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MINIMALITY, (WEIGHTED) INTERPOLATION IN PALEY-WIENER SPACES & CONTROL THEORY

FRÉDÉRIC GAUNARD

Abstract. It is well known from a result by Shapiro-Shields that in the Hardy spaces, a sequence of reproducing kernels is uniformly minimal if and only if it is an unconditional basis in its span. This property which can be reformulated in terms of interpolation and so-called weak interpolation is not true in Paley-Wiener spaces in general. Here we show that the Carleson condition on a sequence Λ together with minimality in Paley-Wiener spaces PW^p_τ of the associated sequence of reproducing kernels implies the interpolation property of Λ in $PW^p_{\tau+\epsilon}$, for every $\epsilon > 0$. With the same technics, using a result of McPhail, we prove a similary result about minimlity and weighted interpolation in $PW^p_{\tau+\epsilon}$.. We apply the results to control theory, establishing that, under some hypotheses, a certain weak type of controllability in time $\tau > 0$ implies exact controllability in time $\tau + \epsilon$, for every $\epsilon > 0$.

1. Introduction

Let X be a Banach space. A sequence $\{\phi_n\}_{n\geq 1}$ of vectors of X is said to be *minimal* in X if $\phi_n \notin \bigvee_{k\neq n} \phi_k := \overline{\text{span}}^X (\phi_k : k \neq n)$, $n \geq 1$, and *uniformly minimal* if moreover

$$\inf_{n\geq 1} \text{dist}\left(\frac{\phi_n}{\|\phi_n\|}, \bigvee_{k\neq n} \phi_k\right) > 0. \tag{1.1}$$

It is well known (see e.g. [Ni02a, p. 93]) that minimality of $\{\phi_n\}_{n\geq 1}$ in X is equivalent to the existence of a sequence $\{\psi_n\}_{n\geq 1} \subset X^\star$ such that $\langle \phi_n, \psi_k\rangle = \delta_{nk}$ and the minimality is said uniform if and only if

$$\sup_{n\geq 1} \|\phi_n\| \cdot \|\psi_n\| < \infty. \tag{1.2}$$

We consider the case where X is a Banach space of analytic functions on a domain Ω. Let $\Lambda = \{\lambda_n\}_{n\geq 1}$ be a sequence of complex numbers

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lying in Ω. We use the terminology *minimal* also for the sequence Λ if there exists a sequence of functions $(f_n)_{n\geq 1}$ of X such that

$$f_n(\lambda_k) = \delta_{nk}, \quad n, k \geq 1,$$

and we say that Λ is a *weak interpolating sequence* in X, which is denoted by $\Lambda \in \mathrm{Int}_w(X)$, if there exists a sequence of functions $(f_n)_{n\geq 1}$ of X such that

$$f_n(\lambda_k) = \delta_{nk} \|k_{\lambda_n}\|_{X^\star}, \ n \geq 1, \text{ and } \sup_{n\geq 1} \|f_n\| < \infty. \tag{1.3}$$

When X is reflexive, this is equivalent to the fact that $\mathcal{K}(\Lambda)$ is uniformly minimal in $X^\star$. Such a sequence Λ could also be called a uniformly minimal sequence in X but we prefer to keep the existing terminology of weak interpolating sequence.

In the case where $X = H^p(\mathbb{C}_a^\pm)$, $1 < p < \infty$, the Hardy space of the half-plane $\mathbb{C}_a^\pm$, we can identify

$$\left(H^p\left(\mathbb{C}_a^\pm\right)\right)^\star \simeq H^q\left(\mathbb{C}_a^\pm\right), \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

and it is known that the reproducing kernel at $\lambda \in \mathbb{C}_a^\pm$ is given by $k_{\lambda_n}(z) = \frac{i}{2\pi}\left(z - \overline{\lambda}_n\right)^{-1}$. We have the estimate

$$\|k_{\lambda_n}\|_{H^q\left(\mathbb{C}_a^\pm\right)} \asymp |\mathrm{Im}(\lambda_n) - a|^{-\frac{1}{p}}.$$

From factorization in Hardy spaces, it can be deduced that the condition (1.3) is equivalent to the so-called *Carleson condition*

$$\inf_{n\geq 1} \prod_{k\neq n} \left| \frac{\lambda_n - \lambda_k}{\lambda_n - \overline{\lambda}_k - 2ia} \right| > 0, \tag{1.4}$$

and by the above observations, this is equivalent to $\mathcal{K}(\Lambda)$ being uniformly minimal in $H^q(\mathbb{C}_a^\pm)$.

In this paper, the spaces X that we consider will only be the Hardy spaces $H^p(\mathbb{C}_a^\pm)$ or the Paley-Wiener spaces PW_τ^p to be defined later.

We say that Λ is an *interpolating sequence* for $X = H^p(\mathbb{C}_a^\pm)$ or PW_τ^p, which is denoted by $\Lambda \in \mathrm{Int}(X)$, if for each sequence $a = (a_n)_{n\geq 1} \in l^p$, there is a fonction $f \in X$ such that

$$f(\lambda_n) = a_n \|k_{\lambda_n}\|_{X^\star}, \quad n \geq 1, \tag{1.5}$$

and a *complete interpolating sequence* for X ($\Lambda \in \mathrm{Int}_c(X)$) if the function satisfying (1.5) is unique. We will give the explicit formula of k_{λ_n} for PW_τ^p and an estimate of $\|k_{\lambda_n}\|_{PW_\tau^q}$ in the next section.

A famous result by Shapiro and Shields ([SS61]) states that in $H^p(\mathbb{C}_a^\pm)$, the Carleson condition (1.4) for a sequence Λ is equivalent to the interpolation property of Λ. It is also known that $\Lambda \in \mathrm{Int}(H^p(\mathbb{C}_a^\pm))$

if and only if $\mathcal{K}(\Lambda)$ is an unconditional basis (or, for $p = 2$, a Riesz basis) in its span in $H^q(\mathbb{C}_a^{\pm})$. We refer to [Ni02b, Section C, Chapter 3] for definitions and details. It appears that in the Hardy spaces, the uniformly minimal sequences are exactly the unconditional sequences.

This property of equivalence between uniform minimality and unconditionnality is not isolated. It turns out to be true in the Bergmann space ([ScS98]), in the Fock spaces and in the Paley Wiener spaces for certain values of p ([ScS00]).

In [AH10], the authors show that uniform minimality implies unconditionality in a bigger space for certain backward shift invariant spaces $K_I^p := H^p \cap I\overline{H_0^p}$ (considered here on the unit circle $\mathbb{T}$) for which the Paley-Wiener spaces are a particular case. We will use here a different approach to obtain a stronger result. More precisely, considering the unit disk, the authors of that paper increase the size of the space K_I^p in two directions: K_J^s, where $s < p$ and J is an inner multiple of I. In our situation of the Paley-Wiener space PW_τ^p, which is isometric to $K_{I_{\mathbb{D}}^\tau}^p$, $I_{\mathbb{D}}^\tau(z) = \exp(2\tau(z+1)/(z-1))$ for $z \in \mathbb{D}$, we still increase the size of the space by taking an inner multiple of $I_{\mathbb{D}}^\tau$ but we keep the same p.

We have already mentioned that Λ is an interpolating sequence for the Hardy space H^p, $1 < p < \infty$, if and only if the sequence $\mathcal{K}(\Lambda)$ is an unconditional basis in its span in H^q (see e.g. [Ni02b] or [Se04]). Hence, the result of Shapiro and Sheilds implies that weak interpolation is equivalent to interpolation in Hardy spaces. A characterization of complete interpolating sequences in PW_τ^p obtained by Lyubarskii and Seip ([LS97]) (involing in particular Carleson's condition and the Muckenhoupt (A_p) condition on the generating function of Λ) implies that Paley-Wiener spaces do not have this property.

Indeed, as shown in [ScS00], the sequence $\Lambda = \{\lambda_n : n \in \mathbb{Z}\}$ defined by

$$\lambda_0 = 0, \quad \lambda_n = n + \frac{\operatorname{sign}(n)}{2\max(p,q)},\ n \in \mathbb{Z} \setminus \{0\},$$

is weak interpolating in PW_π^p and complete, but does not satisfy the conditions of Lyubarskii-Seip's result, and so, Λ is not a (complete) interpolating sequence in PW_π^p. Nevertheless, as we will discuss in Subsection 2.1, a density argument (see [Se95]) allows to show that this sequence is actually an interpolating sequence in a bigger space, i.e. in $PW_{\pi+\epsilon}^p$, for arbitrary $\epsilon > 0$. This is a special case of the main result of this paper.

Theorem 1. *Let $\tau > 0$, $1 < p < \infty$ and Λ be a minimal sequence in PW_τ^p such that for every $a \in \mathbb{R}$, $\Lambda \cap \mathbb{C}_a^{\pm}$ satisfies the Carleson condition (1.4). Then, for every $\epsilon > 0$, Λ is an interpolating sequence in $PW_{\tau+\epsilon}^p$.*

It should be emphasized that surprisingly, we do not need to require uniform minimality here. The Carleson condition allows in a way to compensate this lack of uniformity. As a consequence of this result, we will see that if $\Lambda \in \mathrm{Int}_w\left(PW_\tau^p\right)$, then, for every $\epsilon > 0$, $\Lambda \in \mathrm{Int}\left(PW_{\tau+\epsilon}^p\right)$.

Finally, we recall that a positive measure σ on some half-plane $\mathbb{C}_a^\pm$ is called a *Carleson measure* in $\mathbb{C}_a^\pm$ if

$$\sup_Q \frac{\sigma(Q)}{h} < \infty, \tag{1.6}$$

where the supremum is taken over all the squares Q of the form

$$Q = \left\{z = x + iy \in \mathbb{C}_a^\pm : \ x_0 < x < x_0 + h, \ |y - a| < h\right\},$$

for $x_0 \in \mathbb{R}$ and $h > 0$. It is well known from a result of Carleson (see e.g. [Ga81, pp. 61 and 278]) that (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3), where

(1) The sequence $\Lambda = \{\lambda_n : n \geq 1\} \subset \mathbb{C}_a^\pm$ satisfies the Carleson condition (1.4);

(2) The measure

$$\sigma_\Lambda := \sum_{n\geq 1} |\mathrm{Im}(\lambda_n) - a| \, \delta_{\lambda_n}$$

is a Carleson measure in $\mathbb{C}_a^\pm$;

(3) For every $f \in H^p\left(\mathbb{C}_a^\pm\right)$,

$$\int |f|^p \, d\sigma_\Lambda \lesssim \|f\|_p^p .$$

It is also known that (2) or (3) together with the *uniform pseudo-hyperbolic separation* of Λ in $\mathbb{C}_a^\pm$, which is

$$\inf_{n\neq m} \left| \frac{\lambda_n - \lambda_m}{\lambda_n - \overline{\lambda}_m - 2ia} \right| > 0, \tag{1.7}$$

imply (1). Moreover, if Λ lies in a strip of finite width, *i.e.* $M := \sup_n |\mathrm{Im}(\lambda_n)| < \infty$, the Carleson condition (1.4) in $\mathbb{C}_{\mp 2M}^\pm$ is equivalent to the *uniform separation condition*

$$\inf_{n\neq m} |\lambda_n - \lambda_m| > 0 \tag{1.8}$$

which is, in this case, equivalent to the uniform pseudo-hyperbolic separation since the pseudo-hyperbolic metric defined in $\mathbb{C}_{\mp 2M}^\pm$ by

$$\rho(\lambda, \mu) = \left| \frac{\lambda - \mu}{\lambda - \overline{\mu} - 2iM} \right|$$

is locally equivalent to the euclidian distance.

This paper is organized as follows. The next section will be devoted to interpolation in Paley-Wiener spaces. After having recalled some

properties of these spaces, we discuss links between density and interpolation (in the case of the sequence Λ lying in a strip of finite width) and prove our main result and some consequences.

In the third section, we define and discuss weighted interpolation. Indeed, after having defined weighted interpolation in Hardy and Paley-Wiener spaces, we use a result of McPhail ([McP90]) characterizing the weighted interpolation sequences in $H^p\left(\mathbb{C}_a^{\pm}\right)$ and technics of Theorem 1 to prove that a minimal sequence in PW_τ^p such that its intersection with every half-plane satisfies the McPhail condition is a weighted interpolation sequence in $PW_{\tau+\epsilon}^p$, for every $\epsilon > 0$.

This theorem will be used in the fourth and last section where we consider controllability of linear differential systems, establishing a link between exact and a certain weak type of controllability.

2. Interpolation in Paley-Wiener Spaces

We begin by recalling some facts about Paley-Wiener spaces. For $\tau > 0$, the Paley-Wiener space PW_τ^p consists of all entire functions of exponential type at most τ satisfying

$$\|f\|_p^p = \int_{-\infty}^{+\infty} |f(t)|^p \, dt < \infty.$$

A result, known as Plancherel-Polya inequality (see e.g. [Le96] or [Se04, p.95]) states that if $f \in PW_\tau^p$, then

$$\int_{-\infty}^{+\infty} |f(x+iy)|^p \, dx \leq e^{p\tau|y|} \, \|f\|_p^p \,. \tag{2.1}$$

In particular, it follows that for every $f \in PW_\tau^p$, $z \mapsto e^{\pm i\tau z} f(z)$ belongs to $H^p\left(\mathbb{C}_a^{\pm}\right)$, with same norm as f. It also follows that translation is an isomorphism from PW_τ^p into itself.

Using Cauchy's formula and Cauchy's theorem in an appropriate way, we see that

$$k_\lambda(z) = \frac{\sin \tau \left(z - \overline{\lambda}\right)}{\tau \left(z - \overline{\lambda}\right)}$$

is the reproducing kernel of PW_τ^p associated to λ and we obtain the norm estimate

$$\|k_\lambda\|_{PW_\tau^q} \asymp \left(1 + |\mathrm{Im}(\lambda)|\right)^{-\frac{1}{p}} e^{\tau|\mathrm{Im}(\lambda)|}.$$

This implies a useful pointwise estimate; recalling that for $\frac{1}{p} + \frac{1}{q} = 1$, $(PW_\tau^p)^\star \simeq PW_\tau^q$, we deduce that there exists a constant $C = C(p)$ such that for every $f \in PW_\tau^p$, we have

$$|f(z)| \leq C \, \|f\|_p \left(1 + |\mathrm{Im}\,(z)|\right)^{-\frac{1}{p}} e^{\tau|\mathrm{Im}(z)|}, \quad z \in \mathbb{C}. \tag{2.2}$$

The Paley-Wiener theorem states that

$$L^2(0,2\tau) \simeq \mathcal{F}L^2(-\tau,\tau) = PW_\tau^2,$$

where $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}\phi(z) = \int_{-\tau}^{\tau} \phi(t)e^{-itz}dt.$$

Hence, another approach to interpolation problems in PW_τ^2 is to consider geometric properties of exponentials in $L^2(0,2\tau)$, which is a famous problem with several applications, see e.g. [HNP81] or [AI95].

From the definitions given previously, a sequence Λ is interpolating in PW_τ^p if, for every sequence $a = (a_n)_{n\geq 1} \in l^p$, it is possible to find a function $f \in PW_\tau^p$ such that

$$f(\lambda_n)\left(1+|\mathrm{Im}(\lambda_n)|\right)^{\frac{1}{p}} e^{-\tau|\mathrm{Im}(\lambda_n)|} = a_n, \quad n \geq 1. \tag{2.3}$$

The condition of weak interpolation for Λ in PW_τ^p can be reformulated as the existence of a sequence of functions $(f_n)_{n\geq 1} \subset PW_\tau^p$ such that

$$f_n(\lambda_k)\left(1+|\mathrm{Im}(\lambda_n)|\right)^{\frac{1}{p}} e^{-\tau|\mathrm{Im}(\lambda_n)|} = \delta_{nk}, \quad n \geq 1,$$

and $\sup_{n\geq 1}\|f_n\| < \infty$.

In particular, if $\Lambda \in \mathrm{Int}_w(PW_\tau^p)$, then the Plancherel-Polya inequality (2.1) implies that the sequence $(e^{\pm i\tau\cdot}f_n)_{n\geq 1}$is in $H^p(\mathbb{C}_a^\pm)$, with uniform control of the norm. So, it is easy to see that $\Lambda \cap \mathbb{C}_a^\pm \in \mathrm{Int}_w\left(H^p\left(\mathbb{C}_{a\pm\eta}^\pm\right)\right)$, for every $\eta > 0$, and hence satisfies the Carleson condition in the corresponding half-plane, in view of Shapiro-Shields 's theorem. Moreover, we can affirm that the sequence

$$\Lambda_{a,\eta} := \Lambda \cap \{z \in \mathbb{C} : \ 0 < |\mathrm{Im}(z) - a| < 2\eta\}$$

is uniformly separated, in view of the discussion in the end of the previous section. These two observations imply the following result (for more details, see [Gau11]).

Proposition 2. *If Λ is a weak interpolating sequence in PW_τ^p, then, for every $a \in \mathbb{R}$, the sequence $\Lambda \cap \mathbb{C}_a^\pm$ satisfies the Carleson condition (1.4).*

2.1. **Upper Uniform Density and Interpolation.** In this subsection, we assume that the sequence Λ satisfies

$$M := \sup_{n\geq 1}|\mathrm{Im}(\lambda_n)| < \infty, \tag{2.4}$$

which means that Λ lies in a strip of finite width parallel to the real axis. We define the *upper uniform density* $\mathcal{D}_\Lambda^+$ by

$$\mathcal{D}_\Lambda^+ := \lim_{r\to\infty} \frac{n_\Lambda^+(r)}{r},$$

where

$$n_\Lambda^+(r) := \sup_{x\in\mathbb{R}} |\mathrm{Re}(\Lambda) \cap [x, x+r]|,$$

counting multiplicities.

The reader would have remembered that, from previous remarks, Proposition 2 implies that a weak interpolating sequence in a Paley-Wiener space PW_τ^p satisfies the uniform separation condition.

The next theorem is stated as follows in a paper of Seip ([Se95, Theorem 2.2]) the proof of which is based on a more general result by Beurling ([Be89]).

Theorem 3. *([Se95]) Let Λ be a sequence satisfying (2.4).*

If Λ is uniformly separated and $\mathcal{D}_\Lambda^+ < \frac{\tau}{\pi}$, then $\Lambda \in Int(PW_\tau^p)$. Conversely, if $\Lambda \in Int(PW_\tau^p)$, then, Λ is necessarily uniformly separated and $\mathcal{D}_\Lambda^+ \leq \frac{\tau}{\pi}$.

Corollary 4. *The sequence $\Lambda = \{\lambda_n : n \in \mathbb{Z}\}$ defined by*

$$\lambda_0 = 0 \text{ et } \lambda_n = n + \frac{sign(n)}{2\max(p,q)}, \quad n \neq 0,$$

is interpolating in $PW_{\pi+\epsilon}^p$, for every $\epsilon > 0$.

Proof. We have already mentioned that this sequence is uniformly minimal. The uniform separation condition is obvious. Its upper uniform density is clearly equal to 1. The Corollary now follows from Theorem 3. □

2.2. **Proof of Main Result.** We recall our main theorem, previously stated in the first section.

Theorem. *(Theorem 1) Let $\tau > 0$, $1 < p < \infty$ and Λ be a minimal sequence in PW_τ^p such that for every $a \in \mathbb{R}$, $\Lambda \cap \mathbb{C}_a^\pm$ satisfies the Carleson condition (1.4). Then, for every $\epsilon > 0$, Λ is an interpolating sequence in $PW_{\tau+\epsilon}^p$.*

Proof. Using an idea of Beurling, let $\epsilon > 0$ be fixed and $\phi_\epsilon \in \mathcal{C}^\infty$, with compact support contained in $\left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right)$, be such that $H_\epsilon := c\mathcal{F}\phi_\epsilon$ satisfies $H_\epsilon(0) = 1$. In particular, the Paley-Wiener theorem implies

that H_ϵ is an entire function of exponential type ϵ. Moreover, since ϕ_ϵ belongs to the Schwarz class (and in particular is $\mathcal{C}^1$), we have t

$$|H_\epsilon(x)| \lesssim \frac{1}{1+|x|}, \quad x \in \mathbb{R}.$$

Now, from a a Phragmen-Lindelf principle (see e.g. [Le96, p.39]), we can deduce that

$$|H_\epsilon(z)| \lesssim \frac{e^{\epsilon|\mathrm{Im}(z)|}}{1+|z|}, \quad z \in \mathbb{C}. \tag{2.5}$$

On the other hand, since Λ is minimal in PW^p_τ, there exists a sequence of functions $(f_\lambda)_{\lambda\in\Lambda} \subset PW^p_\tau$ such that $f_n(\lambda_k) = \delta_{nk}$. Let $a = (a_n)_{n\geq 1}$ be a finitely supported sequence and f be the solution of the interpolation problem

$$f(\lambda_n) = a_n, \quad n \geq 1,$$

defined by

$$f(z) = \sum_{n\geq 1} a_n f_n(z) H_\epsilon(z-\lambda_n).$$

(Notice that f is a finite sum of functions belonging to $PW^p_{\tau+\epsilon}$.) From (2.3), it suffices to bound the quantity

$$\inf\left\{\|f-g\|_p : \ g \in PW^p_{\tau+\epsilon},\ g(\lambda) = 0,\ \lambda \in \Lambda\right\}$$

by a constant times the following norm of a

$$\|a\| := \left(\sum_{\lambda\in\Lambda} |a_\lambda|^p (1+|\mathrm{Im}(\lambda)|) e^{-p(\tau+\epsilon)|\mathrm{Im}(\lambda)|}\right)^{\frac{1}{p}}.$$

We split the above sum in two parts: f^+ and f^- corresponding respectively to $\Lambda_0^+ := \Lambda \cap (\mathbb{C}^+ \cup \mathbb{R})$ and $\Lambda \cap \mathbb{C}^-$, and estimate each part separately. Here, we will only estimate the first one, the method is the same for the second one. Let $\eta > 0$ be such that $\{\mathrm{Im}(z) = -\eta\} \cap \Lambda = \emptyset$. The Plancherel-Polya inequality allows us to estimate the norm of $f^+ - g^+$, $g^+ \in PW^p_{\tau+\epsilon}$ and $g^+|\Lambda = 0$, on the axis $\{\mathrm{Im}(z) = -\eta\}$. We consider the Blaschke product associated to $\Lambda^+_{-\eta}$, in the half-plane $\mathbb{C}^+_{-\eta}$

$$B_{-\eta}(z) = \prod_{\lambda_n\in\Lambda^+_{-\eta}} c_{\lambda_n} \frac{z-\lambda_n}{z-\overline{\lambda}_n - 2i\eta}, \quad z \in \mathbb{C}^+_{-\eta},$$

with suitable unimodular coefficients c_{λ_n}. For $\lambda_n \in \Lambda_0^+$, we consider the function

$$G_{\lambda_n,\epsilon} : z \mapsto (z-\lambda_n) H_\epsilon(z-\lambda_n) f_n(z) e^{i(\tau+\epsilon)z}$$

which belongs to $H^\infty\left(\mathbb{C}^+_{-\eta}\right)$ (this follows from (2.1), (2.2) and (2.5)) and vanishes on $\Lambda^+_{-\eta}$ (it actually vanishes on Λ). We recall that $\Lambda^+_{-\eta}$ satisfies the Carleson condition in $\mathbb{C}^+_{-\eta}$. Also, the function $G^0_{\lambda_n,\epsilon} := G_{\lambda_n,\epsilon}/B_{-\eta}$ belongs to $H^\infty\left(\mathbb{C}^+_{-\eta}\right)$. Let B^- be the Blaschke product associated to $\Lambda^-_{-\eta}$ (in $\mathbb{C}^-_{-\eta}$). Observe that

$$\begin{aligned}&\inf\left\{\left\|f^+-g^+\right\|_p:\ g^+\in PW^p_{\tau+\epsilon},\ g^+|\Lambda=0\right\}\\&=\inf\left\{\left\|\sum_{\lambda_n\in\Lambda^+}a_n\frac{G^0_{\lambda_n,\epsilon}}{z-\lambda_n}-g_0^+\right\|_p:\ g_0^+\in Y\right\}\end{aligned}$$

with

$$g_0^+=\frac{g^+}{B_{-\eta}}e^{i(\tau+\epsilon)\cdot}$$

and

$$Y:=H^p_+\cap\overline{B_{-\eta}}\left(K^p_{I^{\tau+\epsilon}}\cap I^{\tau+\epsilon}B^-H^p_-\right)\subset L^p\left(\mathbb{R}\right).$$

By duality arguments inspired by Shapiro and Shields (see [SS61, p. 516] and [Gau11] where we consider the bilinear form $(f,g):=\int_{\mathbb{R}-i\eta}fg$, for $f,g\in L^p\left(\mathbb{R}-i\eta\right)$), and because

$$Y^{\perp_{(\cdot,\cdot)}}=H^q_++Z$$

where Z is such that, for every $h\in Z$, we have

$$\int_{\mathbb{R}-i\eta}\left(\sum_{\lambda_n\in\Lambda^+}a_n\frac{G^0_{\lambda_n,\epsilon}}{z-\lambda_n}\right)hdm=0,$$

it is enough to estimate

$$\sup\left\{N(h):\ h\in H^q\left(\mathbb{C}^+_{-\eta}\right),\|h\|=1\right\},$$

where

$$\begin{aligned}N\left(h\right)&:=\left|\int_{\mathbb{R}-i\eta}\left(\sum_{\lambda_n\in\Lambda^+_0}a_n\frac{G^0_{\lambda_n,\epsilon}}{z-\lambda_n}\right)hdm\right|.\\&=\left|\sum_{\lambda_n\in\Lambda^+_0}a_n\int_{\mathbb{R}}\frac{G^0_{\lambda_n,\epsilon}\left(x-i\eta\right)h\left(x-i\eta\right)}{x-\left(\lambda_n+i\eta\right)}dx\right|.\end{aligned}$$

Now, $z\mapsto G^0_{\lambda_n,\epsilon}\left(z-i\eta\right)h\left(z-i\eta\right)$ is a function in H^q_+ and the Cauchy formula gives

$$N(h)=\left|\sum_{\lambda_n\in\Lambda^+_0}a_nG^0_{\lambda_n,\epsilon}\left(\lambda_n+i\eta-i\eta\right)h\left(\lambda_n+i\eta-i\eta\right)\right|.$$

Moreover, since Λ_0^+ satisfies the Carleson condition in $\mathbb{C}_{-\eta}^+$, we have $\left|\frac{B_{-\eta}}{b_{\lambda_n}}(\lambda_n)\right| \asymp 1$ and since $f_{\lambda_n}(\lambda_n) H_\epsilon(0) = 1$, we can estimate

$$\left|G_{\lambda_n,\epsilon}^0(\lambda_n)\right| \asymp (\mathrm{Im}(\lambda_n) + \eta) e^{-(\tau+\epsilon)\mathrm{Im}(\lambda_n)}, \quad \lambda_n \in \Lambda_0^+.$$

It follows from the triangle inequality and Hlder's inequality that

$$\begin{aligned} N(h) \lesssim & \left(\sum_{\lambda_n \in \Lambda_0^+} |a_n|^p (1 + \mathrm{Im}(\lambda_n)) e^{-p(\tau+\epsilon)\mathrm{Im}(\lambda_n)}\right)^{\frac{1}{p}} \\ & \times \left(\sum_{\lambda_n \in \Lambda_0^+} \mathrm{Im}(\lambda_n + i\eta) \left|\tilde{h}(\lambda_n + i\eta)\right|^q\right)^{\frac{1}{q}}, \end{aligned}$$

where $\tilde{h} = h(\cdot - i\eta) \in H_+^q$. Now, the Carleson condition satisfied by $\Lambda_0^+ + i\eta$ in $\mathbb{C}^+$ gives

$$\left(\sum_{\lambda_n \in \Lambda_0^+} \mathrm{Im}(\lambda_n + i\eta) \left|\tilde{h}(\lambda_n + i\eta)\right|^q\right)^{\frac{1}{q}} \lesssim \|h\| = 1.$$

See (1.6) and properties mentioned thereafter. Finally, we obtain

$$\begin{aligned} & \inf\left\{\left\|f^+ - g\right\| : \ g \in PW_{\tau+\epsilon}^p, \ g|\Lambda = 0\right\} \\ & \lesssim \left(\sum_{\lambda_n \in \Lambda_0^+} |a_n|^p (1 + \mathrm{Im}(\lambda_n)) e^{-p(\tau+\epsilon)\mathrm{Im}(\lambda_n)}\right)^{\frac{1}{p}} \end{aligned}$$

which is the required estimate and ends the proof. □

To conclude this section, we give two immediate corollaries to our main theorem. First, since, by Proposition 2, a weak interpolating sequence in PW_τ^p has to satisfy the Carleson condition in every half-plane $\mathbb{C}_a^\pm$, we can deduce the following result.

Corollary 5. *If $\Lambda \in Int_w(PW_\tau^p)$, then, for every $\epsilon > 0$, Λ is interpolating in $PW_{\tau+\epsilon}^p$.*

We also give a result involving density conditions as a second corollary to our main result, which does not seem easy to prove directly.

Corollary 6. *Let Λ satisfying (2.4) be a weak interpolating sequence in PW_τ^p. Then, $\mathcal{D}_\Lambda^+ \leq \frac{\tau}{\pi}$.*

Proof. It follows from Theorem 1 that Λ is interpolating in $PW^p_{\tau+\epsilon}$, for every $\epsilon > 0$. Thus, Theorem 3 implies that $\mathcal{D}^+_\Lambda \leq \frac{\tau+\epsilon}{\pi}$, for every $\epsilon > 0$, thus $\mathcal{D}^+_\Lambda \leq \frac{\tau}{\pi}$. □

3. Weighted Interpolation and McPhail's Condition

The previous technics can be used to show a more general result. We need to introduce some more definitions. Let X be the Hardy space $H^p(\mathbb{C}^\pm_a)$ or the Paley-Wiener space PW^p_τ, $\Lambda = \{\lambda_n\}_{n\geq 1}$ a sequence of complex numbers lying in the corresponding domain $\mathbb{C}^\pm_a$ or $\mathbb{C}$ and $\omega = (\omega_n)_{n\geq 1}$ a sequence of strictly positive numbers. We say that Λ is $\omega-$*interpolating* in X if for every $(a_n)_{n\geq 1} \in l^p$, there is $f \in X$ such that

$$\omega_n f(\lambda_n) = a_n, \qquad n \geq 1. \tag{3.1}$$

The reader has noticed that the previous definition of interpolation in X is equivalent to $\omega-$interpolation in X, with

$$\omega_n = \|k_{\lambda_n}\|^{-1}_{X^\star}, \qquad n \geq 1.$$

Let $\Lambda \subset \mathbb{C}^\pm_a$. In this section, the sequence Λ is *a priori* not a Carleson sequence. We only assume the Blaschke condition

$$\sum_{n\geq 1} \frac{|\mathrm{Im}(\lambda_n) - a|}{1 + |\lambda_n|^2} < \infty. \tag{3.2}$$

We set

$$\vartheta_n := \prod_{k\neq n} \left| \frac{\lambda_n - \lambda_k}{\lambda_n - \overline{\lambda}_k - 2ia} \right|, \qquad n \geq 1.$$

The couple (Λ, ω) is said to satisfy the *McPhail condition* (M_q), denoted $(\Lambda, \omega) \in (M_q)$, if the measure

$$\nu_{\Lambda,\omega} := \sum_{n\geq 1} \frac{|\mathrm{Im}(\lambda_n) - a|^q}{\omega_n^q \vartheta_n^q} \delta_{\lambda_n} \tag{3.3}$$

is a Carleson measure in $\mathbb{C}^\pm_a$. The following theorem is a special case of McPhail's theorem ([McP90]) and is stated as follows in [JP06].

Theorem 7. *(McPhail)*

Let $1 < p < \infty$, $\Lambda \subset \mathbb{C}^\pm_a$ a sequence satisfying the Blaschke condition (3.2) and $\omega = (\omega_n)_{n\geq 1}$ be a sequence of positive numbers. The following assertions are equivalents.

(i) Λ is $\omega-$interpolating in $H^p(\mathbb{C}^\pm_a)$;

(ii) (Λ, ω) satisfies the McPhail condition (M_q), $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 8. It follows directly from McPhail's Theorem and the Plancherel-Polya inequality that if $\Lambda \subset \mathbb{C}$ is $\omega-$interpolating in PW_τ^p, then for every $a \in \mathbb{R}$, we necessarily have

$$\left(\left(\Lambda \cap \mathbb{C}_a^{\pm}\right), e^{\pm\tau|\mathrm{Im}(\lambda)|}\omega\right) \in (M_q).$$

The following result is a weighted version of Theorem 1. We will only sketch the proof which is analogous to that of our main result.

Theorem 9. *Let $\tau > 0$, $1 < p < \infty$, $\omega = (\omega_n)_{n\geq 1}$ a sequence of strictly positive numbers and Λ be a minimal sequence in PW_τ^p such that for every $a \in \mathbb{R}$,*

$$\left(\left(\Lambda \cap \mathbb{C}_a^{\pm}\right), e^{\pm\tau|Im(\lambda)|}\omega\right) \in (M_q).$$

Then, for every $\epsilon > 0$, Λ is $\omega-$interpolating in $PW_{\tau+\epsilon}^p$.

Proof. As in the proof of the main result of this paper, we fix $\epsilon > 0$ and we take a finitely supported sequence $(a_n)_{n\geq 1}$. We consider the solution of the weighted interpolation problem (3.1) given by

$$f(z) = \sum_{n\geq 1} \frac{a_n}{\omega_n} f_n(z) H_\epsilon(z - \lambda_n).$$

As previously, it is possible to split the sum in two parts that we estimate separately. In order to avoid technical details, let us assume here that Λ lies in the half-plane $\mathbb{C}_1^+$. As before, we set

$$G_{\lambda_n,\epsilon}(z) = e^{i(\tau+\epsilon)z}(z - \lambda_n) f_n(z) H_\epsilon(z - \lambda_n) \in H_+^\infty.$$

If B denotes the Blaschke product associated to Λ, we again write

$$G_{\lambda_n,\epsilon} = B G_{\lambda_n,\epsilon}^0$$

with $G_{\lambda_n,\epsilon}^0$ still in H_+^∞. By duality , we need to estimate

$$\sup\left\{N(h): \quad h \in H_+^q,\ \|h\|_q = 1\right\},$$

where

$$N(h) := \left|\sum_{n\geq 1} \frac{a_n}{\omega_n} \int_{-\infty}^{\infty} \frac{G_{\lambda_n,\epsilon}^0(x)\, h(x)}{x - \lambda_n} dx\right|.$$

The Cauchy formula gives then

$$\begin{aligned}\left|\int_{-\infty}^{\infty} \frac{G_{\lambda_n,\epsilon}^0(x)\, h(x)}{x - \lambda_n} dx\right| &= \left|G_{\lambda_n,\epsilon}^0(\lambda_n)\, h(\lambda_n)\right| \\ &= \frac{|2\mathrm{Im}(\lambda_n)|}{\vartheta_n} e^{-(\tau+\epsilon)|\mathrm{Im}(\lambda_n)|} |h(\lambda_n)|,\end{aligned}$$

and, applying Hlder's inequality, we finally find

$$N(h) \lesssim \left(\sum_{n\geq 1} |a_n|^p\right)^{\frac{1}{p}} \left(\sum_{n\geq 1} \frac{|\mathrm{Im}\,(\lambda_n)|^q}{\vartheta_n^q \omega_n^q} e^{-q(\tau+\epsilon)|\mathrm{Im}(\lambda_n)|} \left|h\left(\lambda_n\right)\right|^q\right)^{\frac{1}{q}}.$$

By assumption, $\nu_{\Lambda,\tilde{\omega}}$, with $\tilde{\omega} = \left(\omega_n e^{\pm\tau|\mathrm{Im}(\lambda_n)|}\right)_n$ (we recall that $\nu_{\Lambda,\tilde{\omega}}$ is defined by (3.3)) is a Carleson measure in $\mathbb{C}^+$ and so

$$\left(\sum_{n\geq 1} \frac{|\mathrm{Im}\,(\lambda_n)|^q}{\vartheta_n^q \omega_n^q e^{q\tau|\mathrm{Im}(\lambda_n)|}} \left|h\left(\lambda_n\right)\right|^q\right)^{\frac{1}{q}} \lesssim \|h\|_q = 1.$$

Since

$$e^{-q\epsilon|\mathrm{Im}(\lambda_n)|} \leq 1,$$

we obtain

$$\sup\left\{N\left(h\right): \quad h \in H_+^q,\ \|h\|_q = 1\right\} \lesssim \|a\|_{l^p},$$

which permits us to end the proof. □

Remark 10. As we have seen in the previous section, the weak interpolation of a sequence Λ in PW_τ^p implies the interpolation property on Λ in $PW_{\tau+\epsilon}^p$, which follows from the fact that a uniformly minimal sequence Λ in the Hardy space is an interpolating sequence in the same space. We wonder if we could have an analog result in the weighted case. More precisely, we say that the sequence $\Lambda \subset \mathbb{C}_a^\pm$ is *uniformly* $\omega-$*minimal* in $H^p\left(\mathbb{C}_a^\pm\right)$ if there exists a sequence $\left(f_n\right)_{n\geq 1}$ of functions of $H^p\left(\mathbb{C}_a^\pm\right)$ such that

$$\omega_n f_n\left(\lambda_k\right) = \delta_{nk}$$

and

$$\sup_{n\geq 1} \|f_n\| < \infty.$$

The question is to know whether a uniformly $\omega-$minimal sequence Λ in $H^p\left(\mathbb{C}_a^\pm\right)$ is necessarily such that the couple (Λ, ω) satisfies the McPhail condition (M_q), $\frac{1}{p} + \frac{1}{q} = 1$.

4. (Weak) Controllability of Linear Differential Systems

We consider linear differential systems of the form

$$\begin{cases} x'(t) &= Ax(t) + Bu(t), \qquad t \geq 0, \\ x(0) &= x_0, \end{cases} \tag{4.1}$$

where A is the generator of a c_0-semigroup $\left(S(t)\right)_{t\geq 0}$ on a Hilbert space H and $B : \mathbb{C} \to H$ is an operator, called the *control operator* which

we *a priori* do not assume bounded. We are thus interested in *rank-1 control*. We refer to [JP06, Part D] and references therein for more details on this terminology and on the subject. Note that those authors also consider unbounded control. We will assume that the semigroup $(S(t))_{t\geq 0}$ is exponentially stable, *i.e.* there exists $\alpha > 0$ such that we can find $M \geq 1$ for which

$$\|S(t)\| \leq Me^{-\alpha t}, \quad t \geq 0. \tag{4.2}$$

Controlling the system (4.1) means to act on the system by means of a suitable *input function* u. More precisely, starting from an initial state $x_0 \in H$, we want the system to attain in time $\tau > 0$ the in advance given final state $x_1 = x(\tau)$. Here the solution x of (4.1) is given by

$$x(t) = S(t)x_0 + \int_0^t S(t-r)\,Bu(r)dr =: S(t)x_0 + \mathcal{B}_t u. \tag{4.3}$$

The operator $\mathcal{B}_t$ is called *controllability operator* and we are interested in the study of its range, the so-called space of reachable states. More precisely, we say that the system (4.1) is *exactly controllable in finite time* $\tau > 0$ (respectively in *infinite time*) if for every $x_0, x_1 \in H$, there is $u \in L^2(0,\tau)$ (respectively $u \in L^2(0,\infty)$) such that $x(0) = x_0$ and $x(\tau) = x_1$ (respectively $\lim_{t\to\infty} x(t) = x_1$) or, equivalently, if $\mathcal{B}_\tau$ (respectively $\mathcal{B}_\infty : u \mapsto \int_0^\infty S(t)Bu(t)dt$) is surjective. It is well known that a bounded compact controllability operator (and in particular a rank one operator) can never cover the whole space H (see [Ni02b, p. 215]).

In all what follows, we will assume that the generator A admits a Riesz basis of (normalized) eigenvectors $(\phi_n)_{n\geq 1}$:

$$A\phi_n = -\lambda_n \phi_n, \quad n \geq 1,$$

and that the sequence of eigenvalues $\Lambda := \{\lambda_n\}_{n\geq 1}$ satisfies the Blaschke condition in the right half-plane

$$\sum_{n\geq 1} \frac{\mathrm{Re}\,(\lambda_n)}{1+|\lambda_n|^2} < \infty.$$

Note that by the exponential stability, Λ indeed lies in the right half-plane.The Riesz basis property gives the representation

$$H = \left\{ x = \sum_{n\geq 1} a_n \phi_n : \sum_{n\geq 1} |a_n|^2 < \infty \right\}. \tag{4.4}$$

We denote by $(\psi_n)_{n\geq 1}$ the biorthogonal family to $(\phi_n)_{n\geq 1}$ (which also forms a Riesz basis of H and satisfies $\|\psi_n\| \asymp \|\phi_n\|^{-1} \asymp 1$). We suppose

that B has the following representation

$$Bv = v\left(\sum_{n\geq 1}\overline{b}_n\phi_n\right), \qquad v\in\mathbb{C},$$

with a sequence $(b_n)_{n\geq 1}\subset\mathbb{C}$. Observe that B does not map $\mathbb{C}$ boundedly in H, but it does map boundedly into some extrapolation space in which the sequence $(\phi_n)_{n\geq 1}$ has dense linear span: for example, we may define

$$H_B := \left\{x := \sum_{n\geq 1}x_n\phi_n : \ \|x\|_B^2 := \sum_{n\geq 1}\frac{|x_n|^2}{n^2\left(1+|b_n|^2\right)} < \infty\right\}.$$

It appears that (4.3) can be written

$$\begin{aligned} x(\tau) &= S(\tau)x_0 + \mathcal{B}_\tau u \\ &= S(\tau)x_0 + \sum_{n\geq 1}\left(\overline{b}_n\int_0^\tau u\left(t\right)e^{-\lambda_n(\tau-t)}dt\right)\phi_n \\ &= S(\tau)x_0 + \sum_{n\geq 1}\left(\overline{b}_n e^{-\frac{\tau}{2}\lambda_n}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\tilde{u}\left(t\right)e^{\lambda_n t}dt\right)\phi_n, \end{aligned}$$

with $\tilde{u} := u\left(\cdot+\frac{\tau}{2}\right)\in L^2\left(-\frac{\tau}{2},\frac{\tau}{2}\right)$. We have already introduced the Fourier transform $\mathcal{F}$ and we have mentioned that $\mathcal{F}L^2\left(-\frac{\tau}{2},\frac{\tau}{2}\right) = PW^2_{\frac{\tau}{2}}$. Hence, if

$$f := \mathcal{F}\tilde{u} = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\tilde{u}(t)e^{-it\cdot}dt \in PW^2_{\frac{\tau}{2}},$$

we have

$$\mathcal{B}_\tau u = \sum_{n\geq 1}\left(\overline{b}_n e^{i\frac{\tau}{2}(i\lambda_n)}f\left(i\lambda_n\right)\right)\phi_n.$$

With the same method, we obtain

$$\mathcal{B}_\infty u = \sum_{n\neq 1}\left(\overline{b}_n g\left(i\lambda_n\right)\right)\phi_n,$$

where $g := \mathcal{F}u$ which belongs to H^2_+ from well known facts about Hardy spaces. Since exact controllability translates to surjectivity of $\mathcal{B}_\tau$ or $\mathcal{B}_\infty$, and by (4.4), we can reformulate exact controllability in terms of a weighted interpolation problem.

Theorem 11. *The following assertions are equivalent.*

(i) The system (4.1) is exactly controllable in finite time $\tau > 0$ (respectively in infinite time);

(ii) The sequence $i\Lambda = \{i\lambda_n\}_{n\geq 1}$ *is* $\omega-$*interpolating in* $PW^2_{\frac{\tau}{2}}$*, with*

$$\omega_n := e^{-\frac{\tau}{2}\mathrm{Im}(i\lambda_n)}\,|b_n|\,, \qquad n \geq 1$$

(respectively $(|b_n|)_{n\geq 1}$ $-$*interpolating in* H^2_+*).*

Remark 12. As a consequence, and in view of Remark 8, an exact controllable system (in finite time τ) has necessarily to be such that $(i\Lambda, (|b_n|)_n)$ satisfies (M_2) in $\mathbb{C}^+$ and hence the system has to be controllable in infinite time.

In [Ni02b, p. 289-290], the author introduces a weaker type of control, called *control for simple oscillations*, requiring that the control operator maps boundedly some Hilbert space $\mathcal{U}$ into H. As already mentioned above, compact (and hence finite rank) control is impossible with such hypotheses so that we have to deal here with unbounded control operators B. Nevertheless, we keep the terminology of [Ni02b] in our situation.

The system (4.1) is said *controllable for simple oscillations in time* $\tau > 0$ if it is possible to find a sequence $(u_n)_{n\geq 1}$ of functions in $L^2(0,\tau)$ such that

$$\mathcal{B}_\tau u_n = \phi_n, \quad n \geq 1. \tag{4.5}$$

Remark 13. Since

$$\langle \mathcal{B}_\tau u,\, \psi_n\rangle = \overline{b}_n e^{i\frac{\tau}{2}(i\lambda_n)} f(i\lambda_n)\,, \qquad n \geq 1,$$

we easily see that (4.5) is equivalent to the minimality of $i\Lambda$ in PW^2_τ.

We can now use Theorems 9, 11 and the previous remark to establish a link between control for simple oscillations at time τ and exact control at time $\tau + \epsilon$. More precisely, we have the following result.

Theorem 14. *Under the above hypotheses, if the system (4.1) is exactly controllable in infinite time (or equivalently if* $(i\Lambda, (|b_n|)_n)$ *satisfies* (M_2) *in* $\mathbb{C}^+$*) and if it is controllable for simple oscillations in time* $\tau > 0$*, then the system is exactly controllable in finite time* $\tau + \epsilon$*, for every* $\epsilon > 0$*.*

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