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► **To cite this version:**

Joo Magueijo, Ali Mozaffari. Simple generalizations of anti-de Sitter spacetime. *Classical and Quantum Gravity*, 2010, 27 (13), pp.135004. 10.1088/0264-9381/27/13/135004 . hal-00629998

HAL Id: hal-00629998

<https://hal.science/hal-00629998>

Submitted on 7 Oct 2011

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Simple generalizations of Anti-de Sitter space-time

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We consider new cosmological solutions which generalize the cosmological patch of the Anti-de Sitter (AdS) space-time, allowing for fluids with equations of state such that $w \neq -1$. We use them to derive the associated full manifolds. We find that these solutions can all be embedded in flat five-dimensional space-time with $- - + +$ signature, revealing deformed hyperboloids. The topology and causal-structure of these spaces is therefore unchanged, and closed time-like curves are identified, before a covering space is considered. However the structure of Killing vector fields is entirely different and so we may expect a different structure of Killing horizons in these solutions.

PACS numbers: 0000000

I. INTRODUCTION

Anti-de Sitter (AdS) space-time [6, 8] is a maximally symmetric solution to Einstein's field equations with a negative cosmological constant Λ . It is one of the simplest solutions to Einstein's gravity and as such it has been a prime test ground for new ideas and toy models in (quantum) gravity. More recently, AdS is best known for its role in the AdS/CFT correspondence [2–5], which conjectures that string theories on a given space are dual to conformal field theories on the conformal boundary of this space. Typically the space in question is the product of AdS with a closed manifold. For example type IIB string theory on $\text{AdS}_5 \times \text{S}^5$ is dual to $\mathcal{N} = 4$ SYM on the 4D boundary of AdS_5 .

A natural question is whether more realistic spacetimes might support extensions of conjectures made (or theorems proved) for AdS. For this reason it is interesting to consider deformations of AdS, i.e. families of solutions to Einstein gravity which contain AdS as a limiting case (allowing, by suitably varying a parameter, to be as close as wanted to AdS).

It is well known that a portion of AdS can be discovered using the formalism of homogeneous and isotropic cosmology. The full manifold can then be inferred by extension. In this paper we consider cosmological solutions that follow from altering the equation of state $w = p/\rho$ (where p is the pressure and ρ is the energy density). AdS follows from $w = -1$, but a variety of solutions with similar properties result from $w < -1/3$. In Section II we derive these solutions and in Section III we use them to infer their associated inextendible manifolds. Finally in Section IV we carry out a preliminary study of the local and global properties of these solutions.

II. COSMOLOGICAL SOLUTIONS

A patch of AdS may be discovered using the formalism of Friedmann-Robertson-Walker (FRW) homogeneous and isotropic cosmology. As is well known, for a con-

stant equation of state $w = p/\rho$, the continuity equation,

$$\dot{\rho} + 3 \frac{\dot{a}}{a} \rho (1 + w) = 0 \quad (1)$$

integrates into $\rho \propto a^{-3(1+w)}$ so that the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} + \frac{\Lambda}{3} \quad (2)$$

becomes

$$\dot{a}^2 = \frac{\Lambda a^2}{3} + C a^\beta - k \quad (3)$$

where $\beta = -(1 + 3w)$ and C is a constant. The cosmological portion of AdS follows from $k = -1$, $w = -1$, so that (3) becomes

$$\dot{a}^2 = 1 + \left(\frac{\Lambda}{3} + C\right) a^2, \quad (4)$$

with the extra condition $C + \Lambda/3 < 0$. Setting $\omega^2 = -(C + \Lambda/3)$ converts (2) into a simple harmonic oscillator equation:

$$\dot{a}^2 = 1 - \omega^2 a^2 \quad (5)$$

solved by $a = \cos \omega t$. The FRW form of the AdS_4 is therefore:

$$ds^2 = -dt^2 + \cos^2(\omega t) d\sigma \quad (6)$$

where $d\sigma$ is the metric on a 3D homogeneous space negatively curved:

$$d\sigma = \frac{dr^2}{1+r^2} + r^2 d\Omega_2 \quad (7)$$

where $d\Omega_2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the area element of a 2-sphere.

In this construction we can use interchangeably a negative cosmological constant or a fluid with $w = -1$ and negative energy. A generalization can be obtained by considering a fluid with negative energy ($C < 0$), but with any $w < -1/3$ (with $\Lambda = 0$ and $k = -1$). To fix

ideas set $C = -1$ (but a generalization is straightforward). Then the Friedman equation becomes

$$\dot{a}^2 = 1 - a^\beta, \quad (8)$$

where $\beta = -(1 + 3w) > 0$. This is no longer a simple harmonic oscillator equation but, availing ourselves of the diffeomorphism invariance of relativity, we may define a time variable η via:

$$dt^2 = \left(\frac{1 - a^2}{1 - a^\beta} \right) d\eta^2. \quad (9)$$

Time coordinate η is not proper time for a cosmological observer, but, rather, a ‘‘harmonic’’ time for which Eq. (8) reduces to

$$a'^2 = 1 - a^2 \quad (10)$$

where $a' = da/d\eta$. Thus $a = \cos \eta$ leading to metric:

$$ds^2 = - \left(\frac{\sin^2 \eta}{1 - \cos^\beta \eta} \right) d\eta^2 + \cos^2 \eta d\sigma^2 \quad (11)$$

so that with these coordinates only g_{00} is modified.

For $\beta = 2$, (i.e. $w = -1$) metric (11) reduces to the AdS metric. For $\beta = 0$ (corresponding to $w = -1/3$, i.e. the Milne universe), the g_{00} component becomes singular. For other $\beta > 0$ ($w < -1/3$) we have obtained generalizations of the AdS solution. They can only be considered in the Friedmann patch of the manifold, and hence it would be interesting to seek their complete manifold, just like for AdS.

For completeness, we include our metric written in terms of proper time t . We note that

$$t = \int \frac{da}{\sqrt{1 - a^\beta}} = a {}_2F_1 \left(\frac{1}{2}, \frac{1}{\beta}; 1 + \frac{1}{\beta}; a^\beta \right) \quad (12)$$

where ${}_2F_1$ is a hypergeometric function. We can thus recast (11) in terms of t , but the result must be expressed in terms of special functions and their inverses.

One may wonder what kind of matter content could lead to these models. Most obviously one could consider a scalar field with a suitable potential. For $K = 0$ an exponential potential leads to solutions with constant w , and it's possible to obtain any $w < -1/3$ (and even a negative energy density) by carefully choosing the sign of the kinetic and potential energy terms [11–13]. Similar solutions might be obtainable for $K = -1$ with an appropriate choice of potential. We defer this study to future work, but note that one often simply postulate the equation of state w for a fluid.

III. THE ASSOCIATED FULL MANIFOLD

The complete extension of the above patch can be obtained by a simple adaptation of the procedure for AdS,

which we briefly review here [6]. Consider a 5D manifold with metric

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2 \quad (13)$$

where there live a series of hyperboloids (see Fig. 1):

$$u^2 + v^2 - x^2 - y^2 - z^2 = \rho^2. \quad (14)$$

We can introduce a system of coordinates $\{t, r, \rho, \theta, \phi\}$ by means of:

$$\begin{aligned} u &= \rho \sin t \\ v &= \rho \cos t \sqrt{1 + r^2} \\ x &= \rho \cos t r \cos \theta \\ y &= \rho \cos t r \sin \theta \cos \phi \\ z &= \rho \cos t r \sin \theta \sin \phi \end{aligned} \quad (15)$$

for which $\rho = \text{constant}$ represents the hyperboloids.

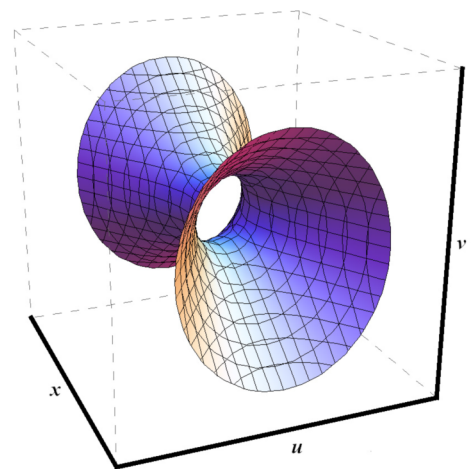


FIG. 1: As is well known AdS space can be represented by an hyperboloid living in 5D flat space with signature $--++++$. Closed time-like curves are evident (see text; but note that any curve in the u, v plane is time-like).

The induced metric on the hyperboloids may be found by writing (13) in terms of these coordinates and setting $d\rho = 0$, an exercise that reveals the metric (6). Thus the portion of the hyperboloids covered by these coordinates are embeddings of the AdS Friedmann patch. For different ρ , different values of ω are recovered.

To find the whole manifold, we note that there are apparent singularities at $t = \pm \frac{1}{2}\pi$ and therefore these coordinates do not cover the whole space [6]. We can reframe the space into a system of static coordinates using relations [1, 2]:

$$\begin{aligned} u &= \rho \sin t' \cosh r' \\ v &= \rho \cos t' \cosh r' \\ x &= \rho \sinh r' \cos \theta \\ y &= \rho \sinh r' \sin \theta \cos \phi \\ z &= \rho \sinh r' \sin \theta \sin \phi. \end{aligned} \quad (16)$$

These cover the full hyperboloid and for $\rho = 1$ the metric is now

$$ds^2 = -\cosh^2 r' dt'^2 + dr'^2 + \sinh^2 r' d\Omega_2. \quad (17)$$

By choosing different values of ρ different Λ may be implemented, but as before, we shall consider $\rho = 1$ to fix ideas in what follows.

A similar construction may be devised for the generalisations in Eq. (11). The manifolds they represent can be embedded in 5D flat space with the same signature. Given that the $d\sigma$ components of the metric remain the same we try

$$x = \rho a \sinh \chi \cos \theta \quad (18)$$

$$y = \rho a \sinh \chi \sin \theta \cos \phi \quad (19)$$

$$z = \rho a \sinh \chi \sin \theta \sin \phi \quad (20)$$

and indeed $\Sigma dx_i^2 = \rho^2 a^2 \sinh^2 \chi d\Omega_2^2 + [d(\rho a \sinh \chi)]^2$, replicating that part of the AdS calculation. Setting $v = \rho a \cosh \chi$ and for some yet to be defined function u_* we find:

$$\begin{aligned} ds^2 &= -du_*^2 - dv^2 + dx_i^2 \\ &= -du_*^2 - \rho^2 da^2 - a^2 d\rho^2 + \rho^2 a^2 d\sigma. \end{aligned} \quad (21)$$

If $u_* = \rho J(\eta)$ then

$$du_*^2 = \rho^2 J'^2 d\eta^2 + 2\rho J J' d\rho d\eta + J^2 d\rho^2 \quad (22)$$

where $J' = dJ/d\eta$. Inserting $a = \cos \eta$ and equating the terms in $d\eta^2$ in (11) and (21) gives:

$$\begin{aligned} \rho^2 J'^2 d\eta^2 &= \rho^2 \left(-\sin^2 \eta + \frac{\sin^2 \eta}{1 - \cos^\beta \eta} \right) d\eta^2 \\ &= \rho^2 \left(\frac{\sin^2 \eta \cos^\beta \eta}{1 - \cos^\beta \eta} \right) d\eta^2. \end{aligned} \quad (23)$$

This reduces to

$$dJ^2 = \left(\frac{\sin^2 \eta \cos^\beta \eta}{1 - \cos^\beta \eta} \right) d\eta^2 \quad (24)$$

which can be explicitly integrated into

$$J(\eta) = \frac{2}{\beta + 2} {}_2F_1 \left(\frac{1}{2} + \frac{1}{\beta}, \frac{1}{2}; \frac{3}{2} + \frac{1}{\beta}; \cos^\beta \eta \right) \cos^{\beta/2+1} \eta \quad (25)$$

Thus (11) is the metric induced on the $\rho = \text{const}$ surfaces (for which $d\rho = 0$) if we take the 5D space metric to be:

$$ds^2 = -du_*^2 - dv^2 + dx^2 + dy^2 + dz^2 \quad (26)$$

The $\rho = \text{constant}$ surfaces can be inferred by analogy, resulting in a deformed hyperboloid of form:

$$\left(\frac{\sin \eta}{J(\eta)} \right)^2 u_*^2 + v^2 - x^2 - y^2 - z^2 = \rho^2 \quad (27)$$

where $u_* = \rho J(\eta)$

We note that for $\beta = 2$ (i.e. $w = -1$) we have $J = \sin \eta$ and we recover the AdS construction. For $\beta = 0$, i.e. Milne space-time (with $w = -1/3$) the construction becomes singular. For other values of $w < -1/3$ the result can be seen most effectively by plotting the resulting deformed hyperboloids. As Figure 2 shows, if $w < -1$ the hyperboloid squashes in the u direction, the more so the smaller the value of w . For $-1 < w < -1/3$, as Fig 3 shows, the hyperboloids expand in the u direction instead, the effect becoming more extreme as $w = -1/3$ is approached. Since $\rho \propto a^{\beta-2}$ there are Ricci singularities as $a \rightarrow 0$. This means that for $a = \cos(\eta) = 0$, i.e. when $\eta = \pm\pi/2$, there are ‘‘point-like’’ singularities, seen as cusps on the hyperboloid. We pick the particular case of $\beta = 1$ to that find

$$J(\eta) \propto \left(\sqrt{(1 - \cos(\eta)) \cos(\eta)} - \sin^{-1}(\sqrt{\cos(\eta)}) \right) \quad (28)$$

which has roots at $\eta = \pm\pi/2$. The hyperboloid construction fails at these points (which are located on the plane $u = \rho, v = z = y = z = 0$). These singular points on the hyperboloid are found for all $0 < \beta < 2$

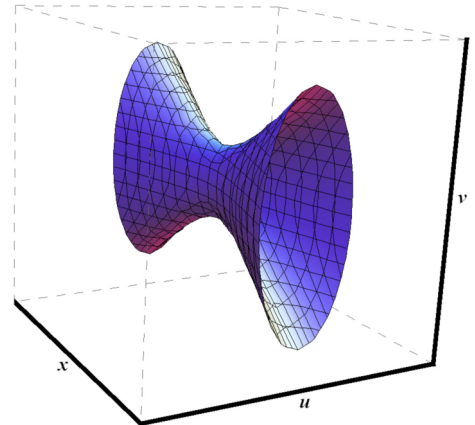


FIG. 2: The full manifold corresponding to the cosmological solution with $w = -5/3$. For all $w < -1$ the hyperboloid squashes along the u direction, the effect becoming more pronounced the smaller the w .

IV. DISCUSSION

These manifolds may prove valuable in assessing conjectures that theorems proved for AdS generalize to more realistic space-times. A close scrutiny of their properties is therefore in order. Here we briefly discuss their most evident properties.

From the embeddings found it's immediately obvious that the new manifolds share with AdS its topology and aspects of the causal structure. In particular they all admit closed time-like curves (any curve in the u, v plane is time-like). From the embedding of AdS we see that the time t' is periodic: t' and $t' + 2\pi$ represent the same point

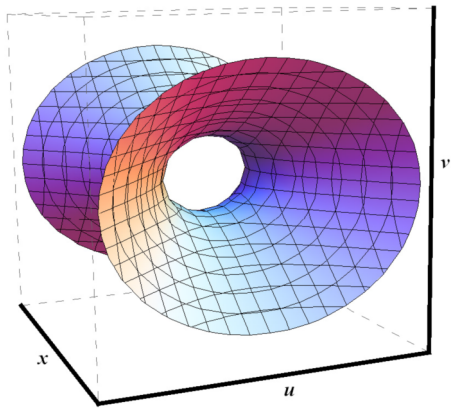


FIG. 3: The full manifold corresponding to the cosmological solution with $w = -2/3$. For $-1 > w > -1/3$ the hyperboloid elongates along the u direction, the more so the closer to $w = -1/3$ (i.e. $\beta = 0$) one gets.

on the hyperboloid. Thus any curve with fixed ρ, θ, ϕ and increasing t' is a closed time-like curve (CTC) (see Fig. (1)). The same feature is present for all deformed hyperboloids. By unwrapping these circles one obtains universal covering spaces, like for AdS [6, 14].

In contrast with these AdS-like features, the structure of Killing vector fields is entirely changed in the new spaces. For any 3+1 metric, there exist up to 10 Killing vector fields: 3 rotational, 3 translational, 3 boosts and 1 time-like vector. All 10 Killing vectors are manifest on the full AdS manifold. In our solutions, the rotations and translations survive, but the time-like and boost isometries are obviously lost.

Within these fields, a null integral hypersurface can sometimes be identified, known as a Killing Horizon. Associated with these there are important geometrical and thermodynamical quantities [9], such as the surface gravity κ , and Hawking's temperature $T = \kappa/2\pi$. Killing horizons in AdS are highly non-trivial, however by considering optical metrics (see [7, 8]) one finds three classes of time-like, orthogonal Killing vectors in the space. Clearly with the loss of isometries these results do not translate into our construction, but its thermodynamical properties should be the subject of a future study.

We have been unable to find the equivalent of static coordinates for the new spaces, and we conjecture that they don't exist (it's not obvious that a simple adaptation of Birkhoff's theorem can be used to prove this). It is easy, however, to adapt the constructions used for building the AdS Penrose diagram, as well and inferring the key causal features. Let us write the metric in terms of conformal time:

$$\xi = \int d\eta \frac{|\sin \eta|}{\cos \eta \sqrt{1 - \cos^\beta \eta}} = \frac{2}{\beta} \tanh^{-1}[\text{sgn}(\eta) \sqrt{1 - \cos^\beta \eta}] \quad (29)$$

A crucial feature of AdS is that light rays may reach $\chi = \infty$ in a finite amount of affine parameter (and proper

time for time-like observers). Thus the Friedmann patch is extendable. This feature is also true here: $\xi(\eta)$ diverges at $\eta = \pm\pi/2$, and although η is not proper time, proper time is convergent at these points.

The metric can be written in terms of ξ by noting that

$$\cos^\beta(\eta) = \text{sech}^2 \frac{\beta\xi}{2} \quad (30)$$

so that:

$$ds^2 = \text{sech}^{\frac{4}{\beta}} \left(\frac{\beta\xi}{2} \right) [-d\xi^2 + d\chi^2 + \sinh^2 \chi d\Omega_2] \quad (31)$$

We can now perform the usual conformal transformation that maps the metric into a diamond inserted in the Einstein Static Universe (ESU). Specifically we set up null coordinates:

$$u = \xi - \chi \quad (32)$$

$$v = \xi + \chi \quad (33)$$

make infinity tangible (and extendable) via:

$$\tan p = \tanh \frac{u}{2} \quad (34)$$

$$\tan q = \tanh \frac{v}{2} \quad (35)$$

and unwrap the new null coordinates into a new space and time:

$$p = \frac{\hat{\xi} - \hat{\chi}}{2} \quad (36)$$

$$q = \frac{\hat{\xi} + \hat{\chi}}{2}. \quad (37)$$

This leads to a metric conformal to the ESU:

$$ds^2 = \Omega^2 (-d\hat{\xi}^2 + d\hat{\chi}^2 + \sin^2 \hat{\chi} d\Omega_2) \quad (38)$$

where the initial cosmological patch corresponds to the diamond $-\pi/4 < p < q < \pi/4$ (see Fig. 4). The conformal factor is:

$$\Omega^2 = \sec(2p) \sec(2q) \text{sech}^{\frac{4}{\beta}} \left(\frac{\beta\xi(p, q)}{2} \right) \quad (39)$$

and for $\beta = 2$ this reduces to $\Omega^2 = \sec^2 \hat{\chi}$, as is well known. For other values of β the expression is more complicated. For example for $\beta = 4$ we find:

$$\Omega^2 = \frac{1}{1 + \sin 2p \sin 2q}. \quad (40)$$

Using the Hopital rule it can be *generally* proved that Ω converges on the null boundaries of the cosmological diamond. It can also be generally proved that the only divergence occurs for $\sin 2p \sin 2q = -1$ (i.e. for $-p = q = \pi/4$ and periodically related points), where the two null boundaries meet. Indeed away from the null

boundaries of the cosmological patches Ω can only diverge if $\eta = i\frac{\pi}{\beta}$. This only has real solutions in $\{p, q\}$ for $\beta = 2$. Thus the Penrose diagram (see Fig. 4) is the whole Einstein static Universe, if no singularities isolate part of it. This is true for $\beta > 2$. Spatial infinity is at $\hat{\chi} = \pi/2$ and $\hat{\xi} = n\pi$. Unlike with AdS $\hat{\chi} = \pi/2$ is no longer \mathcal{I}^\pm . As with AdS, it is impossible to conformally render finite time-like infinity without collapsing spatial distances to a point.

However for $\beta < 2$ there are singularities, associated with a diverging energy density. Homogeneity and isotropy preclude Weyl curvature, so all singularities must be Ricci singularities (since $w \neq 1/3$, a divergence of the Ricci tensor entails a divergent Ricci scalar). But the Ricci curvature is homogeneous in the foliations of constant ξ , and these leaves approach the null hypersurface that bounds the cosmological half-diamond. Therefore the space is singular on these surfaces (which are within a finite affine distance) and the Penrose diagram of the space-time is that depicted in Fig. 5.

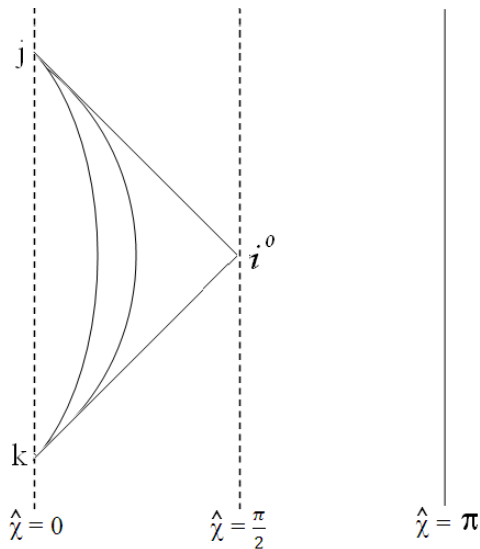


FIG. 4: The Penrose Diagram for our space-time when $\beta > 2$. Here k and j are observers at $\hat{\chi} = 0$; all cosmological observers move from k to j inside the half-diamond depicted (to be repeated up and down the diagram). The diagram can be extended to the whole Einstein static Universe, i.e. up to $\hat{\chi} = \pi$. Points at $\hat{\chi} = \pi/2$ and $\hat{\xi} = n\pi$ represent spatial infinity.

Finally note that a simple coordinate system can be obtained if the angular variables can be ignored. By defining a transformation as above but with (34) and (35) replaced by

$$\tan p = \tanh \frac{\beta u}{4} \quad (41)$$

$$\tan q = \tanh \frac{\beta v}{4} \quad (42)$$

one obtains a metric of form

$$ds^2 = \frac{4 \sec^{\frac{4}{\beta}} \chi}{\beta^2 \cos^{1-\frac{2}{\beta}}(2p) \cos^{1-\frac{2}{\beta}}(2q)} (-d\hat{\xi}^2 + d\hat{\chi}^2 + F(\hat{\xi}, \hat{\chi}) d\Omega_2) \quad (43)$$

Conformal diagrams may be obtained with these coordinates but they hide a complex structure in the angular part of the metric $F(\hat{\xi}, \hat{\chi})$ (with $F = \sin^2 \hat{\chi}$ for $\beta = 2$ only).

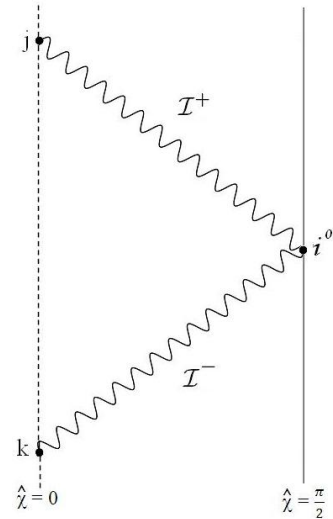


FIG. 5: The Penrose Diagram for our space-time when $\beta < 2$.

V. CONCLUSIONS

In summary we have used the set up of FRW cosmology as a springboard to find simple generalizations of AdS space. They can be seen as $w \neq -1$ FRW solutions, but we extended them to their full manifolds. The embeddings found reveal deformed hyperboloids with the same topology and causal structure as AdS. However the structure of Killing vector fields is entirely modified and should be the subject of further enquiry. We hope that these simple solutions might be useful in assessing the generality of theorems proved for AdS, but conjectured to be true in an adapted form in more realistic spaces.

To conclude we stress that it wouldn't be difficult to generalize our constructions to de Sitter-like space-times (for which all we'd need to do is change the sign of the integration constant C and the signature of the embedding space). More spatial dimensions could also be trivially included. Less trivial is the meaning of our construction when $w > -1/3$. Then the calculations can still be trivially carried out, but they lead to Euclidean metrics. These spaces were not explicitly constructed here but may also be of interest.

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