Equidistribution, counting and arithmetic applications

Jouni Parkkonen Frédéric Paulin

Abstract This short note is an announcement of the results of [PP1] and [PP2].¹

Let M be a finite volume hyperbolic manifold of dimension n at least 2. Let $T^1M \to M$ be the unit tangent bundle of M, where T^1M is endowed with its usual Riemannian metric, whose induced measure is the Liouville measure $\operatorname{vol}_{T^1M}$. Let $(g^t)_{t\in\mathbb{R}}$ be the geodesic flow of M. Let C_0 be a finite volume immersed totally geodesic submanifold of M of dimension k with 0 < k < n, and let $\nu^1 C_0$ be its unit normal bundle, so that $g^t \nu^1 C_0$ is, for every $t \ge 0$, an immersed submanifold of T^1M .

Theorem 1 The induced Riemannian measure of $g^t \nu^1 C_0$ equidistributes to the Liouville measure as $t \to +\infty$:

$$\operatorname{vol}_{g^t\nu^1C_0}/\|\operatorname{vol}_{g^t\nu^1C_0}\| \stackrel{*}{\rightharpoonup} \operatorname{vol}_{T^1M}/\|\operatorname{vol}_{T^1M}\|.$$

This theorem can be deduced from [EM, Theo. 1.2]. Our (short and direct) proof also uses, as in Margulis' equidistribution result for horospheres, the mixing property of the geodesic flow of M.

Let \mathcal{H}_{∞} be a small enough Margulis neighbourhood of an end of M, that is a connected component of the set of points of M at which the injectivity radius of M is at most ϵ_0 , for some $\epsilon_0 > 0$ small enough. We use the above equidistribution theorem, and the fact that the submanifold $g^t \nu^1 C_0$ is locally close to an unstable leaf in $T^1 M$ of the geodesic flow of M, to prove the following counting result.

Theorem 2 The number of common perpendicular locally geodesic arcs between $\partial \mathcal{H}_{\infty}$ and C_0 with length at most t is equivalent, as t tends to $+\infty$, to

$$\frac{\operatorname{Vol}(\mathbb{S}_{n-k-1})\operatorname{Vol}(\mathcal{H}_{\infty})\operatorname{Vol}(C_0)}{\operatorname{Vol}(\mathbb{S}_{n-1})\operatorname{Vol}(M)} e^{(n-1)t} .$$

We refer to [PP1] for the proofs of the above theorems, as well as for references to other works and many geometric complements, and we now give a sample of their arithmetic applications, extracted from [PP1] except for the last corollary.

Counting quadratic irrationals. Let K be a number field and let \mathcal{O}_K be its ring of integers. Endow the set of quadratic irrationals over K with the action by homographies of $PSL_2(\mathcal{O}_K)$, and note that it is not transitive. We denote by α^{σ} the Galois conjugate over K of a quadratic irrational α over K. There are many works (see for instance [Bug]) on the approximation of real or complex numbers by algebraic numbers, and approximating them

¹Keywords: Equidistribution, counting, quadratic irrational, hyperbolic manifold, binary quadratic form, perpendicular geodesic. AMS codes: 37A45, 11R11, 53A35, 22F30, 20H10, 11H06, 53C40, 11E16

by elements in orbits of algebraic numbers under natural group actions for appropriate complexities seems to be interesting.

Starting with $K = \mathbb{Q}$, our first result is a counting result in orbits of real quadratic irrationals over \mathbb{Q} for a natural complexity (see [PP1] for a more algebraic expression in terms of discriminants).

Corollary 1 Let $\alpha_0 \in \mathbb{R}$ be a quadratic irrational over \mathbb{Q} , and let G be a finite index subgroup of $PSL_2(\mathbb{Z})$. Then as s tends to $+\infty$,

$$\operatorname{Card}\{\alpha \in G \cdot \{\alpha_0, \alpha_0^{\sigma}\} \mod \mathbb{Z} : \frac{1}{|\alpha - \alpha^{\sigma}|} \le s\} \sim \frac{24 \ q_G \ \operatorname{argcosh} \frac{|\operatorname{tr} \gamma_0|}{2}}{\pi^2 \left[\operatorname{PSL}_2(\mathbb{Z}) : G\right] \ n_0} \ s \ ,$$

where q_G is the smallest positive integer q such that $z \mapsto z + q$ belongs to G, $\gamma_0 \in G - \{1\}$ fixes α_0 and n_0 is the index of $\gamma_0^{\mathbb{Z}}$ in the stabilizer of $\{\alpha_0, \alpha_0^{\sigma}\}$ in G (and note that q_G, γ_0, n_0 do exist).

For instance, if α_0 is the Golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ (which is reciprocal in Sarnak's terminology) and $G = \text{PSL}_2(\mathbb{Z})$, we get $\text{Card}\{\alpha \in G \cdot \phi \mod \mathbb{Z} : \frac{1}{|\alpha - \alpha^{\sigma}|} \leq s\} \sim \frac{24 \log \phi}{\pi^2} s$. With $\mathbb{H}^2_{\mathbb{R}}$ the upper halfplane model of the real hyperbolic plane, the proof applies Theorem 2 to M the orbifold $G \setminus \mathbb{H}^2_{\mathbb{R}}$, to C_0 the image in M of the geodesic line in $\mathbb{H}^2_{\mathbb{R}}$ with endpoints α_0 and α^{σ}_0 , and to \mathcal{H}_{∞} the image in M of the set of points in $\mathbb{H}^2_{\mathbb{R}}$ with Euclidean height at least 1. The trick is that if a and b are close enough distinct real numbers, then the hyperbolic length of the perpendicular arc between the horizontal line at Euclidean height 1 and the geodesic line with endpoints a and b is exactly $-\log |b-a|$.

Assume K is imaginary quadratic, with discriminant D_K . We proved a general statement analogous to the previous corollary, but we only give here a particular case for ϕ .

Corollary 2 Let \mathfrak{a} be a non zero ideal in \mathcal{O}_K and $\Gamma_0(\mathfrak{a}) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_K) : c \in \mathfrak{a} \right\}$. Assume for simplicity that $D_K \neq -4$ and $\phi^\sigma \notin \Gamma_0(\mathfrak{a}) \cdot \phi$. Then as s tends to $+\infty$, the cardinality of $\{\alpha \in \Gamma_0(\mathfrak{a}) \cdot \{\phi, \phi^\sigma\} \mod \mathcal{O}_K : \frac{1}{|\alpha - \alpha^\sigma|} \leq s\}$ is equivalent to

$$\frac{8\pi^2 k_{\mathfrak{a}} \log \phi}{|D_K| \zeta_K(2) N(\mathfrak{a}) \prod_{\mathfrak{p \text{ prime}}, \mathfrak{p} \mid \mathfrak{a}} \left(1 + \frac{1}{N(\mathfrak{p})}\right)} s^2,$$

with $k_{\mathfrak{a}}$ the smallest $k \in \mathbb{N} - \{0\}$ such that the 2k-th term of the standard Fibonacci sequence belongs to \mathfrak{a} (and note that $k_{\mathfrak{a}}$ does always exist, contrarily to the odd case).

Counting representations of integers by binary forms. Recall that a binary quadratic form $Q(x,y) = ax^2 + bxy + cy^2$ is primitive integral if $a, b, c \in \mathbb{Z}$ are relatively prime, and indefinite non product if its discriminant $D = b^2 - 4ac$ is positive and not a square. Using the well known correspondence between pairs of Galois conjugated quadratic irrationals over \mathbb{Q} and the set of such Q's up to sign, we prove the following counting result for the number of values of a fixed such Q on couples of relatively prime integers satisfying some congruence relations. Let (t, u) be the minimal solution to the Pell-Fermat equation $t^2 - Du^2 = 4$ and $\epsilon = \frac{t+u\sqrt{D}}{2}$ the corresponding fundamental unit.

Corollary 3 Let Q be as above, and let n be an integer at least 3. Then the number of couples $(x, y) \in \mathbb{Z}^2$, relatively prime, with $x \equiv 1 \mod n$ and $y \equiv 0 \mod n$, such that $|Q(x, y)| \leq s$, modulo the linear action of $SL_2(\mathbb{Z})$, is equivalent, as s tends to $+\infty$, to

$$\frac{24 \log \epsilon}{\pi^2 n^2 \sqrt{D}} \prod_{p \text{ prime, } p|n} \left(1 - \frac{1}{p^2}\right)^{-1} s.$$

The final result, for a quadratic imaginary number field K, is proved in [PP2], along with extensions to representations satisfying congruence properties.

Corollary 4 Let $f : (u, v) \mapsto a |u|^2 + 2 \operatorname{Re}(b \, u \, \overline{v}) + c |v|^2$ be a binary Hermitian form, indefinite (that is $\Delta = |b|^2 - ac > 0$) and integral over K (that is $a, c \in \mathbb{Z}, b \in \mathcal{O}_K$). Let $\operatorname{SU}_f(\mathcal{O}_K) = \{g \in \operatorname{SL}_2(\mathcal{O}_K) : f \circ g = g\}$ be the group of automorphs of f. Then the number of orbits under $\operatorname{SU}_f(\mathcal{O}_K)$ of couples (u, v) of relatively prime elements of \mathcal{O}_K such that $|f(u, v)| \leq s$ is equivalent, as s tends to $+\infty$, to

$$\frac{\pi \operatorname{Covol}(\operatorname{SU}_f(\mathcal{O}_K))}{2 |D_K| \zeta_K(2) \Delta} s^2$$

With $\mathbb{H}^3_{\mathbb{R}}$ the upper halfspace model of the real hyperbolic 3-space, the proof applies Theorem 2 to M the orbifold $\mathrm{PSL}_2(\mathcal{O}_K) \setminus \mathbb{H}^3_{\mathbb{R}}$, to C_0 the image in M of the unique hyperbolic plane P(f) in $\mathbb{H}^3_{\mathbb{R}}$ preserved by $\mathrm{PSU}_f(\mathcal{O}_K)$, and to \mathcal{H}_∞ the image in M of the set of points in $\mathbb{H}^3_{\mathbb{R}}$ with Euclidean height at least 1. The trick is that, for every $\gamma \in \mathrm{PSL}_2(\mathcal{O}_K)$, the hyperbolic plane $P(f \circ \gamma)$ is an Euclidean hemisphere whose diameter is $\frac{\sqrt{\Delta}}{f \circ \gamma(1,0)}$, hence whose perpendicular arc to the horizontal plane at Euclidean height 1 has (signed) hyperbolic length log $\frac{f \circ \gamma(1,0)}{\sqrt{\Delta}}$, and that $\mathrm{SL}_2(\mathcal{O}_K)$ acts transitively on the couples of relatively prime elements of \mathcal{O}_K .

References

- [Bug] Y. Bugeaud, Approximation by algebraic numbers, Camb. Tracts Math. 160, Cambridge Univ. Press, 2004.
- [EM] A. Eskin, C. McMullen, Mixing, counting, and equidistribution in Lie groups, Duke Math. J. 71 (1993) 181–209.
- [PP1] J. Parkkonen and F. Paulin. Équidistribution, comptage et approximation par irrationnels quadratiques, Preprint [arXiv:1004.0454].
- [PP2] J. Parkkonen, F. Paulin, On the representations of integers by indefinite binary Hermitian forms, preprint 2010 [arXiv1004.3211].

Department of Mathematics and Statistics, P.O. Box 35 40014 University of Jyväskylä, FINLAND *e-mail: parkkone@maths.jyu.fi*

DMA, UMR 8553 CNRS		Département de mathématique, Bât. 425
Ecole Normale Supérieure, 45 rue d'Ulm	and	Université Paris-Sud 11
75230 PARIS Cedex 05, FRANCE		91405 ORSAY Cedex, FRANCE
e-mail: Frederic.Paulin@ens.fr		$e\text{-}mail:\ frederic.paulin@math.u\text{-}psud.fr$