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# Fermi-Dirac-Fokker-Planck Equation: Well-posedness & Long-time Asymptotics

José A. Carrillo\* Philippe Laurençot<sup>†</sup> & Jesús Rosado<sup>‡</sup>

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## Abstract

A Fokker-Planck type equation for interacting particles with exclusion principle is analysed. The nonlinear drift gives rise to mathematical difficulties in controlling moments of the distribution function. Assuming enough initial moments are finite, we can show the global existence of weak solutions for this problem. The natural associated entropy of the equation is the main tool to derive uniform in time a priori estimates for the kinetic energy and entropy. As a consequence, long-time asymptotics in  $L^1$  are characterized by the Fermi-Dirac equilibrium with the same initial mass. This result is achieved without rate for any constructed global solution and with exponential rate due to entropy/entropy-dissipation arguments for initial data controlled by Fermi-Dirac distributions. Finally, initial data below radial solutions with suitable decay at infinity lead to solutions for which the relative entropy towards the Fermi-Dirac equilibrium is shown to converge to zero without decay rate.

## 1 Introduction

Kinetic equations for interacting particles with exclusion principle, such as fermions, have been introduced in the physics literature in [6, 7, 8, 12, 15, 16, 17, 18, 26] and the review [13]. Spatially inhomogeneous equations appear from formal derivations of generalized

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\*ICREA (Institució Catalana de Recerca i Estudis Avançats) and Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain. E-mail: [carrillo@mat.uab.cat](mailto:carrillo@mat.uab.cat), Internet: <http://kinetic.mat.uab.es/~carrillo/>

<sup>†</sup>Institut de Mathématiques de Toulouse, CNRS UMR 5219 & Université de Toulouse, 118 route de Narbonne, F-31062 Toulouse cedex 9, France. E-mail: [laurenco@mip.ups-tlse.fr](mailto:laurenco@mip.ups-tlse.fr), Internet: <http://www.math.univ-toulouse.fr/~laurenco/>

<sup>‡</sup>Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain. E-mail: [jrosado@mat.uab.cat](mailto:jrosado@mat.uab.cat)

Boltzmann equations and Uehling-Uhlenbeck kinetic equations both for fermionic and bosonic particles. The most relevant questions related to these problems concern their long-time asymptotics and the rate of convergence towards global equilibrium if any.

The spatially inhomogeneous case has recently been studied in [25]. There, the long time asymptotics of these models in the torus is shown to be given by spatially homogeneous Fermi-Dirac distributions when the initial data is not far from equilibrium in a suitable Sobolev space. This nice result is based on techniques developed in previous works [23, 24]. Other related mathematical results for Boltzmann-type models have appeared in [10, 22].

In this work, we focus on the global existence of solutions and the convergence of solutions towards global equilibrium in the spatially homogeneous case without any smallness assumption on the initial data. Preliminary results in the one-dimensional setting were reported in [5]. More precisely, we analyse in detail the following Fokker-Planck equation for fermions, see for instance [13],

$$\frac{\partial f}{\partial t} = \Delta_v f + \operatorname{div}_v [v f(1 - f)], \quad v \in \mathbb{R}^N, t > 0, \quad (1.1)$$

with initial condition  $f(0, v) = f_0(v) \in L^1(\mathbb{R}^N)$ ,  $0 \leq f_0 \leq 1$ , satisfying suitable moment conditions to be specified below. Here,  $f = f(t, v)$  is the density of particles with velocity  $v$  at time  $t \geq 0$ .

This equation has been proposed in order to describe the dynamics of classical interacting particles, obeying the exclusion-inclusion principle in [15]. In fact, equation (1.1) is formally equivalent to

$$\frac{\partial f}{\partial t} = \operatorname{div}_v \left[ f(1 - f) \nabla_v \left( \log \left( \frac{f}{1 - f} \right) + \frac{|v|^2}{2} \right) \right]$$

from which it is easily seen that Fermi-Dirac distributions defined by

$$F^\beta(v) := \frac{1}{1 + \beta e^{\frac{|v|^2}{2}}}$$

with  $\beta \geq 0$  are stationary solutions. Moreover, for each value of  $M \geq 0$ , there exists a unique  $\beta = \beta(M) \geq 0$  such that  $F^{\beta(M)}$  has mass  $M$ , that is,  $\|F^{\beta(M)}\|_1 = M$ . Throughout the paper we shall denote  $F^{\beta(M)}$  by  $F_M$ .

Another striking property of this equation is the existence of a formal Liapunov functional, related to the standard entropy functional for linear and nonlinear Fokker-Planck models [2, 4], given by

$$H(f) := \frac{1}{2} \int_{\mathbb{R}^N} |v|^2 f(v) dv + \int_{\mathbb{R}^N} [(1 - f) \log(1 - f) + f \log(f)] dv.$$

We will show that this functional plays the same role as the H-functional for the spatially homogeneous Boltzmann equation, see for instance [27]. In particular it will be crucial to

characterize long-time asymptotics of (1.1). In fact, the entropy method will be the basis of the main results in this work; more precisely by taking the formal time derivative of  $H(f)$ , we conclude that

$$\frac{d}{dt}H(f) = - \int_{\mathbb{R}^N} f(1-f) \left| v + \nabla_v \log \left( \frac{f}{1-f} \right) \right|^2 dv \leq 0.$$

Therefore, to show the global equilibration of solutions to (1.1) we need to find the right functional setting to show the entropy dissipation. Furthermore, if we succeed in relating functionally the entropy and the entropy dissipation, we will be able to give decay rates towards equilibrium. These are the main objectives of this work. Let us finally mention that these equations are of interest as typical examples of gradient flows with respect to euclidean Wasserstein distance of entropy functionals with nonlinear mobility, see [1, 3] for other examples and related problems.

In section 2, we will show the global existence of solutions for equation (1.1) based on fixed point arguments, estimates involving moment bounds and the conservation of certain properties of the solutions. The suitable functional setting is reminiscent of the one used in equations sharing a similar structure and technical difficulties as those treated in [11, 14]. The main technical obstacle for the Fermi-Dirac-Fokker-Planck equation (1.1) lies in the control of moments. Next, in section 3, we show that the entropy is decreasing for the constructed solutions, and from that, we prove the convergence towards global equilibrium without rate. Again, here the uniform-in-time control of the second moment is crucial. Finally, we obtain an exponential rate of convergence towards equilibrium if the initial data is controlled by Fermi-Dirac distributions and the convergence to zero of the relative entropy when controlled by radial solutions.

## 2 Global Existence of Solutions

In this section, we will show the global existence of solutions to the Cauchy problem to (1.1). We start by proving local existence of solutions together with a characterization of the life-span of these solutions. Later, we show further regularity properties of these solutions with the help of estimates on derivatives. Based on these estimates we can derive further properties of the solutions: conservation of mass, positivity,  $L^\infty$  bounds, comparison principle, moment estimates and entropy estimates. All of these uniform estimates allow us to show that solutions can be extended and thus exist for all times.

### 2.1 Local Existence

We will prove the local existence and uniqueness of solutions using contraction-principle arguments as in [1, 11, 14] for instance. As a first step, let us note that we can write (1.1)

as

$$\frac{\partial f}{\partial t} = \operatorname{div}_v(vf + \nabla_v f) - \operatorname{div}_v(vf^2) \quad (2.1)$$

and, due to Duhamel's formula, we are led to consider the corresponding integral equation

$$f(t, v) = \int_{\mathbb{R}^N} \mathcal{F}(t, v, w) f_0(w) dw - \int_0^t \int_{\mathbb{R}^N} \mathcal{F}(t-s, v, w) (\operatorname{div}_w(wf(s, w)^2)) dw ds \quad (2.2)$$

where  $\mathcal{F}(t, v, w)$  is the fundamental solution for the homogeneous Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = \operatorname{div}_v(vf + \nabla_v f)$$

given by

$$\mathcal{F}(t, v, w) := e^{Nt} M_{\nu(t)}(e^t v - w)$$

with

$$\nu(t) := e^{2t} - 1 \quad \text{and} \quad M_\lambda(\xi) := (2\pi\lambda)^{-\frac{N}{2}} e^{-\frac{|\xi|^2}{2\lambda}}$$

for any  $\lambda > 0$ . Let us define the operator  $\mathcal{F}[g](t, v)$  acting on functions  $g : \mathbb{R}^N \rightarrow \mathbb{R}$  as:

$$\mathcal{F}[g(w)](t, v) := \int_{\mathbb{R}^N} \mathcal{F}(t, v, w) g(w) dw. \quad (2.3)$$

Note that by integration by parts, the expression  $\mathcal{F}[\operatorname{div}_w(wf^2(w))](t, v)$  is equivalent to:

$$\begin{aligned} \int_{\mathbb{R}^N} \left( \frac{e^{Nt}}{(2\pi(e^{2t}-1))^{\frac{N}{2}}} e^{-\frac{|e^t v - w|^2}{2(e^{2t}-1)}} \right) \operatorname{div}_w(wf(w)^2) dw \\ = - \int_{\mathbb{R}^N} \left[ \nabla_w \left( \frac{e^{Nt}}{(2\pi(e^{2t}-1))^{\frac{N}{2}}} e^{-\frac{|e^t v - w|^2}{2(e^{2t}-1)}} \right) \cdot w \right] f(w)^2 dw \\ = - \int_{\mathbb{R}^N} e^{-t} (\nabla_v \mathcal{F}(t, v, w) \cdot w) f(w)^2 dw \\ =: -e^{-t} \nabla_v \mathcal{F}[wf(w)^2](t, v), \end{aligned} \quad (2.4)$$

so that (2.2) becomes

$$f(t, v) = \mathcal{F}[f_0(w)](t, v) + \int_0^t e^{-(t-s)} \nabla_v \mathcal{F}[wf(s, w)^2](t-s, v) ds. \quad (2.5)$$

We will now define a space in which the functional induced by (2.5)

$$\mathcal{T}[f](t, v) := \mathcal{F}[f_0(w)](t, v) + \int_0^t e^{-(t-s)} \nabla_v \mathcal{F}[wf(s, w)^2](t-s, v) ds \quad (2.6)$$

has a fixed point. To this end, we define the spaces  $\Upsilon := L^\infty(\mathbb{R}^N) \cap L^1_1(\mathbb{R}^N) \cap L^p_m(\mathbb{R}^N)$  and  $\Upsilon_T := \mathcal{C}([0, T]; \Upsilon)$  with norms

$$\|f(t)\|_\Upsilon := \max\{\|f(t)\|_\infty, \|f(t)\|_{L^1_1}, \|f(t)\|_{L^p_m}\} \quad \text{and} \quad \|f\|_{\Upsilon_T} := \max_{0 \leq t \leq T} \|f(t)\|_\Upsilon$$

for any  $T > 0$ , where we omit the  $N$ -dimensional euclidean space  $\mathbb{R}^N$  for notational convenience and

$$\|f\|_{L^p_m} := \|(1 + |v|^m)f\|_p \quad \text{and} \quad \|f\|_p := \left( \int_{\mathbb{R}^N} |f|^p dv \right)^{\frac{1}{p}}.$$

**Lemma 2.1** *Let  $p > N$ ,  $p \geq 2$ , and  $m \geq 1$ , and fix  $q$  and  $r$  satisfying*

$$\frac{Np}{N+p} < \frac{p}{2} \leq r \leq \frac{mp}{m+1} < p \quad \text{and} \quad \frac{p}{2} \leq q \leq p. \quad (2.7)$$

*Then*

$$\|\mathcal{T}[f](t)\|_\Upsilon \leq C_1(N, t) \|f_0\|_\Upsilon + C_2(N, p, q, r, t) \|f\|_{\Upsilon_T}^2$$

*for any  $0 \leq t \leq T$  and  $T > 0$ , with*

$$\begin{aligned} C_1(N, t) &:= Ce^{Nt} \\ C_2(N, p, q, r, t) &:= C \max \left\{ \mathcal{I}_1(t), \mathcal{I}_2(t), \int_{e^{-2t}}^1 \chi^{-\frac{3}{2}} (1 - \chi)^{-\frac{1}{2}} d\chi \right\}, \end{aligned}$$

*the functions  $\mathcal{I}_1$  and  $\mathcal{I}_2$  being defined below.*

*Proof.*- Observe first that the properties of  $p$  and  $m$  ensure that we can indeed choose  $q$  and  $r$  satisfying (2.7). Consider next  $0 \leq t \leq T$ . Due to Proposition A.1 from the appendix, and since  $q \leq p \leq 2q$ , we can compute

$$\begin{aligned} \|\mathcal{T}[f](t)\|_\infty &\leq Ce^{Nt} \|f_0\|_\infty + \int_0^t C \frac{e^{N(t-s)}}{\nu(t-s)^{\frac{N}{2q} + \frac{1}{2}}} \|w|f^2(s)\|_q ds \\ &\leq Ce^{Nt} \|f_0\|_\infty + \int_0^t C \frac{e^{N(t-s)}}{\nu(t-s)^{\frac{N}{2q} + \frac{1}{2}}} \|f(s)\|_\infty^{2-\frac{p}{q}} \|f(s)\|_{L^p_m}^{\frac{p}{q}} ds \\ &\leq Ce^{Nt} \|f_0\|_\infty + \int_0^t C \frac{e^{N(t-s)}}{\nu(t-s)^{\frac{N}{2q} + \frac{1}{2}}} ds \|f\|_{\Upsilon_T}^2 \\ &\leq Ce^{Nt} \|f_0\|_\infty + C \mathcal{I}_1(t) \|f\|_{\Upsilon_T}^2, \end{aligned}$$

where the change of variables  $\chi = e^{-2t}$  allows us to write

$$\mathcal{I}_1(t) := \int_{e^{-2t}}^1 \chi^{-\frac{1}{2}(N - \frac{N}{q} - 1) - 1} (1 - \chi)^{-\frac{1}{2}(\frac{N}{q} + 1)} d\chi < \infty$$

by the choice (2.7) of  $q$ . In the same way, since  $r$  satisfies  $(m+1)r \leq mp$  and  $2r \geq p$ , we get

$$\begin{aligned}
\|\mathcal{T}[f](t)\|_{L_m^p} &\leq C e^{\frac{N}{p'}t} \|f_0\|_{L_m^p} + \int_0^t C \frac{e^{\frac{N}{p'}(t-s)}}{\nu(t-s)^{\frac{N}{2}(\frac{1}{r}-\frac{1}{p})+\frac{1}{2}}} \| |w| f^2(s) \|_{L_m^r} ds \\
&\leq C e^{\frac{N}{p'}t} \|f_0\|_{L_m^p} + \int_0^t C \frac{e^{\frac{N}{p'}(t-s)}}{\nu(t-s)^{\frac{N}{2}(\frac{1}{r}-\frac{1}{p})+\frac{1}{2}}} \|f(s)\|_\infty^{2-\frac{p}{r}} \|f(s)\|_{L_m^p}^{\frac{p}{r}} ds \\
&\leq C e^{\frac{N}{p'}t} \|f_0\|_{L_m^p} + \int_0^t C \frac{e^{\frac{N}{p'}(t-s)}}{\nu(t-s)^{\frac{N}{2}(\frac{1}{r}-\frac{1}{p})+\frac{1}{2}}} ds \|f\|_{\Upsilon_T}^2 \\
&\leq C e^{\frac{N}{p'}t} \|f_0\|_{L_m^p} + C \mathcal{I}_2(t) \|f\|_{\Upsilon_T}^2,
\end{aligned}$$

where, using once more the change of variables  $\chi = e^{-2t}$ ,

$$\mathcal{I}_2(t) := \int_{e^{-2t}}^1 \chi^{-\frac{1}{2}[\frac{N}{p'}-(N(\frac{1}{r}-\frac{1}{p})+1)]-1} (1-\chi)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}} d\chi < \infty$$

by the choice (2.7) of  $r$ . Here  $p'$  denotes the conjugate of  $p$ .

Finally we can estimate

$$\|\mathcal{T}[f](t)\|_{L_1^1} \leq C \|f_0\|_{L_1^1} + \int_0^t \frac{C}{\nu(t-s)^{\frac{1}{2}}} \| |w| f^2(s) \|_{L_1^1} ds$$

where by interpolation, we get as  $p \geq 2$  and  $m \geq 1$

$$\begin{aligned}
\| |w| f^2 \|_{L_1^1} &= \int_{\mathbb{R}^N} (1+|w|) |w| f^2 dw \leq \int_{\mathbb{R}^N} (1+|w|)^2 f^2 dw \\
&\leq \left( \int_{\mathbb{R}^N} (1+|w|) f dw \right)^{\frac{p-2}{p-1}} \left( \int_{\mathbb{R}^N} (1+|w|)^p f^p dw \right)^{\frac{1}{p-1}} \\
&\leq \|f\|_{L_1^1}^{\frac{p-2}{p-1}} \|f\|_{L_m^p}^{\frac{p}{p-1}}.
\end{aligned} \tag{2.8}$$

Consequently, we get

$$\|\mathcal{T}[f](t)\|_{L_1^1} \leq C \|f_0\|_{L_1^1} + C \int_{e^{-2t}}^1 \chi^{-\frac{3}{2}} (1-\chi)^{-\frac{1}{2}} d\chi \|f\|_{\Upsilon_T}^2.$$

Collecting all the above estimates completes the proof of Lemma 2.1.  $\square$

We next check the existence of a fixed point of (2.6) in  $\Upsilon_T$ . To this end, we define a sequence  $(f_n)_{n \geq 1}$  by  $f_{n+1} = \mathcal{T}[f_n]$  for  $n \geq 0$ . By Lemma 2.1, we can write

$$\|f_{n+1}(t)\|_{\Upsilon} \leq C_1(N, t) \|f_0\|_{\Upsilon} + C_2(N, p, q, r, t) \|f_n\|_{\Upsilon_T}^2$$

for any  $0 \leq t \leq T$  and  $T > 0$ . Clearly,  $C_1$  and  $C_2$  are increasing functions of the time  $t$  and  $C_2(t)$  tends to 0 as  $t$  does. Thus, for any  $T > 0$

$$\|f_{n+1}\|_{\Upsilon_T} \leq C_1(T) \|f_0\|_{\Upsilon} + C_2(T) \|f_n\|_{\Upsilon_T}^2$$

with  $C_1(T) = C_1(N, T)$  and  $C_2(T) = C_2(N, p, q, r, T)$ , both being increasing functions of  $T$ . We may also assume that  $C_1(T) \geq 1$  without loss of generality.

From now on, we will follow the arguments in [21]. We will first show that if  $T$  is small enough, the functional  $\mathcal{T}$  is bounded in  $\Upsilon_T$ , which will in turn imply the convergence. Let us take  $T > 0$  which verifies

$$0 < \|f_0\|_{\Upsilon} < \frac{1}{4C_1(T)C_2(T)}.$$

We then prove by induction that  $\|f_n\|_{\Upsilon_T} < 2C_1(T)\|f_0\|_{\Upsilon}$  for all  $n$ . It is clear that we have  $\|f_0\|_{\Upsilon} < C_1(T)\|f_0\|_{\Upsilon} < 2C_1(T)\|f_0\|_{\Upsilon}$ . If we suppose that  $\|f_n\|_{\Upsilon_T} < 2C_1(T)\|f_0\|_{\Upsilon}$  for some  $n \geq 1$ , we have

$$\|f_{n+1}\|_{\Upsilon_T} < C_1(T)\|f_0\|_{\Upsilon} + 4C_1^2(T)C_2(T)\|f_0\|_{\Upsilon}^2 < 2C_1(T)\|f_0\|_{\Upsilon},$$

hence the claim. Now, computing the difference between two consecutive iterations of the functional and proceeding with the same estimates as above, we can see for any  $0 \leq t \leq T$  that

$$\begin{aligned} \|f_{n+1} - f_n\|_{\Upsilon_T} &= \left\| \int_0^t e^{-(t-s)} \nabla_v \mathcal{F} [w [f_n^2 - f_{n-1}^2]] (t-s, v) ds \right\|_{\Upsilon_T} \\ &\leq C_2(T) \sup_{[0, T]} \|f_n + f_{n-1}\|_{\infty} \|f_n - f_{n-1}\|_{\Upsilon_T} \\ &\leq C_2(T) \left( \|f_n\|_{\Upsilon_T} + \|f_{n-1}\|_{\Upsilon_T} \right) \|f_n - f_{n-1}\|_{\Upsilon_T} \\ &\leq 4C_1(T)C_2(T)\|f_0\|_{\Upsilon} \|f_n - f_{n-1}\|_{\Upsilon_T} \leq (4C_1(T)C_2(T)\|f_0\|_{\Upsilon})^n \|f_1 - f_0\|_{\Upsilon_T}. \end{aligned}$$

Since  $4C_1(T)C_2(T)\|f_0\|_{\Upsilon} < 1$  we can conclude that there exists a function  $f_*$  in  $\Upsilon_T$  which is a fixed point for  $\mathcal{T}$ , and hence a solution to the integral equation (2.2). It is not difficult to check that the solution  $f \in \Upsilon_T$  to the integral equation is a solution of (1.1) in the sense of distributions defining our concept of solution. We summarize the results of this subsection in the following result.

**Theorem 2.2 (Local Existence)** *Let  $m \geq 1$ ,  $p > N$ ,  $p \geq 2$ , and  $f_0 \in \Upsilon$ . Then there exists  $T > 0$  depending only on the norm of the initial condition  $f_0$  in  $\Upsilon$ , such that (1.1) has a unique solution  $f$  in  $\mathcal{C}([0, T]; \Upsilon)$  with  $f(0) = f_0$ .*



**Remark 2.3** *The previous theorem is also valid for  $f_0 \in (L^\infty \cap L_m^p \cap L^1)(\mathbb{R}^N)$ , with a solution defined in  $\mathcal{C}([0, T]; (L^\infty \cap L_m^p \cap L^1)(\mathbb{R}^N))$  but we will need to have the first moment of the solution bounded in order to be able to extend it to a global in time solution. We thus include here this additional condition.*

**Remark 2.4** *With the same arguments used to prove Theorem 2.2 we can prove an equivalent result for the Bose-Einstein-Fokker-Planck equation*

$$\frac{\partial f}{\partial t} = \operatorname{div}_v [\nabla_v f + v f(1 + f)], \quad v \in \mathbb{R}^N, t > 0.$$

## 2.2 Estimates on Derivatives

Let us now work on estimates on the derivatives. By taking the gradient in the integral equation, we obtain

$$\nabla_v f(t, v) = \nabla_v \mathcal{F}[f(w)](t, v) - \int_0^t \nabla_v \mathcal{F}[\operatorname{div}_w (w f^2(s, w))](t - s, v) ds. \quad (2.9)$$

where  $\nabla_v \mathcal{F}[g](t, v)$  is defined as in (2.4) for the real-valued function  $g$ . Here, we will consider a space with suitable weighted norms for the derivatives

$$X_T := \{f \in \Upsilon_T \mid \nabla_v f \in L_m^p \cap L_1^1 \text{ and } \|f\|_{X_T} < \infty\},$$

for

$$\|f\|_{X_T} = \max \left\{ \|f\|_{\Upsilon_T}, \sup_{0 < t < T} \nu(t)^{\frac{1}{2}} \|\nabla_v f(t)\|_{L_m^p}, \sup_{0 < t < T} \nu(t)^{\frac{1}{2}} \|\nabla_v f(t)\|_{L_1^1}, \right\}$$

where we refer to  $\|\nabla_v f\|_{L_m^p}$  as  $\|\nabla_v f\|_{L_m^p}$  to simplify notation. Let us estimate the  $L_m^p$ - and  $L^1$ -norms of  $\nabla_v f$  using again the results in Proposition A.1 as follows: for  $r \in [1, p)$  satisfying (2.7)

$$\begin{aligned} \|\nabla_v f(t)\|_{L_m^p} &\leq C \frac{e^{\left(\frac{N}{p'}+1\right)t}}{\nu(t)^{\frac{1}{2}}} \|f_0\|_{L_m^p} + \int_0^t \|\nabla_v \mathcal{F}[2f(w \cdot \nabla_w f)] + N f^2\|_{L_m^p} ds \\ &\leq C \frac{e^{\left(\frac{N}{p'}+1\right)t}}{\nu(t)^{\frac{1}{2}}} \|f_0\|_{L_m^p} + C \int_0^t \frac{e^{\left(\frac{N}{p'}+1\right)(t-s)}}{\nu(t-s)^{\frac{1}{2}}} \|f(s)\|_{L_m^p} \|f(s)\|_{\infty} ds \\ &\quad + C \int_0^t \frac{e^{\left(\frac{N}{p'}+1\right)(t-s)}}{\nu(t-s)^{\frac{N}{2}\left(\frac{1}{r}-\frac{1}{p}\right)+\frac{1}{2}}} \|f(w \cdot \nabla_w f)\|_{L_m^r} ds \\ &\leq C \frac{e^{\left(\frac{N}{p'}+1\right)t}}{\nu(t)^{\frac{1}{2}}} \|f_0\|_{L_m^p} + C \|f\|_{\Upsilon_T}^2 \int_{e^{-2t}}^1 \chi^{-\frac{N+2p'}{2p'}} (1-\chi)^{-\frac{1}{2}} ds \\ &\quad + C \sup_{0 < s < T} \left\{ \nu(s)^{1/2} \|f(s)(w \cdot \nabla_w f(s))\|_{L_m^r} \right\} I(t) \end{aligned}$$

where

$$\begin{aligned}
\nu(t)^{\frac{1}{2}} I(t) &\leq \frac{\nu(t)^{\frac{1}{2}}}{2} e^{-t} \int_{e^{-2t}}^1 e^{t(\frac{N+2r'}{r'})} (1-\chi)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{p})+\frac{1}{2}} (\chi - e^{-2t})^{-\frac{1}{2}} d\chi \\
&\leq \frac{\nu(t)^{\frac{1}{2}}}{2} e^{t(\frac{N+r'}{r'})} \left[ \int_{e^{-2t}}^{\frac{1+e^{-2t}}{2}} \left( \frac{1-e^{-2t}}{2} \right)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}} (\chi - e^{-2t})^{-\frac{1}{2}} d\chi \right. \\
&\quad \left. + \int_{\frac{1+e^{-2t}}{2}}^1 (\chi - e^{-2t})^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}} \left( \frac{1-e^{-2t}}{2} \right)^{-\frac{1}{2}} d\chi \right] \\
&\leq C e^{t\frac{N+r'}{r'}} (1 - e^{-2t})^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{p})} \nu(t)^{\frac{1}{2}} \\
&\leq C e^{t(\frac{N+p'}{p'})} \nu(t)^{\frac{1}{2} - \frac{N}{2}(\frac{1}{r}-\frac{1}{p})}
\end{aligned}$$

Note that the right-hand-side of the previous inequality is an increasing function of time taking zero value at  $t = 0$  since  $p > r > Np/(N + p)$ . It remains to estimate  $\|f(w \cdot \nabla_w f)\|_{L_m^r}$ :

$$\|f(w \cdot \nabla_w f)\|_{L_m^r} \leq C \left( \int_{\mathbb{R}^N} f^r |\nabla_w f|^r dw + \int_{\mathbb{R}^N} |w|^{(m+1)r} f^r |\nabla_w f|^r dw \right)^{\frac{1}{r}}$$

Now, we can bound these integrals by using Hölder's inequality to obtain

$$\int_{\mathbb{R}^N} f^r |\nabla_w f|^r dw \leq \left( \int_{\mathbb{R}^N} f^{\frac{pr}{p-r}} dw \right)^{\frac{p-r}{p}} \left( \int_{\mathbb{R}^N} |\nabla_w f|^p dw \right)^{\frac{r}{p}}$$

and

$$\int_{\mathbb{R}^N} |w|^{(m+1)r} f^r |\nabla_w f|^r dw \leq \left( \int_{\mathbb{R}^N} |w|^{\frac{pr}{p-r}} f^{\frac{pr}{p-r}} dw \right)^{\frac{p-r}{p}} \left( \int_{\mathbb{R}^N} |w|^{mp} |\nabla_w f|^p dw \right)^{\frac{r}{p}}.$$

Since  $p < pr/(p - r) \leq mp$  or equivalently  $(m + 1)r/m \leq p < 2r$  by (2.7), we have for any  $0 < t \leq T$

$$\int_{\mathbb{R}^N} f^r |\nabla_w f|^r dw \leq \|f\|_{\infty}^{2r-p} \|f\|_p^{p-r} \|\nabla_w f\|_p^r \leq \frac{\|f\|_{X_T}^{2r}}{\nu(t)^{\frac{r}{2}}}$$

and

$$\int_{\mathbb{R}^N} |w|^{(m+1)r} f^r |\nabla_w f|^r dw \leq \|f\|_{\infty}^{2r-p} \|f\|_{L_m^p}^{p-r} \|\nabla_w f\|_{L_m^p}^r \leq \frac{\|f\|_{X_T}^{2r}}{\nu(t)^{\frac{r}{2}}}.$$

Putting together the above estimates we have shown that,

$$\nu(t)^{1/2} \|f(t)(w \cdot \nabla_w f(t))\|_{L_m^r} \leq C \|f\|_{X_T}^2$$

and

$$\nu(t)^{\frac{1}{2}} \|\nabla_v f(t)\|_{L_m^p} \leq C_1^1(T, N, p) \|f_0\|_{L_m^p} + C_2^1(T, N, p, r) \|f\|_{X_T}^2 \quad (2.10)$$

with  $C_1^1$  and  $C_2^1$  increasing functions of  $T$  and for any  $0 < t \leq T$ . Analogously, we reckon

$$\begin{aligned} \|\nabla_v f(t)\|_{L_1^1} &\leq C \frac{e^t}{\nu(t)^{\frac{1}{2}}} \|f_0\|_{L_1^1} + C \int_0^t \frac{e^{t-s}}{\nu(t-s)^{\frac{1}{2}}} \|f(s)\|_{\infty} \|f(s)\|_{L_1^1} ds \\ &\quad + C \int_0^t \frac{e^{t-s}}{\nu(t-s)^{\frac{1}{2}}} \|f(w \cdot \nabla_w f)(s)\|_{L_1^1} ds \end{aligned}$$

where by taking  $p \geq 2$  and by interpolation as in (2.8), we have

$$\begin{aligned} \|f(w \cdot \nabla_w f)\|_{L_1^1} &\leq \| |w|^{\frac{1}{2}} f \|_2 \| |w|^{\frac{1}{2}} |\nabla_w f| \|_2 \\ &\leq \|f\|_{L_1^1}^{\frac{p-2}{2(p-1)}} \|f\|_{L_m^p}^{\frac{p}{2(p-1)}} \|\nabla_w f\|_{L_1^1}^{\frac{p-2}{2(p-1)}} \|\nabla_w f\|_{L_m^p}^{\frac{p}{2(p-1)}} \\ &\leq \frac{\|f\|_{X_T}^2}{\nu(t)^{1/2}}. \end{aligned}$$

Putting together the last estimates, we deduce

$$\nu(t)^{\frac{1}{2}} \|\nabla_v f(t)\|_{L_1^1} \leq C_1^3(T, N, p) \|f_0\|_{L_1^1} + C_2^3(T, N, p, r) \|f\|_{X_T}^2 \quad (2.11)$$

with  $C_1^3$  and  $C_2^3$  increasing functions of  $T$ , for any  $0 < t \leq T$ . From (2.10) and (2.11) and all the estimates of the previous section, we finally get

$$\|f\|_{X_T} \leq C_1(T, N, p) \|f_0\|_{\Upsilon} + C_2(T, N, p, r) \|f\|_{X_T}^2$$

for any  $T > 0$ . From these estimates and proceeding as at the end of the previous section, it is easy to show that we have uniform estimates in  $X_T$  of the iteration sequence and the convergence of the iteration sequence in the space  $X_T$ . From the uniqueness obtained in the previous section, we conclude that the solution obtained in this new procedure is the same as before and lies in  $X_T$ . Summarizing, we have shown:

**Theorem 2.5** *Let  $m \geq 1$ ,  $p > N$ ,  $p \geq 2$ , and  $f_0 \in \Upsilon$ . Then there exists  $T > 0$  depending only on the norm of the initial condition  $f_0$  in  $\Upsilon$  such that (1.1) has a unique solution in  $\mathcal{C}([0, T]; \Upsilon)$  with  $f(0) = f_0$  and its velocity gradients satisfy that  $t \mapsto \nu(t)^{\frac{1}{2}} |\nabla_v f(t)| \in \mathcal{C}_b((0, T), (L_m^p \cap L^1)(\mathbb{R}^N))$ .*

## 2.3 Properties of the solutions

As (1.1) belongs to the general class of convection-diffusion equation, it enjoys several classical properties which we gather in this section. The proofs of these results use classical

approximation arguments, see [11, 28] for instance. Since these arguments are somehow standard we will only give the detailed proof of the  $L^1$ -contraction property below. Before going into the proofs let us introduce some notation:

For  $\varepsilon > 0$ , we can define a regularized version of the sign function as

$$\text{sign}_\varepsilon(x) = \begin{cases} -1 & \text{if } x \leq -\varepsilon \\ \eta(x) & \text{if } -\varepsilon \leq x \leq \varepsilon \\ 1 & \text{if } x \geq \varepsilon \end{cases},$$

with an increasing and odd function  $\eta \in C^\infty([-\varepsilon, \varepsilon], \mathbb{R})$  such that  $\text{sign}_\varepsilon$  is  $C^\infty$  at  $x = \pm\varepsilon$ . Next let  $\zeta$  be a non-increasing function in  $\mathcal{C}_0^\infty((0, \infty))$  such that  $\zeta(r) = 1$  for  $0 \leq r \leq 1$  and  $\zeta(r) = 0$  for  $r \geq 2$ . We define

$$\zeta_n(x) = \zeta\left(\frac{|x|}{n}\right) \quad \text{for } x \in \mathbb{R}^N \quad \text{and } n \geq 1. \quad (2.12)$$

Note that  $\zeta_n \in \mathcal{C}_0^\infty(\mathbb{R}^N)$  is a cut-off function satisfying  $0 \leq \zeta_n \leq 1$ ,  $\zeta_n(v) = 1$  if  $|v| \leq n$ , and  $\zeta_n(v) = 0$  if  $|v| \geq 2n$ . Furthermore, we clearly have  $|\nabla_v \zeta_n| \leq C/n$  and  $|\Delta_v \zeta_n| \leq C/n^2$ .

**Lemma 2.6 (Positivity and Boundedness)** *Let  $f \in X_T$  be the solution of the Cauchy problem (1.1) with initial condition  $f_0 \in \Upsilon$ . If  $0 \leq f_0 \leq 1$  in  $\mathbb{R}^N$ , then  $0 \leq f(t) \leq 1$  for any  $0 < t \leq T$ .*

**Lemma 2.7 ( $L^1$ -Contraction and Comparison Principle)** *Let  $f \in X_T$  and  $g \in X_T$  be the solutions of the Cauchy problem (1.1) with respective initial data  $f_0 \in \Upsilon$  and  $g_0 \in \Upsilon$ . Then*

$$\|f(t) - g(t)\|_1 \leq \|f_0 - g_0\|_1 \quad (2.13)$$

*for all  $0 < t \leq T$ . Furthermore, if  $f_0 \leq g_0$  then  $f(t, v) \leq g(t, v)$  for all  $0 < t \leq T$  and  $v \in \mathbb{R}^N$ .*

*Proof.*- Since  $f$  and  $g$  solve (1.1),

$$\frac{d}{dt}(f - g) = \Delta_v(f - g) + \nabla_v(v(f - g)) - \nabla_v(v(f^2 - g^2)) \quad (2.14)$$

holds. We will obtain this result from the time evolution of  $|f - g|_\varepsilon$  where  $|\cdot|_\varepsilon$  denotes the primitive vanishing at zero of  $\text{sign}_\varepsilon$ . Multiplying both sides of equation (2.14) by

$\zeta_n(v) \operatorname{sign}_\varepsilon(f - g)$  and integrating over  $\mathbb{R}^N$  we obtain

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^N} \zeta_n(v) |f - g|_\varepsilon dv &\leq - \int_{\mathbb{R}^N} \zeta_n(v) \operatorname{sign}'_\varepsilon(f - g) (v \cdot \nabla_v(f - g)) (f - g) dv \\
&\quad + \int_{\mathbb{R}^N} \zeta_n(v) \operatorname{sign}'_\varepsilon(f - g) (v \cdot \nabla_v(f - g)) (f^2 - g^2) dv \\
&\quad - \int_{\mathbb{R}^N} \nabla_v \zeta_n \operatorname{sign}_\varepsilon(f - g) (\nabla_v(f - g) + v(f - g - (f^2 - g^2))) dv \\
&= - \int_{\mathbb{R}^N} \zeta_n(v) (v \cdot \nabla_v((f - g) \operatorname{sign}_\varepsilon(f - g) - |f - g|_\varepsilon)) dv \\
&\quad + \int_{\mathbb{R}^N} \zeta_n(v) (f + g) (v \cdot \nabla_v((f - g) \operatorname{sign}_\varepsilon(f - g) - |f - g|_\varepsilon)) dv \\
&\quad - \int_{\mathbb{R}^N} \nabla_v \zeta_n \operatorname{sign}_\varepsilon(f - g) (\nabla_v(f - g) + v(f - g - (f^2 - g^2))) dv.
\end{aligned}$$

Integrating by parts, we finally get

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^N} \zeta_n(v) |f - g|_\varepsilon dv &\leq \int_{\mathbb{R}^N} \operatorname{div}_v(v \zeta_n(v)) ((f - g) \operatorname{sign}_\varepsilon(f - g) - |f - g|_\varepsilon) dv \\
&\quad - \int_{\mathbb{R}^N} \operatorname{div}_v(\zeta_n(v) v (f + g)) ((f - g) \operatorname{sign}_\varepsilon(f - g) - |f - g|_\varepsilon) dv \\
&\quad + \frac{1}{n} \int_{\mathbb{R}^N} |\nabla_v(f - g) + v(f - g - (f^2 - g^2))| dv.
\end{aligned}$$

For every  $n$ , the first two integrals become zero as  $\varepsilon \rightarrow 0$ , since  $f$  and  $g$  are in  $X_T$  whence  $f(t), g(t) \in L^1_1 \cap L^\infty(\mathbb{R}^N)$  and  $\nabla_v f(t), \nabla_v g(t) \in L^1_1(\mathbb{R}^N)$  for any  $0 < t \leq T$ , allowing for a Lebesgue dominated convergence argument. We have that  $\nabla_v f + v f(1 - f) \in L^1(\mathbb{R}^N)$  and  $\nabla_v g + v g(1 - g) \in L^1(\mathbb{R}^N)$  for any  $0 < t \leq T$ , and thus the third integral vanishes as  $n \rightarrow \infty$ , getting finally

$$\frac{d}{dt} \int_{\mathbb{R}^N} |f - g| dv \leq 0 \tag{2.15}$$

which concludes the proof of the first assertion of the lemma.  $\square$

Similar arguments show the conservation of mass.

**Lemma 2.8 (Mass Conservation)** *Let  $f \in X_T$  be the solution of the Cauchy problem (1.1) with non-negative initial condition  $f_0 \in \Upsilon$ , then the  $L^1$ -norm of  $f$  is conserved, i.e.  $\|f(t)\|_1 = \|f_0\|_1$  for all  $t \in [0, T]$ .*

Finally, we establish time dependent bounds on moments of the solution to (1.1). More precisely, we will show that moments increase at most as a polynomial of  $t$ . First, let us note that given  $a, b \geq 1$  and  $f \in L^1_{ab}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  then

$$\|f\|_{L^b_a} \leq C \|f\|_{L^1_{ab}}^{\frac{1}{b}} \|f\|_\infty^{1 - \frac{1}{b}}. \tag{2.16}$$

Indeed,

$$\begin{aligned}\|f\|_{L_a^b} &= \left( \int_{\mathbb{R}^N} (1 + |v|^a)^b f^b dv \right)^{\frac{1}{b}} \leq \left( C \int_{\mathbb{R}^N} (1 + |v|^{ab}) f^b dv \right)^{\frac{1}{b}} \\ &\leq \left( C \|f\|_{\infty}^{b-1} \int_{\mathbb{R}^N} (1 + |v|^{ab}) f dv \right)^{\frac{1}{b}} = C \|f\|_{L_{ab}^1}^{\frac{1}{b}} \|f\|_{\infty}^{1-\frac{1}{b}}.\end{aligned}$$

**Lemma 2.9 (Moments Bound)** *Let  $f \in X_T$  be the solution of the Cauchy problem (1.1) with initial condition  $f_0$  satisfying  $0 \leq f_0 \leq 1$ . Assume further that  $f_0 \in L_{mp}^1(\mathbb{R}^N)$ . Then, for  $0 \leq t \leq T$  and  $1 \leq \gamma \leq mp/2$  the  $2\gamma$ -moment of  $f(t)$  is bounded by a polynomial  $P_{[\gamma]}(t)$  of degree  $[\gamma]$ , which depends only on the moments of  $f_0$ . Here and below,  $[\gamma]$  denotes the smallest integer larger or equal than  $\gamma$ .*

As  $(L_{mp}^1 \cap L^\infty)(\mathbb{R}^N) \subset \Upsilon$  by (2.16), the assumption  $f_0 \in L_{mp}^1(\mathbb{R}^N)$  is indeed an additional assumption.

*Proof.*- We will prove Lemma 2.9 by induction on  $\gamma$ . First, we will see that the second moment is bounded (and therefore all  $\gamma_*^{\text{th}}$ -moments with  $0 < \gamma_* \leq 2$ ). Afterwards, we will assume that we can bound the  $2(\gamma - 1)$ -moment and from this induction hipotesis obtain that the  $2\gamma$ -moment of the solution is bounded.

Let  $(\zeta_n)_{n \geq 1}$  be a sequence of smooth cut-off functions as defined in (2.12). We multiply (1.1) by  $|v|^2 \zeta_n(v)$  and integrate over  $\mathbb{R}^N$  to get

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{R}^N} \zeta_n(v) |v|^2 f dv &= \int_{\mathbb{R}^N} \zeta_n(v) |v|^2 \Delta_v f dv + \int_{\mathbb{R}^N} \zeta_n(v) |v|^2 \text{div}_v(v f(1 - f)) dv \\ &\leq \int_{\mathbb{R}^N} [\Delta_v \zeta_n |v|^2 + 4 \nabla_v \zeta_n v + 2N \zeta_n] f dv + \int_{\mathbb{R}^N} |\nabla_v \zeta_n| |v|^3 f(1 - f) dv \\ &\quad - 2 \int_{\mathbb{R}^N} \zeta_n |v|^2 f dv + 2 \int_{\mathbb{R}^N} \zeta_n |v|^2 f^2 dv \\ &\leq 5 \int_{n < |v| < 2n} f dv + 2N \int_{\mathbb{R}^N} \zeta_n f dv + \int_{n < |v| < 2n} |v|^2 f dv.\end{aligned}$$

Now, letting  $n \rightarrow \infty$  and noticing that  $f 1_{\{n < |v| < 2n\}}$  and  $|v|^2 f 1_{\{n < |v| < 2n\}}$  converge pointwise to zero and are bounded by  $f$  and  $|v|^2 f$  respectively with  $f \in X_T$ , we infer from the Lebesgue dominated convergence theorem that the first and the last integrals converge to zero. Finally, integrating in time, we get

$$\int_{\mathbb{R}^N} |v|^2 f(t, v) dv \leq \int_{\mathbb{R}^N} |v|^2 f_0(v) dv + 2NMt \quad (2.17)$$

for all  $0 \leq t \leq T$ . By the conservation of mass and this bound, all moments  $0 < \gamma < 2$  are bounded.

This gives the first step in the induction argument. Now, let us assume that the  $2\gamma - 2$  moment of the solution is bounded by a polynomial of degree  $\gamma - 1$ . Then, for the moment  $2\gamma$  we can see analogously

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^N} \zeta_n(v) |v|^{2\gamma} f dv &= \int_{\mathbb{R}^N} \zeta_n(v) |v|^{2\gamma} \Delta_v f dv + \int_{\mathbb{R}^N} \zeta_n(v) |v|^{2\gamma} \operatorname{div}_v(v f(1-f)) dv \\
&\leq \int_{\mathbb{R}^N} [\Delta_v \zeta_n |v|^{2\gamma} + 4\gamma \nabla_v \zeta_n |v|^{2(\gamma-1)} v + 2\gamma(2(\gamma-1) + N) |v|^{2(\gamma-1)} \zeta_n] f dv \\
&\quad + \int_{\mathbb{R}^N} |\nabla_v \zeta_n| |v|^{2\gamma+1} f(1-f) dv - 2\gamma \int_{\mathbb{R}^N} \zeta_n |v|^{2\gamma} f dv + 2\gamma \int_{\mathbb{R}^N} \zeta_n |v|^{2\gamma} f^2 dv \\
&\leq C \int_{n < |v| < 2n} |v|^{2(\gamma-1)} f dv + 2\gamma(2(\gamma-1) + N) \int_{\mathbb{R}^N} \zeta_n |v|^{2(\gamma-1)} f dv \\
&\quad + \int_{n < |v| < 2n} |v|^{2\gamma} f dv
\end{aligned}$$

and we again let  $n$  go to infinity. If  $2\gamma \leq mp$ , the previous argument ensures that only the second integral remains, and integrating in time, we conclude

$$\int_{\mathbb{R}^N} |v|^{2\gamma} f(t, v) dv \leq \int_{\mathbb{R}^N} |v|^{2\gamma} f_0(v) dv + 2\gamma(2(\gamma-1) + N) \int_0^t \int_{\mathbb{R}^N} |v|^{2(\gamma-1)} f(s, v) dv ds \quad (2.18)$$

for all  $0 \leq t \leq T$ . Whence, by induction,

$$\int_{\mathbb{R}^N} |v|^{2\gamma} f(v, t) dv \leq \int_{\mathbb{R}^N} |v|^{2\gamma} f_0(v) dv + 2\gamma(2(\gamma-1) + N) \int_0^t P_{\lceil \gamma-1 \rceil}(s) ds \quad (2.19)$$

for all  $0 \leq t \leq T$ , defining by induction the polynomial  $P_{\lceil \gamma \rceil}$ .  $\square$

**Remark 2.10** *This lemma could have been stated for  $f_0 \in L^1_\alpha(\mathbb{R}^N)$ , with  $\alpha > 2$ , but we have decided to use this notation to point out that  $\alpha$  shall be obtained as a combination of  $m$  and  $p$  satisfying the conditions of the existence theorem.*

## 2.4 Global existence

Given an initial condition  $f_0 \in L^1_{mp}(\mathbb{R}^N)$ ,  $p > N$ ,  $p \geq 2$ ,  $m \geq 1$  such that  $0 \leq f_0 \leq 1$ , we have  $f_0 \in \Upsilon$  and we have shown in the previous subsections that there exists a unique local solution of (1.1) on an interval  $[0, T)$ . In fact, we can extend this solution to be global in time. If there exists  $T_{max} < \infty$  such that the solution does not exist out of  $(0, T_{max})$ , then the  $\Upsilon$ -norm of it shall go to infinity as  $t$  goes to  $T_{max}$ ; as we will see, that situation cannot happen.

Due to Lemma 2.6, we have that  $0 \leq f(t, v) \leq 1$  for any  $0 \leq t < T$  and any  $v \in \mathbb{R}^N$ , and thus a bound for the  $L^\infty$ -norm of  $f(t)$ . Also, the conservation of the mass in Lemma

2.8 together with the positivity in Lemma 2.6 provide us with a bound for the  $L^1$ -norm. Finally, due to (2.16) and Lemma 2.9 the  $L_m^p$ -norm is also bounded on any finite time interval.

**Theorem 2.11 (Global Existence)** *Let  $f_0 \in L_{mp}^1(\mathbb{R}^N)$ ,  $p > N$ ,  $p \geq 2$ ,  $m \geq 1$  be such that  $0 \leq f_0 \leq 1$ . Then the Cauchy problem (1.1) with initial condition  $f_0$  has a unique solution defined in  $[0, \infty)$  belonging to  $X_T$  for all  $T > 0$ . Also, we have  $0 \leq f(t, v) \leq 1$ , for all  $t \geq 0$  and  $v \in \mathbb{R}^N$  and  $\|f(t)\|_1 = \|f_0\|_1 = M$  for all  $t \geq 0$ .*

**Remark 2.12** *Note that for any  $K > 0$  we can consider (1.1) restricted to the cylinder  $C_K := [0, \infty) \times \{|v| \leq K\}$ . Then, since the solutions to (1.1) we have constructed are bounded, we can show that the solution is in fact  $C^\infty(C_K)$  by applying regularity results from [19] for the Cauchy problem for quasilinear parabolic equations.*

**Corollary 2.13** *If  $f_0 \in L_{mp}^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  is a radially symmetric and non-increasing function (that is,  $f_0(v) = \varphi_0(|v|)$  for some non-increasing function  $\varphi_0$ ), then so is  $f(t)$  for all  $t \geq 0$ , that is,  $f(t, v) = \varphi(t, |v|)$  and  $r \mapsto \varphi(t, r)$  is non-increasing for all  $t \geq 0$ . In addition,  $\varphi$  solves*

$$\frac{\partial \varphi}{\partial t} = \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left( r^{N-1} \frac{\partial \varphi}{\partial r} + r^N \varphi(1 - \varphi) \right) \quad \text{with} \quad \frac{\partial \varphi}{\partial r}(t, 0) = 0 \quad (2.20)$$

and  $\varphi(0, r) = \varphi_0(r)$ .

*Proof.*- The uniqueness part of Theorem 2.11 and the rotational invariance of (1.1) imply that  $f(t)$  is radially symmetric for all  $t \geq 0$ . The other properties are proved by classical arguments, the monotonicity of  $r \mapsto \varphi(t, r)$  being a consequence of the comparison principle applied to the equation solved by  $\partial \varphi / \partial r$ . The latter is obtained from (2.20) after differentiating once with respect to  $r$  and the zero function is a solution to it.  $\square$

## 3 Asymptotic Behaviour

Now that we have shown that under the appropriate assumptions equation (1.1) has a unique solution which is global in time, we are interested in how this solution behaves when the time is large. For that we will define an appropriate entropy functional for the solution and study its properties.

### 3.1 Associated Entropy Functional

In this section, we will show that the solutions constructed above satisfy an additional dissipation property, the entropy decay. For  $g \in \mathcal{Y}$  such that  $0 \leq g \leq 1$ , we define the functional

$$H(g) := S(g) + E(g) \quad (3.1)$$



with the entropy given by

$$S(g) := \int_{\mathbb{R}^N} s(g(v)) \, dv \quad (3.2)$$

where

$$s(r) := (1-r) \log(1-r) + r \log(r) \leq 0, \quad r \in [0, 1], \quad (3.3)$$

and the kinetic energy given by

$$E(g) := \frac{1}{2} \int_{\mathbb{R}^N} |v|^2 g(v) \, dv. \quad (3.4)$$

We first check that  $H(g)$  is indeed well defined and establish a control of the entropy in terms of the kinetic energy.

**Lemma 3.1 (Entropy Control)** *For  $\varepsilon \in (0, 1)$ , there exists a positive constant  $C_\varepsilon$  such that*

$$0 \leq -S(g) \leq \varepsilon E(g) + C_\varepsilon \quad (3.5)$$

for every  $g \in L_2^1(\mathbb{R}^N)$  such that  $0 \leq g \leq 1$ .

*Proof.*- For  $\varepsilon \in (0, 1)$  and  $v \in \mathbb{R}^N$ , we put  $z_\varepsilon(v) := 1/(1 + e^{\varepsilon|v|^2/2})$ . The convexity of  $s$  ensures that

$$\begin{aligned} s(g(v)) - s(z_\varepsilon(v)) &\geq s'(z_\varepsilon(v))(g(v) - z_\varepsilon(v)) \\ -s(z_\varepsilon(v)) + s(g(v)) &\geq \log\left(\frac{z_\varepsilon(v)}{1 - z_\varepsilon(v)}\right) (g(v) - z_\varepsilon(v)) \end{aligned}$$

for  $v \in \mathbb{R}^N$ . Since  $z_\varepsilon(v)/(1 - z_\varepsilon(v)) = e^{-\varepsilon|v|^2/2}$ , we end up with

$$\begin{aligned} -s(g(v)) &\leq \frac{\varepsilon|v|^2}{2} g(v) - s(z_\varepsilon(v)) - \frac{\varepsilon|v|^2}{2} z_\varepsilon(v) \\ &= \frac{\varepsilon|v|^2}{2} g(v) + (1 - z_\varepsilon(v)) \log\left(1 + e^{-\varepsilon|v|^2/2}\right) + z_\varepsilon(v) \log\left(1 + e^{-\varepsilon|v|^2/2}\right) \\ &\leq \frac{\varepsilon|v|^2}{2} g(v) + e^{-\varepsilon|v|^2/2} \end{aligned} \quad (3.6)$$

for  $v \in \mathbb{R}^N$ , where we used  $\log(1 + a) \leq a$  for  $a \geq 0$  and  $0 \leq z_\varepsilon \leq 1$ . Integrating the previous inequality yields (3.5).  $\square$

We next recall that  $F_M$  is the unique Fermi-Dirac equilibrium state satisfying  $\|F_M\|_1 = M := \|f_0\|_1$ ; then we can introduce the next property for  $H$ .

**Lemma 3.2 (Entropy Monotonicity)** *Assume that  $f$  is the solution to the Cauchy problem (1.1) with initial condition  $f_0$  in  $L^1_{mp}(\mathbb{R}^N)$  for some  $p > \max(N, 2)$  and  $m \geq 1$  satisfying  $0 \leq f_0 \leq 1$ . Then, the function  $H$  is a non-increasing function of time satisfying for all  $t > 0$  that*

$$H(f_0) \geq H(f(t)) \geq H(F_M) \quad \text{with} \quad M := \|f_0\|_1. \quad (3.7)$$

*Proof.* - We first give a formal proof of the time monotonicity of  $H(f)$  and supply additional details at the end of the proof. First of all, we observe that we can formulate (1.1) as

$$\frac{\partial f}{\partial t} = \operatorname{div}_v \left[ f(1-f) \nabla_v \left( s'(f) + \frac{|v|^2}{2} \right) \right].$$

We multiply the previous equation by  $s'(f) + |v|^2/2$  and integrate over  $\mathbb{R}^N$  to obtain that

$$\frac{d}{dt} H(f) = - \int_{\mathbb{R}^N} f(1-f) |v + \nabla_v s'(f)|^2 dv \leq 0. \quad (3.8)$$

Consequently, the function  $t \mapsto H(f(t))$  is a non-increasing function of time, whence the first inequality in (3.7). To prove the second inequality, we observe that the convexity of  $s$  entails that

$$\begin{aligned} s(f(t, v)) - s(F_M(v)) &\geq s'(F_M(v))(f(t, v) - F_M(v)) \\ s(F_M(v)) - s(f(t, v)) &\leq \left( \log \beta(M) + \frac{|v|^2}{2} \right) (f(t, v) - F_M(v)) \end{aligned}$$

for  $(t, v) \in [0, \infty) \times \mathbb{R}^N$ . The second inequality in (3.7) now follows from the integration of the previous inequality over  $\mathbb{R}^N$  since  $\|F_M\|_1 = \|f(t)\|_1$  by Lemma 2.8.

We shall point out that, in order to justify the previous computations leading to the time monotonicity of the entropy, one should first start with an initial condition  $f_0^\varepsilon$ ,  $\varepsilon \in (0, 1)$ , given by

$$f_0^\varepsilon(v) = \max \left\{ \min \left\{ f_0(v), \frac{1}{1 + \varepsilon e^{|v|^2/2}} \right\}, \frac{\varepsilon}{\varepsilon + e^{|v|^2/2}} \right\} \in \left[ \frac{\varepsilon}{\varepsilon + e^{|v|^2/2}}, \frac{1}{1 + \varepsilon e^{|v|^2/2}} \right], \quad v \in \mathbb{R}^N.$$

Owing to the comparison principle (Lemma 2.7), the corresponding solution  $f^\varepsilon$  to (1.1) satisfies

$$0 < \frac{\varepsilon}{\varepsilon + e^{|v|^2/2}} \leq f^\varepsilon(t, v) \leq \frac{1}{1 + \varepsilon e^{|v|^2/2}} < 1, \quad (t, v) \in (0, \infty) \times \mathbb{R}^N. \quad (3.9)$$

The previous computations can then be performed on  $f^\varepsilon$  since it is immediately smooth (see remark 2.12) and fast decaying at infinity for all  $t > 0$  by (3.9), and thus  $H(f^\varepsilon(t)) \leq H(f_0^\varepsilon)$  for all  $t \geq 0$ .

Since  $f_0^\varepsilon \rightarrow f_0$  in  $\Upsilon$  and in  $L_{mp}^1(\mathbb{R}^N)$  as  $\varepsilon \rightarrow 0$ , it is not difficult to see that redoing all estimates in subsections 2.1 and 2.2, we have continuous dependence of solutions with respect to the initial data, and thus,  $f^\varepsilon$  converges towards  $f$  in  $X_T$  for any  $T > 0$ . Moreover, we have uniform bounds with respect to  $\varepsilon$  of the moments in finite time intervals using Lemma 2.9. Direct estimates easily show that  $H(f_0^\varepsilon) \rightarrow H(f_0)$  as  $\varepsilon \rightarrow 0$ .

Let us now prove that  $H(f^\varepsilon(t)) \rightarrow H(f(t))$  as  $\varepsilon \rightarrow 0$  for  $t > 0$ . Let us fix  $R > 0$ . Since  $f^\varepsilon(t) \rightarrow f(t)$  in  $L^1(\mathbb{R}^N)$  and we have uniform estimates in  $\varepsilon$  of moments of order  $mp > 2$  then

$$\begin{aligned} \left| \int_{\mathbb{R}^N} |v|^2 (f^\varepsilon(t) - f(t)) \, dv \right| &\leq \int_{|v| \geq R} |v|^2 |f^\varepsilon(t) - f(t)| \, dv + \left| \int_{|v| < R} |v|^2 (f^\varepsilon(t) - f(t)) \, dv \right| \\ &\leq \frac{1}{R^{mp-2}} \int_{|v| \geq R} |v|^{mp} (f^\varepsilon(t) + f(t)) \, dv \\ &\quad + R^2 \|f^\varepsilon(t) - f(t)\|_1 \\ &\leq \frac{C(t)}{R^{mp-2}} + R^2 \|f^\varepsilon(t) - f(t)\|_1. \end{aligned}$$

Since the above inequality is valid for all  $R > 0$ , we conclude that  $E(f^\varepsilon(t)) \rightarrow E(f(t))$  as  $\varepsilon \rightarrow 0$ . Now, taking into account that  $(1 + |v|^2)f^\varepsilon(t) \rightarrow (1 + |v|^2)f(t)$  in  $L^1(\mathbb{R}^N)$ , we deduce that there exists  $h \in L^1(\mathbb{R}^N)$  such that  $|v|^2 f^\varepsilon(t) \leq h$  and  $f^\varepsilon(t) \rightarrow f(t)$  a.e. in  $\mathbb{R}^N$ , for a subsequence that we denote with the same index. Using inequality (3.6), we deduce that

$$0 \leq -s(f^\varepsilon(t, v)) \leq \frac{1}{4}h(v) + e^{-|v|^2/4} \in L^1(\mathbb{R}^N)$$

and that  $-s(f^\varepsilon(t, v)) \rightarrow -s(f(t, v))$  a.e. in  $\mathbb{R}^N$ . Thus, by the Lebesgue dominated convergence theorem, we finally deduce that  $S(f^\varepsilon(t)) \rightarrow S(f(t))$  as  $\varepsilon \rightarrow 0$ . The convergence as  $\varepsilon \rightarrow 0$  of  $S(f^\varepsilon(t))$  to  $S(f(t))$  is actually true for the whole family (and not only for a subsequence) thanks to the uniqueness of the limit. As a consequence, we showed  $H(f^\varepsilon(t)) \rightarrow H(f(t))$  as  $\varepsilon \rightarrow 0$  and passing to the limit  $\varepsilon \rightarrow 0$  in the inequality  $H(f^\varepsilon(t)) \leq H(f_0^\varepsilon)$ , we get the desired result.  $\square$

Now, it is easy to see the existence of a uniform in time bound for the kinetic energy  $E(f(t))$ , or equivalently, of the solutions in  $L_2^1(\mathbb{R}^N)$ . If we take equations (3.1), (3.5) (with  $\varepsilon = 1/2$ ) and (3.7) we get that

$$E(f(t)) = H(f(t)) - S(f(t)) \leq \frac{1}{2}E(f(t)) + C_{1/2} + H(f_0)$$

for  $t \geq 0$  whence

$$E(f(t)) \leq 2(C_{1/2} + H(f_0)). \quad (3.10)$$

## 3.2 Convergence to the Steady State

**Theorem 3.3 (Convergence)** *Let  $f$  be the solution to the Cauchy problem (1.1) with initial condition  $f_0 \in L^1_{mp}(\mathbb{R}^N)$ ,  $p > \max(N, 2)$ ,  $m \geq 1$  satisfying  $0 \leq f_0 \leq 1$ . Then  $\{f(t)\}_{t \geq 0}$  converges strongly in  $L^1(\mathbb{R}^N)$  towards  $F_M$  as  $t \rightarrow \infty$  with  $M := \|f_0\|_1$ .*

For the proof, we first need a technical lemma.

**Lemma 3.4** *Let  $f$  be the solution to the Cauchy problem (1.1) with initial condition  $f_0 \in L^1_{mp}(\mathbb{R}^N)$ ,  $p > \max(N, 2)$ ,  $m \geq 1$  satisfying  $0 \leq f_0 \leq 1$ . If  $A$  is a measurable subset of  $\mathbb{R}^N$ , we have*

$$\int_0^\infty \left( \int_A |vf(1-f) + \nabla_v f| dv \right)^2 dt \leq H(F_M) \sup_{t \geq 0} \left\{ \int_A f(t, v) dv \right\} \quad (3.11)$$

*Proof.*- Owing to the second inequality in (3.7) and the finiteness of  $H(f_0)$ , we also infer from (3.8) that  $(t, v) \mapsto f(1-f)|v + \nabla_v s'(f)|^2$  belongs to  $L^1((0, \infty) \times \mathbb{R}^N)$ . Working again with the regularized solutions  $f^\varepsilon$ , it then follows from Lemma 2.8 and the Cauchy-Schwarz inequality that, if  $A$  is a measurable subset of  $\mathbb{R}^N$ , we can compute

$$\begin{aligned} \int_0^\infty \left( \int_A |vf^\varepsilon(1-f^\varepsilon) + \nabla_v f^\varepsilon| dv \right)^2 dt \\ = \int_0^\infty \left( \int_A \frac{|vf^\varepsilon(1-f^\varepsilon) + \nabla_v f^\varepsilon|}{(f^\varepsilon(1-f^\varepsilon))^{1/2}} (f^\varepsilon(1-f^\varepsilon))^{1/2} dv \right)^2 dt \\ \leq \int_0^\infty \left( \int_A \frac{|vf^\varepsilon(1-f^\varepsilon) + \nabla_v f^\varepsilon|^2}{f^\varepsilon(1-f^\varepsilon)} dv \right) \left( \int_A f^\varepsilon(1-f^\varepsilon) dv \right) dt, \end{aligned}$$

and thus,

$$\begin{aligned} \int_0^\infty \left( \int_A |vf^\varepsilon(1-f^\varepsilon) + \nabla_v f^\varepsilon| dv \right)^2 dt \\ \leq \sup_{t \geq 0} \left\{ \int_A f^\varepsilon(t, v) dv \right\} \int_0^\infty \int_A f^\varepsilon(1-f^\varepsilon) [v + \nabla_v s'(f^\varepsilon)]^2 dv dt \\ \leq H(F_{M^\varepsilon}) \sup_{t \geq 0} \left\{ \int_A f^\varepsilon(t, v) dv \right\}. \end{aligned}$$

Here,  $M^\varepsilon := \|f_0^\varepsilon\|_1$  so that  $F_{M^\varepsilon}$  is the Fermi-Dirac distribution with the mass of the regularized initial condition  $f_0^\varepsilon$ . It is easy to check that  $H(F_{M^\varepsilon}) \rightarrow H(F_M)$  as  $\varepsilon \rightarrow 0$  since  $M^\varepsilon \rightarrow M$  as  $\varepsilon \rightarrow 0$ . Passing to the limit as  $\varepsilon \rightarrow 0$ ,  $f^\varepsilon \rightarrow f$  in  $X_T$  for any  $T > 0$ , and thus we get the conclusion.  $\square$

*Proof of Theorem 3.3.*- We first establish that

$$\{f(t)\}_{t \geq 0} \text{ is bounded in } L_2^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \quad (3.12)$$

From (3.10) and Theorem 2.11, it is straightforward that  $E(f(t))$  is bounded in  $[0, \infty)$ . Recalling the mass conservation, the boundedness of  $\{f(t)\}_{t \geq 0}$  in  $L_2^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  follows.

We next turn to the strong compactness of  $\{f(t)\}_{t \geq 0}$  in  $L^1(\mathbb{R}^N)$ . For that purpose, we put  $R(t, v) := vf(t, v)(1 - f(t, v))$  for  $(t, v) \in (0, \infty) \times \mathbb{R}^N$  and deduce from Theorem 2.11 and (3.12) that

$$\sup_{t \geq 0} (\|R(t)\|_1 + \|R(t)\|_2^2) \leq 2 \sup_{t \geq 0} \int_{\mathbb{R}^N} (1 + |v|^2) f(t, v) dv < \infty. \quad (3.13)$$

Denoting the linear heat semigroup on  $\mathbb{R}^N$  by  $(e^{t\Delta})_{t \geq 0}$ , it follows from (1.1) that  $f$  is given by the Duhamel formula

$$f(t) = e^{t\Delta} f_0 + \int_0^t \nabla_v e^{(t-s)\Delta} R(s) ds, \quad t \geq 0. \quad (3.14)$$

It is straightforward to check by direct Fourier transform techniques that

$$\|e^{t\Delta} g\|_{\dot{H}^\alpha} \leq C(\alpha) \min \{t^{-\alpha/2} \|g\|_2, t^{-(2\alpha+N)/4} \|g\|_1\}$$

for  $t \in (0, \infty)$ ,  $g \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$  and  $\alpha \in [0, 2]$  with

$$\|g\|_{\dot{H}^\alpha} := \left( \int_{\mathbb{R}^N} |\xi|^{2\alpha} |\widehat{g}(\xi)|^2 d\xi \right)^{1/2}$$

and  $\widehat{g}$  being the Fourier transform of  $g$ . Thus, we deduce from (3.14) that, if  $t \geq 1$  and  $\alpha \in ((1 - (N/2))^+, 1)$ , we have

$$\begin{aligned} \|f(t)\|_{\dot{H}^\alpha} &\leq C(\alpha) t^{-(2\alpha+N)/4} \|f_0\|_1 + C(\alpha + 1) \int_0^{t-1} (t-s)^{-(2+2\alpha+N)/4} \|R(s)\|_1 ds \\ &\quad + C(\alpha + 1) \int_{t-1}^t (t-s)^{-(1+\alpha)/2} \|R(s)\|_2 ds \\ &\leq C \left( 1 + \int_1^t s^{-(2+2\alpha+N)/4} ds + \int_0^1 s^{-(1+\alpha)/2} ds \right) \\ &\leq C, \end{aligned}$$

thanks to the choice of  $\alpha$ . Consequently,  $\{f(t)\}_{t \geq 1}$  is also bounded in  $\dot{H}^\alpha$  for  $\alpha \in ((1 - (N/2))^+, 1)$ . Owing to the compactness of the embedding of  $(\dot{H}^\alpha \cap L_2^1)(\mathbb{R}^N)$  in  $L^1(\mathbb{R}^N)$ , we finally conclude that

$$\{f(t)\}_{t \geq 0} \text{ is relatively compact in } L^1(\mathbb{R}^N). \quad (3.15)$$

Consider now a sequence  $\{t_n\}_{n \in \mathbb{N}}$  of positive real numbers such that  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Owing to (3.15), there are a subsequence of  $\{t_n\}$  (not relabelled) and  $g_\infty \in L^1(\mathbb{R}^N)$  such that  $\{f(t_n)\}_{n \in \mathbb{N}}$  converges towards  $g_\infty$  in  $L^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Putting  $f_n(t) = f(t_n + t)$ ,  $t \in [0, 1]$  and denoting by  $g$  the unique solution to (1.1) with initial datum  $g_\infty$ , we infer from the contraction property (2.13) that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, 1]} \|f_n(t) - g(t)\|_1 = 0. \quad (3.16)$$

Next, on one hand, we deduce from the proof of Lemma 3.4 with  $A = \mathbb{R}^N$  that  $(t, v) \mapsto vf(t, v)(1 - f(t, v)) + \nabla_v f(t, v)$  belongs to  $L^2((0, \infty); L^1(\mathbb{R}^N))$ . Since

$$\int_0^1 \left( \int_{\mathbb{R}^N} |vf_n(1 - f_n) + \nabla_v f_n| dv \right)^2 dt = \int_{t_n}^{t_n+1} \left( \int_{\mathbb{R}^N} |vf(1 - f) + \nabla_v f| dv \right)^2 dt,$$

we end up with

$$\lim_{n \rightarrow \infty} \int_0^1 \left( \int_{\mathbb{R}^N} |vf_n(1 - f_n) + \nabla_v f_n| dv \right)^2 dt = 0. \quad (3.17)$$

On the other hand, it follows from the mass conservation and (3.11) that, if  $A$  is a measurable subset of  $\mathbb{R}^N$  with finite measure  $|A|$ , we have

$$\int_0^1 \left( \int_A |vf_n(1 - f_n) + \nabla_v f_n| dv \right)^2 dt \leq H(F_M)|A|,$$

which implies that  $\{vf_n(1 - f_n) + \nabla_v f_n\}_{n \in \mathbb{N}}$  is weakly relatively compact in  $L^1((0, 1) \times \mathbb{R}^N)$  by the Dunford-Pettis theorem. Since  $\{vf_n(1 - f_n)\}_{n \in \mathbb{N}}$  converges strongly towards  $vg(1 - g)$  in  $L^1((0, 1) \times \mathbb{R}^N)$  by (3.12) and (3.16), we conclude that  $\{\nabla_v f_n\}_{n \geq 0}$  is weakly relatively compact in  $L^1((0, 1) \times \mathbb{R}^N)$ . Upon extracting a further subsequence, we may thus assume that  $\{\nabla_v f_n\}_{n \geq 0}$  converges weakly towards  $\nabla_v g$  in  $L^1((0, 1) \times \mathbb{R}^N)$ . Consequently,

$$\int_0^1 \int_{\mathbb{R}^N} |vg(1 - g) + \nabla_v g| dv dt \leq \liminf_{n \rightarrow \infty} \int_0^1 \int_{\mathbb{R}^N} |vf_n(1 - f_n) + \nabla_v f_n| dv dt = 0$$

by (3.17), from which we readily deduce that  $vg(1 - g) + \nabla_v g = 0$  a.e. in  $(0, 1) \times \mathbb{R}^N$ . Since  $\|g(t)\|_1 = M$  for each  $t \in [0, 1]$  by Lemma 2.8 and (3.16), standard arguments allow us to conclude that  $g(t) = F_M$  for each  $t \in [0, 1]$ . We have thus proved that  $F_M$  is the only possible cluster point in  $L^1(\mathbb{R}^N)$  of  $\{f(t)\}_{t \geq 0}$  as  $t \rightarrow \infty$ , which, together with the relative compactness of  $\{f(t)\}_{t \geq 0}$  in  $L^1(\mathbb{R}^N)$ , implies the assertion of Theorem 3.3.  $\square$

By now, we have seen that the solution of (1.1) with initial condition  $f_0$  converges to the Fermi-Dirac distribution  $F_M$  with the same mass as  $f_0$  as  $t \rightarrow \infty$ , but we are also interested in how fast this happens. We will answer that question with the next result, which was already proved in [5] in the one dimensional case, and easily extends to any dimension based on the existence and entropy decay results established above.

**Theorem 3.5 (Entropy Decay Rate)** *Let  $f$  be the solution to the Cauchy problem (1.1) with initial condition  $f_0 \in L^1_{mp}(\mathbb{R}^N)$ ,  $p > \max(N, 2)$ ,  $m \geq 1$  satisfying  $0 \leq f_0 \leq F_{M^*} \leq 1$  for some  $M^*$ . Then*

$$H(f(t)) - H(F_M) \leq (H(f_0) - H(F_M))e^{-2Ct} \quad (3.18)$$

and

$$\|f(t) - F_M\|_1 \leq C_2(H(f_0) - H(F_M))^{1/2}e^{-Ct} \quad (3.19)$$

for all  $t \geq 0$ , where  $C$  depends on  $M^*$  and  $M := \|f_0\|_1$ .

*Proof.* Since  $0 \leq f_0 \leq F_{M^*}$ , then the initial condition satisfies all the hypotheses of Theorems 2.11 and 3.3. In order to show the exponential convergence, we use the same arguments as in [5]. We first remark that the entropy functional  $H$  coincides with the one introduced in [2] for the nonlinear diffusion equation

$$\frac{\partial g}{\partial t} = \operatorname{div}_x [g \nabla_x (x + h(g))] \quad (3.20)$$

for the function  $0 \leq g(t, x) \leq 1$ ,  $x \in \mathbb{R}$ ,  $t > 0$ , where  $h(g) := s'(g) = \log g - \log(1 - g)$ . Let us point out that the relation between the entropy dissipation for the solutions of the nonlinear diffusion equation (3.20), given by

$$-D_0(g) := \frac{d}{dt}H(g) = - \int_{\mathbb{R}^N} g \left| x + \frac{\partial}{\partial x} h(g) \right|^2 dx,$$

and the entropy dissipation for the solutions of (1.1), given by (3.8), is the basic idea of the proof. Indeed, one can check that, once restricted to the range  $f \in (0, 1)$ ,  $h(f)$  verifies the hypotheses of the Generalized Logarithmic Sobolev Inequality [2, Theorem 17]. The Generalized Logarithmic Sobolev Inequality then asserts that

$$H(g) - H(F_M) \leq \frac{1}{2}D_0(g) \quad (3.21)$$

for all integrable positive  $g$  with mass  $M$  for which the right-hand side is well-defined and finite. We can now, by the same regularization argument as before, compare the entropy dissipations  $D(f) := -\frac{d}{dt}H(f)$  of equation (1.1) and  $D_0(f)$  of equation (3.20). Thanks to Lemma 2.7 we have  $f(t, v) \leq F_{M^*}(v) \leq (\beta(M^*) + 1)^{-1}$  a.e. in  $\mathbb{R}^N$ , and thus

$$D(f) = \int_{\mathbb{R}^N} f(1 - f) |v + \nabla_v h(f)|^2 dv \geq C \int_{\mathbb{R}^N} f |v + \nabla_v h(f)|^2 dv \quad (3.22)$$

where  $C = 1 - (\beta(M^*) + 1)^{-1}$ . Applying the Generalized Logarithmic Sobolev Inequality (3.21) to the solution  $f$  and taking into account the previous estimates, we conclude

$$H(f(t)) - H(F_M) \leq (2C)^{-1}D(f(t)). \quad (3.23)$$

Finally, coming back to the entropy evolution:

$$\frac{d}{dt} [H(f(t)) - H(F_M)] = -D(f(t)) \leq -2C [H(f(t)) - H(F_M)],$$

and the result follows from Gronwall's lemma. The convergence in  $L^1$  is obtained by a Csiszár-Kullback type inequality proven in [5, Corollary 4.3], its proof being valid for any space dimension. It is actually a consequence of a direct application of the Taylor theorem to the relative entropy  $H(f) - H(F_M)$  giving:

$$\|f - F_M\|_1^2 \leq 2M(H(f) - H(F_M)).$$

□

### 3.3 Propagation of Moments and Consequences

There is a large gap between Theorem 3.3 which only provides the  $L^1$ -convergence to the equilibrium and Theorem 3.5 which warrants an exponential decay to zero of the relative entropy for a restrictive class of initial data. This last section is devoted to an intermediate result where we prove the convergence to zero of the relative entropy but without a rate for a larger class of initial data than in Theorem 3.5.

**Lemma 3.6 (Time independent bound for Moments)** *Let  $g_0 \in L^1_{mp}(\mathbb{R}^N)$  with  $m \geq 1$ ,  $p > \max(N, 2)$  such that  $0 \leq g_0 \leq 1$ , and assume further that  $g_0$  is a radially symmetric and non-increasing function, i.e., there is a non-increasing function  $\varphi_0$  such that  $g_0(v) = \varphi_0(|v|)$  for  $v \in \mathbb{R}^N$ . Then, for the unique solution  $g$  of the Cauchy problem (1.1) with initial condition  $g_0$ , the control of moments propagates in time, i.e.,*

$$\lim_{R \rightarrow \infty} \sup_{t \geq 0} \int_{\{|v| \geq R\}} |v|^{mp} g(t, v) dv = 0. \quad (3.24)$$

*Proof.* - We have already seen in Corollary 2.13 the existence and uniqueness of  $g$  and that  $g(t, v) = \varphi(t, |v|)$  for  $t \geq 0$  and  $v \in \mathbb{R}^N$  for some function  $\varphi$  such that  $r \mapsto \varphi(t, r)$  is non-increasing. Furthermore, we have that its moments are given by

$$M := \int_{\mathbb{R}^N} g(t, v) dv = N\omega_N \int_0^\infty r^{N-1} \varphi(t, r) dr \quad (3.25)$$

and

$$\int_{\mathbb{R}^N} |v|^{mp} g(t, v) dv = N\omega_N \int_0^\infty r^{N+mp-1} \varphi(t, r) dr \quad (3.26)$$

for  $t \geq 0$ , where  $\omega_N$  denotes the volume of the unit ball of  $\mathbb{R}^N$ .



Next, since  $|v|^{mp}g_0 \in L^1(\mathbb{R}^N)$ , the map  $v \mapsto |v|^{mp}$  belongs to  $L^1(\mathbb{R}^N; g_0(v) dv)$  and a refined version of de la Vallée-Poussin theorem [9, 20] ensures that there is a non-decreasing, non-negative and convex function  $\psi \in \mathcal{C}^\infty([0, \infty))$  such that  $\psi(0) = 0$ ,  $\psi'$  is concave,

$$\lim_{r \rightarrow \infty} \frac{\psi(r)}{r} = \infty \quad \text{and} \quad \int_{\mathbb{R}^N} \psi(|v|^{mp})g_0(v) dv < \infty. \quad (3.27)$$

Observe that, since  $\psi(0) = 0$  and  $\psi'(0) \geq 0$ , the convexity of  $\psi$  and the concavity of  $\psi'$  ensure that for  $r \geq 0$

$$r\psi''(r) \leq \psi'(r) \quad \text{and} \quad \psi(r) \leq r\psi'(r). \quad (3.28)$$

Then, after integration by parts, it follows from (2.20) that

$$\begin{aligned} \frac{1}{mp} \frac{d}{dt} \int_0^\infty \psi(r^{mp}) r^{N-1} \varphi dr &= - \int_0^\infty r^{mp-1} \psi'(r^{mp}) \left( r^{N-1} \frac{\partial \varphi}{\partial r} + r^N \varphi(1 - \varphi) \right) dr \\ &= I_1 + I_2, \end{aligned} \quad (3.29)$$

where

$$\begin{aligned} I_1 &= \int_0^\infty \varphi [(mp + N - 2)r^{mp+N-3}\psi'(r^{mp}) + mpr^{2mp+N-3}\psi''(r^{mp})] dr \\ I_2 &= - \int_0^\infty r^{N+mp-1}\psi'(r^{mp})\varphi(1 - \varphi) dr. \end{aligned}$$

We now fix  $R > 0$  such that  $\omega_N R^N \geq 4M$  and  $R^2 \geq 4(2mp + N - 2)$ , and note that due to the monotonicity of  $\varphi$  with respect to  $r$  and (3.25)-(3.26) the inequality

$$M \geq N\omega_N \int_0^R r^{N-1} \varphi dr \geq \omega_N R^N \varphi(R) \quad (3.30)$$

holds. Therefore, we first use the monotonicity of  $\psi'$  and  $\varphi$  together with (3.30) to obtain

$$\begin{aligned} I_2 &\leq - \int_R^\infty r^{N+mp-1} \psi'(r^{mp}) \varphi(1 - \varphi) dr \leq (\varphi(R) - 1) \int_R^\infty r^{N+mp-1} \psi'(r^{mp}) \varphi dr \\ &\leq \left( \frac{M}{\omega_N R^N} - 1 \right) \int_R^\infty r^{N+mp-1} \psi'(r^{mp}) \varphi dr \leq -\frac{3}{4} \int_R^\infty r^{N+mp-1} \psi'(r^{mp}) \varphi dr \\ &= \frac{3}{4} \int_0^R r^{N+mp-1} \psi'(r^{mp}) \varphi dr - \frac{3}{4} \int_0^\infty r^{N+mp-1} \psi'(r^{mp}) \varphi dr \\ &\leq \frac{3MR^{mp}\psi'(R^{mp})}{4N\omega_N} - \frac{3}{4} \int_0^\infty r^{N+mp-1} \psi'(r^{mp}) \varphi dr. \end{aligned}$$

On the other hand, from (3.25), (3.26), (3.28), (3.30) and the monotonicity of  $\psi'$

$$\begin{aligned}
I_1 &\leq (N + 2mp - 2) \int_0^\infty r^{N+mp-3} \psi'(r^{mp}) \varphi \, dr \\
&\leq (N + 2mp - 2) \psi'(R^{mp}) R^{mp-2} \int_0^R r^{N-1} \varphi \, dr \\
&\quad + \frac{N + 2mp - 2}{R^2} \int_R^\infty r^{N+mp-1} \psi'(r^{mp}) \varphi \, dr \\
&\leq \frac{(N + 2mp - 2) \psi'(R^{mp}) R^{mp-2} M}{N \omega_N} + \frac{1}{4} \int_R^\infty r^{N+mp-1} \psi'(r^{mp}) \varphi \, dr.
\end{aligned}$$

Inserting these bounds for  $I_1$  and  $I_2$  in (3.29) and using (3.28) we end up with

$$\begin{aligned}
\frac{1}{mp} \frac{d}{dt} \int_0^\infty \psi(r^{mp}) r^{N-1} \varphi \, dr \\
&\leq \frac{\psi'(R^{mp}) M R^{mp-2}}{N \omega_N} \left( \frac{3R^2}{4} + N + 2mp - 2 \right) - \frac{1}{2} \int_0^\infty r^{N+mp-1} \psi'(r^{mp}) \varphi \, dr \\
&\leq \frac{\psi'(R^{mp}) M R^{mp-2}}{N \omega_N} \left( \frac{3R^2}{4} + N + 2mp - 2 \right) - \frac{1}{2} \int_0^\infty r^{N-1} \psi(r^{mp}) \varphi \, dr.
\end{aligned}$$

We then use the Gronwall lemma to conclude that there exists  $C > 0$  depending on  $N, M, m, p, g_0$  and  $\psi$  such that

$$\sup_{t \geq 0} \int \psi(|v|^{mp}) g(t, v) \, dv \leq C$$

from which (3.24) readily follows by (3.27).  $\square$

**Theorem 3.7 (Entropy Convergence)** *Let  $f$  be the solution of the Cauchy problem (1.1) with initial condition  $f_0 \in L_{mp}^1(\mathbb{R}^N)$  such that there exists a radially symmetric and non-increasing function  $g_0 \in L_{mp}^1(\mathbb{R}^N)$  with  $0 \leq f_0 \leq g_0 \leq 1$ . Then  $H(f) \rightarrow H(F_M)$  as  $t \rightarrow \infty$  where  $M = \|f_0\|_1$ .*

*Proof.*- Due to [22, Theorem 3] we know that

$$\begin{aligned}
|H(f(t)) - H(F_M)| &\leq C \int_{\mathbb{R}^N} |v|^2 |f(t, v) - F(v)| \, dv \\
&\leq R^2 \|f(t) - F\|_1 + \sup_{t \geq 0} \int_{|v| \geq R} |v|^2 |f(t) - F| \, dv
\end{aligned}$$

Now, Theorem 3.5 and Lemma 3.6 imply that  $H(f(t)) \rightarrow H(F_M)$  as  $t \rightarrow \infty$ .  $\square$

## A $L_m^p$ -bounds for the Fokker-Planck Operator

Here we follow similar arguments as in [14] to show some bounds for  $\|\partial_\alpha \mathcal{F}f(t)\|_{L_m^p}$  which were useful in the fixed point argument in Section 2.1. We recall the well-known Young inequality: Let  $g_1 \in L^r(\mathbb{R}^N)$ ,  $g_2 \in L^q(\mathbb{R}^N)$  with  $1 \leq p, r, q \leq \infty$  and  $\frac{1}{p} + 1 = \frac{1}{r} + \frac{1}{q}$ , then

$$g_1 * g_2 \in L^p(\mathbb{R}^N) \quad \text{and} \quad \|g_1 * g_2\|_p \leq \|g_1\|_r \|g_2\|_q.$$

**Proposition A.1** *Let  $1 \leq q \leq p \leq \infty$ ,  $m \geq 0$  and  $\alpha \in \mathbb{N}^N$ . Then for  $t > 0$ ,*

$$\|\partial_\alpha \mathcal{F}(t)[f]\|_{L_m^p} \leq \frac{C e^{\left(\frac{N}{p} + |\alpha|\right)t}}{\nu(t)^{\frac{N}{2}\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{|\alpha|}{2}}} \|f\|_{L_m^q}. \quad (\text{A.1})$$

*Proof.* - For all  $\alpha \in \mathbb{N}^N$ , we have

$$\begin{aligned} \partial_\alpha \mathcal{F}(t, v)[f] &= \partial^\alpha \int_{\mathbb{R}^N} \left( \frac{e^{tN}}{(2\pi(e^{2t} - 1))^{\frac{N}{2}}} e^{-\frac{|e^t v - w|^2}{2(e^{2t} - 1)}} \right) f(w) \, dw \\ &= \partial^\alpha \int_{\mathbb{R}^N} \left( \frac{e^{2Nt}}{(2\pi(e^{2t} - 1))^{\frac{N}{2}}} e^{-\frac{|e^t(v-w)|^2}{2(e^{2t} - 1)}} \right) f(e^t w) \, dw \\ &= \frac{e^{t(2N + |\alpha|)}}{\nu(t)^{\frac{N + |\alpha|}{2}}} \int_{\mathbb{R}^N} \phi_\alpha \left( \frac{v - w}{e^{-t}\nu(t)^{1/2}} \right) f(e^t w) \, dw \end{aligned} \quad (\text{A.2})$$

where

$$\phi_\alpha(\chi) = \partial_\chi^\alpha (\phi_0)(\chi) = \mathcal{P}_{|\alpha|}(\chi) \phi_0(\chi),$$

$\mathcal{P}_{|\alpha|}(\chi)$  being a polynomial of degree  $|\alpha|$  which we can recursively define by

$$\mathcal{P}_0(\chi) := 1, \quad \mathcal{P}_{|\alpha|}(\chi) := \mathcal{P}'_{|\alpha|-1}(\chi) - \chi \mathcal{P}_{|\alpha|-1}(\chi) \quad \text{and} \quad \phi_0(\chi) := (2\pi)^{-\frac{N}{2}} e^{-\frac{|\chi|^2}{2}}.$$

Since  $1 + |v|^m \leq C(1 + |v - w|^m)(1 + |w|^m)$ , we deduce

$$\begin{aligned} (1 + |v|^m) |(\partial_\alpha \mathcal{F} * f)(t)| &\leq \\ &\leq C \frac{e^{t(2N + |\alpha|)}}{\nu(t)^{\frac{N + |\alpha|}{2}}} \int_{\mathbb{R}^N} (1 + |v - w|^m) \left| \phi_\alpha \left( \frac{v - w}{e^{-t}\nu(t)^{1/2}} \right) \right| (1 + |w|^m) |f(e^t w)| \, dw. \end{aligned} \quad (\text{A.3})$$

Then, we can write

$$\int_{\mathbb{R}^N} (1 + |v - w|^m)^r \left| \phi_\alpha \left( \frac{v - w}{e^{-t}\nu(t)^{1/2}} \right) \right|^r \, dw = C(I + II)$$

with

$$I = \int \mathcal{P}_{|\alpha|}^r \left( \frac{v-w}{e^{-t}\nu(t)^{1/2}} \right) \phi_0 \left( \frac{v-w}{e^{-t}\nu(t)^{1/2}} \right)^r dw = \frac{\nu(t)^{N/2}}{e^{Nt}} \int \mathcal{P}_{|\alpha|}^r(\chi) \phi_0(\chi)^r = C_1 \frac{\nu(t)^{N/2}}{e^{Nt}}$$

and

$$\begin{aligned} II &= \int |v-w|^{mr} \mathcal{P}_{|\alpha|}^r \left( \frac{v-w}{e^{-t}\nu(t)^{1/2}} \right) \phi_0 \left( \frac{v-w}{e^{-t}\nu(t)^{1/2}} \right)^r dw \\ &= \frac{\nu(t)^{(N+mr)/2}}{e^{(N+mr)t}} \int |\chi|^{mr} \mathcal{P}_{|\alpha|}^r(\chi) \phi_0(\chi)^r = C_2 \frac{\nu(t)^{(N+mr)/2}}{e^{(N+mr)t}}. \end{aligned}$$

whence

$$\frac{e^{Nt}}{\nu(t)^{N/2}} \int_{\mathbb{R}^N} (1 + |v-w|^m)^r \left| \phi_\alpha \left( \frac{v-w}{e^{-t}\nu(t)^{1/2}} \right) \right|^r dw \leq C. \quad (\text{A.4})$$

On the other hand, we get

$$\begin{aligned} \left\| (1 + |w|^m) \left| f(e^t w) \right| \right\|_p &= \left( \int (1 + |w|^m)^p \left| f(e^t w) \right|^p dw \right)^{\frac{1}{p}} \\ &= \left( \int e^{-Nt} (1 + |e^{-t}\chi|^m)^p \left| f(\chi) \right|^p d\chi \right)^{\frac{1}{p}} \\ &\leq e^{-\frac{Nt}{p}} \left( \int (1 + |\chi|^m)^p \left| f(\chi) \right|^p d\chi \right)^{\frac{1}{p}}. \end{aligned} \quad (\text{A.5})$$

Putting (A.4) together with (A.5), we can use Young's inequality in (A.3) as before, since  $1 \leq q \leq p$  with  $r$  given by  $\frac{1}{p} + 1 = \frac{1}{r} + \frac{1}{q}$  to get the desired bound.  $\square$

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## References

- [1] M. Burger, M. di Francesco, Y. Dolak, *The Keller-Segel model for chemotaxis with prevention of overcrowding: linear vs. nonlinear diffusion*, SIAM J. Math. Anal. **38** (2006), 1288–1315.
- [2] J. A. Carrillo, A. Jüngel, P. Markowich, G. Toscani, A. Unterreiter, *Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities*, Monatsh. Math. **133** (2001), 1–82.

- [3] J. A. Carrillo, R. J. McCann and C. Villani, *Contractions in the 2-Wasserstein length space and thermalization of granular media*, Arch. Rational Mech. Anal. **179** (2006), 217–263.
- [4] J. A. Carrillo, G. Toscani, *Exponential convergence toward equilibrium for homogeneous Fokker-Planck-type equations*, Math. Methods Appl. Sci. **21** (1998), 1269–1286.
- [5] J. A. Carrillo, J. Rosado, F. Salvarani, *1D nonlinear Fokker-Planck equations for fermions and bosons*, Applied Mathematics Letters **21** (2008), 148–154.
- [6] P.-H. Chavanis, *Generalized thermodynamics and Fokker-Planck equations. Applications to stellar dynamics and two-dimensional turbulence*, Phys. Rev. E **68** (2003), 036108.
- [7] P.-H. Chavanis, *Generalized thermodynamics and kinetic equations: Boltzmann, Landau, Kramers and Smoluchowski*, Phys. A **332** (2004), 89–122.
- [8] P.-H. Chavanis, *Generalized Fokker-Planck equations and effective thermodynamics*, Phys. A **340** (2004), 57–65.
- [9] C. Dellacherie, P. A. Meyer, *Probabilités et Potentiel*, Hermann, Paris, 1975.
- [10] J. Dolbeault, *Kinetic models and quantum effects: a modified Boltzmann equation for Fermi-Dirac particles*, Arch. Ration. Mech. Anal. **127** (1994), 101–131.
- [11] M. Escobedo, E. Zuazua, *Large time behavior for convection-diffusion equations in  $\mathbb{R}^N$* , J. Funct. Anal. **100** (1991), 119–161.
- [12] T. D. Frank, *Classical Langevin equations for the free electron gas and blackbody radiation*, J. Phys. A **37** (2004), 3561–3567.
- [13] T. D. Frank, *Nonlinear Fokker-Planck Equations*, Springer Series in Synergetics, Springer, 2005.
- [14] T. Gallay, C. E. Wayne, *Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on  $\mathbb{R}^2$* , Arch. Ration. Mech. Anal. **163** (2002), 209–258.
- [15] G. Kaniadakis, *Generalized Boltzmann equation describing the dynamics of bosons and fermions*, Phys. Lett. A **203** (1995), 229–234.
- [16] G. Kaniadakis, *H-theorem and generalized entropies within the framework of nonlinear kinetics*, Phys. Lett. A **288** (2001), 283–291.
- [17] G. Kaniadakis, G. Lapenta, P. Quarati, *Stochastic evolution of systems of particles obeying an exclusion principle*, Phys. A **225** (1996), 323–335.

- [18] G. Kaniadakis, P. Quarati, *Kinetic equation for classical particles obeying an exclusion principle*, Phys. Rev. E **48** (1993), 4263–4270.
- [19] O. A. Ladyženskaja, V. A. Solonnikov, N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monogr. **23**, Amer. Math. Soc., Providence, RI, 1968.
- [20] Lê Châu-Hoàn, *Etude de la classe des opérateurs  $m$ -accrétifs de  $L^1(\Omega)$  et accrétifs dans  $L^\infty(\Omega)$*  Thèse de 3<sup>ème</sup> cycle, Université de Paris VI, 1977.
- [21] P. G. Lemarié-Rieusset, *Recent Developments in the Navier-Stokes Problem*, Chapman & Hall-CRC, Boca-Raton, 2002.
- [22] X. Lu, B. Wennberg, *On stability and strong convergence for the spatially homogeneous Boltzmann equation for Fermi-Dirac particles*, Arch. Ration. Mech. Anal. **168** (2003), 1–34.
- [23] C. Mouhot, L. Neumann, *Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus*, Nonlinearity **19** (2006), 969–998.
- [24] L. Neumann, C. Schmeiser, *Convergence to global equilibrium for a kinetic model for fermions*, SIAM J. Math. Anal. **36** (2005), 1652–1663.
- [25] L. Neumann, C. Sparber, *Stability of steady states in kinetic Fokker-Planck equations for bosons and fermions*, Comm. Math. Sci. **5** (2007), 765–777.
- [26] A. Rossani, G. Kaniadakis, *A generalized quasi-classical Boltzmann equation*, Phys. A **277** (2000), 349–358.
- [27] G. Toscani, *Remarks on entropy and equilibrium states*, Appl. Math. Lett. **12** (1999), 19–25.
- [28] J. L. Vázquez, *The Porous Medium Equation: Mathematical Theory*, Oxford Science Publications, 2007.