P$  on the Tate curve has  $x'$ -coordinate

$$x'_P = -ru^{-2} = q^{\frac{2}{5}} + q^{\frac{3}{5}} + 2q^{\frac{4}{5}} - 2q + \dots$$

Since on the Tate curve we have

$$x'(w, q) = \sum_{n \in \mathbb{Z}} \frac{wq^n}{(1 - wq^n)^2} - 2 \sum_{n \geq 1} \frac{nq^n}{1 - q^n} \quad (12.22)$$

we deduce that the value of the parameter  $w$  at the 5-torsion point  $P$  is

$$w(P) = q^{\pm \frac{2}{5}} \bmod \langle q \rangle .$$

We may take either sign in the exponent above because we may choose any of the two isomorphisms corresponding to either possible values for  $u$ . We decide that

$$w(P) = q^{\frac{2}{5}} \bmod \langle q \rangle . \quad (12.23)$$

So the limit curve when  $q \rightarrow 0$  or equivalently when  $b \rightarrow \infty$  is a 5-gon, with a 5-torsion point lying on the second component after the one carrying the origin. So let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be any matrix in  $SL_2(\mathbb{Z})$  such that  $c$  is 2 modulo 5 and  $d$  is 0 modulo 5. For example take

$$\gamma = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

In particular, the cusp  $\gamma(\infty) = 1/2$  has width 5. We set

$$q_\gamma(z) = \exp(2i\pi\gamma^{-1}(z)/5).$$

If we replace  $q^{1/5}$  by  $q_\gamma$  in Equations (12.19), (12.20) and (12.21) we obtain the expansions of  $b^{-1}$ ,  $u^2$  and  $r$  at the cusp  $\gamma(\infty) = 1/2$ . Note in particular that  $u^2$  is a modular function with weight  $-2$  and level 5. By construction, it has no zero and no pole outside the cusps. Similarly  $b$  is a modular function with weight 0 and level 5. By construction, it has no pole outside the cusps.

We now study the situation locally at  $A_0$ . A local parameter at  $A_0$  is  $b$ . We find  $j^{-1} = b^5 + 25b^6 + \dots$  and  $q = b^5 + 25b^6 + \dots$  and we fix an embedding of the local field at  $A_0$  inside  $\mathbb{C}\{\{q\}\}$  by setting

$$b = q^{\frac{1}{5}} - 5q^{\frac{2}{5}} + 15q^{\frac{3}{5}} - 30q^{\frac{4}{5}} + \dots \quad (12.24)$$

Using Equation (12.16) we deduce

$$u^2 = 1 - 6q^{\frac{1}{5}} + 19q^{\frac{2}{5}} - 40q^{\frac{3}{5}} + 55q^{\frac{4}{5}} + \dots \quad (12.25)$$

and

$$r = -q^{\frac{1}{5}} + 4q^{\frac{2}{5}} - 10q^{\frac{3}{5}} + 15q^{\frac{4}{5}} + \dots \quad (12.26)$$

So the coordinate  $x'_P$  of the 5-torsion point  $P$  is

$$x'_P = q^{\frac{1}{5}} + 2q^{\frac{2}{5}} + 3q^{\frac{3}{5}} + 5q^{\frac{4}{5}} + 3q + O(q^{\frac{6}{5}})$$

and the parameter  $w$  at  $P$  can be taken to be  $w(P) = q^{\frac{1}{5}} \bmod \langle q \rangle$  this time. The limit curve when  $q \rightarrow 0$  or equivalently when  $b \rightarrow 0$  is a 5-gon, with a 5-torsion point lying on the first component after the one carrying the origin. So let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be any matrix in  $SL_2(\mathbb{Z})$  such that  $c$  is 1 modulo 5 and  $d$  is 0 modulo 5. For example take

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In particular, the cusp  $\gamma(\infty) = 0$  has width 5. We set

$$q_\gamma(z) = \exp(2i\pi\gamma^{-1}(z)/5).$$

If we replace  $q^{1/5}$  by  $q_\gamma$  in Equations (12.24), (12.25) and (12.26) we obtain the expansions of  $b$ ,  $u^2$  and  $r$  at the cusp  $\gamma(\infty) = 0$ .

We now study the situation locally at  $A_s$ . A local parameter at  $A_s$  is  $b - s$  and

$$j^{-1} = \left(-\frac{1}{2} + \frac{11\sqrt{5}}{50}\right)(b - s) + \left(-\frac{45443}{125} + \frac{4064\sqrt{5}}{25}\right)(b - s)^2 + \dots$$

and

$$b - s = -\frac{125 + 55\sqrt{5}}{2}q - (375 + 170\sqrt{5})q^2 - \frac{3375 + 1495\sqrt{5}}{2}q^3 + \dots$$

and

$$u^2 = -\frac{25 + 11\sqrt{5}}{2} - (200 + 90\sqrt{5})q - \frac{3575 + 1595\sqrt{5}}{2}q^2 + \dots$$

and

$$r = -\frac{7 + 3\sqrt{5}}{2} - (100 + 45\sqrt{5})q - (1300 + 580\sqrt{5})q^2 + \dots$$

and

$$x'_P = -\frac{1}{2} + \frac{\sqrt{5}}{10} - \frac{5 + \sqrt{5}}{2}q + \frac{-15 + \sqrt{5}}{2}q^2 + (-10 + \sqrt{5})q^3 + \dots$$

We note that the coordinate  $x'(P)$  of the 5-torsion point  $P$  is

$$x'(P) = \frac{w}{(1-w)^2} + O(q)$$

where  $w = \exp\left(\frac{4i\pi}{5}\right) = \zeta_5^2$ . So the parameter  $w$  at  $P$  can be taken to be  $w(P) = \zeta_5^2 \bmod \langle q \rangle$  this time. The limit curve when  $b \rightarrow s$  is thus a 1-gon equipped with the 5-torsion point  $\zeta_5^2$  in its smooth locus  $\mathbb{G}_m$ . Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be any matrix in  $\mathrm{SL}_2(\mathbb{Z})$  such that  $c$  is 0 modulo 5 and  $d$  is 2 modulo 5. For example take

$$\gamma = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$

In particular, the cusp  $\gamma(\infty) = 3/5$  has width 1. We set

$$q_\gamma(z) = \exp(2i\pi\gamma^{-1}(z)).$$

If we replace  $q$  by  $q_\gamma$  in the expansions above we obtain the expansions of  $b$ ,  $u^2$  and  $r$  at the cusp  $\gamma(\infty) = 0$ .

We finally study the situation locally at  $A_{\bar{s}}$ . A local parameter at  $A_{\bar{s}}$  is  $b - \bar{s}$  and

$$j^{-1} = \left(-\frac{1}{2} - \frac{11\sqrt{5}}{50}\right)(b - \bar{s}) - \left(\frac{45443}{125} + \frac{4064\sqrt{5}}{25}\right)(b - \bar{s})^2 + \dots$$

and

$$b - \bar{s} = \frac{-125 + 55\sqrt{5}}{2}q + (-375 + 170\sqrt{5})q^2 + \frac{-3375 + 1495\sqrt{5}}{2}q^3 + \dots$$

and

$$u^2 = \frac{-25 + 11\sqrt{5}}{2} + (-200 + 90\sqrt{5})q + \frac{-3575 + 1595\sqrt{5}}{2}q^2 + \dots$$

and

$$r = \frac{-7 + 3\sqrt{5}}{2} + (-100 + 45\sqrt{5})q + (-1300 + 580\sqrt{5})q^2 + \dots$$

and

$$x'_P = -\frac{1}{2} - \frac{\sqrt{5}}{10} + \frac{-5 + \sqrt{5}}{2}q - \frac{15 + \sqrt{5}}{2}q^2 - (10 + \sqrt{5})q^3 + \dots$$

We note that the coordinate  $x'(P)$  of the 5-torsion point  $P$  is

$$x'(P) = \frac{w}{(1-w)^2} + O(q)$$

where  $w = \exp(\frac{2i\pi}{5}) = \zeta_5$ . So the parameter  $w$  at  $P$  can be taken to be  $w(P) = \zeta_5 \bmod \langle q \rangle$  this time. The limit curve when  $b \rightarrow \bar{s}$  is thus a 1-gon equipped with the 5-torsion point  $\zeta_5$  in its smooth locus  $\mathbb{G}_m$ . Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be any matrix in  $SL_2(\mathbb{Z})$  such that  $c$  is 0 modulo 5 and  $d$  is 1 modulo 5. For example take

$$\gamma = \text{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



In particular, the cusp  $\gamma(\infty) = \infty$  has width 1. We set

$$q_{\text{Id}}(z) = \exp(2i\pi z).$$

If we replace  $q$  by  $q_{\text{Id}}$  in the expansions above we obtain the expansions of  $b$ ,  $u^2$  and  $r$  at the cusp  $\infty$ .

Altogether we have proved the following.

**12.3.2 Lemma (Computing expansions of  $b$ ,  $u^2$  and  $r$ )** *There exists a deterministic algorithm that given an integer  $k \geq 1$  computes the  $k$  first terms in the expansions of  $b$ ,  $u^2$  and  $r$  at each of the four cusps of  $X_1(5)$ , at the expense of  $k^\Theta$  elementary operations.*

We recall that the functions  $b$ ,  $u^2$  and  $r$  are defined in Equation (12.15). Further  $b$  is a weight zero and level 5 modular function having no pole outside the cusps. The weight  $-2$  and level 5 modular function  $u^2$  has no zero and no pole outside the cusps. Finally  $r$  is the sum of two level 5 modular functions having no pole outside the cusps. One of weight  $-2$  and one of weight 0. We also notice that the coefficients in the expansions of  $b$ ,  $u^2$  and  $r$  lie in  $\mathbb{Z}$  or  $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ . Lemma 12.4.9 implies that there exists a positive constant  $\Theta$  such that for every integer  $k \geq 1$ , the  $k$ -th coefficient in these expansions has absolute value  $\leq \exp(\Theta\sqrt{k})$ .

### 12.3.3 A plane model for $X_1(5l)$

Let  $l > 5$  be a prime. We call  $X_l = X_1(5l)$  the moduli of elliptic curves with one point of order  $5l$ . The genus of  $X_l$  is  $g_l = (l - 2)^2$ . In this section we define and study a natural homogeneous singular plane model  $C_l$  for this curve. In particular we enumerate the geometric points on  $X_l$  above every singularity of  $C_l$  and we explain how to compute series expansions for affine coordinates at every such branch. Finally we recall how to compute the zeta function of the function field  $\mathbb{F}_p(X_l)$  for every prime integer  $p \notin \{5, l\}$ .

Let  $b$  be an indeterminate and let  $E_b$  be the elliptic curve in Equation (12.10). The field  $\mathbb{Q}(b)$  of rational fractions in  $b$  is the function field  $\mathbb{Q}(X_1(5))$  of the modular curve  $X_1(5)$  over  $\mathbb{Q}$ . The map

$$B_{5l,5,1} : X_l \rightarrow X_1(5)$$

introduced in Section 2.2 defines a degree  $l^2 - 1$  extension  $\mathbb{Q}(X_l)/\mathbb{Q}(b)$ . We construct an explicit model for this extension. The multiplication by  $l$  isogeny

$$[l] : E_b \rightarrow E_b$$

induces a degree  $l^2$  rational function on  $x$ -coordinates:

$$x \mapsto \frac{N(x)}{D(x)}$$

where  $N(x)$  is a monic degree  $l^2$  polynomial in  $\mathbb{Q}(b)[x]$ . Recursion formulae for division polynomials (see [Eng] section 3.6) provide a quick algorithm for computing this polynomial, and also show that the coefficients actually lie in  $\mathbb{Z}[b]$ . The roots of  $N(x)$  are the  $x$ -coordinates of the points  $Q$  on  $E_b$  such that  $[l]Q$  is  $P = (0, 0)$ . If  $l$  is congruent to  $\pm 1$  modulo 5 then  $lP = \pm P$  and  $x$  divides  $N(x)$ . Otherwise  $N(x)$  is divisible by  $x - b$ . Call  $T_l(b, x)$  the quotient of  $N(x)$  by  $x$  or  $x - b$ , accordingly. This is a monic polynomial in  $\mathbb{Z}[b][x]$  with degree  $l^2 - 1$  in  $x$ . As a polynomial in  $x$  we have

$$T_l(b, x) = \sum_{0 \leq k \leq l^2 - 1} a_{l^2 - 1 - k}(b)x^k$$

where  $a_0(b) = 1$ . We call  $d$  be the total degree of  $T_l$ .

Let  $F$  be a field extension of  $\mathbb{Q}(b)$  where  $T_l(b, x) \in \mathbb{Q}(b)[x]$  has a root. A suitable twist of the Tate curve  $E_b$  given by Equation (12.10) has a point of order  $5l$  defined over  $F$ . This proves that the function field extension  $\mathbb{Q}(X_l)/\mathbb{Q}(b)$  can be embedded in  $F/\mathbb{Q}(b)$ . Since the degree of  $T_l(b, x)$  in  $x$  is equal to the degree of the extension  $\mathbb{Q}(X_l)/\mathbb{Q}(b)$  we deduce that the polynomial  $T_l$  is irreducible in  $\mathbb{Q}(b)[x]$  and the quotient field  $\mathbb{Q}(b)[x]/T_l$  is isomorphic to  $\mathbb{Q}(X_l)$ . Since the latter field is a regular extension of  $\mathbb{Q}(b) = \mathbb{Q}(X_1(5))$  we deduce that  $T_l$  is absolutely irreducible.

Let  $C_l \subset \mathbb{P}^2$  be the projective curve with homogeneous equation  $T_l(\frac{b}{z}, \frac{x}{z})z^d$  in the variables  $b, x$  and  $z$ . The map  $B_{5l, 5, 1} : X_1(5l) \rightarrow X_1(5)$  is unramified except at  $b \in \{0, \infty, s, \bar{s}\}$ . So for every point  $R$  on  $X_l$  such that  $b(R) \notin \{0, s, \bar{s}, \infty\}$ , the function  $b - b(R)$  is a uniformizing parameter at  $R$ . Let  $\mathcal{U}$  be the affine open set with equation

$$zb(b^2 - 11bz + z^2) \neq 0.$$

Every point on  $C_l \cap \mathcal{U}$  is smooth and all points on  $X_l$  above points in  $C_l - \mathcal{U}$  are cusps in the modular sense (i.e. the modular invariant has a pole at these points). The smooth point  $R = (b_R, x_R)$  on  $C_l \cap \mathcal{U}$  is the moduli of the curve  $E_{b_R}$  equipped with the unique  $5l$ -torsion point  $Q$  having  $x$ -coordinate  $x_R$  and such that  $[l]Q = P \in E_{b_R}$ .

### 12.3.4 The singularities of $C_l$

We study the cusps of  $X_1(5l)$  that are mapped onto  $A_\infty$  by  $B_{5l,5,1}$ . Set  $\zeta_l = \exp(\frac{2i\pi}{l})$ . Let  $\alpha$  and  $\beta$  be integers such that  $0 \leq \alpha, \beta \leq l-1$ . Let also  $\tilde{\alpha}$  and  $\tilde{\beta}$  be integers such that  $0 \leq \tilde{\alpha}, \tilde{\beta} \leq l-1$  and

$$\tilde{\alpha} \equiv \alpha/5 \pmod{l}$$

and

$$\tilde{\beta} \equiv (\beta - 2)/5 \pmod{l}.$$

We set  $w_Q = \zeta_l^{\tilde{\alpha}} q^{\frac{\tilde{\beta}}{l}} q^{\frac{2}{5l}}$  and observe that

$$w_Q^5 = \zeta_l^\alpha q^{\frac{\beta}{l}} \pmod{\langle q \rangle}$$

and

$$w_Q^l = q^{\frac{2}{5}} \pmod{\langle q \rangle} = w_P$$

according to Equation (12.23). We denote by  $Q$  the point on the Tate curve in (12.12), having  $w$ -coordinate  $w_Q$ . The isomorphism given by Equations (12.14) maps  $Q$  onto a  $5l$ -torsion point on the curve  $E_b$ . This point is called  $Q$  also. The couple  $(E_b, Q)$  defines a point on  $X_1(5l)$  that is mapped onto  $(E_b, P)$  by  $B_{5l,5,1}$ . We substitute  $w$  by  $w_Q$  in expansion (12.22) and find

$$x'_Q = \zeta_l^{\tilde{\alpha}} q^{\frac{\tilde{\beta}}{l}} q^{\frac{2}{5l}} (1 + O(q^{\frac{1}{5l}}))$$

if  $0 \leq \tilde{\beta} \leq \frac{l-1}{2}$  and

$$x'_Q = \zeta_l^{-\tilde{\alpha}} q^{\frac{l-\tilde{\beta}}{l} - \frac{2}{5l}} (1 + O(q^{\frac{1}{5l}}))$$

if  $\frac{l+1}{2} \leq \tilde{\beta} \leq l-1$ . Using Equation (12.14) and the expansions in Equations (12.20) and (12.21) we find

$$x_Q + 1 = \zeta_l^{\tilde{\alpha}} q^{\frac{\tilde{\beta}}{l} + \frac{2}{5l} - \frac{2}{5}} (1 + O(q^{\frac{1}{5l}})) \quad (12.27)$$

if  $0 \leq \tilde{\beta} \leq \frac{l-1}{2}$  and

$$x_Q + 1 = \zeta_l^{-\tilde{\alpha}} q^{\frac{l-\tilde{\beta}}{l} - \frac{2}{5l} - \frac{2}{5}} (1 + O(q^{\frac{1}{5l}})) \quad (12.28)$$

if  $\frac{l+1}{2} \leq \tilde{\beta} \leq l-1$ .

Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be any matrix in  $SL_2(\mathbb{Z})$  such that  $c$  is  $5\tilde{\beta} + 2$  modulo  $5l$  and  $d$  is  $5\tilde{\alpha}$  modulo  $5l$ . In particular, the cusp  $\gamma(\infty) = a/c$  has width  $w = \frac{5l}{\gcd(5l, c)}$ . So  $w$  is  $5$  if  $\beta = 0$  and  $5l$  otherwise. We set

$$q_\gamma(z) = \exp(2i\pi\gamma^{-1}(z)/w).$$

If we replace  $q^{1/w}$  by  $q_\gamma$  in Equation (12.27) or (12.28) above we obtain the expansion of  $x$  at the cusp  $\gamma(\infty) = a/c$ .

The same method applies to cusps of  $X_1(5l)$  that are mapped onto  $A_0$ ,  $A_s$  or  $A_{\bar{s}}$  by  $B_{5l,5,1}$ .

**12.3.5 Lemma (Computing expansions of  $x$ )** *There exists a deterministic algorithm that given an integer  $k \geq 1$  computes the  $k$  first terms in the expansions of  $x$  at each of the cusps of  $X_1(5l)$ , at the expense of  $(kl)^\Theta$  elementary operations.*

We notice that the coefficients in these expansions of  $x$  lie in  $\mathbb{Z}$  or  $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$  and there exists a positive constant  $\Theta$  such that for every integer  $k \geq 1$ , the  $k$ -th coefficient in these expansions has absolute value  $\leq \exp((kl)^\Theta)$ .

So we have a very accurate description of the singularities of  $C_l$  since for every branch of  $X_1(5l)$  above such a singularity we can compute expansions of both coordinates  $b$  and  $x$  as series in the local parameter  $q_\gamma$ .

We shall also need the following result due to Manin, Shokurov, Merel and Cremona [Man1, Mer, Cre, Fre].

**12.3.6 Lemma (Manin, Shokurov, Merel, Cremona)** *For  $l$  a prime and  $p \notin \{5, l\}$  another prime, the zeta function of  $X_l \bmod p$  can be computed in deterministic polynomial time in  $l$  and  $p$ .*

We first compute the action of the Hecke operator  $T_p$  on the space of Manin symbols for the congruence group  $\Gamma_1(5l)$  associated with  $X_l$ . Then, from the Eichler-Shimura identity  $T_p = F_p + p \langle p \rangle / F_p$  we deduce the characteristic polynomial of the Frobenius  $F_p$ .  $\square$

## 12.4 POWER SERIES

In this section we give some notation and we state a few useful elementary properties of power series in several variables. We are mainly interested in relating the size of coefficients in the expansions and the size of the values of the series where it converges. In the course of our calculations we shall encounter three kinds of power series. Expansions of normalized eigenforms have rather small coefficients, according to the Ramanujan conjecture. To deal with such expansions we introduce in Definition 12.4.2 the *type* of a power series. We shall also consider modular functions having no pole outside the cusps. The coefficients in the expansions of such functions may be larger, but they are controlled by the Petersson and Rademacher's inequality as explained in Lemma 12.4.9 below. Even more general modular functions may have quite big coefficients. To deal with this case, we introduce in Definition 12.4.13 the *exp-type* of a power series.

### 12.4.1 The type of a power series

Let  $g \geq 1$  be an integer. The  $L^\infty$  norm of a vector  $\mathbf{x} = (x_1, \dots, x_g) \in \mathbb{C}^g$  is  $\max_k |x_k|$  and it is denoted  $|\mathbf{x}|_\infty$ . The  $L^1$  norm is  $|\mathbf{x}|_1 = \sum_k |x_k|$  and the  $L^2$  norm is  $|\mathbf{x}|_2 = \sqrt{\sum_k |x_k|^2}$ . We denote by  $\|\mathbf{x}\|$  the vector  $(|x_1|, \dots, |x_g|)$ . If  $\mathbf{y} = (y_1, \dots, y_g)$  is another vector in  $\mathbb{C}^g$  we denote by  $\mathbf{x} \star \mathbf{y}$  the componentwise product  $(x_1 y_1, \dots, x_g y_g)$ . We set  $\mathbf{0}_g = (0, \dots, 0) \in \mathbb{C}^g$  and  $\mathbf{1}_g = (1, \dots, 1) \in \mathbb{C}^g$  and  $\mathbf{2}_g = (2, \dots, 2)$ .

If  $\mathbf{n} = (n_1, \dots, n_g) \in \mathbb{N}^g$  we write  $\mathbf{n}!$  for the product  $n_1!n_2! \cdots n_g!$  and  $\mathbf{x}^{\mathbf{n}}$  for the product  $x_1^{n_1} \cdots x_g^{n_g}$ . We note  $P(\mathbf{x}, \mathbf{r}) = \prod_{k=1}^g D(x_k, r_k) \subset \mathbb{C}^g$ , the polydisc with center  $\mathbf{x}$  and polyradius  $\mathbf{r}$ . When  $\mathbf{r} = (r, r, \dots, r)$  we just write  $P(\mathbf{x}, r)$ . If  $\mathbf{x} = (x_1, \dots, x_g) \in \mathbb{R}^g$ , we say that  $\mathbf{x} \geq \mathbf{0}_g$  if and only if  $x_k \geq 0$  for every  $k$ . We say that  $\mathbf{x} > \mathbf{0}_g$  if and only if  $x_k > 0$  for every  $k$ . An entire series  $f$  in the  $g$  variables  $x_1, \dots, x_g$  is a formal sum  $f = \sum_{\mathbf{k}} f_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$  where the index  $\mathbf{k}$  runs over  $\mathbb{N}^g$ .

**12.4.2 Definition (Type of a power series in several variables)** Let  $A$  be a real number  $\geq 1$  and consider  $\mathbf{n} = (n_1, \dots, n_g) \in \mathbb{N}^g$  such that  $\mathbf{n} \geq \mathbf{1}_g$ . We say that  $f$  is of type  $(A, \mathbf{n})$  if for every  $\mathbf{k} \geq \mathbf{0}_g$  we have

$$|f_{\mathbf{k}}| \leq A(\mathbf{k} + \mathbf{1}_g)^{\mathbf{n}} = A \prod_{1 \leq m \leq g} (k_m + 1)^{n_m}.$$

For every  $\mathbf{z} \in P(\mathbf{0}_g, 1)$  we deduce an upper bound for the value of  $f$  at  $\mathbf{z}$ .

$$\begin{aligned} |f(\mathbf{z})| &\leq \sum_{\mathbf{k} \geq \mathbf{0}_g} A(\mathbf{k} + \mathbf{1}_g)^{\mathbf{n}} |\mathbf{z}^{\mathbf{k}}| \leq A \prod_{1 \leq m \leq g} \sum_{k_m \geq 0} (k_m + 1)^{n_m} |z_m|^{k_m} \\ &\leq \frac{\mathbf{n}! A}{\prod_m (1 - |z_m|)^{n_m + 1}} \\ &= \frac{\mathbf{n}! A}{(\mathbf{1}_g - \|\mathbf{z}\|)^{\mathbf{n} + \mathbf{1}_g}}. \end{aligned} \tag{12.29}$$

We check that if  $f$  is of type  $(A, \mathbf{a})$  and  $h$  is of type  $(B, \mathbf{b})$ , then the product  $fh$  is of type

$$(AB, \mathbf{a} + \mathbf{b} + \mathbf{1}_g). \tag{12.30}$$

### 12.4.3 Refocusing a power series

Given an entire series  $f$  of type  $(A, \mathbf{n})$  and a vector

$$\mathbf{c} = (c_1, \dots, c_g) \in P(\mathbf{0}_g, 1)$$

we set

$$\begin{aligned} F_{\mathbf{c}}(\mathbf{y}) &= F_{\mathbf{c}}(y_1, \dots, y_g) \\ &= f(\mathbf{c} + \mathbf{y} \star (\mathbf{1}_g - \|\mathbf{c}\|)) = f((c_m + y_m(1 - |c_m|))_m). \end{aligned}$$

We call  $F_{\mathbf{c}}$  the *refocused* series of  $f$  at  $\mathbf{c}$ . According to [Cou, Lemme 16] this is a series of type  $(A_{\mathbf{c}}, \mathbf{n} + \mathbf{1}_g)$  where

$$A_{\mathbf{c}} = \mathbf{n}!A \exp(g + |\mathbf{n}|_1)2^{g+|\mathbf{n}|_1}(\mathbf{1}_g - \|\mathbf{c}\|)^{-\mathbf{n}-2g}. \quad (12.31)$$

In particular, it converges for  $\mathbf{y} \in P(\mathbf{0}_g, 1)$ .

#### 12.4.4 Bounding the remainder

For any integer  $u \geq 0$  we denote by  $R_u(\mathbf{x})$  the remainder of order  $u$  of the series  $f(\mathbf{x})$ . So

$$f(\mathbf{x}) = \sum_{|\mathbf{k}|_1 \leq u-1} f_{\mathbf{k}}\mathbf{x}^{\mathbf{k}} + R_u(\mathbf{x}).$$

According to [Cou, Lemme 17], if  $f$  is of type  $(A, \mathbf{n})$  and if  $r$  is a real in  $]0, 1[$  and if  $\mathbf{z} \in P(\mathbf{0}_g, r)$  then

$$|R_u(\mathbf{z})| \leq B(u+1)^{(n+1)g}|\mathbf{z}|_{\infty}^u \quad (12.32)$$

where  $n = |\mathbf{n}|_{\infty}$  and

$$B = \frac{\mathbf{n}!2Ag}{(1-r)^{g+|\mathbf{n}|_1}}.$$

Further, if  $\kappa$  is a real in  $]0, 1[$  and if

$$u \geq \max\left(\frac{16(ng)^2}{(\log r)^2}, \frac{2(\log \kappa - \log B)}{\log r}\right)$$

then

$$|R_u(\mathbf{z})| \leq \kappa \text{ for } \mathbf{z} \in P(\mathbf{0}_g, r).$$

#### 12.4.5 The type of a quotient

Let  $f = \sum_{\mathbf{k}} f_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}$  be an entire series in the  $g \geq 2$  variables  $x_1, \dots, x_g$ . Assume that  $f$  has type  $(A, \mathbf{n})$  for some  $A \geq 1$  and  $\mathbf{n} \geq \mathbf{1}_g$ . Assume that  $f$  is divisible by a polynomial  $P(z_1, \dots, z_g)$ . So there exists an entire series  $h = \sum_{\mathbf{k}} h_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}$  such that  $f = Ph$ . We want to estimate the size of coefficients in  $h$ . We shall only need the case when  $P = z_2 - z_1$ . So we

restrict to this special case. Let  $\mathbf{k} = (k_1, \dots, k_g)$ . For every integer  $m$  such that  $2 \leq m \leq g$  we set

$$u_m = \frac{k_m + 1}{k_m + n_m + 1}.$$

We set

$$\hat{u}_1 = \frac{k_1 + 1}{k_1 + n_1 + 1}.$$

If  $|\hat{u}_1 - u_2| < |\frac{\hat{u}_1+1}{2} - u_2|$  we set  $u_1 = \frac{\hat{u}_1+1}{2}$ . Otherwise we set  $u_1 = \hat{u}_1$ . In any case

$$\frac{1}{|1 - u_1|} \leq \frac{2(k_1 + n_1 + 1)}{n_1}$$

and

$$\frac{1}{u_1} \leq \frac{k_1 + n_1 + 1}{k_1 + 1}$$

and

$$\frac{1}{|u_2 - u_1|} \leq \frac{3(k_2 + n_2 + 1)}{n_2}.$$

From Cauchy's integral  $h_{\mathbf{k}}$  is equal to

$$\frac{1}{(2\pi i)^g} \int_{|\zeta_1|=u_1} \dots \int_{|\zeta_g|=u_g} \frac{f(\zeta_1, \zeta_2, \dots, \zeta_g)}{(\zeta_2 - \zeta_1) \prod_{1 \leq m \leq g} \zeta_m^{k_m+1}} d\zeta_1 d\zeta_2 \dots d\zeta_g.$$

Using Equation (12.29) we find

$$\begin{aligned} |f(\zeta_1, \zeta_2, \dots, \zeta_g)| &\leq \frac{\mathbf{n}!A}{\prod_{1 \leq m \leq g} (1 - u_m)^{n_m+1}} \\ &\leq \mathbf{n}!2^{n_1+1}A \prod_{1 \leq m \leq g} \left( \frac{k_m + n_m + 1}{n_m} \right)^{n_m+1}. \end{aligned}$$

So

$$|h_{\mathbf{k}}| \leq \frac{3\mathbf{n}!2^{n_1+1}A(k_2 + n_2 + 1)}{n_2} \prod_{1 \leq m \leq g} \frac{(k_m + n_m + 1)^{k_n+n_m+1}}{(k_m + 1)^{k_m} (n_m)^{n_m+1}}.$$

So  $h$  has type  $(12\mathbf{n}!2^{n_1+|n|_1+g}A \exp(|n|_1), \mathbf{n} + 2 \times \mathbf{1}_g)$ .



Now if we manage to divide  $f$  by  $K$  polynomials of the form  $z_{j_1} - z_{j_2}$  we obtain a series of type

$$(A \exp(8gK(|\mathbf{n}|_\infty + 2K)^2), \mathbf{n} + 2K \times \mathbf{1}_g). \quad (12.33)$$

#### 12.4.6 The type of derivatives

If  $f$  is an entire series in one variable of type  $(A, n)$ , then the derivative  $f'$  of  $f$  is of type

$$(A2^n, n + 1).$$

So the the  $d$ -th derivative of  $f$  is of type

$$(A2^{dn + \frac{d(d-1)}{2}}, n + d). \quad (12.34)$$

#### 12.4.7 The Petersson and Rademacher's inequality

The modular functions that appear in Lemma 12.3.2 are not modular forms: they have poles at the cusps. Since we plan to evaluate these functions at well chosen points  $z$  in the Poincaré upper half plane, we must control the size of the coefficients in the expansions of these functions.

Let  $f : \mathbb{H} \rightarrow \mathbb{C}$  an holomorphic periodic function with integer period  $e$ . So  $f(z + e) = f(z)$  for every  $z \in \mathbb{H}$ . We assume that  $f$  is meromorphic at  $\infty$ . So  $f$  can be written as a series

$$f(z) = \mathbf{F}(q_e) = \sum_{\frac{k}{e} \geq v} a_{\frac{k}{e}} q_e^k$$

where

$$q_e = \exp\left(\frac{2i\pi z}{e}\right)$$

and  $v \in \frac{1}{e}\mathbb{Z}$  is the valuation of  $f$  at  $\infty$  and the  $a_{\frac{k}{e}}$  are the coefficients in the Puiseux expansion of  $f$  at  $\infty$ , and the series  $\mathbf{F}$  converges in the unit disk.

Conversely, to every Puiseux series

$$\sum_{\frac{k}{e} \geq v} a_{\frac{k}{e}} q_e^k$$

with radius of convergence  $\geq 1$  we can associate an holomorphic function  $f : \mathbb{H} \rightarrow \mathbb{C}$  having period  $e$ , and meromorphic at  $\infty$ .

By abuse of notation we say that  $f$  is a Puiseux series with radius  $\geq 1$  and period  $e$ . The set of all such functions is a ring. An example of such a series is provided by Klein's modular invariant  $j(z) = \mathbf{J}(q)$  where  $\mathbf{J}$  is the series in Equation (12.3). In view of Petersson and Rademacher's inequality (12.5) it is natural to state the following lemma.

**12.4.8 Lemma (The Petersson and Rademacher's property)** *Let*

$$f(z) = \sum_{\frac{k}{e} \geq v} a_{\frac{k}{e}} q^{\frac{k}{e}}$$

*be a Puiseux series with radius of convergence  $\geq 1$ . Then the two properties below are equivalent*

1. *There exist two positive constants  $K_1$  and  $K_3$  such that for every integer  $k \geq K_3$  we have*

$$\left| a_{\frac{k}{e}} \right| \leq \exp \left( K_1 \sqrt{\frac{k}{e}} \right). \quad (12.35)$$

2. *There exist two positive constants  $K_2$  and  $K_4$  such that for every  $z = x + iy \in \mathbb{H}$  such that  $y^{-1} \geq K_4$  we have*

$$|f(z)| \leq \exp \left( \frac{K_2}{y} \right). \quad (12.36)$$

*We say that such a Puiseux series is PR (like Petersson and Rademacher). The set of PR series is a ring which is integrally closed in the ring of Puiseux series with radius of convergence  $\geq 1$ .*

This lemma is easily proven: one implication results from Cauchy's formula and the other implication is trivial. Using Lemma 12.4.8 and the Petersson-Rademacher inequality (12.5) we prove the following lemma.

**12.4.9 Lemma (Integral modular forms are PR)** *Let  $N \geq 1$  be an integer. Let  $f : \mathbb{H} \rightarrow \mathbb{C}$  be a modular function of weight 0 for the group  $\Gamma_1(N)$ . Assume that  $f$  has no pole outside the cusps. So  $f$  is holomorphic on  $\mathbb{H}$ , or*

equivalently it belongs to the ring  $\mathbb{C}[Y_1(N)]$  of integral functions on  $Y_1(N)$ . Then the expansion of  $f$  at any cusp is PR.

So the Petersson-Rademacher inequality for the Fourier coefficients of  $j$  holds true for any integral function on  $Y_1(N)$ . We also prove the following lemma concerning the discriminant form.

**12.4.10 Lemma (The discriminant and its inverse are PR)** *Let*

$$\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n$$

*be the discriminant form. The inverse  $\Delta^{-1}$  of  $\Delta$  is a PR series in  $q$ .*

This results from the product formula for  $\Delta$ . We deduce that Lemma 12.4.9 extends to modular functions of arbitrary weight.

**12.4.11 Lemma (Integral modular forms are PR)** *Let  $N \geq 1$  be an integer. Let  $f : \mathbb{H} \rightarrow \mathbb{C}$  be a modular function of weight  $k \in \mathbb{Z}$  for the group  $\Gamma_1(N)$ . Assume that  $f$  has no pole outside the cusps. Then the expansion of  $f$  at any cusp is PR.*

**12.4.12 The exp-type of a power series**

We shall have to deal with series in one variable having bigger coefficients than the ones introduced in Sections 12.4.1 and 12.4.7. The successive derivatives appearing in Lemma 12.11.2 are a good example. We no longer care about convergence then. We just want to control the size of the (logarithms) of the coefficients.

**12.4.13 Definition (Exp-type of a power series in one variable)** *Let  $n$  be an integer  $\geq 2$  and let  $A$  and  $B$  be two real numbers  $\geq 1$ . We say that an entire series  $f = \sum_{k \geq 0} f_k x^k$  is of exp-type  $(A, B, n)$  if for every  $k \geq 0$  we have*

$$|f_k| \leq \exp((Ak + B)^n).$$

*If  $f_1$  is of exp-type  $(A_1, B_1, n_1)$  and  $f_2$  of exp-type  $(A_2, B_2, n_2)$  then the product  $f_1 f_2$  is of exp-type*

$$(A_1 + A_2, B_1 + B_2 + 1, n) \tag{12.37}$$

where  $n$  is the maximum of  $n_1$  and  $n_2$ .

If  $f$  is of exp-type  $(A, B, n)$  and  $k \geq 1$  is an integer, then the  $k$ -th power  $f^k$  is of exp-type  $(kA, kB + k - 1, n)$ . The derivative  $f' = df/dx$  of  $f$  is of exp-type

$$(A, A + B + 1, n). \tag{12.38}$$

If  $f$  is of exp-type  $(A, B, n)$  and  $g(x) = 1/(1 - xf(x))$ , then  $g$  is of exp-type

$$(\sqrt{A + B + 1}, 0, 2n). \tag{12.39}$$

## 12.5 JACOBIANS AND WRONSKIANS OF POWER SERIES

In this section we state and prove an algebraic identity relating Wronskian and Jacobian determinants. This identity will be useful to control the local behaviour of the Jacobi integration map. We first state this identity in its simplest and most natural form in Section 12.5.1. We then state a more general identity in Section 12.5.2. The proofs are given in Sections 12.5.3 and 12.5.4.

### 12.5.1 A special case

We assume that  $g \geq 2$  is an integer and we consider  $g$  entire series  $f_1(x), f_2(x), \dots, f_g(x)$  in one variable  $x$ , with coefficients in  $\mathbb{C}$ . The *Wronskian* associated with  $\mathbf{f} = (f_1, \dots, f_g)$  is the determinant

$$W_{\mathbf{f}}(x) = \frac{1}{\prod_{1 \leq m \leq g-1} m!} \times \begin{vmatrix} f_1(x) & \dots & f_g(x) \\ f_1'(x) & \dots & f_g'(x) \\ \vdots & & \vdots \\ f_1^{(g-1)}(x) & \dots & f_g^{(g-1)}(x) \end{vmatrix}. \tag{12.40}$$

We may also introduce  $g$  indeterminates  $y_1, y_2, \dots, y_g$  and define the *Jacobian* associated with  $\mathbf{f}$  to be the determinant

$$\mathcal{J}_{\mathbf{f}} = \begin{vmatrix} f_1(y_1) & \dots & f_g(y_1) \\ f_1(y_2) & \dots & f_g(y_2) \\ \vdots & & \vdots \\ f_1(y_g) & \dots & f_g(y_g) \end{vmatrix}.$$

Let now  $D = \prod_{k < m} (y_m - y_k)$  be the reduced discriminant. Then the Jacobian  $\mathcal{J}_{\mathbf{f}}$  lies in the ring  $\mathbb{C}[[y_1, \dots, y_g]]$  and is divisible by the reduced discriminant in this ring. Further, the quotient  $\mathcal{J}_{\mathbf{f}}/D$  is congruent to  $W_{\mathbf{f}}(0)$  modulo the maximal ideal of  $\mathbb{C}[[y_1, \dots, y_g]]$ :

$$\frac{\mathcal{J}_{\mathbf{f}}}{D} \equiv W_{\mathbf{f}}(0) \pmod{(y_1, y_2, \dots, y_g)\mathbb{C}[[y_1, \dots, y_g]]}. \quad (12.41)$$

A proof of this identity is given in the next Sections 12.5.3 and 12.5.4.

Now assume that all the series  $f_k(x)$  have type  $(A, n)$ . Then the Jacobian  $\mathcal{J}_{\mathbf{f}}$  has type

$$(g!A^g, n\mathbf{1}_g).$$

We deduce from Equation (12.34) and (12.30) that the Wronskian  $W_{\mathbf{f}}$  has type

$$(A^g \exp(\Theta n g^2 + \Theta g^3), gn - 1 + \frac{g(g+1)}{2}). \quad (12.42)$$

### 12.5.2 A more general identity

Let  $g \geq 2$  be an integer. Let  $\mathbb{K}$  be any field with characteristic zero. Let  $f_1(x), \dots, f_g(x)$  be  $g$  entire series in one variable  $x$  having coefficients in  $\mathbb{K}$ . Let  $\mathbf{f} = (f_1, \dots, f_g)$  be the corresponding vector. Let  $1 \leq n \leq g$  be an integer and consider the  $n$  first derivatives

$$\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \dots, \mathbf{f}^{(n-1)}$$

of  $\mathbf{f}$  with respect to the variable  $x$ . The exterior product

$$W_{\mathbf{f},n} = \frac{\mathbf{f}^{(0)}}{0!} \wedge \frac{\mathbf{f}^{(1)}}{1!} \wedge \dots \wedge \frac{\mathbf{f}^{(n-1)}}{(n-1)!} \in \bigwedge^n (\mathbb{K}[[x]])^g \quad (12.43)$$

is a sort of *partial Wronskian* associated with the vector  $\mathbf{f}$ .

Now let  $S \geq 1$  be an integer and let

$$n = m_1 + m_2 + \cdots + m_S$$

be a partition of  $n$  in  $S$  parts. In particular  $m_s$  is a positive integer for every  $1 \leq s \leq S$ . Let  $y_1, y_2, \dots, y_S$  be  $S$  distinct indeterminates and consider the corresponding *partial Jacobian*

$$\mathcal{J}_{\mathbf{f},(m_s)_{1 \leq s \leq S}} = W_{\mathbf{f},m_1}(y_1) \wedge W_{\mathbf{f},m_2}(y_2) \cdots \wedge W_{\mathbf{f},m_S}(y_S) \quad (12.44)$$

in

$$\bigwedge^n (\mathbb{K}[[y_1, \dots, y_S]])^g.$$

Let

$$D_{(m_s)_{1 \leq s \leq S}} = \prod_{1 \leq s_1 < s_2 \leq S} (y_{s_2} - y_{s_1})^{m_{s_1} m_{s_2}}$$

be the corresponding *partial weighted discriminant*. The following identity is a partial generalization of Equation (12.41)

$$\frac{\mathcal{J}_{\mathbf{f},(m_s)_{1 \leq s \leq S}}}{D_{(m_s)_{1 \leq s \leq S}}} \equiv W_{\mathbf{f},n}(0) \pmod{(y_1, y_2, \dots, y_S)}. \quad (12.45)$$

In particular  $\mathcal{J}_{\mathbf{f},(m_s)_{1 \leq s \leq S}}$  is divisible by the weighted discriminant  $D_{(m_s)_{1 \leq s \leq S}}$  in the  $\mathbb{K}[[y_1, \dots, y_S]]$ -module  $\bigwedge^n (\mathbb{K}[[y_1, \dots, y_S]])^g$ . A proof of this identity given in the next Sections 12.5.3 and 12.5.4.

Now let  $K \geq 1$  be an integer and let

$$g = n_1 + n_2 + \cdots + n_K$$

be a partition of the dimension  $g$ . So  $n_k$  is a positive integer for each  $1 \leq k \leq K$ . We consider  $K$  vectors  $\mathbf{f}_1, \dots, \mathbf{f}_K$  in  $(\mathbb{K}[[x]])^g$ . We introduce  $K$  indeterminates  $x_1, \dots, x_K$  and following Equation (12.43) we define the *total Wronskian* to be

$$W_{(\mathbf{f}_k, n_k)_{1 \leq k \leq K}} = \bigwedge_{1 \leq k \leq K} W_{\mathbf{f}_k, n_k}(x_k) \in \mathbb{K}[[x_1, x_2, \dots, x_K]]. \quad (12.46)$$

For every  $1 \leq k \leq K$  let  $S_k$  be a positive integer and let

$$n_k = m_{k,1} + m_{k,2} + \cdots + m_{k,S_k}$$

be a partition of  $n_k$  into  $S_k$  parts. For every  $1 \leq k \leq K$  and  $1 \leq s \leq S_k$  we introduce the new indeterminate  $y_{k,s}$  and following Equation (12.44) we define the *total Jacobian* to be

$$\mathcal{J}_{(\mathbf{f}_k, (m_{k,s})_{1 \leq s \leq S_k})_{1 \leq k \leq K}} = \bigwedge_{1 \leq k \leq K} \mathcal{J}_{\mathbf{f}_k, (m_{k,s})_{1 \leq s \leq S_k}} \quad (12.47)$$

in

$$\mathbb{K}[[y_{k,s}]_{1 \leq k \leq K; 1 \leq s \leq S_k}].$$

We set

$$\mathbf{n} = (n_k)_{1 \leq k \leq K}$$

and

$$\mathbf{m}_k = (m_{k,s})_{1 \leq s \leq S_k}$$

and

$$\mathbf{m} = (m_{k,s})_{1 \leq k \leq K; 1 \leq s \leq S_k}.$$

Both  $\mathbf{m}$  and  $\mathbf{n}$  are partitions of  $g$ . And  $\mathbf{m}$  is a refinement of  $\mathbf{n}$ . We define the discriminant relative to  $\mathbf{m}$  and  $\mathbf{n}$  to be

$$D_{\mathbf{m}, \mathbf{n}} = \prod_{1 \leq k \leq K} D_{\mathbf{m}_k} = \prod_{1 \leq k \leq K} \prod_{1 \leq s_1 < s_2 \leq S_k} (y_{k,s_2} - y_{k,s_1})^{m_{k,s_1} m_{k,s_2}}.$$

Collecting  $K$  equations like (12.45) we obtain

$$\frac{\mathcal{J}_{(\mathbf{f}_k, \mathbf{m}_k)_{1 \leq k \leq K}}}{D_{\mathbf{m}, \mathbf{n}}} \equiv W_{(\mathbf{f}_k, \mathbf{n}_k)_{1 \leq k \leq K}}(0) \bmod (y_{k,s})_{1 \leq k \leq K; 1 \leq s \leq S_k}. \quad (12.48)$$

This generalization of Equation (12.41) will be useful in Section 12.13.3 when studying the Jacobi map. We now prove it using a formal analogue of the Jacobi map.

### 12.5.3 Proof of Equation (12.45) in a special case

In this section assume that  $S = n$  and  $m_1 = m_2 = \dots = m_S = 1$ . We write  $\mathbf{f}$  as a series in  $x$  with coefficients in  $\mathbb{K}^g$

$$\mathbf{f} = \Psi_0 + \Psi_1 x + \Psi_2 x^2 + \dots \in \mathbb{K}^g[[x]]$$

where

$$\Psi_k = \frac{\mathbf{f}^{(k)}}{k!}(0) \in \mathbb{K}^g$$

for every  $k \geq 0$ . We consider the formal integration

$$\mathbf{F} = \int_0^x \mathbf{f}(x) dx = \Psi_0 x + \frac{\Psi_1}{2} x^2 + \frac{\Psi_2}{3} x^3 + \dots \in \mathbb{K}^g[[x]].$$

We introduce  $S$  new indeterminates  $y_1, \dots, y_S$  and we set

$$\Phi(y_1, \dots, y_S) = \mathbf{F}(y_1) + \dots + \mathbf{F}(y_S) \in \mathbb{K}^g[[y_1, \dots, y_S]]. \quad (12.49)$$

We denote by

$$\mathfrak{m} = (y_1, \dots, y_S) \mathbb{K}[[y_1, \dots, y_S]]$$

the maximal ideal in  $\mathbb{K}[[y_1, \dots, y_S]]$ . For every  $k \geq 1$  we call

$$\nu_k = y_1^k + \dots + y_S^k$$

the  $k$ -th Newton's power sum. We check that

$$\Phi(y_1, \dots, y_S) = \Psi_0 \nu_1 + \frac{\Psi_1}{2} \nu_2 + \dots + \frac{\Psi_{S-1}}{S} \nu_S + R$$

where the remainder  $R$  is a vector in  $(\mathbb{K}[[y_1, \dots, y_S]])^g$  whose coefficients are symmetric functions in the  $(y_s)_{1 \leq s \leq S}$  and belong to  $\mathfrak{m}^{S+1}$ . So these coefficients belong to  $\mathbb{K}[[\nu_1, \dots, \nu_S]]$  and for every  $1 \leq k \leq S$  the partial derivative  $\frac{\partial R}{\partial \nu_k}$  is zero modulo

$$\mathfrak{n} = \mathfrak{m} \cap \mathbb{K}[[\nu_1, \dots, \nu_S]] = (\nu_1, \dots, \nu_S) \mathbb{K}[[\nu_1, \dots, \nu_S]].$$

We deduce that for every  $1 \leq k \leq S$

$$\frac{\partial \Phi}{\partial \nu_k} \equiv \frac{\Psi_{k-1}}{k} \pmod{\mathfrak{n}}. \quad (12.50)$$



On the other hand, it is clear from the definition of  $\Phi$  in Equation (12.49) that for every  $1 \leq s \leq S$

$$\frac{\partial \Phi}{\partial y_s} = \frac{d\mathbf{F}}{dx}(y_s) = \mathbf{f}(y_s). \quad (12.51)$$

Finally, for every  $1 \leq k \leq S$  and  $1 \leq s \leq S$  we have

$$\frac{\partial \nu_k}{\partial y_s} = ky_s^{k-1}$$

and the determinant

$$\left| \frac{\partial \nu_k}{\partial y_s} \right|_{k,s} = S! \prod_{1 \leq s_1 < s_2 \leq S} (y_{s_2} - y_{s_1}). \quad (12.52)$$

Equation (12.45) then follows from Equations (12.50), (12.51) and (12.52) applying the chain rule for derivatives.

#### 12.5.4 Proof of Equation (12.45) in general

We introduce the  $n$  indeterminates  $x_{s,j}$  for  $1 \leq s \leq S$  and  $1 \leq j \leq m_s$ . We put the lexicographic order on these indeterminates and we apply Equation (12.45) to the series  $\mathbf{f}$  and partition  $n = 1 + 1 + \cdots + 1$ . We obtain

$$\begin{aligned} \mathbf{f}(x_{1,1}) \wedge \cdots \wedge \mathbf{f}(x_{S,m_S}) &= (W_{\mathbf{f},n}(0) + O(x_{1,1}, \dots, x_{S,m_S})) \\ &\times \prod_{(1,1) \leq (s_1,j_1) < (s_2,j_2) \leq (S,m_S)} (x_{s_2,j_2} - x_{s_1,j_1}) \end{aligned} \quad (12.53)$$

where  $O(x_{1,1}, \dots, x_{S,m_S})$  stands for any element in the ideal generated by  $x_{1,1}, \dots, x_{S,m_S}$  in  $\mathbb{K}[[x_{1,1}, \dots, x_{S,m_S}]]$ .

We introduce  $S$  new indeterminates  $y_1, \dots, y_S$ . We also introduce  $n$  indeterminates  $z_{s,j}$  for  $1 \leq s \leq S$  and  $1 \leq j \leq m_s$ . For every  $1 \leq s \leq S$  we consider  $\mathbf{f}(y_s + z)$  as a series in  $z$  with coefficients in  $\mathbb{K}[[y_s]]$ . We apply Equation 12.45 to the series  $\mathbf{f}(y_s + z)$  and partition  $m_s = 1 + 1 + \cdots + 1$ . We obtain

$$\begin{aligned} \mathbf{f}(y_s + z_{s,1}) \wedge \cdots \wedge \mathbf{f}(y_s + z_{s,m_s}) &= (W_{\mathbf{f},m_s}(y_s) + O(z_{s,1}, \dots, z_{s,m_s})) \\ &\times \prod_{1 \leq j_1 < j_2 \leq m_s} (z_{s,j_2} - z_{s,j_1}) \end{aligned} \quad (12.54)$$

We now replace  $x_{s,j}$  by  $y_s + z_{s,j}$  in Equation (12.53) and we obtain

$$\begin{aligned} \mathbf{f}(y_1 + z_{1,1}) \wedge \cdots \wedge \mathbf{f}(y_S + z_{S,m_S}) &= (W_{\mathbf{f},n}(0) + O(y_1, \dots, y_S, z_{1,1}, \dots, z_{S,m_S})) \\ &\times \prod_{(1,1) \leq (s_1,j_1) < (s_2,j_2) \leq (S,m_S)} (y_{s_2} - y_{s_1} + z_{s_2,j_2} - z_{s_1,j_1}) \end{aligned} \quad (12.55)$$

We now notice that the left hand side of Equation (12.55) is the wedge product of the left hand sides of the  $S$  equations like (12.54). Further, the discriminants on the right hand sides of the  $S$  equations like (12.54) divide the discriminant in the right hand side of Equation (12.55). So we equate the right hand side of Equation (12.55) and the wedge product of the right hand sides of the  $S$  equations like (12.54). We then divide by the product of the  $S$  small discriminants. We then reduce modulo the ideal generated by the variables  $z_{s,j}$  and we obtain

$$\bigwedge_{1 \leq s \leq S} W_{f, m_s}(y_s) = (W_{f, n}(0) + O(y_1, \dots, y_s)) \times \prod_{1 \leq s_1 < s_2 \leq S} (y_{s_2} - y_{s_1})^{m_{s_1} m_{s_2}}$$

as was to be proven.

## 12.6 A SIMPLE QUANTITATIVE STUDY OF THE JACOBI MAP

In this section we prove some upper and lower bounds for the Jacobi map. Upper bounds are rather trivial but important to control the complexity of the algorithms. Lower bounds are not very surprising either. But they play an important role in the proof of Theorem 12.10.1.

### 12.6.1 Upper bounds for the Jacobi map

We first prove that the Jacobi map is Lipschitz with constant  $l^\Theta$ . Indeed let  $\gamma \in \Xi$  and call  $\phi_\gamma = \phi \circ \mu_\gamma$  the composition of the Jacobi map  $\phi : X \rightarrow J(\mathbb{C})$  with the modular parameterization  $\mu_\gamma : D(0, 1) \rightarrow X$ . So

$$\phi_\gamma : q_\gamma \mapsto \left( \int_{\omega \in \mathcal{B}_{\text{DR}}^1}^{q_\gamma} \omega \right)$$

and we don't need to specify the origin of the integral here. Every  $\omega = f(q)q^{-1}dq$  in  $\mathcal{B}_{\text{DR}}^1$  can be written  $h(q_\gamma)q_\gamma^{-1}dq_\gamma$  where  $h(q_\gamma)$  is the expansion of the modular form  $f$  at the cusp  $\gamma(\infty)$ . It is a consequence of Ramanujan's conjectures proven by Deligne and the explicit formulae by Asai for pseudo-eigenvalues that the series  $h(q_\gamma)q_\gamma^{-1}$  is of type

$$(\Theta, \Theta). \tag{12.56}$$

Therefore if  $q_1$  and  $q_2$  belong to  $D(0, \exp(-\pi/w_\gamma)) \supset F_{w_\gamma}$  the integral  $\int_{q_1}^{q_2} h(q_\gamma) q_\gamma^{-1} dq_\gamma$  is bounded in absolute value by  $|q_2 - q_1|$  times  $l^\Theta$  according to Equation (12.32). So the Jacobi map is Lipschitz with constant

$$\leq l^\Theta. \tag{12.57}$$

We now consider some vector  $\vec{\gamma} = (\gamma_k)_{1 \leq k \leq g} \in \Xi^g$  and we call

$$\phi_{\vec{\gamma}} : D(0, 1)^g \rightarrow \mathbb{C}^{\mathcal{B}_{\text{DR}}^1} / \mathcal{R}$$

the composition of the Jacobi map  $\phi : X^g \rightarrow J(\mathbb{C})$  with the product of the modular parameterizations

$$\mu_{\vec{\gamma}} = \prod_{1 \leq k \leq g} \mu_{\gamma_k} : D(0, 1)^g \rightarrow X^g.$$

For every  $1 \leq k \leq g$  we set  $w_k = w_{\gamma_k}$ . Let  $\mathbf{q} = (q_1, \dots, q_g)$  with  $q_k \in D(0, \exp(-\pi/w_k))$  for every  $1 \leq k \leq g$ . We study the map  $\phi_{\vec{\gamma}}$  locally at  $\mathbf{q}$ . The tangent space to  $D(0, 1)^g \subset \mathbb{C}^g$  at  $\mathbf{q}$  is identified with  $\mathbb{C}^g$  and we denote by  $(\delta_k)_{1 \leq k \leq g}$  its canonical basis. The underlying  $\mathbb{R}$ -vector space has basis  $(\delta_1, \dots, \delta_g, i\delta_1, \dots, i\delta_g)$ . For  $1 \leq k \leq g$  we set  $\delta_{k+g} = i\delta_k$ . Similarly we call  $(e_k)_{1 \leq k \leq g}$  the canonical basis of  $\mathbb{C}^g = \mathbb{C}^{\mathcal{B}_{\text{DR}}^1}$  and set  $e_{k+g} = ie_k$  for  $1 \leq k \leq g$ . So  $(e_k)_{1 \leq k \leq 2g}$  is a basis of the  $\mathbb{R}$ -vector space underlying  $\mathbb{C}^g$ . Let  $D_{\mathbf{q}}\phi_{\vec{\gamma}}$  be the differential of  $\phi_{\vec{\gamma}}$  at  $\mathbf{q}$ . For  $1 \leq k \leq 2g$  let  $\rho_k$  be the image of  $\delta_k$  by this differential. The determinant of  $D_{\mathbf{q}}\phi_{\vec{\gamma}}$

$$\frac{\rho_1 \wedge \dots \wedge \rho_{2g}}{e_1 \wedge \dots \wedge e_{2g}}$$

is the square of the absolute value of the Jacobian determinant

$$\mathcal{J}_{\vec{\gamma}}(\mathbf{q}) = \left| \frac{\omega}{dq_{\gamma_k}}(q_k) \right|_{1 \leq k \leq g, \omega \in \mathcal{B}_{\text{DR}}^1}. \tag{12.58}$$

We shall need an upper bound for the absolute value of  $\mathcal{J}_{\vec{\gamma}}(\mathbf{q})$ . As a series in the  $g$  indeterminates  $q_1, q_2, \dots, q_g$ , the Jacobian  $\mathcal{J}_{\vec{\gamma}}$  has type

$$(\exp(\Theta g^2), \Theta \mathbf{1}_g). \tag{12.59}$$

This results from Equation (12.56) and the definition of  $\mathcal{J}_{\vec{\gamma}}$  in Equation (12.58). If  $\vec{\gamma} = (\gamma, \gamma, \dots, \gamma)$  is the repetition of  $g$  times the same  $\gamma$  in  $\Xi$ ,

we write  $\mathcal{J}_\gamma$  for  $\mathcal{J}_{\vec{\gamma}}$  and we denote by  $W_\gamma(q)$  the Wronskian associated with  $\mathcal{J}_\gamma$ . This is a series in one variable  $q$ . We deduce from Equations (12.42) and (12.56) that  $W_\gamma$  has type

$$(\exp(\Theta g^3), \Theta g^2). \tag{12.60}$$

**12.6.2 Lower bounds for the Jacobi map**

We now bound from below the Wronskian  $W_\gamma(q)$  and the Jacobian  $\mathcal{J}_\gamma(\mathbf{q})$  for some special values of  $q$  and  $\mathbf{q}$ . So we assume that  $\vec{\gamma} = (\gamma, \gamma, \dots, \gamma)$  is the repetition of  $g$  times the same  $\gamma$  in  $\Xi$  and we study the Jacobi map  $\phi : X^g \rightarrow J$  in the neighborhood of  $(\gamma(\infty), \dots, \gamma(\infty))$  where  $\gamma(\infty)$  is the cusp associated with  $\gamma$ . The parameter at the cusp  $\gamma(\infty)$  is the  $q_\gamma$  from Equation (12.2).

We first treat the case when  $\gamma = \text{Id}$  and we write  $\mathcal{J}_\infty$  (resp.  $W_\infty$ ) for  $\mathcal{J}_{\text{Id}}$  (resp.  $W_{\text{Id}}$ ). We denote by

$$G_g = \prod_{1 \leq m \leq g-1} m!$$

the denominator that appears in the definition of the Wronskian in Equation (12.40). The expansions in  $q = q_{\text{Id}}$  of the  $\frac{\omega}{dq}$  for  $\omega \in \mathcal{B}_{\text{DR}}^1$  are entire series with algebraic integer coefficients; and they are permuted by the absolute Galois group of  $\mathbb{Q}$ . So the series  $G_g^2 \times W_\infty(q)^2$  has coefficients in  $\mathbb{Z}$ .

We set  $\eta = \frac{g(g+1)}{2}$ . The product  $W_\infty(q)(dq)^\eta$  is a degree  $\eta$  holomorphic form on  $X$ . Therefore it has  $2(g-1)\eta$  zeros counting multiplicities. Since the  $q$ -valuation  $v$  of  $W_\infty(q)$  is the multiplicity of the cusp  $\infty$  in the divisor of  $W_\infty(q)(dq)^\eta$ , we deduce that  $v \leq 2(g-1)\eta$ . So the series  $G_g^2 \times W_\infty(q)^2$  has valuation  $2v \leq 2(g-1)g(g+1)$  and rational integer coefficients. We deduce from Equations (12.60) and (12.30) that the type of the series  $W_\infty(q)^2$  is

$$(\exp(\Theta g^3), \Theta g^2). \tag{12.61}$$

We write

$$W_\infty(q)^2 = \frac{c}{G_g^2} \times q^{2v} + R_{2v+1}(q)$$

where  $c$  is a non-zero rational integer and  $R_{2v+1}$  is the remainder of order  $2v + 1$ . We can bound this remainder using Equations (12.32) and (12.61).

$$|R_{2v+1}(q)| \leq \exp(\Theta l^6) |q|^{2v+1}.$$

So if  $|q| \leq \exp(-\Theta l^6)$  we have  $|W_\infty(q)^2| \geq \frac{1}{2G_g^2} \times q^{2(g-1)g(g+1)}$  and

$$|W_\infty(q)| \geq \frac{1}{2G_g} \times q^{(g-1)g(g+1)} \geq \exp(-\Theta l^5) \times q^{(g-1)g(g+1)}. \quad (12.62)$$

So we fix such a  $q$ . For example we take

$$q = 10^{-\kappa_1 l^6} \quad (12.63)$$

for some large enough positive constant  $\kappa_1$ . We set

$$\mathbf{q} = q\mathbf{1}_g = (q, \dots, q) \quad (12.64)$$

and  $\mathbf{x} = (x_1, \dots, x_g)$  where  $x_1, \dots, x_g$  are new indeterminates. The Jacobian

$$\mathcal{J}_\infty(\mathbf{q} + \mathbf{x} \star (\mathbf{1}_g - \|\mathbf{q}\|))$$

is an entire series in the  $g$  variables  $x_1, \dots, x_g$ . This is indeed the Jacobian associated with the  $g$  series

$$(f(q + x(1 - |q|)) / (q + x(1 - |q|)))_{f(q)q^{-1}dq \in \mathcal{B}_{\text{DR}}^1}$$

in the variable  $x$ . Equation (12.41) gives us the first non-zero term in the expansion of this series at  $\mathbf{x} = \mathbf{0}_g$ :

$$\begin{aligned} \mathcal{J}_\infty(\mathbf{q} + \mathbf{x} \star (\mathbf{1}_g - \|\mathbf{q}\|)) &= W_\infty(q)(1 - |q|)^{\frac{g(g-1)}{2}} \prod_{k < m} (x_m - x_k) \\ &\quad + R_{\frac{g(g-1)}{2}+1}(\mathbf{x}). \end{aligned}$$

The type of the Jacobian  $\mathcal{J}_\infty(\mathbf{q})$  as a series in  $\mathbf{q}$  is given by Equation (12.59). We deduce from Equation (12.31) that the refocused series  $\mathcal{J}_\infty(\mathbf{q} + \mathbf{x} \star (\mathbf{1}_g - \|\mathbf{q}\|))$  is a series in  $\mathbf{x}$  of type  $(\exp(\Theta g^2), \Theta \mathbf{1}_g)$ . Using Equation (12.32) we deduce that for  $\mathbf{x}$  in  $P(\mathbf{0}_g, \exp(-\pi))$

$$\left| R_{\frac{g(g-1)}{2}+1}(\mathbf{x}) \right| \leq \exp(\Theta g^2) |\mathbf{x}|_\infty^{\frac{g(g-1)}{2}+1}.$$

We set  $s = |\mathbf{x}|_\infty$  and we assume that  $\mathbf{x}$  takes the special form

$$\mathbf{x} = \left( \frac{s}{g}, \frac{2s}{g}, \dots, \frac{(g-1)s}{g}, s \right) \quad (12.65)$$

and  $s \leq \exp(-\pi)$ . Then

$$\begin{aligned} \left| W_\infty(q)(1-|q|)^{\frac{g(g-1)}{2}} \prod_{k < l} (x_l - x_k) \right| &\geq |W_\infty(q)| \left( \frac{s(1-|q|)}{g} \right)^{\frac{g(g-1)}{2}} \\ &\geq \Theta^{-1} |W_\infty(q)| \left( \frac{s}{g} \right)^{\frac{g(g-1)}{2}} \end{aligned}$$

We take

$$s = 10^{-\kappa_2 l^{12}} \quad (12.66)$$

for some large enough positive constant  $\kappa_2$ . Using Equations (12.63) and (12.62) we obtain the following lower bound for the Jacobian

$$|\mathcal{J}_\infty(\mathbf{q} + \mathbf{x}(1-q))| \geq \exp(-\Theta l^{16}) \quad (12.67)$$

when  $q$ ,  $\mathbf{q}$ , and  $\mathbf{x}$  are given by Equations (12.63), (12.64), (12.65), and (12.66). In particular  $|\mathbf{q} + \mathbf{x}(1-q)|_\infty$  can be assumed to be  $\leq \exp(-2\pi)$ . So

$$\max_{\mathbf{q} \in D(0, \exp(-2\pi))^g} |\mathcal{J}_\infty(\mathbf{q})| \geq \exp(-\Theta l^{16}). \quad (12.68)$$

In order to bound from below  $\mathcal{J}_\gamma$  for any  $\gamma \in \Xi$  we observe that  $\mathcal{J}_\gamma$  and  $\mathcal{J}$  are closely related: If  $w$  is the width of the cusp  $\gamma(\infty)$ , there exists a  $w$ -th root of unity  $\zeta_\gamma$  and an algebraic number  $\lambda_\gamma$  of absolute value 1 (the product of all pseudo-eigenvalues) such that the following formal identity in  $\mathbb{C}[[q_1, \dots, q_g]]$  holds true

$$\mathcal{J}_\gamma(q_1, \dots, q_g) = \lambda_\gamma \mathcal{J}_\infty(\zeta_\gamma q_1, \dots, \zeta_\gamma q_g). \quad (12.69)$$

So the lower bound in Equation (12.68) is also valid for every  $\mathcal{J}_\gamma$ .

## 12.7 EQUIVALENCE OF VARIOUS NORMS

The main algorithm in this text (the one in Section 12.8) uses a subroutine that computes the complex roots of an analytic function on a compact set. This problem is well conditioned according to Lemma 5.4.12 of Chapter 5, provided we have a decent lower bound for the maximum of the function in question. In our situation, the analytic functions are derived from quadratic differentials on  $X$ . We need simple conditions for these functions not to be uniformly small in absolute value in the neighborhood of any cusp. The second inequality in Equation (12.71) below provides such a condition. In order to prove this inequality we study Jacobians associated with weight 4 cusp forms on  $X$ , locally at every cusp.

### 12.7.1 Space of quadratic differentials

In this section we shall make use of parabolic modular forms of weight 4 on  $X$ . To every such form  $f(q)$  one can associate a quadratic differential  $\omega = f(q)q^{-2}(dq)^2$ . The divisor of  $\omega$  is related to the divisor of  $f$  by the following relation

$$\text{Div}(\omega) = \text{Div}(f) - 2\text{Cusps}$$

where  $\text{Cusps}$  is the sum of all cusps. Note that  $X_1(5l)$  has no elliptic point. The map  $f(q) \mapsto f(q)q^{-2}(dq)^2$  defines a bijection between the space of weight 4 parabolic forms and the space  $\mathcal{H}^2(\text{Cusps})$  of quadratic differentials with divisor  $\geq -\text{Cusps}$ . We denote by  $g_2$  the dimension of the latter space. This is  $3g - 3$  plus the degree of  $\text{Cusps}$  (the number of cusps).

We shall need a basis  $\mathcal{B}_{\text{quad}}$  for the space  $\mathcal{H}^2(\text{Cusps})$  or equivalently a basis for the space  $S_4(\Gamma_1(5l))$  of weight four cusp forms. We shall again use the standard basis made of normalized newforms of level  $5l$  together with normalized newforms of level  $l$  lifted to level  $5l$  by the two degeneracy maps. We also need the expansion of every form in  $\mathcal{B}_{\text{quad}}$  at every cusp  $\gamma(\infty)$  for  $\gamma \in \Xi$ . More precisely,  $f(q)q^{-2}(dq)^2$  should be rewritten as  $h(q_\gamma)q_\gamma^{-2}(dq_\gamma)^2$  for every  $\gamma$  in  $\Xi$ . As for degree 1 forms, and using the same methods, we can compute the expansion of all quadratic forms in  $\mathcal{B}_{\text{quad}}$  at all cusps in deterministic polynomial time  $(klm)^\Theta$  where  $k$  is the  $q$ -adic accuracy and  $m$  the complex absolute accuracy of coefficients.

We now define several important norms on the space  $\mathcal{H}^2(\text{Cusps})$ . If  $\omega$  is a form in  $\mathcal{H}^2(\text{Cusps})$ , we denote by  $|\omega|_\infty$  the  $L^\infty$  norm in the basis  $\mathcal{B}_{\text{quad}}$ . To every cusp  $\gamma(\infty)$  with  $\gamma \in \Xi$  we associate a norm on  $\mathcal{H}^2(\text{Cusps})$ . We define  $|\omega|_\gamma$  to be the maximum of the modulus of the function  $\omega q_\gamma (dq_\gamma)^{-2}$  for  $|q_\gamma| \leq 1/2$ .

$$|\omega|_\gamma = \max_{|q_\gamma| \leq 1/2} \left| \frac{q_\gamma \omega}{(dq_\gamma)^2} \right|. \quad (12.70)$$

Any two such norms are of course equivalent: their ratios are bounded by a constant. More interestingly, the logarithm of this constant factor is polynomial in the level  $5l$  of  $X$ : for any  $\gamma$  in  $\Xi$  and any  $\omega$  in  $\mathcal{H}^2(\text{Cusps})$  we have

$$l^{-\Theta} \times |\omega|_\gamma \leq |\omega|_\infty \leq \exp(l^\Theta) \times |\omega|_\gamma. \quad (12.71)$$

These inequalities will be proven in Section 12.7.3.

### 12.7.2 Jacobian of weight 4 cusp forms

Remind that we have constructed in Section 12.7.1 a basis  $\mathcal{B}_{\text{quad}}$  for the space  $\mathcal{H}^2(\text{Cusps})$  of quadratic differential forms. If  $\vec{\gamma} \in \Xi^{g_2}$  and  $\mathbf{q} = (q_k)_{1 \leq k \leq g_2}$  we define the quadratic Jacobian

$$\mathcal{J}_{\vec{\gamma}}^{\text{quad}}(\mathbf{q}) = \left| \frac{q_{\gamma_k} \omega}{(dq_{\gamma_k})^2}(q_k) \right|_{1 \leq k \leq g_2, \omega \in \mathcal{B}_{\text{quad}}}. \quad (12.72)$$

It is a consequence of Ramanujan's conjectures proven by Deligne and the explicit formulae by Asai for pseudo-eigenvalues that the series  $\frac{q_\gamma \omega}{(dq_\gamma)^2}$  is of type

$$(\Theta, \Theta). \quad (12.73)$$

for every  $\omega$  in  $\mathcal{B}_{\text{quad}}$  and every  $\gamma$  in  $\Xi$ . So the series in the  $g_2$  variables  $q_1, q_2, \dots, q_{g_2}$  defined by Equation (12.72) is of type

$$(\exp(\Theta l^4), \Theta \mathbf{1}_{g_2}). \quad (12.74)$$

Further, if  $\vec{\gamma} = (\gamma, \dots, \gamma) \in \Xi^{g_2}$  is the repetition of  $g_2$  times the same  $\gamma$ , we write  $\mathcal{J}_\gamma^{\text{quad}}$  for  $\mathcal{J}_{\vec{\gamma}}^{\text{quad}}$  and we denote by  $W_\gamma^{\text{quad}}$  the corresponding Wronskian. This is a series in one variable  $q$  and it is of type



$$(\exp(\Theta l^6), \Theta l^4). \tag{12.75}$$

We need a similar estimate to Equation (12.68) for these quadratic Jacobians  $\mathcal{J}_\gamma^{\text{quad}}$ . We first treat the case when  $\gamma = \text{Id}$  and we write  $\mathcal{J}_\infty^{\text{quad}}$  (resp.  $W_\infty^{\text{quad}}$ ) for  $\mathcal{J}_{\text{Id}}^{\text{quad}}$  (resp.  $W_{\text{Id}}^{\text{quad}}$ ). We denote by

$$G_{g_2} = \prod_{1 \leq m \leq g_2-1} m!$$

the denominator in the definition of  $W_\infty^{\text{quad}}$ .

The expansions in  $q = q_{\text{Id}}$  of the  $\frac{q\omega}{(dq)^2}$  for  $\omega \in \mathcal{H}^2(\text{Cusps})$  are entire series with algebraic integer coefficients; and they are permuted by the absolute Galois group of  $\mathbb{Q}$ . So the series  $G_{g_2}^2 \times W_\infty^{\text{quad}}(q)^2$  has coefficients in  $\mathbb{Z}$ . We set  $\eta_2 = \frac{g_2(g_2+3)}{2}$ . The product  $W_\infty^{\text{quad}}(q)q^{-g_2}(dq)^{\eta_2}$  is a degree  $\eta_2$  form on  $X$  and it is holomorphic outside Cusps. More precisely, it belongs to  $\mathcal{H}^{\eta_2}(g_2\text{Cusps})$ . Therefore it has

$$2(g-1)\eta_2 + g_2 \deg(\text{Cusps}) \leq \Theta l^6$$

zeros counting multiplicities. We deduce that the  $q$ -valuation  $v_2$  of  $W_\infty^{\text{quad}}(q)$  is  $\leq \Theta l^6$ . So the series  $G_{g_2}^2 \times W_\infty^{\text{quad}}(q)^2$  has valuation  $\leq \Theta l^6$  and rational integer coefficients. We deduce from Equations (12.75) and (12.30) that the type of the series  $W_\infty^{\text{quad}}(q)^2$  is

$$(\exp(\Theta l^6), \Theta l^4). \tag{12.76}$$

We write

$$W_\infty^{\text{quad}}(q)^2 = \frac{c}{G_{g_2}^2} q^{2v_2} + R_{2v_2+1}(q)$$

where  $c$  is a non-zero rational integer and  $R_{2v_2+1}$  is the remainder of order  $2v_2 + 1$ . We can bound this remainder using Equations (12.32) and (12.76).

$$|R_{2v_2+1}(q)| \leq \exp(\Theta l^6) |q|^{2v_2+1}.$$

So if  $|q| \leq \exp(-\Theta l^6)$  we have  $|W_\infty^{\text{quad}}(q)^2| \geq \frac{1}{2G_{g_2}^2} |q|^{2v_2}$  and

$$|W_\infty^{\text{quad}}(q)| \geq \exp(-\Theta l^5) |q|^{\Theta l^6}. \tag{12.77}$$

So we fix such a  $q$ . For example we take

$$q = 10^{-\kappa_3 l^6} \tag{12.78}$$

where  $\kappa_3$  is a large enough positive constant. We set

$$\mathbf{q} = q\mathbf{1}_{g_2} = (q, \dots, q) \tag{12.79}$$

and  $\mathbf{x} = (x_1, \dots, x_{g_2})$  where  $x_1, \dots, x_{g_2}$  are new indeterminates. The Jacobian

$$\mathcal{J}_\infty^{\text{quad}}(\mathbf{q} + \mathbf{x} \star (\mathbf{1}_{g_2} - \|\mathbf{q}\|))$$

is an entire series in the  $g_2$  variables  $x_1, \dots, x_{g_2}$ . This is indeed the Jacobian associated with the  $g_2$  series

$$(f(q + x(1 - |q|))/(q + x(1 - |q|)))_{f(q)q^{-2}(dq)^2 \in \mathcal{B}_{\text{quad}}}$$

in the variable  $x$ . Equation (12.41) gives us the first non-zero term in the expansion of this series at  $\mathbf{x} = \mathbf{0}_{g_2}$ :

$$\begin{aligned} \mathcal{J}_\infty^{\text{quad}}(\mathbf{q} + \mathbf{x} \star (\mathbf{1}_{g_2} - \|\mathbf{q}\|)) &= W_\infty^{\text{quad}}(q)(1 - |q|)^{\frac{g_2(g_2-1)}{2}} \prod_{k < l} (x_l - x_k) \\ &\quad + R_{\frac{g_2(g_2-1)}{2}+1}(\mathbf{x}). \end{aligned}$$

The type of the Jacobian  $\mathcal{J}_\infty^{\text{quad}}(\mathbf{q})$  as a series in  $\mathbf{q}$  is given by Equation (12.74). We deduce from Equation (12.31) that the refocused series  $\mathcal{J}_\infty^{\text{quad}}(\mathbf{q} + \mathbf{x} \star (\mathbf{1}_{g_2} - \|\mathbf{q}\|))$  is a series in  $\mathbf{x}$  of type  $(\exp(\Theta l^4), \Theta \mathbf{1}_{g_2})$ . Using Equation (12.32) we deduce that for  $\mathbf{x}$  in  $P(\mathbf{0}_{g_2}, \exp(-\pi))$

$$\left| R_{\frac{g_2(g_2-1)}{2}+1}(\mathbf{x}) \right| \leq \exp(\Theta l^4) |\mathbf{x}|_\infty^{\frac{g_2(g_2-1)}{2}+1}.$$

We set  $s = |\mathbf{x}|_\infty$  and we assume that  $\mathbf{x}$  takes the special form

$$\mathbf{x} = \left( \frac{s}{g_2}, \frac{2s}{g_2}, \dots, \frac{(g_2-1)s}{g_2}, s \right) \tag{12.80}$$

and  $s \leq \exp(-\pi)$ . Then

$$\begin{aligned} \left| W_\infty^{\text{quad}}(q)(1-|q|)^{\frac{g_2(g_2-1)}{2}} \prod_{k < l} (x_l - x_k) \right| &\geq \left| W_\infty^{\text{quad}}(q) \left( \frac{s(1-|q|)}{g_2} \right)^{\frac{g_2(g_2-1)}{2}} \right| \\ &\geq \Theta^{-1} \left| W_\infty^{\text{quad}}(q) \left( \frac{s}{g_2} \right)^{\frac{g_2(g_2-1)}{2}} \right| \end{aligned}$$

We take

$$s = 10^{-\kappa_4 l^{12}} \tag{12.81}$$

where  $\kappa_4$  is a large enough positive constant. Using Equations (12.78) and (12.77) we obtain the following lower bound for the quadratic Jacobian

$$|\mathcal{J}_\infty^{\text{quad}}(\mathbf{q} + \mathbf{x}(1-q))| \geq \exp(-\Theta l^{16}) \tag{12.82}$$

when  $q$ ,  $\mathbf{q}$ , and  $\mathbf{x}$  are given by Equations (12.78), (12.79), (12.80), and (12.81). In particular  $|\mathbf{q} + \mathbf{x}(1-q)|_\infty$  can be assumed to be  $\leq 1/2$ . So

$$\max_{\mathbf{q} \in D(0, 1/2)^{g_2}} |\mathcal{J}_\infty^{\text{quad}}(\mathbf{q})| \geq \exp(-\Theta l^{16}). \tag{12.83}$$

In order to bound from below  $\mathcal{J}_\gamma^{\text{quad}}$  for any  $\gamma \in \Xi$  we observe that if  $w$  is the width of the cusp  $\gamma(\infty)$ , there exists a  $w$ -th root of unity  $\zeta_\gamma$  and an algebraic number  $\lambda_\gamma^{\text{quad}}$  of absolute value 1 (the product of all pseudo-eigenvalues) such that the following formal identity in  $\mathbb{C}[[q_1, \dots, q_g]]$  holds true

$$\mathcal{J}_\gamma^{\text{quad}}(q_1, \dots, q_{g_2}) = \lambda_\gamma^{\text{quad}} \mathcal{J}_\infty^{\text{quad}}(\zeta_\gamma q_1, \dots, \zeta_\gamma q_{g_2}). \tag{12.84}$$

So the lower bound in Equation (12.83) is also valid for every  $\mathcal{J}_\gamma^{\text{quad}}$ .

### 12.7.3 Equivalence of norms on $\mathcal{H}^2(\text{Cusps})$

In Section 12.7.1 we have defined various norms on the space of quadratic differential forms  $\mathcal{H}^2(\text{Cusps})$ . If

$$\omega = f(q)q^{-2}(dq)^2$$

is a quadratic differential form in  $\mathcal{H}^2(\text{Cusps})$ , the norm  $|\omega|_\infty$  is the  $L^\infty$  norm associated with the basis  $\mathcal{B}_{\text{quad}}$ . For every  $\gamma \in \Xi$ , the norm  $|\omega|_\gamma$  is defined by Equation (12.70). We write

$$\omega = f(q)q^{-2}(dq)^2 = h(q_\gamma)q_\gamma^{-2}(dq_\gamma)^2.$$

We must prove both inequalities in Equation (12.71). The first inequality is a trivial consequence of Equations (12.73) and (12.32).

We denote by  $\mathcal{M}_\gamma^{\text{quad}}$  the matrix occurring in the definition of the jacobian  $\mathcal{J}_\gamma^{\text{quad}}$ . So

$$\begin{aligned} \mathcal{M}_\gamma^{\text{quad}}(q_1, q_2, \dots, q_{g_2}) &= \left( \frac{q_\gamma \omega}{(dq_\gamma)^2}(q_k) \right)_{1 \leq k \leq g_2, \omega \in \mathcal{B}_{\text{quad}}} \\ &= \left( \frac{h(q_k)}{q_k} \right)_{1 \leq k \leq g_2, h(q_\gamma) q_\gamma^{-2} (dq_\gamma)^2 \in \mathcal{B}_{\text{quad}}}. \end{aligned}$$

In particular  $\mathcal{J}_\gamma^{\text{quad}}$  is the determinant of  $\mathcal{M}_\gamma^{\text{quad}}$ .

Now let  $q$ ,  $\mathbf{q}$  and  $\mathbf{x}$  be given by Equations (12.78), (12.79), (12.80), and (12.81). We set

$$\mathbf{r} = \mathbf{q} + \mathbf{x}(1 - q) = (r_1, r_2, \dots, r_{g_2}) \in D(0, 1/2)^{g_2}$$

and denote by  $\mathcal{M}_\gamma^{\text{quad}}(\mathbf{r})$  the evaluation of  $\mathcal{M}_\gamma^{\text{quad}}$  at  $\mathbf{r}$ . The entries in  $\mathcal{M}_\gamma^{\text{quad}}(\mathbf{r})$  are bounded above by  $l^\Theta$  in absolute value. Using Equation (12.82) we deduce that the entries in the inverse matrix of  $\mathcal{M}_\gamma^{\text{quad}}(\mathbf{r})$  are bounded in absolute value by  $\exp(\Theta l^{16})$ .

Let  $\omega$  be a form in  $\mathcal{H}^2(\text{Cusps})$  and let  $\mathbf{c} \in \mathbb{C}^{g_2}$  be the coordinate vectors of  $\omega = h(q_\gamma) q_\gamma^{-2} (dq_\gamma)^2$  in the basis  $\mathcal{B}_{\text{quad}}$ . For every  $1 \leq k \leq g_2$  set  $v_k = h(r_k)/r_k$  and let  $\mathbf{v} = (v_1, \dots, v_{g_2})$  be the corresponding vector. We have

$$\mathbf{v}^t = \mathcal{M}_\gamma^{\text{quad}}(\mathbf{r}) \times \mathbf{c}^t$$

where  $\mathbf{v}^t$  is the transposed vector of  $\mathbf{v}$  and  $\mathbf{c}^t$  is the transposed vector of  $\mathbf{c}$ . So

$$|\omega|_\infty = |\mathbf{c}|_\infty \leq g_2 \exp(\Theta l^{16}) \times |\mathbf{v}|_\infty$$

and this is  $\leq \exp(\Theta l^{16}) \times |\omega|_\gamma$  from the definition of  $|\omega|_\gamma$  given in Equation (12.70).

## 12.8 AN ELEMENTARY OPERATION IN THE JACOBIAN

An important prerequisite for the explicit computation in the Jacobian  $J$  is to be able to compute the linear space associated with some divisor on  $X$ .

In this section, we describe an algorithm to solve the following elementary problem: given  $3g - 4$  points  $P_1, P_2, \dots, P_{3g-4}$  in  $X(\mathbb{C})$ , find  $g$  points  $Q_1, \dots, Q_g$  in  $X(\mathbb{C})$  such that

$$Q_1 + \dots + Q_g \sim 2\mathcal{K} - (P_1 + \dots + P_{3g-4})$$

where  $\sim$  stands for linear equivalence of divisors and  $\mathcal{K}$  is the canonical class. This elementary problem will be used as a building block for explicit arithmetic operations in the jacobian  $J$  of  $X$ . We observe that the solution is not always unique. However, the image of  $Q_1 + \dots + Q_g$  by the Jacobi integration map  $\phi' : \text{Sym}^g X \rightarrow J$  is well defined. When doing numerical approximations, it will be convenient to measure the error in  $J(\mathbb{C}) = \mathbb{C}/\Lambda$  in terms of the distance  $d_J$  defined in Equation (12.9).

We shall solve the above problem in two steps. We set

$$P = P_1 + \dots + P_{3g-4}$$

and we first look for a differential quadratic form  $\omega$  in the linear space  $\mathcal{H}^2(-P) \subset \mathcal{H}^2(\text{Cusps})$  using our explicit knowledge of the latter space and linear algebra algorithms. We then compute the divisor  $(\omega)$  of  $\omega$  and output the (effective) difference  $(\omega) - P$ . We now provide details for these two steps.

We denote by  $T = P + \text{Cusps}$  the divisor obtained by adding the cusps to  $P$ . The degree of  $T$  is  $g_2 - 1$ . We write  $T = T_1 + T_2 + \dots + T_{g_2-1}$  where  $T_k = (\gamma_k, q_k)$  for every  $1 \leq k \leq g_2 - 1$ . Let  $\epsilon_1 = \exp(-m_1)$  be a positive real number. We assume that  $m_1 \geq \Theta l$ . We find an  $\epsilon_1$ -simple divisor  $T' = T'_1 + T'_2 + \dots + T'_{g_2-1}$  such that for every  $1 \leq k \leq g_2 - 1$  we have  $T'_k = (\gamma_k, q'_k)$  and

$$|q'_k - q_k| \leq \Theta(g_2 - 1)\epsilon_1.$$

We look for a quadratic form in  $\mathcal{H}^2(\text{Cusps})$  having divisor  $\geq -\text{Cusps} + T'$ . The space of such forms can be described as the kernel of a matrix  $\mathcal{M}$ . Each of the  $g_2 - 1$  lines of  $\mathcal{M}$  corresponds to a point  $T'_k$  for some  $1 \leq k \leq g_2 - 1$ . The  $g_2$  columns of  $\mathcal{M}$  correspond to the  $g_2$  forms in the basis  $\mathcal{B}_{\text{quad}}$ . The entry of  $\mathcal{M}$  at the line corresponding to the point  $T'_k = (\gamma_k, q'_k)$  and column corresponding to the form  $\omega$  in  $\mathcal{B}_{\text{quad}}$  is obtained in the following way: we

consider the expansion of  $\omega$  in the variable  $q_{\gamma_k}$

$$\omega = h(q_{\gamma_k})q_{\gamma_k}^{-2}(dq_{\gamma_k})^2$$

and we evaluate the function  $h(q_{\gamma_k})q_{\gamma_k}^{-1}$  at the value  $q'_k$  of  $q_{\gamma_k}$  corresponding to  $T'_k$ . The entries in  $\mathcal{M}$  are bounded by  $l^\Theta$  in absolute value according to Equations (12.56) and (12.32).

We can't compute  $\mathcal{M}$  exactly. Instead of that, we fix a positive real  $\epsilon_2 = \exp(-m_2)$  and we compute a matrix  $\mathcal{M}'$  with decimal entries in  $\mathbb{Z}[i, 1/10]$  such that the difference  $\mathcal{M}' - \mathcal{M}$  has  $L^\infty$  norm  $\leq \epsilon_2$ . The entries in this matrix  $\mathcal{M}'$  can be chosen to have numerators and denominators bounded in absolute value by  $\exp(\Theta(l + m_2))$ . We find a non-zero vector in the kernel of  $\mathcal{M}'$ , having coefficients in  $\mathbb{Z}[i]$  and bounded in absolute value by  $\exp(\Theta l^2(l + m_2))$ . We divide this vector by its largest coefficient and obtain a vector  $v = (v_k)_{1 \leq k \leq g_2}$  with  $L^\infty$  norm equal to 1. This vector may not lie in the kernel of  $\mathcal{M}$  but  $\mathcal{M}v$  has coefficients bounded by  $g_2 \exp(-m_2)$ . We call  $\nu$  the quadratic differential form in  $\mathcal{H}^2(\text{Cusps})$  having coordinate vector  $v$  in the basis  $\mathcal{B}_{\text{quad}}$ . By definition we have

$$|\nu|_\infty = 1 \tag{12.85}$$

Using Lemma 5.4.12 of Chapter 5 together with Equations (12.71) and (12.85) we show that if  $m_2 \geq l^\Theta$ , then for every  $1 \leq k \leq g_2 - 1$  the form  $\nu$  has a zero  $T''_k = (\gamma_k, q''_k)$  such that

$$|q'_k - q''_k| \leq \exp(-\Theta^{-1}\sqrt{m_2}).$$

If  $m_2 \geq \Theta m_1^2$  then these  $g_2 - 1$  zeros must be pairwise distinct, because each of them is close to some  $T'_k$  and the latter points form an  $\epsilon_1$ -simple divisor. The divisor of  $\nu$  can be written

$$(\nu) = T'' - \text{Cusps} + Q'$$

where  $T'' = \sum_{1 \leq k \leq g_2 - 1} T''_k$  and  $Q'$  is a degree  $g$  effective divisor. We rewrite  $(\nu)$  as

$$(\nu) = P + Q' + \sum_{1 \leq k \leq g_2 - 1} (T''_k - T_k).$$

The image of the error term  $\sum_{1 \leq k \leq g_2 - 1} (T''_k - T_k)$  by  $\phi'$  is small in the torus  $J(\mathbb{C}) = \mathbb{C}/\mathcal{R}$ . More precisely

$$d_J(0, \phi'(\sum_{1 \leq k \leq g_2-1} (T_k'' - T_k))) \leq \exp(-m_1/\Theta)$$

provided  $m_1 \geq l^\Theta$  and  $m_2 \geq m_1^\Theta$ .

So  $Q'$  is a good approximation for the solution  $Q$  to the original problem. Using the algorithm in Theorem 5.4.3 of Chapter 5, we compute an approximation of the divisor of  $\nu$  and output the corresponding approximation  $Q''$  of  $Q'$ .

**12.8.1 Lemma (An elementary operation)** *There is a deterministic algorithm that on input a degree  $3g - 4$  effective divisor  $P = P_1 + \dots + P_{3g-4}$  on  $X_1(5l)$ , returns a degree  $g$  effective divisor  $Q = Q_1 + \dots + Q_g$  such that*

$$Q_1 + \dots + Q_g \sim 2\mathcal{K} - (P_1 + \dots + P_{3g-4})$$

where  $\mathcal{K}$  is the canonical class on  $X_1(5l)$ . The running time is  $(lm)^\Theta$  where  $5l$  is the level and  $m$  the required absolute accuracy of the result.

Remind that the accuracy of the result in the above statement is measured in the torus  $\mathbb{C}/\Lambda$  using the distance  $d_J$  introduced in Equation (12.9).

## 12.9 ARITHMETIC OPERATIONS IN THE JACOBIAN

We fix a degree  $g$  effective divisor  $\Omega$  on  $X$ . We also need an effective degree  $g - 4$  auxiliary divisor  $\Pi$ . For example, we may choose a point  $O$  as origin for the Jacobi integration map (e.g.  $O$  could be the cusp at infinity), and set  $\Omega = gO$  and  $\Pi = (g - 4)O$ . An element in  $\text{Pic}^0(X)$  is given as the class of a divisor  $Q - \Omega$  where  $Q$  is a degree  $g$  effective divisor. Let  $R$  be another degree  $g$  effective divisor. In order to add the class of  $Q - \Omega$  and the class of  $R - \Omega$  we apply Lemma 12.8.1 twice. We first apply it to the divisor  $Q + R + \Pi$ . This is indeed a degree  $3g - 4$  effective divisor. We obtain a degree  $g$  effective divisor  $T$  such that  $T \sim 2\mathcal{K} - Q - R - \Pi$ . We again apply Lemma 12.8.1 to the divisor  $T + \Omega + \Pi$  this time. And we obtain a degree  $g$  effective divisor  $U$  such that  $U + \Omega \sim Q + R$ . So the class of  $U - \Omega$  is the sum of the classes of  $Q - \Omega$  and  $R - \Omega$ .

In order to compute the opposite of the class  $Q - \Omega$ , we apply Lemma 12.8.1 to the divisor  $2\Omega + \Pi$  and obtain a degree  $g$  effective divisor  $R$ , linearly equivalent to  $2\mathcal{K} - \Pi - 2\Omega$ . We apply Lemma 12.8.1 to the divisor  $R + Q + \Pi$  and obtain a degree  $g$  effective divisor  $T$  such that  $T - \Omega \sim -(Q - \Omega)$ .

**12.9.1 Theorem (Arithmetic operations in  $J_1(5l)$ )** *Addition and subtraction in the jacobian of  $X_1(5l)$  can be computed in deterministic time  $(lm)^\Theta$  where  $5l$  is the level and  $m$  the required absolute accuracy of the result.*

Again, the accuracy of the result is measured in the torus  $\mathbb{C}/\Lambda$  using the distance  $d_J$  introduced in Equation (12.9). In particular the error belongs to a group, and when chaining operations in the jacobian, the successive errors add to each other: the error on the result is the sum of the errors on either input plus the error introduced in the current calculation. This observation is particularly useful in conjunction with the fast exponentiation algorithm of Section 5.1: if we multiply a divisor  $Q - \Omega$  by a positive integer  $N$ , assuming that every elementary operation introduces an error  $\leq \epsilon$ , then the error on the final result is  $\leq \Theta \times \epsilon \times N \log N$  so the loss of accuracy is  $\leq \Theta \log N$ .

**12.9.2 Theorem (Fast exponentiation in  $J_1(5l)$ )** *There is a deterministic algorithm that on input two degree  $g$  effective divisors  $\Omega$  and  $Q$  on  $X_1(5l)$  and a positive integer  $N$  outputs a degree  $g$  effective divisor  $R$  such that*

$$R - \Omega \sim N(Q - \Omega).$$

*The algorithm runs in time  $(lm \log N)^\Theta$  where  $5l$  is the level and  $m$  the required absolute accuracy of the result.*

## 12.10 THE INVERSE JACOBI PROBLEM

In this section we are given a degree  $g$  effective origin divisor  $\Omega$  on  $X$  and an element  $x$  in  $\mathbb{C}^{\mathcal{B}_{\text{DR}}^1}$ , and we want to solve the inverse Jacobi problem for

$$x + \Lambda \in \mathbb{C}^{\mathcal{B}_{\text{DR}}^1}/\Lambda = J(\mathbb{C}).$$



So we look for a degree  $g$  effective divisor  $P = P_1 + \dots + P_g$  on  $X$  such that  $\phi'(P - \Omega) = x + \Lambda$ . We note that the solution might not be unique.

The main idea is the following: we start from a family of  $2g$  classes  $b_1 + \Lambda, b_2 + \Lambda, \dots, b_{2g} + \Lambda$  in  $J$  for which the inverse Jacobi problem is already solved: for every  $1 \leq k \leq 2g$ , we know a divisor  $B_k - \Omega$  such that  $\phi'(B_k - \Omega) = b_k + \Lambda$ . We try to approximate  $x$  by an integer combination  $\sum_{1 \leq k \leq 2g} N_k b_k$ . This should not be too difficult if the  $b_k$  are very small and  $\mathbb{R}$ -linearly independent: we compute the coordinates of  $x \in \mathbb{C}^g$  in the  $\mathbb{R}$ -basis made of the  $b_k$  and we round each of these coordinates to the closest integer. Once we have found the  $N_k$  we note that the divisor  $\sum_k N_k B_k$  would be a nice solution to the problem if it were a difference between two effective degree  $g$  divisors. This is not the case of course, but using the algorithms in Theorems 12.9.2 and 12.9.1 we find a degree  $g$  effective divisor  $P$  such that  $P - \Omega$  is linearly equivalent to  $\sum_k N_k B_k$ . We output  $P$  and we are done.

There remains to explain how to find the  $b_k$  and the corresponding  $B_k$ . For every  $1 \leq k \leq 2g$ , we set  $B_k = R'_k - R_k$  where  $R_k$  and  $R'_k$  are two points on  $X$  that are very close. More precisely, we choose the  $g$  first points  $R_k = (\gamma_k, q_k)$  for  $1 \leq k \leq g$ , and we set  $R_{k+g} = R_k$ . We also choose a positive integer  $\Upsilon = \exp(-\chi)$ . We assume that  $\chi \geq l^\Theta$  so  $\Upsilon$  is small. For  $1 \leq k \leq g$ , we set  $R'_k = (\gamma_k, q_k + \Upsilon)$  and  $R'_{k+g} = (\gamma_k, q_k + i\Upsilon)$ . We set

$$b_k = \left( \int_{R_k}^{R'_k} \omega \right)_{\omega \in \mathcal{B}_{\text{DR}}^1} \in \mathbb{C}^{\mathcal{B}_{\text{DR}}^1}.$$

These integrals can be computed efficiently using the same method as for period integrals. We now want to quantify the condition that these  $b_k$  should be  $\mathbb{R}$ -linearly independent. So let  $(e_k)_{1 \leq k \leq g}$  be the canonical basis of  $\mathbb{C}^g$  and set  $e_{k+g} = ie_k$  for  $1 \leq k \leq g$ . So  $(e_k)_{1 \leq k \leq 2g}$  is a basis of the  $\mathbb{R}$ -vector space underlying  $\mathbb{C}^g$ . We shall need a lower bound for the determinant

$$\frac{b_1 \wedge \dots \wedge b_{2g}}{e_1 \wedge \dots \wedge e_{2g}}. \tag{12.86}$$

Since  $\Upsilon$  is going to be small, we derive such a lower bound from the local study of the Jacobi integration map. We call  $\vec{\gamma} \in \Xi^g$  the vector  $(\gamma_k)_{1 \leq k \leq g}$

and we call

$$\phi_{\vec{\gamma}} : D(0, 1)^g \rightarrow \mathbb{C}/\mathcal{R}$$

the composition of the Jacobi map  $\phi : X^g \rightarrow J(\mathbb{C})$  with the product of the modular parameterizations

$$\mu_{\vec{\gamma}} = \prod_{1 \leq k \leq g} \mu_{\gamma_k} : D(0, 1)^g \rightarrow X^g.$$

For every  $1 \leq k \leq g$  we set  $w_k = w_{\gamma_k}$ . Let  $\mathbf{q} = (q_1, \dots, q_g)$  with  $q_k \in D(0, \exp(-\pi/w_k))$  for every  $1 \leq k \leq g$ . We study the map  $\phi_{\vec{\gamma}}$  locally at  $\mathbf{q}$ . The tangent space to  $D(0, 1)^g \subset \mathbb{C}^g$  at  $\mathbf{q}$  is identified with  $\mathbb{C}^g$  and we denote by  $(\delta_k)_{1 \leq k \leq g}$  its canonical basis. The underlying  $\mathbb{R}$ -vector space has basis  $(\delta_1, \dots, \delta_g, i\delta_1, \dots, i\delta_g)$ . For  $1 \leq k \leq g$  we set  $\delta_{k+g} = i\delta_k$ . Let  $D_{\mathbf{q}}\phi_{\vec{\gamma}}$  be the differential of  $\phi_{\vec{\gamma}}$  at  $\mathbf{q}$ . For  $1 \leq k \leq 2g$  let  $\rho_k$  be the image of  $\delta_k$  by this differential. Assuming  $\chi \geq l^\Theta$  and using Equations (12.56) and (12.32) we prove that the coefficients of  $\rho_k$  in the basis  $(e_m)_{1 \leq m \leq 2g}$  have absolute value

$$\leq l^\Theta. \tag{12.87}$$

The determinant of  $D_{\mathbf{q}}\phi_{\vec{\gamma}}$

$$\frac{\rho_1 \wedge \dots \wedge \rho_{2g}}{e_1 \wedge \dots \wedge e_{2g}}$$

is the square of the absolute value of the Jacobian determinant

$$\mathcal{J}_{\vec{\gamma}}(\mathbf{q}) = \left| \frac{\omega}{dq_{\gamma_k}}(q_k) \right|_{1 \leq k \leq g, \omega \in \mathcal{B}_{\text{DR}}^1}. \tag{12.88}$$

We denote  $\lambda$  the opposite of the logarithm of the absolute value of the above determinant, and we call it the *illconditioning* of  $\mathbf{q}$ . We shall see that the inverse Jacobi problem is well conditioned unless the illconditioning is large.

We first observe that we can bound from below the norm of every  $\rho_k$  in terms of  $\lambda$ . Indeed, the determinant of  $D_{\mathbf{q}}\phi_{\vec{\gamma}}$  is bounded from above by the product of the  $L^2$  norms  $\prod_{1 \leq k \leq 2g} |\rho_k|_2$  so

$$\exp(-2\lambda) \leq |\rho_k|_2 \times \prod_{j \neq k} |\rho_j|_2 \leq |\rho_k|_2 \times \exp(\Theta l^3)$$

using Equation (12.87). So

$$|\rho_k|_2 \geq \exp(-2\lambda - \Theta l^3). \quad (12.89)$$

Our next concern is to bound from below the determinant of Equation (12.86) in terms of the illconditioning  $\lambda$ . For every  $1 \leq k \leq 2g$  we notice that  $\Upsilon \rho_k$  is the first order approximation of  $b_k$ . We deduce from Equations (12.56) and (12.32) that

$$|b_k - \Upsilon \rho_k| \leq l^\Theta \Upsilon^2.$$

Using the lower bound (12.89) we deduce that

$$\frac{\Upsilon}{2} |\rho_k|_2 \leq |b_k|_2 \leq \frac{3\Upsilon}{2} |\rho_k|_2 \leq l^\Theta \Upsilon$$

provided  $\chi \geq l^\Theta + 2\lambda$ .

Using multilinearity of the determinant we can bound the difference

$$\frac{b_1 \wedge \dots \wedge b_{2g}}{e_1 \wedge \dots \wedge e_{2g}} - \Upsilon^{2g} \frac{\rho_1 \wedge \dots \wedge \rho_{2g}}{e_1 \wedge \dots \wedge e_{2g}}$$

by

$$2^g \left( \max_{1 \leq k \leq 2g} |b_k|_2 \right)^{2g-1} \max_{1 \leq k \leq 2g} |b_k - \Upsilon \rho_k|_2 \leq 2^g l^{\Theta(2g-1)} l^\Theta \Upsilon^{2g+1}$$

and this is less than half of  $\Upsilon^{2g} \left| \frac{\rho_1 \wedge \dots \wedge \rho_{2g}}{e_1 \wedge \dots \wedge e_{2g}} \right| = \exp(-2\lambda) \Upsilon^{2g}$  as soon as

$$\chi \geq \Theta l^3 + 2\lambda.$$

We deduce that

$$\left| \frac{b_1 \wedge \dots \wedge b_{2g}}{e_1 \wedge \dots \wedge e_{2g}} \right| \geq \frac{1}{2} \exp(-2\lambda - 2g\chi).$$

So we have a lower bound for the determinant of the transition matrix between the basis  $(e_k)_{1 \leq k \leq 2g}$  and the basis  $(b_k)_{1 \leq k \leq 2g}$ . Further the entries in this matrix are bounded by  $l^\Theta \Upsilon$  in absolute value. Therefore the entries in the inverse matrix are bounded by

$$2 \exp(2\lambda + 2g\chi) l^{\Theta g} \Upsilon^{2g-1} \leq \exp(2\lambda + \chi + \Theta l^3)$$

in absolute value. We thus can compute this inverse matrix in time polynomial in  $l$ ,  $\lambda$ ,  $\chi$  and the required absolute accuracy.

In Section 12.2 we have constructed a basis  $\mathcal{B}_{\text{per}}$  for the lattice of periods, consisting of vectors with coordinates bounded by  $\exp(l^\Theta)$  in absolute value in the basis  $(e_k)_{1 \leq k \leq 2g}$ . The coordinates of these periods in the basis  $(b_k)_{1 \leq k \leq 2g}$  are bounded by

$$\exp(l^\Theta + 2\lambda + \chi)$$

in absolute value.

Every point in the fundamental parallelogram associated with the basis  $\mathcal{B}_{\text{per}}$  (i.e. having coordinates in  $[0, 1]$  in this basis) has coordinates  $\leq \exp(2\lambda + \chi + l^\Theta)$  in absolute value in the basis  $(b_k)_{1 \leq k \leq 2g}$ . When we replace the latter coordinates by the closest integer, the induced error is bounded by  $l^\Theta \Upsilon$  according to the  $L^2$  norm for the canonical basis  $(e_k)_{1 \leq k \leq 2g}$ .

According to Equation (12.67) there exists a vector  $\mathbf{q}$  with illconditioning

$$\lambda \leq l^\Theta.$$

This finishes the proof of the following theorem.

**12.10.1 Theorem (Inverse Jacobi problem)** *The exists a deterministic algorithm that takes as input*

- a prime integer  $l$ ,
- an element  $x$  in the tangent space  $\mathbb{C}^{\mathcal{B}_{\text{DR}}^1}$  to  $J_1(5l)$  at the origin (where  $\mathcal{B}_{\text{DR}}^1$  is the basis of  $H_{\text{DR}}^1$  made of normalized newforms of level  $5l$  together with normalized newforms of level  $l$  lifted to level  $5l$  by the two degeneracy maps),
- a degree  $g$  effective (origin) divisor  $\Omega$  on  $X_1(5l)$ ,

and returns an approximation of the degree  $g$  effective divisor

$$P = P_1 + \cdots + P_g$$

on  $X_1(5l)$  such that  $\phi'(P - \Omega) = x + \Lambda$ .

The running time is  $(l \times \log(2 + |x|_\infty) \times m)^\Theta$  where  $5l$  is the level,  $m$  is the required absolute accuracy of the result and  $\log(2 + |x|_\infty)$  is the size of  $x$  i.e. the logarithm of its  $L^\infty$  norm in the canonical basis of  $\mathbb{C}^{\mathcal{B}_{\text{DR}}^1}$ .

We insist that the absolute accuracy in the above theorem is measured in the space  $\mathbb{C}^{\mathcal{B}_{\text{DR}}^1}$  using the  $L^\infty$  norm; or equivalently in the jacobian  $J = J_1(5l)$  using the distance  $d_J$ . However, when  $x$  belongs to the Ramanujan subspace  $W_f$  and assuming the degree  $g$  origin divisor  $\Omega$  is the cuspidal divisor  $D_0$  manufactured in Section 8.1, then there is a unique effective degree  $g$  divisor  $Q$  such that  $\phi'(Q - \Omega) = x + \Lambda$ . We call it the *Ramanujan divisor* associated with  $x$ . In that case, we can and must control the error in  $X^g$ . This error can be expressed as the distance between  $Q = \sum_{1 \leq k \leq g} Q_k$  and the output divisor  $Q' = \sum_{1 \leq k \leq g} Q'_k$ . Assume that  $Q_k = (\gamma_k, q_k)$ . The distance between  $Q$  and  $Q'$  is defined to be the minimum over all permutations  $\sigma$  of  $\{1, 2, \dots, g\}$  of the quantity

$$\max_{1 \leq k \leq g} \left| q_k - q_{\gamma_k}(Q'_{\sigma(k)}) \right|.$$

Sections 12.11 to 12.13 will be mainly devoted to the proof of the theorem below.

### 12.10.2 Theorem (Approximating $V_f$ over the complex numbers)

There exists a deterministic algorithm that takes as input an even integer  $k > 2$ , a prime integer  $l > 6(k - 1)$ , a finite field  $\mathbb{F}$  with characteristic  $l$ , a ring epimorphism  $f : \mathbb{T}(1, k) \rightarrow \mathbb{F}$ , and a cuspidal divisor  $\Omega$  on  $X_1(5l)$  like the divisor  $D_0$  constructed in Section 8.1, and computes complex approximations for every element in  $W_f \subset J_1(5l)$ , the image of  $V_f \subset J_1(l)$  by  $B_{5l, l, 1}^*$ . Here  $V_f \subset J_1(l)$  is defined by Equation (12.1) and we assume that the image of the Galois representation  $\rho_f$  associated with  $f$  contains  $\text{SL}(V_f)$ . The algorithm returns for every element  $x$  in  $W_f$  a complex approximation of the unique degree  $g$  divisor  $Q_x = \sum_{1 \leq n \leq g} Q_{x, n}$  such that  $Q_x - \Omega$  lies in the class represented by  $x$ . Every point  $Q_{x, n}$  is given as a couple  $(\gamma, q)$  where  $\gamma \in \Xi$  and  $q$  is an approximation of the value at  $Q_{x, n}$  of the local analytic parameter  $q_\gamma$  defined by Equation (12.2). The running time of the algorithm is  $\leq (m \times \#V_f)^\Theta$  for some absolute constant  $\Theta$ . Here

$\#V_f$  is the cardinality of the Galois representation  $V_f$  and  $m$  is the required absolute accuracy of the returned approximations for the  $q$  associated with every  $Q_{x,n}$ .

There are two important differences between Theorem 12.10.1 and Theorem 12.10.2. While Theorem 12.10.1 controls the error in  $J_1(5l)$ , Theorem 12.10.2 controls the error in  $X_1(5l)^g$ . Unfortunately, Theorem 12.10.2 only applies to special divisors like the  $Q_x - \Omega$ . For these divisors, one can prove that the inverse Jacobi problem is reasonably well conditioned. Results in Section 8.1 prove that  $Q_x$  is well defined; and using Proposition 11.7.1 in Section 11.7 one can show that computing  $Q_x$  from  $x$  is a well conditioned problem. This will be the purpose of the next three sections.

Another remark concerning notation. We denote by  $\Omega$  the divisor  $D_0$  introduced in section 8.1. And we write  $Q_x$  rather than  $D_x$ . The only reason for this slight change in notation is that many things are already called  $D$  in this chapter, and we want to avoid any possible confusion.

## 12.11 THE ALGEBRAIC CONDITIONING

An important feature of Theorems (12.9.1), (12.9.2), and (12.10.1) is that the error in all these statements is measured in the torus  $J(\mathbb{C})$ . We have seen that this helps controlling the accumulation of errors when we chain computations. However, when solving the inverse Jacobi problem, we want to control the error in  $\text{Sym}^g X$ , at least for the final result of the computation. So we need a theoretical estimate for the error in  $\text{Sym}^g X$  in terms of the error in  $J(\mathbb{C})$ . This will be the main concern of this and the following two sections.

So assume that we are given a degree  $g$  effective origin divisor  $\Omega$  on  $X$  and a vector  $x$  in  $\mathbb{C}^{\mathcal{B}_{\text{DR}}^1}$ , and we look for an effective degree  $g$  divisor  $P$  such that  $\phi'(P - \Omega) = x + \Lambda$ , where  $\Lambda$  is the lattice of periods. We write

$$P = n_1 P_1 + n_2 P_2 + \cdots + n_K P_K \quad (12.90)$$

where the  $(P_k)_{1 \leq k \leq K}$  are pairwise distinct points on  $X$  and the  $(n_k)_{1 \leq k \leq K}$

are positive integers such that

$$n_1 + n_2 + \cdots + n_K = g.$$

We shall make two assumptions.

The *first assumption* is rather essential: we assume that the divisor  $P$  is *non-special* or equivalently that :

$$\dim(\Lambda(P)) = 1. \quad (12.91)$$

A first interesting consequence of this first assumption is that the answer to the inverse Jacobi problem is unique; and the error can be defined as the distance to the unique solution.

The *second assumption* we make is more technical. We assume that Klein's modular function  $j$  does not take the values 0 or 1728 at any of the points  $(P_k)_{1 \leq k \leq K}$ . So

$$j(P_k) \notin \{0, 1728\}.$$

Removing the second assumption would only result in an heavier presentation. By contrast, the first assumption plays a crucial role in the forthcoming calculations. Its meaning is that the Jacobi map  $\phi' : \text{Sym}^g X \rightarrow J$  is a local diffeomorphism at  $P - \Omega$ . Our first task is to reformulate this first assumption in a more algebraic setting. We look for an algebraic variant of the Jacobi determinant in Equation (12.88). Knowing that such an algebraic quantity is non-zero we will bound it from below in Section 12.12.

We assume that we are given a basis  $\mathcal{B}_{\mathbb{Z}}^1$  of  $H_{\text{DR}}^1(X_1(5l))$  such that for every  $\omega = f(q)q^{-1}dq$  in  $\mathcal{B}_{\mathbb{Z}}^1$ , the associated modular form  $f(q)$  has rational integer coefficients and type  $(\exp(\Theta l^4), \Theta)$ . The existence of such a basis is granted by Lemma 12.11.1 below. We stress that we don't try to compute such a basis. We are just happy to know that it exists.

We also need an algebraic uniformizing parameter  $t_Q$  at every point  $Q$  on  $X$  such that  $j(Q) \notin \{0, 1728\}$ . When  $Q$  is not a cusp either, the differential  $dj$  of Klein's function  $j$  has no pole nor zero at  $Q$ . So  $j - j(Q)$  is a uniformizing parameter at  $Q$ . So we set  $t_Q = j - j(Q)$  in that case. When  $Q = \gamma(\infty)$  is a cusp and  $\gamma \in \Xi$ , we set

$$j_\gamma = j \circ W_\gamma^{-1}$$

where  $W_\gamma$  is given in Equation (12.7). We deduce from Equation (12.8) that the automorphism  $W_\gamma^{-1}$  maps the cusp  $\gamma(\infty)$  to the cusp  $\infty$ . So  $j_\gamma$  has a simple pole at  $\gamma(\infty)$  and  $j_\gamma^{-1}$  is a uniformizing parameter at  $\gamma(\infty)$ . So we set  $t_Q = j_\gamma^{-1}$  in that case. We notice that  $j_\gamma$  only depends on the width of  $\gamma(\infty)$ . We call  $j_1 = j$ ,  $j_5$ ,  $j_l$  and  $j_{5l}$  the four corresponding functions. Altogether,  $t_Q$  is one of the following functions:  $j - j(Q)$ ,  $1/j$ ,  $1/j_5$ ,  $1/j_l$  or  $1/j_{5l}$ .

Let  $\omega$  be a form in  $\mathcal{B}_{\mathbb{Z}}^1$  and let  $k$  be an integer such that  $1 \leq k \leq K$  where  $K$  is the number of distinct points in the divisor  $P$  of Equation (12.90). Let  $t_k = t_{P_k}$  be the algebraic uniformizing parameter at  $P_k$  and consider the Taylor expansion of  $\omega/dt_k$  at  $P_k$

$$\omega/dt_k = \xi_{k,0}^\omega + \xi_{k,1}^\omega \times \frac{t_k}{1!} + \xi_{k,2}^\omega \times \frac{t_k^2}{2!} + \dots + \xi_{k,n_k-1}^\omega \times \frac{t_k^{n_k-1}}{(n_k-1)!} + O(t_k^{n_k}). \tag{12.92}$$

We only need the first  $n_k$  terms in this expansion, where  $n_k$  is the multiplicity of  $P_k$  in the divisor  $P$ . We form the matrix

$$\mathcal{M}_P^{\text{alg}} = (\xi_{k,m}^\omega)_{1 \leq k \leq K, 0 \leq m \leq n_k-1; \omega \in \mathcal{B}_{\mathbb{Z}}^1}. \tag{12.93}$$

The lines in  $\mathcal{M}_P^{\text{alg}}$  are indexed by pairs  $(k, m)$  where  $1 \leq k \leq K$  and  $0 \leq m \leq n_k - 1$ . The columns in  $\mathcal{M}_P^{\text{alg}}$  are indexed by forms  $\omega$  in  $\mathcal{B}_{\mathbb{Z}}^1$ . We call  $\mathcal{M}_P^{\text{alg}}$  the *algebraic Wronskian matrix* at  $P$ . The determinant of  $\mathcal{M}_P^{\text{alg}}$  will play an important role in the sequel. We call it the *algebraic conditioning*.

An important feature of the matrix  $\mathcal{M}_P^{\text{alg}}$  is that its entries are algebraic functions evaluated at the points  $P_k$  for  $1 \leq k \leq K$ . Indeed let  $t_k = t_{P_k} \in \{j - j(P_k), 1/j, 1/j_5, 1/j_l, 1/j_{5l}\}$  be the chosen algebraic uniformizing parameter at the point  $P_k$  and set  $\omega^{(0)} = \omega/dt_k$ . For every integer  $m \geq 0$  set  $\omega^{(m+1)} = d\omega^{(m)}/dt_k$ . Then

$$\xi_{k,m}^\omega = \omega^{(m)}(P_k).$$

Lemma 12.11.2 below provides more detailed information about the algebraic dependency between the derivatives  $\omega^{(m)}$  and Klein's function  $j$ .

It is an important consequence of our first assumption in Equation (12.91) that the determinant of the algebraic Wronskian matrix is non-zero

$$\det \mathcal{M}_P^{\text{alg}} \neq 0.$$



In the next Section 12.12 we shall derive a lower bound for this determinant, using the theory of heights. To finish this section, there remains to state and prove the two Lemmas 12.11.1 and 12.11.2. We first construct the basis  $\mathcal{B}_{\mathbb{Z}}^1$ .

**12.11.1 Lemma (A rational basis)** *There is a basis  $\mathcal{B}_{\mathbb{Z}}^1$  of  $H_{\text{DR}}^1(X_1(5l))$  such that for every  $\omega = f(q)q^{-1}dq$  in  $\mathcal{B}_{\mathbb{Z}}^1$ , the associated modular form  $f(q)$  has rational integer coefficients and type*

$$(\exp(\Theta l^4), \Theta).$$

*The transition matrix from the basis  $\mathcal{B}_{\text{DR}}^1$  to the basis  $\mathcal{B}_{\mathbb{Z}}^1$  has algebraic integer entries bounded in absolute value by  $\exp(\Theta l^4)$  and its determinant is the square root of a non-zero rational integer.*

We construct  $\mathcal{B}_{\mathbb{Z}}^1$  from  $\mathcal{B}_{\text{DR}}^1$  using a descent process.

Let  $\omega$  be a differential form in the basis  $\mathcal{B}_{\text{DR}}^1$  and let

$$f(q) = \omega q(dq)^{-1} = \sum_{j \geq 1} f_j q^j$$

be the corresponding modular form. Let  $\mathbb{Z}_f$  be the ring generated by the coefficients of  $f(q)$ . As a  $\mathbb{Z}$ -module,  $\mathbb{Z}_f$  is generated by the  $f_j$  for  $j \leq 4l^2$ . Let  $\mathbb{K}_f$  be the fraction field of  $\mathbb{Z}_f$ . Let  $\mathbb{L}$  be a strict subfield of  $\mathbb{K}_f$ . Let  $\mathbf{a} = (a_j)_{1 \leq j \leq 4l^2}$  be a vector with rational integer coefficients. The associated linear combination  $\sum_{1 \leq j \leq 4l^2} a_j f_j$  belongs to  $\mathbb{L}$  if and only if  $\mathbf{a}$  belongs to a submodule of  $\mathbb{Z}^{4l^2}$  with rank  $< 4l^2$ . The degree  $d_f$  of  $\mathbb{K}_f$  over  $\mathbb{Q}$  is bounded above by  $2g$ . So the number of strict subfields of  $\mathbb{K}_f$  is  $< 2^{2g}$ . So there exists rational integers  $(a_j)_{1 \leq j \leq 4l^2}$  such that  $0 \leq a_j < 2^{2g}$  and  $\theta = \sum_{1 \leq j \leq 4l^2} a_j f_j$  does not belong to any strict subfield of  $\mathbb{K}_f$ . This  $\theta$  is an algebraic integer that generates  $\mathbb{K}_f$  over  $\mathbb{Q}$ . For  $0 \leq k \leq d_f - 1$  we set

$$\text{Tr}(\theta^k f) = \sum_{j \geq 1} \text{Tr}(\theta^k f_j) q^j$$

where  $\text{Tr} : \mathbb{Z}_f \rightarrow \mathbb{Z}$  is the trace map. Since  $|f_j| \leq \Theta(j+1)^\Theta$  and  $|\theta| \leq \exp(\Theta l^2)$  we deduce that  $|\text{Tr}(\theta^k f_j)| \leq \exp(\Theta l^4)(j+1)^\Theta$ . So the series  $\text{Tr}(\theta^k f)$  for  $0 \leq k \leq d_f - 1$  have type  $(\exp(\Theta l^4), \Theta)$ .

We do the same construction for every Galois orbit in  $\mathcal{B}_{\text{DR}}^1$ . We collect all the forms thus obtained. This makes a basis  $\mathcal{B}_{\mathbb{Z}}^1$  of  $H_{\text{DR}}^1(X_1(5l))$  consisting

of forms  $f(q)q^{-1}dq$  where  $f(q)$  is a series with integer coefficients and of type  $(\exp(\Theta l^4), \Theta)$ .  $\square$

Now let's prove some quantitative statement about the algebraic dependency between the successive derivatives  $\omega^{(m)}$  and  $j$ .

**12.11.2 Lemma (An algebraic relation)** *Let  $t$  be one of the functions  $j, 1/j, 1/j_5, 1/j_l, 1/j_{5l}$ . Let  $\omega$  be a form in  $\mathcal{B}_{\mathbb{Z}}^1$  and let  $m \geq 0$  be an integer. Set  $\omega^{(0)} = \omega/dt$  and  $\omega^{(m)} = d^m \omega^{(0)} / (dt)^m$ . There exists a non-zero irreducible polynomial  $E(X, Y) \in \mathbb{Z}[X, Y]$  such that*

$$E(\omega^{(m)}, t) = 0.$$

*Its degree in either variable is*

$$\leq (lm)^\Theta,$$

*and its coefficients are bounded in absolute value by*

$$\exp((lm)^\Theta).$$

All the functions involved belong to the field  $\mathbb{Q}(X_1(5l))$  of modular functions having Puiseux expansion in  $\mathbb{Q}\{\{q\}\}$  at the cusp  $\infty$ . This field is a regular extension of  $\mathbb{Q}(j)$  corresponding to the standard model of  $X_1(5l)$  over  $\mathbb{Q}$ .

We assume that  $t = j$  since the other cases are quite similar. The differential  $dj$  has  $4(l^2 - 1)$  zeros of multiplicity 2 (the points in the fiber of  $j$  above 0) and  $6(l^2 - 1)$  zeros of multiplicity 1 (the points in the fiber of  $j$  above 1728). So  $\omega^{(0)} = \omega/dj$  has degree  $\leq 14l^2$  and less than  $10l^2$  poles. When we differentiate  $\omega^{(0)}$  we increase by one the multiplicity of each pole. So  $d\omega^{(0)}$  has the same poles as  $\omega^{(0)}$  and the total multiplicity of these poles is less than  $24l^2$ . When we divide by  $dj$  we don't add poles but we increase the multiplicities by 1 (in the fiber of  $j$  above 1728) or 2 (in the fiber of  $j$  above 0). The degree of the polar divisor of  $\omega^{(1)}$  is thus  $\leq 38l^2$ . We go on like that and we prove that the degree of  $\omega^{(m)}$  is  $\leq (14 + 24m)l^2 \leq 24(m + 1)l^2$ . The degree of  $j$  is  $12(l^2 - 1)$ . So there is an irreducible polynomial  $E(x, y)$  in  $\mathbb{Z}[x, y]$  such that  $E(\omega^{(m)}, j^{-1}) = 0$  and  $\deg_x E \leq 12(l^2 - 1)$  and  $\deg_y E \leq 24(m + 1)l^2$ .

In order to bound the coefficients in  $E(x, y)$  we consider the expansions of  $j^{-1}$  and  $\omega^{(m)}$  at the cusp  $\infty$ . Remember that

$$j(q) = \frac{1}{q} + 744 + \sum_{k \geq 1} c(k)q^k$$

so

$$-q^2 \frac{dj}{dq} = 1 - q \sum_{k \geq 1} kc(k)q^k.$$

From Equation (12.5) we deduce that  $\sum_{k \geq 1} kc(k)q^k$  has exp-type  $(\Theta, 0, 2)$  in the sense of Section 12.4.12. Using Equation (12.39) we deduce that  $\left(-q^2 \frac{dj}{dq}\right)^{-1}$  has exp-type  $(\kappa_1, 0, 4)$  where  $\kappa_1 \geq 1$  is an absolute constant. We write

$$\omega^{(0)} = -q^2 \frac{\omega}{dq} \left(-q^2 \frac{dj}{dq}\right)^{-1}.$$

The series  $\omega/dq$  has type  $(\exp(\Theta l^4), \Theta)$  and exp-type  $(\Theta, \Theta l^2, 2)$ . Using Equation (12.37) we deduce that  $\omega^{(0)}$  has exp-type  $(\kappa_2, l^2 \kappa_3, 4)$  for some absolute constants  $\kappa_2 \geq 1$  and  $\kappa_3 \geq 1$ . A simple iteration shows that  $\omega^{(m)}$  has exp-type

$$(\kappa_2 + m\kappa_1, l^2 \kappa_3 + m\kappa_2 + \frac{m(m-1)}{2}\kappa_1 + 2m, 4).$$

So if  $a$  is an integer such that  $0 \leq a \leq 12(l^2 - 1)$  then  $(\omega^{(m)})^a$  has exp-type

$$(\Theta l^2(m+1), \Theta l^2(l+m+1)^2, 4).$$

On the other hand,  $j^{-1}$  has exp-type  $(\Theta, 0, 4)$  and if  $b$  is an integer such that  $0 \leq b \leq 24(m+1)l^2$  then  $j^{-b}$  has exp-type

$$(\Theta(m+1)l^2, \Theta(m+1)l^2, 4).$$

So all the monomials  $(\omega^{(m)})^a j^{-b}$  arising in equation  $E(\omega^{(m)}, j^{-1}) = 0$  have exp-type

$$(\Theta l^2(m+1), \Theta l^2(l+m+1)^2, 4)$$

and the coefficients in their  $q$ -expansions up to order

$$\deg(\omega^{(m)}) \times \deg(j^{-1}) \leq \Theta(m+1)l^4$$

are rational integers bounded in absolute value by

$$\exp(\Theta l^{24}(m+1)^8).$$

Since the coefficients in  $E(x, y)$  are solutions of the homogeneous system given by these truncated  $q$ -expansions, they are bounded in absolute value by

$$\exp(\Theta l^{28}(m+1)^9).$$

□

## 12.12 HEIGHTS

In this section we recall basic facts about heights of algebraic numbers and we deduce upper and lower bounds for the determinant of the algebraic Wronskian matrix in Equation (12.93) when the divisor  $P$  is a Ramanujan divisor.

Let  $\bar{\mathbb{Q}} \subset \mathbb{C}$  be the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$ . Let  $\alpha \in \bar{\mathbb{Q}}$  be an algebraic number. The *degree*  $d_\alpha$  of  $\alpha$  is the degree of the field extension  $\mathbb{Q}(\alpha)/\mathbb{Q}$ . Let

$$f(x) = a_{d_\alpha} x^{d_\alpha} + a_{d_\alpha-1} x^{d_\alpha-1} + \cdots + a_0$$

be the unique irreducible polynomial in  $\mathbb{Z}[x]$  such that  $f(\alpha) = 0$  and  $a_{d_\alpha} > 0$ . We say that  $a_{d_\alpha}$  is the *denominator* of  $\alpha$  and we denote it  $\mathfrak{d}_\alpha$ .

Let  $\mathbb{K}$  be a number field containing  $\alpha$ . The multiplicative height of  $\alpha$  with respect to  $\mathbb{K}$  is

$$H_{\mathbb{K}}(\alpha) = \prod_{\sigma} \max(1, |\sigma(\alpha)|) \prod_v \max(1, |\alpha|_v)$$

where the  $\sigma$  in the first product runs over the set of embeddings of  $\mathbb{K}$  into  $\mathbb{C}$  and the  $v$  in the second product runs over the non-archimedean places of  $\mathbb{K}$ . In the special case  $\mathbb{K} = \mathbb{Q}(\alpha)$  we have

$$H_{\mathbb{Q}(\alpha)}(\alpha) = \mathfrak{d}_\alpha \prod_{1 \leq k \leq d_\alpha} \max(1, |\alpha_k|)$$

where the  $\alpha_k$  are the  $d_\alpha$  roots of  $f(x)$ .

The *logarithmic height* of  $\alpha$  with respect to  $\mathbb{K}$  is

$$h_{\mathbb{K}}(\alpha) = \log H_{\mathbb{K}}(\alpha)$$

and the absolute (logarithmic) height of  $\alpha$  is

$$h(\alpha) = \frac{h_{\mathbb{K}}(\alpha)}{\deg(\mathbb{K}/\mathbb{Q})} = \frac{h_{\mathbb{Q}(\alpha)}(\alpha)}{d_\alpha}.$$

Knowing the degree  $d_\alpha$  and absolute height  $h(\alpha)$  of a non-zero algebraic number  $\alpha$ , we deduce the following upper and lower bounds for the absolute value  $|\alpha|$

$$\exp(-d_\alpha \times h(\alpha)) \leq |\alpha| \leq \exp(d_\alpha \times h(\alpha)).$$

Let  $F(x)$  be a degree  $d_F$  polynomial in  $\mathbb{Z}[x]$  and assume that all coefficients in  $F(x)$  are bounded by  $H_F$  in absolute value. Let  $\mathbb{K}$  be a number field and let  $\alpha \in \mathbb{K}$  be an algebraic number. We set  $\beta = F(\alpha)$ .

If  $\sigma$  is any embedding of  $\mathbb{K}$  into  $\mathbb{C}$  we have  $\sigma(\beta) = F(\sigma(\alpha))$  so

$$\max(1, |\sigma(\beta)|) \leq (d_F + 1)H_F \max(1, |\sigma(\alpha)|)^{d_F}.$$

Now, let  $v$  be a non-archimedean valuation of  $\mathbb{K}$ . We have

$$\max(1, |\beta|_v) \leq \max(1, |\alpha|_v)^{d_F}.$$

Forming the product over all  $\sigma$ 's and all  $v$ 's we find that the absolute logarithmic height of  $\beta$  is

$$\leq d_F \times h(\alpha) + \log(d_F + 1) + \log H_F. \quad (12.94)$$

Now assume that  $\alpha$  and  $\beta$  belong to a degree  $d$  extension  $\mathbb{K}$  of  $\mathbb{Q}$  and let  $E(x, y) \in \mathbb{Z}[x, y]$  be a polynomial such that  $E(\alpha, y) \neq 0$  and  $E(\alpha, \beta) = 0$ . Assume that all coefficients in  $E(x, y)$  are bounded by  $H_E$  in absolute value. Call  $d_x$  (resp.  $d_y$ ) the degree of  $E(x, y)$  with respect to the variable  $x$  (resp.  $y$ ).

We write

$$E(x, y) = \sum_{0 \leq k \leq d_y} E_k(x)y^k.$$

We deduce from inequality (12.94) that every  $E_k(\alpha)$  has absolute height

$$\leq d_x \times h(\alpha) + \log(d_x + 1) + \log H_E. \quad (12.95)$$

Let  $K$  be the largest  $k$  such that  $E_k(\alpha) \neq 0$ . Set

$$F(y) = E(\alpha, y) = \sum_{0 \leq k \leq K} E_k(\alpha) y^k.$$

If  $\sigma$  is any embedding of  $\mathbb{K}$  into  $\mathbb{C}$  we call

$${}^\sigma F(y) = E(\sigma(\alpha), y) = \sum_{0 \leq k \leq K} E_k(\sigma(\alpha)) y^k$$

the polynomial obtained by applying  $\sigma$  to all the coefficients in  $F(y)$ . Applying Landau's inequality to  ${}^\sigma F(y)$  we find that

$$\max(1, |\sigma(\beta)|) \leq \frac{\sqrt{d_y + 1} \times \max_{0 \leq k \leq K} (|E_k(\sigma(\alpha))|)}{|E_K(\sigma(\alpha))|}$$

and this is

$$\leq \sqrt{d_y + 1} \times (d_x + 1) H_E \times \max(1, |\sigma(\alpha)|)^{d_x} \times \max(1, |E_K(\sigma(\alpha))|^{-1}).$$

Now, let  $v$  be a non-archimedean valuation of  $\mathbb{K}$ . Applying Gauss' lemma to  $F(y)$  we find that

$$\begin{aligned} \max(1, |\beta|_v) &\leq |E_K(\alpha)|_v^{-1} \times \max(1, |\alpha|_v)^{d_x} \\ &\leq \max(1, |E_K(\alpha)|_v^{-1}) \times \max(1, |\alpha|_v)^{d_x}. \end{aligned}$$

Forming the product over all  $\sigma$ 's and all  $v$ 's we find that the absolute logarithmic height of  $\beta$  is bounded from above by

$$\frac{\log(d_y + 1)}{2} + \log(d_x + 1) + \log H_E + d_x \times h(\alpha) + h(E_K(\alpha)).$$

Using Equation (12.95) we deduce that

$$h(\beta) \leq \frac{\log(d_y + 1)}{2} + 2 \log(d_x + 1) + 2 \log H_E + 2d_x \times h(\alpha).$$

**12.12.1 Lemma (Relating heights of algebraic numbers)** *Let  $\alpha$  and  $\beta$  belong to a degree  $d$  extension of  $\mathbb{Q}$  and let  $E(x, y) \in \mathbb{Z}[x, y]$  be a polynomial such that  $E(\alpha, y) \neq 0$  and  $E(\alpha, \beta) = 0$ . Assume that all coefficients in  $E(x, y)$  are bounded by  $H_E$  in absolute value. Call  $d_x$  (resp.  $d_y$ ) the degree of  $E(x, y)$  with respect to the variable  $x$  (resp.  $y$ ). Then the absolute heights of  $\alpha$  and  $\beta$  are related by the following inequality*

$$h(\beta) \leq \frac{\log(d_y + 1)}{2} + 2 \log(d_x + 1) + 2 \log H_E + 2d_x \times h(\alpha).$$

We now can bound the heights of the entries in the algebraic Wronskian of Section 12.11, at least in the cases we are interested in. Let

$$x + \Lambda \in W_f \subset J(\mathbb{C}) = \mathbb{C}^{\mathcal{B}_{\text{DR}}^1} / \Lambda$$

be a non-zero vector in the Ramanujan subspace  $W_f$  of Theorem 12.10.2. Assume that the degree  $g$  origin divisor  $\Omega$  on  $X_1(5l)$  is the cuspidal divisor  $D_0$  manufactured in Section 8.1. Let  $P$  be the unique degree  $g$  divisor such that

$$\phi'(P - \Omega) = x + \Lambda.$$

As proven in Section 8.1, the two assumptions of Section 12.11 are satisfied in that case: the divisor  $P = n_1 P_1 + n_2 P_2 + \dots + n_K P_K$  is *non-special* and Klein's modular function  $j$  does not take the values 0 or 1728 at any of the points  $(P_k)_{1 \leq k \leq K}$ . Proposition 11.7.1 in Section 11.7 implies that the absolute height of every  $j(P_k)$  is  $\leq l^\Theta$ . Using Lemma 12.11.2 and Lemma 12.12.1 we deduce that every entry in the algebraic Wronskian matrix of Equation (12.93) has absolute height  $\leq l^\Theta$ . Further any such entry generates an extension of  $\mathbb{Q}$  of degree  $\leq (\#V_f)^\Theta$ . So every entry in the algebraic Wronskian has denominator and absolute value bounded by  $\exp((\#V_f)^\Theta)$ . So both the denominator and absolute value of the determinant are  $\leq \exp((\#V_f)^\Theta)$ . So

$$h(\det(\mathcal{M}_P^{\text{alg}})) \leq (\#V_f)^\Theta \tag{12.96}$$

in that case.

Note also that, as an algebraic integer, the algebraic conditioning as degree at most twice  $\#V_f$  because the square of it lies in the definition field of  $x + \Lambda$  and the latter field is a degree  $\#V_f - 1$  extension of  $\mathbb{Q}$ .

We deduce that for all primes  $p$  but a finite number bounded by  $(\#V_f)^\Theta$ , the divisor  $P$  remains non-special when we reduce modulo any place  $\mathfrak{p}$  above  $p$ . Indeed, we can assume that  $p \notin \{5, l\}$  so  $X_1(5l)$  has good reduction modulo  $p$ . We also can assume that  $j(P_k) \notin \{0, 1728\} \pmod{\mathfrak{p}}$  because both the degree and the absolute height of  $j(P_k)$  are  $\leq (\#V_f)^\Theta$ . We also can assume that the  $P_k$  remain pairwise distinct modulo  $\mathfrak{p}$  for the same reason: we just need to exclude less than  $(\#V_f)^\Theta$  primes  $p$ . We can also assume that every  $P_k$  which is not a cusp does not reduce modulo  $\mathfrak{p}$  onto a cusp. We also assume that  $p$  is larger than the genus  $g$  of  $X_1(5l)$  so that the Taylor expansion in Equation (12.92) remains valid modulo  $\mathfrak{p}$ . So the algebraic Wronskian matrix reduces modulo  $\mathfrak{p}$  to the algebraic Wronskian matrix. Excluding a few more primes, but no more than  $(\#V_f)^\Theta$ , we can assume that the algebraic conditioning does not vanish modulo  $\mathfrak{p}$ . So the divisor  $P$  remains non-special.

**12.12.2 Lemma (Reduction modulo  $p$  of a Ramanujan divisor)** *Call  $\Omega$  the cuspidal divisor  $D_0$  on  $X_1(5l)$  introduced in Section 8.1 and let  $W_f$  be the Ramanujan subspace  $W_f$  of Theorem 12.10.2. Let*

$$x \in W_f \subset J_1(5l)$$

*and let  $P$  be the unique divisor on  $X_1(5l)$  such that  $P - \Omega$  lies in the class defined by  $x$ . Then  $P$  is non-special and for all primes  $p$  but a finite number bounded by  $(\#V_f)^\Theta$ , the divisor  $P$  remains non-special modulo any place  $\mathfrak{p}$  above  $p$ .*

We stress that the above lemma is very similar to Theorem 11.7.10.

### 12.13 BOUNDING THE ERROR IN $X^g$

In this section we fill the gap between Theorems 12.10.1 and 12.10.2. For  $x + \Lambda \in W_f \subset J(\mathbb{C})$  and  $P$  the corresponding Ramanujan divisor, we relate the error on  $P$  and the error on  $x$ . We need some control on the Jacobi integration map locally at  $P$ . A first step in this direction is the upper



bound for the height of the algebraic stability, given in Equation (12.96). In Section 12.13.1 we deduce a lower bound for the determinant of the differential of the Jacobi map at  $P$ . This bound implies that the Jacobi map is non-singular on a reasonably large neighborhood of  $P$ , as we show in Section 12.13.3. In Section 12.13.4 we deduce that the inverse of the Jacobi map is well defined and Lipschitz (with reasonably small constant) on a (reasonably large) neighborhood of  $x + \Lambda$ . We shall need the identities between Wronskians and Jacobians of power series proven in Section 12.5. The reason for these algebraic complications is the following: we need an equation for the singular locus in  $X^g$  of the Jacobi integration map. The space  $X^g$  is stratified by the diagonals. The strata correspond to partitions of  $\{1, 2, 3, \dots, g\}$ . We obtain a different equation for the singular locus on every stratum. These various equations are related by algebraic identities between Jacobians and Wronskians.

### 12.13.1 The analytic conditioning of a divisor

Let

$$P = n_1 P_1 + \dots + n_K P_K \quad (12.97)$$

be a degree  $g$  effective divisor on  $X$  where the  $n_k$  are positive integers for  $1 \leq k \leq K$ . For every  $k$  we write

$$P_k = (\gamma_k, q_k)$$

where  $\gamma_k \in \Xi$  and  $q_k \in F_{w_k} \subset D(0, \exp(-\pi/w_k))$ , where  $w_k$  is the width of the cusp  $\gamma_k(\infty)$ . Let

$$\phi_{(n_k)_{1 \leq k \leq K}} : \prod_{1 \leq k \leq K} \text{Sym}^{n_k} X \rightarrow J(\mathbb{C}) = \mathbb{C}^{\mathcal{B}_{\text{DR}}^1} / \Lambda$$

be the relevant Jacobi integration map in this context. We stress that this map is different from the maps introduced before. Its initial set is a sort of *semi-symmetric product* i.e. something between the Cartesian product  $X^g$  and the full symmetric product  $\text{Sym}^g X$ . In order to write down the differential of this map at the divisor  $P$ , we need a local system of coordinates on  $\prod_{1 \leq k \leq K} \text{Sym}^{n_k} X$ . We shall use partial Newton power sums. Let

$$R = (\gamma, q_\gamma(R))$$

be a point on  $X$ . For every  $n \geq 1$  and  $m \geq 1$  we define the  $m$ -th power sum  $\nu_{R,n,m}$  on  $\text{Sym}^n X$  to be the function that takes the value

$$(q_\gamma(Q_1) - q_\gamma(R))^m + (q_\gamma(Q_2) - q_\gamma(R))^m + \cdots + (q_\gamma(Q_n) - q_\gamma(R))^m$$

at  $\{Q_1, \dots, Q_n\}$ .

Now let us come back to the divisor  $P$  in Equation (12.97). The functions

$$(\nu_{P_k, n_k, m})_{1 \leq k \leq K; 1 \leq m \leq n_k}$$

form a local system of coordinates on  $\prod_{1 \leq k \leq K} \text{Sym}^{n_k} X$  at the divisor  $P$ . The matrix of the differential at  $P$  of the integration map  $\phi_{(n_k)_{1 \leq k \leq K}}$  in the bases  $(d\nu_{P_k, n_k, m})_{1 \leq k \leq K; 1 \leq m \leq n_k}$  and  $\mathcal{B}_{\text{DR}}^1$  is

$$\mathcal{M}_P^{\text{ana}} = \frac{1}{\prod_{1 \leq k \leq K} \prod_{1 \leq m \leq n_k} m!} (\psi_{k,m}^\omega)_{1 \leq k \leq K; 0 \leq m \leq n_k - 1} \quad (12.98)$$

where  $\psi_{k,m}^\omega$  is the  $m$ -th derivative of  $\omega/dq_{\gamma_k}$  with respect to  $q_{\gamma_k}$ , evaluated at  $P_k = (\gamma_k, q_k)$ . The determinant of this matrix is called the *analytic conditioning* of the divisor  $P$ . This analytic conditioning and the algebraic conditioning defined in Section 12.11 after Equation (12.93) differ by simple factors we should not be afraid of.

Now, we assume that we are in the context of Theorem 12.10.2. We call  $\Omega$  the cuspidal divisor  $D_0$  constructed in Section 8.1 and we assume that the class of  $P - \Omega$  corresponds to a point  $x$  in the Ramanujan subspace  $W_f \subset J$ . Equation (12.96) implies that the algebraic conditioning of  $P$  is  $\geq \exp(-(\#V_f)^\Theta)$ . In order to obtain a similar lower bound for the analytic conditioning, we must bound from below the complementary factors.

The factor  $1/\prod_{1 \leq k \leq K} \prod_{1 \leq m \leq n_k} m!$  is  $\geq \exp(-\Theta l^\tau)$ .

Lemma 12.11.1 implies that the factor coming from the change of bases from  $\mathcal{B}_{\text{DR}}^1$  to  $\mathcal{B}_{\mathbb{Z}}^1$  is  $\geq \exp(-l^\Theta)$ .

There are also factors due to the change of coordinates. Assume for example that  $P_k$  is not a cusp. So the algebraic parameter at  $P_k$  is  $j - j(P_k)$ . The analytic parameter at  $P_k$  is  $q_{\gamma_k} - q_k$ . The extra factor in the analytic conditioning is thus

$$\left( \frac{dj}{dq_{\gamma_k}}(P_k) \right)^{\frac{n_k(n_k+1)}{2}}$$

Lemma 12.13.2 below ensures that there exists a constant  $K_1$  such that this factor is  $\geq \exp(-(\#V_f)^{K_1})$  provided there exists a constant  $K_2$  such that  $|j(P_k)|$  and  $|j(P_k) - 1728|$  are both  $\geq \exp(-(\#V_f)^{K_2})$ . But this latter condition is met because  $j(P_k)$  is not  $\{0, 1728\}$ , and both its degree and logarithmic height are  $\leq (\#V_f)^\Theta$ .

Assume now that  $P_k$  is a cusp. Then the algebraic parameter at  $P$  is the function  $1/j_{w_k}$  introduced in Section 12.11 where  $w_k \in \{1, 5, l, 5l\}$  is the width of the cusp  $P_k = \gamma_k(\infty)$ . And the derivative  $d(1/j_{w_k})/dq_{\gamma_k}$  is just 1. So the analytic conditioning at a Ramanujan  $l$ -torsion divisor is

$$\geq \exp(-(\#V_f)^\Theta). \tag{12.99}$$

To finish this section, there remains to state and prove Lemma 12.13.2.

**12.13.2 Lemma (Lower bounds for  $d\mathbf{J}/dx$ )** *For every positive real number  $K_1$  there exists a positive real number  $K_2$  such that the following statement is true.*

*Let  $L \geq 2$  be an integer and let  $\mathbf{J}(x)$  be Klein's series given in Equation (12.3). Let  $q \in D(0, 1)$  be a complex number in the unit disk such that  $1 - |q| \geq L^{-K_1}$  and  $|\mathbf{J}(q)| \geq \exp(-L^{K_1})$  and  $|\mathbf{J}(q) - 1728| \geq \exp(-L^{K_1})$ . Then  $|\frac{d\mathbf{J}}{dx}(q)| \geq \exp(-L^{K_2})$ .*

We first note that  $\frac{d\mathbf{J}}{dx}(q)$  only vanishes if  $\mathbf{J}(q) \in \{0, 1728\}$ . So we just want to prove that if  $\frac{d\mathbf{J}}{dx}$  is small at  $q$  then  $q$  is close to a zero of it. We would like to apply Lemma 5.4.12 to the series  $-x^2\mathbf{J}'(x) = -x^2\frac{d\mathbf{J}}{dx}$ . But this series is not of type  $(A, n)$  because its coefficients are a bit too large. So we set  $R = (1 + |q|)/2$  and we set  $\mathfrak{J}(x) = -x^2R^2 \times \mathbf{J}'(Rx)$ . From the hypothesis in the lemma there exists a positive real  $K_3$  such that  $\log R^n \leq -n \times L^{-K_3}$ . From Equation (12.5) there exists an absolute constant  $\kappa$  such that the coefficient of  $x^n$  in  $-x^2 \times \mathbf{J}'(x)$  is  $\leq \exp(\kappa\sqrt{n})$ . This implies that the series  $\mathfrak{J}(x)$  has type  $(\exp(L^{K_4}), K_4)$  for some  $K_4 \geq 1$ . We apply Lemma 5.4.12 of Chapter 5 to the series  $\mathfrak{J}(x)$  at  $q/R$  and we are done.  $\square$

**12.13.3 The neighborhood of a non-special divisor**

The singular locus of the Jacobi integration map  $\phi : X^g \rightarrow J$  is a strict closed subset. So every non-special effective degree  $g$  divisor has a neigh-

neighborhood in  $\text{Sym}^g X$  consisting of non-special divisors. In this section, we provide a quantified version of this statement. We first need to define simple neighborhoods of the divisor  $P$  in Equation (12.97). Let  $\epsilon$  be a positive real number. We call  $\mathcal{P}_\epsilon$  the set of degree  $g$  effective divisors  $P'$  that can be written  $P' = \sum_{1 \leq k \leq K} \sum_{1 \leq m \leq n_k} P'_{k,m}$  where  $P'_{k,m} = (\gamma_k, q'_{k,m})$  and  $|q'_{k,m} - q_k| \leq \epsilon$ .

We assume that  $P$  is non-special. We expect that if  $\epsilon$  is small enough, then  $P'$  is non-special as well. In order to write down the analytic conditioning of  $P'$  we must take multiplicities into account. So we rewrite  $P'$  as

$$P' = \sum_{1 \leq k \leq K} \sum_{1 \leq s \leq S_k} m_{k,s} P'_{k,s},$$

where  $\mathbf{m}_k = (m_{k,s})_{1 \leq s \leq S_k}$  is a partition of  $n_k$  into  $S_k$  non-empty parts. In particular

$$n_k = m_{k,1} + m_{k,2} + \cdots + m_{k,S_k}.$$

For every  $1 \leq k \leq K$  we call

$$x_k = q_{\gamma_k} - q_k$$

the local analytic parameter at  $P_k$ . We call  $\mathbf{f}_k$  the vector in  $(\mathbb{C}[[x_k]])^{\mathcal{B}_{\text{DR}}^1}$  defined by

$$\mathbf{f}_k = \left( \text{Taylor} \left( \frac{\omega}{dq_{\gamma_k}}, q_k \right) \right)_{\omega \in \mathcal{B}_{\text{DR}}^1} \in (\mathbb{C}[[x_k]])^{\mathcal{B}_{\text{DR}}^1} \quad (12.100)$$

where  $\text{Taylor} \left( \frac{\omega}{dq_{\gamma_k}}, q_k \right)$  is the Taylor expansion of  $\frac{\omega}{dq_{\gamma_k}}$  at  $q_k$  in the parameter  $x_k$ .

For every  $1 \leq k \leq K$  and  $1 \leq s \leq S_k$  we write

$$P'_{k,s} = (\gamma_k, q'_{k,s})$$

and we set  $y_{k,s} = q'_{k,s} - q_k$ .

The analytic conditioning of  $P$  is the Wronskian in Equation (12.46) evaluated at  $(0, 0, \dots, 0) \in \mathbb{C}^K$  and divided by  $\mathbf{n}! = \prod_{1 \leq k \leq K} n_k!$

The analytic conditioning of  $P'$  is the Jacobian in Equation (12.47) divided by  $\mathbf{m}! = \prod_{1 \leq k \leq K} \prod_{1 \leq s \leq S_k} m_{k,s}!$  More precisely, the Jacobian in

Equation (12.47) is  $\mathbf{m}!$  times the Taylor expansion of the analytic conditioning in the parameters  $(y_{k,s})_{1 \leq k \leq K; 1 \leq s \leq S_k}$ .

These two conditionings are related by Equation (12.48). The conditioning at  $P'$  is divisible by the determinant  $D_{\mathbf{m},\mathbf{n}}$  relating the two partitions. And the quotient specializes to the conditioning at  $P$  times  $\mathbf{n}!/\mathbf{m}!$

Let  $S = \sum_{1 \leq k \leq K} S_k$ . Then  $\mathcal{J}_{(\mathbf{f}_k, \mathbf{m}_k)_{1 \leq k \leq K}}$  is a series in the  $S$  variables  $(y_{k,s})_{1 \leq k \leq K; 1 \leq s \leq S_k}$ .

We assume that  $q_k$  and  $q'_{k,m}$  belong to  $D(0, \exp(-\pi/w_k))$ .

Using Equation (12.56) about the size of coefficients in modular forms, the type of refocused series as described in Section 12.4.3, the type of derivatives given in Section 12.4.6, the type of quotient series given in Section 12.4.5, we can prove that the series

$$\frac{\mathcal{J}_{(\mathbf{f}_k, \mathbf{m}_k)_{1 \leq k \leq K}}}{D_{\mathbf{m},\mathbf{n}}}$$

has type  $(\exp(l^\Theta), l^\Theta \mathbf{1}_g)$ .

We denote by  $\lambda$  the opposite of the logarithm of the absolute value of the analytic conditioning of  $P$ . We call it the *analytic illconditioning* of  $P$ . It generalizes the illconditioning introduced in Section 12.10. The only difference is that here we take multiplicities into account. Using Equation (12.48) and inequality (12.32) on bounding the remainder, we now prove that if

$$-\log \epsilon \geq l^\Theta + \lambda \tag{12.101}$$

then  $P'$  is a non-special divisor also. According to Equation (12.99) the analytic illconditioning  $\lambda$  is  $\leq (\#V_f)^\Theta$ . So any divisor in the neighborhood  $\mathcal{P}_\epsilon$  is non-special, provided  $\epsilon \leq \exp(-(\#V_f)^\Theta)$ .

### 12.13.4 Relating the direct and inverse error

We go on with the notation in the previous Section 12.13.3. We have a divisor  $P$  as in Equation (12.97). We denote by  $\mathcal{S}_{(n_k)_{1 \leq k \leq K}}$  or just  $\mathcal{S}$  the map

$$(R_1, \dots, R_g) \mapsto (\{R_1, \dots, R_{n_1}\}, \{R_{n_1+1}, \dots, R_{n_1+n_2}\}, \dots, \{R_{n_1+n_2+\dots+n_{K-1}+1}, \dots, R_g\})$$

and we check that the following diagram commutes:

$$\begin{array}{ccc}
 X^g & & \\
 \downarrow \mathcal{S}_{(n_k)_{1 \leq k \leq K}} & \searrow \phi & \\
 \prod_{1 \leq k \leq K} \text{Sym}^{n_k} X & \xrightarrow{\phi_{(n_k)_{1 \leq k \leq K}}} & J
 \end{array}$$

We can see the divisor  $P$  as a point on the semi-symmetric product  $\prod_{1 \leq k \leq K} \text{Sym}^{n_k} X$ . We call  $U \in X^g$  the unique  $g$ -uple such that  $\mathcal{S}(U) = P$ .

We assume that  $P$  is a non-special divisor. The map  $\phi_{(n_k)_{1 \leq k \leq K}}$  is thus a local diffeomorphism at  $P$ . However, the map  $\phi$  is not a local diffeomorphism at  $U$  because  $\mathcal{S}$  is not, unless the partition  $\mathbf{n}$  is  $(1, 1, \dots, 1)$ . We shall allow ourselves to write  $\phi'$  instead of  $\phi_{(n_k)_{1 \leq k \leq K}}$  in this section.

The previous Section 12.13.3 provides an explicit analytic description of the maps  $\phi$  (resp.  $\phi'$ ) at  $U$  (resp.  $P$ ). We have local coordinates  $(z_{k,m})_{1 \leq k \leq K; 1 \leq m \leq n_k}$  at  $U \in X^g$ . If  $R$  is a  $g$ -uple in  $X^g$  we write

$$R = (R_{k,m})_{1 \leq k \leq K; 1 \leq m \leq n_k}$$

and we set

$$z_{k,m}(R) = q_{\gamma_k}(R_{k,m}) - q_k.$$

For every  $1 \leq k \leq K$  we define the series  $\mathbf{F}_k \in (\mathbb{C}[[x_k]])^{\mathcal{B}_{\text{DR}}^1}$  to be the formal integral

$$\mathbf{F}_k(x_k) = \int_0^{x_k} \mathbf{f}_k(x_k) dx_k = \mathbf{F}_{k,1} \times x_k + \mathbf{F}_{k,2} \times x_k^2 + \dots$$

where  $\mathbf{f}_k$  is the vector defined in Equation (12.100). This is the Taylor expansion of the Jacobi map at the point  $P_k$ . The Taylor expansion of  $\phi$  at  $U$  is

$$\text{Taylor}(\phi, U) = \sum_{1 \leq k \leq K} \sum_{1 \leq m \leq n_k} \mathbf{F}_k(z_{k,m})$$

and it lies in

$$(\mathbb{C}[[z_{k,m}]_{1 \leq k \leq K; 1 \leq m \leq n_k}]]^{\mathcal{B}_{\text{DR}}^1}.$$

We stress that for every  $\omega$  in  $\mathcal{B}_{\text{DR}}^1$ , the corresponding coordinate in the above Taylor expansion is a series of type

$$(l^\Theta, \Theta \times \mathbf{1}_g) \tag{12.102}$$

because it is mostly the expansion of a modular form at a point which is not too close to the boundary of the unit disk.

We now split this Taylor expansion in two pieces. We write

$$\text{Taylor}(\phi, U) = T_1 + T_2$$

where  $T_1$  is a sort of principal part

$$\begin{aligned} T_1 &= \sum_{1 \leq k \leq K} \sum_{1 \leq m \leq n_k} \sum_{1 \leq j \leq n_k} \mathbf{F}_{k,j} \times z_{k,m}^j \\ &= \sum_{1 \leq k \leq K} \sum_{1 \leq j \leq n_k} \mathbf{F}_{k,j} \sum_{1 \leq m \leq n_k} z_{k,m}^j \\ &= \sum_{1 \leq k \leq K} \sum_{1 \leq j \leq n_k} \mathbf{F}_{k,j} \times \nu_{P_k, n_k, j}(\{R_{k,1}, R_{k,2}, \dots, R_{k, n_k}\}) \\ &= \sum_{1 \leq k \leq K} \sum_{1 \leq j \leq n_k} \mathbf{F}_{k,j} \times \nu_{P_k, n_k, j}(\mathcal{S}(R)), \end{aligned} \tag{12.103}$$

and  $T_2 = \text{Taylor}(\phi, U) - \phi_1$  is the corresponding remainder. It is clear from Equation (12.103) that  $T_1$  can be written

$$T_1 = T_1' \circ \mathcal{S}$$

where  $T_1'$  is the first order term (the differential) in the Taylor expansion of  $\phi'$  at  $P = \mathcal{S}(U)$ . We write  $T_2 = T_2' \circ \mathcal{S}$  where  $T_2'$  is the corresponding remainder. Our goal now is to prove that in many circumstances the principal term  $T_1'$  dominates the remainder  $T_2'$ . In order to control how close is  $R$  to  $U$  (or equivalently how close is  $\mathcal{S}(R)$  to  $P$ ) we set

$$\begin{aligned} \epsilon(R) &= \max_{1 \leq k \leq K; 1 \leq m \leq n_k} |z_{k,m}(R_{k,m})| \\ &= \max_{1 \leq k \leq K; 1 \leq m \leq n_k} |q_{\gamma_k}(R_{k,m}) - q_k| \end{aligned}$$

and

$$\eta(\mathcal{S}(R)) = \max_{1 \leq k \leq K; 1 \leq j \leq n_k} |\nu_{P_k, n_k, j}(\{R_{k,1}, R_{k,2}, \dots, R_{k, n_k}\})|$$

and we assume that  $\eta(\mathcal{S}(R))$  is  $\leq \exp(-l^\Theta)$ .

We first bound the remainder  $T_2(R) = T'_2(\mathcal{S}(R))$  from above in terms of  $\eta(\mathcal{S}(R))$ . Using Buckholtz inequality (5.10) we can bound the coordinates  $z_{k,m}(R) = q_{\gamma_k}(R_{k,m}) - q_k$  for every  $1 \leq k \leq K$  and  $1 \leq m \leq n_k$  in terms of the bound  $\eta(\mathcal{S}(R))$  on Newton power functions:

$$|z_{k,m}(R)| \leq 5\eta(\mathcal{S}(R))^{1/n_k}$$

so

$$|z_{k,m}(R)|^{n_k+1} \leq 5^{g+1} \times \eta(\mathcal{S}(R))^{(n_k+1)/n_k} \leq 5^{g+1} \times \eta(\mathcal{S}(R))^{1+1/g}.$$

Using the type estimate in Equation (12.102) and Equation (12.32) we deduce that

$$|T'_2(\mathcal{S}(R))|_\infty \leq \exp(l^\Theta) \times \eta(\mathcal{S}(R))^{1+1/g}. \quad (12.104)$$

We now bound from below  $T_1(R) = T'_1(\mathcal{S}(R))$  in terms of  $\eta(\mathcal{S}(R))$ . This time we need some hypothesis on the divisor  $P$ . We assume that we are in the context of Theorem 12.10.2. We call  $\Omega$  the cuspidal divisor  $D_0$  constructed in Section 8.1 and we assume that the class of  $P - \Omega$  corresponds to a point  $x$  in the Ramanujan subspace  $W_f \subset J$ . We note that  $T'_1$  is a linear map and its matrix in the bases  $(\nu_{P_k, n_k, j})_{1 \leq k \leq K; 1 \leq j \leq n_k}$  and  $\mathcal{B}_{\text{DR}}^1$  is the matrix  $\mathcal{M}_P^{\text{ana}}$  of Equation (12.98). The coefficients in this matrix are  $\leq l^\Theta$  in absolute value. The determinant of this matrix is the analytic conditioning. According to Equation (12.99) the analytic conditioning at such a Ramanujan  $l$ -torsion divisor is  $\geq \exp(-(\#V_f)^\Theta)$ . We deduce that

$$|T'_1(\mathcal{S}(R))|_\infty \geq \exp(-(\#V_f)^\Theta) \times \eta(\mathcal{S}(R)). \quad (12.105)$$

We deduce from Equations (12.105) and (12.104) that

$$|T'_1(\mathcal{S}(R))|_\infty \geq 2 |T'_2(\mathcal{S}(R))|_\infty$$

as soon as  $\epsilon(R) \leq \exp(-(\#V_f)^\Theta)$ .



Using Lemma 12.13.5 below we deduce that there exist two absolute positive constants  $\kappa_1$  and  $\kappa_2$  such that if

$$|\phi(R) - \phi'(P)|_\infty \leq \exp(-(\#V_f)^{\kappa_1})$$

then there exists a divisor  $S$  in  $\prod_{1 \leq k \leq K} \text{Sym}^{n_k} X$  such that

$$\eta(S) \leq \exp((\#V_f)^{\kappa_2}) \times |\phi(R) - \phi'(P)|_\infty$$

and  $\phi'(S) = \phi(R)$ .

Using again that the conditioning of  $P$  is  $\geq \exp(-(\#V_f)^\Theta)$  together with the estimates in Section 12.13.3, e.g. Equation (12.101), we deduce that if the constant  $\kappa_1$  has been chosen big enough, then this divisor  $S$  is non-special. Therefore  $S = \mathcal{S}(R)$  and we have proven that

$$\eta(\mathcal{S}(R)) \leq \exp((\#V_f)^{\kappa_2}) \times |\phi(R) - \phi'(P)|_\infty$$

therefore

$$\epsilon(R) \leq \exp((\#V_f)^{\kappa_3}) \times |\phi(R) - \phi'(P)|_\infty$$

for some absolute positive constant  $\kappa_3$ .

This relation between the error in  $J$  and the error in  $X^g$  finishes the proof of Theorem 12.10.2.

To finish this section there remains to state and prove Lemma 12.13.5.

**12.13.5 Lemma (Perturbation of a non-singular linear map)** *Let  $g \geq 1$  be a positive integer and let  $\bar{P}(\mathbf{0}_g, \mathbf{1}_g) \in \mathbb{C}^g$  be the closed polydisk with polyradius  $\mathbf{1}_g$  and centered at the origin. Let  $T : \bar{P}(\mathbf{0}_g, \mathbf{1}_g) \rightarrow \mathbb{C}^g$  be a continuous function and let  $T_1 : \mathbb{C}^g \rightarrow \mathbb{C}^g$  be a linear function. Set  $T_2 = T - T_1$  and assume that for every  $R = (R_1, \dots, R_g)$  in  $\bar{P}(\mathbf{0}_g, \mathbf{1}_g)$  we have  $|T_1(R)|_\infty > 2|T_2(R)|_\infty$ . Let  $K > 0$  be a positive real number such that for every  $R$  in  $\mathbb{C}^g$*

$$|T_1(R)|_\infty \geq K \times |R|_\infty.$$

*Then the image by  $T$  of the polydisk  $\bar{P}(\mathbf{0}_g, \mathbf{1}_g)$  contains the polydisk  $\bar{P}(\mathbf{0}_g, \frac{K}{2} \times \mathbf{1}_g)$  with polyradius  $K/2 \times \mathbf{1}_g$ .*

For every real number  $t \in [0, 1]$  we denote by  $\mathcal{S}_t$  the image by  $T_1 + tT_2$  of the  $L^\infty$ -sphere with radius 1. Then  $\mathcal{S}_t$  is contained in

$$\mathbb{C}^g - \bar{P}(\mathbf{0}_g, \frac{K}{2} \times \mathbf{1}_g)$$

for every  $t \in [0, 1]$ . So  $\mathcal{S}_1$  is homologous to  $\mathcal{S}_0$  in  $\mathbb{C}^g - \bar{P}(\mathbf{0}_g, \frac{K}{2} \times \mathbf{1}_g)$  and its class in

$$H_{2g-1}(\mathbb{C}^g - \bar{P}(\mathbf{0}_g, \frac{K}{2} \times \mathbf{1}_g), \mathbb{Z})$$

is non-zero. So for every  $x$  in  $\bar{P}(\mathbf{0}_g, \frac{K}{2} \times \mathbf{1}_g)$ , the class of  $\mathcal{S}_1$  in  $H_{2g-1}(\mathbb{C}^g - \{x\}, \mathbb{Z})$  is non-zero either. Assume now that  $x$  does not belong to the image by  $T$  of the polydisk  $\bar{P}(\mathbf{0}_g, \mathbf{1}_g)$ . Then  $\mathcal{S}_1$  is the boundary of  $T(\bar{P}(\mathbf{0}_g, \mathbf{1}_g)) \subset \mathbb{C}^g - \{x\}$  and its class in  $H_{2g-1}(\mathbb{C}^g - \{x\}, \mathbb{Z})$  is trivial. A contradiction.  $\square$

## 12.14 FINAL RESULT OF THIS CHAPTER

We now state the final result of this chapter.

### 12.14.1 Theorem (Approximating $V_f$ over the complex numbers)

*There exists a deterministic algorithm that takes as input an even integer  $k > 2$ , a prime integer  $l > 6(k - 1)$ , a finite field  $\mathbb{F}$  with characteristic  $l$ , a ring epimorphism  $f : \mathbb{T}(1, k) \rightarrow \mathbb{F}$ , and a cuspidal divisor  $\Omega$  on  $X_1(5l)$  like the divisor  $D_0$  constructed in Section 8.1, and computes complex approximations for every element in  $W_f \subset J_1(5l)$ , the image of  $V_f \subset J_1(l)$  by  $B_{5l, l, 1}^*$ . Here  $V_f \subset J_1(l)$  is defined by Equation (12.1) and we assume that the image of the Galois representation  $\rho_f$  associated with  $f$  contains  $\text{SL}(V_f)$ . The algorithm returns for every element  $x$  in  $W_f$  the unique degree  $g$  effective divisor  $Q_x$  such that  $Q_x - \Omega$  lies in the class represented by  $x$ . More precisely, the algorithm returns the cuspidal part  $Q_x^{\text{cusp}}$  of  $Q_x$  and a complex approximation of its finite part  $Q_x^{\text{fin}} = \sum_{1 \leq n \leq d_x} Q_{x,n}$ . Every point  $Q_{x,n}$  is given by complex approximations of its affine coordinates  $(b(Q_{x,n}), x(Q_{x,n}))$  in the plane model  $C_l$  of  $X_1(5l)$  given in Section 12.3.3. The running time of the algorithm is  $\leq (m \times \#V_f)^\Theta$  for some absolute constant  $\Theta$ . Here  $\#V_f$  is the cardinality of the Galois representation  $V_f$  and  $m$  is the required absolute accuracy.*

There are only two differences between this Theorem 12.14.1 and the previous Theorem 12.10.2. Firstly we claim that we can separate the cuspidal and the finite part of  $Q_x$ . Secondly we return algebraic coordinates  $b$  and  $x$  for the points  $Q_{x,i}$  rather than analytic ones.

Indeed, Theorem 12.10.2 gives us for every point  $P$  in the support of  $Q_x$  an analytic coordinate  $(\gamma, q)$ . We want to decide if the point  $P$  in question is equal to  $\gamma(\infty)$ . Let  $j$  be Klein's modular function. We know that both the degree and the logarithmic height of  $j(P)$  are  $\leq (\#V_f)^\Theta$ . So there exists an absolute constant  $\kappa_1$  such that if  $1/j(P)$  is  $\leq \exp(-(\#V_f)^{\kappa_1})$  then it is zero. But the Petersson-Rademacher inequality tells us that there exists an  $\epsilon > 0$  such that if  $x$  is a complex number bounded by  $\epsilon$  in absolute value then

$$|\mathbf{J}(x)| \geq \frac{1}{2|x|}$$

where  $\mathbf{J}(x)$  is Klein's series given in Equation (12.3). Using Equation (12.4) we deduce that there exists an absolute constant  $\kappa_2$  such that if  $q$  is  $\leq \exp(-(\#V_f)^{\kappa_2})$  then  $q = 0$  and the point  $P = (\gamma, q)$  is the cusp  $\gamma(\infty)$ . This explains why we can recognize cusps.

Now assume that  $P = (\gamma, q)$  is a point in the support of some  $Q_x$  and assume that  $P$  is not a cusp. Then  $|q| \geq \exp(-(\#V_f)^{\kappa_2})$  and in order to compute  $b(P)$  and  $x(P)$  we just substitute  $q_\gamma$  by  $q$  in the expansions for  $x$  and  $b$  at  $\gamma(\infty)$  computed in Section 12.3.

## Chapter Thirteen

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### Computing $V_f$ modulo $p$

J.-M. Couveignes

In this chapter we address the problem of computing in the group of  $l^k$ -torsion rational points in the jacobian variety of algebraic curves over finite fields, with an application to computing modular representations.

Let  $p$  be a prime and let  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$  be the field with  $p$  elements. Let  $\overline{\mathbb{F}_p}$  be an algebraic closure of  $\mathbb{F}_p$ . For any power  $q$  of  $p$  we call  $\mathbb{F}_q \subset \overline{\mathbb{F}_p}$  the field with  $q$  elements. Let  $\mathbb{A}^2 \subset \mathbb{P}^2$  be the affine and projective planes over  $\mathbb{F}_q$ . Let  $C \subset \mathbb{P}^2$  be a plane projective geometrically integral curve over  $\mathbb{F}_q$ . Let  $X$  be its smooth projective model and let  $J$  be the jacobian variety of  $X$ . We note  $g$  the genus of  $X$  and  $d$  the degree of  $C$ . We assume that we are given the numerator of the zeta function of the function field  $\mathbb{F}_q(X)$ . So we know the characteristic polynomial of the Frobenius endomorphism  $F_q$  of  $J$ . This is a monic degree  $2g$  polynomial  $\chi(x)$  with integer coefficients.

Let  $l \neq p$  be a prime integer and let  $k \geq 1$  be an integer. We look for a *nice generating set* for the group  $J[l^k](\mathbb{F}_q)$  of  $l^k$ -torsion points in  $J(\mathbb{F}_q)$ . By *nice* we mean that the generating set  $(g_i)_{1 \leq i \leq I}$  should induce a decomposition of  $J[l^k](\mathbb{F}_q)$  as a direct product  $\prod_{1 \leq i \leq I} \langle g_i \rangle$  of cyclic subgroups with non-decreasing orders. Given such a generating set and an  $\mathbb{F}_q$ -endomorphism of  $J$ , we also want to describe the action of this endomorphism on  $J[l^k](\mathbb{F}_q)$  by an  $I \times I$  integer matrix.

By an algorithm in this paper we usually mean a probabilistic Las Vegas algorithm. In some places we shall give deterministic algorithms or probabilistic Monte-Carlo algorithms, but this will be stated explicitly. See Section 5.1 for a reminder of computational complexity theory. The main reason for using probabilistic Turing machines is that we shall need to construct generating sets for the Picard group of curves over finite fields. Solv-

ing such a problem in the deterministic world is out of reach at this time. See Section 5.1.

In Section 13.1 we recall how to compute in the Picard group  $J(\mathbb{F}_q)$ . Section 13.2 gives a naive algorithm for picking random elements in this group. Pairings are useful when looking for relations between divisor classes. So we recall how to compute pairings in Section 13.3. Section 13.4 is concerned with characteristic subspaces for the action of Frobenius inside the  $l^\infty$ -torsion of  $J(\overline{\mathbb{F}}_p)$ . In Section 13.5 we look for a convenient surjection from  $J(\mathbb{F}_q)$  onto its  $l^k$ -torsion subgroup. We use the Kummer exact sequence and the structure of the ring generated by the Frobenius endomorphism. In Section 13.6 we give an algorithm that, on input a degree  $d$  plane projective curve over  $\mathbb{F}_q$ , plus some information on its singularities, and the zeta function of its function field, returns a nice generating set for the group of  $l^k$ -torsion points inside  $J(\mathbb{F}_q)$  in probabilistic polynomial time in  $\log q$ ,  $d$  and  $l^k$ . In Section 13.7 we apply the general algorithms in Section 13.6 to the modular curve  $X_1(5l)$  in order to compute explicitly a modular representations  $V_f$  modulo  $l$ . Such a representation modulo  $l$  can be realized as a subgroup  $W_f$  inside the  $l$ -torsion of  $J_1(5l)/\mathbb{Q}$ . The idea is to compute the reduction modulo  $p$  of the group scheme  $W_f$  as a subgroup of  $J_1(5l)/\mathbb{F}_p$ , for many small primes  $p$ . One can then lift using the Chinese Remainder Theorem, as will be explained in Section 14.8.

*Remark.* The symbol  $\Theta$  in this chapter stands for a positive effective absolute constant. So any statement containing this symbol becomes true if the symbol is replaced in every occurrence by some large enough real number.

### 13.1 BASIC ALGORITHMS FOR PLANE CURVES

In this section, we recall elementary results about computing in the Picard group of an algebraic curve over a finite field. See [Hac, Vol, Die] for a more detailed treatment.

#### 13.1.1 Finite fields

We should first explain how finite fields are represented. The prime field  $\mathbb{F}_p$  is just  $\mathbb{Z}/p\mathbb{Z}$  so elements in it are represented as integers in  $[0, p - 1[$ .

The base field  $\mathbb{F}_q$  is given as  $\mathbb{F}_p[x]/f(x)$  where  $f(x)$  is an irreducible unitary polynomial with degree  $a$  in  $\mathbb{F}_p[x]$  where  $p$  is the characteristic and  $q = p^a$ . A finite extension of  $\mathbb{F}_q$  is given as  $\mathbb{F}_q[y]/h(y)$  where  $h(y)$  is a unitary irreducible polynomial in  $\mathbb{F}_q[y]$ . We shall never use two extensions of  $\mathbb{F}_q$  simultaneously. Remind polynomial factoring in  $\mathbb{F}_q[x]$  is Las Vegas probabilistic polynomial time in  $\log q$  and the degree of the polynomial to be factored [Ga-Ge, Chapter 14].

### 13.1.2 Plane projective curves and their smooth model

We now explain how curves are represented in this paper. To start with, a projective plane curve  $C$  over  $\mathbb{F}_q$  is given by a degree  $d$  homogeneous polynomial  $E(x, y, z)$  in the three variables  $x, y$  and  $z$ , with coefficients in  $\mathbb{F}_q$ . The curve  $C$  is assumed to be absolutely integral. By a *point* on  $C$  we mean a geometric point: an element of  $C(\overline{\mathbb{F}}_q)$  where  $\overline{\mathbb{F}}_q = \overline{\mathbb{F}}_p$  is the algebraic closure of  $\mathbb{F}_p$  fixed in the introduction. Any  $\overline{\mathbb{F}}_q$ -point on  $C$  can be represented by its affine or projective coordinates.

Let  $X$  be a smooth model of  $C$  and let  $X \rightarrow C$  be the desingularization map. If  $P \in X(\overline{\mathbb{F}}_q)$  is a geometric point on  $X$  above a singular point  $S$  on  $C$ , we say that  $P$  is a *singular branch*. The *conductor*  $\mathfrak{C}$  is an effective divisor on  $X$ . It is the closed subscheme of  $X$  defined by the sheaf  $\text{Ann}_{\mathcal{O}_C}(\mathcal{O}_X/\mathcal{O}_C)$ . Every multiplicity in  $\mathfrak{C}$  is even. Some authors call  $\mathfrak{C}$  the *adjunction divisor*. Its support consists of all singular branches. The conductor expresses the local behaviour of the map  $X \rightarrow C$ . See [Ser1, IV.1], [Gor]. We have  $\deg(\mathfrak{C}) = 2\delta$  where

$$\delta = \frac{(d-1)(d-2)}{2} - g$$

is the difference between the arithmetic genus of  $C$  and the geometric genus of  $X$ . Since  $\delta \leq (d-1)(d-2)/2$ , the support of  $\mathfrak{C}$  contains at most  $(d-1)(d-2)/2$  geometric points in  $X(\overline{\mathbb{F}}_q)$ . So the field of definition of any singular branch on  $X$  is an extension of  $\mathbb{F}_q$  with degree  $\leq (d-1)(d-2)/2$ . A modern reference for singularities of plane curves is [Cas, Section 5.8].

The smooth model  $X$  of  $C$  is not given as a projective variety. Indeed, we shall only need a nice local description of  $X$  above every singularity of

$C$ . This means that we need a list of all singular points on  $C$ , and a list (a labelling) of all points in  $X(\overline{\mathbb{F}}_q)$  lying above every singularity of  $C$  (the singular branches), and a uniformizing parameter at every such branch. We also need the Laurent series expansions of affine plane coordinates in terms of all these uniformizing parameters.

More precisely, let  $P \in X(\overline{\mathbb{F}}_q)$  be a geometric point above a singular point  $S$ , and let  $v$  be the corresponding valuation. The field of definition of  $P$  is an extension field  $\mathbb{F}_P$  of  $\mathbb{F}_q$  with degree  $\leq (d-1)(d-2)/2$ . Let  $x_S$  and  $y_S$  be affine coordinates that vanish at the singular point  $S$  on  $C$ . We need a local parameter  $t_P$  at  $P$  and expansions  $x_S = \sum_{k \geq v(x_S)} a_k t_P^k$  and  $y_S = \sum_{k \geq v(y_S)} b_k t_P^k$  with coefficients in  $\mathbb{F}_P$ .

Because these expansions are not finite, we just assume that we are given a black box that on input a positive integer  $n$  returns the first  $n$  terms in all these expansions. In all the cases we shall be interested in, this black box will contain a Turing machine that answers in time polynomial in  $n$  and  $\log q$  and the genus  $g$ . This is the case for curves with ordinary multiple points for example. We have shown in Section 12.3 that this is also the case for the standard plane model of modular curves  $X_1(5l)$ . Using general normalization and factorization algorithms [Die, Section 2] one may show that this is indeed the case for all plane curves without any restriction, but this is beyond the scope of this text.

We may also assume that we are given the conductor  $\mathfrak{C}$  of  $C$  as a combination of singular branches with even coefficients. The following algorithms still work if the conductor is replaced by any divisor  $\mathfrak{D}$  that is greater than the conductor and has polynomial degree in  $d$ . Such a divisor can be found easily: the singular branches on  $X$  are supposed to be known already, and the multiplicities are bounded above by  $(d-1)(d-2)/2$ .

### 13.1.3 Divisors, forms, and functions

Smooth  $\overline{\mathbb{F}}_q$ -points on  $C$  are represented by their affine or projective coordinates. Labelling for the branches above singular points is given in the description of  $X$ . So we know how to represent divisors on  $X$ . For any integer  $h \geq 0$  the  $\mathbb{F}_q$ -linear space  $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(h))$  of degree  $h$  homogeneous polynomials in  $x, y,$  and  $z$  has dimension  $(h+1)(h+2)/2$ . A basis for it is

made of all monomials of the form  $x^a y^b z^c$  with  $a, b, c \in \mathbb{N}$  and  $a+b+c = h$ . We denote by  $\mathcal{O}_X(h)$  the pullback of  $\mathcal{O}_{\mathbb{P}^2}(h)$  to  $X$ . Let  $F$  be a degree  $h$  form on  $\mathbb{P}^2$  having non-zero pullback  $F_X$  on  $X$ . Let  $\Delta = (F_X)$  be the divisor of this restriction. The map  $f \mapsto \frac{f}{F_X}$  is a bijection from  $H^0(X, \mathcal{O}_X(h))$  to the linear space  $H^0(X, \mathcal{O}_X(\Delta))$  associated with  $\Delta$ .

We assume that we are given a divisor  $\mathfrak{D}$  bigger than the conductor  $\mathfrak{C}$ . We assume that the degree of  $\mathfrak{D}$  is  $\leq d^\ominus$ . We have explained in the previous Section 13.1.2 how to find such a divisor. The dimension of  $H^0(X, \mathcal{O}_X(h)(-\mathfrak{D}))$  is at least  $dh + 1 - g - \deg(\mathfrak{D})$  and is equal to this number when it exceeds  $g - 1$ . This is the case if

$$h \geq d + \frac{\deg \mathfrak{D}}{d}.$$

The dimension of  $H^0(X, \mathcal{O}_X(h)(-\mathfrak{D}))$  is greater than  $2g$  if

$$h \geq \frac{3d}{2} + \frac{\deg \mathfrak{D}}{d}.$$

We take  $h$  to be the smallest integer fulfilling this condition. The composite map  $X \rightarrow C \hookrightarrow \mathbb{P}^2$  induces a map

$$\rho_h : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(h)) \rightarrow H^0(X, \mathcal{O}_X(h)).$$

The image of  $\rho_h$  contains  $H^0(X, \mathcal{O}_X(h)(-\mathfrak{D}))$ . This is known as Noether's residue theorem [Gor, Theorem 7]. It will be convenient to describe  $H^0(X, \mathcal{O}_X(h)(-\mathfrak{D}))$  as a quotient

$$0 \rightarrow \text{Ker } \rho_h \rightarrow \rho_h^{-1}(H^0(X, \mathcal{O}_X(h)(-\mathfrak{D}))) \xrightarrow{\rho_h} H^0(X, \mathcal{O}_X(h)(-\mathfrak{D})) \rightarrow 0.$$

We need linear equations for

$$\rho_h^{-1}(H^0(X, \mathcal{O}_X(h)(-\mathfrak{D}))) \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(h)).$$

We consider a generic homogeneous form

$$F(x, y, z) = \sum_{a+b+c=h} \epsilon_{a,b,c} x^a y^b z^c$$

of degree  $h$  in  $x, y$  and  $z$ . For every branch  $P$  above a singular point  $S \in C$  (assuming for example that  $S$  has non-zero  $z$ -coordinate) we replace in  $F(\frac{x}{z}, \frac{y}{z}, 1)$  the affine coordinates  $x = \frac{x}{z}$  and  $y = \frac{y}{z}$  by their expansions



as series in the local parameter  $t_P$  at this branch. We ask the resulting series in  $t_P$  to have valuation at least the multiplicity of  $P$  in the divisor  $\mathcal{D}$ . Every singular branch thus produces linear equations in the  $\epsilon_{a,b,c}$ . The collection of all such equations defines the subspace  $\rho_h^{-1}(H^0(X, \mathcal{O}_X(h)(-\mathcal{D})))$ .

A basis for the subspace  $\text{Ker } \rho_h$  of  $\rho_h^{-1}(H^0(X, \mathcal{O}_X(h)(-\mathcal{D})))$  consists of all  $x^a y^b z^c E(x, y, z)$  with  $a + b + c = h - d$ . We fix a supplementary space  $M_C$  to  $\text{Ker } \rho_h$  in  $\rho_h^{-1}(H^0(X, \mathcal{O}_X(h)(-\mathcal{D})))$  and we assimilate  $H^0(X, \mathcal{O}_X(h)(-\mathcal{D}))$  to it.

Given a homogeneous form in three variables one can compute its divisor on  $X$  using resultants and the given expansions of affine coordinates in terms of the local parameters at every singular branch. A function is given as a quotient of two forms.

### 13.1.4 The Brill-Noether algorithm

The linear space  $M_C$  computed in the previous paragraph is isomorphic to  $H^0(X, \mathcal{O}_X(h)(-\mathcal{D}))$  via the map  $\rho_h$ . This space allows us to compute in the group  $J(\mathbb{F}_q)$  of  $\mathbb{F}_q$ -points in the jacobian of  $X$ . We fix an effective  $\mathbb{F}_q$ -divisor  $\Omega$  with degree  $g$  on  $X$ . This  $\Omega$  will play the role of origin: a point  $\alpha \in J(\mathbb{F}_q)$  is represented by a divisor  $A - \Omega$  in the corresponding linear equivalence class, where  $A$  is an effective  $\mathbb{F}_q$ -divisor on  $X$  with degree  $g$ . Given another point  $\beta \in J(\mathbb{F}_q)$  by a similar divisor  $B - \Omega$ , we can compute the space  $H^0(X, \mathcal{O}_X(h)(-\mathcal{D} - A - B))$  which is non-trivial and pick a non-zero form  $F_1$  in it. The divisor of  $F_1$  is  $(F_1) = A + B + \mathcal{D} + R$  where  $R$  is an effective divisor with degree  $hd - 2g - \text{deg}(\mathcal{D})$ . The linear space  $H^0(X, \mathcal{O}_X(h)(-\mathcal{D} - R - \Omega))$  has dimension at least 1. We pick a non-zero form  $F_2$  in it. It has divisor  $(F_2) = \mathcal{D} + R + \Omega + D$  where  $D$  is effective with degree  $g$ . And  $D - \Omega$  is linearly equivalent to  $A - \Omega + B - \Omega$ .

In order to invert the class  $\alpha$  of  $A - \Omega$  we pick a non-zero form  $F_1$  in  $H^0(X, \mathcal{O}_X(h)(-\mathcal{D} - 2\Omega))$ . The divisor of  $F_1$  is  $(F_1) = 2\Omega + \mathcal{D} + R$  where  $R$  is an effective divisor with degree  $hd - 2g - \text{deg}(\mathcal{D})$ . The linear space  $H^0(X, \mathcal{O}_X(h)(-\mathcal{D} - R - A))$  has dimension at least 1. We pick a non-zero form  $F_2$  in it. It has divisor  $(F_2) = \mathcal{D} + R + A + B$  where  $B$  is effective with degree  $g$ . And  $B - \Omega$  is linearly equivalent to  $-(A - \Omega)$ .

### 13.1.5 Lemma (Arithmetic operations in the jacobian) Let $C/\mathbb{F}_q$ be a

degree  $d$  plane projective absolutely integral curve. Let  $g$  be the geometric genus of  $C$ . Assume that we are given the smooth model  $X$  of  $C$  and a  $\mathbb{F}_q$ -divisor with degree  $g$  on  $X$ , denoted  $\Omega$ . We assume that  $\Omega$  is given as a difference between two effective divisors with degrees bounded by  $d^\Theta$ . This  $\Omega$  serves as an origin. Arithmetic operations in the Picard group  $\text{Pic}^0(X/\mathbb{F}_q)$  can be performed in time polynomial in  $\log q$  and  $d$ . This includes addition, subtraction and comparison of divisor classes.

If  $\Omega$  is not effective, we use Lemma 13.1.6 below to compute a non-zero function  $f$  in  $H^0(X, \mathcal{O}_X(\Omega))$  and we write  $\Omega' = (f) + \Omega$ . This is an effective divisor with degree  $g$ . We replace  $\Omega$  by  $\Omega'$  and finish as in the paragraph before Lemma 13.1.5  $\square$

We now recall the principle of the Brill-Noether algorithm for computing complete linear series. Remind functions in  $\mathbb{F}_q(X)$  are represented as quotients of forms.

**13.1.6 Lemma (Brill-Noether)** *There exists an algorithm that on input a degree  $d$  plane projective absolutely integral curve  $C/\mathbb{F}_q$  and the smooth model  $X$  of  $C$  and two effective  $\mathbb{F}_q$ -divisors  $A$  and  $B$  on  $X$ , computes a basis for  $H^0(X, \mathcal{O}_X(A - B))$  in time polynomial in  $d$  and  $\log q$  and the degrees of  $A$  and  $B$ .*

We assume that  $\deg(A) \geq \deg(B)$ , otherwise  $H^0(X, \mathcal{O}_X(A - B)) = 0$ . Let  $a$  be the degree of  $A$ . We let  $h$  be the smallest integer such that

$$hd - g + 1 > a + \deg \mathfrak{D}.$$

The space  $H^0(X, \mathcal{O}_X(h)(-\mathfrak{D} - A))$  is non-zero. It is contained in the image of the map

$$\rho_h : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(h)) \rightarrow H^0(X, \mathcal{O}_X(h))$$

so that we can represent it as a subspace of  $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(h))$ . We pick a non-zero form  $f$  in  $H^0(X, \mathcal{O}_X(h)(-\mathfrak{D} - A))$  and compute its divisor  $(f) = \mathfrak{D} + A + D$ . The space  $H^0(X, \mathcal{O}_X(h)(-\mathfrak{D} - B - D))$  is contained in the image of the map  $\rho_h$  so that we can represent it as a subspace of  $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(h))$ . We compute forms  $\gamma_1, \gamma_2, \dots, \gamma_k$  in  $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(h))$  such

that their images by  $\rho_h$  provide a basis for  $H^0(X, \mathcal{O}_X(h)(-\mathcal{D} - B - D))$ . A basis for  $H^0(X, \mathcal{O}_X(A - B))$  is made of the functions  $\frac{\gamma_1}{f}, \frac{\gamma_2}{f}, \dots, \frac{\gamma_k}{f}$ .  $\square$

We deduce an explicit moving lemma for divisors.

**13.1.7 Lemma (Moving divisor lemma I)** *There exists an algorithm that on input a degree  $d$  plane projective absolutely integral curve  $C/\mathbb{F}_q$  and the smooth model  $X$  of  $C$  and a degree zero divisor  $D = D^+ - D^-$  and an effective divisor  $A$  with degree  $< q$  on  $X$  computes a divisor  $E = E^+ - E^-$  linearly equivalent to  $D$  and disjoint to  $A$  in time polynomial in  $d$  and  $\log q$  and the degrees of  $D^+$ , and  $A$ . Further the degree of  $E^+$  and  $E^-$  can be taken to be  $\leq 2gd$ .*

Let  $O$  be an  $\mathbb{F}_q$ -rational divisor on  $X$  such that  $1 \leq \deg(O) \leq d$  and disjoint to  $A$ . We may take  $O$  to be a well chosen fiber of some plane coordinate function on  $X$ . We compute the linear space  $H^0(X, \mathcal{O}_X(D^+ - D^- + 2gO))$ . The subspace of  $H^0(X, \mathcal{O}_X(D^+ - D^- + 2gO))$  consisting of functions  $f$  such that  $(f) + D^+ - D^- + 2gO$  is not disjoint to  $A$  is contained in a union of at most  $\deg(A) < q$  hyperplanes. We conclude invoking Lemma 13.1.8 below.  $\square$

There remains to state and prove the

**13.1.8 Lemma (Solving inequalities)** *Let  $q$  be a prime power,  $d \geq 2$  and  $n \geq 1$  two integers and let  $H_1, \dots, H_n$  be hyperplanes inside  $V = \mathbb{F}_q^d$ , each given by a linear equation. Assume that  $n < q$ . There exists a deterministic algorithm that finds a vector in  $U = V - \bigcup_{1 \leq k \leq n} H_k$  in time polynomial in  $\log q, d$  and  $n$ .*

This is proved by lowering the dimension  $d$ . For  $d = 2$  we pick any affine line  $L$  in  $V$  not containing the origin. We observe that there are at least  $q - n$  points in  $U \cap L = L - \bigcup_{1 \leq k \leq n} L \cap H_k$ . We enumerate points in  $L$  until we find one which is not in any  $H_k$ . This requires at most  $n + 1$  trials.

Assume now that  $d$  is bigger than 2. Hyperplanes in  $V$  are parametrized by the projective space  $\mathbb{P}(\hat{V})$  where  $\hat{V}$  is the dual of  $V$ . We enumerate points in  $\mathbb{P}(\hat{V})$  until we find a hyperplane  $K$  distinct from every  $H_k$ . We compute a basis for  $K$  and an equation for every  $H_k \cap K$  in this basis. This way, we have lowered the dimension by 1.  $\square$

We can strengthen a bit the moving divisor algorithm by removing the condition that  $A$  has degree  $< q$ . Indeed, in case this condition is not met, we call  $\alpha$  the smallest integer such that  $q^\alpha > \deg(A)$  and we set  $\beta = \alpha + 1$ . We apply Lemma 13.1.7 after base change to the field with  $q^\alpha$  elements and find a divisor  $E_\alpha$ . We call  $e_\alpha$  the norm of  $E_\alpha$  from  $\mathbb{F}_{q^\alpha}$  to  $\mathbb{F}_q$ . It is equivalent to  $\alpha D$ . We similarly construct a divisor  $e_\beta$  that is equivalent to  $(\alpha + 1)D$ . We return the divisor  $E = e_\beta - e_\alpha$ . We observe that we can take  $\alpha \leq 1 + \log_q \deg(A)$  so the degree of the positive part  $E^+$  of  $E$  is  $\leq 6gd(\log_q(\deg(A)) + 1)$ .

**13.1.9 Lemma (Moving divisor lemma II)** *There exists an algorithm that on input a degree  $d$  plane projective absolutely integral curve  $C/\mathbb{F}_q$  and the smooth model  $X$  of  $C$  and a degree zero  $\mathbb{F}_q$ -divisor  $D = D^+ - D^-$  and an effective divisor  $A$  on  $X$  computes a divisor  $E = E^+ - E^-$  linearly equivalent to  $D$  and disjoint to  $A$  in time polynomial in  $d$  and  $\log q$  and the degrees of  $D^+$ , and  $A$ . Further the degree of  $E^+$  and  $E^-$  can be taken to be  $\leq 6gd(\log_q(\deg(A)) + 1)$ .*

### 13.2 A FIRST APPROACH TO PICKING RANDOM DIVISORS

Given a finite field  $\mathbb{F}_q$  and a plane projective absolutely integral curve  $C$  over  $\mathbb{F}_q$  with projective smooth model  $X$ , we call  $J$  the jacobian of  $X$  and we consider two related problems: picking a random element in  $J(\mathbb{F}_q)$  with (close to) uniform distribution and finding a generating set for (a large subgroup of)  $J(\mathbb{F}_q)$ . Let  $g$  be the genus of  $X$ . We assume that we are given a degree 1 divisor  $O = O^+ - O^-$  where  $O^+$  and  $O^-$  are effective,  $\mathbb{F}_q$ -rational and have degree bounded by  $\Theta g^\Theta$  for some positive constant  $\Theta$ .

We know from [Mu-St-Th, Theorem 2] that the group  $\text{Pic}^0(X/\mathbb{F}_q)$  is generated by the classes  $[\mathfrak{p} - \deg(\mathfrak{p})O]$  where  $\mathfrak{p}$  runs over the set of prime divisors of degree  $\leq 1 + 2 \log_q(4g - 2)$ . For the convenience of the reader we quote this result as a lemma.

**13.2.1 Lemma (Müller, Stein, Thiel)** *Let  $K$  be an algebraic function field of one variable over  $\mathbb{F}_q$ . Let  $N \geq 0$  be an integer. Let  $g$  be the genus of  $K$ . Let  $\chi : \text{Div}(K) \rightarrow \mathbb{C}^*$  be a character of finite order which is non-trivial*

when restricted to  $\text{Div}^0$ . Assume that  $\chi(\mathfrak{B}) = 1$  for every prime divisor  $\mathfrak{B}$  of degree  $\leq N$ . Then

$$N < 2 \log_q(4g - 2).$$

If  $q < 4g^2$ , the number of prime divisors of degree  $\leq 1 + 2 \log_q(4g - 2)$  is bounded by  $\Theta g^\Theta$ . So we can compute easily a small generating set for  $J(\mathbb{F}_q)$ . In the rest of this section, we will assume that the size  $q$  of the field is greater than or equal to  $4g^2$ . This condition ensures the existence of a  $\mathbb{F}_q$ -rational point.

Picking efficiently and provably random elements in  $J(\mathbb{F}_q)$  with uniform distribution seems difficult to us. We first give here an algorithm for efficiently constructing random divisors with a distribution that is far from uniform but still sufficient to construct a generating set for a large subgroup of  $J(\mathbb{F}_q)$ . Once given generators, picking random elements becomes much easier.

Let  $r$  be the smallest prime integer bigger than  $30$ ,  $2g - 2$  and  $d$ . We observe  $r$  is less than  $\max(4g - 4, 2d, 60)$ . The set  $\mathcal{P}(r, q)$  of  $\mathbb{F}_q$ -places with degree  $r$  on  $X$  has cardinality

$$\#\mathcal{P}(r, q) = \frac{\#X(\mathbb{F}_{q^r}) - \#X(\mathbb{F}_q)}{r}.$$

So

$$(1 - 10^{-2}) \frac{q^r}{r} \leq \#\mathcal{P}(r, q) \leq (1 + 10^{-2}) \frac{q^r}{r}.$$

Indeed,  $|\#X(\mathbb{F}_{q^r}) - q^r - 1| \leq 2gq^{\frac{r}{2}}$  and  $|\#X(\mathbb{F}_q) - q - 1| \leq 2gq^{\frac{1}{2}}$ . So

$$\left| \#\mathcal{P}(r, q) - \frac{q^r}{r} \right| \leq \frac{4g + 3}{r} q^{\frac{r}{2}} \leq 8q^{\frac{r}{2}}$$

and  $8rq^{-\frac{r}{2}} \leq r2^{3-\frac{r}{2}} \leq 10^{-2}$  since  $r \geq 31$ .

Since we are given a degree  $d$  plane model  $C$  for the curve  $X$ , we have a degree  $d$  map  $x : X \rightarrow \mathbb{P}^1$ . This is the composition of the desingularization map  $X \rightarrow C$  with the restriction to  $C$  of the rational map  $[x, y, z] \rightarrow [x, z]$ . Since  $d < r$ , the function  $x$  maps  $\mathcal{P}(r, q)$  to the set  $\mathcal{U}(r, q)$  of monic prime polynomials of degree  $r$  over  $\mathbb{F}_q$ . The cardinality of  $\mathcal{U}(r, q)$  is  $\frac{q^r - q}{r}$  so

$$(1 - 10^{-9}) \frac{q^r}{r} \leq \#\mathcal{U}(r, q) \leq \frac{q^r}{r}.$$

The fibers of the map  $x : \mathcal{P}(r, q) \rightarrow \mathcal{U}(r, q)$  have cardinality between 0 and  $d$ . We can pick a random element in  $\mathcal{U}(r, q)$  with uniform distribution in the following way: we pick a random monic polynomial of degree  $r$  with coefficients in  $\mathbb{F}_q$ , with uniform distribution. We check whether it is irreducible. If it is, we output it. Otherwise we start again. This is polynomial time in  $r$  and  $\log q$ .

Given a random element in  $\mathcal{U}(r, q)$  with uniform distribution, we can compute the fiber of  $x : \mathcal{P}(r, q) \rightarrow \mathcal{U}(r, q)$  above it and, provided this fiber is non-empty, pick a random element in it with uniform distribution. If the fiber is empty, we pick another element in  $\mathcal{U}(r, q)$  until we find a non-empty fiber. At least one in every  $d \times (0.99)^{-1}$  fibers is non-empty. We thus define a distribution  $\mu$  on  $\mathcal{P}(r, q)$  and prove the following.

**13.2.2 Lemma (A very rough measure)** *There is a unique measure  $\mu$  on  $\mathcal{P}(r, q)$  such that all non-empty fibers of the map  $x : \mathcal{P}(r, q) \rightarrow \mathcal{U}(r, q)$  have the same measure, and all points in a given fiber have the same measure. There exists a probabilistic algorithm that picks a random element in  $\mathcal{P}(r, q)$  with distribution  $\mu$  in time polynomial in  $d$  and  $\log q$ . For every subset  $Z$  of  $\mathcal{P}(r, q)$  the measure  $\mu(Z)$  is related to the uniform measure  $\frac{\#Z}{\#\mathcal{P}(r, q)}$  by*

$$\frac{\#Z}{d\#\mathcal{P}(r, q)} \leq \mu(Z) \leq \frac{d\#Z}{\#\mathcal{P}(r, q)}.$$

Now let  $\mathcal{D}(r, q)$  be the set of effective  $\mathbb{F}_q$ -divisors with degree  $r$  on  $X$ . Since we have assumed that  $q \geq 4g^2$  we know that  $X$  has at least one  $\mathbb{F}_q$ -rational point. Let  $\Omega$  be a degree  $r$  effective divisor on  $X/\mathbb{F}_q$ . We associate to every  $\alpha$  in  $\mathcal{D}(r, q)$  the class of  $\alpha - \Omega$  in  $J(\mathbb{F}_q)$ . This defines a surjection

$$\phi_r : \mathcal{D}(r, q) \rightarrow J(\mathbb{F}_q)$$

with all its fibers having cardinality  $\#\mathbb{P}^{r-g}(\mathbb{F}_q)$ . So the set  $\mathcal{D}(r, q)$  has cardinality  $\frac{q^{r-g+1}-1}{q-1} \#J(\mathbb{F}_q)$ . So

$$\#\mathcal{P}(r, q) \leq \#\mathcal{D}(r, q) \leq q^{r-g} \frac{1 - \frac{1}{q^{r-g+1}}}{1 - \frac{1}{q}} q^g \left(1 + \frac{1}{\sqrt{q}}\right)^{2g}.$$

Since  $q \geq 4g^2$  we have

$$\#\mathcal{D}(r, q) \leq 2eq^r.$$

Assume that  $G$  is a finite group and  $\psi$  an epimorphism of groups

$$\psi : J(\mathbb{F}_q) \rightarrow G.$$

We look for some divisor  $\Delta \in \mathcal{D}(r, q)$  such that

$$\psi(\phi_r(\Delta)) \neq 0 \in G.$$

Since all the fibers of  $\psi \circ \phi_r$  have the same cardinality, the fiber above 0 has at most

$$\frac{2eq^r}{\#G}$$

elements. So the number of prime divisors  $\Delta \in \mathcal{P}(r, q)$  such that  $\psi(\phi_r(\Delta))$  is not 0 is at least

$$q^r \left( \frac{0.99}{r} - \frac{2e}{\#G} \right).$$

We assume that  $\#G$  is at least  $12r$ . Then at least half of the divisors in  $\mathcal{P}(r, q)$  are not mapped onto 0 by  $\psi \circ \phi_r$ . The  $\mu$ -measure of the subset consisting of these elements is at least  $\frac{1}{2d}$ . So if we pick a random  $\Delta$  in  $\mathcal{P}(r, q)$  with  $\mu$ -measure as in Lemma 13.2.2, the probability of success is at least  $\frac{1}{2d}$ . If we make  $2d$  trials, the probability of success is  $\geq 1 - \exp(-1) \geq \frac{1}{2}$ .

**13.2.3 Lemma (Finding non-zero classes)** *There exists a probabilistic (Monte-Carlo) algorithm that takes as input*

1. *a degree  $d$  and geometric genus  $g$  plane projective absolutely integral curve  $C$  over  $\mathbb{F}_q$ , such that  $q \geq 4g^2$ ,*
2. *the smooth model  $X$  of  $C$ ,*
3. *a degree 1 divisor  $O = O^+ - O^-$  where  $O^+$  and  $O^-$  are effective,  $\mathbb{F}_q$ -rational and have degree bounded by  $\Theta g^\Theta$  for some positive constant  $\Theta$ ,*
4. *an epimorphism  $\psi : \text{Pic}^0(X/\mathbb{F}_q) \rightarrow G$  (that need not be computable) such that the cardinality of  $G$  is at least  $\max(48g, 24d, 720)$ ,*

*and outputs a sequence of  $2d$  elements in  $\text{Pic}^0(X/\mathbb{F}_q)$  such that at least one of them is not in the kernel of  $\psi$  with probability  $\geq \frac{1}{2}$ . The algorithm is polynomial time in  $d$  and  $\log q$ .*

As a special case we take  $G = G_0 = J(\mathbb{F}_q)$  and  $\psi = \psi_0$  the identity. Applying Lemma 13.2.3 we find a sequence of elements in  $J(\mathbb{F}_q)$  out of which one at least is non-zero (with high probability). We take  $G_1$  to be the quotient of  $G$  by the subgroup generated by these elements and  $\psi_1$  the quotient map. Applying the lemma again we construct another sequence of elements in  $J(\mathbb{F}_q)$  out of which one at least is not in  $G_0$  (with high probability). We go on like that and produce a sequence of subgroups in  $J(\mathbb{F}_q)$  that increase with constant probability until the index in  $J(\mathbb{F}_q)$  becomes smaller than  $\max(48g, 24d, 720)$ . Note that every step in this method is probabilistic: it succeeds with some probability, that can be made very high (exponentially close to 1) while keeping a polynomial overall complexity.

**13.2.4 Lemma (Finding an almost generating set)** *There exists a probabilistic (Monte-Carlo) algorithm that takes as input*

1. *a degree  $d$  and geometric genus  $g$  plane projective absolutely integral curve  $C$  over  $\mathbb{F}_q$ , such that  $q \geq 4g^2$ ,*
2. *the smooth model  $X$  of  $C$ ,*
3. *a degree 1 divisor  $O = O^+ - O^-$  where  $O^+$  and  $O^-$  are effective,  $\mathbb{F}_q$ -rational and have degree bounded by  $\Theta g^\Theta$  for some positive constant  $\Theta$ ,*

*and outputs a sequence of elements in  $\text{Pic}^0(X/\mathbb{F}_q)$  that generate a subgroup of index at most*

$$\max(48g, 24d, 720)$$

*with probability  $\geq \frac{1}{2}$ . The algorithm is polynomial time in  $d$  and  $\log q$ .*

Note that we do not catch the whole group  $J(\mathbb{F}_q)$  of rational points but a subgroup  $\mathcal{G}$  with index at most  $\iota = \max(48g, 24d, 720)$ . This is a small but annoying gap. In the sequel we shall try to compute the  $l$ -torsion of the group  $J(\mathbb{F}_q)$  of rational points. Because of the small gap in the above lemma, we may miss some  $l$ -torsion points if  $l$  is smaller than  $\iota$ . However, let  $k$  be an integer such that  $l^k > \iota$ . And let  $x$  be a point of order  $l$  in  $J(\mathbb{F}_q)$ . Assume that there exists a point  $y$  in  $J(\mathbb{F}_q)$  such that  $x = l^{k-1}y$ . The group



$\langle y \rangle$  generated by  $y$  and the group  $\mathcal{G}$  have non-trivial intersection because the product of their orders is bigger than the order of  $J(\mathbb{F}_q)$ . Therefore  $x$  belongs to  $\mathcal{G}$ .

Our strategy for computing  $J(\mathbb{F}_q)[l]$  will be to find a minimal field extension  $\mathbb{F}_Q$  of  $\mathbb{F}_q$  such that all points in  $J(\mathbb{F}_q)[l]$  are divisible by  $l^{k-1}$  in  $J(\mathbb{F}_Q)$ . We then shall apply the above lemma to  $J(\mathbb{F}_Q)$ . To finish with, we shall have to compute  $J(\mathbb{F}_q)$  as a subgroup of  $J(\mathbb{F}_Q)$ . To this end, we shall use the Weil pairing.

### 13.3 PAIRINGS

Let  $n$  be a prime to  $p$  integer and  $J$  a jacobian variety over  $\mathbb{F}_q$ . The Weil pairing relates the full  $n$ -torsion subgroup  $J(\overline{\mathbb{F}}_q)[n]$  with itself. It can be defined using Kummer theory and is geometric in nature. The Tate-Lichtenbaum-Frey-Rück pairing is more arithmetic and relates the  $n$ -torsion  $J(\mathbb{F}_q)[n]$  in the group of  $\mathbb{F}_q$ -rational points and the quotient  $J(\mathbb{F}_q)/nJ(\mathbb{F}_q)$ . In this section, we quickly review the definitions and algorithmic properties of these pairings, following work by Weil, Lang, Menezes, Okamoto, Vanstone, Frey and Rück.

We first recall the definition of Weil pairing following [Lan3]. Let  $k$  be an algebraically closed field with characteristic  $p$ . For every abelian variety  $A$  over  $k$ , we denote by  $Z_0(A)_0$  the group of 0-cycles with degree 0 and by  $S : Z_0(A)_0 \rightarrow A$  the summation map, that associates to every 0-cycle of degree 0 the corresponding sum in  $A$ . Let  $V$  and  $W$  be two projective non-singular integral varieties over  $k$ , and let  $\alpha : V \rightarrow A$  and  $\beta : W \rightarrow B$  be the canonical maps into their Albanese varieties. Let  $D$  be a correspondence on  $V \times W$ . Let  $n \geq 2$  be a prime to  $p$  integer. Let  $\mathfrak{a}$  (resp.  $\mathfrak{b}$ ) be a 0-cycle of degree 0 on  $V$  (resp.  $W$ ) and let  $a = S(\alpha(\mathfrak{a}))$  (resp.  $b = S(\beta(\mathfrak{b}))$ ) be the associated point in  $A$  (resp.  $B$ ). Assume that  $na = nb = 0$ . The Weil pairing  $e_{n,D}(a, b)$  is defined in [Lan3, VI, §4, Theorem 10]. It is an  $n$ -th root of unity in  $k$ . It is linear in  $a, b$  and  $D$ .

Assume now that  $V = W = X$  is a smooth projective integral curve over  $k$  and assume that  $A = B = J$  is its jacobian and

$$\alpha = \beta = \phi : X \rightarrow J$$

is the Jacobi map (once an origin on  $X$  has been chosen). If we take  $D$  to be the diagonal on  $X \times X$  we define a pairing  $e_{n,D}(a, b)$  that will be denoted  $e_n(a, b)$  or  $e_{n,X}(a, b)$ . It does not depend on the origin for the Jacobi map. It is non-degenerate.

The jacobian  $J$  comes with a principal polarization i.e. an isomorphism  $\lambda : J \rightarrow \hat{J}$  between  $J$  and its dual  $\hat{J}$ . If  $\alpha$  is an endomorphism  $\alpha : J \rightarrow J$ , we denote by  ${}^t\alpha$  its transpose  ${}^t\alpha : \hat{J} \rightarrow \hat{J}$ . If  $D$  is a divisor on  $J$  that is algebraically equivalent to zero, the image by  ${}^t\alpha$  of the linear equivalence class of  $D$  is the linear equivalence class of the inverse image  $\alpha^{-1}(D)$ . See [Lan3, V, §1]. The Rosati dual of  $\alpha$  is defined to be  $\alpha^* = \lambda^{-1} \circ {}^t\alpha \circ \lambda$ . The map  $\alpha \rightarrow \alpha^*$  is an involution, and  $\alpha^*$  is the adjoint of  $\alpha$  for the Weil pairing

$$e_{n,X}(a, \alpha(b)) = e_{n,X}(\alpha^*(a), b) \tag{13.1}$$

according to [Lan3, VII, §2, Proposition 6].

If  $Y$  is another smooth projective integral curve over  $k$  and  $K$  its jacobian and  $f : X \rightarrow Y$  a non-constant map with degree  $d$ , and  $f^* : K \rightarrow J$  the associated map between jacobians, then for  $a$  and  $b$  of order dividing  $n$  in  $K$  one has

$$e_{n,X}(f^*(a), f^*(b)) = e_{n,Y}(a, b)^d.$$

The Frey-Rück pairing can be constructed from the Lichtenbaum version of Tate's pairing [Lic] as was shown in [Fr-Ru]. Let  $q$  be a power of  $p$ . Let again  $n \geq 2$  be a prime to  $p$  integer and let  $X$  be a smooth projective absolutely integral curve over  $\mathbb{F}_q$ . Let  $g$  be the genus of  $X$ . We assume that  $n$  divides  $q - 1$ . Let  $J$  be the jacobian of  $X$ . The Frey-Rück pairing

$$\{, \}_n : J(\mathbb{F}_q)[n] \times J(\mathbb{F}_q)/nJ(\mathbb{F}_q) \rightarrow \mathbb{F}_q^*/(\mathbb{F}_q^*)^n$$

is defined as follows. We take a class of order dividing  $n$  in  $J(\mathbb{F}_q)$ . Such a class can be represented by an  $\mathbb{F}_q$ -divisor  $D$  with degree 0. We take a class in  $J(\mathbb{F}_q)$  and pick a degree zero  $\mathbb{F}_q$ -divisor  $E$  in this class, that we assume to be disjoint to  $D$ . The pairing evaluated at the classes  $[D]$  and  $[E] \bmod n$  is

$$\{[D], [E] \bmod n\}_n = f(E) \bmod (\mathbb{F}_q^*)^n$$

where  $f$  is any function with divisor  $nD$ . This is a non-degenerate pairing.

We now explain how one can compute the Weil pairing, following work by Menezes, Okamoto, Vanstone, Frey and Rück. The Tate-Lichtenbaum-Frey-Rück pairing can be computed similarly. As usual, we assume that we are given a degree  $d$  plane model  $C$  for  $X$ . Assume that  $\mathfrak{a}$  and  $\mathfrak{b}$  have disjoint support (otherwise we may replace  $\mathfrak{a}$  by some linearly equivalent divisor using the explicit moving Lemma 13.1.7.) We compute a function  $f_{\mathfrak{a}}$  with divisor  $n\mathfrak{a}$ . We similarly compute a function  $f_{\mathfrak{b}}$  with divisor  $n\mathfrak{b}$ . Then

$$e_n(a, b) = \frac{f_{\mathfrak{b}}(\mathfrak{a})}{f_{\mathfrak{a}}(\mathfrak{b})}.$$

This algorithm is polynomial in the degree  $d$  of  $C$  and the order  $n$  of the divisors, provided the initial divisors  $\mathfrak{a}$  and  $\mathfrak{b}$  are given as differences between effective divisors with polynomial degree in  $d$ .

Using an idea that appears in a paper by Menezes, Okamoto and Vanstone [Me-Ok-Va] in the context of elliptic curves, and in [Fr-Ru] for general curves, one can make this algorithm polynomial in  $\log n$  in the following way. We write  $\mathfrak{a} = \mathfrak{a}_0 = \mathfrak{a}_0^+ - \mathfrak{a}_0^-$  where  $\mathfrak{a}_0^+$  and  $\mathfrak{a}_0^-$  are effective divisors. Let  $f_{\mathfrak{a}}$  be the function computed in the above simple minded algorithm. One has  $(f_{\mathfrak{a}}) = n\mathfrak{a}_0^+ - n\mathfrak{a}_0^-$ . We want to express  $f_{\mathfrak{a}}$  as a product of small degree functions. We use a variant of fast exponentiation. Using Lemma 13.1.7 we compute a divisor  $\mathfrak{a}_1 = \mathfrak{a}_1^+ - \mathfrak{a}_1^-$  and a function  $f_1$  such that  $\mathfrak{a}_1$  is disjoint to  $\mathfrak{b}$  and  $(f_1) = \mathfrak{a}_1 - 2\mathfrak{a}_0$  and such that the degrees of  $\mathfrak{a}_1^+$  and  $\mathfrak{a}_1^-$  are  $\leq 6gd(\log_q(\deg(\mathfrak{b})) + 1)$ . We go on and compute, for  $k \geq 1$  an integer, a divisor  $\mathfrak{a}_k = \mathfrak{a}_k^+ - \mathfrak{a}_k^-$  and a function  $f_k$  such that  $\mathfrak{a}_k$  is disjoint to  $\mathfrak{b}$  and  $(f_k) = \mathfrak{a}_k - 2\mathfrak{a}_{k-1}$  and such that the degrees of  $\mathfrak{a}_k^+$  and  $\mathfrak{a}_k^-$  are  $\leq 6gd(\log_q(\deg(\mathfrak{b})) + 1)$ . We write the base 2 expansion of  $n = \sum_k \epsilon_k 2^k$  with  $\epsilon_k \in \{0, 1\}$ . We compute the function  $\Phi$  with divisor  $\sum_k \epsilon_k \mathfrak{a}_k$ . We claim that the function  $f_{\mathfrak{a}}$  can be written as a product of the  $f_k$ , for  $k \leq \log_2 n$ , and  $\Phi$  with suitable integer exponents bounded by  $n$  in absolute value. Indeed we write  $F_1 = f_1$ ,  $F_2 = f_2 f_1^2$ ,  $F_3 = f_3 f_2^2 f_1^4$  and so on. We have  $(F_k) = \mathfrak{a}_k - 2^k \mathfrak{a}$  and

$$\Phi \prod_k F_k^{-\epsilon_k}$$

has divisor  $n\mathfrak{a}$  so is the  $f_{\mathfrak{a}}$  we were looking for.

**13.3.1 Lemma (Computing the Weil pairing)** *There exists an algorithm that on input a prime to  $q$  integer  $n \geq 2$  and a degree  $d$  absolutely integral plane projective curve  $C$  over  $\mathbb{F}_q$  and its smooth model  $X$  and two  $\mathbb{F}_q$ -divisors on  $X$ , denoted  $\mathfrak{a} = \mathfrak{a}^+ - \mathfrak{a}^-$  and  $\mathfrak{b} = \mathfrak{b}^+ - \mathfrak{b}^-$ , with degree 0, and order dividing  $n$  in the jacobian, computes the Weil pairing  $e_n(\mathfrak{a}, \mathfrak{b})$  in time polynomial in  $d, \log q, \log n$  and the degrees of  $\mathfrak{a}^+, \mathfrak{a}^-, \mathfrak{b}^+, \mathfrak{b}^-$ , the positive and negative parts of  $\mathfrak{a}$  and  $\mathfrak{b}$ .*

**13.3.2 Lemma (Computing Tate-Lichtenbaum-Frey-Rück pairings)**

*There exists an algorithm that on input an integer  $n \geq 2$  dividing  $q - 1$  and a degree  $d$  absolutely integral plane projective curve  $C$  over  $\mathbb{F}_q$  and its smooth model  $X$  and two  $\mathbb{F}_q$ -divisors on  $X$ , denoted  $\mathfrak{a} = \mathfrak{a}^+ - \mathfrak{a}^-$  and  $\mathfrak{b} = \mathfrak{b}^+ - \mathfrak{b}^-$ , with degree 0, and such that the class of  $\mathfrak{a}$  has order dividing  $n \geq 2$  in the jacobian, computes the Tate-Lichtenbaum-Frey-Rück pairing  $\{\mathfrak{a}, \mathfrak{b}\}_n$  in time polynomial in  $d, \log q, \log n$  and the degrees of  $\mathfrak{a}^+, \mathfrak{a}^-, \mathfrak{b}^+, \mathfrak{b}^-$ , the positive and negative parts of  $\mathfrak{a}$  and  $\mathfrak{b}$ .*

### 13.4 DIVISIBLE GROUPS

Let  $\mathbb{F}_q$  be a finite field with characteristic  $p$  and let  $X$  be a projective smooth absolutely integral algebraic curve over  $\mathbb{F}_q$ . Let  $g$  be the genus of  $X$  and let  $l \neq p$  be a prime integer. We assume that  $g \geq 1$ . Let  $J$  be the jacobian of  $X$  and let  $\text{End}(J/\mathbb{F}_q)$  be the ring of endomorphisms of  $J$  over  $\mathbb{F}_q$ . Let  $F_q$  be the Frobenius endomorphism. In this section we study the action of  $F_q$  on  $l^k$ -torsion points of  $J$ . We first consider the whole  $l^k$ -torsion group. We then restrict to some well chosen subgroups where this action is more amenable.

Let  $\chi(x)$  be the characteristic polynomial of  $F_q \in \text{End}(J/\mathbb{F}_q)$ . The Rosati dual to  $F_q$  is  $q/F_q$ . Let

$$\mathcal{O} = \mathbb{Z}[x]/\chi(x)$$

and  $\mathcal{O}_l = \mathbb{Z}_l[x]/\chi(x)$ . We set

$$\varphi_q = x \bmod \chi(x) \in \mathcal{O}.$$

Mapping  $\varphi_q$  onto  $F_q$  defines an epimorphism from the ring  $\mathcal{O}$  onto  $\mathbb{Z}[F_q]$ . In order to control the degree of the field of definition of  $l^k$ -torsion points

we shall bound the order of  $\varphi_q$  in  $(\mathcal{O}/l^k\mathcal{O})^*$ . We set

$$\mathcal{U}_1 = (\mathcal{O}/l\mathcal{O})^* = (\mathbb{F}_l[x]/\chi(x))^*.$$

Let the prime factorization of  $\chi(x) \bmod l$  be  $\prod_i \chi_i(x)^{e_i}$  with  $\deg(\chi_i) = f_i$ . The order of  $\mathcal{U}_1$  is  $\prod_i l^{(e_i-1)f_i}(l^{f_i} - 1)$ . Let  $\gamma$  be the smallest integer such that  $l^\gamma$  is bigger than or equal to  $2g$ . Then the exponent of the group  $\mathcal{U}_1$  divides

$$A_1 = l^\gamma \prod_i (l^{f_i} - 1).$$

We set

$$B_1 = \prod_i (l^{f_i} - 1)$$

and

$$C_1 = l^\gamma.$$

There is a unique polynomial  $M_1(x) \in \mathbb{Z}[x]$  with degree  $< 2g$  such that

$$\frac{\varphi_q^{A_1} - 1}{l} = M_1(\varphi_q) \in \mathcal{O}.$$

Now for every positive integer  $k$ , the element  $\varphi_q$  belongs to the unit group

$$\mathcal{U}_k = (\mathcal{O}/l^k\mathcal{O})^*$$

of the quotient algebra  $\mathcal{O}/l^k\mathcal{O} = \mathbb{Z}[x]/(l^k, \chi(x))$ . The prime factorization of  $\chi(x) \bmod l$  is lifted modulo  $l^k$  as  $\prod_i \Xi_i(x)$  with  $\Xi_i$  monic and  $\deg(\Xi_i) = e_i f_i$ , and the order of  $\mathcal{U}_k$  is  $\prod_i l^{f_i(k e_i - 1)}(l^{f_i} - 1)$ . The exponent of the latter group divides

$$A_k = A_1 l^{k-1}.$$

So we set

$$B_k = B_1 = \prod_i (l^{f_i} - 1)$$

and

$$C_k = C_1 l^{k-1} = l^{k-1+\gamma}.$$

There is a unique polynomial  $M_k(x) \in \mathbb{Z}[x]$  with degree  $< \deg(\chi)$  such that

$$\frac{\varphi_q^{A_k} - 1}{l^k} = M_k(\varphi_q) \in \mathcal{O}.$$

For every integer  $N \geq 2$  we can compute  $M_k(x) \bmod N$  from  $\chi(x)$  in probabilistic polynomial time in  $\log q, \log l, \log N, k, g$ . Indeed we first factor  $\chi(x) \bmod l$  then compute the  $\chi_i$  and the  $e_i$  and  $f_i$ . We compute

$$x^{A_k} \bmod (\chi(x), l^k N)$$

using the fast exponentiation algorithm presented in Section 5.1. We remove 1 and divide by  $l^k$ .

**13.4.1 Lemma (Frobenius and  $l$ -torsion)** *Let  $k$  be a positive integer and  $l \neq p$  a prime. Let  $\chi(x)$  be the characteristic polynomial of the Frobenius  $F_q$  of  $J/\mathbb{F}_q$ . Let  $e_i$  and  $f_i$  be the multiplicities and inertiae in the prime decomposition of  $\chi(x) \bmod l$ . Let  $\gamma$  be the smallest integer such that  $l^\gamma$  is bigger than or equal to  $2g$ . Let  $B = \prod_i (l^{f_i} - 1)$ . Let  $C_k = l^{k-1+\gamma}$  and let  $A_k = BC_k$ . The  $l^k$ -torsion in  $J$  splits completely over the degree  $A_k$  extension of  $\mathbb{F}_q$ . There is a degree  $< 2g$  polynomial  $M_k(x) \in \mathbb{Z}[x]$  such that*

$$F_q^{A_k} = 1 + l^k M_k(F_q).$$

*For every integer  $N$  one can compute such a  $M_k(x) \bmod N$  from  $\chi(x)$  in probabilistic polynomial time in  $\log q, \log l, \log N, k, g$ .*

In order to state sharper results it is convenient to introduce  $l$ -divisible subgroups inside the  $l^\infty$ -torsion of a jacobian  $J$ , that may or may not correspond to subvarieties. We now see how to define such subgroups and control their rationality properties.

**13.4.2 Lemma (Divisible group)** *Let  $\Pi : J[l^\infty] \rightarrow J[l^\infty]$  be a group homomorphism whose restriction to its image  $\mathbb{G}$  is a bijection. Multiplication by  $l$  is then a surjection from  $\mathbb{G}$  onto itself. We denote by  $\mathbb{G}[l^k]$  the  $l^k$ -torsion in  $\mathbb{G}$ . There is an integer  $w$  such that  $\mathbb{G}[l^k]$  is a free  $\mathbb{Z}/l^k\mathbb{Z}$  module of rank  $w$  for every  $k$ . We assume that  $\Pi$  commutes with the Frobenius endomorphism*

$F_q$ . We then say  $\mathbb{G}$  is the divisible group associated with  $\Pi$ . From Tate's theorem [Tat1]  $\Pi$  is induced by some endomorphism in  $\text{End}(J/\mathbb{F}_q) \otimes_{\mathbb{Z}} \mathbb{Z}_l$  and we can define  $\Pi^*$  the Rosati dual of  $\Pi$  and denote by  $\mathbb{G}^* = \text{Im}(\Pi^*)$  the associated divisible group, that we call the adjoint of  $\mathbb{G}$ .

*Remark.* The dual  $\mathbb{G}^*$  does not only depend on  $\mathbb{G}$ . It may depend on  $\Pi$  also. This will not be a problem for us.

*Remark.* We may equivalently define  $\Pi^*$  as the dual of  $\Pi$  for the Weil pairing. See Equation (13.1).

We now give an example of divisible group. Let  $F(x) = F_1(x)$  and  $G(x) = G_1(x)$  be two monic coprime polynomials in  $\mathbb{F}_l[x]$  such that

$$\chi(x) = F_1(x)G_1(x) \pmod{l}.$$

According to Bezout's theorem we have two polynomials  $H_1(x)$  and  $K_1(x)$  in  $\mathbb{F}_l[x]$  such that

$$F_1H_1 + G_1K_1 = 1$$

and  $\deg(H_1) < \deg(G_1)$  and  $\deg(K_1) < \deg(F_1)$ .

From Hensel's lemma, for every positive integer  $k$  there exist four polynomials  $F_k(x)$ ,  $G_k(x)$ ,  $H_k(x)$  and  $K_k(x)$  in  $(\mathbb{Z}/l^k\mathbb{Z})[x]$  such that  $F_k$  and  $G_k$  are monic and

$$\chi(x) = F_k(x)G_k(x) \pmod{l^k}$$

and

$$F_kH_k + G_kK_k = 1 \pmod{l^k}$$

and  $\deg(H_k) < \deg(G_1)$  and  $\deg(K_k) < \deg(F_1)$  and  $F_1 = F_k \pmod{l}$ ,  $G_1 = G_k \pmod{l}$ ,  $H_1 = H_k \pmod{l}$ ,  $K_1 = K_k \pmod{l}$ .

The sequences  $(F_k)_k$ ,  $(G_k)_k$ ,  $(H_k)_k$ ,  $(K_k)_k$  converge in  $\mathbb{Z}_l[x]$  to  $F_0$ ,  $G_0$ ,  $H_0$ ,  $K_0$ . If we substitute  $F_q$  for  $x$  in  $F_0H_0$  we obtain a map

$$\Pi_G : J[l^\infty] \rightarrow J[l^\infty]$$

and similarly, if we substitute  $F_q$  for  $x$  in  $G_0K_0$  we obtain a map  $\Pi_F$ . It is clear that  $\Pi_F^2 = \Pi_F$  and  $\Pi_G^2 = \Pi_G$  and  $\Pi_F + \Pi_G = 1$  and  $\Pi_F\Pi_G = 0$ .

We call  $\mathbb{G}_F = \text{Im}(\Pi_F)$  and  $\mathbb{G}_G = \text{Im}(\Pi_G)$  the associated supplementary  $l$ -divisible groups.

**13.4.3 Definition (Characteristic subspaces)** For every non-trivial monic factor  $F(x)$  of  $\chi(x) \bmod l$  such that the cofactor  $G = \chi/F \bmod l$  is prime to  $F$ , we write  $\chi = F_0 G_0$  the corresponding factorization in  $\mathbb{Z}_l[x]$ . The  $l$ -divisible group

$$\mathbb{G}_F = \text{Im}(\Pi_F)$$

is called the  $F_0$ -torsion in  $J[l^\infty]$  and is denoted  $J[l^\infty, F_0]$ . It is the characteristic subspace of  $F_q$  associated with the factor  $F$ . If  $F = (x - 1)^e$  is the largest power of  $x - 1$  dividing  $\chi(x) \bmod l$  we abbreviate  $\mathbb{G}_{(x-1)^e} = \mathbb{G}_1$ . If  $F = (x - q)^e$  then we write similarly  $\mathbb{G}_{(x-q)^e} = \mathbb{G}_q$ .

We notice that there exists a unit  $u$  in  $\text{End}(J/\mathbb{F}_q) \otimes_{\mathbb{Z}} \mathbb{Z}_l$  such that the Rosati dual  $\Pi_1^*$  of  $\Pi_1$  is

$$\Pi_1^* = \Pi_q \circ u.$$

Therefore

$$\mathbb{G}_q = \mathbb{G}_1^*$$

and the restriction of the Weil pairing to  $\mathbb{G}_1[l^k] \times \mathbb{G}_q[l^k]$  is non-degenerate for any integer  $k \geq 1$ .

We now compute fields of definitions for torsion points inside such divisible groups. The action of  $F_q$  on the  $l^k$ -torsion  $\mathbb{G}_F[l^k] = J[l^k, F_0]$  inside  $\mathbb{G}_F$  factors through the ring  $\mathcal{O}_l/(l^k, F_0(\varphi_q)) = \mathbb{Z}_l[x]/(l^k, F_0)$ . We deduce the following.

**13.4.4 Lemma (Frobenius and  $F_0$ -torsion)** *Let  $k$  be a positive integer and  $l \neq p$  a prime. Let  $\chi(x)$  be the characteristic polynomial of the Frobenius  $F_q$  of  $J$ . Let  $\chi = FG \bmod l$  with  $F$  and  $G$  monic coprime. Let  $e_i$  and  $f_i$  be the multiplicities and inertiae in the prime decomposition of  $F(x) \bmod l$ . Let  $\gamma$  be the smallest integer such that  $l^\gamma$  is bigger than or equal to  $2g$ . Let  $B(F) = \prod_i (l^{f_i} - 1)$ . Let  $C_k(F) = l^{k-1+\gamma}$  and  $A_k(F) = B(F)C_k(F)$ .*



The  $l^k$ -torsion in  $\mathbb{G}_F$  splits completely over the degree  $A_k(F)$  extension of  $\mathbb{F}_q$ . There is a degree  $< \deg(F)$  polynomial  $M_k(x) \in \mathbb{Z}_l[x]$  such that

$$\Pi_F \circ F_q^{A_k(F)} = \Pi_F + l^k \Pi_F \circ M_k(F_q).$$

For every power  $N$  of  $l$ , one can compute such an  $M_k(x)$  modulo  $N$  from  $\chi(x)$  and  $F(x)$  in probabilistic polynomial time in  $\log q, \log l, \log N, k, g$ .

If we take for  $F$  the largest power of  $x - 1$  dividing  $\chi(x) \bmod l$  in the above lemma, we can take  $B(F) = 1$  so  $A_k(F)$  is an  $l$  power  $\leq 2gl^k$ .

If we take for  $F$  the largest power of  $x - q$  dividing  $\chi(x) \bmod l$  in the above lemma, we have  $B(F) = l - 1$  so  $A_k(F)$  is  $\leq 2g(l - 1)l^k$ .

So the characteristic spaces associated with the eigenvalues 1 and  $q$  split completely over small degree extensions of  $\mathbb{F}_q$ .

### 13.5 THE KUMMER MAP

Let  $X$  be a smooth projective absolutely integral curve over  $\mathbb{F}_q$  of genus  $g$  and  $J$  the jacobian of  $X$ . Let  $n \geq 2$  be an integer dividing  $q - 1$ . We assume that  $g \geq 1$ . In this section, we construct a convenient surjection from  $J(\mathbb{F}_q)$  to  $J(\mathbb{F}_q)[n]$ . If  $P$  is in  $J(\mathbb{F}_q)$  we take some  $R \in J(\overline{\mathbb{F}_q})$  such that  $nR = P$  and form the 1-cocycle  $(\sigma R - R)_\sigma$  in  $H^1(\mathbb{F}_q, J[n])$ . Using the Weil pairing we deduce an element

$$S \mapsto (e_n(\sigma R - R, S))_\sigma$$

in

$$\text{Hom}(J[n](\mathbb{F}_q), H^1(\mathbb{F}_q, \mu_n)) = \text{Hom}(J[n](\mathbb{F}_q), \text{Hom}(\text{Gal}(\mathbb{F}_q), \mu_n)).$$

The map that sends  $P \bmod nJ(\mathbb{F}_q)$  to  $S \mapsto (e_n(\sigma R - R, S))_\sigma$  is injective because the Frey-Rück pairing is non-degenerate. We observe that  $\text{Hom}(\text{Gal}(\mathbb{F}_q), \mu_n)$  is isomorphic to  $\mu_n$  because giving an homomorphism from  $\text{Gal}(\mathbb{F}_q)$  to  $\mu_n$  is equivalent to giving the image of the Frobenius generator  $F_q$ . We obtain a bijection

$$T_{n,q} : J(\mathbb{F}_q)/nJ(\mathbb{F}_q) \rightarrow \text{Hom}(J[n](\mathbb{F}_q), \mu_n)$$

that we call the *Tate map*. It maps  $P$  onto  $S \mapsto e_n({}^{F_q}R - R, S)$ . If  $J[n]$  splits completely over  $\mathbb{F}_q$  we set  $K_{n,q}(P) = {}^{F_q}R - R$  and define a bijection

$$K_{n,q} : J(\mathbb{F}_q)/nJ(\mathbb{F}_q) \rightarrow J[n](\mathbb{F}_q) = J[n]$$

that we call the *Kummer map*.

**13.5.1 Definition (The Kummer map)** Let  $J/\mathbb{F}_q$  be a jacobian and  $n \geq 2$  a prime to  $p$  integer. Assume that  $J[n]$  splits completely over  $\mathbb{F}_q$ . For  $P$  in  $J(\mathbb{F}_q)$  we choose any  $R$  in  $J(\overline{\mathbb{F}_q})$  such that  $nR = P$  and we set  $K_{n,q}(P) = {}^{F_q}R - R$ . This defines a bijection

$$K_{n,q} : J(\mathbb{F}_q)/nJ(\mathbb{F}_q) \rightarrow J[n](\mathbb{F}_q) = J[n].$$

We now assume that

$$n = l^k$$

is a power of some prime integer  $l \neq p$ . We still make the (strong !) assumption that  $J[n]$  splits completely over  $\mathbb{F}_q$ . We want to compute the Kummer map  $K_{n,q}$  explicitly. Let  $P$  be an  $\mathbb{F}_q$ -rational point in  $J$ . Let  $R$  be such that  $nR = P$ . Since  $F_q - 1$  annihilates  $J[n]$ , there is an  $\mathbb{F}_q$ -endomorphism  $\kappa$  of  $J$  such that  $F_q - 1 = n\kappa$ . We note that  $\kappa$  belongs to  $\mathbb{Z}[F_q] \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[F_q]$  and therefore commutes with  $F_q$ . We have

$$\kappa(P) = (F_q - 1)(R) = K_{n,q}(P)$$

and  $\kappa(P)$  is  $\mathbb{F}_q$ -rational. So we can compute  $K_{n,q}(P)$  without computing  $R$ . We don't need to divide  $P$  by  $n$ .

The Kummer map will show very useful but its definition requires that  $J[n]$  splits completely over  $\mathbb{F}_q$ . If this is not the case, we must base change to some extension of  $\mathbb{F}_q$ .

Let  $\chi(x)$  be the characteristic polynomial of  $F_q$  and let  $B = \prod_i (l^{f_i} - 1)$  where the  $f_i$  are the degrees of the irreducible factors of  $\chi(x) \bmod l$ . Let  $l^\gamma$  be the smallest power of  $l$  that is bigger than or equal to  $2g$ . Let  $C_k = l^{\gamma+k-1}$  and  $A_k = BC_k$ . Set  $Q = q^{A_k}$ . From Lemma 13.4.1 there is a polynomial  $M_k(x)$  such that

$$F_Q = 1 + l^k M_k(F_q).$$

So for  $P$  an  $\mathbb{F}_Q$ -rational point in  $J$  and  $R$  such that  $nR = P$ , the Kummer map  $K_{n,Q}$  applied to  $P$  is

$$M_k(F_q)(P) = (F_Q - 1)(R) = K_{n,Q}(P)$$

and this is an  $\mathbb{F}_Q$ -rational point.

**13.5.2 Lemma (Computing the Kummer map)** *Let  $J/\mathbb{F}_q$  be a jacobian. Let  $g \geq 1$  be its dimension. Let  $l \neq p$  be a prime integer and  $n = l^k$  a power of  $l$ . Let  $\chi(x)$  be the characteristic polynomial of  $F_q$  and let  $B = \prod_i (l^{f_i} - 1)$  where the  $f_i$  are the degrees of the irreducible factors of  $\chi(x) \bmod l$ . Let  $l^\gamma$  be the smallest power of  $l$  that is bigger than or equal to  $2g$ . Let  $C_k = l^{\gamma+k-1}$  and  $A_k = BC_k$ . Set  $Q = q^{A_k}$  and observe that  $n$  divides  $Q - 1$  because  $J[n]$  splits completely over  $\mathbb{F}_Q$ . There exists an endomorphism  $\kappa \in \mathbb{Z}[F_q]$  of  $J$  such that  $n\kappa = F_Q - 1$  and for every  $\mathbb{F}_Q$ -rational point  $P$  and any  $R$  with  $nR = P$  one has  $\kappa(P) = (F_Q - 1)(R) = K_{n,Q}(P)$ . This endomorphism  $\kappa$  induces a bijection between  $J(\mathbb{F}_Q)/nJ(\mathbb{F}_Q)$  and  $J[n](\mathbb{F}_Q) = J[n]$ . Given  $\chi(x)$  and a positive integer  $N$  one can compute  $\kappa \bmod N$  as a polynomial in  $F_q$  with coefficients in  $\mathbb{Z}/N\mathbb{Z}$  in probabilistic polynomial time in  $g, \log l, \log q, k, \log N$ .*

This lemma is not of much use in practice because the field  $\mathbb{F}_Q$  is too big. On the other hand, we may not be interested in the whole  $n$ -torsion in  $J$  but just a small piece in it, namely the  $n$ -torsion of a given divisible group.

So let  $l \neq p$  be a prime integer and  $\mathbb{G}$  an  $l$ -divisible group in  $J[l^\infty]$ . Let

$$\Pi \in \text{End}(J/\mathbb{F}_q) \otimes_{\mathbb{Z}} \mathbb{Z}_l$$

be a projection onto  $\mathbb{G}$ . So

$$\Pi : J[l^\infty] \rightarrow \mathbb{G}.$$

We assume that  $\Pi^2 = \Pi$ . Let  $n = l^k$  and let  $Q$  be a power of  $q$  such that  $\mathbb{G}[n]$  splits completely over  $\mathbb{F}_Q$ . Let  $P$  be an  $\mathbb{F}_Q$ -rational point in  $\mathbb{G}$ . Let  $R \in \mathbb{G}(\overline{\mathbb{F}_q})$  be such that  $nR = P$ . We set

$$K_{\mathbb{G},n,Q}(P) = {}^{F_Q}R - R$$

and define an isomorphism

$$K_{\mathbb{G},n,Q} : \mathbb{G}(\mathbb{F}_Q)/n\mathbb{G}(\mathbb{F}_Q) \rightarrow \mathbb{G}(\mathbb{F}_Q)[n] = \mathbb{G}[n].$$

In order to make this construction explicit, we now assume that there exists some  $\kappa \in \mathbb{Z}_l[F_q]$  such that

$$(F_Q - 1 - n\kappa)\Pi = 0.$$

Lemma 13.4.4 provides us with such a  $Q$  and such a  $\kappa$  when  $\mathbb{G} = J[l^\infty, F_0]$  is some characteristic subspace.

We now can compute this new Kummer map  $K_{\mathbb{G},n,Q}$ . Let  $P$  be an  $\mathbb{F}_Q$ -rational point in  $\mathbb{G}$ . Let  $R \in \mathbb{G}$  be such that  $nR = P$ . From

$$(F_Q - 1 - n\kappa)\Pi(R) = 0 = (F_Q - 1 - n\kappa)(R)$$

we deduce that  $K_{\mathbb{G},n,Q}(P) = \kappa(P)$ . Hence the following lemma.

**13.5.3 Lemma (The Kummer map for a divisible group)** *Let  $J/\mathbb{F}_q$  be a jacobian. Let  $g$  be its dimension. Let  $l \neq p$  be a prime integer and  $n = l^k$  a power of  $l$ . We assume that  $g \geq 1$ . Let  $\chi(x)$  be the characteristic polynomial of  $F_q$ . Assume that  $\chi(x) = F(x)G(x) \pmod{l}$  with  $F$  and  $G$  monic coprime polynomials in  $\mathbb{F}_l[x]$  and let  $\mathbb{G}_F$  be the associated divisible group. Let  $B = \prod_i (l^{f_i} - 1)$  where the  $f_i$  are the degrees of the irreducible factors of  $F(x) \pmod{l}$ . Let  $l^\gamma$  be the smallest power of  $l$  that is bigger than or equal to  $2g$ . Let  $C_k = l^{k-1+\gamma}$  and  $A_k = BC_k$ . Set  $Q = q^{A_k}$ . From Lemma 13.4.4 there exists an endomorphism  $\kappa \in \mathbb{Z}_l[F_q]$  such that*

$$\Pi_F(n\kappa - F_Q + 1) = 0$$

and for every  $\mathbb{F}_Q$ -rational point  $P \in \mathbb{G}_F$  and any  $R \in \mathbb{G}_F$  with  $nR = P$  one has

$$\kappa(P) = (F_Q - 1)(R) = K_{\mathbb{G},n,Q}(P).$$

*This endomorphism  $\kappa$  induces a bijection between  $\mathbb{G}_F(\mathbb{F}_Q)/n\mathbb{G}_F(\mathbb{F}_Q)$  and  $\mathbb{G}_F[n](\mathbb{F}_Q) = \mathbb{G}_F[n]$ . Given  $\chi(x)$  and  $F(x)$  and a power  $N$  of  $l$ , one can compute  $\kappa \pmod{N}$  as a polynomial in  $F_q$  with coefficients in  $\mathbb{Z}/N\mathbb{Z}$  in probabilistic polynomial time in  $g, \log l, \log q, k, \log N$ . Any  $N \geq n(4Q)^g$  suffices for the purpose of computing  $\kappa(P)$ .*

### 13.6 LINEARIZATION OF TORSION CLASSES

Let  $C$  be a degree  $d$  plane projective absolutely integral curve  $C$  over  $\mathbb{F}_q$  with geometric genus  $g \geq 1$ , and assume that we are given the smooth model  $X$  of  $C$ . We also assume that we are given a degree 1 divisor  $O = O^+ - O^-$  where  $O^+$  and  $O^-$  are effective,  $\mathbb{F}_q$ -rational and have degree bounded by  $\Theta g^\Theta$  for some constant  $\Theta$ .

Let  $J$  be the jacobian of  $X$ . We assume that  $l \neq p$  is a prime integer that divides  $\#J(\mathbb{F}_q)$ . Let  $n = l^k$  be a power of  $l$ . We want to describe  $J(\mathbb{F}_q)[l^k]$  by generators and relations.

If  $x_1, x_2, \dots, x_I$  are elements in a finite commutative group  $G$  we let  $\mathcal{R}$  be the kernel of the map  $\xi : \mathbb{Z}^I \rightarrow G$  defined by  $\xi(a_1, \dots, a_I) = \sum_i a_i x_i$ . We call  $\mathcal{R}$  the *lattice of relations* between the  $x_i$ . We first give a very general and rough algorithm for computing relations in any finite commutative group.

**13.6.1 Lemma (Finding relations in blackbox groups)** *Let  $G$  be a finite and commutative group and let  $x_1, x_2, \dots, x_I$  be elements in  $G$ . A basis for the lattice of relations between the  $x_i$  can be computed at the expense of  $3I\#G$  operations (or comparisons) in  $G$ .*

We first compute and store all the multiples of  $x_1$ . So we list  $0, x_1, 2x_1, \dots$  until we find the first multiple  $e_1 x_1$  that is equal to zero. This gives us the relation  $r_1 = (e_1, 0, \dots, 0) \in \mathcal{R}$ . This first step requires at most  $o = \#G$  operations in  $G$  and  $o$  comparisons.

We then compute successive multiples of  $x_2$  until we find the first one  $e_2 x_2$  that is in  $L_1 = \{0, x_1, \dots, (e_1 - 1)x_1\}$ . This gives us a second relation  $r_2$ . The couple  $(r_1, r_2)$  is a basis for the lattice of relations between  $x_1$  and  $x_2$ . Using this lattice, we compute the list  $L_2$  of elements in the group generated by  $x_1$  and  $x_2$ . This second step requires at most  $2o$  operations and  $e_1 e_2 \leq o$  comparisons.

We then compute successive multiples of  $x_3$  until we find the first one  $e_3 x_3$  that is in  $L_2$ . This gives us a third relation  $r_3$ . The triple  $(r_1, r_2, r_3)$  is a basis for the lattice of relations between  $x_1, x_2$  and  $x_3$ . Using this lattice, we compute the list  $L_3$  of elements in the group generated by  $x_1, x_2$  and  $x_3$ . This third step requires at most  $2o$  operations and  $o$  comparisons. And we go on like this.  $\square$

We stress that the algorithm above is far from efficient and will not be very useful unless the group  $G$  is very small.

We now come back to the computation of generators and relations for  $J(\mathbb{F}_q)[l^k]$ . Let  $B = l - 1$ . Let  $l^\gamma$  be the smallest power of  $l$  that is bigger than or equal to  $2g$  and let  $A_k = Bl^{\gamma+k-1}$ . We set  $Q_k = q^{A_k}$ . Let  $F$  be the largest power of  $x - 1$  dividing the characteristic polynomial  $\chi(x)$  of  $F_q$ . Definition 13.4.3 and Lemma 13.5.3 provide us with two surjective maps

$$\Pi_1 : J(\mathbb{F}_{Q_k})[l^\infty] \rightarrow \mathbb{G}_1(\mathbb{F}_{Q_k})$$

and

$$K_{\mathbb{G}_1, l^k, Q_k} : \mathbb{G}_1(\mathbb{F}_{Q_k}) \rightarrow \mathbb{G}_1[l^k].$$

If we now take for  $F$  the largest power of  $x - q$  dividing  $\chi(x)$ , Definition 13.4.3 and Lemma 13.5.3 give two surjective maps

$$\Pi_q : J(\mathbb{F}_{Q_k})[l^\infty] \rightarrow \mathbb{G}_q(\mathbb{F}_{Q_k})$$

and

$$K_{\mathbb{G}_q, l^k, Q_k} : \mathbb{G}_q(\mathbb{F}_{Q_k}) \rightarrow \mathbb{G}_q[l^k].$$

Remember that  $\mathbb{G}_q = \mathbb{G}_1^*$  and the restriction of the Weil pairing to  $\mathbb{G}_1[l^k] \times \mathbb{G}_q[l^k]$  is non-degenerate. We use this pairing to build a presentation for  $\mathbb{G}_1[l^k]$  and  $\mathbb{G}_q[l^k]$  simultaneously. The motivation for this approach is that generators for  $\mathbb{G}_1[l^k]$  provide relations for  $\mathbb{G}_q[l^k]$ , and conversely.

If  $Q_k \geq 4g^2$ , we use Lemma 13.2.4 to produce a sequence  $\gamma_1, \dots, \gamma_I$  of elements in  $J(\mathbb{F}_{Q_k})$  that generate (with high probability) a subgroup of index at most

$$\iota = \max(48g, 24d, 720).$$

If  $Q_k \leq 4g^2$  we use Lemma 13.2.1 to produce a sequence  $\gamma_1, \dots, \gamma_I$  of elements in  $J(\mathbb{F}_{Q_k})$  that generate it.

Let  $N$  be the largest divisor of  $\#J(\mathbb{F}_{Q_k})$  which is prime to  $l$ .

We set

$$\alpha_i = K_{\mathbb{G}_1, l^k, Q_k}(\Pi_1(N\gamma_i))$$

and

$$\beta_i = K_{\mathbb{G}_q, l^k, Q_k}(\Pi_q(N\gamma_i)).$$

The group  $\mathcal{A}_k$  generated by the  $\alpha_i$  has index at most  $\iota$  in  $\mathbb{G}_1[l^k]$ . The group  $\mathcal{B}_k$  generated by the  $\beta_i$  has index at most  $\iota$  in  $\mathbb{G}_q[l^k]$ . Let  $l^\delta$  be smallest power of  $l$  that is bigger than  $\iota$  and assume that  $k > \delta$ . Then

$$\mathbb{G}_1[l^{k-\delta}] \subset \mathcal{A}_k.$$

We now explain how to compute the lattice of relations between given elements  $\rho_1, \dots, \rho_J$  in  $\mathbb{G}_1[l^k]$ . We denote by  $\mathcal{R}$  this lattice. We recall that the restriction of the Weil pairing to  $\mathbb{G}_1[l^k] \times \mathbb{G}_q[l^k]$  is a non-degenerate pairing

$$e_{l^k} : \mathbb{G}_1[l^k] \times \mathbb{G}_q[l^k] \rightarrow \mu_{l^k}.$$

We fix an isomorphism between the group

$$\mu_{l^k}(\overline{\mathbb{F}}_q) = \mu_{l^k}(\mathbb{F}_{Q_k})$$

of  $l^k$ -th roots of unity and  $\mathbb{Z}/l^k\mathbb{Z}$ . Having chosen the preimage of  $1 \pmod{l^k}$ , computing this isomorphism is a problem called *discrete logarithm*. We can compute this discrete logarithm by exhaustive search at the expense of  $O(l^k)$  operations in  $\mathbb{F}_{Q_k}$ . There exist more efficient algorithms, but we don't need them for our complexity estimates.

We regard the matrix  $(e_{l^k}(\rho_j, \beta_i))$  as a matrix with  $I$  rows,  $J$  columns and coefficients in  $\mathbb{Z}/l^k\mathbb{Z}$ . This matrix defines a morphism from  $\mathbb{Z}^J$  to  $(\mathbb{Z}/l^k\mathbb{Z})^I$  whose kernel is a lattice  $\mathcal{R}'$  that contains  $\mathcal{R}$ . The index of  $\mathcal{R}$  in  $\mathcal{R}'$  is at most  $\iota$ . Indeed  $\mathcal{R}'/\mathcal{R}$  is isomorphic to the orthogonal subspace to  $\mathcal{B}_k$  inside  $\langle \rho_1, \dots, \rho_J \rangle \subset \mathbb{G}_1[l^k]$ . So it has order  $\leq \iota$ . We then compute a basis of  $\mathcal{R}'$ . This boils down to computing the kernel of an  $I \times (J + I)$  integer matrix with entries bounded by  $l^k$ . This can be done by putting this matrix in Hermite normal form (see [Coh, 2.4.3]). The complexity is polynomial in  $I, J$  and  $k \log l$ . See [Hav], [Coh, 2.4.3] and [Kal].

Once given a basis of  $\mathcal{R}'$ , the sublattice  $\mathcal{R}$  can be computed using Lemma 13.6.1 at the expense of  $\leq 3J\iota$  operations.

We apply this method to the generators  $(\alpha_i)_i$  of  $\mathcal{A}_k$ . Once given the lattice  $\mathcal{R}$  of relations between the  $\alpha_i$  it is a matter of linear algebra to find a basis

$(b_1, \dots, b_w)$  for  $\mathcal{A}_k[l^{k-\delta}] = \mathbb{G}_1[l^{k-\delta}]$ . The latter group is a rank  $w$  free module over  $\mathbb{Z}/l^{k-\delta}\mathbb{Z}$  and is acted on by the  $q$ -Frobenius  $F_q$ . For every  $b_j$  we can compute the lattice of relations between  $F_q(b_j), b_1, b_2, \dots, b_w$  and deduce the matrix of  $F_q$  with respect to the basis  $(b_1, \dots, b_w)$ . From this matrix we deduce a nice generating set for the kernel of  $F_q - 1$  in  $\mathbb{G}_1[l^{k-\delta}]$ . This kernel is  $J[l^{k-\delta}](\mathbb{F}_q)$ . We deduce the following.

**13.6.2 Theorem (Computing the  $l^k$ -torsion in the Picard group)** *There is a probabilistic Monte-Carlo algorithm that on input*

1. *a degree  $d$  and geometric genus  $g$  plane projective absolutely integral curve  $C$  over  $\mathbb{F}_q$ ,*
2. *the smooth model  $X$  of  $C$ ,*
3. *a degree 1 divisor  $O = O^+ - O^-$  where  $O^+$  and  $O^-$  are effective,  $\mathbb{F}_q$ -rational and have degree bounded by  $\Theta g^\Theta$  for some positive constant  $\Theta$ ,*
4. *a prime  $l$  different from the characteristic  $p$  of  $\mathbb{F}_q$  and a power  $n = l^k$  of  $l$ ,*
5. *the zeta function of  $X$ ,*

*outputs a set  $g_1, \dots, g_I$  of divisor classes in the Picard group of  $X/\mathbb{F}_q$ , such that the  $l^k$ -torsion  $\text{Pic}(X/\mathbb{F}_q)[l^k]$  is the direct product of the  $\langle g_i \rangle$ , and the orders of the  $g_i$  form a non-decreasing sequence. Every class  $g_i$  is given by a divisor  $G_i - gO$  in the class, where  $G_i$  is a degree  $g$  effective  $\mathbb{F}_q$ -divisor on  $X$ .*

*The algorithm runs in probabilistic polynomial time in  $d, \log q$  and  $l^k$ . It outputs the correct answer with probability  $\geq \frac{1}{2}$ . Otherwise, it may return either nothing or a strict subgroup of  $\text{Pic}(X/\mathbb{F}_q)[l^k]$ .*

*If one is given a degree zero  $\mathbb{F}_q$ -divisor  $D = D^+ - D^-$  of order dividing  $l^k$ , one can compute the coordinates of the class of  $D$  in the basis  $(g_i)_{1 \leq i \leq I}$  in polynomial time in  $d, \log q, l^k$  and the degree of  $D^+$ . These coordinates are integers  $x_i$  such that  $\sum_{1 \leq i \leq I} x_i g_i = [D]$ .*



### 13.7 COMPUTING $V_f$ MODULO $p$

In this section, we apply the general algorithm given in Section 13.6 to the plane curve  $C_l$  constructed in Section 12.3.3 and we compute Ramanujan divisors modulo  $p$ . So we assume that we are given an even integer  $k > 2$ , a prime integer  $l > 6(k - 1)$ , a finite field  $\mathbb{F}$  with characteristic  $l$ , and a ring epimorphism  $f : \mathbb{T}(1, k) \rightarrow \mathbb{F}$ . More precisely, we are given the images  $f(T_i)$  for  $i \leq k/12$ . We want to compute the associated Galois representation  $V_f \subset J_1(l)$  or rather its image  $W_f \subset J_1(5l)$  by  $B_{5l, l, 1}^* : J_1(l) \rightarrow J_1(5l)$ . We will assume that the image of  $\rho_f$  contains  $\mathrm{SL}(V_f)$ . We set  $X_l = X_1(5l)$  and  $J_l = J_1(l)$  and we denote by  $g_l$  the genus of  $X_l$ .

Let  $p \notin \{5, l\}$  a prime integer. We explain how to compute divisors on  $X_l/\mathbb{F}_p$  associated to every element  $x$  in  $W_f/\mathbb{F}_p \subset J/\mathbb{F}_p$ . The definition field  $\mathbb{F}_q$  for such divisors can be predicted from the characteristic polynomial of the Frobenius endomorphism  $F_p$  acting on  $V_f$ . So the strategy is to pick random  $\mathbb{F}_q$ -points in the  $l$ -torsion of the jacobian  $J_l$  and to project them onto  $W_f$  using Hecke operators.

The covering map  $B_{5l, l, 1} : X_l \rightarrow X_1(l)$  has degree 24. We call it  $\pi$ . It induces two morphisms  $\pi^* : J_1(l) \rightarrow J_l$  and  $\pi_* : J_l \rightarrow J_1(l)$  such that the composite map  $\pi_* \circ \pi^*$  is multiplication by 24 in  $J_1(l)$ . We denote by  $\mathcal{A}_l \subset J_l$  the image of  $\pi^*$ . This is a subvariety of  $J_l$  isogenous to  $J_1(l)$ . The restriction of  $\pi^* \circ \pi_*$  to  $\mathcal{A}_l$  is multiplication by 24. The maps  $\pi^*$  and  $\pi_*$  induce Galois equivariant bijections between the  $N$ -torsion subgroups  $J_1(l)[N]$  and  $\mathcal{A}_l[N]$  for every integer  $N$  which is prime to 6.

Using Theorem 2.5.12 we derive from  $f$  a finite set  $(t_1, \dots, t_r)$  of elements in  $\mathbb{T}(l, 2)$  with  $r = (l^2 - 1)/6$  and

$$V_f = \bigcap_{1 \leq i \leq r} \ker(t_i, J_1(l)(\overline{\mathbb{Q}})[l]). \quad (13.2)$$

and  $W_f \subset \mathcal{A}_l \subset J_l$  is the image of  $V_f$  by  $\pi^*$ .

We choose an integer  $s$  such that  $24s$  is congruent to 1 modulo  $l$ . For every integer  $n \geq 2$  we note  $T_n \in \mathbb{T}(l, 2)$  the  $n$ -th Hecke operator with weight 2 and level  $l$ . We can see  $T_n$  as endomorphism of  $J_1(l)$ . We set

$\hat{T}_n = [s] \circ \pi^* \circ T_n \circ \pi_*$ . We notice that

$$\hat{T}_n \circ \pi^* = \pi^* \circ T_n$$

on  $J_1(l)[l]$ . This way, the map  $\pi^* : J_1(l) \rightarrow J_l$  induces a Galois equivariant bijection of  $\mathbb{T}(l, 2)$ -modules between  $J_1(l)[l]$  and  $\mathcal{A}_l[l]$ . And  $W_f = \pi^*(V_f)$  is the subspace in  $\mathcal{A}_l[l]$  cut out by all  $\hat{T}_n - \tau(n)$ . We notice that  $\pi^*$ ,  $\pi_*$ ,  $T_n$ , and  $\hat{T}_n$  can be seen as correspondences as well as morphisms between jacobians. The following lemma states that the Hecke action on divisors can be efficiently computed.

**13.7.1 Lemma (Computing the Hecke action)** *Let  $l$  and  $p$  be primes such that  $p \notin \{5, l\}$ . Let  $n \geq 2$  be an integer. Let  $q$  be a power of  $p$  and let  $D$  be an effective  $\mathbb{F}_q$ -divisor of degree  $\deg(D)$  on  $X_l \bmod p$ . The divisor  $\pi^* \circ \pi_*(D)$  can be computed in polynomial time in  $l$ ,  $\deg(D)$  and  $\log q$ . The divisor  $\pi^* \circ T_n \circ \pi_*(D)$  can be computed in polynomial time in  $l$ ,  $\deg(D)$ ,  $n$  and  $\log q$ .*

If  $n$  is prime to  $l$ , we define the Hecke operator  $T(n, n)$  as an element in the ring of correspondences on  $X_1(l)$  tensored by  $\mathbb{Q}$ . See [Lan4, VII, §2]. From [Lan4, VII, §2, Theorem 2.1] we have  $T_l^i = (T_l)^i$  and  $T_n^i = T_{n^{i-1}}T_n - nT_{n^{i-2}}T(n, n)$  if  $n$  is prime and  $n \neq l$ . And of course  $T_{n_1}T_{n_2} = T_{n_1n_2}$  if  $n_1$  and  $n_2$  are coprime. So it suffices to explain how to compute  $T_l$  and also  $T_n$  and  $T(n, n)$  for  $n$  prime and  $n \neq l$ .

Let  $x = (E, u)$  be a point on  $Y_1(l) \subset X_1(l)$  representing an elliptic curve  $E$  with one  $l$ -torsion point  $u$ . Let  $n$  be an integer. The Hecke operator  $T_n$  maps  $x$  onto the sum of all  $(E_I, I(u))$ , where  $I : E \rightarrow E_I$  runs over the set of all isogenies of degree  $n$  from  $E$  such that  $I(u)$  still has order  $l$ . If  $n$  is prime to  $l$ , the Hecke operator  $T(n, n)$  maps  $x$  onto  $\frac{1}{n^2}$  times  $(E, nu)$ . So we can compute the action of these Hecke correspondences on points  $x = (E, u)$  using Vélu's formulae [Vel].

There remains to treat the case of cusps.

We call  $\sigma_\beta$  for  $1 \leq \beta \leq \frac{l-1}{2}$  the cusp on  $X_1(l)$  corresponding to the  $l$ -gon equipped with an  $l$ -torsion point on the  $\beta$ -th component. The corresponding Tate curves  $\mathbb{C}^*/q$  have an  $l$ -torsion point  $w = \zeta_l^* q^{\frac{\beta}{l}}$  where the star runs over the set of all residues modulo  $l$ . There are  $l$  Tate curves at every such cusp.

We call  $\mu_\alpha$  for  $1 \leq \alpha \leq \frac{l-1}{2}$  the cusps on  $X_1(l)$  corresponding to a 1-gon equipped with the  $l$ -torsion point  $\zeta_l^\alpha$  in its smooth locus  $\mathbb{G}_m$ . The Tate curve at  $\mu_\alpha$  is the Tate curves  $\mathbb{C}^*/q$  with  $l$ -torsion point  $w = \zeta_l^\alpha$ . One single Tate curve here: no ramification.

For  $n$  prime and  $n \neq l$  we have

$$T_n(\sigma_\beta) = \sigma_\beta + n\sigma_{n\beta}$$

and

$$T_n(\mu_\alpha) = n\mu_\alpha + \mu_{n\alpha},$$

where  $n\alpha$  in  $\mu_{n\alpha}$  (resp.  $n\beta$  in  $\sigma_{n\beta}$ ) should be understood as a class in  $(\mathbb{Z}/l\mathbb{Z})^*/\{1, -1\}$ .

Similarly

$$T_l(\sigma_\beta) = \sigma_\beta + 2l \sum_{1 \leq \alpha \leq \frac{l-1}{2}} \mu_\alpha$$

and

$$T_l(\mu_\alpha) = l\mu_\alpha.$$

And of course, if  $n$  is prime to  $l$ , then

$$T(n, n)(\sigma_\beta) = \frac{1}{n^2} \sigma_{n\beta}$$

and

$$T(n, n)(\mu_\alpha) = \frac{1}{n^2} \mu_{n\alpha}.$$

All together, one can compute the effect of  $T_n$  on cusps for all  $n$ . For the sake of completeness, we also give the action of the diamond operator  $\langle n \rangle$  on cusps. If  $n$  is prime to  $l$  then  $\langle n \rangle(\sigma_\beta) = \sigma_{n\beta}$  and  $\langle n \rangle(\mu_\alpha) = \mu_{n\alpha}$ .

□

We can now state the following theorem.

**13.7.2 Theorem (Computing  $V_f$  modulo  $p$ )** *There is a probabilistic (Las Vegas) algorithm that takes as input an even integer  $k > 2$ , a prime integer  $l > 6(k - 1)$ , a finite field  $\mathbb{F}$  with characteristic  $l$ , a ring epimorphism*

$f : \mathbb{T}(1, k) \rightarrow \mathbb{F}$ , a cuspidal divisor  $\Omega$  on  $X_1(5l)$  as constructed in Section 8.1, and a prime  $p \notin \{5, l\}$ , and computes the reduction modulo  $p$  of every element in  $W_f \subset J_1(5l)$ . Here  $V_f \subset J_1(l)$  is defined by Equation (13.2) and  $W_f$  the image of  $V_f \subset J_1(l)$  by  $B_{5l, l, 1}^*$ , and we assume that the image of the Galois representation  $\rho_f$  associated with  $f$  contains  $\mathrm{SL}(V_f)$ . The algorithm returns for every element  $x$  in  $W_f$  a degree  $g$  effective divisor  $Q_{x,p}$  on  $X_1(5l)/\mathbb{F}_p$  such that  $Q_{x,p} - \Omega$  lies in the class represented by  $x$  modulo  $p$ . The running time of the algorithm is  $\leq (p \times \#V_f)^\Theta$  for some absolute constant  $\Theta$ .

*Remark.* It has been proven in Section 8.1 that the divisor  $Q_x$  on  $X_1(5l)/\mathbb{Q}$  associated to  $x \in W_f \subset J_1(5l)(\overline{\mathbb{Q}})[l]$  is non-special. According to Lemma 12.12.2 and Theorem 11.7.10, for every prime  $p$  but a finite number bounded by  $l^\Theta$ , the divisor  $Q_x$  remains non-special modulo  $p$  for every  $x$  in  $W_f$ , and is equal to the divisor  $Q_{x,p}$  returned by the algorithm above.

To prove Theorem 13.7.2 we notice that Section 12.3.3 gives us a plane model for  $X_l \bmod p$  and a resolution of its singularities. From Lemma 12.3.6 we obtain the zeta function of  $X_l \bmod p$ . The characteristic polynomial of  $F_p$  acting on the 2-dimensional  $\mathbb{F}$ -vector space  $V_f$  is  $X^2 - f(T_p)X + p^{k-1} \bmod l$ . Since we know  $f(T_i)$  for  $2 \leq i \leq k/12$  we deduce  $f(T_p)$  using Manin-Drinfeld-Shokurov theory. Knowing the characteristic polynomial of  $F_p$ , we deduce the order of  $F_p$  acting on  $V_f \bmod p$ . We deduce some small enough splitting field  $\mathbb{F}_q$  for  $V_f \bmod p$ . We then apply Theorem 13.6.2 and obtain a basis for the  $l$ -torsion in the Picard group of  $X_l/\mathbb{F}_q$ . The same theorem allows us to compute the matrix of the endomorphism  $\pi^* \circ \pi_*$  in this basis. We deduce a  $\mathbb{F}_l$ -basis for the image  $\mathcal{A}[l](\mathbb{F}_q)$  of  $\pi^* \circ \pi_*$ . Using Theorem 13.6.2 again, we now write down the matrices of the Hecke operators  $\hat{T}_n$  in this basis for all  $n \leq (l^2 - 1)/6$ . It is then a matter of linear algebra to compute a basis for the intersection of the kernels of all  $(t_i)_{1 \leq i \leq (l^2 - 1)/6}$  in  $\mathcal{A}[l](\mathbb{F}_q)$ . The algorithm is Las Vegas rather than Monte-Carlo because we can check the result, the group  $W_f$  having known cardinality  $(\#\mathbb{F})^2$ .  $\square$



## Chapter Fourteen

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### Computing the residual Galois representations

B. Edixhoven

In this chapter we first combine the results of Chapters 11 and 12 in order to work out the strategy of Chapter 3 in the setup of Section 8.2. This gives the main result, Theorem 14.1.1: a deterministic polynomial time algorithm, based on computations with complex numbers. The crucial transition from approximations to exact values is done in Section 14.4, and the proof of Theorem 14.1.1 is finished in Section 14.7. In Section 14.8 we replace the complex computations with the computations over finite fields from Chapter 13, and give a probabilistic (Las Vegas type) polynomial time variant of the algorithm in Theorem 14.1.1.

#### 14.1 MAIN RESULT

For positive integers  $k$  and  $n$  we have defined, in Section 2.4,  $\mathbb{T}(n, k)$  as the  $\mathbb{Z}$ -algebra in  $\text{End}_{\mathbb{C}}(S_k(\Gamma_1(n)))$  generated by the Hecke operators  $T_m$  ( $m \geq 1$ ) and the  $\langle a \rangle$  ( $a$  in  $(\mathbb{Z}/n\mathbb{Z})^\times$ ). Theorem 2.5.10 says that  $\mathbb{T}(n, k)$  is generated as  $\mathbb{Z}$ -module by the Hecke operators  $T_i$  with  $1 \leq i \leq k \cdot [\text{SL}_2(\mathbb{Z}) : \Gamma_1(n)]/12$ . In particular,  $\mathbb{T}(1, k)$  is generated as  $\mathbb{Z}$ -module by the  $T_i$  with  $i \leq k/12$ . For each  $k$  and  $n$ , each surjective ring morphism  $\mathbb{T}(n, k) \rightarrow \mathbb{F}$  gives rise to a Galois representation  $\rho_m : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$ . Theorem 2.5.18 says that if  $n = 1$  and the characteristic  $l$  of  $\mathbb{F}$  satisfies  $l > 6(k - 1)$ , then  $\rho_m$  is reducible, or has image containing  $\text{SL}_2(\mathbb{F})$ .

**14.1.1 Theorem** *There is a deterministic algorithm that on input a positive integer  $k$ , a finite field  $\mathbb{F}$ , and a surjective ring morphism  $f : \mathbb{T}(1, k) \rightarrow \mathbb{F}$*

such that the associated Galois representation  $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$  is reducible or has image containing  $\text{SL}_2(\mathbb{F})$ , computes  $\rho$  in time polynomial in  $k$  and  $\#\mathbb{F}$ . The morphism  $f$  is given by the images of  $T_1, \dots, T_{\lfloor k/12 \rfloor}$ . More explicitly, the algorithm gives:

1. a Galois extension  $K$  of  $\mathbb{Q}$ , given as a  $\mathbb{Q}$ -basis  $e$  and the products  $e_i e_j$  (i.e., the  $a_{i,j,k}$  in  $\mathbb{Q}$  such that  $e_i e_j = \sum_k a_{i,j,k} e_k$  are given);
2. a list of the elements  $\sigma$  of  $\text{Gal}(K/\mathbb{Q})$ , where each  $\sigma$  is given as its matrix with respect to  $e$ ;
3. an injective morphism  $\rho$  from  $\text{Gal}(K/\mathbb{Q})$  into  $\text{GL}_2(\mathbb{F})$ , making  $\mathbb{F}^2$  into a semi-simple representation of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ ,

such that  $K$  is unramified outside  $l$ , with  $l$  the characteristic of  $\mathbb{F}$ , and such that for all prime numbers  $p$  different from  $l$  we have:

$$\text{trace}(\rho(\text{Frob}_p)) = f(T_p) \quad \text{and} \quad \det(\rho(\text{Frob}_p)) = p^{k-1} \quad \text{in } \mathbb{F}.$$

**14.1.2 Remark** Of course, we do not only prove *existence* of such an algorithm, but we actually *describe* one in the proof. We cannot claim that we really *give* such an algorithm because we did not make all constants in our estimates explicit, e.g., those in Theorem 11.7.6 and Theorem 11.7.10.

**Proof** As this proof is rather long, we divide it into sections.

## 14.2 REDUCTION TO IRREDUCIBLE REPRESENTATIONS

Let  $k$ ,  $\mathbb{F}$ ,  $l$  and  $f$  be as in Theorem 14.1.1. By definition, the associated representation  $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$  is semi-simple, unramified outside  $\{l\}$  and has  $\det \circ \rho = \chi_l^{k-1}$ , with  $\chi_l: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{F}_l^\times$  the mod  $l$  cyclotomic character. Hence  $\rho$  is reducible if and only if it is of the form  $\chi_l^i \oplus \chi_l^j$ , for some  $i$  and  $j$  in  $\mathbb{Z}/(l-1)\mathbb{Z}$  with  $i+j = k-1$  in  $\mathbb{Z}/(l-1)\mathbb{Z}$ . The following result gives us an effective way to decide if  $\rho$  is reducible, and to determine  $\rho$  in that case. More precisely, using the standard algorithms based on modular symbols this proposition reduces the proof of Theorem 14.1.1 to the case where  $\rho$  is irreducible.

**14.2.1 Proposition** *In this situation, if  $l = 2$ , then  $\rho \cong 1 \oplus 1$ , and if  $l = 3$ , then  $\rho \cong 1 \oplus \chi_3$ . Assume now that  $l \geq 5$ . Let  $i$  and  $j$  be in  $\mathbb{Z}/(l-1)\mathbb{Z}$  such that  $i + j = k-1$ . Then  $\rho$  is isomorphic to  $\chi_l^i \oplus \chi_l^j$  if and only if for all prime numbers  $p \neq l$  with  $p \leq (l^2-1)/12$  we have  $f(T_p) = p^i + p^j$  in  $\mathbb{F}$ .*

**Proof** The statements about  $l=2$  and  $l=3$  are proved in Théorème 3 of [Ser4] (see also Theorem 3.4 of [Edi1]). As  $\mathbb{T}(1, k)$  is not zero (it has  $\mathbb{F}$  as quotient),  $k$  is even.

The idea for the rest of the proof is to use suitable cuspidal eigenforms over  $\mathbb{F}_l$  whose associated Galois representations give all the  $\chi_l^i \oplus \chi_l^j$  with  $i+j$  odd, and then to apply Proposition 2.5.16.

For  $a$  in  $\mathbb{Z}$  even such that  $4 \leq a \leq l-3$  or  $a = l+1$ , let  $\overline{E}_a$  be the element of  $M_a(1, \mathbb{F}_l)$  with  $a_1(\overline{E}_a)=1$  and  $T_p(\overline{E}_a) = (1+p^{a-1})\overline{E}_a$  for all primes  $p$ . These  $\overline{E}_a$  can be obtained by reducing the Eisenstein series  $E_a$  modulo  $l$ , after multiplication by  $B_k/2k$  (see Example 2.2.3). We cannot use  $E_{l-1}$  because its reduction has constant  $q$ -expansion 1, and so it cannot be normalised as we need. The  $\overline{E}_a$  are eigenforms, and we have  $\rho_{\overline{E}_a} = 1 \oplus \chi_l^{a-1}$ . We note that the  $\chi_l^{a-1}$  give all powers of  $\chi_l$  except  $\chi_l^{-1}$ . But we do have  $1 \oplus \chi_l^{-1} = \chi_l^{-1} \otimes \rho_{\overline{E}_{l+1}}$ . We also note that, for each  $a$ ,  $\overline{E}_a$  and  $\theta^{l-1}\overline{E}_a$  give the same Galois representation. Therefore, all  $\chi_l^i \oplus \chi_l^j$  with  $i+j$  odd are associated with suitable cuspidal eigenforms. The proof is then finished by invoking Proposition 2.5.16. □

### 14.3 REDUCTION TO TORSION IN JACOBIANS

Let  $k, \mathbb{F}, l$  and  $f$  be as in Theorem 14.1.1, such that the representation  $\rho$  attached to  $f$  is irreducible. The following proposition is a special case of Theorem 3.4 of [Edi1].

**14.3.1 Proposition** *In this situation, there is a  $k'$  in  $\mathbb{Z}_{>0}$  and a surjective ring morphism  $f': \mathbb{T}(1, k') \rightarrow \mathbb{F}$  and an  $i$  in  $\mathbb{Z}/(l-1)\mathbb{Z}$  such that  $2 \leq k' \leq l+1$  and  $\rho \cong \rho' \otimes \chi_l^i$ , where  $\rho': \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$  is the Galois representation attached to  $f'$ .*

Such a  $k', f'$  and  $i$  can be computed in time polynomial in  $k$  and  $l$  as follows. First, one computes the  $\mathbb{F}_l$ -algebras  $\mathbb{F}_l \otimes \mathbb{T}(1, k')$  for  $2 \leq k' \leq l+1$ .



Then, for each  $i$  in  $\mathbb{Z}/(l-1)\mathbb{Z}$  and for each  $k'$ , one checks if there exists an  $\mathbb{F}_l$ -linear map from  $\mathbb{F}_l \otimes \mathbb{T}(1, k')$  to  $\mathbb{F}$  that sends, for all  $m \leq (l^2-1)/12$  with  $l$  not dividing  $m$ ,  $T_m$  to  $m^{-i}f(T_m)$ , and if so, if it is an algebra morphism. The previous proposition guarantees that such  $k'$ ,  $f'$  and  $i$  do exist. Proposition 2.5.16 guarantees that  $\rho \cong \rho' \otimes \chi_l^i$ .

Using standard algorithms for linear algebra over  $\mathbb{Q}$ , the computation of  $\rho$  is reduced to that of  $\rho'$ . By Theorem 2.5.12,  $\rho'$  is realised in  $J_1(l)(\overline{\mathbb{Q}})[l]$  as the intersection of the kernels of a set of elements of  $\mathbb{F}_l \otimes \mathbb{T}(l, 2)$  that can be computed in time polynomial in  $k$  and  $l$ . Hence, the proof of Theorem 14.1.1 is reduced to the case where  $\rho$  is irreducible, and is realised in  $J_1(l)(\overline{\mathbb{Q}})[l]$  as described in Theorem 2.5.12.

#### 14.4 COMPUTING THE $\mathbb{Q}(\zeta_l)$ -ALGEBRA CORRESPONDING TO $V$

We recall the situation. The representation  $\rho$  of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  attached to the surjective ring morphism  $f_k: \mathbb{T}(1, k) \rightarrow \mathbb{F}$  has image containing  $\text{SL}_2(\mathbb{F})$ , and is also attached to a surjective ring morphism  $f_2: \mathbb{T}(2, l) \rightarrow \mathbb{F}$ . In particular,  $\rho$  is realised on the two-dimensional  $\mathbb{F}$ -vector space  $V$  in  $J_1(l)(\overline{\mathbb{Q}})[l]$  consisting of all elements annihilated by  $\ker(f_2)$ . We note that  $l > 5$ . As in Section 8.2 we let  $X_l$  be the modular curve  $X_1(5l)$ , over  $\mathbb{Q}$ , and we embed  $V$  in the Jacobian  $J_l$  of  $X_l$  via pullback by the standard map from  $X_l$  to  $X_1(l)$ . We take a cuspidal divisor  $D_0$  on  $X_{l, \mathbb{Q}(\zeta_l)}$  as in Theorem 8.1.7. We have, for each  $x \in V$ , a unique effective divisor  $D_x$  of degree  $g_l$  (the genus of  $X_l$ ) on  $X_{l, \overline{\mathbb{Q}}}$ , such that in  $J_l(\overline{\mathbb{Q}})$  we have  $x = [D_x - D_0]$ . We write each  $D_x$  as  $D_x^{\text{fin}} + D_x^{\text{cusp}}$ , where  $D_x^{\text{cusp}}$  is supported on the cusps and  $D_x^{\text{fin}}$  is disjoint from the cusps. We write  $D_x = \sum_{i=1}^{g_l} Q_{x,i}$ , with  $Q_{x,i}$  in  $X_l(\overline{\mathbb{Q}})$ , such that  $D_x^{\text{fin}} = \sum_{i=1}^{d_x} Q_{x,i}$ .

Theorem 12.10.2 says that we have an analytic description of  $V$  inside  $J_l(\mathbb{C})$ , and, that for every embedding  $\sigma: \mathbb{Q}(\zeta_l) \rightarrow \mathbb{C}$ , complex approximations  $D_{\sigma,x}$  of the  $D_x$  can be computed in polynomial time in  $\#\mathbb{F}$  and the required accuracy (the number of digits on the right of the decimal point). Each such approximation is given as a sum of  $g_l$  points  $Q_{\sigma,x,i}$  in  $X_l(\mathbb{C})$ , the numbering of which by  $i \in \{1, \dots, g_l\}$  is completely arbitrary, i.e., unrelated to each other when  $\sigma$  varies. Similarly, complex approximations of all  $b_l(Q_{x,i})$  or  $(1/b_l)(Q_{x,i})$  (one of which two has absolute value  $< 2$ ) and of

all  $x'_l(Q_{x,i})$  or  $(1/x'_l)(Q_{x,i})$  can be computed in polynomial time in  $\#\mathbb{F}$  and the required accuracy. We denote such approximations by  $b_l(Q_{\sigma,x,i})$ , etc.

We compute such approximations, and also of  $j(Q_{x,i})$  or  $(1/j)(Q_{x,i})$ , for all  $\sigma$ ,  $x$  and  $i$ , with accuracy a sufficiently large absolute constant times  $l^{15} \cdot (\#\mathbb{F})^6$ . Here, and the rest of this section, we will use the  $O$ -notation without making the implied “absolute” constants explicit.

Using these approximations we will first decide for which  $(\sigma, x, i)$  the point  $Q_{\sigma,x,i}$  approximates a cusp or not. For  $x = 0$  we have  $D_x = D_0$  and so all  $Q_{0,i}$  are cusps. Recall that the cusps are precisely the poles of the rational function  $j$ . Hence a necessary for  $Q_{\sigma,x,i}$  to approximate a cusp is that  $(1/j)(Q_{\sigma,x,i})$  is small. Let  $x$  in  $V$  be non-zero, and  $i$  in  $\{1, \dots, g_l\}$  such that  $j(Q_{x,i}) \neq 0$ . By Proposition 11.7.1, the expression for  $j$  in  $b$  in Proposition 8.2.8, and Lemma 4.2.3, we have  $h((1/j)(Q_{x,i})) = O(l^{12})$ . The degree of  $(1/j)(Q_{x,i})$  over  $\mathbb{Q}$  is at most  $l^2 \cdot (\#\mathbb{F})^2$ . By Lemma 4.2.5, we have, for all  $\sigma: \overline{\mathbb{Q}} \rightarrow \mathbb{C}$ , that if  $(1/j)(Q_{x,i}) \neq 0$ , then  $|\sigma((1/j)(Q_{x,i}))| \geq \exp(-O(l^{14}(\#\mathbb{F})^2))$ . We conclude that the  $Q_{\sigma,x,i}$  for which  $|\sigma((1/j)(Q_{\sigma,x,i}))| < \exp(-O(l^{14}(\#\mathbb{F})^2))$  are the ones that approximate cusps. This gives us the correct value of the integers  $d_x$ , and, after renumbering the  $Q_{\sigma,x,i}$ , approximations  $D_{\sigma,x}^{\text{fin}} = \sum_{i=1}^{d_x} Q_{\sigma,x,i}$  of the  $D_x^{\text{fin}}$ .

The next step is to get an integer  $n$  with  $0 \leq n \leq l^4 \cdot (\#\mathbb{F})^4$  such that the function  $f_l := b_l + nx'_l$  separates the various  $Q_{x,i}$  with  $x \in V$  and  $i \leq d_x$  that are distinct. We do this using just one embedding  $\sigma$  of  $\mathbb{Q}(\zeta_l)$  in  $\mathbb{C}$ . For  $x$  in  $V$  and  $i \leq d_x$  we have, by Proposition 11.7.1,  $h((b_l(Q_{x,i}), x'_l(Q_{x,i}))) = O(l^{12})$ . For  $x$  and  $y$  in  $V$ , the field over which they are both defined has degree at most  $l \cdot (\#\mathbb{F})^3$  over  $\mathbb{Q}$ . By Lemma 4.2.5, we conclude that  $Q_{\sigma,x,i}$  and  $Q_{\sigma,y,j}$  approximate the same point if and only if  $|b_l(Q_{\sigma,x,i}) - b_l(Q_{\sigma,y,j})| < \exp(-O(l^{13} \cdot (\#\mathbb{F})^3))$  and  $|x'_l(Q_{\sigma,x,i}) - x'_l(Q_{\sigma,y,j})| < \exp(-O(l^{13} \cdot (\#\mathbb{F})^3))$ . We observe that the required approximations can indeed be computed within the required time because the height bounds from Proposition 11.7.1 imply that  $|b_l(Q_{\sigma,x,i})| < O(l^{14} \cdot (\#\mathbb{F})^2)$ , i.e., also on the left of the decimal point there are not too many digits. An integer  $n$  as above does not give a suitable  $f_l$  if and only if there are  $Q_{\sigma,x,i}$  and  $Q_{\sigma,y,j}$  that approximate different points and still  $|f_l(Q_{\sigma,x,i}) - f_l(Q_{\sigma,y,j})| < \exp(-O(l^{13} \cdot (\#\mathbb{F})^3))$ . Trying the possible

$n$  one by one until we have a suitable one gives us a function  $f_l$  as desired.

Now that we have our function  $f_l$ , we continue, as explained at the end of Section 8.2, by computing an integer  $m$  with  $0 \leq m \leq l^2 \cdot (\#\mathbb{F})^4$  such that the function:

$$a_m: V \rightarrow \overline{\mathbb{Q}}, \quad x \mapsto \prod_{i=1}^{d_x} (m - f_l(Q_{x,i}))$$

is injective, and hence a generator of the  $\mathbb{Q}(\zeta_l)$ -algebra  $A_{\mathbb{Q}(\zeta_l)}$  corresponding to  $V$ . As in the previous step, we do this using just one embedding  $\sigma$  of  $\mathbb{Q}(\zeta_l)$  in  $\mathbb{C}$ . We estimate the loss in accuracy in computing a product  $\prod_{i \leq d_x} (m - f_l(Q_{\sigma,x,i}))$ . For  $x$  in  $V$  and  $i \leq d_x$  we have  $h(f_l(Q_{x,i})) = O(l^{12})$ . Hence, in the product, we have  $|f_l(Q_{\sigma,x,i})| = O(l^{14} \cdot (\#\mathbb{F})^2)$ . As there are at most  $l^2$  factors, the loss of accuracy is at most  $O(l^{16} \cdot (\#\mathbb{F})^2)$  digits. Hence, from our approximations  $f_l(Q_{\sigma,x,i})$  we get approximations  $a_{\sigma,m}(x)$  of the  $a_m(x)$  with accuracy  $O(l^{15} \cdot (\#\mathbb{F})^3)$ . For all candidates  $m$  and all  $x$  in  $V$ , that  $h(a_m(x)) = O(l^{14})$ , and the degree of  $a_m(x)$  over  $\mathbb{Q}$  is at most  $(\#\mathbb{F})^2$ . We conclude that a candidate  $m$  is not suitable if and only if there are distinct  $x$  and  $y$  in  $V$  with  $|a_{\sigma,m}(x) - a_{\sigma,m}(y)| < \exp(O(-l^{14} \cdot (\#\mathbb{F})^2))$ . Trying one by one gives us a suitable  $m$ .

We denote by  $a = a_{D_0, f_l, m}$  the generator of  $A_{\mathbb{Q}(\zeta_l)}$  that we obtained by finding suitable  $n$  and  $m$ . We will now compute the minimal polynomial of  $a$  over  $\mathbb{Q}(\zeta_l)$ :

$$(14.4.1) \quad P = \prod_{x \in V} (T - a(x)), \quad P = \sum P_j T^j, \quad P_j \in \mathbb{Q}(\zeta_l).$$

As  $h(a(x)) = O(l^{14})$ , and, for  $j$  in  $\{0, \dots, \#V\}$ ,  $P_j$  is, up to a sign, an elementary symmetric polynomial in the  $a(x)$ , Lemma 4.2.4 gives that  $h(P_j) = O(l^{14} \cdot (\#\mathbb{F})^2)$ . We write the  $P_j$  in the  $\mathbb{Q}$ -basis  $(1, \zeta_l, \dots, \zeta_l^{l-2})$  of  $\mathbb{Q}(\zeta_l)$ :

$$P_j = \sum_{i < l-1} P_{j,i} \zeta_l^i, \quad P_{j,i} \in \mathbb{Q}.$$

Then, for each  $j$ ,  $(P_{j,0}, \dots, P_{j,l-2})$  is the unique solution in  $\overline{\mathbb{Q}}$  of the system

of linear equations, indexed by the  $\sigma$  in  $\text{Gal}(\mathbb{Q}(\zeta_l)/\mathbb{Q})$ :

$$\sum_{i < l-1} \sigma(\zeta_l^i) P_{j,i} = \sigma(P_j).$$

Applying Lemma 4.2.6 and Cramer's rule give:

$$(14.4.2) \quad h(P_{j,i}) \leq l \log l + l \cdot h(P_j) = O(l^{15} \cdot (\#\mathbb{F})^2).$$

So, in order to deduce the  $P_{j,i}$  from approximations as in Proposition 3.2 the accuracy we need is  $O(l^{15} \cdot (\#\mathbb{F})^2)$ . We estimate the loss of accuracy in the evaluation of the product in (14.4.1). We already know that  $h(a(x)) = O(l^{14})$ , and that the degree of  $a(x)$  over  $\mathbb{Q}$  is  $(\#\mathbb{F})^2$ . Therefore, for all embeddings  $\sigma$  of  $\mathbb{Q}(\zeta_l)$  into  $\mathbb{C}$ , we have  $|a_\sigma(x)| = \exp(O(l^{14} \cdot (\#\mathbb{F})^2))$ . As there are  $(\#\mathbb{F})^2$  factors in (14.4.1), the loss of accuracy is at most  $O(l^{14} \cdot (\#\mathbb{F})^4)$ . We conclude that our approximations  $P_\sigma = \prod_{x \in V} (T - a_\sigma(x))$  at all  $\sigma$  are accurate enough to get good enough approximations of the  $P_{j,i}$  such that Proposition 3.2 gives us the exact values of the  $P_{j,i}$ . So, finally, we know  $A_{\mathbb{Q}(\zeta_l)}$  explicitly as:

$$A_{\mathbb{Q}(\zeta_l)} = \mathbb{Q}(\zeta_l)[T]/(P) = \bigoplus_{0 \leq i < (\#\mathbb{F})^2} \mathbb{Q}(\zeta_l) \cdot T^i.$$

We remark that the definition of  $a$  directly implies that  $a(0) = 1$ . Hence  $P$  has a factor  $T - 1$ . Under our assumption that the image of  $\rho$  contains  $\text{SL}_2(\mathbb{F})$  the polynomial  $P/(T - 1)$  is irreducible over  $\mathbb{Q}(\zeta_l)$ .

### 14.5 COMPUTING THE VECTOR SPACE STRUCTURE

The addition map  $+: V \times V \rightarrow V$  corresponds to a morphism of  $\mathbb{Q}$ -algebras  $+^*: A \rightarrow A \otimes_{\mathbb{Q}} A$ , called co-addition. We will now explain how to compute the co-addition over  $\mathbb{Q}(\zeta_l)$ , i.e., the morphism of  $\mathbb{Q}(\zeta_l)$ -algebras:

$$+^*: A_{\mathbb{Q}(\zeta_l)} \longrightarrow A_{\mathbb{Q}(\zeta_l)} \otimes_{\mathbb{Q}(\zeta_l)} A_{\mathbb{Q}(\zeta_l)} = \mathbb{Q}(\zeta_l)[U, V]/(P(U), P(V)).$$

To give this morphism is equivalent to give the image of our generator  $a$  of the previous section. This image can be written uniquely as a polynomial in  $U$  and  $V$  of degree less than  $(\#\mathbb{F})^2$  in each variable. Hence, there are

unique  $\mu_{i,j}$  in  $\mathbb{Q}(\zeta_l)$ , for  $i$  and  $j$  in  $\{0, \dots, (\#\mathbb{F})^2 - 1\}$ , such that for all  $x$  and  $y$  in  $V$  we have, in  $\overline{\mathbb{Q}}$ :

$$(14.5.1) \quad a(x + y) = \sum_{i,j} \mu_{i,j} \cdot a(x)^i a(y)^j.$$

We view (14.5.1) as an inhomogeneous system of  $(\#\mathbb{F})^4$  linear equations in the  $\mu_{i,j}$ . Then our bound  $h(a(x)) = O(l^{14})$ , together with Cramer's rule and Lemma 4.2.6 give  $h(\mu_{i,j}) = O(l^{14} \cdot (\#\mathbb{F})^6)$ . Writing  $\mu_{i,j} = \sum_{k < l-1} \mu_{i,j,k} \zeta_l^k$ , we have  $h(\mu_{i,j,k}) = O(l^{15} \cdot (\#\mathbb{F})^6)$ . Hence our approximations  $a_\sigma(x)$  are sufficiently precise to deduce the exact values of the  $\mu_{i,j,k}$ .

We also want to compute the multiplication map  $\mathbb{F} \times V \rightarrow V$ . That is, for each  $\lambda$  in  $\mathbb{F}$  we want to know the map  $(\lambda \cdot)^*$  from  $A_{\mathbb{Q}(\zeta_l)}$  to itself that it induces. For  $\lambda = 0$  this is the map that sends  $a$  to 1. Let now  $\lambda$  be in  $\mathbb{F}^\times$ . Then there are unique  $\alpha_i$  in  $\mathbb{Q}(\zeta_l)$ , for  $i$  in  $\{0, \dots, (\#\mathbb{F})^2 - 1\}$  such that  $(\lambda \cdot)^*(a) = \sum_i \alpha_i \cdot a^i$ . These  $\alpha_i$  uniquely determined by the following equalities in  $\overline{\mathbb{Q}}$ , for all  $x$  in  $V$ :

$$(14.5.2) \quad a(\lambda \cdot x) = \sum_i \alpha_i \cdot a(x)^i.$$

Arguments as above for the addition show that our approximations  $a_\sigma(x)$  allow us to get the exact values of the  $\alpha_i$ .

### 14.6 DESCENT TO $\mathbb{Q}$

At this moment, we finally have to pay the price for working with a divisor  $D_0$  on  $X_{l, \mathbb{Q}(\zeta_l)}$  and not on  $X_l$  itself. We have  $A_{\mathbb{Q}(\zeta_l)} = \mathbb{Q}(\zeta_l) \otimes A$ , hence we have a semi-linear action of  $\text{Gal}(\mathbb{Q}(\zeta_l)/\mathbb{Q})$  on the  $\mathbb{Q}(\zeta_l)$ -algebra  $A_{\mathbb{Q}(\zeta_l)}$ : for  $\tau$  in  $\text{Gal}(\mathbb{Q}(\zeta_l)/\mathbb{Q})$ ,  $\lambda$  in  $\mathbb{Q}(\zeta_l)$ , and  $x$  in  $A_{\mathbb{Q}(\zeta_l)}$  we have  $\tau(\lambda x) = \tau(\lambda) \cdot \tau(x)$ . The  $\mathbb{Q}$ -algebra  $A$  is precisely the subset of  $A_{\mathbb{Q}(\zeta_l)}$  of elements of that are fixed by this action.

In order to understand what the action of  $\text{Gal}(\mathbb{Q}(\zeta_l)/\mathbb{Q})$  does with our generator  $a$  of  $A_{\mathbb{Q}(\zeta_l)}$ , we must include the divisor  $D_0$  into its notation: we will write  $a_{D_0}$  for it. Then, for each  $\tau$  in  $\text{Gal}(\mathbb{Q}(\zeta_l)/\mathbb{Q})$  we have  $\tau(a_{D_0}) = a_{\tau D_0}$ , where  $a_{\tau D_0}$  is defined as  $a$ , but with the divisor  $D_0$  re-

placed by  $\tau D_0$ . For each  $\sigma: \mathbb{Q}(\zeta_l) \rightarrow \mathbb{C}$ , and for each  $x$  in  $V \subset J_l(\mathbb{C})$ , we have the approximations  $a_{\sigma, D_0}(x)$  of  $a_{D_0}(x)$ , and  $a_{\sigma\tau, D_0}(x)$  of  $a_{\tau D_0}(x)$ .

We take a generator  $\tau$  of  $\text{Gal}(\mathbb{Q}(\zeta_l)/\mathbb{Q}) = \mathbb{F}_l^\times$ . There are unique  $c_i$  in  $\mathbb{Q}(\zeta_l)$ , for  $i$  in  $\{0, \dots, (\#\mathbb{F})^2 - 1\}$ , such that  $\tau(a) = \sum_i c_i \cdot a^i$ . These  $c_i$  are uniquely determined by the following system of equalities in  $\overline{\mathbb{Q}}$ , indexed by the  $x$  in  $V$ :

$$(14.6.1) \quad a_{\tau D_0}(x) = \sum_{0 \leq i < (\#\mathbb{F})^2} c_i \cdot a_{D_0}(x)^i.$$

Lemma 4.2.6 implies that for all  $i$  we have  $h(c_i) = O(l^{14} \cdot (\#\mathbb{F})^6)$ , and hence, writing  $c_i = \sum_{j < l-1} c_{i,j} \zeta_l^j$ ,  $h(c_{i,j}) = O(l^{15} \cdot (\#\mathbb{F})^6)$ . We conclude that our approximations  $a_{\sigma, D_0}(x)$  of  $a_{D_0}(x)$ , and  $a_{\sigma\tau, D_0}(x)$  of  $a_{\tau D_0}(x)$ , for all  $x$  and  $\sigma$ , are sufficiently accurate to get the exact values of the  $c_i$ .

Linear algebra over  $\mathbb{Q}$  gives us then  $A$ , in terms of a  $\mathbb{Q}$ -basis with multiplication table, and with the maps  $+^*: A \rightarrow A \otimes A$  and  $(\lambda \cdot)^*: A \rightarrow A$  that correspond to the  $\mathbb{F}$ -vector space structure on  $V$ .

### 14.7 EXTRACTING THE GALOIS REPRESENTATION

We finish our computation of the Galois representation  $\rho$  as indicated in Chapter 3. We view  $V \times V$  as  $\text{Hom}_{\mathbb{F}}(\mathbb{F}^2, V)$ . This gives a right-action by  $\text{GL}_2(\mathbb{F})$  on  $V \times V$ , hence a left-action on  $A \otimes A$ . This action can be expressed in the co-addition and the  $\mathbb{F}^\times$ -action. We let  $B$  be the  $\mathbb{Q}$ -algebra corresponding to the subset  $\text{Isom}_{\mathbb{F}}(\mathbb{F}^2, V)$  of  $\text{Hom}_{\mathbb{F}}(\mathbb{F}^2, V)$ . To compute  $B$ , as a factor of  $A \otimes A$ , we compute its idempotent, i.e., the element of  $A \otimes A$  that is 1 on  $\text{Isom}_{\mathbb{F}}(\mathbb{F}^2, V)$  and 0 on its complement, as follows. In  $A_{\mathbb{Q}(\zeta_l)} = \mathbb{Q}(\zeta_l)[T]/((T-1)P_l)$  we have the idempotent  $a_1 = P_1/P_1(1)$  which, as function on  $V$ , is the characteristic function of  $\{0\}$ . Then  $a_1$  is an element of  $A$ . Let  $a_2 = 1 - a_1$  in  $A$ . Then  $a_2$  is the characteristic function of  $V - \{0\}$ . We let  $a_3$  be the element of  $A \otimes A$  obtained by taking the product of the  $g \cdot (a_2 \otimes 1)$ , where  $g$  ranges through  $\text{GL}_2(\mathbb{F})$ . Then  $a_3$  is the characteristic function of  $\text{Isom}_{\mathbb{F}}(\mathbb{F}^2, V)$ . We compute  $B = (A \otimes A)/(1 - a_3)$  by linear algebra over  $\mathbb{Q}$ , in terms of a basis with a multiplication table, and with the  $\text{GL}_2(\mathbb{F})$ -action.

We factor the algebra  $B$  as a product of fields, using a polynomial time

factoring algorithm over  $\mathbb{Q}$  (see [Le-Le-Lo] and [Len2]). Each factor  $K$  of  $B$  then gives us an explicit realisation of  $\rho$ , as explained in Chapter 3: let  $G \subset \mathrm{GL}_2(\mathbb{F})$  be the stabiliser of a chosen factor  $K$ ; then  $G = \mathrm{Gal}(K/\mathbb{Q})$  and the inclusion  $\rho$  is a representation from  $G$  to  $\mathrm{GL}_2(\mathbb{F})$ . This finishes the proof of Theorem 14.1.1.  $\square$

**14.7.1 Remark** In the factorisation of the algebra  $B$  above, our assumption that  $\mathrm{im}\rho$  contains  $\mathrm{SL}_2(\mathbb{F})$  implies that the idempotents lie in the sub-algebra  $B^G$  of invariants by the subgroup  $G$  of  $\mathrm{GL}_2(\mathbb{F})$  consisting of the  $g$  with  $\det(g)$  a  $k-1$ th power in  $\mathbb{F}_l^\times$ . This subalgebra is a product of copies of  $\mathbb{Q}$ .

If  $\mathbb{F} = \mathbb{F}_l$ , factoring  $B$  can be avoided by twisting  $\rho$  by a suitable power of  $\chi_l$ . Indeed,  $\rho' := \rho \otimes \chi_l^{1-k/2}$  has image  $\mathrm{GL}_2(\mathbb{F}_l)$ , and  $\rho$  can then be obtained as  $\rho' \otimes \chi_l^{k/2-1}$ .

## 14.8 A PROBABILISTIC VARIANT

In this section we give a probabilistic Las Vegas type algorithm, based on the results of Chapter 13, that computes the representation  $\rho$  as in Theorem 14.1.1, in probabilistic running time polynomial in  $k$  and  $\#\mathbb{F}$ . A nice feature of computations over finite fields is that there is no loss of accuracy, as in the previous sections where computations with complex numbers were used. On the other hand, information obtained modulo varying primes is a bit harder to combine, and here the point of view of Galois theory that we have taken, relating sets with Galois action to algebras, is very convenient.

Let  $f: \mathbb{T}(1, k) \rightarrow \mathbb{F}$  be as in Theorem 14.1.1, as well as  $l$  and the representation  $\rho: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_2(\mathbb{F})$ . Sections 14.2 and 14.3 apply without any change. So we can now put ourselves in the situation as in the beginning of Section 14.4. Then  $\mathrm{im}\rho$  contains  $\mathrm{SL}_2(\mathbb{F})$ ,  $\rho$  is realised on a two-dimensional  $\mathbb{F}$ -vector space in  $J_l(\overline{\mathbb{Q}})[l]$ , and for each  $x$  in  $V$  there is a unique effective divisor  $D_x$  of degree  $g_l$  on  $X_{l, \overline{\mathbb{Q}}}$  such that  $x = [D_x - D_0]$  in  $J_l(\overline{\mathbb{Q}})$ . For each  $x$  in  $V$ , we write  $D_x = D_x^{\mathrm{fin}} + D_x^{\mathrm{cusp}}$  as before, with  $D_x^{\mathrm{fin}} = \sum_{i=1}^{d_x} Q_{x,i}$  and  $D_x = \sum_{i=1}^{g_l} Q_{x,i}$ .

Using Theorem 13.7.2, we try to compute the reductions  $D_{x, \mathbb{F}_q}$  over suitable extensions of the residue fields of  $\mathbb{Z}[\zeta_l]$  at successive prime numbers  $p$  up to  $c \cdot l^{15} \cdot (\#\mathbb{F})^6$ , with  $c$  a suitable absolute constant, skipping 5 and  $l$ .

These fields  $\mathbb{F}_q$  have degree at most  $l \cdot (\#\mathbb{F})^3$  over their prime field  $\mathbb{F}_p$ . If some  $D_{x, \mathbb{F}_q}$  is not unique, this will be detected by our computations, and we throw the corresponding prime  $p$  away. By Theorem 11.7.10, at most  $O(l^{12} \cdot (\#\mathbb{F})^2)$  primes  $p$  are thrown away. For the  $V$ -good primes  $p \leq B$ , with  $V$ -good defined as in Theorem 11.7.10, we then have computed all  $D_{x, \mathbb{F}_q}$ .

We split each such  $D_{x, \mathbb{F}_q}$  in a cuspidal part  $D_{x, \mathbb{F}_q}^{\text{cusp}}$  and a non-cuspidal part  $D_{x, \mathbb{F}_q}^{\text{fin}}$ . By Theorem 11.7.10, there are at most  $O(l^{12} \cdot (\#\mathbb{F})^2)$  primes  $p$  where at some  $\mathbb{F}_q$ , the sum  $\sum_{x \in V} \deg D_{x, \mathbb{F}_q}^{\text{fin}}$  is less than  $\sum_{x \in V} d_x$ . (In fact, as  $\text{im} \rho$  contains  $\text{SL}_2(\mathbb{F})$ , all  $d_x$  for  $x \neq 0$  are equal, but our argument does not need this.) This means that we have computed the unordered list of  $d_x$ . We discard the primes  $p$  where for some  $\mathbb{F}_q$  the sum  $\sum_{x \in V} \deg D_{x, \mathbb{F}_q}^{\text{fin}}$  is less than  $\sum_{x \in V} d_x$ . For the remaining primes, we have computed the  $D_{x, \mathbb{F}_q}^{\text{fin}}$  for all  $x$  in  $V$ .

We want an integer  $n$  such that  $f_l := b_l + nx'_l$  separates the  $D_x^{\text{fin}}$ . Let  $p$  be the smallest  $V$ -good prime with  $p > l^4 \cdot (\#\mathbb{F})^4$ . We compute the  $D_{x, \mathbb{F}_q}^{\text{fin}}$  over some  $\mathbb{F}_q$  at  $p$ . We view the effective divisor  $D_{\mathbb{F}_q} := \sum_{x \in V} D_{x, \mathbb{F}_q}^{\text{fin}}$  as a closed subscheme of  $X_{l, \mathbb{F}_q}$ . Then  $f_l$  as above embeds  $D_{\mathbb{F}_q}$  into  $\mathbb{A}_{\mathbb{F}_q}^1$  if and only if it is injective on the geometric points of  $D_{\mathbb{F}_q}$  and has non-zero derivative at the multiple points of  $D_{\mathbb{F}_q}$ . As the degree of  $D_{\mathbb{F}_q}$  is at most  $l^2 \cdot (\#\mathbb{F})^2$ , this excludes at most  $l^4 \cdot (\#\mathbb{F})^4$  elements of  $\mathbb{F}_p$ , and the algorithms of Chapter 13 let us compute these in polynomial time. We choose  $n$  in  $\{0, \dots, l^4 \cdot (\#\mathbb{F})^4\}$  such that  $f_l$  embeds  $D_{\mathbb{F}_q}$  into  $\mathbb{A}_{\mathbb{F}_q}^1$ . Then, by Nakayama's lemma,  $f_l$  embeds  $\sum_{x \in V} D_x^{\text{fin}}$  into  $\mathbb{A}_{\mathbb{Q}}^1$ , and hence separates the  $D_x^{\text{fin}}$ .

The next step is to compute an integer  $m$  in  $\{0, l^2 \cdot (\#\mathbb{F})^4\}$  such that the function:

$$a_m : V \rightarrow \overline{\mathbb{Q}}, \quad x \mapsto \prod_{i=1}^{d_x} (m - f_l(Q_{x,i}))$$

is injective, and hence a generator of the  $\mathbb{Q}(\zeta_l)$ -algebra  $A_{\mathbb{Q}(\zeta_l)}$  corresponding to  $V$ . Let  $p$  and  $\mathbb{F}_q$  be as in the preceding paragraph. For  $x \in V$ , write  $D_{x, \mathbb{F}_q}^{\text{fin}} = \sum_{i=1}^{d_x} Q_{x,i, \mathbb{F}_q}$ . We take  $m$  in  $\{0, \dots, l^2 \cdot (\#\mathbb{F})^4\}$  such that the elements  $\prod_{i=1}^{d_x} (m - f_l(Q_{x,i, \mathbb{F}_q}))$  of  $\mathbb{F}_q$ , for  $x \in V$ , are all distinct. Then  $a := a_m$  has the desired property.



The minimal polynomial  $P$  in  $\mathbb{Q}(\zeta_l)[T]$  of  $a$  over  $\mathbb{Q}(\zeta_l)$  is given as:

$$P = \prod_{x \neq 0} (T - a(x)), \quad P = \sum_j P_j T^j, \quad P_j \in \mathbb{Q}(\zeta_l).$$

We have seen in (14.4.2) that, when writing  $P_j = \sum_{i < l-1} P_{j,i} \zeta_l^i$ , we have  $h(P_{j,i}) = O(l^{15} \cdot (\#\mathbb{F})^2)$ . Our construction shows that all  $P_{j,i}$  are integral at all primes  $p$  that are  $V$ -good. From the  $D_{x, \mathbb{F}_q}^{\text{fin}}$  that we have computed, we get the images of the  $P_{j,i}$  in  $\mathbb{F}_p$ , for all  $p$  that are  $V$ -good, up to  $c \cdot l^{15} \cdot (\#\mathbb{F})^2$ . Proposition 3.7 then gives us the  $P_{j,i}$ . So, at this point, we have computed the  $\mathbb{Q}(\zeta_l)$ -algebra  $A_{\mathbb{Q}(\zeta_l)}$ .

To compute the  $\mathbb{F}$ -vector space structure, we proceed as in Section 14.5. For each  $V$ -good prime  $p$  up to our bound  $c \cdot l^{15} \cdot (\#\mathbb{F})^6$ , we compute at each  $\mathbb{F}_q$  over  $p$  the images in  $\mathbb{F}_q$  of the  $\mu_{i,j}$  from the linear system (14.5.1), over  $\mathbb{F}_q$ , and then the  $\mu_{i,j,k}$  in  $\mathbb{F}_p$ . Proposition 3.7 and the height bound  $h(\mu_{i,j,k}) = O(l^{15} \cdot (\#\mathbb{F})^6)$  give us the  $\mu_{i,j,k}$ . The  $\alpha_i$  as in (14.5.2) can be computed in similarly, as well as the  $c_i$  as in (14.6.1). From here on, the computation then proceeds as in Section 14.7.

## Chapter Fifteen

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### Computing coefficients of modular forms

B. Edixhoven

In this chapter we apply our main result on the computation of Galois representations attached to modular forms of level one to the computation of coefficients of modular forms. In Section 15.1 we treat the case of the discriminant modular form, i.e., the computation of Ramanujan's  $\tau$ -function at primes. In Section 15.2 we deal with the more general case of forms of level one and arbitrary weight  $k$ , reformulated as the computation of Hecke operators  $T_n$  as  $\mathbb{Z}$ -linear combinations of the  $T_i$  with  $i < k/12$ . In Section 15.3 we give an application to theta functions of even, unimodular positive definite quadratic forms over  $\mathbb{Z}$ .

#### 15.1 COMPUTING $\tau(p)$ IN TIME POLYNOMIAL IN $\log p$

We recall that Ramanujan's  $\tau$ -function is defined by the following identity of formal power series with integer coefficients:

$$x \prod_{n \geq 1} (1 - x^n)^{24} = \sum_{n \geq 1} \tau(n) x^n.$$

**15.1.1 Theorem** *There exists a deterministic algorithm that on input a prime number  $p$  gives  $\tau(p)$ , in running time polynomial in  $\log p$ .*

**Proof** Deligne has proved in [Del1] and [Del2] that for all prime numbers  $p$  we have  $|\tau(p)| < 2p^{11/2}$ . Therefore, it suffices to compute  $\tau(p) \bmod l$  for all primes  $l < x$ , if the product of these  $l$  is at least  $4p^{11/2}$ . Analytic number theory (see for example [Ten], I.2.6, Corollary 10.1) tells us that we can take  $x = O(\log p)$ , hence the proof is reduced to showing that there is a

deterministic algorithm that computes  $\tau(p) \bmod l$  for prime numbers  $p$  and  $l$  in time polynomial in  $\log p$  and  $l$ . Of course, the slightly weaker but much more elementary bound  $|\tau(p)| = O(n^6)$  in [Miy, Cor. 2.1.6] also suffices for our purposes.

We take an algorithm as in Theorem 14.1.1, and we apply it with  $k = 12$ . We have  $\mathbb{T}(1, 12) = \mathbb{Z}$ , and for each  $n$  in  $\mathbb{Z}_{\geq 1}$ , the element  $T_n$  of  $\mathbb{T}(1, k)$  is the integer  $\tau(n)$ . We must now show that from the output of the algorithm we can compute  $\tau(p) \bmod l$  deterministically, in time polynomial in  $\log p$  and  $l$ . For  $l$  prime, we let  $\rho_l: \text{Gal}(K_l/\mathbb{Q}) \hookrightarrow \text{GL}_2(\mathbb{F}_l)$  denote the Galois representation attached to  $\Delta$ . As the discriminant of the ring of integers  $A$  of  $K_l$  is a power of  $l$ , Theorem 1.4 of [Bu-Le] gives the existence of a deterministic polynomial time algorithm that, given  $K_l$ , produces  $A$ , given by a  $\mathbb{Z}$ -basis. The maximal order  $A$  is preserved by the action of  $\text{Gal}(K_l/\mathbb{Q})$ . Then  $\text{Gal}(K_l/\mathbb{Q})$  acts on the étale  $\mathbb{F}_p$ -algebra  $\overline{A} := A/pA$ , and  $\text{Hom}(\overline{A}, \overline{\mathbb{F}}_p)$  is a  $\text{Gal}(K_l/\mathbb{Q})$ -torsor. Moreover,  $\overline{A}$  is the product of its finitely many residue fields  $\overline{A}/m$ , where  $m$  ranges through the maximal ideals of  $\overline{A}$ .

We let  $\text{Frob}$  denote the absolute Frobenius endomorphism of  $\overline{A}$ ; it sends  $a$  to  $a^p$ , it is an automorphism and it induces the absolute Frobenius automorphism on each of the residue fields. The matrix of  $\text{Frob}$  can be computed in time polynomial in  $l$  and  $\log p$ .

The Frobenius element  $\sigma_m$  attached to a maximal ideal  $m$  of  $\overline{A}$  is the unique element  $\sigma$  of  $\text{Gal}(K_l/\mathbb{Q})$  that fixes  $m$  and induces the absolute Frobenius on  $\overline{A}/m$ . For varying  $m$ , the  $\sigma_m$  form the Frobenius conjugacy class (at  $p$ ) in  $\text{Gal}(K_l/\mathbb{Q})$ .

For each  $\sigma$  in  $\text{Gal}(K_l/\mathbb{Q})$  we let  $\overline{A}_\sigma$  be the quotient of  $\overline{A}$  by the ideal generated by the image of  $\text{Frob} - \sigma$ . Such an  $\overline{A}_\sigma$  can be computed in polynomial time. The  $\sigma$  in the Frobenius conjugacy class are precisely those  $\sigma$  for which  $\overline{A}_\sigma$  is non-zero. We can try the  $\sigma$  one by one until we have found a  $\sigma$  in the Frobenius conjugacy class. Then we apply the map  $\rho_l$  given by Theorem 14.1.1 to it, and get an element  $\rho_l(\sigma)$  of  $\text{GL}_2(\mathbb{F}_l)$ . The trace of  $\rho_l(\sigma)$  is then  $\tau(p) \bmod l$ .  $\square$

**15.2 COMPUTING  $T_n$  FOR LARGE  $n$  AND LARGE WEIGHT**

In this section we prove the following two results, for which we first recall some notation. For positive integers  $k$  and  $N$  we have defined, in Section 2.4,  $\mathbb{T}(N, k)$  as the  $\mathbb{Z}$ -algebra in  $\text{End}_{\mathbb{C}}(S_k(\Gamma_1(N)))$  generated by the Hecke operators  $T_n$  ( $n \geq 1$ ) and the  $\langle a \rangle$  ( $a$  in  $(\mathbb{Z}/N\mathbb{Z})^\times$ ). By Theorem 2.5.10,  $\mathbb{T}(N, k)$  is generated as  $\mathbb{Z}$ -module by the Hecke operators  $T_i$  with  $1 \leq i \leq k \cdot [\text{SL}_2(\mathbb{Z}) : \Gamma_1(N)]/12$ . Just before (2.4.9) we have defined  $M_k(\Gamma_1(N), \mathbb{Z})$  as the sub  $\mathbb{Z}$ -module of  $M_k(\Gamma_1(N))$  consisting of the  $f$  with all  $a_i(f)$  in  $\mathbb{Z}$ .

**15.2.1 Theorem** *One can compute, on input two positive integers  $k$  and  $n$ , and the factorisation of  $n$  into prime factors, the element  $T_n$  of the Hecke algebra  $\mathbb{T}(1, k)$ , by computing the Galois representations attached to sufficiently many maximal ideals of  $\mathbb{T}(1, k)$  as in Theorem 14.1.1. The computation gives  $T_n$  as  $\mathbb{Z}$ -linear combination of the  $T_i$  with  $i \leq k/12$ . This algorithm is deterministic. For fixed  $k$ , it has running time polynomial in  $\log n$ . If the generalised Riemann hypothesis (GRH) holds for all number fields that are quotients of  $\mathbb{Q} \otimes \mathbb{T}(1, k)$ 's, then the algorithm has running time polynomial in  $k$  and  $\log n$ .*

**15.2.2 Corollary** *Assume GRH. There exists a deterministic algorithm that on input the weight  $k \geq 0$  and the coefficients  $a_i(f)$  with  $0 \leq i \leq k/12$  of a modular form  $f$  in  $M_k(\text{SL}_2(\mathbb{Z}), \mathbb{Z})$ , and a positive integer  $n$  together with its factorisation into primes, computes  $a_n(f)$  in running time polynomial in  $k$ ,  $\log n$ , and the maximum of the  $\log(1 + |a_i(f)|)$  with  $i \leq k/12$ .*

The principle of the proof of Theorem 15.2.1 is first to reduce the computation of  $T_n$  to that of the  $T_p$  for the primes  $p$  that divide  $n$ , using the identities implicit in (2.2.11). For  $p$  prime,  $T_p$  is computed from its images in sufficiently many residue fields  $\mathbb{T}(1, k)/m$ , using the LLL-algorithm for lattice reduction. Such an image is computed as the trace of a Frobenius element at  $p$  of the Galois representation  $\rho_m$  attached to  $m$ . The Galois representation is computed as in Theorem 14.1.1. The problem in doing all this is to keep the residue fields small, because the computation of  $\rho_m$  takes time polynomial in  $k$  and  $\#(\mathbb{T}(1, k)/m)$ . Here it makes a big difference if one

assumes GRH or not. Assuming GRH, there are sufficiently many  $m$  with  $\mathbb{T}(1, k)/m$  of size polynomial in  $k$  and  $\log p$  so that  $T_p$  can be reconstructed from its images in these  $\mathbb{T}(1, k)/m$ . If we do not assume GRH, then we cannot rule out the possibility that all “small” primes (small in terms of the discriminant of  $\mathbb{T}(1, k)$ ) are completely inert in  $\mathbb{T}(1, k)$ . Before we give the proof of Theorem 15.2.1, where the details of the algorithm are given, and the analysis of the running time, we state and prove some preliminary results. Corollary 15.2.2 will be deduced from Theorem 15.2.1, using some elementary properties of the Eisenstein series  $E_k$ , and the interpretation of the  $\mathbb{Z}$ -module  $S_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$  as the  $\mathbb{Z}$ -dual of  $\mathbb{T}(1, k)$ .

We start with a simple result that is well known.

**15.2.3 Proposition** *Let  $k \geq 4$  be an even integer. Let  $n = (k - 14)/12$  if  $k \equiv 2 \pmod{12}$ , and  $n = \lfloor k/12 \rfloor$  otherwise. Then  $\mathbb{T}(1, k)$  is free of rank  $n$  as  $\mathbb{Z}$ -module, and the  $T_i$  with  $1 \leq i \leq n$  form a  $\mathbb{Z}$ -basis for  $\mathbb{T}$ .*

**Proof** Theorem 2.5.10 tells us that  $\mathbb{T} := \mathbb{T}(1, k)$  is generated as  $\mathbb{Z}$ -module by the  $T_i$  with  $1 \leq i \leq k/12$ , but we will not use this. What we do use is that the pairing:

$$S_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z}) \times \mathbb{T} \longrightarrow \mathbb{Z}, \quad (f, t) \mapsto a_1(tf)$$

is perfect (see (2.4.9)). What we must show is then that  $S_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$  is free of rank  $n$  as  $\mathbb{Z}$ -module, and that the maps  $a_i: f \mapsto a_i(f)$ , with  $1 \leq i \leq n$ , form a  $\mathbb{Z}$ -basis of the dual of  $S_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$ . As  $\Delta$  is  $q$  times a unit in  $\mathbb{Z}[[q]]$ , we have:

$$S_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z}) = \Delta \cdot M_{k-12}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z}).$$

According to Swinnerton-Dyer ([Swi, §3]):

$$(15.2.4) \quad M(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z}) = \mathbb{Z}[E_4, \Delta] \oplus E_6 \cdot \mathbb{Z}[E_4, \Delta],$$

with  $E_4$  and  $\Delta$  algebraically independent, and  $E_6^2 = E_4^3 - 1728\Delta$ . It follows that  $S_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$  is free of rank  $n$ . Suitable monomials in  $E_4, \Delta$  and  $E_6$  show that  $(a_1, \dots, a_n)$  is a  $\mathbb{Z}$ -basis of  $S_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})^\vee$ .  $\square$

We note that we do not absolutely need the previous proposition, because from the set of generators  $T_i$  with  $1 \leq i \leq k/12$  as given by Theorem 2.5.10

one can also compute a  $\mathbb{Z}$ -basis in time polynomial in  $k$ , and this is what one will probably do in the case of arbitrary level and weight. But in this case of level one we have chosen to be more explicit.

We consider the Hecke algebras  $\mathbb{T}(1, k)$  as lattices in the  $\mathbb{R}$ -algebras  $\mathbb{T}(1, k)_{\mathbb{R}} := \mathbb{R} \otimes \mathbb{T}(1, k)$ . As all  $T_m$  are selfadjoint as operators on  $S_k(\mathrm{SL}_2(\mathbb{Z}))$  with respect to the Petersson inner product (see Section 2.2), all their eigenvalues are real. As the level is 1, all normalised eigenforms are newforms; let us write them as  $f_1, \dots, f_n$ , and view them as ring morphisms  $f_i: \mathbb{T}(1, k) \rightarrow \mathbb{R}$ . Then the map:

$$f: \mathbb{T}(1, k)_{\mathbb{R}} \longrightarrow \mathbb{R}^n, \quad t \mapsto (f_1(t), \dots, f_n(t)),$$

is an isomorphism of  $\mathbb{R}$ -algebras. The trace form on the  $\mathbb{Z}$ -algebra  $\mathbb{T}(1, k)$  induces the trace form of the  $\mathbb{R}$ -algebra  $\mathbb{R}^n$ , i.e., the standard inner product on  $\mathbb{R}^n$ . We equip  $\mathbb{T}(1, k)_{\mathbb{R}}$  with the standard volume form, i.e., the one for which a unit cube has volume 1. Our first goal is now to get a bound for the absolute discriminant of  $\mathbb{T}(1, k)$ , or, equivalently, over the volume of the quotient  $\mathbb{T}(1, k)_{\mathbb{R}}/\mathbb{T}(1, k)$ .

**15.2.5 Proposition** *Let  $k \geq 12$  be an even integer. Then we have:*

$$\log \mathrm{Vol}(\mathbb{T}(1, k)_{\mathbb{R}}/\mathbb{T}(1, k)) = \frac{1}{2} \log |\mathrm{discr} \mathbb{T}(1, k)| \leq \frac{k^2}{24} \log k.$$

**Proof** As  $T_1, \dots, T_n$  is a  $\mathbb{Z}$ -basis of  $\mathbb{T}(1, k)$ , we have:

$$\mathrm{Vol}(\mathbb{T}(1, k)_{\mathbb{R}}/\mathbb{T}(1, k)) = |\det(f(T_1), \dots, f(T_n))|.$$

Using Deligne's bound  $|a_p(f_j)| \leq 2p^{(k-1)/2}$  of [Del2] as we used the Weil bounds in the proof of Lemma 11.1.1 we get:

$$(15.2.6) \quad \begin{aligned} |(f(T_i))_j| &= |a_i(f_j)| \leq \sigma_0(i) i^{(k-1)/2} \\ &\leq 2 \cdot i^{1/2} i^{(k-1)/2} = 2 \cdot i^{k/2}. \end{aligned}$$

Hence the square of the length of  $f(T_i)$  is at most  $4ni^k$ . This implies:

$$|\det(f(T_1), \dots, f(T_n))| \leq \prod_{i=1}^n \left( 2 \cdot i^{k/2} \cdot \sqrt{n} \right) = 2^n \cdot (n!)^{k/2} \cdot n^{n/2}.$$

Hence:

$$\begin{aligned} \log \text{Vol}(\mathbb{T}(1, k)_{\mathbb{R}}/\mathbb{T}(1, k)) &\leq n \log 2 + \frac{k}{2} \log(n!) + \frac{n}{2} \log n \\ &\leq n \log 2 + \frac{kn}{2} \log n - \frac{kn}{2} + \frac{k}{2} + \frac{n}{2} \log n, \end{aligned}$$

where we have used that  $\log n! \leq n \log n - n + 1$ . Simple estimates for  $k > 24$  and direct checks in the remaining cases give the result (the only non-trivial case being  $k = 24$ , in which case the discriminant equals  $2^6 3^2 144169$ ).  $\square$

The next ingredient to be used in our proof of Theorem 15.2.1 comes from analytic number theory: an *effective* prime number theorem for number fields, under the assumption of GRH. The effectivity just alluded to means that for all real numbers  $x$  in a specified interval such as  $(2, \infty)$  an estimate for the number of prime ideals of norm at most  $x$  of an arbitrary number field  $K$  is given, whereas the usual prime number theorem is, for each  $K$ , a different asymptotic statement.

**15.2.7 Theorem (Weinberger)** *For  $K$  a number field and  $x$  a real number let  $\pi(x, K)$  denote the number of maximal ideals  $m$  of the ring of integers  $O_K$  of  $K$  with  $\#(O_K/m) \leq x$ . For  $x > 2$  in  $\mathbb{R}$  let  $\text{li } x = \int_2^x (1/\log y) dy$ . Then there exists  $c_1$  in  $\mathbb{R}$  such that for every number field  $K$  for which GRH holds, and for every  $x > 2$  one has:*

$$|\pi(x, K) - \text{li } x| \leq c_1 \sqrt{x} \log \left( \left| \text{discr}(O_K) x^{\dim_{\mathbb{Q}} K} \right| \right).$$

Weinberger states this result in [Wei4] (it is the lemma on page 181, and says that it is proved in more generality in [Wei3] (probably on page 328, just after (4.4)).

**15.2.8 Corollary** *There exist  $c_2$  and  $c_3$  in  $\mathbb{R}$  such that for every number field  $K$  for which GRH holds and for every  $x$  in  $\mathbb{R}$  such that:*

$$\begin{aligned} x &> c_2 \cdot (\log |\text{discr}(O_K)|)^2 \cdot (\log(1 + \log |\text{discr}(O_K)|))^2, \quad \text{and} \\ x &> c_3 \cdot (\dim_{\mathbb{Q}} K)^2 \cdot (1 + \log \dim_{\mathbb{Q}} K)^4 \end{aligned}$$

we have:

$$\pi(x, K) \geq \frac{1}{2} \frac{x}{\log x}.$$

**Proof** Let  $K$  be a number field for which GRH holds, and let  $x \in \mathbb{R}_{>2}$ . We write  $d_K$  for  $|\text{discr}(O_K)|$  and  $n_K$  for  $\dim_{\mathbb{Q}} K$ . Then Theorem 15.2.7 says that:

$$\begin{aligned} \pi(x, K) &\geq \text{li}(x) - c_1 x^{1/2} \log(d_K x^{n_K}) \\ &\geq \frac{x-2}{\log x} - c_1 x^{1/2} \log(d_K x^{n_K}) \\ &= \frac{x}{\log x} \left( 1 - \frac{2}{x} - \frac{c_1(\log x)(\log d_K)}{x^{1/2}} - \frac{c_1 n_K (\log x)^2}{x^{1/2}} \right). \end{aligned}$$

In order to estimate the last term of the previous line, one uses the substitutions  $x = y^4$  and  $y = n_K^{1/2} z$ . For the one but last term, one uses  $x = y^2$  and  $y = 2c_1(\log d_K)z$  if  $d_K \neq 1$ .  $\square$

In order to find an element of  $\mathbb{T}(1, k)$  from sufficiently many of its images modulo maximal ideals, we will need a lower bound on the length of a shortest non-zero vector in the intersection of the maximal ideals. The following lemma gives a general, well known lower bound in the context of orders in number fields.

**15.2.9 Lemma** *Let  $K$  be a number field,  $n$  its dimension as  $\mathbb{Q}$ -vector space,  $A \subset K$  an order, i.e., a subring of finite index in the ring of integers of  $K$ , and  $I \subset A$  a non-zero ideal. We equip  $K_{\mathbb{R}} := \mathbb{R} \otimes K$  with the inner product induced from the standard inner product on  $\mathbb{C}^n$ , where  $K_{\mathbb{R}}$  is embedded in  $\mathbb{C}^n$  via all distinct  $\sigma_i: K \rightarrow \mathbb{C}$ . We consider  $A$  and  $I$  as a lattices in  $K_{\mathbb{R}}$ . Then we have:*

$$\mu_1(I) \geq \sqrt{n} \cdot \#(A/I)^{1/n},$$

where  $\mu_1(I)$  is the length of a shortest element of  $I - \{0\}$ .

**Proof** Let  $x \in I$  be non-zero. Then  $A \cdot x \subset I$ , and we have:

$$|x_1| \cdots |x_n| = |\text{N}_{K/\mathbb{Q}}(x)| = \#(A/A \cdot x) \geq \#(A/I).$$

The inequality between geometric and arithmetic mean gives:

$$\left(|x_1|^2 \cdots |x_n|^2\right)^{1/n} \leq \frac{|x_1|^2 + \cdots + |x_n|^2}{n} = \frac{\|x\|^2}{n}.$$



Combining the last two inequalities gives:

$$\|x\| \geq \sqrt{n} \cdot |N_{K/\mathbb{Q}}(x)|^{1/n} \geq \#(A/I)^{1/n},$$

which finishes the proof.  $\square$

The next result summarises the standard approach for using the LLL-algorithm for the “closest or nearest vector problem”. For more details on lattice reduction we refer to [Le-Le-Lo], [Len3] and [Coh, §2.6].

**15.2.10 Proposition** *Let  $n \geq 0$  and let  $L$  be a free  $\mathbb{Z}$ -module of finite rank  $n$ , equipped with a positive symmetric bilinear form  $b: L \times L \rightarrow \mathbb{Z}$ . We view  $L$  as a lattice in the  $\mathbb{R}$ -vector space  $L_{\mathbb{R}}$  equipped with the inner product given by  $b$ , and for  $x$  in  $L_{\mathbb{R}}$  we put  $\|x\| := (b(x, x))^{1/2}$ . Let  $L'$  be a submodule of finite index of  $L$ , and let  $\mu_1(L')$  be the length of a shortest non-zero element of  $L'$ . Let  $t$  be an element of  $L$  such that:*

$$\|t\| < 2^{-(n+1)/2} \mu_1(L').$$

*Let  $e = (e_1, \dots, e_n)$  be an “LLL-reduced basis” of  $L'$ : if  $e^* = (e_1^*, \dots, e_n^*)$  denotes the orthogonal  $\mathbb{R}$ -basis of  $L_{\mathbb{R}}$  obtained from  $e$  by letting  $e_i^*$  be the orthogonal projection of  $e_i$  to the orthogonal complement of the subspace of  $L_{\mathbb{R}}$  generated by  $\{e_j \mid j < i\}$  (i.e., by the Gram-Schmidt orthogonalisation process), and  $\mu_{i,j} := b(e_i, e_j^*)/b(e_j^*, e_j^*)$ , then we have:*

$$\begin{aligned} |\mu_{i,j}| &\leq \frac{1}{2} \quad \text{for } 1 \leq j < i \leq n, \text{ and} \\ \|e_i^*\|^2 &\geq \left( \frac{3}{4} - \mu_{i,i-1}^2 \right) \|e_{i-1}^*\|^2 \quad \text{for } 1 < i \leq n. \end{aligned}$$

*Then  $t$  is the shortest element of  $t + L'$ , and for any  $x$  in  $t + L'$  we recover  $t$  as follows:*

- put  $x_n := x$ ;
- for  $i$  going down from  $n$  to 1 let  $x_{i-1} := x_i - [b(x_i, e_i^*)/b(e_i^*, e_i^*)]e_i$ , where, for  $y$  in  $\mathbb{Q}$ ,  $[y]$  denotes the largest of the (one or two) integers nearest to  $y$ ;
- then  $t = x_0$ .

**Proof** We claim that the orthogonal block:

$$B := \left\{ \sum_i \lambda_i e_i^* \mid -1/2 \leq \lambda_i < 1/2 \right\}$$

is a fundamental domain for  $L'$  acting on  $L_{\mathbb{R}}$  by translations. Indeed, for  $x$  in  $L_{\mathbb{R}}$  and the  $x_i$  as above,  $x_0$  is in  $B$  and  $x - x_0$  is in  $L'$ , and moreover,  $B$  and  $L_{\mathbb{R}}/L'$  have the same volume, namely  $\prod_i \|e_i^*\|$ . By the defining properties of an LLL-reduced basis we have, for  $i$  in  $\{2, \dots, n\}$ , that  $\|e_i^*\|^2 \geq (1/2)\|e_{i-1}^*\|^2$ . Also, as  $e_1^* = e_1$ , we have  $\|e_1^*\| \geq \mu_1(L')$ . It follows that for all  $i$ :

$$\|e_i^*\| \geq 2^{-(i-1)/2} \|e_1\| \geq 2^{-(i-1)/2} \mu_1(L'),$$

and, in particular:

$$\|e_i^*\| \geq 2^{-(n-1)/2} \mu_1(L').$$

Hence, for any  $x$  in  $L_{\mathbb{R}}$  not in  $B$  we have:

$$\|x\| \geq \min_i \|e_i^*\|/2 \geq 2^{-(n+1)/2} \mu_1(L').$$

□

We can now finally prove Theorem 15.2.1. We split it in three parts: description of the algorithm, proof of its correctness, and running time analysis.

**Proof** [of Theorem 15.2.1]

**Description of the algorithm**

Let  $k$  and  $n$  be given, with the factorisation of  $n = \prod_p p^{v_p(n)}$  in prime factors. Let  $\mathbb{T} := \mathbb{T}(1, k)$ , and let  $r$  be its rank. Proposition 15.2.3 tells us that  $r \leq k/12$  and that  $(T_1, \dots, T_r)$  is a  $\mathbb{Z}$ -basis of  $\mathbb{T}$ . Using modular symbols algorithms (see Chapter 8 of [Ste2]), one computes the  $\mathbb{Z}$ -algebra structure of  $\mathbb{T}$ , i.e., one computes  $T_i T_j$  as linear combinations of the  $T_l$ , with  $1 \leq i, j, l \leq r$ . The identity (2.2.11) gives us:

$$T_n = \prod_{p|n} T_{p^{v_p(n)}}, \quad T_{p^i} = T_p T_{p^{i-1}} - p^{k-1} T_{p^{i-2}},$$

where  $p$  is prime and  $i \geq 2$ . Hence the computation of  $T_n$  is reduced to that of the  $T_p$  for the primes  $p$  dividing  $n$ .

Let now  $p$  be a prime dividing  $n$ . Using a factorisation algorithm as in [Le-Le-Lo], we factor the  $\mathbb{Q}$ -algebra  $\mathbb{T}_{\mathbb{Q}} = \prod_i K_i$  as a product of fields. In fact, in all cases that we know of, the number of factors  $K_i$  is at most one (see [Fa-Ja]). For each  $i$ , let  $A_i$  be the image of  $\mathbb{T}$  in  $K_i$ . The  $A_i$  are computed as  $\mathbb{Z}$ -algebra, and the surjections  $\mathbb{T} \rightarrow A_i$  are described by the images of the  $T_j$  for  $j \leq r$ . These morphisms embed  $\mathbb{T}$  into  $\prod_i A_i$ . The computation of  $T_p$  is reduced to that of its images in the  $A_i$ .

Let  $K$  be one of the factors  $K_i$ , let  $A$  be  $A_i$ , let  $n_K := \dim_{\mathbb{Q}} K$  and let  $d_A := |\text{discr}(A)|$ . We define:

$$(15.2.11) \quad B_A := c \cdot ((\log d_A)^2 + n_K^2 + n_K \cdot k \cdot \log p),$$

for  $c$  a suitable absolute constant. For all primes  $l$  with:

$$6(k-1) < l < B_A \cdot (\log B_A)^4$$

we compute the maximal ideals of  $A/lA$ , and order them by their norms:  $\#(A/m_1A) \leq \#(A/m_2A)$ , etc. Then we take  $j$  minimal such that:

$$(15.2.12) \quad \#(A/m_1 \cdots m_j) > \left(2^{(n_K+1)/2} \cdot 2 \cdot p^{(k-1)/2}\right)^{n_K}.$$

For each of the  $m_i$  with  $i \leq j$ , we compute the Galois representation  $\rho_{m_i}$  as in Theorem 14.1.1, and a Frobenius element  $\rho_{m_i}(\text{Frob}_p)$  at  $p$  as described in Section 15.1. Then we have the images of  $T_p$  in all  $\mathbb{T}/m_i$  for  $1 \leq i \leq j$ .

We compute a  $\mathbb{Z}$ -basis  $e' = (e'_1, \dots, e'_{n_K})$  of  $m_1 \cdots m_j$ , starting from the  $\mathbb{Z}$ -basis of  $A$  that is part of the description of  $A$ , adding the congruence condition modulo the  $m_i$ 's one by one. The LLL-algorithm gives us an LLL-reduced basis  $e$  of  $m_1 \cdots m_j$ .

Let  $\overline{T}_p$  denote the image of  $T_p$  in  $\prod_{i \leq j} A/m_i$ . We compute a preimage  $T'_p$  in  $A$  of  $\overline{T}_p$ , adding the congruence conditions one by one. We compute  $T_p$  itself with the algorithm of Proposition 15.2.10.

### Correctness of the algorithm

We will now show that the computation works, and that it gives the correct result, if the constant  $c$  in (15.2.11) is large enough.

The first thing that we have to show is that there are distinct maximal ideals  $m_1, \dots, m_j$  of  $A$ , whose residue characteristics are between  $6(k-1)$  and  $B_A \cdot (\log B_A)^4$ , not equal to  $p$ , and such that (15.2.12) holds. We note that for each  $l$  in that range there is at least one maximal ideal  $m$  in  $A/lA$ , and that  $\#(A/m)$  is at least  $l$ . According to Corollary 10.1 in Section I.2.6 of [Ten], for sufficiently large real numbers  $x$ , the sum  $\sum_{l \leq x} \log(l)$  is at least  $x/2$ . So, indeed, taking all  $l$  between  $6(k-1)$  and a constant times  $n_K \cdot k \cdot \log p$  implies that the required  $m_i$  exist. We observe that the choice of  $B_A$  is sufficiently large for this.

The second point where an argument is needed is that the  $\rho_{m_i}$  are either reducible, or have image containing  $\mathrm{SL}_2(A/m_i)$ , so that they can be computed by the algorithm of Theorem 14.1.1. But this is guaranteed by Theorem 2.5.18.

The third point is that in the situation where we invoke the algorithm of Proposition 15.2.10, the assumptions of that proposition hold. That means that we must check that  $\|T_p\|$  is strictly less than  $2^{-(n_K+1)/2} \cdot \mu_1(m_1 \cdots m_j)$ . By Deligne's bound of [Del2], we know that  $\|T_p\| \leq \sqrt{n_K} \cdot 2 \cdot p^{(k-1)/2}$ . Lemma 15.2.9 says that  $\mu_1(m_1 \cdots m_j) \geq \sqrt{n_K} \cdot (\#(A/m_1 \cdots m_j))^{1/n_K}$ . Hence the required inequality follows from (15.2.12). This finishes the proof of the correctness of the algorithm.

### Running time analysis

We show the two claims on the running time. We also indicate at the appropriate places that the algorithm is deterministic.

Let  $k$ , and  $n = \prod_p p^{v_p(n)}$  be given. The computation of the Hecke algebra  $\mathbb{T} := \mathbb{T}(1, k)$ , i.e., of the products  $T_i T_j$  as  $\mathbb{Z}$ -linear combinations of the  $T_l$ , with  $1 \leq i, j, l \leq r$ , using modular symbols as in Chapter 8 of [Ste2], is deterministic and is done in time polynomial in  $k$ . Multiplication of two elements of  $\mathbb{T}$  can be done in time polynomial in  $k$  and the maximum of the heights of the coordinates of the elements with respect to the  $\mathbb{Z}$ -basis  $T = (T_1, \dots, T_r)$ . Lemma 4.2.7 and (15.2.6) give that the heights of the coordinates of  $T_m$  (with  $m \in \mathbb{Z}_{>0}$ ) with respect to  $T$  are  $O(k^3(\log(km)))$ . Hence the computation of  $T_n$  from the  $T_p$  for  $p$  dividing  $n$  is done in time polynomial in  $k$  and  $\log n$ . The number of primes dividing  $n$  is at most

$\log_2 n$ , and each of them is at most  $n$ . Let  $p$  be a prime dividing  $n$ .

Factorisation as in [Le-Le-Lo] is a deterministic polynomial time algorithm, hence we do get the factorisation  $\mathbb{T}_\mathbb{Q} = \prod_i K_i$  and the surjections  $\mathbb{T} \rightarrow A_i \subset K_i$  in time polynomial in  $k$ .

The number of factors  $K_i$  is at most  $r \leq k/12$ . Let  $A$  be one of the  $A_i$ . As the morphism  $\mathbb{T} \rightarrow \prod_i A_i$  is injective, we have:

$$\prod_i \text{discr}(A_i) \leq \text{discr}(\mathbb{T}) \leq (k^2 \log k)/12.$$

Therefore, the number  $B_A$  as in (15.2.11) is  $O(k^2 \cdot ((k \log k)^2 + \log n))$ .

For  $l$  a prime number, the computation of the maximal ideals of  $A/lA$  can be done deterministically in time polynomial in  $l$  and  $n_K$  as follows (compare with Algorithm 3.4.10 of [Coh], “Berlekamp for small primes”). Take  $i \in \mathbb{Z}_{\geq 1}$  such that  $l^i \geq n_K$ . The product  $I$  of the maximal ideals of  $A/lA$  is the subspace of  $x$  with  $x^{l^i} = 0$ . This can be computed with linear algebra over  $\mathbb{F}_l$ . One computes, again with linear algebra, the sub-algebra  $(A/lA)'$  of elements  $x$  with  $x^l = x$ . This algebra is isomorphic to  $\mathbb{F}_l^d$  for some  $d$ . If  $d > 1$ , then for any  $x$  in  $(A/lA)'$  that is not in  $\mathbb{F}_l$  there is an  $a$  in  $\mathbb{F}_l$  such that  $(x - a)^{l-1}$  is a non-trivial idempotent in  $(A/lA)'$ . Repeating this procedure splits  $(A/lA)'$  completely. The maximal ideal corresponding to an elementary idempotent  $\iota$  of  $(A/lA)'$  is then  $(1 - \iota)I$ .

We conclude that the computation of all maximal ideals of the  $A/lA$ , for all primes  $l$  with  $6(k - 1) < l < B_A \cdot (\log B_A)^4$  can be done deterministically in time polynomial in  $k$  and  $\log n$ . We can order them by their norms during their computation. Then we have  $m_1, \dots, m_j$  such that (15.2.12) holds. As  $\#(A/m_i) \geq 2$  for all  $i$ , we have:

$$j \leq n_K \cdot ((n_K + 3)/2 + (k - 1)(\log_2 p)/2) = O(k^2 \log n).$$

Let  $m$  be one of the  $m_i$ , where  $i \leq j$ . Then  $\#(A/m) \leq l^{n_K}$ , where  $l$  is the largest prime with  $l < B_A \cdot (\log B_A)^4$ . A rough estimate gives that there is an absolute constant such that:

$$\#(A/mA) \leq (c \cdot k^5 \log n)^{k/6}.$$

The running time for computing the Galois representation  $\rho_m$ , is therefore bounded by a fixed power of  $(c \cdot k^5 \log n)^{k/6}$ . For fixed  $k$ , this is of polynomial size in  $\log n$ . Assuming GRH for  $K$ , we get a much smaller estimate

for  $\#(A/m)$  as follows. We use the fact that if  $a$  is in  $\mathbb{R}_{>e}$ , and  $x > 2a \log a$ , then  $x/\log x > a$  (to prove this, use that  $x \mapsto x/\log x$  is increasing on  $\mathbb{R}_{>e}$ , and that  $e/\log e > 2$ ). We let  $x = B_A \cdot (\log B_A)^4$ . Then  $x$  satisfies the two hypotheses of Corollary 15.2.8 (assuming the constant  $c$  of (15.2.11) large enough with respect to  $c_2$  and  $c_3$ ), and therefore (here we use GRH!)  $\pi(x, K) \geq x/2 \log x$ . But then  $\pi(x, K) \geq B_A/2$ , i.e., there are at least  $B_A/2$  maximal ideals of  $O_K$  with  $\#(O_K/m) \leq x$ . Now  $O_K/A$  is finite, of order at most  $d_A^{1/2}$ , hence there are at most  $(1/2) \cdot \log_2 d_A$  distinct primes dividing  $\#(O_K/A)$ , and hence at most  $(n_K/2) \cdot \log_2 d_A$  maximal ideals of  $A$  where  $A$  and  $O_K$  differ. Similarly, there are at most  $6(k-1)n_K$  maximal ideals of  $A$  with residue characteristic at most  $6(k-1)$ . The definition of  $B_A$  in equation (15.2.11) implies that:

$$\frac{x}{2 \log x} - \frac{n_K}{2 \log_2 d_A} - 6kn_K > n_K \left( \frac{n_K + 1}{2} + 1 + \frac{k-1}{2} \log_2 p \right),$$

assuming the absolute constant  $c$  large enough. Let  $I$  be the product of the maximal ideals  $m$  in  $A$  with  $\#(A/m) \leq x$  and with residue characteristic at least  $6(k-1)$ . Using that for all  $m$  containing  $I$  we have  $\#(A/m) \geq 2$  we get:

$$\#(A/I) > \left( 2^{(n_K+1)/2} \cdot 2 \cdot p^{(k-1)/2} \right)^{n_K}.$$

The definition of  $j$ , see (15.2.12), implies that for every  $i \leq j$ , we have  $\#(A/m_i) \leq x$ . As we have already noticed above,  $x$  is of size polynomial in  $k$  and  $\log n$ . Summarising: if we assume GRH for  $K$ , then running time for the computation of  $\rho_m$  is polynomial in  $k$  and  $\log n$ .

The computation of a Frobenius element  $\rho_m(\text{Frob}_p)$ , given  $\rho_m$ , is deterministic and has running time polynomial in  $k$  and  $\log n$ ; see the proof of Theorem 15.1.1.

At this point we have computed the images of  $T_p$  in the  $A/m_i$ , for all  $i \leq j$ . The computation of a  $\mathbb{Z}$ -basis  $e'$  of  $m_1 \cdots m_j$  can be done in time polynomial in  $k$  and  $\log n$ . One starts with the basis of  $A$  that is part of its description, and computes successively  $\mathbb{Z}$ -bases of  $m_1$ ,  $m_1 m_2$ , etc. At the  $i$ th step, the maximum of the absolute values of the coordinates of the elements of our temporary basis with respect to the basis of  $A$  gets at most  $n_K l_i$  times larger, where  $l_i$  is the characteristic of  $A/m_i$ . It follows that the

absolute values of the coordinates of the  $e'_i$  with respect to the basis of  $A$  are at most  $n_K^j \cdot \#(A/m_1 \cdots m_j)$ , which is of size polynomial in  $k$  and  $\log n$ .

The LLL -algorithm gives us an LLL-reduced basis  $e$  of  $m_1 \cdots m_j$  in time polynomial in  $k$  and  $\log n$ , see [Le-Le-Lo], [Len3] or [Coh, §2.6].

A preimage  $T'_p$  in  $A$  of the image  $\overline{T_p}$  of  $T_p$  in  $\prod_{i \leq j} A/m_i$  can be computed in time polynomial in  $k$  and  $\log n$  as follows. One lifts the image of  $T_p$  in  $A/m_1$  to an element  $T_{p,1}$  of  $A$ , with small coordinates with respect to the  $\mathbb{Z}$ -basis of  $A$ , then one adjusts  $T_{p,1}$  with an element of  $m_1$  with small coordinates with respect to the  $\mathbb{Z}$ -basis that was already computed, to get the correct image in  $A/m_2$ , etc. The size of the coordinates of  $T'_p$  with respect to the  $\mathbb{Z}$ -basis of  $A$  is polynomial in  $k$  and  $\log n$ .

The algorithm of Proposition 15.2.10 then computes  $T_p$  for us, in time polynomial in  $k$  and  $\log n$ .  $\square$

**Proof** [of Corollary 15.2.2] Let  $k \geq 0$ , and let  $f$  be in  $M_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$ . We are given the integers  $k$  and the  $a_i(f)$  for  $0 \leq i \leq k/12$ , and a positive integer  $n$  together with its factorisation into primes. We are to compute  $a_n(f)$ .

If  $M_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$  is zero, then so is  $f$ , and we have  $a_n(f) = 0$ . If  $k = 0$ , then  $f$  is a constant power series, hence  $a_n(f) = 0$ . So, we may and do assume that  $k \geq 4$ , and that  $k$  is even.

Let  $E_k$  be the Eisenstein form of weight  $k$  and level one normalised as in Example 2.2.3. Note that the coefficients  $a_i(E_k)$  are rational numbers, not necessarily integers. Let  $g := f - a_0(f) \cdot E_k$  in  $S_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q})$ . Then we have:

$$f = a_0(f) \cdot E_k + g, \quad \text{hence} \quad a_n(f) = a_0(f)a_n(E_k) + a_n(g).$$

As the factorisation of  $n$  into primes is given, we can compute the coefficient:

$$a_n(E_k) = -\frac{2k}{B_k} \sigma_{k-1}(n) = -\frac{2k}{B_k} \prod_{p|n} \left(1 + p^{k-1} + \cdots + p^{(k-1)v_p(n)}\right)$$

in time polynomial in  $k$  and  $\log n$ . Hence it remains to compute  $a_n(g)$ .

Let  $\mathbb{T}$  denote the Hecke algebra  $\mathbb{T}(1, k)$  acting on  $S_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$ . Recall that its rank is at most  $k/12$ . Via the perfect pairing from (2.4.9):

$$S_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z}) \times \mathbb{T} \longrightarrow \mathbb{Z}, \quad (h, t) \mapsto a_1(th)$$

we view  $g$  as the element in  $\text{Hom}_{\mathbb{Z}\text{-Mod}}(\mathbb{T}, \mathbb{Q})$  that sends  $t$  to  $a_1(tg)$ . We then have, by (2.2.12):

$$a_r(g) = g(T_r) \quad \text{for all } r \geq 1.$$

The  $T_i$  with  $1 \leq i \leq \text{rank}(\mathbb{T})$  form a  $\mathbb{Z}$ -basis for  $\mathbb{T}$ , by Proposition 15.2.3. The products  $T_i T_j$  of these basis elements can be computed in time polynomial in  $k$ .

The coefficients  $a_i(E_k)$  for  $0 \leq i \leq k/12$  can be computed in time polynomial in  $k$  from the standard formulas as given in Example 2.2.3. The coefficients  $a_i(f)$  for  $0 \leq i \leq k/12$  are given. Hence the  $g(T_i) = a_i(g)$  with  $1 \leq i \leq \text{rank}(\mathbb{T})$  are computed in time polynomial in  $k$  and the maximum of the  $\log(1 + |a_i(f)|)$  with  $0 \leq i \leq k/12$ . Theorem 15.2.1 says that  $T_n$  can be computed as  $\mathbb{Z}$ -linear combination of the  $T_i$  with  $1 \leq i \leq \text{rank}(\mathbb{T})$ , in time polynomial in  $k$  and  $\log n$ . Applying  $g$ , viewed as element of  $\text{Hom}_{\mathbb{Z}\text{-Mod}}(\mathbb{T}, \mathbb{Q})$ , to this linear combination gives  $a_n(g) = g(T_n)$ , in the required time.  $\square$

### 15.3 AN APPLICATION TO QUADRATIC FORMS

In this section we apply our results on the computation of coefficients of modular forms to theta functions of even, unimodular positive definite quadratic forms over  $\mathbb{Z}$ . According to Corollary 2.3.6, such a theta function is a modular form on  $\text{SL}_2(\mathbb{Z})$ , of weight equal to half the dimension of the space of the quadratic form.

In view of Example 2.3.10, Theorem 15.1.1 has the following consequence.

**15.3.1 Theorem** *There is a deterministic algorithm that on input an integer  $m > 0$  together with its factorisation into primes gives the representation number  $r_L(m)$  where  $L$  is the Leech lattice, in running time polynomial in  $\log m$ .*

Let us now turn to arbitrary even unimodular lattices. We can then apply Corollary 15.2.2 to prove the following result.



**15.3.2 Theorem** *Assume GRH. There is a deterministic algorithm that, on input the rank  $n_L$  and the integers  $r_L(i)$  for  $1 \leq i \leq n_L/24$  of an even unimodular lattice  $(L, b)$ , and an integer  $m > 0$  together with its factorisation into primes, computes  $r_L(m)$  in running time polynomial in  $n_L$  and  $\log(m)$ .*

**Proof** Let  $n_L$  and the  $r_L(i)$  for  $1 \leq i \leq n_L/24$  be given. As  $L$  is even and unimodular,  $n_L$  is even (in fact, it is a multiple of 8, see [Ser5, VII, §6]). Let  $k := n_L/2$ . Then  $\theta_L$  is in  $M_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$ . The coefficients  $a_i(\theta_L) = r_L(2i)$  with  $0 \leq i \leq k/12$  are given to us. The fact that the open balls in  $L_{\mathbb{R}}$  centered at the  $x \in L$  and with radius  $2^{-1/2}$  do not overlap gives that for  $i$  with  $0 \leq i \leq k/12$  we have  $\log(1 + r_L(i)) \leq k \log(1 + k/6)$ . Corollary 15.2.2 then means that  $r_L(m) = a_{m/2}(\theta_L)$  can be computed in the required time.  $\square$

Let us point out that Theorem 15.3.2 can be applied to the orthogonal direct sum  $(L, b)$  of even unimodular lattices  $(L_i, b_i)$  for which the  $r_{L_i}(j)$  can be computed in time polynomial in  $n_{L_i}$  and  $j$ . Indeed:

$$\theta_{L_1} \cdots \theta_{L_r} = \sum_{x \in L} q^{(b_1(x_1, x_1) + \cdots + b_r(x_r, x_r))/2} = \sum_{x \in L} q^{b(x, x)/2} = \theta_L.$$

The coefficients  $a_i(\theta_L)$  with  $0 \leq i \leq n_L/24$  can be computed in time polynomial in  $n_L$  by computing the product of the images of the  $\theta_{L_i}$  in  $\mathbb{Z}[[q]]/(q^{n_L/24+1})$ .

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## *Epilogue*

Theorems 14.1.1 and 15.2.1 will certainly be generalised to spaces of cusp-forms of arbitrarily varying level and weight. This is currently being worked out for the probabilistic variants, and at least for square free levels (and for level two times a square free number for reasons that will become clear below). We describe some consequences of this current work. The reader will understand what the consequences are of deterministic generalisations to arbitrary levels and weights.

### **Computation of Galois representations**

Peter Bruin will give, in his forthcoming PhD thesis (Summer 2010), a probabilistic algorithm that on input positive integers  $k$  and  $N$ , with  $N$  square free, and a surjective morphism  $f: \mathbb{T}(2N, k) \rightarrow \mathbb{F}$  to a finite field, computes  $\rho_f$  with probabilistic running time polynomial in  $k$ ,  $N$  and  $\#\mathbb{F}$ . An important new ingredient in his method, suggested by Couveignes, is to be more flexible concerning the choice of the divisor  $D_0$  of Section 8.2. His algorithm computes, for each  $x$  in  $V$ , the smallest integer  $d_x$  such that  $h^0(X_1(2N)_{\overline{\mathbb{Q}}}, \mathcal{L}_x(d_x \cdot \infty)) = 1$ , and then represents  $x$  by the unique effective divisor  $D_x$  of degree  $d_x$  on  $X_1(2N)_{\overline{\mathbb{Q}}}$  such that  $x = [D_x - d_x \cdot \infty]$  in  $J_1(2N)(\overline{\mathbb{Q}})$ . This generalises the probabilistic variant of Theorem 14.1.1 that is described in Section 14.8.

### **Computing coefficients of modular forms**

Concerning Theorem 15.2.1, Peter Bruin will obtain a probabilistic algorithm that, assuming GRH, on input positive integers  $k$ ,  $N$  and  $n$ , with  $N$  squarefree, together with the factorisation of  $n$  into prime factors, the element  $T_n$  of the Hecke algebra  $\mathbb{T}(2N, k)$  in probabilistic running time poly-

nomial in  $k$ ,  $N$  and  $\log n$ .

### Lattices, sums of squares

This last result, applied to theta functions of lattices as in Section 15.3, will have the following remarkable consequence that, assuming GRH, there is a probabilistic algorithm that on input positive integers  $k$  and  $n$ , together with the factorisation of  $n$  into prime factors, computes the number:

$$r_{\mathbb{Z}^{2k}}(n) = \#\{x \in \mathbb{Z}^{2k} \mid x_1^2 + \cdots + x_{2k}^2 = n\}$$

in time polynomial in  $k$  and  $\log n$ .

For some small values of  $k$  there are well known explicit formulas for the representation numbers  $r_{\mathbb{Z}^{2k}}(n)$ , owing their existence to the fact that there are no non-CM cuspidal eigenforms on  $\Gamma_1(4)$  of weight  $k$ . In order to give these formulas, let  $\chi: \mathbb{Z} \rightarrow \mathbb{C}$  be the map obtained from the character  $(\mathbb{Z}/4\mathbb{Z})^\times = \{1, -1\} \subset \mathbb{C}^\times$ , by extending it by zero to  $\mathbb{Z}/4\mathbb{Z}$  and composing the result with the reduction map  $\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z}$ . Then we have:

$$\begin{aligned} r_{\mathbb{Z}^2}(n) &= 4 \sum_{d|n} \chi(d), \\ r_{\mathbb{Z}^4}(n) &= 8 \sum_{2 \nmid d|n} d + 16 \sum_{2 \nmid d|(n/2)} d, \\ r_{\mathbb{Z}^6}(n) &= 16 \sum_{d|n} \chi(n/d) d^2 - 4 \sum_{d|n} \chi(d) d^2, \\ r_{\mathbb{Z}^8}(n) &= 16 \sum_{d|n} d^3 - 32 \sum_{d|(n/2)} d^3 + 256 \sum_{d|(n/4)} d^3, \\ r_{\mathbb{Z}^{10}}(n) &= (4/5) \sum_{d|n} \chi(d) d^4 + (64/5) \sum_{d|n} \chi(n/d) d^4 \\ &\quad + (8/5) \sum_{d \in \mathbb{Z}[i], |d|^2=n} d^4. \end{aligned}$$

For the history of these formulas, featuring, among others, Fermat, Euler, Lagrange, Legendre, Gauss, Jacobi and Liouville, we refer the reader to [Mil2] and to Chapter 20 of [Ha-Wr].

In her forthcoming masters thesis (Summer 2010), Ila Varma will show that there are no other values of  $k$  for which the theta function  $\theta_{\mathbb{Z}^{2k}} = \theta_{\mathbb{Z}}^{2k}$ ,

where  $\mathbb{Z}^{2k}$  is equipped with the standard inner product, is a linear combination of Eisenstein series and cuspforms coming from Hecke characters.

We conclude that, even in the absence of formulas as above, one will be able to compute the numbers  $r_{\mathbb{Z}^{2k}}(n)$  as fast as if such formulas existed. Often, theta functions are considered to be modular forms of which the coefficients are easy to compute, and hence useful for computing Hecke operators and coefficients of eigenforms. However, it seems that for coefficients  $a_n(f)$  with  $n$  large, this will be the other way around, from now on.

### Point counting on modular curves

Another consequence of Peter Bruin's results mentioned above is that, again assuming GRH, there will be a probabilistic algorithm that on input a positive square free number  $n$  and a finite field  $\mathbb{F}_q$  computes the number  $\#X_1(n)(\mathbb{F}_q)$  in time polynomial in  $n$  and  $\log q$ . Indeed, this is a matter of computing the element  $T_p$  (where  $p$  is the prime dividing  $q$ ) in the Hecke algebra  $\mathbb{T}(n, 2)$  acting on the space  $S_2(\Gamma_1(n), \mathbb{Z})$ .

### Point counting in a more general context

The methods that we have used in this book can also be tried outside the context of modular forms. Let us consider, for example, a smooth surface  $S$  in  $\mathbb{P}_{\mathbb{Q}}^3$  of degree at least 5, say. Letting  $l$  be a prime number, one has the cohomology groups  $H^i(S_{\overline{\mathbb{Q}}, \text{et}}, \mathbb{F}_l)$  for  $0 \leq i \leq 4$ , being finite dimensional  $\mathbb{F}_l$ -vector spaces with  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -action. It seems reasonable to suspect that, again, there is an algorithm that on input a prime  $l$  computes these cohomology groups, with their  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -action, in time polynomial in  $l$ . Once such an algorithm is known, one also has an algorithm that, on input a prime  $p$  of good reduction of  $S$ , gives the number  $\#S(\mathbb{F}_p)$  of  $\mathbb{F}_p$ -valued points of  $S$  in time polynomial in  $\log p$ . This result would be of interest because the known  $p$ -adic algorithms for finding such numbers have running time exponential in  $\log p$ .

In this case, we choose a Lefschetz fibration from a blow-up of  $S$  to  $\mathbb{P}_{\mathbb{Q}}^1$ , and use the derived direct images of the constant sheaf  $\mathbb{F}_{l,S}$  under this fibration to compute the étale cohomology of  $S_{\overline{\mathbb{Q}}}$  with  $\mathbb{F}_l$ -coefficients. The most complicated contribution then comes from the first derived image  $\mathcal{F}_l$ ,

which is a locally constant sheaf of  $\mathbb{F}_l$ -vector spaces of dimension  $r$ , say, with  $r$  independent of  $l$ , on the open part  $U$  of  $\mathbb{P}_{\mathbb{Q}}^1$  over which the fibration is smooth. This open part  $U$  is independent of  $l$  as well, and it is the analog of the open part of the  $j$ -line over which all modular curves are unramified.

For each  $l$  let  $V_l := \underline{\text{Isom}}_U(\mathbb{F}_l^r, \mathcal{F}_l)$ . These  $V_l$  play the role of the covers  $X_1(l)$  of the  $j$ -line, as, by definition, they trivialise the sheaves  $\mathcal{F}_l$ . Each cover  $V_l \rightarrow U$  is finite Galois with group  $G = \text{GL}_r(\mathbb{F}_l)$ , and  $H^1(U_{\overline{\mathbb{Q}}, \text{et}}, \mathcal{F}_l)$  is closely related to  $H^1(V_{l, \overline{\mathbb{Q}}, \text{et}}, \mathbb{F}_l^r)$  which sits in the  $l$ -torsion of the Jacobian of the smooth projective model  $\overline{V}_l$  of  $V_l$ . It is our hope that methods as in this book (height bounds, approximations) can show that there is a polynomial algorithm for computing these cohomology groups.

In [Ed-dJ-Sc] a first step in this program is taken, by proving, in the function field case, that, for varying  $l$ , the height of  $\overline{V}_l$  is bounded by a fixed power of  $l$ .

### Modular forms of half integral weight

Much to our regret, we have nothing to say about modular forms of half integer weight. Nevertheless, it would be very interesting to be able to compute coefficients of such modular forms, as they encode interesting arithmetic information, such as class numbers.

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