Poncelet's theorem and Billiard knots

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October 3, 2011

Abstract

Let D be any elliptic right cylinder. We prove that every type of knot can be realized as the trajectory of a ball in D. This proves a conjecture of Lamm and gives a new proof of a conjecture of Jones and Przytycki. We use Jacobi's proof of Poncelet's theorem by means of elliptic functions.

keywords: Poncelet's theorem, Jacobian elliptic functions, Billiard knots , Lissajous knots, Cylinder knots

Mathematics Subject Classification 2000: 57M25,

Contents

1 Introduction

The Poncelet closure theorem is one of the most beautiful theorems in geometry. It says that if there exists a closed polygon inscribed in a conic E and circumscribed about another conic, then there exist infinitely such polygons, one with a vertex at any given point of E.

When the conics are concentric circles the proof is very simple, each Poncelet polygon is obtained by rotating any one of them. What makes Poncelet's theorem great, is that it is impossible to generalize this simple proof.

Poncelet's proof ([Po]) uses "pure" projective geometry, see [Be, DB, Sa] for modern proofs along these lines. Shortly after the publication of Poncelet's book, Jacobi ([Ja]) gave a proof by means of Jacobian elliptic functions. He discovered what is now called a uniformization of the problem by an elliptic curve. Most modern developments and generalizations follow Jacobi's proof, see [Lau, GH, BKOR, Sc, LT] .

A beautiful example of Poncelet polygonal lines is given by elliptic billiards. If a segment of a billiard trajectory in an ellipse E does not intersect the focal segment $[F_1F_2]$ of E, then there exists an ellipse C called a caustic, such that the trajectory is a Poncelet polygonal line inscribed in E and circumscribed about C, see [St, T, LT].

On the other hand, Jones and Przytycki defined billiard knots as periodic billiard trajectories without self-intersections in a three-dimensional billiard. They proved that billiard knots in a cube are very special knots, the Lissajous knots. They also conjectured that every knot is a billiard knot in some convex polyhedron. ([JP], see also [La2, C, BHJS, BDHZ, P]).

Lamm and Obermeyer [La1, LO] proved that not all knots are billiard knots in a cylinder. Then Lamm conjectured that there exists an elliptic cylinder containing all knots as billiard knots ([La1, O]). It is easy to see that Lamm's conjecture implies the conjecture of Jones and Przytycki: if K is a billiard knot in a convex set, then it is also a billiard knot in the polyhedron delimited by the tangent planes. Dehornoy constructed in [D] (see also [O]) a billiard which contains all knots, but this billiard is not convex.

In this paper we will use Jacobi's method to study billiard trajectories in a right cylinder with an elliptic basis. We obtain a proof of Lamm's conjecture. Our result is more precise:

Theorem 20 *Let E be an ellipse which is not a circle, and let D be the elliptic cylinder $D = E \times [0, 1]$. Every knot (or link) is a billiard knot (or link) in D.*

Billiard trajectories in an ellipse are introduced in section 2. We show that an elementary theorem of Poncelet implies the existence of a caustic. We also show that Poncelet polygons in a pair of nested ellipses are projections of torus knots. Then, by a theorem of Manturov ([M]), we deduce that every knot has a projection which is a billiard trajectory in an ellipse.

In section 3, we recall the basic definitions and properties of the Jacobian elliptic functions $\mathrm{sn}(z)$ and $\mathrm{cn}(z)$. Then we give the Hermite–Laurent version of Jacobi's proof, which is based on Jacobi's uniformization lemma.

In section 4, we use Jacobi's lemma to compute the coordinates of the crossings and vertices of a billiard trajectory in E. We deduce that if the number of sides of a periodic billiard trajectory is odd, then it is generally completely irregular. This means that if we start at any vertex, 1 and the arc lengths of the other vertices and crossings are linearly independent over $\mathbf{Q}$. We also see how our proofs generalize in the link case.

In section 5, we use Kronecker's density theorem to obtain our main result. The same strategy was used in [KP2] to give an elementary proof of the Jones–Przytycki conjecture. There is another application of Kronecker's theorem to the construction of knots in [KP1].

2 Billiard trajectories in an ellipse

The study of billiard trajectories in an ellipse was introduced by Birkhoff in 1927 ([Bi]), see also [T] for a modern exposition of the subject.

2.1 Some elementary facts

The following elementary theorem is due to Poncelet ([Po, Be]).

Theorem 1 (The second little Poncelet theorem)
Let E be an ellipse (or a hyperbola) with foci F_1 and F_2. Let PM_1 and PM_2 be the tangents to E at the points M_1 and M_2. Then the angles $\widehat{M_1PM_2}$ and $\widehat{F_1PF_2}$ have the same bisectors.

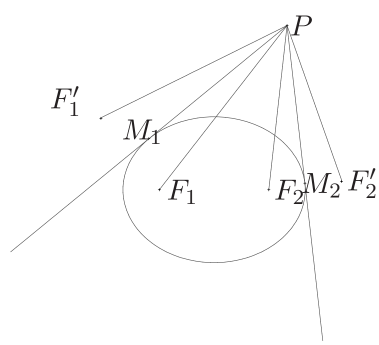

Figure 1: The second little Poncelet theorem.

Proof. Suppose that E is an ellipse (the case of a hyperbola is similar).

Reflect F_1 in PM_1 to F_1', and F_2 in PM_2 to F_2'. As PM_1 is a bisector of $\widehat{F_1M_1F_2}$, we see that F_1', M_1, F_2 are collinear and $F_1'F_2 = M_1F_1 + M_1F_2$ is the major axis of E. We deduce that $F_1'F_2 = F_1F_2'$. Consequently, the triangles $F_1'PF_2$ and F_1PF_2' are congruent, because their sides are of equal length. This implies $\widehat{F_1'PF_2} = \widehat{F_1PF_2'}$, and then

$$\widehat{F_1'PF_1} = \widehat{F_1'PF_2} - \widehat{F_1PF_2} = \widehat{F_2'PF_1} - \widehat{F_1PF_2} = \widehat{F_2'PF_2},$$

which concludes the proof. □

Theorem 2 *Suppose that some segment of a billiard trajectory in an ellipse does not intersect the focal segment $[F_1F_2]$. Then the billiard trajectory remains forever tangent to a fixed confocal ellipse called the caustic.*

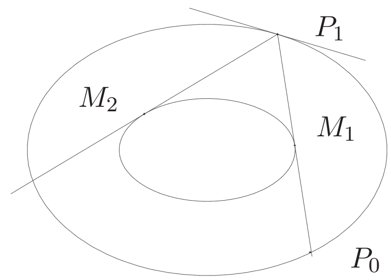

Figure 2: Existence of an elliptic caustic.

Proof. Let P_0P_1 be a segment of a billiard trajectory in an ellipse E, and suppose that P_0P_1 does not intersect $[F_1F_2]$.

Reflect F_1 in P_0P_1 to F_1', and consider the ellipse $C = \{MF_1 + MF_2 = F_2F_1'$. We see that $M_1 = F_2F_1' \bigcap P_0P_1$ belongs to E, and since P_0P_1 is a bisector of $\widehat{F_1M_1F_2}$, it is the tangent to C at M_1.

Draw P_1M_2 the second tangent to C. By the second little Poncelet theorem, the angles $\widehat{M_1P_1M_2}$ and $\widehat{F_1P_1F_2}$ have the same bisectors. Hence P_1M_2 is the second segment of our billiard trajectory, and is tangent to C.

□

Remark 3 *When some segment contains only one focus, then every segment contains a focus, and there is no caustic. When some segment intersects the interior of the focal segment then there is a caustic, which is a hyperbola with foci F_1 and F_2. But the tangency points need not be at a finite distance. This fact will be illustrated by the following example.*

Example

Consider the points $P_0 = (-1,-1)$, $P_1 = (1,1)$, $P_2 = (1,-1)$, and $P_3 = (-1,1)$. The trajectory $P_0P_1P_2P_3$ is a periodic billiard trajectory in the ellipse $E = \{x^2 + 2y^2 = 3\}$. The caustic is the hyperbola $C = \{x^2 - y^2 = 1\}$, and P_0P_1, P_2P_3 are tangent to C at infinity.

By Poncelet's theorem, if we apply the Poncelet construction from another point $Q_0 \in E$, then we obtain another periodic billiard trajectory $Q_0Q_1Q_2Q_3$. Moreover, the diagonals Q_1Q_3 and Q_0Q_2 remain forever parallel to the x-axis (this is a consequence of a theorem of Darboux that we will see later). Consequently, all these Poncelet quadrilaterals are symmetric with respect to the y-axis.

If we start from $Q_1 = Q_3 = A = (0, \sqrt{3/2})$, then the quadrilateral degenerates into a trajectory of the form $A\,B\,A\,C\,A$, where $B = (\sqrt{5/3}, -\sqrt{2/3})$, $C = (-\sqrt{5/3}, -\sqrt{2/3})$.

The preceding example shows that a billiard trajectory $A_0, \ldots, A_n$ such that $A_n = A_0$ needs not be n-periodic. Moreover, a billiard trajectory in an ellipse is not necessarily a

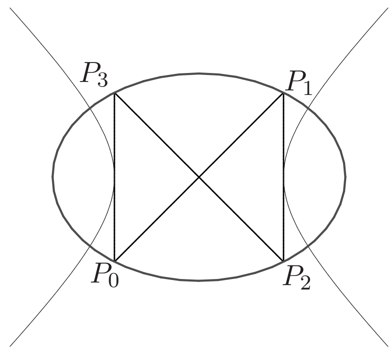

Figure 3: A billiard quadrilateral in an ellipse. The caustic is a hyperbola.

Poncelet polygonal line. Consider for example an ellipse E with foci F_1, F_2 and a chord AB such that AB is the internal bisector of $\widehat{F_1BF_2}$. The polygonal line ABA is a billiard trajectory in E, but generally it is neither a Poncelet polygonal line, nor a periodic billiard trajectory.

Happily, these defects cannot occur for billiard trajectories that do not intersect the focal segment.

Corollary 4 *Let $P_0, P_1, \dots, P_{n-1}$, $P_n = P_0$ be a billiard trajectory in an ellipse E such that P_0P_1 does not intersect the focal segment $[F_1F_2]$. Then it is a periodic billiard trajectory inscribed in E and circumscribed about a confocal ellipse C.*

Proof. Since $P_{n-1}P_0$ does not intersect the focal segment, its reflection at P_0 cannot be P_0P_{n-1}. Since it is another tangent to C through P_0, it must be P_0P_1, and then $P_{n+1} = P_1$. □

Now, we shall prove the existence of billiard polygons with n sides and rotation number p. Following an idea due to Chasles and Birkhoff ([Be, Bi]), we will obtain them as polygons of maximum perimeter among the n-polygons of rotation number p inscribed in E. We shall need the following classic lemma

Lemma 5 *Let A, B be (not necessarily distinct) points of an ellipse E. If the function $f(M) = MA + MB$, $M \in E$ has a local maximum at $C \in E$, then CB is the reflection of CA in the normal to E at C.*

Proof. See [LT]. □

Proposition 6 *Let P_0 be a point of an ellipse E. Let n and p be coprime integers such that $n \geq 2p + 1$. There exists a billiard trajectory in E, of period n, winding number p and starting at P_0.*

Proof. We shall consider the following maximum problem. The domain of definition of the function to maximize is $\mathcal{A} = \{(\alpha_1, \dots, \alpha_n) \in [0, \pi]^n, \sum \alpha_i = 2\pi p\}$, it is a compact set. For $\alpha \in \mathcal{A}$ and $P_0 \in E$, let us define the inscribed polygon $P_0, P_1, \dots, P_n = P_0$ by the angular condition $(\widehat{\overrightarrow{FP_{i-1}}, \overrightarrow{FP_i}}) = \alpha_i$, where F denotes a focus of E. We want to maximize the perimeter of this polygon, which is $f(\alpha) = P_0P_1 + P_1P_2 + \cdots + P_{n-1}P_0$.

Since $\mathcal{A}$ is compact and f continuous, this maximum exists. By lemma 5, it is a billiard trajectory at the points $P_1, \dots, P_{n-1}$.

Let us show that no segment $P_{i-1}P_i$ intersects the open focal segment (FF') of E. If it was the case, then all segments of the trajectory would intersect (FF'), hence the winding number of this trajectory about F would be zero, which is impossible by the definition of our polygon.

If some segment was the major axis, then all segments would be this axis, and all $\alpha_i = \pi$, which is impossible since $\sum \alpha_i = 2p\pi < n\pi$.

If some segment contains only one focus, then it is known that the billiard trajectory converges to the major axis and is not periodic (see [St, Fr]).

We deduce that P_0P_1 does not intersect the focal segment $[FF']$ of E. By lemma 5 and corollary 4, we conclude that $P_0, P_1, \dots P_{n-1}$ is a periodic billiard trajectory in E.

The exact period d of our trajectory is a divisor of n, and we have $n = d\,u$. By the angular condition, u is a divisor of p. Since n and p are coprime $u = 1$ thus n is the exact period of our trajectory.

□

Remark 7 *This does not not prove that the caustics C_n do not depend on the initial point P_0. This is true by Poncelet's theorem, which we shall prove later. Using a theorem of Graves* [Be] *, it can be shown that all the Poncelet polygons in two confocal ellipses have the same perimeter.*

2.2 Poncelet polygons and toric braids

A toric braid is a braid corresponding to the closed braid obtained by projecting the standardly embedded torus knot into the xy-plane. A toric braid is a braid of the form $\tau_{p,n} = \Big(\sigma_1\,\sigma_2\,\cdots\sigma_{p-1}\Big)^n$, where $\sigma_1, \dots, \sigma_{p-1}$ are the standard generators of the full braid group B_p.

Remark 8 *Let E and C be nested ellipses such that there exists a Poncelet polygon inscribed in E and circumscribed about C. Every Poncelet polygon is the projection of a torus knot of type $T(n,p)$, $n \geq 2p+1$. More precisely, if we cut the elliptic annulus delimited by E and C along a half-tangent, then we see that such a polygon is ambient isotopic to the projection of the closure of the toric braid $\tau_{p,n}$. Consequently, it is also ambient isotopic to the star polygon $\left\{{n \atop p}\right\}$, see* [KP2].

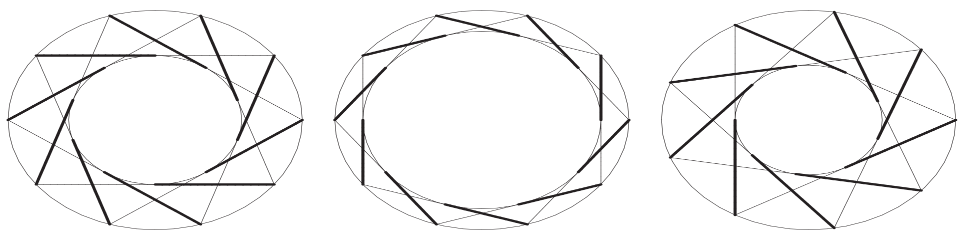

Figure 4: Some Poncelet polygons (or unions of Poncelet polygons) in nested ellipses. They are projections of the toric braids $\tau_{3,10}$, $\tau_{2,10}$ and $\tau_{3,9}$, and are denoted $\left\{{10 \atop 3}\right\}$, $\left\{{10 \atop 2}\right\}$ and $\left\{{9 \atop 3}\right\}$.

We shall need the following results on braids, due to Manturov [M]. A quasitoric braid of type $B(p,n)$ is a braid obtained by changing some crossings in the toric braid $\tau_{p,n}$.

Manturov's theorem tells us that every knot (or link) is realized as the closure of a quasitoric braid ([M]). More precisely, he proved that any μ-component link can be realized as the closure of a quasitoric braid of type $B(p\mu, n\mu)$ where $(p,n)=1$, p even and n odd.

The quasitoric braids form a subgroup of the full braid group, hence there exist trivial quasitoric braids of arbitrarily great length. Hence we can suppose $n \geq 2p+1$ in Manturov's theorem. Using this theorem, proposition 6, and Poncelet's theorem, we obtain the main result of this section.

Theorem 9 *Let E be an ellipse. Every μ-component link has a projection which is the union of μ billiard trajectories in E with the same odd period, and with the same caustic C.*

3 Jacobi's proof of Poncelet's theorem

We shall only need the following properties of elliptic functions, see [WW] for proofs.

3.1 The Jacobian elliptic functions sn z, cn z and dn z.

They depend on the choice of a parameter k, $0<k<1$, called the elliptic modulus.

The Jacobi amplitude $\varphi = \mathrm{am}(z)$ is defined by inverting the elliptic integral

$$z = \int_0^{\varphi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}.$$

It verifies $\mathrm{am}(u+2nK) = \mathrm{am}(u) + n\pi$, where $\mathrm{am}(K) = \frac{\pi}{2}$, and $n \in \mathbf{Z}$.

The Jacobian elliptic functions are defined for z real by

$$\mathrm{sn}\, z = \sin\bigl(\mathrm{am}(z)\bigr), \quad \mathrm{cn}\, z = \cos\bigl(\mathrm{am}(z)\bigr), \quad \mathrm{dn}\, z = \sqrt{1-k^2 \mathrm{sn}^2 z},$$

and can be extended to meromorphic functions on $\mathbf{C}$. When $k=0$, these functions degenerate into the ordinary circular function $\sin z$ and $\cos z$. But, contrarily to the circular functions, they are doubly periodic functions with periods $4K \in \mathbf{R}$, and $4iK' \in i\mathbf{R}$, and they have poles. For example, the poles of $\operatorname{sn} z$ are congruent to iK' (mod.$2K, 2iK'$), its zeros are the points congruent to 0 (mod.$2K, 2iK'$), and its exact periods are $4K,\ 2iK'$. The zeros of $\operatorname{cn} z$ are the points congruent to K (mod.$2K, 2iK'$). We have $\operatorname{sn}(z+2K) = -\operatorname{sn} z$, and $\operatorname{cn}(z+2K) = -\operatorname{cn} z$. We also have $\operatorname{sn}(K+iK') = k^{-1}$, which implies that the zeros of $\operatorname{dn} z$ are the points congruent to $K+iK'$ (mod.$2K, 2iK'$).

We have the following addition formulas

$$\operatorname{sn}(x+y) = \frac{\operatorname{sn} x \operatorname{cn} y \operatorname{dn} y + \operatorname{sn} y \operatorname{cn} x \operatorname{dn} x}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y}, \quad \operatorname{cn}(x+y) = \frac{\operatorname{cn} x \operatorname{cn} y - \operatorname{sn} x \operatorname{sn} y \operatorname{dn} x \operatorname{dn} y}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y}$$

When $k = 0$, these formulas degenerate into the usual addition formulas for the circular functions.

In the next section we will use the following formula due to Jacobi ([WW] p.529).

$$\sin\big(\operatorname{am}(u+v) + \operatorname{am}(u-v)\big) = \frac{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \ \operatorname{sn}^2 v}$$

3.2 Jacobi's uniformisation

The next result is a variant of Jacobi's uniformization of the Poncelet problem. It is due to Hermite and Laurent ([Lau]).

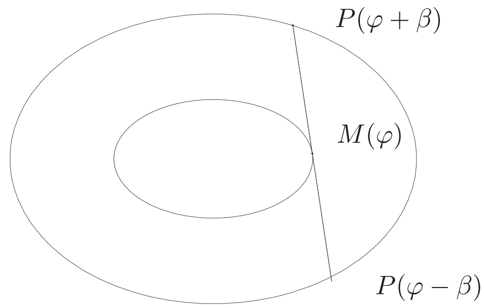

Figure 5: Jacobi's lemma.

Lemma 10 *Let E and C be the ellipses defined by*

$$E = \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\},\ a > b > 1, \quad C = \left\{x^2 + y^2 = 1\right\}.$$

Let us parameterize E by $P(\psi) = (a \operatorname{cn} \psi,\ b \operatorname{sn} \psi)$, and C by $M(\varphi) = (\operatorname{cn} \varphi,\ \operatorname{sn} \varphi)$, where the elliptic modulus k is defined by $k^2(a^2-1) = (a^2-b^2)$. Let β be a real number such that $\operatorname{cn} \beta = 1/a$.

Then the tangent to C at $M(\varphi)$ intersects E at $P(\varphi-\beta)$ and $P(\varphi+\beta)$.

Proof. We have $\mathrm{dn}^2\beta = 1 - k^2\,\mathrm{sn}^2\beta = b^2/a^2$, hence $\mathrm{dn}\,\beta = b/a$.

Let us show that $P(\varphi+\beta)$ belongs to the tangent to C at $M(\varphi)$. The equation of this tangent is $x\,\mathrm{cn}\,\varphi + y\,\mathrm{sn}\,\varphi = 1$. Let us compute $S = a\,\mathrm{cn}(\varphi+\beta)\,\mathrm{cn}\,\varphi + b\,\mathrm{sn}(\varphi+\beta)\,\mathrm{sn}\,\varphi$.

Using the addition formulas we obtain

$$S\,(1 - k^2\,\mathrm{sn}^2\varphi\,\mathrm{sn}^2\beta) = \mathrm{cn}^2\varphi + \mathrm{sn}^2\varphi\,\mathrm{dn}^2\beta = \mathrm{cn}^2\varphi + \mathrm{sn}^2\varphi\,(1 - k^2\,\mathrm{sn}^2\beta) = 1 - k^2\,\mathrm{sn}^2\varphi\,\mathrm{sn}^2\beta.$$

Consequently, $S = 1$, and $P(\varphi+\beta)$ belongs to the tangent to C at $M(\varphi)$. Changing β to $-\beta$, we see that $P(\varphi-\beta)$ also belongs to this tangent. □

Remark 11 *By affinity, Jacobi's lemma extends easily in the case of two nested ellipses with the same two axis, meeting transversally in $P_2(\mathbf{C})$. When this pair of ellipses is affinely equivalent to a pair of concentric circles, the elliptic parametrizations degenerate into the usual circular ones.*

3.3 Proof of Poncelet's closure theorem

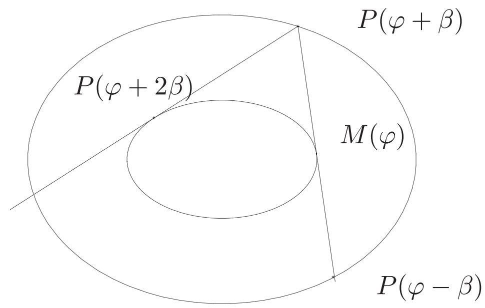

Figure 6: Proof of Poncelet's closure theorem.

We shall now present the Hermite–Laurent proof of Poncelet's theorem for a pair of confocal ellipses. Since any pair of conics meeting transversally in $P_2(\mathbf{C})$ is projectively equivalent to a pair of confocal ellipses ([LT]), we obtain a proof of the generic case of Poncelet's theorem. For the nongeneric cases see [Be, Sa], and for the original proof of Jacobi see [BKOR]. *Proof.* Let $P_0, P_1, \ldots P_{n-1}P_0$ be a Poncelet polygon inscribed in E and circumscribed about C. Let P_jP_{j+1} be tangent to C at M_j. We will use the Jacobi parametrizations of E and C. If $M_0 = M(\varphi)$, then by Jacobi's lemma we can suppose $P_1 = P(\varphi+\beta)$. Using Jacobi's lemma again, we have $M_1 = M(\varphi+2\beta)$, and by induction $M_j = M(\varphi+2j\beta)$.

Since the polygon closes after n steps, we have $M_n = M_0$, or $M(\varphi+2n\beta) = M(\varphi)$.

That means $\mathrm{am}(\varphi+2n\beta) = \mathrm{am}\,\varphi + 2q\pi = \mathrm{am}(\varphi+4qK)$ by the properties of the Jacobi amplitude. Consequently we obtain $2n\beta = 4qK$, or $\beta = 2qK/n$.

Now, let us consider a Poncelet polygon starting from an arbitrary point $M'_0 = M(\varphi')$ of C. By Jacobi's lemma we have $M'_n = M(\varphi' + 2n\beta) = M(\varphi' + 4qK) = M(\varphi') = M'_0$.

Consequently, we see that every Poncelet polygonal line closes after n steps. □

Corollary 12 *Let E and C be confocal ellipses, and let $\mathcal{P}$ be a Poncelet polygon with an even number of sides. Then $\mathcal{P}$ possesses a central symmetry.*

Proof. We use the same notation as before. We have $\beta = 2qK/n$, where $n = 2h$ is the number of sides of $\mathcal{P}$. The numbers n and q are coprime, hence q is odd. For every φ, we have

$$M_h(\varphi) = M(\varphi+2h\beta) = M(\varphi+2qK) = M(\varphi+2K) = \big(\mathrm{cn}(\varphi+2K),\ \mathrm{sn}(\varphi+2K)\big) = -M_0(\varphi).$$

□

By projective equivalence, this implies a remarkable theorem of Darboux ([Da]).

Theorem 13 *Let $\mathcal{P}$ be a Poncelet polygon with an even number of sides inscribed in a conic E and circumscribed about another conic C. Then the diagonals of $\mathcal{P}$ pass through a point, which is the same for every Poncelet polygon.*

4 Irregularity of Poncelet odd polygons

Most regularity properties of a polygon can be expressed by rational linear relations between some of its segments. Let us parameterize a (crossed) polygon by arc length, starting at a vertex P_0. We shall say that this polygon is completely irregular if 1 and the arc lengths of its crossings and vertices (except P_0) are linearly independent over $\mathbf{Q}$.

The purpose of this section is to prove that if E and C is a pair of confocal ellipses possessing a Poncelet polygon with an odd number of sides, then there exists a completely irregular Poncelet polygon. We will give an analogous result for unions of finitely many Poncelet polygons.

4.1 Two lemmas on elliptic functions

We shall use elliptic functions to compute the arc lengths of the crossings and vertices of Poncelet polygons. We shall need the following two technical lemmas.

Lemma 14 *Let n and p be coprime integers, with n odd. For every integer j, let us define the function $f_j(z) = \mathrm{sn}^2(z+j\theta)+r^2$, where $r^2 > 0$ and $\theta = 4pK/n$. Then, if $h \not\equiv j \pmod n$, the functions $f_j(z)$ and $f_h(z)$ do not possess any common zero.*

Proof. First, let us study the zeros of the elliptic function $g(z) = \operatorname{sn} z + ir,\ r > 0$. By considering its restriction to the y-axis, we see that there exists a pure imaginary α, such that $g(\alpha) = 0$. Since we have $\operatorname{sn}(2K - \alpha) = \operatorname{sn}\alpha$, we see that $2K - \alpha$ is another zero of $g(z)$. As $g(z)$ is an elliptic function of order two, its zeros are the points congruent to α or $2K - \alpha$ (mod. $4K, 2iK'$). By parity, we deduce that the zeros of $f_j(z)$ are the numbers which are congruent to $\pm\alpha - j\theta$, or $2K \pm \alpha - j\theta$, (mod. $4K, 2iK'$).

If we had $\alpha - j\theta \equiv \alpha - h\theta$, or $\alpha - j\theta \equiv 2K + \alpha - h\theta$ (mod.$4K$, $2iK'$), then we would deduce $(h-j)\theta \equiv 0 \pmod{2K}$. This implies that $2(h-j)p/n$ is an integer, which is impossible since n is odd, $(n,p) = 1$, and $h \not\equiv j \pmod{n}$.

If we had $\alpha - j\theta \equiv -\alpha - h\theta$ or $\alpha - j\theta \equiv 2K - \alpha - h\theta$, (mod.$4K$, $2iK'$), then we would have $2\alpha \equiv ((j-h)\theta$ (mod. $2K, 2iK'$). Taking the real parts, we would obtain $(j-h)\theta \equiv 0 \pmod{2K}$ which is impossible.

Consequently, $\alpha - j\theta$ cannot be a zero of $f_h(z)$. The proof that the other zeros of $f_j(z)$ cannot be zeros of $f_h(z)$ is entirely similar. □

Remark 15 *As the proof shows it, the condition n odd is necessary in lemma 14.*

Lemma 16 *Let n and p be coprime integers. For $j \not\equiv 0 \pmod{n}$, let us define the functions $D_j(z)$ and $F_j(z)$ by*

$$D_j(z) = \operatorname{sn}(z + j\theta)\operatorname{cn} z - \operatorname{cn}(z + j\theta)\operatorname{sn} z, \quad F_j(z) = \frac{\operatorname{sn}(z + j\theta) - \operatorname{sn} z}{D_j(z)}, \ \textit{where } \theta = \frac{4pK}{n}.$$

Then, for every integer j there exists a complex number α_j such that $F_j(\alpha_j) = \infty$, and if $h \not\equiv j \pmod{n}$, then $F_h(\alpha_j) \neq \infty$.

Proof. We have

$$D_j(z) = \sin\bigl(\operatorname{am}(z + j\theta) - \operatorname{am} z\bigr) = \sin\bigl(\operatorname{am}(z + j\theta) + \operatorname{am}(-z)\bigr)$$

Now, using the Jacobi formula for $\sin\bigl(\operatorname{am}(u+v) + \operatorname{am}(u-v)\bigr)$, we obtain

$$D_j(z) = \frac{2\operatorname{sn}(j\beta)\operatorname{cn}(j\beta)\operatorname{dn}(z + j\beta)}{1 - k^2\operatorname{sn}^2(j\beta)\operatorname{sn}^2(z + j\beta)}, \quad \text{where } \beta = \frac{\theta}{2}$$

Let $\alpha_j = -j\beta + K + iK'$. We have $\operatorname{dn}(\alpha_j + j\beta) = \operatorname{dn}(K + iK') = 0$. Since $\operatorname{dn}^2 z + k^2\operatorname{sn}^2 z = 1$, we obtain $\operatorname{sn}^2(\alpha_j + j\beta) = 1/k^2$, and then $D_j(\alpha_j) = 0$.

The numerator of $F_j(\alpha_j)$ is

$$N(\alpha_j) = \operatorname{sn}(\alpha_j + j\theta) - \operatorname{sn}(\alpha_j) = \operatorname{sn}(K + iK' + j\beta) - \operatorname{sn}(K + iK' - j\beta).$$

Using the addition formula for the function $\operatorname{sn} z$, we obtain

$$N(\alpha_j) = 2\frac{\operatorname{sn}(K + iK')\operatorname{cn}(j\beta)\operatorname{dn}(j\beta)}{1 - k^2\operatorname{sn}^2(K + iK')\operatorname{sn}^2(j\beta)}.$$

Since $\operatorname{sn}(K+iK') = k^{-1}$, we obtain $N(\alpha_j) = 2k^{-1}\dfrac{\operatorname{dn}(j\beta)}{\operatorname{cn}(j\beta)} \neq 0$, and then $F_j(\alpha_j) = \infty$.

On the other hand, if $h \not\equiv j \pmod n$, we have $\alpha_j + h\beta = K + iK' + 2(h-j)pK/n$.

First, we see that $\alpha_j + h\beta \not\equiv K + iK'$ (mod.$2K, 2iK'$), which implies that $\operatorname{dn}(\alpha_j + h\beta) \neq 0$.

We also see that $\alpha_j + h\beta \not\equiv iK'$ (mod.$2K, 2iK'$), which implies that $\operatorname{sn}(\alpha_j + h\beta) \neq \infty$. We conclude that $D_h(\alpha_j) \neq 0$.

Let us show that if $\operatorname{sn} z = \infty$, then $F_h(z) \neq \infty$.

Since the functions $\operatorname{sn} z$ and $\operatorname{sn}(z + h\theta)$ do not have common poles, $\operatorname{sn}(z + h\theta) \neq \infty$.

On the other hand, as $\operatorname{sn}^2 z + \operatorname{cn}^2 z = 1$, we obtain

$$\frac{\operatorname{cn}^2 z}{\operatorname{sn}^2 z} = -1, \text{ and then } F_h(z) = \frac{-1}{\operatorname{sn}(z+h\theta)\dfrac{\operatorname{cn} z}{\operatorname{sn} z} - \operatorname{cn}(z+h\theta)}$$

If we had $F_h(z) = \infty$, then

$$\operatorname{sn}(\mathrm{z} + \mathrm{h}\theta)\frac{\operatorname{cn} \mathrm{z}}{\operatorname{sn} \mathrm{z}} = \operatorname{cn}(\mathrm{z} + \mathrm{h}\theta),$$

whence $\operatorname{sn}^2(z+h\theta) = -\operatorname{cn}^2(z+h\theta) \neq \infty$, and $\operatorname{sn}^2(z+h\theta) + \operatorname{cn}^2(z+h\theta) = 0$, which is impossible.

Similarly, we see that if $\operatorname{sn}(z+h\theta) = \infty$, then $F_h(z) \neq \infty$.

Now, let us prove that $F_h(\alpha_j) \neq \infty$. We have $D_h(\alpha_j) \neq 0$, and we have proved that we can suppose $\operatorname{sn}(\alpha_j) \neq \infty$ and $\operatorname{sn}(\alpha_j + h\theta) \neq \infty$, then

$$F_h(\alpha_j) = \frac{\operatorname{sn}(\alpha_j + h\theta) - \operatorname{sn}\alpha_j}{D_h(\alpha_j)} \neq \infty.$$

□

4.2 Irregular Poncelet polygons with an odd number of sides

Proposition 17 *Let E and C be confocal ellipses such that there exists a Poncelet polygon $\mathcal{P}$ inscribed in E and circumscribed about C. We suppose that the number of sides of $\mathcal{P}$ is odd. Then there exists a Poncelet polygon satisfying the following condition.*

If the arc lengths t_i of the vertices and crossings are measured from a vertex P_0, then the numbers 1 and t_i, $t_i \neq 0$ are linearly independent over $\mathbf{Q}$.

Proof.

$$\text{Let } E = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\},\ a > b > 1, \text{ and } \ C = \{x^2 + \frac{y^2}{c^2} = 1\},\ c < 1$$

be our ellipses. The condition on the eccentricity of C means that $2c^2 > 1$.

Let us consider the Jacobi parametrizations of E and C by means of elliptic functions, and let $\theta = 4pK/n$. To each real number φ corresponds a Poncelet polygon $\mathcal{P}_\varphi$ through

$M(\varphi) = \big(\operatorname{cn}\varphi,\, c\operatorname{sn}\varphi\big)$. Let us denote $\varphi_j = \varphi + j\theta,\ M_j = M(\varphi_j)$, and let ℓ_j be the tangent to C at M_j. The equation of ℓ_j is

$$x\operatorname{cn}\varphi_j + \frac{y}{c}\operatorname{sn}\varphi_j = 1.$$

Let $Q_{h,j} = \ell_h \bigcap \ell_{h+j},\ j \not\equiv 0 \pmod n$. The abcissa $x_{h,j}$ of $Q_{h,j}$ is

$$x_{h,j} = \frac{-\operatorname{sn}\varphi_h + \operatorname{sn}(\varphi_h + j\theta)}{\operatorname{sn}(\varphi_h + j\theta)\operatorname{cn}\varphi_h - \operatorname{cn}(\varphi_h + j\theta)\operatorname{sn}\varphi_h} = F_j(\varphi_h)$$

where F_j is the function defined in lemma 16. The abcissa of $P_h = Q_{h,-1} = Q_{h-1,1}$ will also be denoted by $x_h = x_{h,-1}$. The distance $P_hQ_{h,j}$ is $|d_{h,j}|$ where

$$d_{h,j} = d_{h,j}(\varphi) = \frac{\sqrt{1-c^2}}{\operatorname{sn}\varphi_h}\sqrt{\operatorname{sn}^2\varphi_h + \frac{c^2}{1-c^2}}\,\Big(x_h - x_{h,j}\Big)$$

Since $c^2/(1-c^2) > 1$, the function $d_{h,j}(\varphi)$ is meromorphic in a neighborhood of the real axis.

Our first step is to prove that the functions 1 and $d_{h,j}(\varphi),\ j \not\equiv -1 \pmod n$ are linearly independent over $\mathbf{C}$.

Let $\lambda_{h,j}$ and λ be complex numbers such that $\sum_{h=1}^{n}\sum_{j=1}^{n-2}\lambda_{h,j}d_{h,j} = \lambda$, or

$$\sum_{h=1}^{n}\frac{\sqrt{1-c^2}}{\operatorname{sn}\varphi_h}\sqrt{\operatorname{sn}^2\varphi_h + \frac{c^2}{1-c^2}}\left(\sum_{j=1}^{n-2}\lambda_{h,j}(x_h - x_{h,j})\right) = \lambda.$$

Since $c^2/(1-c^2) > 0$, we see by lemma 14 that the functions $f_h(\varphi) = \sqrt{\operatorname{sn}^2\varphi_h + c^2/(1-c^2)}$ do not possess any common zero. Hence, in the neighborhood of a zero of $f_h(\varphi)$ this function is not meromorphic, while the other functions are.

This implies that for every $h = 1\ldots n$ we have

$$\sum_{j=1}^{n-2}\lambda_{h,j}\big(x_h - x_{h,j}\big) = 0,\ \text{and then } \lambda = 0.$$

Using our expressions of the abcissas $x_{h,j}$, we obtain the following relation between meromorphic functions

$$\sum_{j=1}^{n-2}\lambda_{h,j}\big(F_{-1}(z) - F_j(z)\big) = 0.$$

By lemma 16, for every integer $j \neq 0$ there exists a number α_j such that $F_j(\alpha_j) = \infty$, and $F_h(\alpha_j) \neq \infty$ if $h \not\equiv j \pmod n$. Letting $z = \alpha_j$, we obtain $\lambda_{h,j} = 0$, which concludes the proof of the linear independence of our functions.

Now, we shall prove that for most $\varphi \in \mathbf{R}$, the numbers $d_{h,j}(\varphi)$ and 1 are linearly independent over $\mathbf{Q}$.

For every nonzero collection of rational numbers $\Lambda = (\lambda, \lambda_{h,j})$, let us define the function F_Λ by $F_\Lambda(\varphi) = \lambda - \sum_{h,j} \lambda_{h,j}\, d_{h,j}(\varphi)$. By our first step, this function is not identically zero, and it is meromorphic in a neighborhood of $\mathbf{R}$. Therefore, the set of its real zeros is countable. Consequently, the set of all real numbers φ such that 1 and the numbers $d_{h,j}(\varphi)$ are linearly dependent over $\mathbf{Q}$ is countable. By cardinality, we deduce that the complementary set is not countable, hence nonempty. Consequently, there exists a real φ such that 1 and the numbers $|d_{h,j}(\varphi)|$ are linearly independent over $\mathbf{Q}$.

Now, let us parameterize our Poncelet polygon by arc length, starting from P_0 for $t_0 \in \mathbf{Q}$. The arc length $t_{h,j}$ of $Q_{h,j}$ is

$$\begin{aligned} t_{h,j} &= t_0 + d(P_0, P_1) + d(P_1, P_2) + \ldots + d(P_{h-1}, P_h) + d(P_h, Q_{h,j}) \\ &= t_0 + |d_{0,1}| + |d_{1,1}| + |d_{2,1}| + \ldots + |d_{h-1,1}| + |d_{h,j}|. \end{aligned}$$

The result follows from the independence of the numbers 1 and $|d_{h,j}|$. □

We shall also need an analogous result for links.

Proposition 18 *Let E and C be confocal ellipses such that there exists a polygon of an odd number of sides inscribed in E and circumscribed about C.*

For any integer μ, there exist μ Poncelet polygons $\mathcal{P}^{(0)}, \mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(\mu-1)}$ satisfying the following condition:

for each such polygon, if t_i are the arc lengths corresponding to its vertices , its crossings, an its intersections with the other polygons, then the numbers 1 and t_i, $i \neq 0$ are linearly independent over $\mathbf{Q}$.

Proof. Let $\tau = \dfrac{\theta}{\mu}$, and let us denote $M_h = M(\varphi + h\tau) \in C$, and ℓ_h the tangent to C at M_h. Let us consider the Poncelet polygons $\mathcal{P} = \mathcal{P}^{(0)},\ \mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(\mu-1)}$ through the points $M_0, M_1, \ldots, M_{\mu-1}$. The polygon $\mathcal{P}$ is tangent to C at the points $M_0, M_\mu, M_{2\mu}, \ldots, M_{(n-1)\mu}$. The vertices and crossings of $\mathcal{P}$ are the points $Q_{h,j} = \ell_h \bigcap \ell_{h+j}$, where $h \equiv 0 \pmod{\mu}$.

Just as before, it can be proved that the distances 1 and $|d_{h,j}(\varphi)|,\ h \equiv 0,\ j \not\equiv 0,\ j \not\equiv -1 \pmod{\mu}$ are linearly independent over $\mathbf{Q}$, except for a countable set of numbers φ.

Consequently, the number 1 and the arc lengths t_i, $i \neq 0$ of the crossings and vertices of $\mathcal{P}$ are linearly independent over $\mathbf{Q}$ except on a countable set of values of φ.

By cardinality, we can suppose that the same property is true for each polygon $\mathcal{P}^{(j)}$, $j = 0, \ldots, \mu - 1$, which proves our result. □

5 Proof of the theorem

We will use Kronecker's theorem (see [HW, Theorem 443]):

Theorem 19 *If $\theta_1, \theta_2, \ldots, \theta_k, 1$ are linearly independent over $\mathbf{Q}$, then the set of points $\big((n\theta_1), \ldots, (n\theta_k)\big)$ is dense in the unit cube. Here (x) denotes the fractional part of x.*

Now, we can prove our main theorem.

Theorem 20 *Let E be an ellipse which is not a circle, and let D be the elliptic cylinder $D = E \times [0,1]$. Every knot (or link) is a billiard knot (or link) in D.*

Proof. First, we consider knots. By theorem 9 there exists a knot isotopic to K, whose projection on the xy-plane is a billiard trajectory of odd period in the ellipse E. If $t_0, t_1, \ldots, t_k$ are the arc lengths corresponding to the vertices and crossings, we can suppose by proposition 17 that the numbers $t_1, \ldots, t_k$, and 1 are linearly independent over $\mathbf{Q}$. Using a dilatation, we can suppose that the total length of the trajectory is 1.

Let us consider the polygonal curve defined by $(x(t), y(t), z(t))$, where $z(t)$ is the saw-tooth function $z(t) = 2|(nt+\varphi)-1/2|$ depending on the integer n and on the real number φ. If the heights $z(P_j)$ of the vertices are such that $z(P_j) \neq 0,\ z(P_j) \neq 1$, then it is a periodic billiard trajectory in the elliptic cylinder $\mathbf{D} = E \times [0,1]$ (see [JP, La2, LO, P, KP1]). If we set $\varphi = 1/2 + z_0/2,\ z_0 \in (0,1)$, we have $z(0) = z_0$. Now, using Kronecker's theorem, there exists an integer n such that the numbers $z(t_i)$ are arbitrarily close to any chosen collection of heights, which completes our proof.

The case of μ-component links is similar. First, by theorem 9, we find a diagram that is the union of μ Poncelet polygons with the same odd number of sides. Then, by proposition 18 and Kronecker's theorem, we parameterize each component so that the heights of the vertices and crossings are close to any chosen numbers. □

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