# Equidistribution, counting and arithmetic applications 

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Abstract This short note is an announcement of the results of [PP1] and [PP2]. ${ }^{1}$
Let $M$ be a finite volume hyperbolic manifold of dimension $n$ at least 2 . Let $T^{1} M \rightarrow M$ be the unit tangent bundle of $M$, where $T^{1} M$ is endowed with its usual Riemannian metric, whose induced measure is the Liouville measure $\operatorname{vol}_{T^{1} M}$. Let $\left(g^{t}\right)_{t \in \mathbb{R}}$ be the geodesic flow of $M$. Let $C_{0}$ be a finite volume immersed totally geodesic submanifold of $M$ of dimension $k$ with $0<k<n$, and let $\nu^{1} C_{0}$ be its unit normal bundle, so that $g^{t} \nu^{1} C_{0}$ is, for every $t \geq 0$, an immersed submanifold of $T^{1} M$.

Theorem 1 The induced Riemannian measure of $g^{t} \nu^{1} C_{0}$ equidistributes to the Liouville measure as $t \rightarrow+\infty$ :

$$
\operatorname{vol}_{g^{t} \nu^{1} C_{0}} /\left\|\operatorname{vol}_{g^{t} \nu^{1} C_{0}}\right\| \xrightarrow{*} \operatorname{vol}_{T^{1} M} /\left\|\operatorname{vol}_{T^{1} M}\right\| .
$$

This theorem can be deduced from [EM, Theo. 1.2]. Our (short and direct) proof also uses, as in Margulis' equidistribution result for horospheres, the mixing property of the geodesic flow of $M$.

Let $\mathcal{H}_{\infty}$ be a small enough Margulis neighbourhood of an end of $M$, that is a connected component of the set of points of $M$ at which the injectivity radius of $M$ is at most $\epsilon_{0}$, for some $\epsilon_{0}>0$ small enough. We use the above equidistribution theorem, and the fact that the submanifold $g^{t} \nu^{1} C_{0}$ is locally close to an unstable leaf in $T^{1} M$ of the geodesic flow of $M$, to prove the following counting result.

Theorem 2 The number of common perpendicular locally geodesic arcs between $\partial \mathcal{H}_{\infty}$ and $C_{0}$ with length at most $t$ is equivalent, as tends to $+\infty$, to

$$
\frac{\operatorname{Vol}\left(\mathbb{S}_{n-k-1}\right) \operatorname{Vol}\left(\mathcal{H}_{\infty}\right) \operatorname{Vol}\left(C_{0}\right)}{\operatorname{Vol}\left(\mathbb{S}_{n-1}\right) \operatorname{Vol}(M)} e^{(n-1) t}
$$

We refer to [PP1] for the proofs of the above theorems, as well as for references to other works and many geometric complements, and we now give a sample of their arithmetic applications, extracted from [PP1] except for the last corollary.

Counting quadratic irrationals. Let $K$ be a number field and let $\mathcal{O}_{K}$ be its ring of integers. Endow the set of quadratic irrationals over $K$ with the action by homographies of $\mathrm{PSL}_{2}\left(\mathcal{O}_{K}\right)$, and note that it is not transitive. We denote by $\alpha^{\sigma}$ the Galois conjugate over $K$ of a quadratic irrational $\alpha$ over $K$. There are many works (see for instance [Bug]) on the approximation of real or complex numbers by algebraic numbers, and approximating them

[^0]by elements in orbits of algebraic numbers under natural group actions for appropriate complexities seems to be interesting.

Starting with $K=\mathbb{Q}$, our first result is a counting result in orbits of real quadratic irrationals over $\mathbb{Q}$ for a natural complexity (see [PP1] for a more algebraic expression in terms of discriminants).

Corollary 1 Let $\alpha_{0} \in \mathbb{R}$ be a quadratic irrational over $\mathbb{Q}$, and let $G$ be a finite index subgroup of $\mathrm{PSL}_{2}(\mathbb{Z})$. Then as $s$ tends to $+\infty$,

$$
\operatorname{Card}\left\{\alpha \in G \cdot\left\{\alpha_{0}, \alpha_{0}^{\sigma}\right\} \bmod \mathbb{Z}: \frac{1}{\left|\alpha-\alpha^{\sigma}\right|} \leq s\right\} \sim \frac{24 q_{G} \operatorname{argcosh} \frac{\left|\operatorname{tr} \gamma_{0}\right|}{2}}{\pi^{2}\left[\operatorname{PSL}_{2}(\mathbb{Z}): G\right] n_{0}} s,
$$

where $q_{G}$ is the smallest positive integer $q$ such that $z \mapsto z+q$ belongs to $G, \gamma_{0} \in G-\{1\}$ fixes $\alpha_{0}$ and $n_{0}$ is the index of $\gamma_{0}{ }^{\mathbb{Z}}$ in the stabilizer of $\left\{\alpha_{0}, \alpha_{0}^{\sigma}\right\}$ in $G$ (and note that $q_{G}, \gamma_{0}, n_{0}$ do exist).

For instance, if $\alpha_{0}$ is the Golden ratio $\phi=\frac{1+\sqrt{5}}{2}$ (which is reciprocal in Sarnak's terminology) and $G=\operatorname{PSL}_{2}(\mathbb{Z})$, we get $\operatorname{Card}\left\{\alpha \in G \cdot \phi \quad \bmod \mathbb{Z}: \frac{1}{\left|\alpha-\alpha^{\sigma}\right|} \leq s\right\} \sim$ $\frac{24 \log \phi}{\pi^{2}}$ s. With $\mathbb{H}_{\mathbb{R}}^{2}$ the upper halfplane model of the real hyperbolic plane, the proof applies Theorem 2 to $M$ the orbifold $G \backslash \mathbb{H}_{\mathbb{R}}^{2}$, to $C_{0}$ the image in $M$ of the geodesic line in $\mathbb{H}_{\mathbb{R}}^{2}$ with endpoints $\alpha_{0}$ and $\alpha_{0}^{\sigma}$, and to $\mathcal{H}_{\infty}$ the image in $M$ of the set of points in $\mathbb{H}_{\mathbb{R}}^{2}$ with Euclidean height at least 1 . The trick is that if $a$ and $b$ are close enough distinct real numbers, then the hyperbolic length of the perpendicular arc between the horizontal line at Euclidean height 1 and the geodesic line with endpoints $a$ and $b$ is exactly $-\log |b-a|$.

Assume $K$ is imaginary quadratic, with discriminant $D_{K}$. We proved a general statement analogous to the previous corollary, but we only give here a particular case for $\phi$.

Corollary 2 Let $\mathfrak{a}$ be a non zero ideal in $\mathcal{O}_{K}$ and $\Gamma_{0}(\mathfrak{a})=\left\{ \pm\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \operatorname{PSL}_{2}\left(\mathcal{O}_{K}\right)\right.$ : $c \in \mathfrak{a}\}$. Assume for simplicity that $D_{K} \neq-4$ and $\phi^{\sigma} \notin \Gamma_{0}(\mathfrak{a}) \cdot \phi$. Then as $s$ tends to $+\infty$, the cardinality of $\left\{\alpha \in \Gamma_{0}(\mathfrak{a}) \cdot\left\{\phi, \phi^{\sigma}\right\} \bmod \mathcal{O}_{K}: \frac{1}{\left|\alpha-\alpha^{\sigma}\right|} \leq s\right\}$ is equivalent to

$$
\frac{8 \pi^{2} k_{\mathfrak{a}} \log \phi}{\left|D_{K}\right| \zeta_{K}(2) N(\mathfrak{a}) \prod_{\mathfrak{p} \text { prime, } \mathfrak{p} \mid \mathfrak{a}}\left(1+\frac{1}{N(\mathfrak{p})}\right)} s^{2},
$$

with $k_{\mathfrak{a}}$ the smallest $k \in \mathbb{N}-\{0\}$ such that the $2 k$-th term of the standard Fibonacci sequence belongs to $\mathfrak{a}$ (and note that $k_{\mathfrak{a}}$ does always exist, contrarily to the odd case).

Counting representations of integers by binary forms. Recall that a binary quadratic form $Q(x, y)=a x^{2}+b x y+c y^{2}$ is primitive integral if $a, b, c \in \mathbb{Z}$ are relatively prime, and indefinite non product if its discriminant $D=b^{2}-4 a c$ is positive and not a square. Using the well known correspondence between pairs of Galois conjugated quadratic irrationals over $\mathbb{Q}$ and the set of such $Q$ 's up to sign, we prove the following counting result for the number of values of a fixed such $Q$ on couples of relatively prime integers satisfying some congruence relations. Let $(t, u)$ be the minimal solution to the Pell-Fermat equation $t^{2}-D u^{2}=4$ and $\epsilon=\frac{t+u \sqrt{D}}{2}$ the corresponding fundamental unit.

Corollary 3 Let $Q$ be as above, and let $n$ be an integer at least 3. Then the number of couples $(x, y) \in \mathbb{Z}^{2}$, relatively prime, with $x \equiv 1 \bmod n$ and $y \equiv 0 \bmod n$, such that $|Q(x, y)| \leq s$, modulo the linear action of $\mathrm{SL}_{2}(\mathbb{Z})$, is equivalent, as s tends to $+\infty$, to

$$
\frac{24 \log \epsilon}{\pi^{2} n^{2} \sqrt{D}} \prod_{p \text { prime }, p \mid n}\left(1-\frac{1}{p^{2}}\right)^{-1} s
$$

The final result, for a quadratic imaginary number field $K$, is proved in [PP2], along with extensions to representations satisfying congruence properties.

Corollary 4 Let $f:(u, v) \mapsto a|u|^{2}+2 \operatorname{Re}(b u \bar{v})+c|v|^{2}$ be a binary Hermitian form, indefinite (that is $\Delta=|b|^{2}-a c>0$ ) and integral over $K$ (that is $a, c \in \mathbb{Z}, b \in \mathcal{O}_{K}$ ). Let $\mathrm{SU}_{f}\left(\mathcal{O}_{K}\right)=\left\{g \in \mathrm{SL}_{2}\left(\mathcal{O}_{K}\right): f \circ g=g\right\}$ be the group of automorphs of $f$. Then the number of orbits under $\mathrm{SU}_{f}\left(\mathcal{O}_{K}\right)$ of couples $(u, v)$ of relatively prime elements of $\mathcal{O}_{K}$ such that $|f(u, v)| \leq s$ is equivalent, as $s$ tends to $+\infty$, to

$$
\frac{\pi \operatorname{Covol}\left(\mathrm{SU}_{f}\left(\mathcal{O}_{K}\right)\right)}{2\left|D_{K}\right| \zeta_{K}(2) \Delta} s^{2}
$$

With $\mathbb{H}_{\mathbb{R}}^{3}$ the upper halfspace model of the real hyperbolic 3 -space, the proof applies Theorem 2 to $M$ the orbifold $\mathrm{PSL}_{2}\left(\mathcal{O}_{K}\right) \backslash \mathbb{H}_{\mathbb{R}}^{3}$, to $C_{0}$ the image in $M$ of the unique hyperbolic plane $P(f)$ in $\mathbb{H}_{\mathbb{R}}^{3}$ preserved by $\operatorname{PSU}_{f}\left(\mathcal{O}_{K}\right)$, and to $\mathcal{H}_{\infty}$ the image in $M$ of the set of points in $\mathbb{H}_{\mathbb{R}}^{3}$ with Euclidean height at least 1. The trick is that, for every $\gamma \in \operatorname{PSL}_{2}\left(\mathcal{O}_{K}\right)$, the hyperbolic plane $P(f \circ \gamma)$ is an Euclidean hemisphere whose diameter is $\frac{\sqrt{\Delta}}{f \circ \gamma(1,0)}$, hence whose perpendicular arc to the horizontal plane at Euclidean height 1 has (signed) hyperbolic length $\log \frac{f \circ \gamma(1,0)}{\sqrt{\Delta}}$, and that $\mathrm{SL}_{2}\left(\mathcal{O}_{K}\right)$ acts transitively on the couples of relatively prime elements of $\mathcal{O}_{K}$.

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