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# NONUNIVERSAL SELF-SIMILARITY IN A COAGULATION-ANNIHILATION MODEL WITH CONSTANT KERNELS

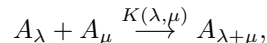
PHILIPPE LAURENÇOT AND HENRY VAN ROESSEL

ABSTRACT. The large time dynamics of a two species coagulation-annihilation system with constant coagulation and annihilation rates is studied analytically when annihilation is complete. A scaling behaviour is observed which varies with the parameter coupling the annihilation of the two species, and, which is nonuniversal in the sense that it varies, in some cases, with the initial conditions as well. The latter actually occurs when either the coupling parameter is equal to one or the initial number of particles is the same for the two species.

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## 1. INTRODUCTION

The irreversible growth of particles by the successive merger of clusters of particles occurs in many fields of science, such as polymer chemistry, colloid science, cloud dynamics and star formation, and the kinetics of coagulation/aggregation models has been the subject of extensive research over the past decades. The aggregation process can be represented schematically as follows:



where  $A_\lambda$  denotes a cluster of size  $\lambda \in [0, \infty)$  and  $K(\lambda, \mu) = K(\mu, \lambda) \geq 0$  is the coagulation kernel giving the rate at which clusters of size  $\lambda$  and  $\mu$  coalesce. The most popular mean-field model describing such phenomena is Smoluchowski's coagulation equation [13]

$$\frac{\partial a}{\partial t}(t, \lambda) = \frac{1}{2} \int_0^\lambda K(\lambda - \mu, \mu) a(t, \lambda - \mu) a(t, \mu) d\mu - a(t, \lambda) \int_0^\infty K(\lambda, \mu) a(t, \mu) d\mu, \quad (1)$$

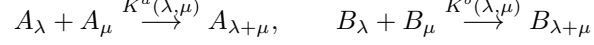
where  $a(t, \lambda)$  represents the number density of particles of size  $\lambda$  at time  $t$ .

The dynamical properties of Smoluchowski's coagulation equation with homogeneous kernels have been thoroughly investigated in recent years (see [1, 3, 4, 8, 10, 14, 15], and the references therein) and strongly depend on the homogeneity exponent of the kernel. Recall that a kernel is said to be homogeneous of degree (or exponent)  $\sigma \in \mathbb{R}$  if  $K(\xi\lambda, \xi\mu) = \xi^\sigma K(\lambda, \mu)$  for all  $\xi, \lambda, \mu > 0$ . In particular, if  $K$  does not increase faster than  $(\lambda\mu)^{1/2}$ , that is if the kernel is homogeneous of degree not exceeding one, a universal scaling behaviour (depending only on the homogeneity of the coagulation kernel) has been predicted in the physical literature [10, 14] for large times and initial conditions decaying sufficiently rapidly as  $\lambda \rightarrow \infty$  with mathematical proofs having been supplied in a few particular cases [5, 7, 11, 12].

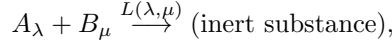
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*Date:* March 5, 2018.

It is of interest to see whether taking into account additional effects will lead to a deviation from this universal scaling behaviour. For example, consider two species,  $A$  and  $B$ , which coagulate amongst themselves



with  $K^a(\lambda, \mu) = K^a(\mu, \lambda) \geq 0$  and  $K^b(\lambda, \mu) = K^b(\mu, \lambda) \geq 0$  being their respective coagulation kernels and, when they come together, completely annihilate each other according to



with  $L(\lambda, \mu) \geq 0$  being the annihilation kernel. For the discrete model with the computation of particular solutions for the specific choice  $K^a = K^b = 2$  and  $L = J$  ( $= \text{const}$ ), and with monodisperse initial conditions, it has been found in [2, 9] that this coupling of coagulation and annihilation in a two-species population of particles shows evidence of a nonuniversal scaling behaviour.

The purpose of this paper is to extend the formal analysis that Ben-Naim and Krapivsky [2] gave for the discrete case of coagulation–annihilation with monodisperse initial conditions. We give a complete and rigorous classification of the scaling behaviour for the continuous case of coagulation–annihilation with arbitrary non-negative initial conditions.

With constant coagulation and annihilation kernels  $K^a(\lambda, \mu) = K^b(\lambda, \mu) = k > 0$  and  $L(\lambda, \mu) = l > 0$ , the mean-field model describing the time evolution of the particle size distributions  $a(t, \lambda)$  and  $b(t, \lambda)$  of particles of species  $A$  and  $B$  with size  $\lambda \in [0, \infty)$  at time  $t \geq 0$  is:

$$\begin{aligned} \frac{\partial a}{\partial t}(t, \lambda) &= \frac{1}{2} \int_0^\lambda k a(t, \lambda - \mu) a(t, \mu) d\mu - a(t, \lambda) \left( \int_0^\infty k a(t, \mu) d\mu + \int_0^\infty l b(t, \mu) d\mu \right), \\ \frac{\partial b}{\partial t}(t, \lambda) &= \frac{1}{2} \int_0^\lambda k b(t, \lambda - \mu) b(t, \mu) d\mu - b(t, \lambda) \left( \int_0^\infty k b(t, \mu) d\mu + \int_0^\infty l a(t, \mu) d\mu \right). \end{aligned}$$

If we rescale the time  $t \mapsto 2t/k$  and introduce a parameter  $J := 2l/k$ , which measures the relative strength of the annihilation kernel to the coagulation kernel, we may write this system more conveniently with a single free parameter as:

$$\frac{\partial a}{\partial t}(t, \lambda) = \int_0^\lambda a(t, \lambda - \mu) a(t, \mu) d\mu - a(t, \lambda)(2A(t) + JB(t)), \quad (2)$$

$$\frac{\partial b}{\partial t}(t, \lambda) = \int_0^\lambda b(t, \lambda - \mu) b(t, \mu) d\mu - b(t, \lambda)(2B(t) + JA(t)), \quad (3)$$

where

$$A(t) := \int_0^\infty a(t, \lambda) d\lambda, \quad B(t) := \int_0^\infty b(t, \lambda) d\lambda, \quad t \geq 0,$$

represent the total particle number of species  $A$  and  $B$  present at time  $t$  respectively. Now we study the system of equations (2) and (3) with initial conditions:

$$a(0, \lambda) = a_0(\lambda) \geq 0, \quad b(0, \lambda) = b_0(\lambda) \geq 0, \quad \lambda \in (0, \infty).$$

Let us denote the total initial particle numbers and masses of species  $A$  and  $B$  respectively by:

$$\begin{aligned} A_0 &:= A(0) = \int_0^\infty a_0(\lambda) d\lambda, & B_0 &:= B(0) = \int_0^\infty b_0(\lambda) d\lambda, \\ M_0^a &:= \int_0^\infty \lambda a_0(\lambda) d\lambda, & M_0^b &:= \int_0^\infty \lambda b_0(\lambda) d\lambda, \end{aligned}$$

with the ratio of initial particle concentrations given by  $\vartheta := B_0/A_0$ . Then without loss of generality, we may assume  $A_0 \geq B_0$  (i.e.  $\vartheta \in [0, 1]$ ) and we have the following:

**Main Result.** *For nonnegative initial conditions  $a_0, b_0 \in L^1(0, \infty; (1+x) dx)$ , the solution  $(a, b)$  to Eqs. (2) and (3) exhibits the following self-similar scaling behaviour:*

(1) *If  $J > 1$  then*

$$\lim_{t \rightarrow \infty} t^2 a(t, \lambda t) = \alpha e^{-\alpha \lambda}, \quad \text{with } \alpha := \frac{1}{(1-\vartheta)^{J/(J-1)} M_0^a}, \quad (4)$$

*while  $b$  seems not to have a self-similar scaling behaviour.*

(2) *If  $J = 1$  then setting*

$$\kappa_a := \frac{A_0}{A_0 + B_0}, \quad \text{and} \quad \kappa_b := \frac{B_0}{A_0 + B_0}$$

*we have*

$$\lim_{t \rightarrow \infty} t^{1+\kappa_a} a(t, \lambda t^{\kappa_a}) = \kappa_a \alpha e^{-\alpha \lambda}, \quad \text{with } \alpha := \frac{A_0}{(A_0 + B_0)^{\kappa_a} M_0^a}, \quad (5)$$

$$\lim_{t \rightarrow \infty} t^{1+\kappa_b} b(t, \lambda t^{\kappa_b}) = \kappa_b \beta e^{-\beta \lambda}, \quad \text{with } \beta := \frac{B_0}{(A_0 + B_0)^{\kappa_b} M_0^b}. \quad (6)$$

(3) *If  $J \in (0, 1)$  then*

$$\lim_{t \rightarrow \infty} ((J+1)t)^{(J+2)/(J+1)} a(t, \lambda ((J+1)t)^{1/(J+1)}) = \alpha e^{-\alpha \lambda}, \quad (7)$$

$$\lim_{t \rightarrow \infty} ((J+1)t)^{(J+2)/(J+1)} b(t, \lambda ((J+1)t)^{1/(J+1)}) = \beta e^{-\beta \lambda}, \quad (8)$$

*with*

$$\alpha := A_0^{J/(J+1)} \vartheta^{J/(1-J^2)}, \quad \beta := A_0^{J/(J+1)} \vartheta^{-J^2/(1-J^2)}. \quad \blacksquare$$

As in the case of Smoluchowski's coagulation equation (1) with constant coagulation kernel, the specific structure of (2)-(3) allows us to make use of the Laplace transform to cast the problem as a dynamical system which can be solved explicitly in terms of  $A$ ,  $B$ , and the initial conditions. A thorough study of the large time behaviour of  $A$  and  $B$  then leads to the identification of the scaling exponents and the pointwise convergence of re-scaled Laplace transforms. According to Feller [6] this turns out to guarantee the pointwise convergence of a re-scaled version of the cumulative distribution function of  $a$  and/or  $b$  defined by

$$(t, \lambda) \mapsto \int_0^\lambda a(t, \mu) d\mu, \quad (t, \lambda) \mapsto \int_0^\lambda b(t, \mu) d\mu.$$

This in turn implies the stated convergence on  $a$  and  $b$ , but in a weak sense.

## 2. FORMULATION AS A DYNAMICAL SYSTEM

Define Laplace transforms:

$$F(t, x) := \int_0^\infty e^{-\lambda x} a(t, \lambda) d\lambda, \quad G(t, x) := \int_0^\infty e^{-\lambda x} b(t, \lambda) d\lambda.$$

Then we get

$$\begin{aligned} \frac{\partial F}{\partial t}(t, x) &= F^2(t, x) - (2A(t) + JB(t)) F(t, x), \\ \frac{\partial G}{\partial t}(t, x) &= G^2(t, x) - (2B(t) + JA(t)) G(t, x), \end{aligned}$$

with initial conditions

$$F(0, x) = F_0(x) := \int_0^\infty e^{-\lambda x} a_0(\lambda) d\lambda, \quad G(0, x) = G_0(x) := \int_0^\infty e^{-\lambda x} b_0(\lambda) d\lambda.$$

These equations are easily solved yielding:

$$F(t, x) = \frac{F_0(x)e^{-P(t)}}{1 - F_0(x) \int_0^t e^{-P(\tau)} d\tau}, \quad G(t, x) = \frac{G_0(x)e^{-Q(t)}}{1 - G_0(x) \int_0^t e^{-Q(\tau)} d\tau}, \quad (9)$$

where

$$P(t) := \int_0^t [2A(\tau) + JB(\tau)] d\tau, \quad Q(t) := \int_0^t [2B(\tau) + JA(\tau)] d\tau. \quad (10)$$

It also follows from (2) and (3) that  $A$  and  $B$  satisfy the following dynamical system:

$$\frac{dA}{dt} = -A^2 - JAB, \quad A(0) = A_0, \quad (11)$$

$$\frac{dB}{dt} = -B^2 - JAB, \quad B(0) = B_0. \quad (12)$$

It is clear that the origin ( $A = B = 0$ ) is the only physically relevant critical point and, as the solution below indicates, it is globally stable. Indeed, make a change of variable:

$$u = A - B, \quad v = AB,$$

to get

$$\frac{du}{dt} = -u\sqrt{u^2 + 4v}, \quad u(0) = u_0 := A_0 - B_0, \quad (13)$$

$$\frac{dv}{dt} = -(J+1)v\sqrt{u^2 + 4v}, \quad v(0) = v_0 := A_0B_0. \quad (14)$$

Clearly, if  $u_0 = 0$ , then  $u(t) = 0$  for all  $t$ . Otherwise, for  $u_0 > 0$ , we may divide one equation by the other, to get

$$\frac{dv}{du} = (J+1)\frac{v}{u},$$

which leads to

$$\frac{v}{v_0} = \left(\frac{u}{u_0}\right)^{J+1}. \quad (15)$$

Thus we have

$$\frac{du}{dt} = -u\sqrt{u^2 + \kappa u^{J+1}}, \quad \text{where} \quad \kappa = \frac{4v_0}{u_0^{J+1}},$$

which gives

$$\int_{u(t)}^{u_0} \frac{d\xi}{\xi \sqrt{\xi^2 + \kappa \xi^{J+1}}} = t \quad \text{for all } t \geq 0. \quad (16)$$

Clearly  $u$  is a positive and decreasing function of time and we have

$$\lim_{t \rightarrow \infty} u(t) = 0. \quad (17)$$

### 3. THE ASYMMETRIC CASE: $A_0 > B_0$

**3.1. The case  $J > 1$ .** In this case, we have  $\sqrt{\xi^2 + \kappa \xi^{J+1}} \sim \xi$  as  $\xi \rightarrow 0$ . Consequently, by (16) and (17), we have

$$t \sim \int_{u(t)}^{u_0} \frac{d\xi}{\xi^2} \sim \frac{1}{u(t)} \quad \text{as } t \rightarrow \infty,$$

whence

$$u(t) \sim \frac{1}{t} \quad \text{and} \quad v(t) \sim \frac{\kappa}{4t^{J+1}} \quad \text{as } t \rightarrow \infty$$

by (15). In particular,  $v(t)/u(t)^2 \rightarrow 0$  as  $t \rightarrow \infty$  and, since

$$A = \frac{u + \sqrt{u^2 + 4v}}{2} \quad \text{and} \quad B = \frac{-u + \sqrt{u^2 + 4v}}{2}, \quad (18)$$

we end up with

$$A(t) \sim \frac{1}{t} \quad \text{and} \quad B(t) \sim \frac{\kappa}{4t^J} \quad \text{as } t \rightarrow \infty. \quad (19)$$

We next introduce for  $t \geq 0$

$$\mathcal{A}(t) := \exp \left\{ \int_0^t A(\tau) d\tau \right\}, \quad \mathcal{B}(t) := \exp \left\{ \int_0^t B(\tau) d\tau \right\}, \quad (20)$$

and note that

$$e^{P(t)} = \mathcal{A}(t)^2 \mathcal{B}(t)^J, \quad e^{Q(t)} = \mathcal{A}(t)^J \mathcal{B}(t)^2, \quad (21)$$

the functions  $P$  and  $Q$  being defined in (10). Since  $e^P$  and  $e^Q$  are involved in the formulae giving the Laplace transforms  $F$  and  $G$  of  $a$  and  $b$ , we study their behaviour for large times or equivalently that of  $\mathcal{A}$  and  $\mathcal{B}$ . To this end, we observe that (11) and (12) also read  $d \ln A/dt = -(A + JB)$  and  $d \ln B/dt = -(JA + B)$ , whence, after integration,

$$\frac{A_0}{A(t)} = \mathcal{A}(t) \mathcal{B}(t)^J \quad \text{and} \quad \frac{B_0}{B(t)} = \mathcal{A}(t)^J \mathcal{B}(t), \quad t \geq 0. \quad (22)$$

Therefore,

$$\mathcal{A}(t)^{J^2-1} = \frac{A(t)}{A_0} \frac{B_0^J}{B(t)^J}, \quad \mathcal{B}(t)^{J^2-1} = \frac{A_0^J}{A(t)^J} \frac{B(t)}{B_0}, \quad (23)$$

and we deduce from (19) that

$$\begin{cases} \mathcal{A}(t) \sim A_0 (1 - \vartheta)^{J/(J-1)} t \\ \mathcal{B}(t) \sim (1 - \vartheta)^{-1/(J-1)} \end{cases} \quad \text{as } t \rightarrow \infty. \quad (24)$$

Coming back to  $e^P$  and  $e^Q$ , it follows from (21) and (24) that

$$\begin{cases} e^{P(t)} \sim A_0^2 (1 - \vartheta)^{J/(J-1)} t^2 \\ e^{Q(t)} \sim A_0^J (1 - \vartheta)^{(J^2-2)/(J-1)} t^J \end{cases} \quad \text{as } t \rightarrow \infty. \quad (25)$$

In particular, both  $e^{-P}$  and  $e^{-Q}$  belong to  $L^1(0, \infty)$  and we infer from (9) (with  $x = 0$ ), (21), (22), and (24) that

$$1 - A_0 \int_0^t e^{-P(\tau)} d\tau = \frac{A_0}{A(t)e^{P(t)}} = \frac{1}{\mathcal{A}(t)} \longrightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (26)$$

$$1 - B_0 \int_0^t e^{-Q(\tau)} d\tau = \frac{B_0}{B(t)e^{Q(t)}} = \frac{1}{\mathcal{B}(t)} \longrightarrow (1 - \vartheta)^{1/(J-1)} \quad \text{as } t \rightarrow \infty. \quad (27)$$

Consequently,

$$\int_0^\infty e^{-P(\tau)} d\tau = \frac{1}{A_0} \quad \text{and} \quad \int_0^\infty e^{-Q(\tau)} d\tau = \frac{1 - (1 - \vartheta)^{1/(J-1)}}{B_0}. \quad (28)$$

We are now in a position to identify the large time behaviour of  $F$ . To this end, we observe that, according to (26) and (28), we have

$$\begin{aligned} 1 - F_0(x) \int_0^t e^{-P(\tau)} d\tau &= \frac{F_0(0) - F_0(x)}{A_0} + F_0(x) \int_t^\infty e^{-P(\tau)} d\tau \\ &= \frac{F_0(0) - F_0(x)}{A_0} + \frac{F_0(x)}{A_0} \frac{1}{\mathcal{A}(t)} \end{aligned}$$

for  $(t, x) \in [0, \infty)^2$ . Since  $a_0 \in L^1(0, \infty; (1+x)dx)$ , it follows from (24) that the two terms of the right-hand side of the above identity are of the same order  $1/t$  if we scale  $x$  by  $t$ , that is,

$$\begin{aligned} 1 - F_0\left(\frac{x}{t}\right) \int_0^t e^{-P(\tau)} d\tau &= \frac{1}{A_0} \left[ F_0(0) - F_0\left(\frac{x}{t}\right) + F_0\left(\frac{x}{t}\right) \frac{1}{\mathcal{A}(t)} \right] \\ &\sim \frac{1}{A_0} \left[ -\frac{x}{t} F_0'(0) + \frac{1}{(1 - \vartheta)^{J/(J-1)} t} \right] \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Recalling (9), we infer from (25) and the above equivalence that, for all  $x \geq 0$ ,

$$\lim_{t \rightarrow \infty} tF\left(t, \frac{x}{t}\right) = \frac{1}{1 + (1 - \vartheta)^{J/(J-1)} M_0^\alpha x} \quad \text{as } t \rightarrow \infty. \quad (29)$$

Next, because of the non-zero limit in (27), it does not seem possible to find a scaling behaviour for  $G$ , an observation already made in Ben-Naim and Krapivsky [2]. Still, it is clear from (19) that species  $B$  is disappearing at a faster rate than species  $A$ .

**3.2. The case  $J = 1$ .** In this case Eq. (16) yields

$$u(t) = \frac{u_0}{\varphi(t)}, \quad v(t) = \frac{v_0}{\varphi(t)^2},$$

where

$$\varphi(t) := 1 + (A_0 + B_0)t.$$

This leads to

$$A(t) = \frac{A_0}{\varphi(t)}, \quad B(t) = \frac{B_0}{\varphi(t)}.$$

Finally, we have

$$F(t, x) = \frac{A_0 F_0(x) \varphi(t)^{-1 - \kappa_a}}{A_0 - F_0(x) [1 - \varphi(t)^{-\kappa_a}]}, \quad G(t, x) = \frac{B_0 G_0(x) \varphi(t)^{-1 - \kappa_b}}{B_0 - G_0(x) [1 - \varphi(t)^{-\kappa_b}]},$$

where

$$\kappa_a := \frac{A_0}{A_0 + B_0}, \quad \kappa_b := \frac{B_0}{A_0 + B_0}.$$

Thus we have

$$\begin{aligned} \lim_{t \rightarrow \infty} t F \left( t, \frac{x}{t^{\kappa_a}} \right) &= \lim_{t \rightarrow \infty} \left\{ \frac{t}{\varphi(t)} \cdot \frac{A_0 F_0 \left( \frac{x}{t^{\kappa_a}} \right)}{F_0 \left( \frac{x}{t^{\kappa_a}} \right) + \left[ A_0 - F_0 \left( \frac{x}{t^{\kappa_a}} \right) \right] \varphi(t)^{\kappa_a}} \right\} \\ &= \frac{A_0^2}{(A_0 + B_0)[A_0 + (A_0 + B_0)^{\kappa_a} M_0^a x]}. \end{aligned}$$

Similarly,

$$\lim_{t \rightarrow \infty} t G \left( t, \frac{x}{t^{\kappa_b}} \right) = \frac{B_0^2}{(A_0 + B_0)[B_0 + (A_0 + B_0)^{\kappa_b} M_0^b x]}.$$

*Example.* For initial conditions given by

$$a_0(\lambda) = A_0 e^{-\lambda}, \quad b_0(\lambda) = B_0 e^{-\lambda},$$

the exact solution is easily computed

$$a(t, \lambda) = \frac{A_0}{\varphi(t)^{1+\kappa_a}} e^{-\lambda/\varphi(t)^{\kappa_a}}, \quad b(t, \lambda) = \frac{B_0}{\varphi(t)^{1+\kappa_b}} e^{-\lambda/\varphi(t)^{\kappa_b}}.$$

This is clearly a self-similar solution to (2)-(3).

**3.3. The case  $J \in (0, 1)$ .** In that case, we have  $\sqrt{\xi^2 + \kappa \xi^{J+1}} \sim \sqrt{\kappa} \xi^{(J+1)/2}$  as  $\xi \rightarrow 0$ . Consequently, by (16) and (17), we have

$$t \sim \frac{1}{\sqrt{\kappa}} \int_{u(t)}^{u_0} \frac{d\xi}{\xi^{(J+3)/2}} \sim \frac{2}{\sqrt{\kappa}(J+1)} \frac{1}{u(t)^{(J+1)/2}} \quad \text{as } t \rightarrow \infty,$$

whence

$$u(t) \sim \left( \frac{2}{\sqrt{\kappa}(J+1)t} \right)^{2/(J+1)} \quad \text{and} \quad v(t) \sim \left( \frac{1}{(J+1)t} \right)^2 \quad \text{as } t \rightarrow \infty$$

by (15). Since  $u(t)^2/v(t) \rightarrow 0$  as  $t \rightarrow \infty$ , we deduce from (18) that

$$A(t) \sim \frac{1}{(J+1)t} \quad \text{and} \quad B(t) \sim \frac{1}{(J+1)t} \quad \text{as } t \rightarrow \infty. \quad (30)$$

Recalling that the functions  $\mathcal{A}$  and  $\mathcal{B}$  are defined in (20), we infer from (23) and (30) that

$$\begin{cases} \mathcal{A}(t) \sim \vartheta^{J/(J^2-1)} (A_0 (J+1)t)^{1/(J+1)} \\ \mathcal{B}(t) \sim \vartheta^{-1/(J^2-1)} (A_0 (J+1)t)^{1/(J+1)} \end{cases} \quad \text{as } t \rightarrow \infty. \quad (31)$$

Coming back to  $e^P$  and  $e^Q$ , it follows from (21) and (31) that

$$\begin{cases} e^{P(t)} \sim \vartheta^{J/(J^2-1)} (A_0 (J+1)t)^{(J+2)/(J+1)} \\ e^{Q(t)} \sim \vartheta^{(J^2-2)/(J^2-1)} (A_0 (J+1)t)^{(J+2)/(J+1)} \end{cases} \quad \text{as } t \rightarrow \infty. \quad (32)$$

Once again, both  $e^{-P}$  and  $e^{-Q}$  belong to  $L^1(0, \infty)$  and we infer from (9) (with  $x = 0$ ), (21), (22), and (31) that

$$1 - A_0 \int_0^t e^{-P(\tau)} d\tau = \frac{A_0}{A(t)e^{P(t)}} = \frac{1}{\mathcal{A}(t)} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (33)$$

$$1 - B_0 \int_0^t e^{-Q(\tau)} d\tau = \frac{B_0}{B(t)e^{Q(t)}} = \frac{1}{\mathcal{B}(t)} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (34)$$



Consequently,

$$\int_0^\infty e^{-P(\tau)} d\tau = \frac{1}{A_0} \quad \text{and} \quad \int_0^\infty e^{-Q(\tau)} d\tau = \frac{1}{B_0}. \quad (35)$$

We are now in a position to identify the large time behaviour of  $F$  and  $G$ . To this end, we observe that, according to (33) and (35), we have

$$\begin{aligned} 1 - F_0(x) \int_0^t e^{-P(\tau)} d\tau &= \frac{F_0(0) - F_0(x)}{A_0} + F_0(x) \int_t^\infty e^{-P(\tau)} d\tau \\ &= \frac{F_0(0) - F_0(x)}{A_0} + \frac{F_0(x)}{A_0} \frac{1}{\mathcal{A}(t)} \end{aligned}$$

for  $(t, x) \in [0, \infty)^2$ . Since  $a_0 \in L^1(0, \infty; (1+x)dx)$ , it follows from (31) that the two terms of the right-hand side of the above identity are of the same order  $1/t^{1/(J+1)}$  if we scale  $x$  by  $t^{J+1}$ , that is, setting  $s(t) := ((J+1)t)^{1/(J+1)}$ ,

$$\begin{aligned} 1 - F_0\left(\frac{x}{s(t)}\right) \int_0^t e^{-P(\tau)} d\tau &= \frac{1}{A_0} \left[ F_0(0) - F_0\left(\frac{x}{s(t)}\right) + F_0\left(\frac{x}{s(t)}\right) \frac{1}{\mathcal{A}(t)} \right] \\ &\sim \frac{1}{A_0} \left[ -\frac{x}{s(t)} F_0'(0) + \frac{A_0^{J/(J+1)}}{\vartheta^{J/(J^2-1)} s(t)} \right] \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Recalling (9), we infer from (32) and the above equivalence that, for all  $x \geq 0$ ,

$$\lim_{t \rightarrow \infty} s(t)^{J+1} F\left(t, \frac{x}{s(t)}\right) = \frac{1}{1 + \vartheta^{J/(J^2-1)} A_0^{-J/(J+1)} M_0^a x} \quad (36)$$

Similarly, by (34) and (35), we have

$$\begin{aligned} 1 - G_0(x) \int_0^t e^{-Q(\tau)} d\tau &= \frac{G_0(0) - G_0(x)}{B_0} + G_0(x) \int_t^\infty e^{-Q(\tau)} d\tau \\ &= \frac{G_0(0) - G_0(x)}{B_0} + \frac{G_0(x)}{B_0} \frac{1}{\mathcal{B}(t)} \end{aligned}$$

for  $(t, x) \in [0, \infty)^2$ . Since  $b_0 \in L^1(0, \infty; (1+x)dx)$ , it follows from (31) that the two terms of the right-hand side of the above identity are of the same order  $1/t^{1/(J+1)}$  if we scale  $x$  by  $t^{J+1}$ , that is, keeping the notation  $s(t) = ((J+1)t)^{1/(J+1)}$ ,

$$\begin{aligned} 1 - G_0\left(\frac{x}{s(t)}\right) \int_0^t e^{-Q(\tau)} d\tau &= \frac{1}{B_0} \left[ G_0(0) - G_0\left(\frac{x}{s(t)}\right) + G_0\left(\frac{x}{s(t)}\right) \frac{1}{\mathcal{B}(t)} \right] \\ &\sim \frac{1}{\vartheta A_0} \left[ -\frac{x}{s(t)} G_0'(0) + \frac{A_0^{J/(J+1)} \vartheta^{J^2/(J^2-1)}}{s(t)} \right] \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Recalling (9), we infer from (32) and the above equivalence that, for all  $x \geq 0$ ,

$$\lim_{t \rightarrow \infty} s(t)^{J+1} G\left(t, \frac{x}{s(t)}\right) = \frac{1}{1 + \vartheta^{J^2/(1-J^2)} A_0^{-J/(J+1)} M_0^b x}. \quad (37)$$

#### 4. THE SYMMETRIC CASE: $A_0 = B_0$

Clearly, in this case we have  $u(t) = 0$  by (13) and it follows from (14) and (18) that

$$v(t) = \frac{A_0^2}{\varphi(t)^2}, \quad A(t) = B(t) = \frac{A_0}{\varphi(t)}, \quad \text{where } \varphi(t) := 1 + (J+1)A_0 t.$$

This leads to

$$F(t, x) = \frac{A_0 F_0(x) \varphi(t)^{-1}}{(A_0 - F_0(x)) \varphi(t)^{1/(J+1)} + F_0(x)}.$$

Thus we have

$$\lim_{t \rightarrow \infty} t F \left( t, \frac{x}{t^{1/(J+1)}} \right) = \frac{A_0}{(J+1)[A_0 + [(J+1)A_0]^{1/(J+1)} M_0^a x]}. \quad (38)$$

Similarly

$$\lim_{t \rightarrow \infty} t G \left( t, \frac{x}{t^{1/(J+1)}} \right) = \frac{A_0}{(J+1)[A_0 + [(J+1)A_0]^{1/(J+1)} M_0^b x]}.$$

Comparing (29) and (38) reveals another dependence of the scaling exponents on the initial conditions for  $J > 1$ , since the value of the scaling exponents for species  $A$  vary according to whether  $\vartheta = 1$  or  $\vartheta < 1$ . This again demonstrates the sensitivity of the scaling behaviour of this coagulation-annihilation model with respect to the initial conditions already observed for  $J = 1$  and  $\vartheta \in (0, 1)$ .

## 5. CONCLUSION

In this paper we examined the dynamics of a two species coagulation-annihilation system with constant coagulation and annihilation rates and complete annihilation. It was found that the long time self-similar behaviour, or lack thereof, of the system is primarily governed by a parameter  $J$  which represents the relative strength of the coupling between the annihilation and the coagulation kernels.

It is clear from the main result that  $J = 1$  is a threshold value for the coupling parameter  $J$  at which a crossover from one scaling regime to another takes place, with a nonuniversal scaling behaviour dividing the two. More precisely, for  $J > 1$ , the scaling behaviour of  $a$ , the species initially being more abundant, is that of Smoluchowski's coagulation equation Eq. (1) with constant coagulation kernel  $K = 2$  [10], and the influence of the species  $B$  being only reflected by the dependence of the constant  $\alpha$  upon  $\vartheta$ . In fact, an asymptotic simplification occurs in that case and the dynamics of (2)-(3) is governed by the binary coagulation of species  $A$  alone. The fact that  $b$  appears not to have a scaling behaviour has already been noted in Ben-Naim and Krapivsky [2] for monodisperse initial conditions and is shown here to be true in general. Next, the nonuniversality of the scaling behaviour for  $J = 1$  is obvious as the scaling exponents  $\kappa_a$  and  $\kappa_b$  vary with the initial conditions and the scaling exponents are solely determined by the initial particle numbers  $A_0$  and  $B_0$ . This shows in particular that, during the time evolution, the details of the initial distributions of particles of the two species fade away, only retaining the two "macroscopic" parameters  $A_0$  and  $B_0$ . Still, in contrast with the case  $J > 1$ , both  $a$  and  $b$  have a self-similar behaviour. Finally, when  $J \in (0, 1)$ , there is a balance between coagulation and annihilation and both  $a$  and  $b$  have the same scaling behaviour.

Let us finally emphasize that, as already pointed out in [2], the scaling behaviour is also not truly universal for  $J > 1$  as a different scaling behaviour appears for symmetric initial data  $A_0 = B_0$ , see Section 4. This can be expected from the convergence (4) as the right hand side converges to zero as  $\vartheta$  approaches 1.

A related model which seems to exhibit nonuniversal scaling behaviors is the coagulation-annihilation model with constant kernels and incomplete annihilation

which corresponds to the elementary reactions

$$A_\lambda + B_\mu \longrightarrow \begin{cases} A_{\lambda-\mu} & \text{if } \lambda > \mu, \\ \text{inert substance} & \text{if } \lambda = \mu, \\ B_{\mu-\lambda} & \text{if } \lambda < \mu. \end{cases}$$

Some explicit solutions have been constructed in [9] which show evidence of nonuniversal scaling but a thorough analysis is still lacking and will be investigated in a forthcoming work. It is indeed yet unclear whether the approach developed in this paper or a modification thereof could apply to that case as well.

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