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# Equidistribution, counting and arithmetic applications

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**Abstract** This short note is an announcement of the results of [PP1] and [PP2].<sup>1</sup>

Let  $M$  be a finite volume hyperbolic manifold of dimension  $n$  at least 2. Let  $T^1M \rightarrow M$  be the unit tangent bundle of  $M$ , where  $T^1M$  is endowed with its usual Riemannian metric, whose induced measure is the Liouville measure  $\text{vol}_{T^1M}$ . Let  $(g^t)_{t \in \mathbb{R}}$  be the geodesic flow of  $M$ . Let  $C_0$  be a finite volume immersed totally geodesic submanifold of  $M$  of dimension  $k$  with  $0 < k < n$ , and let  $\nu^1 C_0$  be its unit normal bundle, so that  $g^t \nu^1 C_0$  is, for every  $t \geq 0$ , an immersed submanifold of  $T^1M$ .

**Theorem 1** *The induced Riemannian measure of  $g^t \nu^1 C_0$  equidistributes to the Liouville measure as  $t \rightarrow +\infty$ :*

$$\text{vol}_{g^t \nu^1 C_0} / \|\text{vol}_{g^t \nu^1 C_0}\| \xrightarrow{*} \text{vol}_{T^1M} / \|\text{vol}_{T^1M}\| .$$

This theorem can be deduced from [EM, Theo. 1.2]. Our (short and direct) proof also uses, as in Margulis' equidistribution result for horospheres, the mixing property of the geodesic flow of  $M$ .

Let  $\mathcal{H}_\infty$  be a small enough Margulis neighbourhood of an end of  $M$ , that is a connected component of the set of points of  $M$  at which the injectivity radius of  $M$  is at most  $\epsilon_0$ , for some  $\epsilon_0 > 0$  small enough. We use the above equidistribution theorem, and the fact that the submanifold  $g^t \nu^1 C_0$  is locally close to an unstable leaf in  $T^1M$  of the geodesic flow of  $M$ , to prove the following counting result.

**Theorem 2** *The number of common perpendicular locally geodesic arcs between  $\partial \mathcal{H}_\infty$  and  $C_0$  with length at most  $t$  is equivalent, as  $t$  tends to  $+\infty$ , to*

$$\frac{\text{Vol}(\mathbb{S}_{n-k-1}) \text{Vol}(\mathcal{H}_\infty) \text{Vol}(C_0)}{\text{Vol}(\mathbb{S}_{n-1}) \text{Vol}(M)} e^{(n-1)t} .$$

We refer to [PP1] for the proofs of the above theorems, as well as for references to other works and many geometric complements, and we now give a sample of their arithmetic applications, extracted from [PP1] except for the last corollary.

**Counting quadratic irrationals.** Let  $K$  be a number field and let  $\mathcal{O}_K$  be its ring of integers. Endow the set of quadratic irrationals over  $K$  with the action by homographies of  $\text{PSL}_2(\mathcal{O}_K)$ , and note that it is not transitive. We denote by  $\alpha^\sigma$  the Galois conjugate over  $K$  of a quadratic irrational  $\alpha$  over  $K$ . There are many works (see for instance [Bug]) on the approximation of real or complex numbers by algebraic numbers, and approximating them

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<sup>1</sup>**Keywords:** Equidistribution, counting, quadratic irrational, hyperbolic manifold, binary quadratic form, perpendicular geodesic. **AMS codes:** 37A45, 11R11, 53A35, 22F30, 20H10, 11H06, 53C40, 11E16

by elements in orbits of algebraic numbers under natural group actions for appropriate complexities seems to be interesting.

Starting with  $K = \mathbb{Q}$ , our first result is a counting result in orbits of real quadratic irrationals over  $\mathbb{Q}$  for a natural complexity (see [PP1] for a more algebraic expression in terms of discriminants).

**Corollary 1** *Let  $\alpha_0 \in \mathbb{R}$  be a quadratic irrational over  $\mathbb{Q}$ , and let  $G$  be a finite index subgroup of  $\mathrm{PSL}_2(\mathbb{Z})$ . Then as  $s$  tends to  $+\infty$ ,*

$$\mathrm{Card}\{\alpha \in G \cdot \{\alpha_0, \alpha_0^\sigma\} \bmod \mathbb{Z} : \frac{1}{|\alpha - \alpha^\sigma|} \leq s\} \sim \frac{24 q_G \operatorname{argcosh} \frac{|\operatorname{tr} \gamma_0|}{2}}{\pi^2 [\mathrm{PSL}_2(\mathbb{Z}) : G] n_0} s,$$

where  $q_G$  is the smallest positive integer  $q$  such that  $z \mapsto z + q$  belongs to  $G$ ,  $\gamma_0 \in G - \{1\}$  fixes  $\alpha_0$  and  $n_0$  is the index of  $\gamma_0^{\mathbb{Z}}$  in the stabilizer of  $\{\alpha_0, \alpha_0^\sigma\}$  in  $G$  (and note that  $q_G, \gamma_0, n_0$  do exist).

For instance, if  $\alpha_0$  is the Golden ratio  $\phi = \frac{1+\sqrt{5}}{2}$  (which is reciprocal in Sarnak's terminology) and  $G = \mathrm{PSL}_2(\mathbb{Z})$ , we get  $\mathrm{Card}\{\alpha \in G \cdot \phi \bmod \mathbb{Z} : \frac{1}{|\alpha - \alpha^\sigma|} \leq s\} \sim \frac{24 \log \phi}{\pi^2} s$ . With  $\mathbb{H}_{\mathbb{R}}^2$  the upper halfplane model of the real hyperbolic plane, the proof applies Theorem 2 to  $M$  the orbifold  $G \backslash \mathbb{H}_{\mathbb{R}}^2$ , to  $C_0$  the image in  $M$  of the geodesic line in  $\mathbb{H}_{\mathbb{R}}^2$  with endpoints  $\alpha_0$  and  $\alpha_0^\sigma$ , and to  $\mathcal{H}_\infty$  the image in  $M$  of the set of points in  $\mathbb{H}_{\mathbb{R}}^2$  with Euclidean height at least 1. The trick is that if  $a$  and  $b$  are close enough distinct real numbers, then the hyperbolic length of the perpendicular arc between the horizontal line at Euclidean height 1 and the geodesic line with endpoints  $a$  and  $b$  is exactly  $-\log |b - a|$ .

Assume  $K$  is imaginary quadratic, with discriminant  $D_K$ . We proved a general statement analogous to the previous corollary, but we only give here a particular case for  $\phi$ .

**Corollary 2** *Let  $\mathfrak{a}$  be a non zero ideal in  $\mathcal{O}_K$  and  $\Gamma_0(\mathfrak{a}) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_K) : c \in \mathfrak{a} \right\}$ . Assume for simplicity that  $D_K \neq -4$  and  $\phi^\sigma \notin \Gamma_0(\mathfrak{a}) \cdot \phi$ . Then as  $s$  tends to  $+\infty$ , the cardinality of  $\{\alpha \in \Gamma_0(\mathfrak{a}) \cdot \{\phi, \phi^\sigma\} \bmod \mathcal{O}_K : \frac{1}{|\alpha - \alpha^\sigma|} \leq s\}$  is equivalent to*

$$\frac{8\pi^2 k_{\mathfrak{a}} \log \phi}{|D_K| \zeta_K(2) N(\mathfrak{a}) \prod_{\mathfrak{p} \text{ prime, } \mathfrak{p}|\mathfrak{a}} \left(1 + \frac{1}{N(\mathfrak{p})}\right)} s^2,$$

with  $k_{\mathfrak{a}}$  the smallest  $k \in \mathbb{N} - \{0\}$  such that the  $2k$ -th term of the standard Fibonacci sequence belongs to  $\mathfrak{a}$  (and note that  $k_{\mathfrak{a}}$  does always exist, contrarily to the odd case).

**Counting representations of integers by binary forms.** Recall that a binary quadratic form  $Q(x, y) = ax^2 + bxy + cy^2$  is primitive integral if  $a, b, c \in \mathbb{Z}$  are relatively prime, and indefinite non product if its discriminant  $D = b^2 - 4ac$  is positive and not a square. Using the well known correspondence between pairs of Galois conjugated quadratic irrationals over  $\mathbb{Q}$  and the set of such  $Q$ 's up to sign, we prove the following counting result for the number of values of a fixed such  $Q$  on couples of relatively prime integers satisfying some congruence relations. Let  $(t, u)$  be the minimal solution to the Pell-Fermat equation  $t^2 - Du^2 = 4$  and  $\epsilon = \frac{t+u\sqrt{D}}{2}$  the corresponding fundamental unit.

**Corollary 3** *Let  $Q$  be as above, and let  $n$  be an integer at least 3. Then the number of couples  $(x, y) \in \mathbb{Z}^2$ , relatively prime, with  $x \equiv 1 \pmod n$  and  $y \equiv 0 \pmod n$ , such that  $|Q(x, y)| \leq s$ , modulo the linear action of  $\mathrm{SL}_2(\mathbb{Z})$ , is equivalent, as  $s$  tends to  $+\infty$ , to*

$$\frac{24 \log \epsilon}{\pi^2 n^2 \sqrt{D}} \prod_{p \text{ prime}, p|n} \left(1 - \frac{1}{p^2}\right)^{-1} s.$$

The final result, for a quadratic imaginary number field  $K$ , is proved in [PP2], along with extensions to representations satisfying congruence properties.

**Corollary 4** *Let  $f : (u, v) \mapsto a|u|^2 + 2\mathrm{Re}(bu\bar{v}) + c|v|^2$  be a binary Hermitian form, indefinite (that is  $\Delta = |b|^2 - ac > 0$ ) and integral over  $K$  (that is  $a, c \in \mathbb{Z}, b \in \mathcal{O}_K$ ). Let  $\mathrm{SU}_f(\mathcal{O}_K) = \{g \in \mathrm{SL}_2(\mathcal{O}_K) : f \circ g = g\}$  be the group of automorphs of  $f$ . Then the number of orbits under  $\mathrm{SU}_f(\mathcal{O}_K)$  of couples  $(u, v)$  of relatively prime elements of  $\mathcal{O}_K$  such that  $|f(u, v)| \leq s$  is equivalent, as  $s$  tends to  $+\infty$ , to*

$$\frac{\pi \mathrm{Covol}(\mathrm{SU}_f(\mathcal{O}_K))}{2 |D_K| \zeta_K(2) \Delta} s^2.$$

With  $\mathbb{H}_{\mathbb{R}}^3$  the upper halfspace model of the real hyperbolic 3-space, the proof applies Theorem 2 to  $M$  the orbifold  $\mathrm{PSL}_2(\mathcal{O}_K) \backslash \mathbb{H}_{\mathbb{R}}^3$ , to  $C_0$  the image in  $M$  of the unique hyperbolic plane  $P(f)$  in  $\mathbb{H}_{\mathbb{R}}^3$  preserved by  $\mathrm{PSU}_f(\mathcal{O}_K)$ , and to  $\mathcal{H}_{\infty}$  the image in  $M$  of the set of points in  $\mathbb{H}_{\mathbb{R}}^3$  with Euclidean height at least 1. The trick is that, for every  $\gamma \in \mathrm{PSL}_2(\mathcal{O}_K)$ , the hyperbolic plane  $P(f \circ \gamma)$  is an Euclidean hemisphere whose diameter is  $\frac{\sqrt{\Delta}}{f \circ \gamma(1,0)}$ , hence whose perpendicular arc to the horizontal plane at Euclidean height 1 has (signed) hyperbolic length  $\log \frac{f \circ \gamma(1,0)}{\sqrt{\Delta}}$ , and that  $\mathrm{SL}_2(\mathcal{O}_K)$  acts transitively on the couples of relatively prime elements of  $\mathcal{O}_K$ .

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