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VOLUME ENTROPY RIGIDITY OF NON-POSITIVELY CURVED SYMMETRIC SPACES

FRANÇOIS LEDRAPPIER

To Werner Ballmann for his 60th birthday

Abstract. We characterize symmetric spaces of non-positive curvature by the equality case of general inequalities between geometric quantities.

1. Introduction

Let \((M, g)\) be a closed connected Riemannian manifold, and \(\pi: (\widetilde{M}, \tilde{g}) \rightarrow (M, g)\) its universal cover endowed with the lifted Riemannian metric. We denote \(p(t, x, y), t \in \mathbb{R}_+, x, y \in \tilde{M}\) the heat kernel on \(\tilde{M}\), the fundamental solution of the heat equation \(\frac{\partial u}{\partial t} = \text{Div} \nabla u\) on \(\tilde{M}\). Since we have a compact quotient, all the following limits exist as \(t \to \infty\) and are independent of \(x \in \tilde{M}\):

\[
\lambda_0 = \inf_{f \in C_0^2(\tilde{M})} \frac{\int |\nabla f|^2}{\int |f|^2} = \lim_{t \to \infty} \frac{1}{t} \ln p(t, x, x)
\]

\[
\ell = \lim_{t \to \infty} \frac{1}{t} \int d(x, y)p(t, x, y)d\text{Vol}(y)
\]

\[
h = \lim_{t \to \infty} \frac{1}{t} \int p(t, x, y) \ln p(t, x, y)d\text{Vol}(y)
\]

\[
v = \lim_{t \to \infty} \frac{1}{t} \ln \text{Vol}B_{\tilde{M}}(x, t),
\]

where \(B_{\tilde{M}}(x, t)\) is the ball of radius \(t\) centered at \(x\) in \(\tilde{M}\) and \(\text{Vol}\) is the Riemannian volume on \(\tilde{M}\).

All these numbers are nonnegative. Recall \(\lambda_0\) is the Rayleigh quotient of \(\tilde{M}\), \(\ell\) the linear drift, \(h\) the stochastic entropy and \(v\) the volume entropy. There is the following relation:

\[
4\lambda_0 \overset{(a)}{\leq} h \overset{(b)}{\leq} \ell v \overset{(c)}{\leq} v^2.
\]

See [L1] for (a), [Gu] for (b). Inequality (c) is shown in [L3] as a corollary of (b) and (2):

\[
\ell^2 \leq h
\]

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If $(\widetilde{M}, g)$ is a locally symmetric space of nonpositive curvature, all five numbers $4\lambda_0, \ell^2, h, \ell v$ and $v^2$ coincide and are positive unless $(\widetilde{M}, g)$ is $\mathbb{R}^n$ (Eucl.). Our result is a partial converse:

**Theorem 1.1.** Assume $(M, g)$ has nonpositive curvature. With the above notation, any of the equalities

$$\ell = v, \quad h = v^2 \quad \text{and} \quad 4\lambda_0 = v^2$$

hold if, and only if, $(\widetilde{M}, \widetilde{g})$ is a symmetric space.

As recalled in [L3], Theorem 1.1 is known in negative curvature and follows from [K], [BFL], [FL], [BCG] and [L1]. The other possible converses are delicate: even for negatively curved manifolds, in dimension greater than two, it is not known that $h = \ell v$ holds only for locally symmetric spaces. This is equivalent to a conjecture of Sullivan (see [L2] for a discussion). Sullivan conjecture holds for surfaces of negative curvature ([L1], [Ka]). It is not known either whether $4\lambda_0 = h$ holds only for locally symmetric spaces. This would follow from the hypothetical $4\lambda_0 = \ell^2$ by the arguments of this note.

We assume henceforth that $(M, g)$ has nonpositive sectional curvature. Given a geodesic $\gamma$ in $M$, Jacobi fields along $\gamma$ are vector fields $t \mapsto J(t) \in T_{\gamma(t)}M$ which describe infinitesimal variation of geodesics around $\gamma$. By nonpositive curvature, the function $t \mapsto \|J(t)\|$ is convex. Jacobi fields along $\gamma$ form a vector space of dimension $2 \dim M$. The rank of the geodesic $\gamma$ is the dimension of the space of Jacobi fields such that $t \mapsto \|J(t)\|$ is a constant function on $\mathbb{R}$. The rank of a geodesic $\gamma$ is at least one because of the trivial $t \mapsto \dot{\gamma}(t)$ which describes the variation by sliding the geodesic along itself. The rank of the manifold $M$ is the smallest rank of geodesics in $M$. Using rank rigidity theorem ([B1], [BS]), we reduce in section 2 the proof of Theorem 1.1 to proving that if $(M, g)$ is rank one, equality in (2) implies that $(\widetilde{M}, \widetilde{g})$ is a symmetric space. For this, we show in section 3 that equality in (2) implies that $(M, g)$ is asymptotically harmonic (see the definition below). This uses the Dirichlet property at infinity (Ballmann [B2]). Finally, it was recently observed by A. Zimmer ([Z]) that asymptotically harmonic universal covers of rank one manifolds are indeed symmetric spaces.

2. Generalities and reduction of Theorem 1.1

We recall the notations and results from Ballmann’s monograph [B3] about the Hadamard manifold $(\widetilde{M}, \widetilde{g})$ that we use. The space $\widetilde{M}$ is homeomorphic to a ball. The covering group $G := \pi_1(M)$ satisfies the duality condition ([B3] page 45).

2.1. **Boundary at infinity.** Two geodesic rays $\gamma, \gamma'$ in $\widetilde{M}$ are said to be asymptotic if $\sup_{t \geq 0} d(\gamma(t), \gamma'(t)) < \infty$. The set of classes of asymptotic unit speed geodesic rays is called the boundary at infinity $\widetilde{M}(\infty)$. $\widetilde{M} \cup \widetilde{M}(\infty)$ is endowed with the topology of a compact space where $\widetilde{M}(\infty)$ is a sphere and where, for each unit speed geodesic ray $\gamma$, $\gamma(t) \to [\gamma]$
as \( t \to \infty \). The action of the group \( G \) on \( \widetilde{M}(\infty) \) is the continuous extension of its action on \( \widetilde{M} \). For any \( x, \xi \in \widetilde{M} \times \widetilde{M}(\infty), \) there is a unique unit speed geodesic \( \gamma_{x,\xi} \) such that \( \gamma_{x,\xi}(0) = x \) and \( [\gamma_{x,\xi}] = \xi \). The mapping \( \xi \mapsto \dot{\gamma}_{x,\xi}(0) \) is a homeomorphism \( \pi_x^{-1} \) between \( \widetilde{M}(\infty) \) and the unit sphere \( S_x \widetilde{M} \) in the tangent space at \( x \) to \( \widetilde{M} \). We will identify \( SM \) with \( \widetilde{M} \times \widetilde{M}(\infty) \) by \( (x, v) \mapsto (x, \pi_x v) \). Then the quotient \( SM \) is identified with the quotient of \( \widetilde{M} \times \widetilde{M}(\infty) \) under the diagonal action of \( G \).

Fix \( x_0 \in \widetilde{M} \) and \( \xi \in \widetilde{M}(\infty) \). The Busemann function \( b_\xi \) is the function on \( \widetilde{M} \) given by:

\[
b_\xi(x) = \lim_{y \to \xi} d(y, x) - d(y, x_0).
\]

Clearly, \( b_\xi(gx) = b_\xi(x) + b_\xi(gx_0) \). Moreover, the function \( x \mapsto b_\xi(x) \) is of class \( C^2 \) ([HI]). It follows that the function \( \Delta_x b_\xi \) satisfies \( \Delta_x b_\xi = \Delta_x b_\xi \) and therefore defines a function \( B \) on \( G \setminus (\widetilde{M} \times \widetilde{M}(\infty)) = SM \). It follows from the argument of [HI] that the function \( B \) is continuous on \( SM \) (see [B3], Proposition 2.8, page 69).

2.2. Jacobi fields. Let \((x, v)\) be a point in \( T\widetilde{M} \). Tangent vectors in \( T_{x,v}T\widetilde{M} \) correspond to variations of geodesics and can be represented by Jacobi fields along the unique geodesic \( \gamma_{x,v} \) with initial value \( \gamma(0) = x, \dot{\gamma}(0) = v \). A Jacobi field \( J(t), t \in \mathbb{R} \) along \( \gamma_{x,v} \) is uniquely determined by the values of \( J(0) \) and \( J'(0) \). We describe tangent vectors in \( T_{x,v}T\widetilde{M} \) by the associated pair \((J(0), J'(0))\). The metric on \( T_{x,v}T\widetilde{M} \) is given by \( \| (J_0, J'_0) \|^2 = \| J_0 \|^2 + \| J'_0 \|^2 \). Assume \((x, v) \in SM \). A vertical vector in \( T_{x,v}SM \) is a vector tangent to \( S_x \widetilde{M} \). It corresponds to a pair \((0, J'(0))\), with \( J'(0) \) orthogonal to \( v \). Horizontal vectors correspond to pairs \((J(0), 0)\). In particular, let \( X \) be the vector field on \( SM \) such that the integral flow of \( X \) is the geodesic flow. The geodesic spray \( X_{x,v} \) is the horizontal vector associated to \((v, 0)\). The orthogonal space to \( X \) is preserved by the differential \( Dg_t \) of the geodesic flow. More generally, the Jacobi fields representation of \( TT\widetilde{M} \) satisfies \( D_{x,v}g_t(J(0), J'(0)) = (J(t), J'(t)) \).

For any vector \( Y \in T_{x,v}T\widetilde{M} \), there is a unique vector \( Z = S_{x,v}Y \) such that the Jacobi field \( J \) with \( J(0) = Y, J'(0) = Z \) satisfies \( \| J(t) \| \leq C \) for \( t \geq 0 \) ([B3], Proposition 2.8 (i)). The mapping \( S_{x,v} : T_x \widetilde{M} \to T_x \widetilde{M} \) is linear and selfadjoint. The vectors \((Y, SY)\) describe variations of asymptotic geodesics and the subspace \( E_{x,v}^s \subset T_{x,v}T\widetilde{M} \) they generate corresponds to \( TW_{x,v}^s \), where \( W_{x,v}^s \) is the set of initial vectors of geodesics asymptotic to \( \gamma_{x,v} \), is identified with \( \widetilde{M} \times \pi_x(v) \) in \( \widetilde{M} \times \widetilde{M}(\infty) \). Observe that \( S_{x,v} \dot{\gamma}_{x,v}(0) = 0 \) and that the operator \( S_{x,v} \) preserves \( (\dot{\gamma}_{x,v}(0))^\perp \). Recall from [B3], Proposition 3.2 page 71, that, for \( Y \in (\dot{\gamma}_{x,v}(0))^\perp \), with \( \pi_x v = \xi \),

\[
D_Y(\nabla b_\xi) = -S_{x,v}Y,
\]

and therefore \( \Delta_x b_\xi = -\text{Tr} S_{x,v} \) with \( \pi_x(v) = \xi \).

Similarly, there is a selfadjoint linear operator \( U_{x,v} : T_x \widetilde{M} \to T_x \widetilde{M} \) such that the Jacobi field \( J \) with \( J(0) = Y, J'(0) = UY \) satisfies \( \| J(t) \| \leq C \) for \( t \leq 0 \). The subspace \( E_{x,v}^u \subset
If $T_{x,v}T\tilde{M}$ they generate corresponds to $T\tilde{M}$, where $W^u_x$ is the set of opposite vectors to vectors in $W^s_{x,-v}$. By definition, $S^\perp_{x,0} = -U_{x,-v}(0)$, so that we also have:

$$B(x,v) := -Tr S_{x,v} = Tr U_{x,-v}.$$  

We have Ker $S = Ker U$ and $Y \in Ker S$ if, and only if, the Jacobi field $J(t)$ with $J(0) = Y, J'(0) = 0$ is bounded for all $t \in \mathbb{R}$. The rank of the geodesic $\gamma_{x,v}$ therefore is $\kappa = \dim Ker S$ and the geodesic $\gamma_{x,v}$ is of rank one only if $\det((U - S)|_{\langle(-x,v)(0^+\rangle}) = 0$.

Recall that $SM$ is identified with the quotient of $\tilde{M} \times \tilde{M}(\infty)$ under the diagonal action of $G$. Clearly, for $g \in G$, $g(W^s_{x,v}) = W^s_{Dg(x,v)}$ so that the $W^s$ define a foliation $W^s$ on $SM$.

The leaves of the foliation $W^s$ are quotient of $\tilde{M}$, they are naturally endowed with the Riemannian metric induced from $\tilde{g}$.

2.3. Proof of Theorem 1.1. We continue assuming that $(\tilde{M}, \tilde{g})$ has nonpositive curvature. By the Rank Rigidity Theorem (see [B3]), $(\tilde{M}, \tilde{g})$ is of the form

$$(\tilde{M}_0 \times \tilde{M}_1 \times \cdots \times \tilde{M}_j \times \tilde{M}_{j+1} \times \cdots \times \tilde{M}_k, \tilde{g})^1,$$

where $\tilde{g}$ is the product metric $\tilde{g}^2 = (\tilde{g_0})^2 + (\tilde{g_1})^2 + \cdots + (\tilde{g}_j)^2 + (\tilde{g}_{j+1})^2 + \cdots + (\tilde{g}_k)^2,$

$(\tilde{M}_0, \tilde{g}_0)$ is Euclidean, $(\tilde{M}_i, \tilde{g}_i)$ is an irreducible symmetric space of rank at least two for $i = 1, \cdots, j$ and a rank-one manifold for $i = j + 1, \cdots, k$. If the $(\tilde{M}_i, \tilde{g}_i), i = j + 1, \cdots, k$, are all symmetric spaces of rank one, then $(\tilde{M}, \tilde{g})$ is a symmetric space. Moreover in that case, all inequalities in (1) are equalities: this is the case for irreducible symmetric spaces (all numbers are 0 for Euclidean space; for the other spaces, $4\lambda_0$ and $\nu^2$ are classically known to coincide ([(O)]) and we have:

$$4\lambda_0(\tilde{M}) = \sum_i 4\lambda_0(\tilde{M}_i), \quad \nu^2(\tilde{M}) = \sum_i \nu^2(\tilde{M}_i).$$

To prove Theorem 1.1, it suffices to prove that if $\ell^2 = h$, all $\tilde{M}_i$ in the decomposition are symmetric spaces. This is already true for $i = 0, 1, \cdots, j$. It remains to show that $(\tilde{M}_i, \tilde{g}_i)$ are symmetric spaces for $i = j + 1, \cdots, k$. Eberlein showed that each one of the spaces $(\tilde{M}_i, \tilde{g}_i)$ admits a cocompact discrete group of isometries (see [Kn], Theorem 3.3). This shows that the linear drifts $\ell_i$ and the stochastic entropies $h_i$ exist for each one of the spaces $(\tilde{M}_i, \tilde{g}_i)$. Moreover, we clearly have

$$\ell^2 = \sum_i \ell_i^2, \quad h = \sum h_i.$$ 

Therefore Theorem 1.1 follows from

**Theorem 2.1.** Assume $(M, g)$ is a closed connected rank one manifold of nonpositive curvature and that $\ell^2 = h$. Then $(\tilde{M}, \tilde{g})$ is a symmetric space.

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1With a clear convention for the cases when $\dim \tilde{M}_0 = 0, j = 0$ or $k = j$. 

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Therefore Theorem 1.1 follows from

**Theorem 2.1.** Assume $(M, g)$ is a closed connected rank one manifold of nonpositive curvature and that $\ell^2 = h$. Then $(\tilde{M}, \tilde{g})$ is a symmetric space.
A Hadamard manifold \( \tilde{M} \) is called asymptotically harmonic if the function \( B(= \Delta_x b) \) is constant on \( SM \). Theorem 2.1 directly follows from two propositions:

**Proposition 2.2.** Assume \((M, g)\) is a closed connected rank one manifold of nonpositive curvature and that \( \ell^2 = h \). Then \((\tilde{M}, \tilde{g})\) is asymptotically harmonic.

**Proposition 2.3.** [\(Z\), Theorem 1.1] Assume \((M, g)\) is a closed connected rank one manifold of nonpositive curvature such that \((\tilde{M}, \tilde{g})\) is asymptotically harmonic. Then, \((\tilde{M}, \tilde{g})\) is a symmetric space.

### 3. Proof of Proposition 2.2

We consider the foliation \( W \) of subsection 2.2. Recall that the leaves are endowed with a natural Riemannian metric. We write \( \Delta^W \) for the associated Laplace operator on functions which are of class \( C^2 \) along the leaves of \( W \). A probability measure \( m \) on \( SM \) is called harmonic if it satisfies, for any \( C^2 \) function \( f \), we have:

\[
\int_{SM} \Delta^W f \, dm = 0.
\]

Let \( M \) be a closed connected manifold such that \( \ell^2 = h \). In \([L3]\) it is shown that then, there exists a harmonic probability measure \( m \) on \( SM \) such that, at \( m \)-a.e. \((x, v)\), \( B(x, v) = \ell \). Since \( B \) is a continuous function, Proposition 2.2 follows from Theorem 3.1.

**Theorem 3.1.** Let \((M, g)\) be a closed connected rank one manifold of nonpositive curvature, \( W \) the stable foliation on \( SM \) endowed with the natural metric as above. Then, there is only one harmonic probability measure \( m \) and the support of \( m \) is the whole space \( SM \).

**Proof.** Let \( m \) be a \( W \) harmonic probability measure on \( SM \). Then, there is a unique \( G \)-invariant measure \( \mathcal{m} \) and the support of \( m \) is the whole space \( SM \).

To reprove it: on the one hand, the measure \( \mathcal{m} \) projects on \( M \) as a \( G \)-invariant measure satisfying \( \int \Delta f \, dm = 0 \). The projection of \( \mathcal{m} \) on \( M \) is proportional to Volume, gives measure 1 to fundamental domains and formula (3) is the desintegration formula. On the other hand, if one projects \( \mathcal{m} \) first on \( M(\infty) \), there is a probability measure \( \nu \) on \( M(\infty) \) such that

\[
\int f(x, \xi) \, d\mathcal{m}(x, \xi) = \frac{1}{\text{Vol}M} \int_M \left( \int_{M(\infty)} f(x, \xi) \, d\nu_x(\xi) \right) \, dx,
\]

where the family \( x \mapsto \nu_x \) is a family of probability measures on \( M(\infty) \) such that, for all \( \varphi \) continuous on \( M(\infty) \), \( x \mapsto \int \varphi(\xi) \, d\nu_x(\xi) \) is a harmonic function on \( M \) and the measure \( dx \) is the Riemannian volume on \( M \). The claim follows from \([Ga]\). For convenience, let us reprove it: on the one hand, the measure \( \mathcal{m} \) projects on \( M \) as a \( G \)-invariant measure satisfying \( \int \Delta f \, dm = 0 \). The projection of \( \mathcal{m} \) on \( M \) is proportional to Volume, gives measure 1 to fundamental domains and formula (3) is the desintegration formula. On the other hand, if one projects \( \mathcal{m} \) first on \( M(\infty) \), there is a probability measure \( \nu \) on \( M(\infty) \) such that

\[
\int f(x, \xi) \, d\mathcal{m}(x, \xi) = \int_{M(\infty)} \left( \int_M f(x, \xi) \, dm_\xi(dx) \right) \, d\nu(\xi).
\]
For \( \nu \)-a.e. \( \xi \), the measure \( m_\xi \) is a harmonic measure on \( \tilde{M} \); therefore, for \( \nu \)-a.e. \( \xi \), there is a positive harmonic function \( k_\xi(x) \) such that \( m_\xi = k_\xi(x) \text{Vol} \). Comparing the two expressions for \( \int f d\tilde{m} \), we see that the measure \( \nu_x \) is given by
\[
\nu_x = k_\xi(x) \nu
\]
and \( x \mapsto \int_{\tilde{M}(\infty)} \varphi(\xi) d\nu_x(\xi) \) is indeed a harmonic function.

The \( G \)-invariance of \( \tilde{m} \) implies that, for all \( g \in G \), \( g_*\nu_x = \nu_{gx} \). In particular, the support of \( \nu \) is \( G \)-invariant. By [E] (see [B3], page 48), the support of \( \nu \) is the whole \( \tilde{M}(\infty) \) and therefore the support of \( m \) is the whole \( SM \). This result would be sufficient for proving Proposition 2.2, but using discretization, we are going to identify the measure \( \nu_x \) on \( \tilde{M}(\infty) \) as the hitting measure of the Brownian motion on \( \tilde{M} \) starting from \( x \). This shows Theorem 3.1.

Fix \( x_0 \in \tilde{M} \). The discretization procedure of Lyons and Sullivan ([LS]) associates to the Brownian motion on \( \tilde{M} \) a probability measure \( \mu \) on \( G \) such that \( \mu(g) > 0 \) for all \( g \) and that any bounded harmonic function \( F \) on \( \tilde{M} \) satisfies
\[
F(x_0) = \sum_{g \in G} F(gx_0) \mu(g).
\]
Recall that for all \( \varphi \) continuous on \( \tilde{M}(\infty) \), \( x \mapsto \nu_x(\varphi) \) is a harmonic function and that \( \nu_{gx} = g_*\nu_x \). It follows that the measure \( \nu_{x_0} \) is stationary for \( \mu \), i.e. it satisfies:
\[
\nu_{x_0} = \sum_{g \in G} g_*\nu_{x_0} \mu(g).
\]
Since the support of \( \mu \) generates \( G \) as a semigroup (actually, it is already the whole \( G \)), there is only one stationary probability measure on \( \tilde{M}(\infty) \) (see [B3], Theorem 4.11 page 58). We know one already: the hitting measure \( m_{x_0} \) of the Brownian motion on \( \tilde{M} \) starting from \( x_0 \). This shows that \( \nu_{x_0} = m_{x_0} \). Since \( x_0 \) was arbitrary in the above reasoning, we have \( \nu_x = m_x \) for all \( x \in \tilde{M} \) and the measure \( \tilde{m} \) is given by:
\[
\int f(x,\xi) d\tilde{m}(x,\xi) = \frac{1}{\text{Vol}M} \int_{\tilde{M}} \left( \int_{\tilde{M}(\infty)} f(x,\xi) dm_x(\xi) \right) dx.
\]

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References


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