



# Self-stabilizing processes in multi-wells landscape in $R^d$ - Convergence

Julian Tugaut

## ► To cite this version:

Julian Tugaut. Self-stabilizing processes in multi-wells landscape in  $R^d$  - Convergence. 2011. hal-00628086v2

**HAL Id: hal-00628086**

**<https://hal.science/hal-00628086v2>**

Preprint submitted on 10 Aug 2012

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Self-stabilizing processes in multi-wells landscape in $\mathbb{R}^d$ - Convergence\*

Julian Tugaut  
Fakultät für Mathematik  
Universität Bielefeld  
D-33615 Bielefeld  
Germany  
Email: jtugaut@math.uni-bielefeld.de

## Abstract

Self-stabilizing processes are inhomogeneous diffusions in which the law of the process intervenes in the drift. If the external force is the gradient of a convex potential, it has been proved that the process converges toward the unique invariant probability as the time goes to infinity. However, in a previous article, we established that the diffusion may admit several invariant probabilities, provided that the external force derives from a non-convex potential. We here provide results about the limiting values of the family  $\{\mu_t; t \geq 0\}$ ,  $\mu_t$  being the law of the diffusion. Moreover, we establish the weak convergence under an additional hypothesis.

**Key words and phrases:** Self-interacting diffusion, Free-energy, McKean-Vlasov stochastic differential equations, Multi-wells potential, Granular media equation

**2000 AMS subject classifications:** Primary 60H10, 35B40 ; secondary 35K55, 60J60, 60G10

## Introduction

The aim of the work is the study about the long-time behaviour of a special class of inhomogeneous diffusions. With potentials  $V$  and  $F$ , which hypotheses are given further, we are interested in the weak convergence (as the time  $t$  goes to infinity) of the so-called self-stabilizing diffusion,

$$\begin{cases} X_t = X_0 + \sigma B_t - \int_0^t \nabla V(X_s) ds - \int_0^t \nabla F * \mu_s(X_s) ds, \\ \mu_s = \mathcal{L}(X_s). \end{cases} \quad (\text{I})$$

---

\*Supported by the DFG-funded CRC 701, Spectral Structures and Topological Methods in Mathematics, at the University of Bielefeld.

The star denotes the convolution. The drift in the previous equation does depend on the own law of the diffusion. Hence, it is non-linear, in the sense of McKean, see [McK67]. Let us notice that  $X_t$ , the diffusion at time  $t$ , and the probability measure  $\mu_t$  do depend on  $\sigma$ . Nevertheless, the parameter  $\sigma$  is fixed, with the exception of a statement which holds in the small-noise limit, so we skip this dependence in the notation for the comfort of the reading.

In this model,  $(B_t)_{t \geq 0}$  is a  $d$ -dimensional Wiener process. The diffusion coefficient  $\sigma$  is not assumed to be small. The function  $V$  is the so-called confining potential. This terminology comes from the effect of  $V$  on the diffusion  $X$ . Indeed, intuitively, it forces the diffusion to be close to the minimizers of  $V$ .  $F$  is called the interacting potential because the term  $\nabla F * \mu_s(X_s)$  corresponds to an interaction between all the trajectories:

$$\nabla F * \mu_s(X_s(\omega_0)) = \int_{\omega \in \Omega} \nabla F(X_s(\omega_0) - X_s(\omega)) d\mathbb{P}(\omega),$$

where  $(\Omega, \mathcal{F}, \mathbb{P})$  is the underlying probability space. This interaction can be seen as the hydrodynamical limit of a mean-field system. This aspect is developed further in the introduction.

The subject of the current work is the set of the adherence values of the family  $\{\mu_t; t \geq 0\}$ . In a particular case, we establish the weak convergence of the self-stabilizing diffusion (I) to an invariant probability measure.

We here make some smoothness assumptions on both potentials  $V$  and  $F$ . More precisely, the potential  $F$  is a radial polynomial function. This hypothesis allows us to simplify the description of the invariant probabilities. Indeed, the existence problem of an invariant probability thus is equivalent to a fixed point problem in finite dimension.

Let us notice that there exist some previous articles with interaction  $F$  which does not satisfy such assumptions. In the one-dimensional case, without external force, by taking the Heaviside step function as interacting potential  $F$ , Equation (I) is the Burgers' equation, that is

$$X_t(\omega_0) = \sigma B_t(\omega_0) - \int_0^t \mathbb{P}\left\{\omega \in \Omega \mid X_s(\omega) = X_s(\omega_0)\right\} ds,$$

see [SV79]. By taking  $F := \delta_0$  instead of the Heaviside step function, it is the Oelschläger equation,

$$X_t(\omega_0) = \sigma B_t(\omega_0) - \int_0^t \frac{d}{dx} \Big|_{x=X_s(\omega_0)} \mathbb{P}\left\{\omega \in \Omega \mid X_s(\omega) = x\right\} ds,$$

studied in [Oel85]. The hypotheses on  $V$  and  $F$  are given at the end of the introduction.

The existence problem of a solution to Equation (I) has been investigated by two different methods. The first one consists in considering the existence question as a fixed point problem in a functional space. Indeed, to solve Equation (I) is

equivalent to solve the system of two equations,

$$\begin{cases} b(t, x) = \nabla V(x) + \mathbb{E} \left[ \nabla F \left( x - X_t^{(b)} \right) \right], \\ X_t^{(b)} = X_0 + \sigma B_t - \int_0^t b(s, X_s^{(b)}) ds. \end{cases}$$

See [BRTV98, HIP08, Szn91] for the details. The second method consists in looking at the law  $\mu_t$  as the hydrodynamical limit of the empirical measure of a mean-field interacting particle system. See for example [Mél96].

It is well known that  $\mu_t$ , the law at time  $t$  of the unique strong solution  $X$  to Equation (I), admits a  $\mathcal{C}^\infty$ -continuous density with respect to the Lebesgue measure, provided that  $t > 0$ , see [McK67]. Let us denote this density by  $u_t$ . Furthermore,  $u_t$  satisfies the so-called granular media equation,

$$\frac{\partial}{\partial t} u_t = \operatorname{div} \left\{ \frac{\sigma^2}{2} \nabla u_t + u_t (\nabla V + \nabla F * u_t) \right\}. \quad (\text{II})$$

The non-linear partial differential equation (II) is a useful tool for characterizing the invariant probabilities of Diffusion (I) and its long time behaviour, see [BRTV98, BRV98, Tam84, Tam87, Ver06].

If the confining potential  $V$  is not convex, at least in the one-dimensional case, our previous results in [HT10a, HT10b, HT12, Tug11a] provide the exact number of the invariant probabilities and their small-noise behaviour under easily checked assumptions. If the phase space is  $\mathbb{R}^d$ , it has been proved in [Tug11b] that Diffusion (I) admits several invariant probabilities when the interaction is sufficiently strong. Consequently, the weak convergence of  $\mu_t$  in the long-time behaviour is more delicate than if the external force corresponds to the gradient of a convex potential.

In the one-dimensional case, the study has been made with convex confining potential by several technics. In [BRV98], with  $V$  identically equal to 0, the authors focus on the semi-group associated to Equation (II). They use an ultracontractivity property (from [KKR93]), a Poincaré inequality (with Bakry-Émery criteria) and a comparison lemma for stochastic processes (from [KS91]). With a non-convex confining potential, the results on [KKR93] still hold and we can replace Bakry-Émery criteria by Muckenhoupt theorem in the one-dimensional case (see [ABC<sup>+</sup>00]) or by the results in [BBCG08]. Nonetheless, the comparison lemma requires the convexity assumption.

Another method consists in considering a diffusion  $X^1$  in a mean-field system of  $N$  particles as an approximation of the self-stabilizing diffusion  $X$ . Under simple hypotheses, Cattiaux, Guillin and Malrieu establish in [CGM08] a uniform propagation of chaos, that is the convergence to 0 of  $\sup_{t \geq 0} \mathbb{E} \left\{ \|X_t - X_t^1\|^2 \right\}$  as

the number of particles  $N$  goes to infinity. Thus, the convergence in long-time of the mean-field system implies the one of the non-linear process  $X$ . Nevertheless, the non-uniqueness of the invariant probabilities pointed out in [Tug11b] implies that such a uniform propagation of chaos does not hold.

Still for the convex case, we also refer the reader to [Mal03, Tam84, Mal01,

HS87, AMTU01, CMV03].

The method used in [Tug10], for the non-convex case, is based on [BCCP98]. By combining the results in [HT10a, HT10b, HT12] with the work of [BCCP98, CMV03] about the free-energy, we have been able to prove the convergence without assuming that the center of mass is fixed.

The technic used in this work is similar but we do not assume that the set of the invariant probabilities is discrete. We give results on the adherence values of the family  $\{\mu_t; t \geq 0\}$ . To do so, we use a bijection between  $\mathcal{S}_\sigma$ , the set of the invariant probabilities, and a closed subset of  $\mathbb{R}^k$ . The integer  $k$  here does depend on the dimension and on the degree of the polynomial function  $F$ . Furthermore, if the set of the invariant probabilities is discrete, the probability measure  $\mu_t$  converges weakly to an invariant probability  $\mu^\sigma$  as the time  $t$  goes to infinity.

As written previously, the diffusion  $X$  corresponds to the hydrodynamical limit of a particle in a continuous mean-field system. This mean-field system associated to the self-stabilizing process (I) is a classical diffusion in  $(\mathbb{R}^d)^N$ , that is

$$\begin{cases} X_t^1 = X_0^1 + \sigma B_t^1 - \int_0^t \nabla V(X_s^1) ds - \frac{1}{N} \sum_{j=1}^N \int_0^t \nabla F(X_s^1 - X_s^j) ds, \\ \vdots \\ X_t^i = X_0^i + \sigma B_t^i - \int_0^t \nabla V(X_s^i) ds - \frac{1}{N} \sum_{j=1}^N \int_0^t \nabla F(X_s^i - X_s^j) ds, \\ \vdots \\ X_t^N = X_0^N + \sigma B_t^N - \int_0^t \nabla V(X_s^N) ds - \frac{1}{N} \sum_{j=1}^N \int_0^t \nabla F(X_s^N - X_s^j) ds, \end{cases}$$

the  $d$ -dimensional Wiener processes  $B^1, \dots, B^N$  being independent and the random variables  $X_0^i$  being iid with common law  $\mu_0$ . The link between Diffusion (I) and this particle system is called the propagation of chaos. Indeed, the larger is  $N$ , the more independent (chaotic) are the particles. Moreover, each one tends to behave like a self-stabilizing diffusion, see [Szn91, BRTV98, M  l96, CGM08]. Let us also give some references about the propagation of chaos with different hypotheses about the dynamic or about the phase space: [Gra90, Gra92, Der03, JM08].

As noticed previously, Diffusion (I) is similar to the particle  $X^1$ . However, the whole particle system can be seen as an homogeneous diffusion  $\mathcal{X}$  in  $(\mathbb{R}^d)^N$  satisfying

$$\mathcal{X}_t = \mathcal{X}_0 + \sigma \mathcal{B}_t - N \int_0^t \nabla \Upsilon_0^N(\mathcal{X}_s) ds,$$

where the  $i$ th coordinate of  $\mathcal{X}_t$  (resp.  $\mathcal{B}_t$ ) is  $X_t^i$  (resp.  $B_t^i$ ) and

$$\Upsilon_0^N(\mathcal{X}) := \frac{1}{N} \sum_{j=1}^N V(\mathcal{X}_j) + \frac{1}{2N^2} \sum_{i=1}^N \sum_{j=1}^N F(\mathcal{X}_i - \mathcal{X}_j),$$

for all  $\mathcal{X} \in \left(\mathbb{R}^d\right)^N$ . Taking  $\sigma$  equal to 0 yields

$$\frac{d}{dt}\mathbb{E}\{\Upsilon_0^N(\mathcal{X})\} = -\mathbb{E}\left\{\|\nabla\Upsilon_0^N(\mathcal{X})\|^2\right\}.$$

The equivalent of this potential  $\Upsilon_0^N$  for the flow defined by Equation (II), that is after taking the hydrodynamical limit ( $N \rightarrow +\infty$ ), is the energy  $\Upsilon_0$  defined by

$$\Upsilon_0(\mu) := \int_{\mathbb{R}^d} V(x)\mu(dx) + \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} F(x-y)\mu(dx)\mu(dy).$$

Nevertheless, the diffusion coefficient  $\sigma$  is positive. Consequently, the energy is not the good quantity to consider and we need to take into account the entropy  $S(\mu)$ , defined as it follows. If  $\mu$  is absolutely continuous with respect to the Lebesgue measure,  $S(\mu)$  is equal to  $-\int_{\mathbb{R}^d} u(x)\log(u(x))dx$  if it is well defined,  $u$  being the density of  $\mu$ . Otherwise,  $S(\mu) := -\infty$ .

We subtract the dissipated heat, that is the temperature  $\frac{\sigma^2}{2}$  multiplied by the entropy  $S(\mu)$ , to the energy. This new quantity, which corresponds to the free-energy is denoted by  $\Upsilon_\sigma(\mu)$ . If  $\mu$  is not absolutely continuous with respect to the Lebesgue measure,  $\Upsilon_\sigma(\mu)$  is equal to infinity. If  $\mu$  is absolutely continuous with respect to the Lebesgue measure, with density equal to  $u$ , we have

$$\begin{aligned} \Upsilon_\sigma(\mu) := & \frac{\sigma^2}{2} \int_{\mathbb{R}^d} u(x)\log(u(x))dx \\ & + \int_{\mathbb{R}^d} V(x)u(x)dx + \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} F(x-y)u(x)u(y)dx dy. \end{aligned} \quad (\text{III})$$

The free-energy functional  $\Upsilon_\sigma$  plays the role of a Lyapunov function. Indeed, the free-energy is non-increasing along the trajectories of the flow  $(\mu_t)_{t \geq 0}$ . See [CMV03] for a proof of this statement.

We finish the introduction by giving the set of assumptions of the work, some definitions and the three main results (Theorem A and Corollaries B and C). Then, first section presents some previous results and some notations which are used in the whole work. Some basic but essential results also are given. Second section is dedicated to the keystones of the work and an example. The proof of Theorem A is made in third section. The ones of Corollaries B and C are provided in section four. Finally, we postpone two technical but essential propositions in the annex.

**Assumption (M).** We say that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M) if

(M-1)  $V$  is a smooth function on  $\mathbb{R}^d$ .

(M-2) there exists a compact subset  $\mathcal{K}$  of  $\mathbb{R}^d$  such that  $\nabla^2 V(x) > 0$ , for all  $x \notin \mathcal{K}$ . Moreover,  $\lim_{\|x\| \rightarrow +\infty} \nabla^2 V(x) = +\infty$ .

(M-3) the gradient  $\nabla V$  is slowly increasing: there exist  $m \in \mathbb{N}^*$ ,  $C > 0$  and a function  $\mathcal{R}$  from  $\mathbb{R}^d$  to  $\mathbb{R}^d$  such that  $\nabla V(x) = C\|x\|^{2m-2}x + \mathcal{R}(x)$ , for all

$x \in \mathbb{R}^d$ . Here, the function  $\mathcal{R}$  satisfies  $\lim_{\|x\| \rightarrow +\infty} \mathcal{R}(x) \|x\|^{-(2m-1)} = 0$ .

(M-4) there exists an even polynomial function  $G$  on  $\mathbb{R}$  such that  $F(x) = G(\|x\|)$ . And,  $\deg(G) = 2n \geq 2$ .

(M-5) The function  $G$  is convex.

(M-6) The  $8q^2$ th moment of the measure  $\mu_0$  is finite with  $q := \min\{m, n\}$ .

(M-7) The measure  $\mu_0$  admits a  $\mathcal{C}^\infty$ -continuous density  $u_0$  with respect to the Lebesgue measure. And, the entropy  $S(\mu_0) = -\int_{\mathbb{R}^d} u_0(x) \log(u_0(x)) dx$  is finite.

By Theorem 2.13 in [HIP08], we know that Equation (I) admits a unique strong solution on  $\mathbb{R}_+$ . Moreover, there exists  $M_0 > 0$  satisfying

$$\max_{1 \leq j \leq 8q^2} \sup_{t \geq 0} \mathbb{E} \left[ \|X_t\|^j \right] \leq M_0. \quad (\text{IV})$$

We immediately deduce that the family  $\{\mu_t; t \geq 0\}$  is tight.

**Definition.** By  $\mathcal{A}_\sigma$  (resp.  $\mathcal{S}_\sigma$ ), we denote the set of the limiting values of the family  $\{\mu_t; t \geq 0\}$  (resp. the set of the invariant probabilities of Diffusion (I)).

**Definition.** We say that a set  $\mathcal{D}$  of measures on  $\mathbb{R}^d$  is discrete if for any  $\nu \in \mathcal{D}$ , there exists a neighbourhood  $\mathcal{V}$  of  $\nu$  for the topology of the weak convergence such that  $\mathcal{D} \cap \mathcal{V} = \{\nu\}$ . In a similar way, we say that  $\mathcal{D}$  is path-connected if it is path-connected for the topology of the weak convergence.

We now give the three main results of the work.

**Theorem A.** Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M).

Thus, the set  $\mathcal{A}_\sigma$  is either a single element  $\mu^\sigma \in \mathcal{S}_\sigma$  either a path-connected subset of  $\mathcal{S}_\sigma$ . Moreover, for all  $\mu \in \mathcal{A}_\sigma$ , we have  $\Upsilon_\sigma(\mu) = L_\sigma := \lim_{t \rightarrow +\infty} \Upsilon_\sigma(\mu_t)$ .

From this theorem, we deduce the following statements.

**Corollary B.** Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M). We also assume that the set  $\mathcal{S}_\sigma \cap \Upsilon_\sigma^{-1}(\{\lambda\})$  is discrete, for any  $\lambda \in \mathbb{R}$ .

Thus, the probability measure  $\mu_t$  converges weakly to an invariant probability  $\mu^\sigma \in \mathcal{S}_\sigma$ , as  $t$  goes to infinity.

**Corollary C.** Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M). We also admit that

- we are in the synchronized case, that is  $G''(0) + \nabla^2 V(x) > 0$ , for all  $x \in \mathbb{R}^d$ .
- the confining potential  $V$  is a polynomial function with degree  $\deg(V) = 2m > 2n = \deg(F)$ .

By  $\mathcal{G} := \{x \in \mathbb{R}^d \mid \nabla V(x) = 0\}$ , we denote the set of the critical points of  $V$ . Thus, the following limit holds:

$$\limsup_{\sigma \rightarrow 0} \inf_{a \in \mathcal{G}} \limsup_{t \rightarrow +\infty} \int_{\mathbb{R}^d} \|x - a\|^2 \mu_t(dx) = 0.$$

# 1 Preliminaries

In this paragraph, we present the material and the notations that we use in the work.

**Definition 1.1.** *Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M). For all  $t \geq 0$ , we introduce the two functions*

$$\xi(t) := \Upsilon_\sigma(\mu_t) \quad \text{and} \quad \eta_t(x) := \frac{\sigma^2}{2} \frac{\nabla u_t(x)}{u_t(x)} + \nabla V(x) + \nabla F * u_t(x).$$

With these notations, the granular media equation (II) becomes

$$\frac{\partial}{\partial t} u_t = \operatorname{div}(\eta_t u_t).$$

Thus, if  $\eta_t$  is identically equal to 0 then  $\mu_t$  is an invariant probability of Diffusion (I). We remind the well known entropy dissipation.

**Proposition 1.2.** *Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M). Thus, for all  $t, s \geq 0$ , we have the inequality*

$$\xi(t+s) \leq \xi(t) \leq \xi(0) < +\infty.$$

Moreover,  $\xi$  is differentiable and it verifies

$$\xi'(t) \leq - \int_{\mathbb{R}^d} \|\eta_t(x)\|^2 u_t(x) dx.$$

See [CMV03] for a proof of this statement.

Under Assumption (M-7), the probability measure  $\mu_t$  is absolutely continuous with respect to the Lebesgue measure, for all  $t \geq 0$ . Consequently, the weak convergence of  $\mu_t$  is equivalent to a problem of convergence in a functional space.

**Definition 1.3.** *Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M). We define the functional space  $\mathcal{M}$  as the set of the functions  $f$  which satisfy the three following hypotheses.*

1. For all  $x \in \mathbb{R}^d$ ,  $f(x) > 0$ .
2. The integral of  $f$  on the whole space  $\mathbb{R}^d$  is equal to 1.
3. With  $M_0$  defined in Inequality (IV), we have  $\max_{1 \leq j \leq 8q^2} \int_{\mathbb{R}^d} \|x\|^j f(x) dx \leq M_0$ .



By reminding that  $u_t$  is the density of the measure  $\mu_t$  with respect to the Lebesgue measure, we remark that  $u_t \in \mathcal{M}$ , for all  $t \geq 0$ .

By proceeding like in Lemma 1.4 in [Tug10], we obtain  $\int_{\mathbb{R}^d} u(x) \log(u(x)) dx \geq C - \int_{\mathbb{R}^d} \|x\|^2 u(x) dx$ , where  $C$  is a constant. Consequently, for any measure  $\mu$  absolutely continuous with respect to the Lebesgue measure with density  $u \in \mathcal{M}$ , we have the lower-bound

$$\begin{aligned} \Upsilon_\sigma(\mu) &= \frac{\sigma^2}{2} \int_{\mathbb{R}^d} u(x) \log(u(x)) dx + \int_{\mathbb{R}^d} V(x) u(x) dx + \frac{1}{2} \int_{\mathbb{R}^d} F * u(x) u(x) dx \\ &\geq C - M_0 + \inf_{z \in \mathbb{R}^d} V(z) > -\infty. \end{aligned}$$

We skip the details. Due to this minoration and the monotonicity of  $\xi$ , we obtain the next lemma.

**Lemma 1.4.** *Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M). Thus, there exists  $L_\sigma \in \mathbb{R}$  such that  $\Upsilon_\sigma(\mu_t)$  converges to  $L_\sigma$  as the time  $t$  goes to infinity.*

We now remind Proposition 2.1 in [Tug11b].

**Proposition 1.5.** *Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M). Thus, there exist an invariant probability  $\mu^\sigma$  and a sequence  $(t_k)_{k \in \mathbb{N}}$  which converges to infinity such that  $\mu_{t_k}$  converges weakly to  $\mu^\sigma$ , as  $k$  goes to infinity.*

The particular form of the potential  $F$  allows us to write explicitly what is the function  $x \mapsto F * u(x)$  for all  $u \in \mathcal{M}$ . We remind Lemma 1.1 in [Tug11b].

**Lemma 1.6.** *Let us assume that the interacting potential  $F$  satisfy (M4)–(M5). Let  $\mu$  be a probability measure absolutely continuous with respect to the Lebesgue measure with density  $u \in \mathcal{M}$ . Thus, the quantity  $F * \mu(x) = F * u(x)$  is well defined and we have a simple development,*

$$F * u(x) = \sum_{k=1}^n \sum_{p_1=0}^k \sum_{p_2=0}^{k-p_1} \sum_{\tau \in \llbracket 1; d \rrbracket^{\llbracket 1; k-p_1-p_2 \rrbracket}} C_{k,p_1,p_2}^\tau(\mu) \|x\|^{2p_1} \nu_{k-p_1-p_2}^\tau(x),$$

$$\text{with } C_{k,p_1,p_2}^\tau(\mu) := \frac{G^{(2k)}(0)}{(2k)!} \frac{k!(-2)^{k-p_1-p_2}}{p_1!p_2!(k-p_1-p_2)!} \int_{\mathbb{R}^d} \|y\|^{2p_2} \nu_{k-p_1-p_2}^\tau(y) \mu(dy)$$

$$\text{and } \nu_l^\tau(x) := \prod_{i=1}^l x_{\tau(i)}, \quad \forall \tau \in \llbracket 1; d \rrbracket^{\llbracket 1; l \rrbracket}.$$

This lemma implies that any invariant probability  $\mu$  of Diffusion (I) is parametrised by a finite number of moments, that are quantities of the form  $\int_{\mathbb{R}^d} x_1^{l_1} \times \cdots \times x_d^{l_d} \mu(dx)$ , with  $(l_1, \dots, l_d) \in \mathbb{N}^d$ . Notations related to the moments are given subsequently. Let us first give an essential property on the invariant probabilities.

**Lemma 1.7.** *Let us assume that the confining potential  $V$  and the interacting potential  $F$  satisfy the set of assumptions (M).*

*Thus, an invariant probability of Diffusion (I) is uniquely determined by its moments. In other words, if a measure  $\nu$  has the same moments than a stationary measure  $\mu^\sigma$ , we deduce  $\nu = \mu^\sigma$ .*

*Proof.* Let  $\mu^\sigma$  be an invariant probability. By Lemma 2.1 in [Tug11b], it is absolutely continuous with respect to the Lebesgue measure and its density  $u^\sigma$  satisfies the implicit equation,

$$u^\sigma(x) = \frac{\exp\left[-\frac{2}{\sigma^2}(V(x) + F * u^\sigma(x))\right]}{\int_{\mathbb{R}^d} \exp\left[-\frac{2}{\sigma^2}(V(y) + F * u^\sigma(y))\right] dy}.$$

Consequently, for all  $r > 0$ , we have

$$\int_{\mathbb{R}^d} e^{\frac{2}{\sigma^2} r \|x\|} \mu^\sigma(dx) = \frac{\int_{\mathbb{R}^d} \exp\left[-\frac{2}{\sigma^2}(V(x) + F * u^\sigma(x) - r\|x\|)\right] dx}{\int_{\mathbb{R}^d} \exp\left[-\frac{2}{\sigma^2}(V(x) + F * u^\sigma(x))\right] dx}.$$

Since  $F$  is convex and since  $\nabla^2 V(x) > 0$  if  $\|x\|$  is sufficiently large, we deduce that  $\int_{\mathbb{R}^d} e^{\frac{2}{\sigma^2} r \|x\|} \mu^\sigma(dx) < +\infty$ . Consequently, the serie  $\sum_{k=0}^{\infty} \frac{\nu_k}{k!} x^k$ , with  $\nu_k := \int_{\mathbb{R}^d} \|x\|^k \mu^\sigma(dx)$ , has a positive convergence radius. After applying the Fourier criteria, the statement holds.  $\square$

From now on, we use the following notations, about the moments.

**Definition 1.8. 1.** *For all  $l := (l_1, \dots, l_d) \in \mathbb{N}^d$  and for all  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ , we denote*

$$x^l := \prod_{j=1}^d x_j^{l_j}.$$

**2.** *For all  $l \in \mathbb{N}^d$ , we introduce the linear function  $\lambda_l$  from  $\mathcal{M}$  (see Definition 1.3) to  $\mathbb{R}$ , defined by*

$$\lambda_l(u) := \int_{\mathbb{R}^d} x^l u(x) dx.$$

This linear function  $\lambda_l$  corresponds to the moment with order  $l$  of the measures.

**3.** *For all  $p \in \mathbb{N}^*$ , by  $\zeta(p)$ , we denote the subset of  $\mathbb{N}^d$  which contains the elements  $l := (l_1, \dots, l_d) \in \mathbb{N}^d$  such that  $\sum_{i=1}^d l_i \leq 2p$ .*

**4.** *For all  $p \geq n$ , by  $\Lambda_p$ , we denote the application from  $\mathcal{M}$  to  $\mathbb{R}^{\zeta(p)}$  defined by*

$$\Lambda_p(u) := (\lambda_l(u))_{l \in \zeta(p)}.$$

In other words, the linear function  $\Lambda_p$  corresponds to the vector of the moments of total order less than  $2p$ .

By rewriting Lemma 1.6 with the previous notations, we deduce the following statement.

**Lemma 1.9.** *Let us assume that the confining potential  $V$  and the interacting potential  $F$  satisfy (M1)–(M5).*

*Thus, there exists a unique family of linear functions  $\{\mathcal{X}_l; l \in \zeta(n)\}$  from  $\mathbb{R}^{\zeta(n)}$  to  $\mathbb{R}$  such that*

$$F * u(x) = \sum_{l \in \zeta(n)} \mathcal{X}_l(\Lambda_n(u)) x^l,$$

*for any function  $u \in \mathcal{M}$ .*

The implicit equation (see Lemma 2.1 in [Tug11b]) satisfied by any invariant probability  $\mu^\sigma$  with density  $u^\sigma$  implies

$$\lambda_l(u^\sigma) = \frac{\int_{\mathbb{R}^d} x^l \exp \left[ -\frac{2}{\sigma^2} \left( V(x) + \sum_{l' \in \zeta(n)} \mathcal{X}_{l'}(\Lambda_n(u^\sigma)) x^{l'} \right) \right] dx}{\int_{\mathbb{R}^d} \exp \left[ -\frac{2}{\sigma^2} \left( V(x) + \sum_{l' \in \zeta(n)} \mathcal{X}_{l'}(\Lambda_n(u^\sigma)) x^{l'} \right) \right] dx},$$

for any  $l \in \mathbb{N}^d$ . Consequently, any invariant probability is uniquely associated to a vector of  $\mathbb{R}^{\zeta(n)}$ .

**Definition 1.10. 1.** *For all  $p \geq n$ , for all  $m \in \mathbb{R}^{\zeta(p)}$  and for all  $l_0 \in \zeta(n)$ , we use the notation*

$$\mathcal{X}_{l_0}(m) := \mathcal{X}_{l_0} \left( (m_l)_{l \in \zeta(n)} \right).$$

*It is the natural extension of the function  $\mathcal{X}_l$  to the space  $\mathbb{R}^{\zeta(p)} \supset \mathbb{R}^{\zeta(n)}$ .*

**2.** *For all  $p \geq n$  and for all  $l \in \mathbb{N}^d$ , by  $\varphi_l$ , we denote the function from  $\mathbb{R}^{\zeta(p)}$  to  $\mathbb{R}$  defined by*

$$\varphi_l(m) := \frac{\int_{\mathbb{R}^d} x^l \exp \left[ -\frac{2}{\sigma^2} \left( V(x) + \sum_{l' \in \zeta(n)} \mathcal{X}_{l'}(m) x^{l'} \right) \right] dx}{\int_{\mathbb{R}^d} \exp \left[ -\frac{2}{\sigma^2} \left( V(x) + \sum_{l' \in \zeta(n)} \mathcal{X}_{l'}(m) x^{l'} \right) \right] dx} - m_l,$$

for all  $m := (m_l)_{l \in \zeta(p)} \in \mathbb{R}^{\zeta(p)}$ . The definition of  $\varphi_l$  is consistent, due to the first point of the definition. This is why we write  $\varphi_l$  instead of  $\varphi_{l,p}$ .

**3.** *For all  $p \geq n$ , by  $\Phi_p$ , we denote the application from  $\mathbb{R}^{\zeta(p)}$  to itself defined by*

$$\Phi_p(m) := (\varphi_l(m))_{l \in \zeta(p)}.$$

Consequently, a function  $u^\sigma \in \mathcal{M}$  is the density of an invariant probability  $\mu^\sigma$  if and only if it satisfies

$$\Phi_n(\Lambda_n(u^\sigma)) = 0.$$

Since the invariant probabilities are uniquely determined by their moments (according to Lemma 1.7), there is a bijection between  $\mathcal{S}_\sigma$  and the set of points  $m \in \mathbb{R}^{\zeta(n)}$  such that  $\Phi_n(m) = 0$ .

To finish the preliminaries, we give a natural definition.

**Definition 1.11.** *Let  $\mu$  be a probability measure on  $\mathbb{R}^d$  absolutely continuous with respect to the Lebesgue measure with density  $u \in \mathcal{M}$ . For all  $p \geq n$  and for all  $l \in \mathbb{N}^d$ , we denote  $\lambda_l(\mu) := \lambda_l(u)$  and  $\Lambda_p(\mu) := \Lambda_p(u)$ .*

## 2 First results

In this paragraph, we establish the important statements which are used in next section for proving Theorem A. We remind that  $\mathcal{A}_\sigma$  is the set of the adherence values of the family  $\{\mu_t; t \geq 0\}$  and  $\mathcal{S}_\sigma$  is the set of the invariant probabilities of Diffusion (I).

It has already been proved that the free-energy of an adherence value is less than the limit of the free-energy, see Proposition 2.5 in [Tug10]. We now prove that equality holds under some additional hypotheses.

**Proposition 2.1.** *Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M). We admit the existence of an increasing sequence  $(t_k)_{k \in \mathbb{N}}$  which goes to infinity such that  $\xi'(t_k)$  goes to 0 and  $\mu_{t_k}$  goes to  $\mu^\sigma \in \mathcal{A}_\sigma$  as  $k$  goes to infinity. Thus, the inequality  $\Upsilon_\sigma(\mu^\sigma) = L_\sigma := \lim_{t \rightarrow +\infty} \Upsilon_\sigma(\mu_t)$  holds.*

The proof is similar to the one of Proposition 1.8 in [Tug10].

*Proof.* From now on, by  $u_{t_k}$  (resp.  $u^\sigma$ ), we denote the density with respect to the Lebesgue measure of the probability measure  $\mu_{t_k}$  (resp.  $\mu^\sigma$ ). The convergence from the term  $\int_{\mathbb{R}^d} V(x)u_{t_k}(x)dx + \frac{1}{2} \int_{\mathbb{R}^d} (F * u_{t_k}(x)) u_{t_k}(x)dx$  toward  $\int_{\mathbb{R}^d} V(x)u^\sigma(x)dx + \frac{1}{2} \int_{\mathbb{R}^d} (F * u^\sigma(x)) u^\sigma(x)dx$  is implied by the convergence hypothesis of  $\mu_{t_k}$  to  $\mu^\sigma$ . Hence, we focus on the entropy term.

**Step 1.** *We here aim to prove the existence of a constant  $C > 0$  such that  $u_{t_k}(x) \leq C$  for all  $k \in \mathbb{N}^*$  and for all  $x \in \mathbb{R}^d$ .*

For doing so, we bound the integral of  $\|\nabla u_{t_k}\|$  on  $\mathbb{R}^d$ . The triangular inequality provides

$$\begin{aligned} \int_{\mathbb{R}^d} \|\nabla u_{t_k}(x)\| dx &\leq \frac{2}{\sigma^2} \int_{\mathbb{R}^d} \|\eta_{t_k}(x)\| u_{t_k}(x) dx \\ &\quad + \frac{2}{\sigma^2} \int_{\mathbb{R}^d} \|\nabla V(x) + \nabla F * u_{t_k}(x)\| u_{t_k}(x) dx, \end{aligned}$$

where  $t \mapsto \eta_t$  has been introduced in Definition 1.1. The growth property on  $\nabla V$  and  $\nabla F$  and Inequality (IV) yield

$$\int_{\mathbb{R}^d} \|\nabla V(x) + \nabla F * u_{t_k}(x)\| u_{t_k}(x) dx \leq C_1 \int_{\mathbb{R}^d} (1 + \|x\|^{2q}) u_{t_k}(x) dx \leq C_2,$$

where  $C_2 > 0$  is a constant. By using Cauchy-Schwarz inequality and the entropy dissipation, we obtain

$$\int_{\mathbb{R}^d} \|\eta_{t_k}(x)\| u_{t_k}(x) dx \leq \sqrt{\int_{\mathbb{R}^d} \|\eta_{t_k}(x)\|^2 u_{t_k}(x) dx} \leq \sqrt{-\xi'(t_k)}.$$

The quantity  $\sqrt{-\xi'(t_k)}$  tends to 0 as  $k$  goes to infinity so it is bounded. Finally, it leads to the existence of a positive constant  $C_3$  such that

$$\int_{\mathbb{R}^d} \|\nabla u_{t_k}(x)\| dx \leq C_3.$$

Consequently,  $u_{t_k}(x) \leq u_{t_k}(0) + C$  for all  $x \in \mathbb{R}^d$  and for all  $k \in \mathbb{N}$ . Nonetheless, the sequence  $(u_{t_k}(0))_{k \in \mathbb{N}}$  converges so it is bounded. Hence, there exists a constant  $C_4$  such that  $u_{t_k}(x) \leq C_4$  for all  $k \in \mathbb{N}$  and  $x \in \mathbb{R}^d$ .

**Step 2.** The application  $x \mapsto u_{t_k}(x) \log(u_{t_k}(x))$  is uniformly lower-bounded with respect to  $k$ . We can then apply the Lebesgue theorem which provides the convergence as  $k$  goes to infinity of the integral term

$$\int_{\mathbb{R}^d} u_{t_k}(x) \log(u_{t_k}(x)) \mathbb{1}_{\{|x| \leq R\}} dx$$

to  $\int_{\mathbb{R}^d} u^\sigma(x) \log(u^\sigma(x)) \mathbb{1}_{\{|x| \leq R\}} dx$  for any  $R \geq 0$ .

**Step 3.** The other integral is split into three terms.

**Step 3.1.** The first one is

$$I_1(k) := \int_{\mathbb{R}^d} u_{t_k}(x) \log(u_{t_k}(x)) \mathbb{1}_{\{|x| > R; u_{t_k}(x) \geq 1\}} dx.$$

Due to Step 1, Inequality (IV) and Markov inequality, we obtain

$$I_1(k) \leq \log(C) \mu_{t_k}(\mathbb{B}(0; R)^c) \leq \frac{\log(C) M_0}{R^2}.$$

**Step 3.2.** The second term is defined as

$$I_2(k) := \int_{\mathbb{R}^d} u_{t_k}(x) \log(u_{t_k}(x)) \mathbb{1}_{\{|x| > R; u_{t_k}(x) < 1; u_{t_k}(x) \geq e^{-|x|}\}} dx < 0.$$

We bound it in the following way:

$$|I_2(k)| = -I_2(k) \leq \int_{\mathbb{B}(0; R)^c} |x| u_{t_k}(x) dx \leq \frac{M_0}{R^2}.$$

**Step 3.3.** We now look at the third term,

$$I_3(k) := \int_{\mathbb{R}^d} u_{t_k}(x) \log(u_{t_k}(x)) \mathbb{1}_{\{|x| > R; u_{t_k}(x) < 1; u_{t_k}(x) < e^{-|x|}\}} dx < 0.$$

We introduce the function  $\gamma(x) := \sqrt{x} \log(x) \mathbb{1}_{\{x < 1\}}$ . We have  $-C \leq \gamma(x) \leq 0$  for all  $x \in \mathbb{R}$ , where  $C$  is a positive constant. This provides

$$|I_3(k)| = -I_3(k) \leq - \int_{\mathbb{B}(0; R)^c} \gamma(u_{t_k}(x)) e^{-\frac{1}{2}|x|} dx \leq \Theta(R),$$

$\Theta$  being a decreasing function from  $\mathbb{R}_+$  to itself such that  $\lim_{R \rightarrow +\infty} \Theta(R) = 0$ .

**Step 4.** Let  $\epsilon$  be a positive real, arbitrarily small. By taking  $R$  sufficiently large, we have the upper-bound

$$\sup_{k \in \mathbb{N}^*} \max \{I_1(k); I_2(k); I_3(k)\} < \frac{\epsilon}{9}.$$

Moreover, for  $R$  large enough, we have the inequality

$$\int_{\mathbb{R}^d} u^\sigma(x) \log(u^\sigma(x)) \mathbb{1}_{\{|x| \geq R\}} dx < \frac{\epsilon}{3}.$$

The two previous inequalities do not depend on  $k$ . Then, by taking  $k$  sufficiently large, the following upper-bound holds:

$$\left| \int_{\mathbb{R}^d} u_{t_k}(x) \log(u_{t_k}(x)) \mathbb{1}_{\{|x| \leq R\}} dx - \int_{\mathbb{R}^d} u^\sigma(x) \log(u^\sigma(x)) \mathbb{1}_{\{|x| \leq R\}} dx \right| < \frac{\epsilon}{3}.$$

This implies the inequality  $|\Upsilon_\sigma(\mu_{t_k}) - \Upsilon_\sigma(\mu^\sigma)| < \epsilon$ . In other words, the free-energy  $\Upsilon_\sigma(\mu_{t_k})$  converges to  $\Upsilon_\sigma(\mu^\sigma)$ . Due to the monotonicity of the free-energy along the trajectories of Equation (II),  $\Upsilon_\sigma(\mu_t)$  converges to  $L_\sigma$  which implies  $\Upsilon_\sigma(\mu^\sigma) = L_\sigma$ .  $\square$

Let us stress that the equality  $\Upsilon_\sigma(u_\sigma) = L_\sigma$  does not *a priori* hold without the hypothesis  $\lim_{k \rightarrow +\infty} \xi'(t_k) = 0$ . It is used for proving the inequality  $I_1(k) \leq \frac{\log(C)M_0}{R^2}$ , see Step 3.1 and Step 1.

In the following, we show that  $\mathcal{A}_\sigma$ , the set of the adherence values is included into  $\mathcal{S}_\sigma$ , the set of the invariant probabilities. The method that we use requires the following statement. *For each probability measure  $\mu$  on  $\mathbb{R}^d$  with  $\mu \notin \mathcal{S}_\sigma$ , there exist  $p \geq n$  and an hypercube  $\mathcal{C}$  in  $\mathbb{R}^{\zeta(p)}$  which contains  $\Lambda_p(\mu)$  and which has empty intersection with the set  $\Lambda_p(\mathcal{S}_\sigma)$ .*

**Proposition 2.2.** *Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M). Let  $\nu$  be a measure with density  $u \in \mathcal{M}$ . We also assume  $\nu \notin \mathcal{S}_\sigma$ . Thus, there exist  $p \in \mathbb{N}^*$  and  $\rho > 0$  such that*

$$\inf_{\mu \in \mathcal{S}_\sigma} \max_{l \in \zeta(p)} |\lambda_l(\mu) - \lambda_l(\nu)| \geq \rho.$$

*Proof.* According to Proposition A.2, for all  $N_0 \in \mathbb{N}^*$  and for all  $t > 0$ , we have

$$\int_{\mathbb{R}^d} \|x\|^{2N_0} \mu_t(dx) < +\infty.$$

Moreover,  $\sup_{t \geq 1} \int_{\mathbb{R}^d} \|x\|^{2N_0} \mu_t(dx) < +\infty$ . Consequently, we can define the integral term  $\int_{\mathbb{R}^d} x^l \mu_t(dx)$ , for any  $l \in \zeta(p)$ , for all  $p \geq n$  and for all  $t > 0$ .

Let us proceed a *reductio ad absurdum* by assuming that for all  $k \geq n$ , there exists  $\mu_k^\sigma \in \mathcal{S}_\sigma$  such that

$$\max_{l \in \zeta(k)} |\lambda_l(\mu_k^\sigma) - \lambda_l(\nu)| \leq \frac{1}{k}.$$

We deduce that the sequence  $(\lambda_l(\mu_k^\sigma))_{k \in \mathbb{N}}$  is bounded for all  $l \in \zeta(n)$ . In other words, the sequence  $(\Lambda_n(\mu_k^\sigma))_{k \in \mathbb{N}}$  is bounded in the space  $\mathbb{R}^{\zeta(n)}$ . Consequently,

we can extract a subsequence  $(\phi(k))_{k \in \mathbb{N}}$  such that the sequence  $\left(\Lambda_n \left(\mu_{\phi(k)}^\sigma\right)\right)_{k \in \mathbb{N}}$

converges to a vector  $m_\infty^\sigma \in \mathbb{R}^{\zeta(n)}$ .

Since the family  $\{\mu_t; t \geq 0\}$  is tight, Prokhorov theorem allows us to assume, without any loss of generality, that the measure  $\mu_{\phi(k)}^\sigma$  converges weakly to a measure  $\mu_\infty^\sigma$  as  $k$  goes to infinity.

However,  $\Phi_n(\Lambda_n(\mathcal{S}_\sigma)) = \{0\}$ . The set  $\Lambda_n(\mathcal{S}_\sigma)$  thus is closed. It implies that  $\Phi_n(m_\infty^\sigma) = 0$ . Hence, the measure  $\mu_\infty^\sigma$  satisfies  $\Phi_n(\Lambda_n(\mu_\infty^\sigma)) = 0$ . Consequently,  $\mu_\infty^\sigma$  is an invariant probability of Diffusion (I).

Nevertheless, we have

$$|\lambda_l(\mu_\infty^\sigma) - \lambda_l(\nu)| = 0$$

for all  $l \in \zeta(p)$  and  $p \geq n$ . In other words, the invariant probability  $\mu_\infty^\sigma$  has the same moments than  $\nu$ . Since  $\mu_\infty^\sigma$  is uniquely determined by its moments, we deduce that  $\nu = \mu_\infty^\sigma$  so  $\nu$  is an invariant probability. This is an absurdity.  $\square$

Let us point out that in the previous proof, the equality  $|\lambda_l(\mu_\infty^\sigma) - \lambda_l(\nu)| = 0$  must be verified for all  $l \in \mathbb{N}^d$  and not only for the elements  $l$  in  $\zeta(n)$ .

We remind that we do not require  $\mathcal{S}_\sigma$  to be discrete in the current work. Let us give an example of a confining potential  $V$  and an interacting potential  $F$  which satisfy Assumptions (M1)–(M5) such that  $\mathcal{S}_\sigma$ , the set of the invariant probabilities, is not discrete.

**Proposition 2.3.** *Let  $d$  be an integer at least equal to 2. For all  $x \in \mathbb{R}^d$ , we take  $V(x) := \frac{r(x)^4}{4} - \frac{r(x)^2}{2}$  and  $F(x) := \frac{\alpha}{2} r(x)^2$ , with  $\alpha > 0$  and  $r(x) := \sqrt{\sum_{i=1}^d x_i^2}$ . Thus, for  $\sigma$  sufficiently small, there exists a path-connected set of invariant probabilities which is not a single element.*

*Proof.* First of all, for any radial measure  $\mu$ , we have  $\int_{\mathbb{R}^d} x \mu(dx) = 0$ . Consequently, for all  $\sigma > 0$ , there is exactly one radial invariant probability, that is

$$\mu_0^\sigma(dx) := \frac{\exp\left[-\frac{2}{\sigma^2} \left(V(x) + \frac{\alpha}{2} \sum_{i=1}^d x_i^2\right)\right]}{\exp\left[-\frac{2}{\sigma^2} \left(V(y) + \frac{\alpha}{2} \sum_{i=1}^d y_i^2\right)\right]} dy.$$

By a simple computation, we can prove that  $\Upsilon_\sigma(\mu_0^\sigma)$  goes to  $-\frac{(\max\{1-\alpha; 0\})^2}{4}$  as  $\sigma$  goes to 0. Also, we can prove that the free-energy of the non-radial measure,

$$\nu := \frac{\exp\left[-\frac{2}{\sigma^2} \left(V(x) + \frac{\alpha}{2} \sum_{i=1}^d x_i^2 - \alpha x_1\right)\right]}{\int_{\mathbb{R}^d} \exp\left[-\frac{2}{\sigma^2} \left(V(y) + \frac{\alpha}{2} \sum_{i=1}^d y_i^2 - \alpha y_1\right)\right] dy} dx,$$

goes to  $-\frac{1}{4} < -\frac{(\max\{1-\alpha; 0\})^2}{4}$  as  $\sigma$  goes to 0. By considering Diffusion (I) with  $\mu_0 := \nu$ , we deduce, after applying the entropy dissipation (Proposition 1.2) and Proposition 1.5, that there exists an invariant probability  $\nu^\sigma$  such that  $\Upsilon_\sigma(\nu^\sigma) \leq \Upsilon_\sigma(\nu) < \Upsilon_\sigma(\mu_0^\sigma)$  so  $\nu^\sigma$  is not radial. We then introduce

$$m^\sigma := \int_{\mathbb{R}^d} x \nu^\sigma(dx).$$

By definition, we have

$$m^\sigma = \frac{\int_{\mathbb{R}^d} x \exp \left[ -\frac{2}{\sigma^2} \left( V(x) + \frac{\alpha}{2} \sum_{i=1}^d x_i^2 - \alpha \langle m^\sigma ; x \rangle \right) \right] dx}{\int_{\mathbb{R}^d} \exp \left[ -\frac{2}{\sigma^2} \left( V(x) + \frac{\alpha}{2} \sum_{i=1}^d x_i^2 - \alpha \langle m^\sigma ; x \rangle \right) \right] dx}.$$

For all  $\Theta = (\vartheta_1, \dots, \vartheta_{d-1}) \in [0; 2\pi]^{d-1}$ , we consider the vector  $m_\Theta^\sigma \in \mathbb{R}^d$  which  $i$ th coordinate equal to

$$m_\Theta^\sigma(i) := \|m^\sigma\|^2 \prod_{j=1}^{i-1} \sin(\vartheta_j) \times \cos(\vartheta_i),$$

with the convention  $\vartheta_d := 0$ . Both functions  $F$  and  $V$  are radials. Consequently, for all  $\Theta \in [0; 2\pi]^{d-1}$ , we have the equality

$$m_\Theta^\sigma = \frac{\int_{\mathbb{R}^d} x \exp \left[ -\frac{2}{\sigma^2} \left( V(x) + \frac{\alpha}{2} \sum_{i=1}^d x_i^2 - \alpha \langle m_\Theta^\sigma ; x \rangle \right) \right] dx}{\int_{\mathbb{R}^d} \exp \left[ -\frac{2}{\sigma^2} \left( V(x) + \frac{\alpha}{2} \sum_{i=1}^d x_i^2 - \alpha \langle m_\Theta^\sigma ; x \rangle \right) \right] dx}.$$

By  $\nu_\Theta^\sigma$ , we denote the invariant probability associated to this first moment,

$$\nu_\Theta^\sigma(dx) = \frac{\exp \left[ -\frac{2}{\sigma^2} \left( V(x) + \frac{\alpha}{2} \sum_{i=1}^d x_i^2 - \alpha \langle m_\Theta^\sigma ; x \rangle \right) \right]}{\int_{\mathbb{R}^d} \exp \left[ -\frac{2}{\sigma^2} \left( V(y) + \frac{\alpha}{2} \sum_{i=1}^d y_i^2 - \alpha \langle m_\Theta^\sigma ; y \rangle \right) \right] dy} dx.$$

Hence, the path-connected set  $\{\nu_\Theta^\sigma ; \Theta \in [0; 2\pi]^{d-1}\}$  is not a single element (provided that  $d - 1 \geq 1$ ) and, by construction, it is included into  $\mathcal{S}_\sigma$ .  $\square$

### 3 Proof of Theorem A

The aim of this paragraph is to prove Theorem A. We remind that  $\mathcal{A}_\sigma$  and  $\mathcal{S}_\sigma$  respectively are the set of the limiting values of the family  $\{\mu_t ; t \geq 0\}$  and the set of the invariant probabilities of Diffusion (I).

**Theorem A.** *Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M).*

*Thus, the set  $\mathcal{A}_\sigma$  is either a single element  $\mu^\sigma \in \mathcal{S}_\sigma$  either a path-connected subset of  $\mathcal{S}_\sigma$ . Moreover, for all  $\mu^\sigma \in \mathcal{A}_\sigma$ ,  $\Upsilon_\sigma(\mu^\sigma) = L_\sigma := \lim_{t \rightarrow +\infty} \Upsilon_\sigma(\mu_t)$ .*

#### 3.1 Outline of the proof

We here provide the ideas of the proof. The details are postponed in next subsections. By Proposition 1.5, we know that there exists a probability measure  $\mu_0^\sigma$  such that

- the measure  $\mu_0^\sigma$  is in  $\mathcal{A}_\sigma$ .



- the measure  $\mu_0^\sigma$  is invariant for Diffusion (I).
- the free-energy of  $\mu_0^\sigma$  is equal to  $L_\sigma := \lim_{t \rightarrow 0} \Upsilon_\sigma$ .

If  $\mathcal{A}_\sigma = \{\mu_0^\sigma\}$ , the proof is achieved. We assume from now on that  $\#\mathcal{A}_\sigma > 1$ . The proof now consists in establishing that

1. the set  $\mathcal{A}_\sigma$  is included into  $\mathcal{S}_\sigma$ .
2. the set  $\mathcal{A}_\sigma$  is path-connected.
3. the free-energy is constant on  $\mathcal{A}_\sigma$ .

**Step 1.** *We proceed a reductio ad absurdum in order to prove the first statement. The details are in Subsection 3.2.*

We assume the existence of  $\nu \in \mathcal{A}_\sigma$  such that  $\nu \notin \mathcal{S}_\sigma$ . According to Proposition 2.2, there exists a closed set with non-empty interior  $\mathcal{H}$  which contains  $\nu$  and which has an empty intersection with  $\mathcal{S}_\sigma$ .

Since  $\nu$  is an adherence value, we can prove the existence of a smooth function with compact support  $\phi$ , a constant  $\rho > 0$  and two increasing sequences  $(r_k)_{k \in \mathbb{N}}$  and  $(s_k)_{k \in \mathbb{N}}$  such that  $r_k < s_k$  and for all  $t \in [r_k; s_k]$ , we have

$$\rho = \int_{\mathbb{R}^d} \phi(x) \mu_{r_k}(dx) \leq \int_{\mathbb{R}^d} \phi(x) \mu_t(dx) \leq \int_{\mathbb{R}^d} \phi(x) \mu_{s_k}(dx) = 2\rho.$$

Moreover, for all  $t \in [r_k; s_k]$ ,  $\mu_t \in \mathcal{H}$ . By applying Proposition A.1, we construct an invariant probability  $\nu^\sigma \in \mathcal{S}_\sigma \cap \mathcal{H}$ . This is absurd.

**Step 2.** *The details of the proof of the second statement are in Subsection 3.3.*

We use the previous result: all the limiting values are invariant probabilities. Since the function  $\Lambda_n$  is a bijection from  $\mathcal{S}_\sigma$  to  $\Lambda_n(\mathcal{S}_\sigma) \subset \mathbb{R}^{\zeta(n)}$ , we deduce that  $\Lambda_n$  is a bijection from  $\mathcal{A}_\sigma$  to  $\mathcal{C}_\sigma := \Lambda_n(\mathcal{A}_\sigma) \subset \mathbb{R}^{\zeta(n)}$ .

Due to the continuity of the functions  $t \mapsto \int_{\mathbb{R}^d} x^l \mu_t(dx)$ , we deduce that  $\mathcal{C}_\sigma$  is path-connected. This implies that  $\mathcal{A}_\sigma$  also is path-connected, according to Lemma 1.9.

**Step 3.** *The proof of the third point is made in Subsection 3.4.*

We proceed a *reductio ad absurdum* by assuming the existence of  $\nu \in \mathcal{A}_\sigma \subset \mathcal{S}_\sigma$  such that  $\Upsilon_\sigma(\nu) \neq L_\sigma$ . We remind that  $\Lambda_n$  is a bijection from  $\mathcal{A}_\sigma$  to  $\mathcal{C}_\sigma \subset \mathbb{R}^{\zeta(n)}$ . Due to the continuity of  $\Lambda_n$  and  $\Upsilon_\sigma$ , there exists a closed set  $\mathcal{D}_\sigma \subset \mathbb{R}^{\zeta(n)}$  which contains the point  $\Lambda_n(\nu)$  in its interior and such that  $\Upsilon_\sigma(\mu) \neq L_\sigma$  for all the probability measures  $\mu$  satisfying  $\Lambda_n(\mu) \in \mathcal{D}_\sigma$ .

We now use similar arguments than the ones in Step 1. There exist a smooth function with compact support  $\phi$ , a constant  $\rho > 0$  and two increasing sequences  $(r_k)_{k \in \mathbb{N}}$  and  $(s_k)_{k \in \mathbb{N}}$  such that  $r_k < s_k$  and for all  $t \in [r_k; s_k]$ , we have

$$\rho = \int_{\mathbb{R}^d} \phi(x) \mu_{r_k}(dx) \leq \int_{\mathbb{R}^d} \phi(x) \mu_t(dx) \leq \int_{\mathbb{R}^d} \phi(x) \mu_{s_k}(dx) = 2\rho.$$

Moreover, for all  $t \in [r_k; s_k]$ ,  $\Lambda_n(\mu_t) \in \mathcal{D}_\sigma$ . By applying Proposition A.1, we construct an invariant probability  $\nu^\sigma \in \mathcal{S}_\sigma$  with free-energy equal to  $L_\sigma := \lim_{t \rightarrow +\infty} \Upsilon_\sigma(\mu_t)$  and such that  $\Lambda_n(\nu^\sigma) \in \mathcal{D}_\sigma$ . By construction of the set  $\mathcal{D}_\sigma$ , this is impossible.

### 3.2 The set $\mathcal{A}_\sigma$ is included into $\mathcal{S}_\sigma$

We proceed a *reductio ad absurdum* by assuming the existence of a measure  $\nu \in \mathcal{A}_\sigma$  such that  $\nu$  is not an invariant probability. According to Proposition 2.2, there exist  $p \in \mathbb{N}^*$  and  $\rho > 0$  such that

$$\inf_{\mu \in \mathcal{S}_\sigma} \max_{l \in \zeta(p)} |\lambda_l(\mu) - \lambda_l(\nu)| \geq \rho.$$

Let us define  $\mathcal{H}_\kappa^p \subset \mathbb{R}^{\zeta(p)}$  the hypercube of center  $\Lambda_p(\nu)$  with radius equal to  $\kappa > 0$ ,

$$\mathcal{H}_\kappa^p := \left\{ \mathcal{X} \in \mathbb{R}^{\zeta(p)} \mid \max_{l \in \zeta(p)} |\mathcal{X}_l - \lambda_l(\nu)| \leq \kappa \right\}.$$

It is a closed bounded set with non-empty interior. Let us define  $\mathcal{H}_\kappa := \Lambda_p^{-1}(\{\mathcal{H}_\kappa^p\})$ . By construction,  $\nu \in \mathcal{H}_{\frac{\rho}{4}}$  and  $\mathcal{H}_{\frac{\rho}{2}} \cap \mathcal{S}_\sigma = \emptyset$ . Particularly,  $\mu_0^\sigma \notin \mathcal{H}_{\frac{\rho}{2}}$  where  $\mu_0^\sigma$  has been introduced in Subsection 3.1.

Nonetheless,  $\mu_0^\sigma$  and  $\nu$  are adherence values of the family  $\{\mu_t; t \geq 0\}$ . Consequently, there exist two increasing sequences  $(r_k)_{k \in \mathbb{N}}$  and  $(s_k)_{k \in \mathbb{N}}$  such that for all  $k \in \mathbb{N}$ ,  $\mu_{r_k} \in \partial \mathcal{H}_{\frac{\rho}{4}}$ ,  $\mu_{s_k} \in \partial \mathcal{H}_{\frac{\rho}{2}}$  and for all  $r_k < t < s_k$ , we have

$$\frac{\rho}{4} < |\lambda_l(\mu_t) - \lambda_l(\nu)| < \frac{\rho}{2}, \quad \text{for any } l \in \zeta(p).$$

By construction of the sequences  $(r_k)_{k \in \mathbb{N}}$  and  $(s_k)_{k \in \mathbb{N}}$ , there exists  $l(k) \in \zeta(p)$  such that  $\lambda_{l(k)}(\mu_{r_k}) = \lambda_{l(k)}(\nu) + \vartheta_k^1 \frac{\rho}{4}$  and  $\lambda_{l(k)}(\mu_{s_k}) = \lambda_{l(k)}(\nu) + \vartheta_k^2 \frac{\rho}{2}$  where  $\vartheta_k^1 := \pm 1$  and  $\vartheta_k^2 \in [-1; 1]$ .

Since  $\#\zeta(p) < +\infty$ , we can extract two subsequences (we continue to write  $r_k$  and  $s_k$  for the comfort of the reading) such that there exist  $l_0 \in \zeta(p)$  and  $\vartheta \in \{-1; 1\}$ , all independent from the index  $k$ , which satisfy

$$\vartheta \lambda_{l_0}(\mu_{r_k}) = \vartheta \lambda_{l_0}(\nu) + \frac{\rho}{4}, \quad \vartheta \lambda_{l_0}(\mu_{s_k}) = \vartheta \lambda_{l_0}(\nu) + \frac{\rho}{2}$$

and for all  $t \in [r_k; s_k]$ , we have

$$\vartheta \lambda_{l_0}(\nu) + \frac{\rho}{4} \leq \vartheta \lambda_{l_0}(\mu_t) \leq \vartheta \lambda_{l_0}(\nu) + \frac{\rho}{2}.$$

Moreover, for all  $t \in [r_k; s_k]$ ,  $\mu_t \in \mathcal{H}_{\frac{\rho}{2}}$ . Without any loss of generality, we assume that  $\vartheta = 1$ .

We apply Proposition A.1 with a smooth function with compact support which is equal to  $x^{l_0}$  if  $\|x\| \leq R$  and equal to 0 if  $\|x\| \geq R + 1$ .

By taking  $R$  sufficiently large, we deduce the existence of a measure  $\nu^\sigma \in \mathcal{H}_{\frac{\rho}{2}} \cap \mathcal{A}_\sigma \cap \mathcal{S}_\sigma$ . However,  $\mathcal{S}_\sigma \cap \mathcal{H}_{\frac{\rho}{2}} = \emptyset$ . This is an absurdity.

### 3.3 The set $\mathcal{A}_\sigma$ is path-connected

According to the previous paragraph, the set  $\mathcal{A}_\sigma$  is included into  $\mathcal{S}_\sigma$ , the set of the invariant probabilities. We now consider the application  $\Lambda_n$  from the set of

the probability measures to  $\mathbb{R}^{\zeta(n)}$ .

By  $\mathcal{C}_\sigma$ , we denote the set  $\Lambda_n(\mathcal{A}_\sigma)$ . Due to the continuity of the application  $t \mapsto \Lambda_n(\mu_t)$ , we deduce that the set of the limiting values of the family  $\{\Lambda_n(\mu_t) ; t \geq 0\}$  is path-connected. In other words, the set  $\mathcal{C}_\sigma$  is path-connected.

Consequently, for all the measures  $\mu_0$  and  $\mu_1$  in  $\mathcal{A}_\sigma$ , there exists an application from  $[0; 1]$  to the set of the probability measures,  $\vartheta \mapsto \mu_\vartheta$  such that for all  $l \in \zeta(n)$ , the functions  $\vartheta \mapsto \lambda_l(\mu_\vartheta)$  are continuous.

According to Lemma 1.9, we deduce that for all  $x \in \mathbb{R}^d$ , the function  $\vartheta \mapsto V(x) + F * \mu_\vartheta(x)$  is continuous. We know that for all  $\vartheta \in [0; 1]$ , the measure  $\mu_\vartheta$  is absolutely continuous with respect to the Lebesgue measure with a density  $u_\vartheta$  satisfying

$$u_\vartheta(x) = \frac{\exp \left[ -\frac{2}{\sigma^2} (V(x) + F * u_\vartheta(x)) \right]}{\int_{\mathbb{R}^d} \exp \left[ -\frac{2}{\sigma^2} (V(y) + F * u_\vartheta(y)) \right] dy}.$$

Hence, for any smooth function with compact support  $\phi$ , the application  $\vartheta \mapsto \int_{\mathbb{R}^d} \phi(x) \mu_\vartheta(dx)$  is continuous. In other words, the set  $\mathcal{A}_\sigma$  is path-connected.

### 3.4 The free-energy is constant on $\mathcal{A}_\sigma$

We introduce  $L_\sigma := \lim_{t \rightarrow +\infty} \Upsilon_\sigma(\mu_t)$ . By Proposition 2.5 in [Tug10] (which can easily be adapted to the general dimensional case), we know that for any  $\mu \in \mathcal{A}_\sigma$ , we have the inequality  $\Upsilon_\sigma(\mu) \leq L_\sigma$ . In other words, the free-energy of a limiting value is less than  $L_\sigma$ , the limit of the free-energy.

Let us proceed a *reductio ad absurdum* by assuming that there exist  $\nu \in \mathcal{A}_\sigma$  and  $\eta > 0$  satisfying  $\Upsilon_\sigma(\nu) = L_\sigma - 2\eta$ . We use similar arguments than the ones of the previous subsections.

Due to the continuity of  $\Lambda_n$  and to Lemma 1.9, we deduce that there exists  $\rho > 0$  sufficiently small such that  $\Upsilon_\sigma(\mu) \leq L_\sigma - \eta$  for all  $\mu \in \mathcal{S}_\sigma \cap \mathcal{H}_\rho$  where  $\mathcal{H}_\rho$  is defined like in Subsection 3.2, as the set of the probability measures  $\mu$  such that  $|\lambda_l(\mu) - \lambda_l(\nu)| \leq \rho$ , for all  $l \in \zeta(n)$ .

Nevertheless, the measure  $\mu_0^\sigma \in \mathcal{A}_\sigma$  introduced in Subsection 3.1 satisfies the equality  $\Upsilon_\sigma(\mu_0^\sigma) = L_\sigma$ . Hence,  $\mu_0^\sigma \notin \mathcal{H}_\rho$ .

We now proceed exactly like in Subsection 3.2. We obtain the existence of two increasing sequences  $(r_k)_{k \in \mathbb{N}}$  and  $(s_k)_{k \in \mathbb{N}}$ , of  $l_0 \in \zeta(n)$  and  $\vartheta \in \{-1; 1\}$ , all independent from the index  $k$ , which satisfy

$$\vartheta \lambda_{l_0}(\mu_{r_k}) = \vartheta \lambda_{l_0}(\nu) + \frac{\rho}{4}, \quad \vartheta \lambda_{l_0}(\mu_{s_k}) = \vartheta \lambda_{l_0}(\nu) + \frac{\rho}{2}$$

and for all  $t \in [r_k; s_k]$ , we have

$$\vartheta \lambda_{l_0}(\nu) + \frac{\rho}{4} \leq \vartheta \lambda_{l_0}(\mu_t) \leq \vartheta \lambda_{l_0}(\nu) + \frac{\rho}{2}.$$

Moreover, for all  $t \in [r_k; s_k]$ ,  $\mu_t \in \mathcal{H}_\rho$ . Without any loss of generality, we assume that  $\vartheta = 1$ . We apply Proposition A.1 with a smooth function with compact

support equal to  $x^{l_0}$  if  $\|x\| \leq R$  and equal to 0 if  $\|x\| \geq R+1$ . By taking  $R$  sufficiently large, we deduce the existence of a measure  $\nu^\sigma \in \mathcal{H}_{\frac{\varepsilon}{2}} \cap \mathcal{A}_\sigma$  such that  $\Upsilon_\sigma(\nu^\sigma) = L_\sigma$ . Nonetheless, by construction of  $\mathcal{H}_{\frac{\varepsilon}{2}}$ , we have  $\Upsilon_\sigma(\nu^\sigma) \leq L_\sigma - \eta$ . This is an absurdity.

## 4 Proofs of the Corollaries

We here provide the proofs of the two corollaries. We first remind Corollary B.

**Corollary B.** *Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M). We also assume that the set  $\mathcal{S}_\sigma \cap \Upsilon_\sigma^{-1}(\{\lambda\})$  is discrete, for all  $\lambda \in \mathbb{R}$ .*

*Thus, the measure  $\mu_t$  converges weakly to an invariant probability  $\mu^\sigma \in \mathcal{S}_\sigma$  as  $t$  goes to infinity.*

*Proof.* According to Theorem A, the set of the limiting values  $\mathcal{A}_\sigma$  is a path-connected subset of  $\mathcal{S}_\sigma$  in which the free-energy is constant. Due to the hypothesis of Corollary B, any path-connected subset of  $\mathcal{S}_\sigma$  with constant free-energy is a single element. This achieves the proof.  $\square$

We now present a result which holds in the small-noise limit.

**Corollary C.** *Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M). We also admit that*

- *we are in the synchronized case, that is  $G''(0) + \nabla^2 V(x) > 0$ , for all  $x \in \mathbb{R}^d$ .*
- *the confining potential  $V$  is a polynomial function with degree  $\deg(V) = 2m > 2n = \deg(F)$ .*

By  $\mathcal{G} := \{x \in \mathbb{R}^d \mid \nabla V(x) = 0\}$ , we denote the set of the critical points of  $V$ . Thus,

$$\limsup_{\sigma \rightarrow 0} \inf_{a \in \mathcal{G}} \limsup_{t \rightarrow +\infty} \int_{\mathbb{R}^d} \|x - a\|^2 \mu_t(dx) = 0.$$

*Proof. Step 1.* First of all, we introduce  $\mathcal{S}_0$ , the set of the probability measures  $\mu^0$  such that there exist a decreasing sequence  $(\sigma_k)_{k \in \mathbb{N}}$  which goes to 0 and a sequence  $(\mu^{\sigma_k})_{k \in \mathbb{N}}$  of invariant probabilities of Diffusion (I) which converges weakly to  $\mu^0$ , as  $k$  goes to infinity.

**Step 2.** By proceeding exactly like in [Tug11b] (Proposition 3.10, Lemma 3.3, Theorem 3.7 and Proposition 3.8 hold under the set of assumptions (M)), we obtain that the set  $\mathcal{S}_0$  is included into  $\{\delta_a; a \in \mathcal{G}\}$ .

**Step 3.** We deduce that for any  $\rho > 0$ , there exists  $\sigma_0 > 0$  sufficiently small such that for all  $\sigma < \sigma_0$  and for all  $\nu \in \mathcal{S}_\sigma$ , we have  $\min_{a \in \mathcal{G}} \int_{\mathbb{R}^d} \|x - a\|^2 \nu(dx) \leq \rho$ .

**Step 4.** Theorem A tells us that  $\mathcal{A}_\sigma \subset \mathcal{S}_\sigma$ . This achieves the proof.  $\square$

Let us remark that we do not have *a priori* the existence of  $a \in \mathcal{G}$  such that

$$\limsup_{\sigma \rightarrow 0} \limsup_{t \rightarrow +\infty} \int_{\mathbb{R}^d} \|x - a\|^2 \mu_t(dx) = 0.$$

Indeed, the basins of attraction *a priori* are not independent of  $\sigma > 0$ .

## A Useful technical results

We here present the proposition which is used several times in the proof of Theorem A for constructing elements of  $\mathcal{S}_\sigma$  satisfying two contradictory hypotheses.

**Proposition A.1.** *Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M). We also assume the existence of two polynomial functions  $\mathcal{P}$  and  $\mathcal{Q}$ , a smooth function  $\phi$  from  $\mathbb{R}^d$  to  $\mathbb{R}$  with compact support such that  $|\phi(x)| \leq \mathcal{P}(\|x\|)$  and  $\|\nabla \phi(x)\|^2 \leq \mathcal{Q}(\|x\|)$ ,  $\kappa > 0$  and two sequences  $(r_k)_{k \in \mathbb{N}}$  and  $(s_k)_{k \in \mathbb{N}}$  which go to infinity such that for all  $r_k \leq t \leq s_k < r_{k+1}$ , we have*

$$\kappa = \int_{\mathbb{R}^d} \phi(x) u_{r_k}(x) dx \leq \int_{\mathbb{R}^d} \phi(x) u_t(x) dx \leq \int_{\mathbb{R}^d} \phi(x) u_{s_k}(x) dx = 2\kappa.$$

*Thus, there exists  $\nu^\sigma \in \mathcal{A}_\sigma \cap \mathcal{S}_\sigma$  which density  $v^\sigma$  with respect to the Lebesgue measure verifies  $\int_{\mathbb{R}^d} \phi(x) v^\sigma(x) dx \in [\kappa; 2\kappa]$ . Moreover,  $\Upsilon_\sigma(\nu^\sigma) = \lim_{t \rightarrow +\infty} \Upsilon_\sigma(\mu_t)$ .*

**Proof. Outline.** We proceed exactly like in the proof of Proposition 2.1 in [Tug11b]. We just need to find a sequence  $(q_k)_{k \in \mathbb{N}}$  which tends to infinity as  $k$  goes to infinity and which verifies the two following conditions:

- $\xi'(q_k)$  converges to 0.
- $r_k \leq q_k \leq s_k$ .

For doing so, we remark that the inequality  $\xi'(s) \leq 0$  and the convergence of  $\int_t^\infty \xi'(s) ds$  to 0 implies that  $\sum_{p=k}^\infty \int_{r_p}^{s_p} \xi'(s) ds$  vanishes when  $k$  is going to infinity. However, to obtain the existence of such a sequence  $(q_k)_{k \in \mathbb{N}}$ , we need to prove that  $\liminf_{k \rightarrow +\infty} (s_k - r_k) > 0$ . This is done like in the proof of Proposition A.1 in [Tug10].

**Step 1.** We begin to prove that  $\liminf_{k \rightarrow +\infty} (s_k - r_k) > 0$ . We introduce the function

$$\Phi(t) := \int_{\mathbb{R}^d} \phi(x) u_t(x) dx.$$

This function is well defined since  $|\phi(x)|$  is bounded by  $\mathcal{P}(\|x\|)$ . The derivation of  $\Phi$ , the application of Equation (II) and an integration by parts lead to

$$\begin{aligned} \Phi'(t) &= - \int_{\mathbb{R}^d} \left\langle \nabla \phi(x); \left\{ \frac{\sigma^2}{2} \nabla u_t(x) + u_t(x) (\nabla V(x) + \nabla F * u_t(x)) \right\} \right\rangle dx \\ &= - \int_{\mathbb{R}^d} \langle \nabla \phi(x); \eta_t(x) \rangle u_t(x) dx. \end{aligned}$$

The Cauchy-Schwarz inequality implies

$$|\Phi'(t)| \leq \sqrt{-\xi'(t)} \sqrt{\int_{\mathbb{R}^d} \|\nabla \phi(x)\|^2 u_t(x) dx},$$

by reminding that  $\xi(t) := \Upsilon_\sigma(\mu_t)$ . The quantity  $\|\nabla \phi(x)\|^2$  is bounded by  $\mathcal{Q}(\|x\|)$  and  $\int_{\mathbb{R}^d} \|x\|^{2N_0} u_t(x) dx$  is uniformly bounded with respect to  $t \geq 1$  for all  $N_0 \in \mathbb{N}$  according to Proposition A.2. So, there exists  $C > 0$  such that  $\int_{\mathbb{R}^d} \|\nabla \phi(x)\|^2 u_t(x) dx \leq C^2$  for all  $t \geq 1$ . We deduce

$$|\Phi'(t)| \leq C \sqrt{|\xi'(t)|}. \quad (\text{V})$$

By definition of the two sequences  $(r_k)_{k \in \mathbb{N}}$  and  $(s_k)_{k \in \mathbb{N}}$ , we have

$$\Phi(s_k) - \Phi(r_k) = \kappa.$$

Combining this identity with Inequality (V) and the monotonicity of  $\xi$  yields

$$C \int_{r_k}^{s_k} \sqrt{-\xi'(t)} dt \geq \kappa.$$

We apply the Cauchy-Schwarz inequality and we obtain

$$C \sqrt{s_k - r_k} \sqrt{\xi(r_k) - \xi(s_k)} \geq \kappa.$$

Moreover,  $\xi(t)$  converges as  $t$  goes to infinity (see Lemma 1.4). It implies the convergence of  $\xi(r_k) - \xi(s_k)$  to 0 when  $k$  goes to infinity. Consequently,  $s_k - r_k$  converges to infinity so

$$\liminf_{k \rightarrow +\infty} s_k - r_k > 0.$$

**Step 2.** By Lemma 1.4,  $\Upsilon_\sigma(\mu_t) - L_\sigma = -\int_t^\infty \xi'(s) ds$  converges to 0. As  $\xi'$  is non-positive, we deduce that  $\sum_{k=N}^\infty \int_{r_k}^{s_k} \xi'(s) ds$  also converges to 0 when  $N$  goes to infinity. As  $\liminf_{k \rightarrow +\infty} s_k - r_k > 0$ , we deduce the existence of an increasing sequence  $q_k \in [r_k; s_k]$  which goes to infinity and such that  $\xi'(q_k)$  converges to 0 as  $k$  goes to infinity. Furthermore,  $\int_{\mathbb{R}} \phi(x) u_{q_k}(x) dx \in [\kappa; 2\kappa]$ , for all  $k \in \mathbb{N}$ .

**Step 3.** By proceeding similarly as in the proof of Theorem 2.7 in [Tug11b], we extract a subsequence of  $(q_k)_{k \in \mathbb{N}}$  (we continue to write it  $q_k$  for simplifying the reading) such that  $\mu_{q_k}$  converges weakly to an invariant probability  $\nu^\sigma$ . Moreover, its density  $v^\sigma$  with respect to the Lebesgue measure verifies  $\int_{\mathbb{R}^d} \phi(x) v^\sigma(x) dx \in [\kappa; 2\kappa]$ .

**Step 4.** Since  $\xi'(q_k)$  goes to 0 as  $k$  goes to infinity, we can apply Proposition 2.1. Thus, we deduce that  $\Upsilon_\sigma(\nu^\sigma) = L_\sigma := \lim_{t \rightarrow +\infty} \Upsilon_\sigma(\mu_t)$ .  $\square$

Now, we explain why all the moments of  $\mu_t$  are finite for all  $t > 0$ . This is essential for making the separation between an element of  $\mathcal{S}_\sigma \cap \mathcal{A}_\sigma$  and any other elements of  $\mathcal{A}_\sigma$  and it also is used in the previous proposition.

**Proposition A.2.** *Let us assume that the confining potential  $V$ , the interacting potential  $F$  and the initial law  $\mu_0$  satisfy the set of assumptions (M).*

*Thus, for all  $t > 0$ , for all  $p \in \mathbb{N}^*$ , we have  $\mathbb{E}[||X_t||^p] < +\infty$ . Moreover,  $\sup_{t \geq 1} \mathbb{E}[||X_t||^p] < +\infty$*

*Proof. Step 1.* If  $\mathbb{E}[||X_0||^p] < +\infty$ , then  $\mathbb{E}[||X_t||^p] < +\infty$  for all  $t > 0$ , see Theorem 2.13 in [HIP08]. We now assume that there exists  $p_0 \in \mathbb{N}^*$  such that  $\mathbb{E}[||X_0||^{2p_0}] = +\infty$ . Let us introduce  $l_0 := \min \left\{ l \geq 0 \mid \mathbb{E}[||X_0||^{2l}] = +\infty \right\}$ . We know that  $\mathbb{E}[||X_t||^{2l_0-2}] < +\infty$  for all  $t \geq 0$ , see [HIP08].

**Step 2.** Let  $T$  be a positive real. Let us prove that  $\mathbb{E}[||X_T||^{2l_0}] < +\infty$ . We proceed a *reductio ad absurdum* by assuming that  $\mathbb{E}[||X_T||^{2l_0}] = +\infty$ . This implies directly  $\mathbb{E}[||X_t||^{2l_0}] = +\infty$  for all  $t \in [0, T]$ .

For all  $t \in [0, T]$ , the application  $x \mapsto \nabla F * \mu_t(x)$  is a polynomial function and its parameters are the moments of  $\mu_t$  with total order less than  $2n$ . These moments are uniformly bounded with respect to the time  $t$  by Inequality (IV),

$$\sup_{1 \leq j \leq 8q^2} \sup_{t \in [0, T]} \mathbb{E} \left\{ ||X_t||^j \right\} \leq M_0.$$

Nevertheless, the principal term of  $\nabla F * \mu_t$  does not depend on the moment and is of degree  $2n - 1$ . We thus deduce the existence of a function  $\mathcal{R}_t$  and a positive constant  $\kappa_{2q-1}$  such that

$$\nabla V(x) + \nabla F * \mu_t(x) = \kappa_{2q-1} ||x||^{2q-2} x + \mathcal{R}_t(x),$$

with  $|\mathcal{R}_t(x)| \leq \mathcal{P}_t(x)$  for all  $x \in \mathbb{R}^d$ ,  $\mathcal{P}_t$  being a polynomial function with degree at most  $2q - 2$ . Moreover, the polynomial function  $\mathcal{P}_t$  depends in  $t$  by the moments of  $\mu_t$  with total order less than  $2n$ . According to Inequality (IV), there exists a positive constant  $C$  such that  $|\mathcal{P}_t(x)| \leq C(1 + ||x||^{2q-2})$ .

Let  $l$  be a positive integer. We denote  $\phi(x) := ||x||^{2l}$ . Simple computations lead to

$$\langle \nabla \phi(x); \nabla V(x) + \nabla F * \mu_t(x) \rangle - \frac{\sigma^2}{2} \Delta \phi(x) \geq C_l \left( ||x||^{2l+2q-2} - 1 \right)$$

where  $C_l$  is a positive constant. Consequently, for all  $\omega \in \Omega$ , we have the inequality

$$||X_T(\omega)||^{2l} \leq ||X_0(\omega)||^{2l} + \mathcal{M}_T(\omega) + C_l T - C_l \int_0^T ||X_t(\omega)||^{2l+2q-2} dt,$$

$(\mathcal{M}_t)_{t \geq 0}$  being a martingale. We choose  $l := l_0 + 1 - q < l_0$ . Taking the

expectation yields

$$0 \leq \mathbb{E} \left[ \|X_T\|^{2l_0+2-2q} \right] \leq \mathbb{E} \left[ \|X_0\|^{2l_0+2-2q} \right] \\ + C_{l_0+1-q} T - C_{l_0+1-q} \int_0^T \mathbb{E} \left[ \|X_t\|^{2l_0} \right] dt.$$

Nonetheless,  $\mathbb{E} \left[ \|X_t\|^{2l_0} \right] = +\infty$  for all  $0 \leq t \leq T$ . This implies  $0 \leq -\infty$ . This is absurd. Consequently, for all  $T > 0$ ,  $\mathbb{E} \left[ \|X_T\|^{2l_0} \right] < +\infty$ .

**Step 3.** Let  $T$  be a positive real and  $k \in \mathbb{N}$  such that  $k \geq 0$ . We remind the integer  $l_0 := \min \left\{ l \geq 0 \mid \mathbb{E} \left[ \|X_0\|^{2l} \right] = +\infty \right\}$ .

We introduce  $t_i := \frac{i}{k+1}T$  for all  $1 \leq i \leq k+1$ . We apply Step 2 to  $t_1$  and we deduce  $\mathbb{E} \left[ \|X_{t_1}\|^{2l_0} \right] < +\infty$ . By an inductive argument, we deduce the inequality  $\mathbb{E} \left[ \|X_{t_i}\|^{2l_0+2i-2} \right] < +\infty$ , for all  $1 \leq i \leq k+1$ . In particular,  $\mathbb{E} \left[ \|X_{t_{k+1}}\|^{2l_0+2k} \right] < +\infty$ , that is  $\mathbb{E} \left[ \|X_T\|^{2l_0+2k} \right] < +\infty$ . This inequality holds for all  $k \geq 0$  so the probability measure  $\mu_T$  satisfies  $\int_{\mathbb{R}^d} \|x\|^{2p} \mu_T(dx) < +\infty$  for all  $p \in \mathbb{N}^*$ .

**Step 4.** In particular, it holds with  $T := 1$ . Thank to Theorem 2.13 in [HIP08], we obtain directly  $\sup_{t \geq 1} \mathbb{E} \left[ \|X_t\|^p \right] < +\infty$ .  $\square$

**Acknowledgments:** *This work has been motivated by the natural question about what happens in the  $d$ -dimensional case. Consequently, I would like to thank Arnaud Guillin and Patrick Cattiaux for having suggested me to work on it on Friday 13th May 2011.*

*Finalement, un très grand merci à Manue et à Sandra pour tout.*

## References

- [ABC<sup>+</sup>00] Cécile Ané, Sébastien Blachère, Djalil Chafaï, Pierre Fougères, Ivan Gentil, Florent Malrieu, Cyril Roberto, and Grégory Scheffer. *Sur les inégalités de Sobolev logarithmiques*, volume 10 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2000. With a preface by Dominique Bakry and Michel Ledoux.
- [AMTU01] Anton Arnold, Peter Markowich, Giuseppe Toscani, and Andreas Unterreiter. On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Comm. Partial Differential Equations*, 26(1-2):43–100, 2001.
- [BBCG08] D. Bakry, F. Barthe, P. Cattiaux, and A. Guillin. A simple proof of the Poincaré inequality for a large class of probability measures including the log-concave case. *Electronic Communications in Probability*, 13:60–66, 2008.



- [BCCP98] D. Benedetto, E. Caglioti, J. A. Carrillo, and M. Pulvirenti. A non-Maxwellian steady distribution for one-dimensional granular media. *J. Statist. Phys.*, 91(5-6):979–990, 1998.
- [BRTV98] S. Benachour, B. Roynette, D. Talay, and P. Vallois. Nonlinear self-stabilizing processes. I. Existence, invariant probability, propagation of chaos. *Stochastic Process. Appl.*, 75(2):173–201, 1998.
- [BRV98] S. Benachour, B. Roynette, and P. Vallois. Nonlinear self-stabilizing processes. II. Convergence to invariant probability. *Stochastic Process. Appl.*, 75(2):203–224, 1998.
- [CGM08] P. Cattiaux, A. Guillin, and F. Malrieu. Probabilistic approach for granular media equations in the non-uniformly convex case. *Probab. Theory Related Fields*, 140(1-2):19–40, 2008.
- [CMV03] José A. Carrillo, Robert J. McCann, and Cédric Villani. Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates. *Rev. Mat. Iberoamericana*, 19(3):971–1018, 2003.
- [Der03] Azzouz Dermoune. Propagation and conditional propagation of chaos for pressureless gas equations. *Probab. Theory Related Fields*, 126(4):459–476, 2003.
- [Gra90] Carl Graham. Nonlinear limit for a system of diffusing particles which alternate between two states. *Appl. Math. Optim.*, 22(1):75–90, 1990.
- [Gra92] Carl Graham. McKean-Vlasov Itô-Skorohod equations, and nonlinear diffusions with discrete jump sets. *Stochastic Process. Appl.*, 40(1):69–82, 1992.
- [HIP08] Samuel Herrmann, Peter Imkeller, and Dierk Peithmann. Large deviations and a Kramers’ type law for self-stabilizing diffusions. *Ann. Appl. Probab.*, 18(4):1379–1423, 2008.
- [HS87] Richard Holley and Daniel Stroock. Logarithmic Sobolev inequalities and stochastic Ising models. *J. Statist. Phys.*, 46(5-6):1159–1194, 1987.
- [HT10a] S. Herrmann and J. Tugaut. Non-uniqueness of the invariant probabilities for self-stabilizing processes. *Stochastic Process. Appl.*, 120(7):1215–1246, 2010.
- [HT10b] S. Herrmann and J. Tugaut. Stationary measures for self-stabilizing processes: asymptotic analysis in the small noise limit. *Electron. J. Probab.*, 15:2087–2116, 2010.
- [HT12] S. Herrmann and J. Tugaut. Self-stabilizing processes: uniqueness problem for stationary measures and convergence rate in the small noise limit. To appear in *ESAIM Probability and statistics*, 2012.
- [JM08] Benjamin Jourdain and Florent Malrieu. Propagation of chaos and Poincaré inequalities for a system of particles interacting through their CDF. *Ann. Appl. Probab.*, 18(5):1706–1736, 2008.
- [KS91] I. Karatzas, and S. E. Shreve. *Brownian motion and stochastic calculus.*, Springer-Verlag, second edition (1991).
- [KKR93] Otared Kavian, Gérard Kerkycharian, and Bernard Roynette. Quelques remarques sur l’ultracontractivité. *J. Funct. Anal.*, 111(1):155–196, 1993.

- [Mal01] F. Malrieu. Logarithmic Sobolev inequalities for some nonlinear PDE's. *Stochastic Process. Appl.*, 95(1):109–132, 2001.
- [Mal03] Florent Malrieu. Convergence to equilibrium for granular media equations and their Euler schemes. *Ann. Appl. Probab.*, 13(2):540–560, 2003.
- [McK67] H. P. McKean, Jr. Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967)*, pages 41–57. Air Force Office Sci. Res., Arlington, Va., 1967.
- [Mél96] Sylvie Méléard. Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models. In *Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995)*, volume 1627 of *Lecture Notes in Math.*, pages 42–95. Springer, Berlin, 1996.
- [Oel85] Karl Oelschläger. A law of large numbers for moderately interacting diffusion processes. *Z. Wahrsch. Verw. Gebiete*, 69(2):279–322, 1985.
- [SV79] Daniel W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1979.
- [Szn91] Alain-Sol Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.
- [Tam84] Yozo Tamura. On asymptotic behaviors of the solution of a nonlinear diffusion equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 31(1):195–221, 1984.
- [Tam87] Yozo Tamura. Free energy and the convergence of distributions of diffusion processes of McKean type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 34(2):443–484, 1987.
- [Tug10] J. Tugaut. Convergence to the equilibria for self-stabilizing processes in double well landscape. *To appear in The Annals of Probability available on [http://www.imstat.org/aop/future\\_papers.htm](http://www.imstat.org/aop/future_papers.htm)*, 2010.
- [Tug11a] J. Tugaut. Phase transitions of McKean-Vlasov processes in double-wells landscape. *available on <http://hal.archives-ouvertes.fr/hal-00573046>*, 2011.
- [Tug11b] J. Tugaut. Self-stabilizing processes in multi-wells landscape in  $\mathbb{R}^d$  - Invariant probabilities. *To appear in Journal of Theoretical Probability available on <http://www.springerlink.com/content/q228286j00661936>*, 2011.
- [Ver06] A. Yu. Veretennikov. On ergodic measures for McKean-Vlasov stochastic equations. In *Monte Carlo and quasi-Monte Carlo methods 2004*, pages 471–486. Springer, Berlin, 2006.