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Extension of Boolean algebra by a Bayesian operator: application to the definition of a Deterministic Bayesian Logic

Frédéric Dambreville

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$[x]x = x \Rightarrow x \in \{\emptyset, \Omega\}$ — *Free of itself, it is the all or the none.*

Abstract

This work contributes to the domains of Boolean algebra and of Bayesian probability, by proposing an algebraic extension of Boolean algebras, which implements an operator for the Bayesian conditional inference and is closed under this operator. It is known since the work of Lewis (Lewis' triviality) that it is not possible to construct such conditional operator within the space of events. Nevertheless, this work proposes an answer which complements Lewis' triviality, by the construction of a conditional operator outside the space of events, thus resulting in an algebraic extension. In particular, it is proved that any probability defined on a Boolean algebra may be extended to its algebraic extension in compliance with the multiplicative definition of the conditional probability. In the last part of this paper, a new *bivalent* logic is introduced on the basis of this algebraic extension, and basic properties are derived.

Keywords: Boolean algebra, Bayesian inference, Lewis' triviality, Logic

1 Introduction

Many implementations of practical problems make apparent the logical nature of conditional probabilities, which are kinds of inference operators. This fact typically led to the development of various Bayesian approach for manipulating uncertain logical information (Bayesian networks, Bayesian logic, . . .) The interpretation of conditional probabilities as logical inferences naturally introduced the question of the definition of conditionals directly at the propositional level: *is it possible to define conditional probabilities of events as probabilities of conditional events?* A negative answer to this question was given by Lewis' triviality [1], which implies that it is not possible to define a conditional operator within the space of unconditional events – *c.f.* property 1. However, Lewis' triviality does not forbid the construction of conditional operators by means of an extension of the space of event. In accordance with this observation, the purpose of this paper is to prove the following main theorem:

Theorem 1 (Bayesian extension of countable Boolean algebra). *Let $(B_{\text{bool}}, \cap, \cup, \sim, \emptyset, \Omega)$ be a countable Boolean algebra. Then there is a septuple $(B_{\text{ayes}}, \cap, \cup, \sim, \emptyset, \Omega, [\])$ such that:*

- B_{ayes} , considered as $(B_{\text{ayes}}, \cap, \cup, \sim, \emptyset, \Omega)$, is a Boolean algebra,
- B_{ayes} is countable,
- There is an injective Boolean morphism $\mu : B_{\text{bool}} \rightarrow B_{\text{ayes}}$,
- The operator $[\]$ is such that:
 - $z \mapsto [x]z$ is a Boolean automorphism of B_{ayes} ,
 - $x \subset y$ implies $[x]y = \Omega$ or $x = \emptyset$,
 - $x \cap [x]y = x \cap y$,
 - $[\sim x][x]y = [x][x]y = [x]y$,

for all $x, y \in B_{\text{ayes}}$.

- Given any probability distribution P_{bool} defined on B_{bool} , there is a probability distribution P_{ayes} defined on B_{ayes} such that $P_{\text{ayes}} \circ \mu = P_{\text{bool}}$ and:

$$P_{\text{ayes}}(x \cap y) = P_{\text{ayes}}([x]y)P_{\text{ayes}}(x) \quad \text{for all } x, y \in B_{\text{ayes}}. \quad (1)$$

This result, extending the structure of Boolean space, provides an example of algebraic construction of a Bayesian space, which is closed under the conditional operator $[\]$. It is noticed that there are in the domain of *conditional event algebra* interesting examples of algebraic construction of *external* conditional proposition – for example [2]. However, nested conditional propositions, obtained from known closures of such algebras under the conditional operator, are not compliant with the conditional relation (1). Lewis' triviality is now recalled.

Lewis' triviality.

Property 1. *Let (Ω, \mathcal{F}) be a measurable space. Let be defined $[\] : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ such that $P[x]yP(x) = P(x \cap y)$ for all $x, y \in \mathcal{F}$ and P a probability distribution on \mathcal{F} .*

Let $x, y \in \mathcal{F}$ and a probability distribution P such that $P(x \cap y) > 0$ and $P(\sim x \cap y) > 0$. Then $P(x \cap y) = P(x)P(y)$.

Proof. Define $P_x(y) = P([x]y)$ and $P_{\sim x}(y) = P([\sim x]y)$. Then:

$$P_x([y]x) = \frac{P_x(x \cap y)}{P_x(y)} = \frac{\frac{P(x \cap y)}{P(x)}}{\frac{P(x \cap y)}{P(x)}} = 1 \quad \text{and} \quad P_{\sim x}([y]x) = \frac{P_{\sim x}(x \cap y)}{P_{\sim x}(y)} = \frac{\frac{P(\sim x \cap x \cap y)}{P(\sim x)}}{\frac{P(\sim x \cap y)}{P(\sim x)}} = 0.$$

Then:

$$\frac{P(x \cap y)}{P(y)} = P([y]x) = P(x)P_x([y]x) + P(\sim x)P_{\sim x}([y]x) = P(x) + 0 = P(x).$$

□

In particular, the existence of $[\]$ implies that it is impossible to have $x \subset y$ such that $0 < P(x) < P(y) < 1$. This result is irrelevant.

As explained previously, the triviality is based on the hypothesis that $[x]y \in \mathcal{F}$, which makes possible the previous computation of $P_x([y]x)$ and $P_{\sim x}([y]x)$. In theorem 1 however, the conditional operator is constructed outside of the measured space, and the triviality is thus avoided.

This presentation consists of two sections. The main section 2 establishes a proof of theorem 1. It is based on the construction of an algebraic extension of Boolean algebras, by introducing a conditional operator compatible with a probability extension. Based on this work, a notion of Bayesian algebra is introduced. As a corollary, section 3 deals with the logical interpretation of these Bayesian algebras.

Section 4 concludes.

2 Proof of the main theorems

A proof of main theorem 1 is derived throughout this section. It is first recalled in section 2.1 and section 2.2 some useful tools, which will be instrumental for the construction of a Bayesian algebra. The construction of the algebra is done in section 2.3, and section 2.4 addresses the problem of probability extension on this algebra. At last, section 2.5 compiles these results, thus achieving the proof of the main theorem.

Conventions.

- For the sack of simplicity, a structured set $(E, *_1, \dots, *_n)$ – E is a set, $*_i$ is an operator, a relation, a constant, \dots – will be simply denoted E ,
- Notations $1 : n$, $x_{1:n}$ and $i = 1 : n$ stand respectively for the sequences $1, \dots, n$, x_1, \dots, x_n and the relation $i \in \{1, \dots, n\}$,
- Notations $x, y, z \in E$ stands for $x \in E$, $y \in E$ and $z \in E$.

2.1 Direct limit

Hereinafter, direct limits will be quite useful tools for constructing Bayesian extensions of Boolean algebras. Thorough references on direct limits may be found in [3] and in [4].

2.1.1 Basic notions

A basic introduction to direct limits is done now. These known results are presented without proofs.

Definition 2 (directed set). *A (partially) ordered set (I, \leq) is a directed set if there is $k \in I$ such that $k \geq i$ and $k \geq j$ for all $i, j \in I$.*

Definition 3 (direct system). *Let (I, \leq) be a directed set. Let $(E_i)_{i \in I}$ be a sequence of structured set of same nature, and let $\mu_{i,j} : E_i \rightarrow E_j$ be a morphism defined for all $i \leq j$*

with the properties $\mu_{i,i} = \text{id}_{E_i}$ and $\mu_{j,k} \circ \mu_{i,j} = \mu_{i,k}$ for all $i \leq j \leq k$. The pair $(E_i, \mu_{i,j})_{\substack{i,j \in I \\ i \leq j}}$ is called a direct system.

Property 4. Let $(E_i, \mu_{i,j})_{\substack{i,j \in I \\ i \leq j}}$ be a direct system. Let \approx be a relation defined on $\bigsqcup_{i \in I} E_i$, the disjoint union of the set E_i , by:

$$x_i \approx y_j \text{ if and only if there is } k \geq i, j \text{ such that } \mu_{i,k}(x_i) = \mu_{j,k}(y_j).$$

The relation \approx is an equivalence relation. Moreover, this relation is compatible with the structures of $(E_i)_{i \in I}$.

Definition 5 (direct limit). The direct limit $\varinjlim E_i$ of a direct system $(E_i, \mu_{i,j})_{\substack{i,j \in I \\ i \leq j}}$ is defined by:

$$\varinjlim E_i = \bigsqcup_{i \in I} E_i / \approx,$$

the set of classes of equivalence of $\bigsqcup_{i \in I} E_i$. It is defined the canonical mapping μ_i , which maps the elements of E_i to their equivalence class:

$$\mu_i(x_i) = x_i / \approx \text{ for all } i \in I \text{ and } x_i \in E_i.$$

Property 6 (structure inheritance). The direct limit $\varinjlim E_i$ inherits their structure from $(E_i)_{i \in I}$. Moreover, μ_i are morphisms such that:

$$\mu_i = \mu_j \circ \mu_{i,j} \text{ for all } i, j \in I \text{ such that } i \leq j. \quad (2)$$

2.1.2 Direct limit of partially defined structures

Knowledge of the structure is not always fully available. In some case, it may be worthwhile to build intermediate structures partially, and to infer complete structure by passing to the limit. This section introduces a method for doing that.

In this section, it is assumed that $I = \mathbb{N}$, and that $(E_i, \mu_{i,j})_{\substack{i,j \in \mathbb{N} \\ i \leq j}}$ is a direct system, such that E_i is countable for all $i \in \mathbb{N}$. It is also defined a surjective mapping $u_i = u(i, \cdot) : j \in \mathbb{N} \mapsto u(i, j) \in E_i$ for all $i \in \mathbb{N}$.

Definition 7 (Cantor pairing function). The Cantor pairing function is the mapping $\gamma : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ defined by $\gamma(i, j) = \frac{1}{2}(i+j)(i+j+1) + i$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$.

Property 8 (bijection and inverse). The Cantor pairing function is a bijection and its inverse is defined for all $n \in \mathbb{N}$ by:

$$\gamma^{-1}(n) = (c_\gamma(n), r_\gamma(n)), \text{ where } \begin{cases} w = \left\lfloor \frac{\sqrt{8n+1} - 1}{2} \right\rfloor \\ r_\gamma(n) = w - n + \frac{w^2 + w}{2}, \\ c_\gamma(n) = n - \frac{w^2 + w}{2}. \end{cases} \quad (3)$$

Moreover, it is noticed that $c_\gamma(n) \leq n$.

Definition 9. It is defined $D_i \subset E_i \times E_i$ and $(\mu \times \mu)_{i,j} : D_i \longrightarrow D_j$ such that:

- $(\mu \times \mu)_{i,j}(x_i, y_i) = (\mu_{i,j}(x_i), \mu_{i,j}(y_i))$ for all $(x_i, y_i) \in D_i$,
- $D_{i+1} \supseteq (\mu \times \mu)_{i,i+1}(D_i) \cup \left\{ (x_{i+1}, \mu_{c_\gamma(i),i+1} \circ u \circ \gamma^{-1}(i)) \mid x_{i+1} \in E_{i+1} \right\}$.

It is interesting here to explain the meaning of $\mu_{c_\gamma(i),i+1} \circ u \circ \gamma^{-1}(i)$. Define the pair $(k, l) = (c_\gamma(n), r_\gamma(n)) = \gamma^{-1}(i)$. Then $u \circ \gamma^{-1}(i) = y_k$ where $y_k = u_k(l) \in E_k$. In other words, $u \circ \gamma^{-1}(i)$ just does the choice of a set E_k and of an element $y_k \in E_k$. Then, $\mu_{c_\gamma(i),i+1} \circ u \circ \gamma^{-1}(i) = \mu_{k,i+1}(y_k)$ where $y_k = u_k(l) \in E_k$. In other words, $\mu_{c_\gamma(i),i+1} \circ u \circ \gamma^{-1}(i)$ just does the choice of a set E_k and of a mapped element $\mu_{k,i+1}(y_k) \in \mu_{k,i+1}(E_k)$. And it is known from the definition of the Cantor pairing function and the surjection u , that this choice will be done for all k and $y_k \in E_k$. This property will ensure the following lemma:

Lemma 10. $(D_i, (\mu \times \mu)_{i,j})_{\substack{i,j \in \mathbb{N} \\ i \leq j}}$ is a direct system and:

$$\lim_{\longrightarrow} D_i = \lim_{\longrightarrow} E_i \times \lim_{\longrightarrow} E_i. \quad (4)$$

Proof. By definition, $D_i \subset E_i \times E_i$ for all $i \in \mathbb{N}$. Thus, $\lim_{\longrightarrow} D_i \subset \lim_{\longrightarrow} E_i \times \lim_{\longrightarrow} E_i$. Let $x, y \in \lim_{\longrightarrow} E_i$. Then, there is $k, j \in \mathbb{N}$, $y_k \in E_k$ and $x_j \in E_j$ such that $\mu_k(y_k) = y$ and $\mu_j(x_j) = x$. Without loss of generality, it is possible to choose $j \leq k + 1$. Let $i = \gamma(k, l)$, with $l \in u_k^{-1}(y_k)$. Then $i + 1 \geq k + 1 \geq j$ and $u \circ \gamma^{-1}(i) = y_k$. As a consequence, $(\mu_{j,i+1}(x_j), \mu_{k,i+1}(y_k)) \in D_{i+1}$ and $(x, y) \in \lim_{\longrightarrow} D_i$. \square

Property 11. Let be defined the mappings $\varphi_i : D_i \rightarrow E_i$ for all $i \in \mathbb{N}$. Assume that $\mu_{i,j} \circ \varphi_i = \varphi_j \circ (\mu \times \mu)_{i,j}$ for all $i \leq j$. Then, there is $\varphi : \lim_{\longrightarrow} E_i \times \lim_{\longrightarrow} E_i \longrightarrow \lim_{\longrightarrow} E_i$ such that $\varphi \circ (\mu \times \mu)_i = \mu_i \circ \varphi_i$ for all $i \in \mathbb{N}$.

Proof. Let $(x, y) \in \lim_{\longrightarrow} E_i \times \lim_{\longrightarrow} E_i$. Take the smallest $s \in \mathbb{N}$ such that $(x, y) = (\mu \times \mu)_s(x_s, y_s)$, where $(x_s, y_s) \in D_s$. Then, $\varphi(x, y)$ is defined by $\varphi(x, y) = \mu_s \circ \varphi_s(x_s, y_s)$. The equality $\varphi \circ (\mu \times \mu)_i = \mu_i \circ \varphi_i$ is then implied by the definition. \square

2.2 Boolean algebra

All along this paper, the notion of Boolean algebra is widely used and it is assumed that the reader is familiar with basic definitions and properties. An introduction to these notions is found in [5] or in [6].

2.2.1 Definition

Let $(E, \cap, \cup, \sim, \Omega, \emptyset)$ be a sextuple, where E is a set, $\Omega, \emptyset \in E$ and \cap, \cup, \sim are respectively binary, binary and unary operators on E .

Definition 12 (Boolean algebra). $(E, \cap, \cup, \sim, \Omega, \emptyset)$ is a Boolean algebra if:

- \cap and \cup are commutative, associative and mutually distributive,

- (absorption) $x \cap (x \cup y) = x$ and $x \cup (x \cap y) = x$ for all $x, y \in E$,
- (complements) $x \cap \sim x = \emptyset$ and $x \cup \sim x = \Omega$ for all $x \in E$.

Definition 13 (generating subset). *Let $F \subset E$. Then, F is called a generating subset of the Boolean algebra E if there is only one subalgebra of E containing F –de facto, this subalgebra is E itself.*

Definition 14 (partition). *Let $F \subset E \setminus \{\emptyset\}$ be a finite set such that $x \cap y = \emptyset$ for all $x, y \in F$ and $\bigcup_{y \in F} y = \Omega$. Then F is called a partition of E .*

Property 15 (generating partition). *Let $(E, \cap, \cup, \sim, \Omega, \emptyset)$ be a finite Boolean algebra. Then there is a unique partition $F \subset E$ which is a generating subset of E . This generating partition is denoted σE , subsequently.*

Definition 16 (Boolean morphism). *Let $(E, \cap, \cup, \sim, \Omega, \emptyset)$ and $(F, \cap, \cup, \sim, \Omega, \emptyset)$ be Boolean algebras. A morphism $\mu : E \rightarrow F$ is a mapping from E to F such that $\mu(\sim x) = \sim \mu(x)$ and $\mu(x \cap y) = \mu(x) \cap \mu(y)$ for all $x, y \in E$.*

Property 17 (direct limit). *Let (I, \leq) be a directed set. Let $(E_i, \cap_i, \cup_i, \sim_i, \emptyset_i, \Omega_i)$ be a Boolean algebra defined for all $i \in I$, and let $\mu_{i,j} : \Omega_i \rightarrow \Omega_j$ be Boolean morphism for all $i \leq j$. Then, $(E_i, \mu_{i,j})_{\substack{i,j \in I \\ i \leq j}}$ is a directed system and its direct limit $E = \lim_{\substack{\longrightarrow \\ i \in I}} E_i$ is a Boolean algebra characterized by $\mu_i(x_i) \cap \mu_i(y_i) = \mu_i(x_i \cap_i y_i)$ and $\sim \mu_i(x_i) = \mu_i(\sim_i x_i)$ for all $i \in I, x_i, y_i \in E_i$.*

2.2.2 Probability on Boolean algebras

It is given the Boolean algebra $(E, \cap, \cup, \sim, \emptyset, \Omega)$.

Definition 18. *A non negative mapping $P : E \rightarrow \mathbb{R}_+$ is a probability distribution on E if $P(\emptyset) = 0$, $P(\Omega) = 1$ and $P(x \cap y) + P(x \cup y) = P(x) + P(y)$ for all $x, y \in E$.*

Notation 19. *The set of probability distributions defined on a Boolean algebra E is denoted $\mathbb{P}(E)$.*

Property 20 (probability mass function). *Assume that E is finite. Then:*

- $\sum_{x \in \sigma E} P(x) = 1$ for all $P \in \mathbb{P}(E)$,
- If there is a mapping $p : \sigma E \rightarrow \mathbb{R}_+$ such that $\sum_{x \in \sigma E} p(x) = 1$, then there is a unique probability density $P \in \mathbb{P}(E)$ such that $P(x) = p(x)$ for all $x \in \sigma E$. p is called the probability mass of the distribution P .

For the purpose of this paper, we introduce notion of *tangible Boolean algebra*

Definition 21 (tangible Boolean algebra). *The Boolean algebra E is tangible, if there is a probability distribution $P \in \mathbb{P}(E)$ such that $P(x) > 0$ for all $x \in E \setminus \{\emptyset\}$.*

Example 22. *The free Boolean algebra generated by I is a tangible Boolean algebra even when I is not countable, since it is possible to define a probability distribution P , by setting:*

$$P\left(\bigcap_{p \in J} p\right) = 2^{-\text{card}J} \quad \text{for all finite set } J \subset I,$$

and this probability is non zero for all elements except \emptyset .

The power set 2^I is tangible if and only if I is countable. In particular, for all $P \in \mathbb{P}(2^I)$, it is derived:

$$\max_{\substack{J \subset I \\ J \text{ is finite}}} \sum_{x \in J} P(x) \leq 1,$$

and as a consequence, $\{x \in I / P(x) > 0\}$ is countable.

This example illustrated the fact that the notion of tangible Boolean algebra is not directly related to the cardinality of the algebra. The characterization of Boolean algebras which admit strictly positive finite measure is still an open question [7].

Notation 23 (cropping). *Let F be a subset of E , and $x, y \in E$. The cropping of F by x is the set:*

$$F[x] = \{y \in F / y \subset x\}.$$

The cropping of F by a pair (x, y) is the product set:

$$F[x, y] = F[x] \times F[y].$$

Lemma 24. *Assume that $E = \{x_i / i \in \mathbb{N}\}$. Let E_k be defined as the Boolean subalgebra of E generated by $\{x_i / i \in \mathbb{N} \text{ and } i < k\}$. Then E_k is finite and $E = \bigcup_{k \in \mathbb{N}} E_k$ and $E_k \subset E_{k+1}$.*

Property 25. *If E is denumerable, then E is tangible.*

Proof. Let $(E_i)_{i \in \mathbb{N}}$ be a sequence of finite Boolean algebra such that $E = \bigcup_{i \in \mathbb{N}} E_i$ and $E_i \subset E_{i+1}$ for all $i \in \mathbb{N}$. Now, define $P_i \in \mathbb{P}(E_i)$ for all $i \in \mathbb{N}$ inductively by:

- For all $\omega_0 \in \sigma E_0$, define $P_0(\omega_0) = \frac{1}{\text{card}\sigma E_0}$,
- For all $\omega_i \in \sigma E_i$ and any $\omega_{i+1} \in \sigma E_{i+1}[\omega_i]$, define $P_{i+1}(\omega_{i+1}) = \frac{P_i(\omega_i)}{\text{card}\sigma E_{i+1}[\omega_i]}$.

It is clear that $P_i(x) > 0$ for all $i \in E_i$ and $x \in E_i \setminus \{\emptyset\}$. Moreover, $P_j(x_i) = P_i(x_i)$ for all $i, j \in \mathbb{N}$ such that $j \geq i$ and all $x_i \in E_i$. Then, it is possible to define $P \in \mathbb{P}(E)$ by $P(x) = P_i(x)$ for all $i \in \mathbb{N}$ and $x_i \in E_i$. Then P is positive except for \emptyset . \square

Property 26. *Assume a family $(P_\epsilon)_{\epsilon \in]0,1]}$ of probability distributions on E such that $\epsilon \mapsto P_\epsilon(x)$ is a rational function for all $x \in E$. Then $\lim_{\epsilon \rightarrow 0_+} P_\epsilon(x)$ exists for all $x \in E$, and the function P , defined for all $x \in E$ by $P(x) = \lim_{\epsilon \rightarrow 0_+} P_\epsilon(x)$, is a probability distribution on E .*

Proof. Let $x, y \in E$. Since P_ϵ is a probability distribution, the rational function $\epsilon \mapsto P_\epsilon(x)$ is bounded by $[0, 1]$. As a consequence, $\lim_{\epsilon \rightarrow 0^+} P_\epsilon(x)$ exists.

Since $P_\epsilon(\emptyset) = 0$, $P_\epsilon(\Omega) = 1$ and $P_\epsilon(x \cap y) + P_\epsilon(x \cup y) = P_\epsilon(x) + P_\epsilon(y)$ for all $\epsilon \in]0, 1]$, it comes that $P(\emptyset) = 0$, $P(\Omega) = 1$ and $P(x \cap y) + P(x \cup y) = P(x) + P(y)$. \square

Property 27. *Assume that E is tangible. Let P be a probability distribution on E . Then, there is a family $(P_\epsilon)_{\epsilon \in]0, 1]}$ of probability distributions on E such that $\epsilon \mapsto P_\epsilon(x)$ is a rational function, $P_\epsilon(x) > 0$ and $P_\epsilon(x) = \lim_{\epsilon \rightarrow 0^+} P_\epsilon(x)$ for all $x \in E$ and for all $\epsilon \in]0, 1]$.*

Proof. Let Q be a probability distribution on E such that $Q(x) > 0$ for all $x \in E \setminus \{\emptyset\}$. The family defined by $P_\epsilon = \epsilon Q + (1 - \epsilon)P$ will satisfy the property. \square

2.3 Bayesian model

2.3.1 Partial Bayesian model

In this section, a recursive construction of partial Bayesian models is done, and a global model is deduced as a direct limit. At this time, we will not explain about the Bayesian nature of this model; this task will be done in the next sections.

The following definitions will be instrumental:

Definition 28. *For all (ω, ν) , it is defined $T(\omega, \nu) = (\nu, \omega)$. For all set of pairs x , it is defined $T(x) = \{(\nu, \omega) / (\omega, \nu) \in x\}$ and $(\text{id} \cup T)(x) = x \cup T(x)$.*

Initial construction. It is defined a *finite* Boolean algebra $(E_0, \cap, \cup, \sim_0, \emptyset, \Omega_0)$, where Ω_0 is a set and E_0 is a set of subsets of Ω_0 . Then, \cap, \cup, \emptyset are respectively the set intersection, union and the empty set, and \sim_0 is the set complement defined by $\sim_0 x = \Omega_0 \setminus x$ for all $x \in E_0$. It is defined $\mu_{0,0} = \text{id}_{E_0}$. It is defined $D_0 = \emptyset$, $\varphi_0 : D_0 \rightarrow E_0$ (trivially empty), $D_0(y) = \emptyset$ for all $y \in E_0$. It is defined $u_0 = u(0, \cdot) : j \in \mathbb{N} \mapsto u(0, j) \in E_0$, a surjective mapping.

Inductive construction. Assume that $(E_i, \cap, \cup, \sim_i, \emptyset, \Omega_i)$, D_i , $\varphi_i : D_i \rightarrow E_i$ $\mu_{i,j}$ and $u_i = u(i, \cdot)$ are constructed, and E_i is a finite set, for all $i \leq j \leq n$. Let $b_n = \mu_{c_\gamma(n), n} \circ u \circ \gamma^{-1}(n) \in E_n$. Then, it is defined by induction:

Definition 29 (case $\in \{\emptyset, \Omega_n\}$). *If $b_n \in \{\emptyset, \Omega_n\}$, then:*

- Set $E_{n+1} = E_n$, $\Omega_{n+1} = \Omega_n$ and $\sim_{n+1} = \sim_n$,
- Set $\mu_{i,n+1} = \mu_{i,n}$ for all $i \leq n$, and $\mu_{n+1,n+1} = \text{id}_{E_{n+1}}$,
- Define D_{n+1} and φ_{n+1} by:
 - $D_{n+1} = D_n \cup \{(x, \emptyset) / x \in D_{n+1}\} \cup \{(x, \Omega_{n+1}) / x \in D_{n+1}\}$,
 - $\varphi_{n+1}(x, y) = \varphi_n(x, y)$ for $(x, y) \in D_n$,
 - $\varphi_{n+1}(x, \emptyset) = \varphi_{n+1}(x, \Omega_{n+1}) = x$ for $x \in E_{n+1}$.

- For all $y \in E_{n+1}$, define $D_{n+1}(y) = \{x \in E_{n+1} / (x, y) \in D_{n+1}\}$. It is noticed that $D_{n+1}(\sim_n y) = D_{n+1}(y)$ by construction.

Definition 30 (case $\notin \{\emptyset, \Omega_n\}$). If $b_n \notin \{\emptyset, \Omega_n\}$, then:

- Define:

$$Z_{n+1} = \left\{ (x, y) \in E_n[\sim_n b_n] \times E_n[b_n] / (x, \sim_n b_n), (y, b_n) \in D_n \right. \\ \left. \text{and } x \cap \varphi_n(y, b_n) = y \cap \varphi_n(x, \sim_n b_n) = \emptyset \right\},$$

- Define $\bar{\emptyset}_{n+1} = \bigcup_{(x,y) \in Z_{n+1}} (\text{id} \cup T)(x \times y)$,

- Define $E_{n+1} = \bigcup_{k \in \mathbb{N}} E_{n+1}^k$ and E_{n+1}^k by:

$$E_{n+1}^k = \left\{ \bigcup_{i=1}^k \left((x_i \times y_i) \setminus \bar{\emptyset}_{n+1} \right) / \forall i, (x_i, y_i) \in (\text{id} \cup T)(E_n[b_n] \times E_n[\sim_n b_n]) \right\},$$

- Define $\Omega_{n+1} = (\text{id} \cup T)(b_n \times \sim_n b_n) \setminus \bar{\emptyset}_{n+1}$, and $\sim_{n+1} x = \Omega_{n+1} \setminus x$ for all $x \in E_{n+1}$,
- Set $\mu_{i,n+1}(x) = \left(\left((\mu_{i,n}(x) \cap b_n) \times \sim_n b_n \right) \cup \left((\mu_{i,n}(x) \cap \sim_n b_n) \times b_n \right) \right) \setminus \bar{\emptyset}_{n+1}$ for all $i \leq n$ and $x \in E_i$; set $\mu_{n+1,n+1} = \text{id}_{E_{n+1}}$,
- Define D_{n+1} and φ_{n+1} by:

$$- D_{n+1} = (\mu \times \mu)_{n,n+1}(D_n) \cup \bigcup_{x \in E_{n+1}} \left\{ (x, b_{n:n+1}), (x, \sim_{n+1} b_{n:n+1}) \right\},$$

$$- \varphi_{n+1} \left((\mu \times \mu)_{n,n+1}(x, y) \right) = \mu_{n,n+1} \circ \varphi_n(x, y) \text{ for } (x, y) \in D_n,$$

$$- \varphi_{n+1}(x, b_{n:n+1}) = (\text{id} \cup T)(x \cap b_{n:n+1}) \text{ for all } x \in E_{n+1} \text{ such that } (x, b_{n:n+1}) \notin (\mu \times \mu)_{n,n+1}(D_n),$$

$$- \varphi_{n+1}(x, \sim_{n+1} b_{n:n+1}) = (\text{id} \cup T)(x \cap \sim_{n+1} b_{n:n+1}) \text{ for all } x \in E_{n+1} \text{ such that } (x, \sim_{n+1} b_{n:n+1}) \notin (\mu \times \mu)_{n,n+1}(D_n),$$

where $b_{n:n+1} = \mu_{n,n+1}(b_n) = (b_n \times \sim_n b_n) \setminus \bar{\emptyset}_{n+1}$.

- For all $y \in E_{n+1}$, define $D_{n+1}(y) = \{x \in E_{n+1} / (x, y) \in D_{n+1}\}$. It is noticed that $D_{n+1}(\sim_n y) = D_{n+1}(y)$ by construction.

Property 31. E_{n+1} is finite.

Proof. Immediate induction from the definition. □

It is interesting here to explain this construction. First at all, the idea is to model the conditioning $[X]Y$ by $\varphi(y, x)$ where x, y are respective models of X, Y . This model is of course partial at this step. The definition 29 just models the hypothesis $[\perp]X \equiv [\top]X \equiv X$ in the implied Bayesian logic. We will explain later the reason of this hypothesis. But one have to consider that $P(x|\emptyset)$ is an undefined conditional statement in regards to the classical definition $P(x|\emptyset)P(\emptyset) = P(x \cap \emptyset)$, and thus, it is possible to make the choice $P(x|\emptyset) = P(x)$. It is difficult to explain the origin of the definition 30, without a discussion about the logic. This is done in later section. However, one could say that the construction of E_{n+1} from E_n is the analog of the probabilistic trivial equation:

$$P(x) = \sum_{\omega \in x \cap b} \sum_{v \in \sim b} p(\omega)p(v|\sim b) + \sum_{\omega \in x \cap \sim b} \sum_{v \in b} p(\omega)p(v|b).$$

Typically, for $(x, y) \in E_n[b_n] \times E_n[\sim_n b_n]$, the elements $x \times y$ and $y \times x$ are the respective models of $X \wedge [\neg b]Y$ and of $Y \wedge [b]X$. However, one has to take into account logical relations implied from previous constructions, and especially exclusions like $X \wedge [\neg b]Y \equiv \perp$ or $Y \wedge [b]X \equiv \perp$. The set $\bar{\emptyset}_{n+1}$ is a compilation of such exclusions, and has to be removed from the models construction.

Property 32. *Let $i \in \mathbb{N}$ and $x \in E_i$. Then $D_i(x) = D_i(\sim_i x)$ and:*

$$\sim_i y, y \cup z, \varphi_i(y, x), \varphi_i(y, \sim_i x) \in D_i(x),$$

for all $y, z \in D_i(x)$.

Proof. Immediate induction from the definition. □

Property 33 (transitive mapping). $\mu_{j,k} \circ \mu_{i,j} = \mu_{i,k}$ for all $i, j, k \in \mathbb{N}$ such that $i \leq j \leq k$.

Proof. True for $k = 0$.

Now, assume the property for $k \leq n$.

By definition, it is clear that $\mu_{n+1,n+1} \circ \mu_{i,n+1} = \mu_{i,n+1}$.

Now assume $j \leq n$. Since $\mu_{j,n}(\mu_{i,j}(x)) = \mu_{i,n}(x)$, it follows from the definition of $\mu_{\cdot,n+1}$ that $\mu_{j,n+1}(\mu_{i,j}(x)) = \mu_{i,n+1}(x)$. □

Lemma 34. *Let A, B, C, D be sets. Then:*

- $A \cap D = B \cap C = \emptyset$ implies $(A \cup C) \cap (B \cup D) = (A \cap B) \cup (C \cap D)$,
- $A \subset B, C \subset D$ and $B \cap D = \emptyset$ imply $(B \cup D) \setminus (A \cup C) = (B \setminus A) \cup (D \setminus C)$.

Property 35 (Boolean morphism). *Let $i, j \in \mathbb{N}$ such that $i \leq j$. Let $x, y \in E_i$. Then $\mu_{i,j}(x \cap y) = \mu_{i,j}(x) \cap \mu_{i,j}(y)$, $\mu_{i,j}(\Omega_i) = \Omega_j$ and $\mu_{i,j}(\sim_i x) = \sim_j \mu_{i,j}(x)$. As a consequence, $\mu_{i,j}$ is a Boolean morphism.*

Proof. The properties are obviously true for $j = 0$.

Assume the properties for $j \leq n$. Then:

$$\begin{aligned} \mu_{i,n+1}(x_i \cap y_i) &= \mu_{n,n+1} \circ \mu_{i,n}(x_i \cap y_i) = \mu_{n,n+1}(\mu_{i,n}(x_i) \cap \mu_{i,n}(y_i)), \\ \mu_{i,n+1}(\Omega_i) &= \mu_{n,n+1} \circ \mu_{i,n}(\Omega_i) = \mu_{n,n+1}(\Omega_n), \\ \mu_{i,n+1}(\sim_i x_i) &= \mu_{n,n+1} \circ \mu_{i,n}(\sim_i x_i) = \mu_{n,n+1}(\Omega_n \setminus \mu_{i,n}(x_i)), \end{aligned}$$

for all $x_i, y_i \in E_i$.

It is recalled that $\mu_{n,n+1}(y_n) = \left(\left((y_n \cap b_n) \times \sim_n b_n \right) \cup \left((y_n \cap \sim_n b_n) \times b_n \right) \right) \setminus \bar{\emptyset}_{n+1}$ for all $y_n \in E_n$. It is first deduced $\mu_{i,n+1}(\Omega_i) = \Omega_{n+1}$ from the definition of Ω_{n+1} . Now from $b_n \cap \sim_n b_n = \emptyset$ and lemma 34, it is deduced $\mu_{i,n+1}(\sim_i x_i) = \sim_{n+1} \mu_{i,n+1}(x_i)$ and $\mu_{i,n+1}(x_i \cap y_i) = \mu_{i,n+1}(x_i) \cap \mu_{i,n+1}(y_i)$ for all $x_i, y_i \in E_i$. \square

Property 36 (commutation). $\mu_{i,j} \circ \varphi_i = \varphi_j \circ (\mu \times \mu)_{i,j}$ for all $i, j \in \mathbb{N}$ such that $i \leq j$.

An immediate consequence of the definition.

Lemma 37. Let $i \in \mathbb{N}$ such that $b_i \notin \{\emptyset, \Omega_i\}$. Then, for all $x, y \in E_{i+1}$, it is proved:

- (a) $\varphi_{i+1}(x, b_{i:i+1}) = (\text{id} \cup T)(x \cap b_{i:i+1})$,
- (b) $\varphi_{i+1}(x, \sim_{i+1} b_{i:i+1}) = (\text{id} \cup T)(x \cap \sim_{i+1} b_{i:i+1})$,
- (c) $\varphi_{i+1}(b_{i:i+1}, b_{i:i+1}) = \Omega_{i+1}$,
- (d) $\varphi_{i+1}(\sim_{i+1} b_{i:i+1}, \sim_{i+1} b_{i:i+1}) = \Omega_{n+1}$,
- (e) $\varphi_{i+1}(x \cap y, b_{i:i+1}) = \varphi_{i+1}(x, b_{i:i+1}) \cap \varphi_{i+1}(y, b_{i:i+1})$,
- (f) $\varphi_{i+1}(x \cap y, \sim_{i+1} b_{i:i+1}) = \varphi_{i+1}(x, \sim_{i+1} b_{i:i+1}) \cap \varphi_{i+1}(y, \sim_{i+1} b_{i:i+1})$,
- (g) $\varphi_{i+1}(\sim_{i+1} x, b_{i:i+1}) = \sim_{i+1} \varphi_{i+1}(x, b_{i:i+1})$,
- (h) $\varphi_{i+1}(\sim_{i+1} x, \sim_{i+1} b_{i:i+1}) = \sim_{i+1} \varphi_{i+1}(x, \sim_{i+1} b_{i:i+1})$,
- (i) $b_{i:i+1} \cap \varphi_{i+1}(x, b_{i:i+1}) = x \cap b_{i:i+1}$,
- (j) $\sim_{i+1} b_{i:i+1} \cap \varphi_{i+1}(x, \sim_{i+1} b_{i:i+1}) = x \cap \sim_{i+1} b_{i:i+1}$,
- (k) $\varphi_{i+1}(\varphi_{i+1}(x, b_{i:i+1}), b_{i:i+1}) = \varphi_{i+1}(\varphi_{i+1}(x, b_{i:i+1}), \sim_{i+1} b_{i:i+1}) = \varphi_{i+1}(x, b_{i:i+1})$,
- (l) $\varphi_{i+1}(\varphi_{i+1}(x, \sim_{i+1} b_{i:i+1}), b_{i:i+1}) = \varphi_{i+1}(\varphi_{i+1}(x, \sim_{i+1} b_{i:i+1}), \sim_{i+1} b_{i:i+1}) = \varphi_{i+1}(x, \sim_{i+1} b_{i:i+1})$.

Proof. The proofs are done by induction on i . The results are trivial for $i = 0$.

Now assume that the result is true for $i \leq n - 1$.

Preliminary remark. Consider the greatest $k < n$ such that $\mu_{k,n}(b_k) \in \{b_n, \sim_n b_n\}$, if it exists. Then:

$$D_n(b_n) = D_n(\sim_n b_n) \subset \mu_{k+1,n}(D_{k+1}(b_{k:k+1})) = \mu_{k+1,n}(D_{k+1}(\sim_{k+1} b_{k:k+1})),$$

and from induction hypothesis, the properties (a) to (l) do hold for all $x_{k+1}, y_{k+1} \in D_{k+1}(b_{k:k+1})$. Then, it is proved the properties for $i = n + 1$ in that order:

Proof of (a). The only difficult point is for $x = \mu_{n,n+1}(x_n)$ such that $x_n \in D_n(b_n)$. Then:

$$\varphi_{n+1}(x, b_{n:n+1}) = \left(\left((\varphi_n(x_n, b_n) \cap b_n) \times \sim_n b_n \right) \cup \left((\varphi_n(x_n, b_n) \cap \sim_n b_n) \times b_n \right) \right) \setminus \bar{\emptyset}_{n+1}.$$

Since $\varphi_n(x_n, b_n) \cap b_n = x \cap b_n$ from the preliminary remark, it is deduced:

$$\left(\left(\varphi_n(x_n, b_n) \cap b_n \right) \times \sim_n b_n \right) \setminus \bar{\emptyset}_{n+1} = \left((x_n \cap b_n) \times \sim_n b_n \right) \setminus \bar{\emptyset}_{n+1} = x \cap b_{n:n+1} .$$

Now, let consider the second component $\left(\left(\varphi_n(x_n, b_n) \cap \sim_n b_n \right) \times b_n \right) \setminus \bar{\emptyset}_{n+1}$.
From the preliminary remark, it is deduced

$$\varphi_n(\sim_n x_n \cap b_n, b_n) \cap \varphi_n(x_n, b_n) \cap \sim_n b_n = \emptyset ,$$

and

$$\begin{aligned} \sim_n x_n \cap b_n \cap \varphi_n(\varphi_n(x_n, b_n) \cap \sim_n b_n, \sim_n b_n) &= \sim_n x_n \cap b_n \cap \varphi_n(\varphi_n(x_n, b_n), \sim_n b_n) \\ &= \sim_n x_n \cap b_n \cap \varphi_n(x_n, b_n) = \sim_n x_n \cap b_n \cap x_n = \emptyset . \end{aligned}$$

As a consequence, $(\varphi_n(x_n, b_n) \cap \sim_n b_n, \sim_n x_n \cap b_n) \in Z_{n+1}$. Similarly, it is shown $(\varphi_n(\sim_n x_n, b_n) \cap \sim_n b_n, x_n \cap b_n) \in Z_{n+1}$. As a consequence:

$$\begin{aligned} \left(\left(\varphi_n(x_n, b_n) \cap \sim_n b_n \right) \times b_n \right) \setminus \bar{\emptyset}_{n+1} &= \left(\left(\varphi_n(x_n, b_n) \cap \sim_n b_n \right) \times (x_n \cap b_n) \right) \setminus \bar{\emptyset}_{n+1} \\ &= (\sim_n b_n \times (x_n \cap b_n)) \setminus \bar{\emptyset}_{n+1} = T(x \cap b_{n:n+1}) . \end{aligned}$$

Thus the result.

Proof of (b). Proof is similar to (b).

Proof of (c). From (a), it is deduced:

$$\varphi_{n+1}(b_{n:n+1}, b_{n:n+1}) = (\text{id} \cup T)(b_{n:n+1}) = (\text{id} \cup T)(b_n \times \sim_n b_n) \setminus \bar{\emptyset}_{n+1} = \Omega_{n+1} .$$

Proof of (d). Proof is similar to (d).

Proof of (e), (f), (g) and (h). Immediate corollaries of (a) and (b).

Proof of (i). Since $T(b_{n:n+1}) = \sim_{n+1} b_{n:n+1}$ by definition, it comes:

$$b_{n:n+1} \cap \varphi_{n+1}(x, b_{n:n+1}) = b_{n:n+1} \cap (\text{id} \cup T)(x \cap b_{n:n+1}) = x \cap b_{n:n+1} .$$

Proof of (j). Proof is similar to (i).

Proof of (k). It is first deduced:

$$\begin{aligned} \varphi_{n+1}(\varphi_{n+1}(x, b_{n:n+1}), b_{n:n+1}) &= \varphi_{n+1}(b_{n:n+1} \cap \varphi_{n+1}(x, b_{n:n+1}), b_{n:n+1}) \\ &= \varphi_{n+1}(b_{n:n+1} \cap x, b_{n:n+1}) = \varphi_{n+1}(x, b_{n:n+1}) . \end{aligned}$$

Now, by applying property (a), it comes:

$$\begin{aligned} \varphi_{n+1}(\varphi_{n+1}(x, b_{n:n+1}), \sim_{n+1} b_{n:n+1}) &= (\text{id} \cup T)((\text{id} \cup T)(x \cap b_{n:n+1}) \cap \sim_{n+1} b_{n:n+1}) \\ &= (\text{id} \cup T)(T(x \cap b_{n:n+1})) = (\text{id} \cup T)(x \cap b_{n:n+1}) = \varphi_{n+1}(x, b_{n:n+1}) . \end{aligned}$$

Proof of (l). Proof is similar to (k). □

Property 38 (conditional Boolean morphism). *Let $i \in \mathbb{N}$, $x \in E_i$ and $y, z \in D_i(x)$. Then $\varphi_i(y \cup z, x) = \varphi_i(y, x) \cup \varphi_i(z, x)$ and $\varphi_i(\sim_i y, x) = \sim_i \varphi_i(y, x)$.*

Proof. An immediate induction from definition 29, property 36 and lemma 37. □

Property 39 (reflexive conditioning). *Let $i \in \mathbb{N}$. Then:*

- *If $(\emptyset, \emptyset) \in D_i$, then $\varphi_i(\emptyset, \emptyset) = \emptyset$,*
- *If $(x, x) \in D_i \setminus \{(\emptyset, \emptyset)\}$, then $\varphi_i(x, x) = \Omega_i$.*

Proof. An immediate induction from definition 29, property 36 and lemma 37. □

Property 40 (conditional inference). *Let $i \in \mathbb{N}$. Let $x \in E_i$ and $y \in D_i(x)$. Then $x \cap \varphi_i(y, x) = x \cap y$.*

Proof. An immediate induction from definition 29, property 36 and lemma 37. □

Property 41 (independence). *Let $i \in \mathbb{N}$. Let $x \in E_i$ and $y \in D_i(x)$. Then:*

$$\varphi_i(\varphi_i(y, x), x) = \varphi_i(\varphi_i(y, x), \sim_i x) = \varphi_i(y, x).$$

Proof. An immediate induction from definition 29, property 36 and lemma 37. □

Lemma 42. *Let $i \in \mathbb{N}$, $x \in E_i$ and $y, z \in D_i(x)$. Then:*

$$x \cap y \cap \varphi_i(z, \sim_i x) = \emptyset \text{ implies } \sim_i x \cap z \cap \varphi_i(y, x) = \emptyset.$$

Proof. From $x \cap y \cap \varphi_i(z, x) = \emptyset$, it is deduced:

$$\begin{aligned} \sim_i x \cap z \cap \varphi_i(y, x) &= \sim_i x \cap \varphi_i(z, \sim_i x) \cap \varphi_i(x \cap y, x) \\ &= \sim_i x \cap \varphi_i(\varphi_i(z, \sim_i x), x) \cap \varphi_i(x \cap y, x) \\ &= \sim_i x \cap \varphi_i(x \cap y \cap \varphi_i(z, \sim_i x), x) = \sim_i x \cap \varphi_i(\emptyset, x) = \emptyset \end{aligned}$$

□

Lemma 43. *Let $i \in \mathbb{N}$. Let $x \in E_i[\sim_i b_i]$ and $y \in E_i[b_i]$ such that:*

$$\mu_{i,i+1}(y) \cap \varphi_{i+1}(\mu_{i,i+1}(x), \sim_{i+1} b_{i:i+1}) = \emptyset$$

or:

$$\mu_{i,i+1}(x) \cap \varphi_{i+1}(\mu_{i,i+1}(y), b_{i:i+1}) = \emptyset.$$

Then:

$$x = \emptyset \text{ or } y = \emptyset \text{ or } \exists(t, u) \in Z_{i+1}, x \times y \subset t \times u.$$

Proof. Denote $x' = \mu_{i,i+1}(x)$, $y' = \mu_{i,i+1}(y)$ and $b' = b_{i:i+1}$. From lemma 42, it is deduced:

$$y' \cap \varphi_{i+1}(x', \sim_{i+1} b') = \emptyset \text{ and } x' \cap \varphi_{i+1}(y', b') = \emptyset.$$

By applying the definitions to $y' \cap \varphi_{i+1}(x', \sim_{i+1} b') = \emptyset$, it comes:

$$y \times x = (y \times \sim_i b_i) \cap (b_i \times x) \subset \bar{\emptyset}_{n+1} = \bigcup_{(t,u) \in Z_{n+1}} ((t \times u) \cup (u \times t)).$$

At this step, it is noticed that $\bar{\emptyset}_{i+1} = \emptyset$ implies $x = \emptyset$ or $y = \emptyset$, the third conclusion being refuted. Now, it is assumed that $x \neq \emptyset$ and $y \neq \emptyset$. It is thus deduced $y \times x = \bigcup_{(t,u) \in Z_{i+1}} (u \times t)$. For all $\omega \in y$, let $J_\omega = \{(t, u) \in Z_{i+1} / \omega \in u\}$. It comes:

$$\{\omega\} \times x \subset \left(\bigcap_{(t,u) \in J_\omega} u \right) \times \left(\bigcup_{(t,u) \in J_\omega} t \right).$$

Define $u_\omega = \bigcap_{(t,u) \in J_\omega} u$ and $t_\omega = \bigcup_{(t,u) \in J_\omega} t$. Since $\varphi_i(t, \sim_i b_i)$ is defined for all $(t, u) \in Z_{i+1}$, then $\varphi_i(t_\omega, \sim_i b_i)$ is defined and:

$$u_\omega \cap \varphi_i(t_\omega, \sim_i b_i) = \bigcup_{(t,u) \in J_\omega} \left(\left(\bigcap_{(t,u) \in J_\omega} u \right) \cap \varphi_i(t, \sim_i b_i) \right) \subset \bigcup_{(t,u) \in J_\omega} (u \cap \varphi_i(t, \sim_i b_i)) = \emptyset.$$

As a consequence:

$$\left(\bigcup_{\omega \in y} u_\omega \right) \cap \varphi_i \left(\bigcap_{\omega \in y} t_\omega, \sim_i b_i \right) \subset \bigcup_{\omega \in y} (u_\omega \cap \varphi_i(t_\omega, \sim_i b_i)) = \emptyset.$$

By applying lemma 42, it is deduced:

$$x \times y \subset \left(\bigcap_{\omega \in y} t_\omega \right) \times \left(\bigcup_{\omega \in y} u_\omega \right), \quad \text{with} \quad \left(\bigcap_{\omega \in y} t_\omega, \bigcup_{\omega \in y} u_\omega \right) \in Z_{i+1}.$$

□

Property 44 (injective mapping). $\mu_{i,j}$ is injective for all $i, j \in \mathbb{N}$ such that $i \leq j$.

Proof. It is equivalent to prove that $\mu_{i,i+1}$ is injective for all $i \in \mathbb{N}$.

In the case $b_i \in \{\emptyset, \Omega_i\}$, then $\mu_{i,i+1} = \text{id}_{E_i}$ by definition and is injective.

Assume $b_i \notin \{\emptyset, \Omega_i\}$. Let $x \in E_i$ such that $\mu_{i,i+1}(x) = \emptyset$. Then:

$$\mu_{i,i+1}(x \cap b_i) \cap \varphi_{i+1}(\sim_{i+1} b_{i:i+1}, \sim_{i+1} b_{i:i+1}) = \emptyset.$$

By applying lemma 43, it is deduced the disjunction of three possible conclusions:

$$\sim_i b_i = \emptyset \text{ or } x \cap b_i = \emptyset \text{ or } \exists(t, u) \in Z_{i+1}, \sim_i b_i \times (x \cap b_i) \subset t \times u.$$

It is hypothesized that $\sim_i b_i \neq \emptyset$. Then, first conclusion is refuted. Assume third conclusion. Then necessarily, $D_i(b_i) \neq \emptyset$, and then $\sim_i b_i \in D_i(b_i)$ and $\varphi_i(\sim_i b_i, \sim_i b_i) = \Omega_i$. Let $(t, u) \in Z_{i+1}$ be such that $\sim_i b_i \times (x \cap b_i) \subset t \times u$. Then $\sim_i b_i \subset t$ and $u = u \cap \varphi_i(t, \sim_i b_i) = \emptyset$ by applying property 39. At last, $x \cap b_i = \emptyset$, in any cases. Similarly, it is proved $x \cap \sim_i b_i = \emptyset$ and it is deduced $x = \emptyset$.

At this point, it is proved that $\mu_{i,i+1}(x) = \emptyset$ implies $x = \emptyset$. Since moreover, $\mu_{i,i+1}$ is a Boolean morphism, it is necessarily injective. □

Property 45. $D_{i+1} \supseteq (\mu \times \mu)_{i,i+1}(D_i) \cup \left\{ (x_{i+1}, \mu_{c_\gamma(i), i+1} \circ u \circ \gamma^{-1}(i)) / x_{i+1} \in E_{i+1} \right\}$ for all $i \in \mathbb{N}$.

A direct consequence of the definition $b_i = \mu_{c_\gamma(i), i} \circ u \circ \gamma^{-1}(i)$.

2.3.2 Bayesian model

It happens that $(E_i, \mu_{i:j})_{\substack{i,j \in \mathbb{N} \\ i \leq j}}$ and $(D_i, (\mu \times \mu)_{i:j})_{\substack{i,j \in \mathbb{N} \\ i \leq j}}$ are directed systems. Moreover, $E_i, \mu_{i,j}, D_i, \varphi_i : D_i \rightarrow E_i$ and $u_i = u(i, \cdot)$ match the hypothesis of property 11. Then it is deduced the following property.

Property 46. *Let $(E, \cap, \cup, \sim, \emptyset, \Omega)$ be the direct limit of $((E_i, \mu_{i,j})_{\substack{i,j \in \mathbb{N} \\ i \leq j}})$ and let $\mu_i : E_i \rightarrow E$ be the canonical mapping. Then:*

- μ_i is an injective Boolean morphism,
- For all $(x, y) \in E \times E$, there is $i \in \mathbb{N}$ and $(x_i, y_i) \in D_i$ such that $(x, y) = (\mu_i(x_i)\mu_i(y_i))$,
- There is a mapping $\varphi : E \times E \rightarrow E$ defined by:

$$\varphi(\mu_i(x_i)\mu_i(y_i)) = \mu_i \circ \varphi_i(x_i, y_i) \quad \text{for all } i \in \mathbb{N} \text{ and } (x_i, y_i) \in D_i,$$

- The mapping φ is such that:
 - $z \mapsto \varphi(z, x)$ is a Boolean automorphism of E ,
 - $x \subset y$ implies $\varphi(y, x) = \Omega$ or $x = \emptyset$,
 - $x \cap \varphi(y, x) = x \cap y$,
 - $\varphi(\varphi(y, x), x) = \varphi(\varphi(y, x), \sim x) = \varphi(y, x)$,

for all $x, y \in E$.

Proof. Immediate consequence of properties 11, 38, 40, 41, 44, and 45 □

Notation 47. *From now on, it is defined:*

$$E_{i:} = \mu_i(E_i) \quad \text{and} \quad b_{i:} = \mu_i(b_i).$$

2.4 Probability extension

2.4.1 Probability extension of a finite Boolean algebra

Let Π_ϵ be *any* probability distribution defined on E_0 , such that $\Pi_\epsilon(x) > 0$ when $x \neq \emptyset$, and $\epsilon \mapsto \Pi_\epsilon(x)$ is a rational function for all $x \in E_0$ and $\epsilon \in]0, 1]$.

The probability distribution $P_{i,\epsilon} \in \mathbb{P}(E_{i:})$ are constructed by induction for all $\epsilon \in]0, 1]$ and $i \in \mathbb{N}$.

Initial construction. Let $P_{0,\epsilon}$ be defined by $P_{0,\epsilon} \circ \mu_0 = \Pi_\epsilon$.

Inductive construction. It is assumed that the probability distributions $P_{i,\epsilon}$ are defined for all $\epsilon \in]0, 1]$ and $i \leq n$. The following property about the generating partitions of E_{n+1} : is instrumental.

Property 48 (generating partition).

- If $b_n \in \{\emptyset, \Omega\}$, then $\sigma E_{n+1} = \sigma E_n$.
- If $b_n \notin \{\emptyset, \Omega\}$, then $\sigma E_{n+1} = \bigcup_{x \in \{b_n, \sim b_n\}} \{\omega \cap \varphi(v, x) / (\omega, v) \in \sigma E_n.[\sim x, x]\}$.

Proof. The case $b_n \in \{\emptyset, \Omega\}$ being obvious, it is assumed $b_n \notin \{\emptyset, \Omega\}$. Let $\omega_1, \omega_2 \in \sigma E_n.[b_n]$ and $v_1, v_2 \in \sigma E_n.[\sim b_n]$. Then:

$$(\omega_1, v_1) \neq (\omega_2, v_2) \text{ implies } \begin{cases} (\omega_1 \cap \varphi(v_1, \sim b_n)) \cap (\omega_2 \cap \varphi(v_2, \sim b_n)) = \emptyset, \\ (v_1 \cap \varphi(\omega_1, b_n)) \cap (v_2 \cap \varphi(\omega_2, b_n)) = \emptyset. \end{cases}$$

Now:

$$\bigcup_{(\omega, v) \in \sigma E_n.[\sim b_n, b_n]} \omega \cap \varphi(v, b_n) = \sim b_n \text{ and } \bigcup_{(\omega, v) \in \sigma E_n.[\sim b_n, b_n]} v \cap \varphi(\omega, \sim b_n) = b_n.$$

Then the lemma is concluded as a direct consequence of lemma 43, which implies:

case 0 $b_i \neq b_n$ for all $i < n$.

$$v \cap \varphi(\omega, \sim b_n) \neq \emptyset \iff \omega \cap \varphi(v, b_n) \neq \emptyset \iff (\omega, v) \in \sigma E_n.[\sim b_n, b_n],$$

case 1 It is defined the greatest $k < n$ such that $b_k = b_n$.

$$v \cap \varphi(\omega, \sim b_n) \neq \emptyset \iff \omega \cap \varphi(v, b_n) \neq \emptyset \iff (\omega, v) \in \bigcup_{(y, z) \in \sigma E_{k+1}.[\sim b_n, b_n]} \sigma E_n.[y, z].$$

But it happens that:

$$\sigma E_n.[\sim b_n, b_n] = \bigcup_{(y, z) \in \sigma E_{k+1}.[\sim b_n, b_n]} \sigma E_n.[y, z].$$

□

Then, the functions $P_{\epsilon, n+1}$ are constructed by means of the partitions.

Definition 49. Let $\epsilon \in]0, 1]$.

- If $b_n \in \{\emptyset, \Omega\}$, then $P_{\epsilon, n+1} = P_{\epsilon, n}$.
- If $b_n \notin \{\emptyset, \Omega\}$, then:

$$P_{\epsilon, n+1}(\omega \cap \varphi(v, x)) = \frac{P_{\epsilon, n}(\omega)P_{\epsilon, n}(v)}{P_{\epsilon, n}(x)}$$

for all $x \in \{b_n, \sim b_n\}$ and $(\omega, v) \in \sigma E_n.[\sim x, x]$.

Property 50. $P_{\epsilon,n+1}$ is a probability distribution on E_{n+1} : and $P_{\epsilon,n+1}(x) > 0$ for all $x \in E_{n+1} \setminus \{\emptyset\}$.

Proof. $P_{\epsilon,n+1} > 0$ (for non empty propositions) by definition. Now, it is shown that the total probability is 1. The case $b_n \in \{\emptyset, \Omega\}$ being obvious, it is assumed $b_n \notin \{\emptyset, \Omega\}$. It is first deduced for $x \neq \emptyset$:

$$\sum_{(\omega,v) \in \sigma E_n: [\sim x, x]} \frac{P_{\epsilon,n}(\omega)P_{\epsilon,n}(v)}{P_{\epsilon,n}(x)} = \sum_{\omega \in \sigma E_n: [\sim x]} P_{\epsilon,n}(\omega) \frac{\sum_{v \in \sigma E_n: [x]} P_{\epsilon,n}(v)}{P_{\epsilon,n}(x)} = P_{\epsilon,n}(\sim x).$$

As a consequence:

$$\sum_{\substack{x \in \{b_n, \sim b_n\} \\ (\omega,v) \in \sigma E_n: [\sim x, x]}} \frac{P_{\epsilon,n}(\omega)P_{\epsilon,n}(v)}{P_{\epsilon,n}(x)} = 1.$$

□

Properties.

Property 51. For all $i \in \mathbb{N}$ and $x \in E_i$, $\epsilon \mapsto P_{\epsilon,i}(x)$ is a rational function.

Proof. Obvious from the definition. □

Property 52. $P_{\epsilon,i} \subset P_{\epsilon,j}$, i.e. $P_{\epsilon,j}(x) = P_{\epsilon,i}(x)$ for all $x \in E_i$, for all $i, j \in \mathbb{N}$ such that $i \leq j$, and $\epsilon \in]0, 1]$.

Proof. It is equivalent to prove this result for $j = i+1$. The case $b_i \in \{\emptyset, \Omega\}$ being obvious, it is assumed $b_i \notin \{\emptyset, \Omega\}$.

Let $x \in E_i$. Then $x = \bigcup_{\substack{y \in \{b_i, \sim b_i\} \\ \omega \in \sigma E_i: [x \cap y]}} \omega$, and:

$$x = \bigcup_{\substack{y \in \{b_i, \sim b_i\} \\ \omega \in \sigma E_i: [x \cap \sim y]}} \left(\omega \cap \bigcup_{v \in \sigma E_i: [y]} \varphi(v, y) \right) = \bigcup_{\substack{y \in \{b_i, \sim b_i\} \\ (\omega,v) \in \sigma E_i: [x \cap \sim y, y]}} (\omega \cap \varphi(v, y)).$$

It is thus derived:

$$P_{\epsilon,i+1}(x) = \sum_{\substack{y \in \{b_i, \sim b_i\} \\ (\omega,v) \in \sigma E_i: [x \cap \sim y, y]}} \frac{P_{\epsilon,i}(\omega)P_{\epsilon,i}(v)}{P_{\epsilon,i}(y)} = \sum_{\substack{y \in \{b_i, \sim b_i\} \\ \omega \in \sigma E_i: [x \cap \sim y]}} P_{\epsilon,i}(\omega) \frac{\sum_{v \in \sigma E_i: [y]} P_{\epsilon,i}(v)}{P_{\epsilon,i}(y)} = P_{\epsilon,i}(x).$$

□

As an immediate consequence of the previous constructions, it is deduced:

Property 53 (limit). For all $\epsilon \in]0, 1]$, it is defined the probability distribution $P_\epsilon \in \mathbb{P}(E)$ by:

$$P_\epsilon(x) = P_{\epsilon,i}(x), \text{ for all } x \in E_i.$$

These probabilities verify:

- $\epsilon \mapsto P_\epsilon(x)$ is a rational function for all $x \in E$,
- $P_\epsilon(x) = \Pi_\epsilon(x)$ for all $\epsilon \in]0, 1]$ and $x \in E_o$,
- $P_\epsilon(x) > 0$ for all $\epsilon \in]0, 1]$ and $x \in E \setminus \{\emptyset\}$,
- $P_\epsilon(\sim x \cap y \cap \varphi(x \cap y, x)) = \frac{P_\epsilon(\sim x \cap y)P_\epsilon(x \cap y)}{P_\epsilon(x)}$, for all $y \in E$ and $x \in E \setminus \{\emptyset, \Omega\}$.

Conclusion. As a direct consequence of the previous construction, it is deduced the following property: property:

Property 54 (extension of a finite Bayesian algebra). *For all finite Boolean algebra E_o , there are a countable Boolean algebra E , an injective Boolean morphism $\mu: E_o \rightarrow E$ and an operator $\varphi: E \times E \rightarrow E$ such that:*

- For all $x, y \in E$:
 - $z \mapsto \varphi(z, x)$ is a Boolean automorphism of E ,
 - $x \subset y$ implies $\varphi(y, x) = \Omega$ or $x = \emptyset$,
 - $x \cap \varphi(y, x) = x \cap y$,
 - $\varphi(\varphi(y, x), x) = \varphi(\varphi(y, x), \sim x) = \varphi(y, x)$,
- If F is a Boolean algebra such that $E_o \subset F$ and $x, y \in F \Rightarrow \varphi(x, y) \in F$, then $E = F$, (i.e. E is generated by φ and E_o)
- For all family $(\Pi_\epsilon)_{\epsilon \in]0, 1]} \in \mathbb{P}(E_o)^{]0, 1]}$ such that:
 - $\epsilon \mapsto \Pi_\epsilon(x)$ is a rational function for all $x \in E_o$,
 - $\Pi_\epsilon(x) > 0$ for all $\epsilon \in]0, 1]$ and $x \in E_o \setminus \{\emptyset\}$,

there is a family $(P_\epsilon)_{\epsilon \in]0, 1]} \in \mathbb{P}(E)^{]0, 1]}$ defined by $P_\epsilon \circ \mu = \Pi_\epsilon$ and:

$$P_\epsilon(\sim x \cap y \cap \varphi(x \cap y, x)) = \frac{P_\epsilon(\sim x \cap y)P_\epsilon(x \cap y)}{P_\epsilon(x)}, \quad \text{for all } y \in E \text{ and } x \in E \setminus \{\emptyset, \Omega\}, \quad (5)$$

for all $\epsilon \in]0, 1]$.

Then $\epsilon \mapsto P_\epsilon(x)$ is a rational function for all $x \in E$, and $P_\epsilon(x) > 0$ for all $\epsilon \in]0, 1]$ and $x \in E \setminus \{\emptyset\}$.

It is noticed, as shown in previous section, that the induction (5) is sufficient for defining P_ϵ .

2.4.2 Probability extension of a tangible Boolean algebra

Let E_o be a tangible Boolean algebra. Let $\mathbb{B}(E_o)$ the set of *finite* Boolean subalgebra. It is noticed that $(\mathbb{B}(E_o), \subset)$ is a directed set. By applying property 54, there are for each $F \in \mathbb{B}(E_o)$ a countable Boolean algebra \overline{F} , an injective Boolean morphism $\mu_{o,F} : F \rightarrow \overline{F}$ and an operator $\varphi_F : \overline{F} \times \overline{F} \rightarrow \overline{F}$, which verify the characteristics of property 54. Then, it is defined, for all $F, G \in \mathbb{B}(E_o)$ such that $F \subset G$, the mappings $\mu_{F,G} : \overline{F} \rightarrow \overline{G}$ by induction:

$$\begin{cases} \mu_{F,G} \circ \mu_{o,F} = \mu_{o,G} , \\ \mu_{F,G}(x \cap y) = \mu_{F,G}(x) \cap \mu_{F,G}(y) \text{ and } \mu_{F,G}(\sim x) = \sim \mu_{F,G}(x) , \\ \mu_{F,G}(\varphi_F(x, y)) = \varphi_G(\mu_{F,G}(x), \mu_{F,G}(y)) . \end{cases}$$

Then $\mu_{F,G}$ are Boolean morphisms by construction, such that $\mu_{G,H} \circ \mu_{F,G} = \mu_{F,G}$ and $\mu_{F,F} = \text{id}_{\overline{F}}$ for all $F \leq G \leq H$. As a consequence, $(\overline{F}, \mu_{F,G})_{F,G \in \mathbb{B}(E_o)}$ is a direct system.

Let $E = \varinjlim \overline{F} - E$ is a Boolean algebra and $\mu_F : \overline{F} \rightarrow E$ be the canonical Boolean morphism defined for all $F \in \mathbb{B}(E_o)$. Then $\mu_F \circ \mu_{o,F} \subset \mu_G \circ \mu_{o,G}$, for all $F, G \in \mathbb{B}(E_o)$ such that $F \subset G$. Define the Boolean morphism $\mu_o : E_o \rightarrow E$ by $\mu_o = \bigcup_{F \in \mathbb{B}(E_o)} \mu_{o,F}$. Since $\mu_{o,F}$ is injective for all $F \in \mathbb{B}(E_o)$, it is deduced by construction that μ_o is injective. By construction, it is also true that $\mu_F \circ \varphi_F = \mu_G \circ \varphi_G \circ (\mu \times \mu)_{F,G}$ for all $F, G \in \mathbb{B}(E_o)$ such that $F \subset G$. Define the mapping $\varphi : E \times E \rightarrow E$ by $\varphi \circ (\mu \times \mu)_F = \varphi_F$ for all $F \in \mathbb{B}(E_o)$. Then φ inherits the characteristics of the mappings φ_F :

- $z \mapsto \varphi(z, x)$ is a Boolean automorphism of E ,
- $x \subset y$ implies $\varphi(y, x) = \Omega$ or $x = \emptyset$,
- $x \cap \varphi(y, x) = x \cap y$ and $\varphi(\varphi(y, x), x) = \varphi(\varphi(y, x), \sim x) = \varphi(y, x)$,

for all $x, y \in E$.

Now, let $(\Pi_\epsilon)_{\epsilon \in]0,1]} \in \mathbb{P}(E_o)^{]0,1]}$ be such that:

- $\epsilon \mapsto \Pi_\epsilon(x)$ is a rational function for all $x \in E_o$,
- $\Pi_\epsilon(x) > 0$ for all $\epsilon \in]0,1]$ and $x \in E_o \setminus \{\emptyset\}$,

Since \overline{F} , $\mu_{o,F}$ and φ_F are defined by proposition 54 for all $F \in \mathbb{B}(E_o)$, then there is a family $(P_{F,\epsilon})_{\epsilon \in]0,1]} \in \mathbb{P}(\overline{F})^{]0,1]}$ defined by:

$$P_{F,\epsilon} \circ \mu_{o,F}(x) = \Pi_\epsilon(x) \text{ for all } x \in F ,$$

and:

$$P_{F,\epsilon}(\sim x \cap y \cap \varphi_F(x \cap y, x)) = \frac{P_{F,\epsilon}(\sim x \cap y) P_{F,\epsilon}(x \cap y)}{P_{F,\epsilon}(x)} , \text{ for all } y \in \overline{F} \text{ and } x \in \overline{F} \setminus \{\emptyset, \Omega\} ,$$

for all $F \in \mathbb{B}(E_o)$ and $\epsilon \in]0,1]$. The definitions are the same modulo a morphism, and it is thus deduced:

$$P_{G,\epsilon} \circ \mu_{F,G} = P_{F,\epsilon} \text{ for all } \epsilon \in]0,1] \text{ and } F, G \in \mathbb{B}(E_o) \text{ such that } F \subset G .$$

Then, it is defined $(P_\epsilon)_{\epsilon \in]0,1]} \in \mathbb{P}(E)^{]0,1]}$ by setting:

$$P_\epsilon \circ \mu_F = P_{F,\epsilon}, \quad \text{for all } F \in \mathbb{B}(E_o) \text{ and } \epsilon \in]0,1].$$

The distributions P_ϵ inherits the characteristics of $P_{F,\epsilon}$, so that for all $\epsilon \in]0,1]$:

$$P_\epsilon(\sim x \cap y \cap \varphi(x \cap y, x)) = \frac{P_\epsilon(\sim x \cap y)P_\epsilon(x \cap y)}{P_\epsilon(x)}, \quad \text{for all } y \in E \text{ and } x \in E \setminus \{\emptyset, \Omega\},$$

$P_\epsilon(x) > 0$ for all $x \in E \setminus \{\emptyset\}$ and $P_\epsilon \circ \mu_o = \Pi_\epsilon$. Moreover $\epsilon \mapsto P_\epsilon(x)$ is a rational function for all $x \in E$. The following property is thus deduced, which generalizes property 54.

Property 55 (extension of a tangible Boolean algebra). *For all tangible Boolean algebra E_o , there is a tangible Boolean algebra E , an injective Boolean morphism $\mu : E_o \rightarrow E$ and an operator $\varphi : E \times E \rightarrow E$ such that:*

- *If E_o is finite, then E is countable; If E_o is infinite, then $\text{card}E_o = \text{card}E$,*
- *For all $x, y \in E$:*
 - *$z \mapsto \varphi(z, x)$ is a Boolean automorphism of E ,*
 - *$x \subset y$ implies $\varphi(y, x) = \Omega$ or $x = \emptyset$,*
 - *$x \cap \varphi(y, x) = x \cap y$,*
 - *$\varphi(\varphi(y, x), x) = \varphi(\varphi(y, x), \sim x) = \varphi(y, x)$,*
- *If F is a Boolean algebra such that $E_o \subset F$ and $x, y \in F \Rightarrow \varphi(x, y) \in F$, then $E = F$, (i.e. E is generated by φ and E_o)*
- *For all family $(\Pi_\epsilon)_{\epsilon \in]0,1]} \in \mathbb{P}(E_o)^{]0,1]}$ such that:*
 - *$\epsilon \mapsto \Pi_\epsilon(x)$ is a rational function for all $x \in E_o$,*
 - *$\Pi_\epsilon(x) > 0$ for all $\epsilon \in]0,1]$ and $x \in E_o \setminus \{\emptyset\}$,*

there is a family $(P_\epsilon)_{\epsilon \in]0,1]} \in \mathbb{P}(E)^{]0,1]}$ defined by $P_\epsilon \circ \mu = \Pi_\epsilon$ and:

$$P_\epsilon(\sim x \cap y \cap \varphi(x \cap y, x)) = \frac{P_\epsilon(\sim x \cap y)P_\epsilon(x \cap y)}{P_\epsilon(x)}, \quad \text{for all } y \in E \text{ and } x \in E \setminus \{\emptyset, \Omega\}, \tag{6}$$

for all $\epsilon \in]0,1]$.

Then $\epsilon \mapsto P_\epsilon(x)$ is a rational function for all $x \in E$, and $P_\epsilon(x) > 0$ for all $\epsilon \in]0,1]$ and $x \in E \setminus \{\emptyset\}$.

2.5 Main theorem

2.5.1 Definitions of Bayesian structures

Definition 56 (Bayesian algebra). *The septuple $(E, \cap, \cup, \sim, \emptyset, \Omega, [\])$ is called a Bayesian algebra if:*

- E , considered as $(E, \cap, \cup, \sim, \emptyset, \Omega)$, is a Boolean algebra,
 - The operator $[\]$ is such that:
 - Bool:** The conditional mapping $\rho_x : z \mapsto [x]z$ is a Boolean automorphism of E ,
 - Def:** $x \subset y$ implies $[x]y = \Omega$ or $x = \emptyset$,
 - Inf:** $x \cap [x]y = x \cap y$,
 - Ind:** $[x][x]y = [\sim x][x]y = [x]y$,
- for all $x, y \in E$.

Definition 57 (Bayesian morphism). *Let $(E, \cap, \cup, \sim, \emptyset, \Omega, [\])$ and $(F, \cap, \cup, \sim, \emptyset, \Omega, [\])$ be two Bayesian algebras. A mapping $\mu : E \rightarrow F$ is a Bayesian morphism if:*

- $\mu(x \cap y) = \mu(x) \cap \mu(y)$,
- $\mu(\sim x) = \sim \mu(x)$,
- $\mu([x]y) = [\mu(x)]\mu(y)$,

for all $x, y \in E$.

The intuition behind. It is not difficult to understand the characteristic **Bool**, which implies a Boolean behavior of the conditioned proposition; it is related to the fact that conditional probabilities are actually probabilities. The characteristic **Inf** deals with the fact that a conditional proposition is a kind of inference; for example, $P(y|x) = 1$ and $P(x) = 1$ imply $P(x \cap y) = 1$. The characteristic **Def** deals with definition of a conditional proposition; it is related to the fact that $x \subset y$ and $P(x) > 0$ imply $P(Y|x) > 0$, the case $P(x)$ being *undefined*. The characteristic **Ind** is related to the notion of probabilistic independence, which may be defined as $P(y|x) = P(y)$ (y is independent of x in regards to P). Actually, the relation $[x]y = y$ means an algebraic independence of y in regards to x . In particular, **Ind** is related to the probability relation: $P(y|x) = P(y)$ implies $P(y|\sim x) = P(y)$.

2.5.2 Probability distribution on a Bayesian algebra

Definition 58. *Let $(E, \cap, \cup, \sim, \emptyset, \Omega, [\])$ be a Bayesian algebra. A non negative mapping $P : E \rightarrow \mathbb{R}_+$ is a probability distribution on the Bayesian algebra E if:*

- $P(\emptyset) = 0$ and $P(\Omega) = 1$,
- $P(x \cap y) + P(x \cup y) = P(x) + P(y)$,

- $P(x \cap y) = P([x]y)P(x)$,

for all $x, y \in E$.

Notation 59. The set of probability distributions defined on a Bayesian algebra E is denoted $\mathbf{IP}(E)$.

While a Bayesian algebra E is a Boolean algebra, a probability defined on the Boolean algebra E is not necessary a probability defined on the Bayesian algebra E .

2.5.3 Properties

Property 60. Let $(E, \cap, \cup, \sim, \emptyset, \Omega, [\])$ be a Bayesian algebra. Let $x, y \in E$. Then $\{[x]x, [x] \sim x\} = \{\emptyset, \Omega\}$ and $[[x]x]y = [[x] \sim x]y = [\emptyset]y = [\Omega]y = y$.

Proof. Let $x \in E$. From **Inf**, it is deduced $[\Omega]x = x$. Then $[\emptyset]x = [\emptyset][\Omega]x = [\Omega]x = x$ from **Ind**. As a consequence, $\{[\emptyset]\emptyset, [\emptyset] \sim \emptyset\} = \{\emptyset, \Omega\}$. Now, **Def** imply $x = \emptyset$ or $[x]x = \Omega$. As a consequence, $[x]x \in \{\emptyset, \Omega\}$, and $\{[x]x, [x] \sim x\} = \{\emptyset, \Omega\}$, by applying **Bool**. The second point is then immediate. \square

Definition 61 (independence). Let $(E, \cap, \cup, \sim, \emptyset, \Omega, [\])$ be a Bayesian algebra and $x, y \in E$. Then y is said to be independent of –or free of– x if $[x]y = y$.

Corollary 62 (self independence). Let $(E, \cap, \cup, \sim, \emptyset, \Omega, [\])$ be a Bayesian algebra and $x \in E$. If x is independent of x , i.e. $[x]x = x$, then $x \in \{\emptyset, \Omega\}$.

Property 63 (total probability). Let $(E, \cap, \cup, \sim, \emptyset, \Omega, [\])$ be a Bayesian algebra. Let $(x_i)_{i=1:s} \in E^s$ be such that $\bigcup_{i=1:s} x_i = E$. Then $y = \bigcup_{i=1:s} (x_i \cap [x_i]y)$ for all $y \in E$, and

$\prod_{i=1:s} \rho_{x_i} : y \mapsto ([x_i]y)_{i=1:s}$ is an injective Boolean morphism from E to E^s .

Assume moreover that $x_i \cap x_j = \emptyset$ for all $i \neq j$, and let $P \in \mathbf{IP}(E)$. Then:

$$P(y) = \sum_{i=1:s} P(x_i)P([x_i]y).$$

Proof. It is derived $y = \bigcup_{i=1:s} (y \cap x_i) = \bigcup_{i=1:s} (x_i \cap [x_i]y)$. Then:

$$\left\{ \begin{array}{l} \prod_{i=1:s} \rho_{x_i}(y) = \prod_{i=1:s} \rho_{x_i}(z) \text{ implies } y = \bigcup_{i=1:s} (x_i \cap [x_i]y) = \bigcup_{i=1:s} (x_i \cap [x_i]z) = z, \\ \forall i, j \in \mathbb{N}, i < j \Rightarrow x_i \cap x_j = \emptyset \text{ implies } P(y) = \sum_{i=1:s} P(x_i \cap [x_i]y). \end{array} \right.$$

\square

Property 64 (Reinterpretation of the triviality). Let $(E, \cap, \cup, \sim, \emptyset, \Omega, [\])$ be a Bayesian algebra. Let $x \in E$ and $P \in \mathbf{IP}(E)$. Define $P_x = P \circ \rho_x$ and $P_{\sim x} = P \circ \rho_{\sim x}$. Then $P_x \in \mathbb{P}(E)$ and $P_{\sim x} \in \mathbb{P}(E)$, since ρ_x and $\rho_{\sim x}$ are Boolean morphisms. In general, however, ρ_x and $\rho_{\sim x}$ are not Bayesian morphisms, and $P_x, P_{\sim x} \notin \mathbf{IP}(E)$. More precisely, such hypotheses imply trivialities:

- If ρ_x is a Bayesian morphism, then $x \in \{\emptyset, \Omega\}$,
- (Lewis' triviality) If $P_x \in \mathbf{IP}(E)$ and $P_{\sim x} \in \mathbf{IP}(E)$, then $P(x \cap y) = P(x)P(y)$ for all $y \in E$ such that $P(x \cap y) > 0$ and $P(\sim x \cap y) > 0$.

Proof.

First point. Assume that ρ_x is a Bayesian morphism. Then:

$$[\sim x]y = [x][\sim x]y = \rho_x([\sim x]y) = [\rho_x(\sim x)]\rho_x(y) = [[x] \sim x][x]y = [x]y .$$

Then $x = (x \cap [x]x) \cup (\sim x \cap [\sim x]x) = (x \cap [x]x) \cup (\sim x \cap [x]x) = [x]x \in \{\emptyset, \Omega\}$.

Second point. Assume x, y such that $P(x \cap y) > 0$ and $P(\sim x \cap y) > 0$. Then $P_x(y) \neq 0$, $P_{\sim x}(y) \neq 0$. Assume moreover that $P_x, P_{\sim x} \in \mathbf{IP}(E)$. Then:

$$\begin{aligned} \frac{P(x \cap y)}{P(y)} &= P([y]x) = P(x)P([x][y]x) + P(\sim x)P([\sim x][y]x) = P(x)P_x([y]x) \\ &+ P(\sim x)P_{\sim x}([y]x) = P(x)\frac{P_x(x \cap y)}{P_x(y)} + P(\sim x)\frac{P_{\sim x}(x \cap y)}{P_{\sim x}(y)} = P(x) + 0 = P(x) . \end{aligned}$$

□

The triviality implies constraints on the probability distribution, which exclude most practical distributions. As a consequence, the hypothesis $P_x \in \mathbf{IP}(E)$ is irrelevant in general. The following extension theorem actually confirms that this hypothesis is generally false.

2.5.4 Extension theorem

Definition 65 (Bayesian extension of a Boolean algebra). *Let $(B_{\text{bool}}, \cap, \cup, \sim, \emptyset, \Omega)$ be a Boolean algebra. Then, a Bayesian algebra $(B_{\text{ayes}}, \cap, \cup, \sim, \emptyset, \Omega, [\])$ is a Bayesian extension of B_{bool} if:*

- There is an injective Boolean morphism $\mu : B_{\text{bool}} \rightarrow B_{\text{ayes}}$,
- Given any $P_{\text{bool}} \in \mathbb{P}(B_{\text{bool}})$, there is $P_{\text{ayes}} \in \mathbf{IP}(B_{\text{ayes}})$ such that $P_{\text{ayes}} \circ \mu = P_{\text{bool}}$.

By definition, a Bayesian extension of a Boolean algebra will avoid Lewis' triviality.

Lemma 66. *Let $(E, \cap, \cup, \sim, \emptyset, \Omega, [\])$ be a Bayesian algebra. Assume $P \in \mathbb{P}(E)$ such that:*

$$P(x)P(\sim x \cap y \cap [x](x \cap y)) = P(\sim x \cap y)P(x \cap y) , \quad \text{for all } x, y \in E .$$

Then $P \in \mathbf{IP}(E)$.

Proof. Let $x, y \in E$. Since $P(\emptyset) = 0$, it is deduced:

$$\begin{aligned} P(x)P([x]y) &= P(x)P\left([x](x \cap (\sim x \cup y))\right) = P(x)\left(P\left(x \cap [x](x \cap (\sim x \cup y))\right)\right. \\ &\quad \left.+ P\left(\sim x \cap (\sim x \cup y) \cap [x](x \cap (\sim x \cup y))\right)\right) = P(x)P(x \cap y) \\ &\quad + P(\sim x \cap (\sim x \cup y))P(x \cap (\sim x \cup y)) = P(x \cap y) . \end{aligned}$$

□

Theorem 2 (Bayesian extension of tangible Boolean algebra). *Let $(B_{\text{bool}}, \cap, \cup, \sim, \emptyset, \Omega)$ be a tangible Boolean algebra. Then, there is a Bayesian extension $(B_{\text{ayes}}, \cap, \cup, \sim, \emptyset, \Omega, [\])$ of B_{bool} , such that:*

- B_{ayes} is a tangible,
- If B_{bool} is finite, then B_{ayes} is countable; If B_{bool} is infinite, then $\text{card}B_{\text{ayes}} = \text{card}B_{\text{bool}}$,

Proof. By applying property 55, there is a tangible Boolean algebra B_{ayes} , an injective Boolean morphism $\mu : B_{\text{bool}} \rightarrow B_{\text{ayes}}$ and an operator $\varphi : B_{\text{ayes}} \times B_{\text{ayes}} \rightarrow B_{\text{ayes}}$ such that:

- $z \mapsto \varphi(z, x)$ is a Boolean automorphism of B_{ayes} ,
- $x \subset y$ implies $\varphi(y, x) = \Omega$ or $x = \emptyset$,
- $x \cap \varphi(y, x) = x \cap y$,
- $\varphi(\varphi(y, x), x) = \varphi(\varphi(y, x), \sim x) = \varphi(y, x)$,

for any $x, y \in E$. As a consequence, $(B_{\text{ayes}}, \cap, \cup, \sim, \emptyset, \Omega, \varphi)$ is a Bayesian algebra.

Now, assume $\Pi \in \mathbb{P}(B_{\text{bool}})$. Since B_{bool} is tangible, and by applying property 27, there is $(\Pi_\epsilon)_{\epsilon \in]0,1]}$ such that $\Pi_\epsilon(x) > 0$ for all $\epsilon \in]0,1]$ and $x \in E \setminus \{\emptyset\}$, $\epsilon \mapsto \Pi_\epsilon(x)$ is a rational function and $\lim_{\epsilon \rightarrow 0^+} \Pi_\epsilon(x) = \Pi(x)$ for all $x \in B_{\text{bool}}$. By property 55, there is $(P_\epsilon)_{\epsilon \in]0,1]} \in \mathbb{P}(B_{\text{ayes}})^{]0,1]}$ defined by $P_\epsilon \circ \mu = \Pi_\epsilon$ and:

$$P_\epsilon(\sim x \cap y \cap \varphi(x \cap y, x)) = \frac{P_\epsilon(\sim x \cap y)P_\epsilon(x \cap y)}{P_\epsilon(x)}, \quad \text{for all } y \in B_{\text{ayes}} \text{ and } x \in B_{\text{ayes}} \setminus \{\emptyset, \Omega\},$$

for all $\epsilon \in]0,1]$. Since $\epsilon \mapsto P_\epsilon(x)$ is rational, it is possible to define $P(x) = \lim_{\epsilon \rightarrow 0^+} P_\epsilon(x)$ for all $x \in B_{\text{ayes}}$. Then, it is clear that $P \in \mathbb{P}(B_{\text{ayes}})$, $P \circ \mu = \Pi$ and:

$$P(x)P(\sim x \cap y \cap \varphi(x \cap y, x)) = P(\sim x \cap y)P(x \cap y), \quad \text{for all } x, y \in B_{\text{ayes}}.$$

By applying lemma 66, $P \in \mathbf{IP}(B_{\text{ayes}})$.

The property on the cardinality of B_{ayes} is derived from property 55, and it is obvious that B_{ayes} is tangible thanks to P_ϵ . \square

Next section presents an application of the notion of Bayesian algebra to the domain of logic.

3 Toward a Deterministic Bayesian Logic

It is known that Boolean algebra are the models for classical propositional logic. Similarly, the Bayesian algebras have a logical interpretation. In this section, the *Deterministic Bayesian Logic* (DBL) is introduced concisely as a logical abstraction of Bayesian algebras. Since Bayesian algebras are Boolean algebras, this logic also implements classical logical operators. But in addition, DBL implements a Bayesian operator, while being bivalent. This logic has been introduced in previous works [8].

3.1 Language of Deterministic Bayesian Logic

Notation 67. It is defined \mathcal{P} , a set of atomic propositions.

Definition 68 (Language of DBL). The set \mathcal{L} of Bayesian propositions is defined inductively by:

1. $[\perp] \in \mathcal{L}$ and $[p] \in \mathcal{L}$ for all $p \in \mathcal{P}$,
2. $[X \rightarrow Y] \in \mathcal{L}$ for all $[X] \in \mathcal{L}$ and $[Y] \in \mathcal{L}$,
3. $[X|Y] \in \mathcal{L}$ for all $[X] \in \mathcal{L}$ and $[Y] \in \mathcal{L}$,
4. $[X_1|\dots|X_n] \in \mathcal{L}$ for all $[X_1], \dots, [X_n] \in \mathcal{L}$.

The set of classical propositions, $\mathcal{L}_C \subset \mathcal{L}$, is defined inductively by step 1 and 2.

Notation 69. Are defined $\neg X \triangleq X \rightarrow \perp$, $X \vee Y \triangleq \neg X \rightarrow Y$, $X \wedge Y \triangleq \neg(\neg X \vee \neg Y)$, $\top \triangleq \neg \perp$ and $X \leftrightarrow Y \triangleq (X \rightarrow Y) \wedge (Y \rightarrow X)$. The Greek uppercase letters $\Gamma, \Delta, \Lambda, \Pi$ are notations for sequences of propositions like $X_1|\dots|X_n$ (without $[]$).

Explanation of language format \perp is called the *contradiction* and \top is called the *tautology*. \rightarrow, \neg, \vee and \wedge are called respectively the *classical inference*, the *negation*, the *disjunction* and the *conjunction*. $[]$ is called the *conditional modality*. The delimiters $[|]$ are interpreted as meta-disjunctions:

A proposition of the form $[X]$ is interpreted as (*X is a tautology*), while $[X|Y|Z]$ is interpreted as (*X is a tautology*) OR (*Y is a tautology*) OR (*Z is a tautology*).

3.2 Axioms and rules of Deterministic Bayesian Logic

Definition 70. The axioms and rules of DBL are:

- *Classical axioms*,
- *Modus Ponens (MP)*.

From $[\Gamma|X]$ and $[\Delta|X \rightarrow Y]$, it is deduced $[\Gamma|\Delta|Y]$,

- *Meta-permutation ($\mu\mathbf{P}$)*. Let $\sigma : \{1 : n\} \rightarrow \{1 : n\}$ be a bijective mapping.

From $[X_1|\dots|X_n]$, it is deduced $[X_{\sigma(1)}|\dots|X_{\sigma(n)}]$,

- *Meta-contraction ($\mu\mathbf{C}$)*. From $[\Gamma|X|X]$, it is deduced $[\Gamma|X]$,
- *Meta-weakening ($\mu\mathbf{W}$)*. From $[\Gamma]$, it is deduced $[\Gamma|X]$,

- *Bayesian axioms and rules:*

Cd : From $[\Gamma|X \rightarrow Y]$, it is deduced $[\Gamma|\neg X|[X]Y]$,

K : $[[X](Y \rightarrow Z) \rightarrow ([X]Y \rightarrow [X]Z)]$,

Cd⁻¹: $[[X]Y \rightarrow (X \rightarrow Y)]$,

Neg: $[[X]\neg Y \leftrightarrow \neg[X]Y]$,

Ind: From $[\Gamma|Y \leftrightarrow \neg X]$ and $[\Gamma|[X]Z \leftrightarrow Z]$, it is deduced $[\Gamma|[Y]Z \leftrightarrow Z]$,

for all $X, Y, Z, X_{1:n} \in \mathcal{L}$.

The well known *Modus Ponens* rule infers a conclusion from a hypothesis and an inference. Since the propositions are meta-disjunctions of simple propositions, the Modus Ponens concatenates these meta-disjunctions. The meta-permutation, meta-contraction and meta-weakening are rules for the meta-disjunction. These rules should not be confused with the structural rules of sequent calculus. The Bayesian rule **Cd** is a logical counterpart to the algebraic characteristic **Def**. The axioms **K** and **Neg** are together logical counterparts to the algebraic characteristic **Bool**. The axioms **Cd⁻¹** is a logical counterpart to the algebraic characteristic **Inf**. At last, the axiom **Ind** is a logical counterpart to the algebraic characteristic **Ind**.

The conditional operator may be seen as some kind of modality [9]. In particular, it is noticed that axiom **K** is literally the *distribution axiom* of modal logic. The axiom **Cd** may be compared as a weakened variant of the *necessitation rule* N; by the way, it is equivalent to N for the modality $[\top]$ as soon as we assume that $\neg\top$ is not a tautology (consistency). Similarly, **Cd⁻¹** identifies to the *reflexivity axiom* for the modality $[\top]$. Nevertheless, the axioms of DBL depart from the usual modal axioms, in order to deal with the Boolean nature of the modalities, which are expressed typically by axioms **K** and **Neg**. Moreover, rule **Cd** is a quite unusual rule, since it infers a meta-disjunction. By doing so, it implies a coherent behavior in regards to the Bayesian interpretation of the modality, but also introduces a notion of *independence* which is encoded within the rules of DBL. This notion of independence will be discussed in section 3.6.

Definition 71 (theorems). *A proposition $[\Gamma] \in \mathcal{L}$ is proved in DBL, if it is deduced by a sequence of axioms and rules of DBL. If $[\Gamma] \in \mathcal{L}$ is proved in DBL, then it is denoted $\vdash [\Gamma]$. A proposition proved in DBL is called a theorem of DBL.*

3.3 Model

Definition 72 (semantic). *Let E be a Bayesian algebra. Let $\alpha : \mathcal{P} \rightarrow E$ be a mapping; α is called an assignment mapping from \mathcal{P} to E . Define the mapping $\bar{\alpha} : \mathcal{L} \rightarrow E$ by:*

- $\bar{\alpha}(\perp) = \emptyset$ and $\bar{\alpha}(p) = \alpha(p)$ for all $p \in \mathcal{P}$,
- $\bar{\alpha}(X \rightarrow Y) = \sim \bar{\alpha}(X) \cup \bar{\alpha}(Y)$ for all $X, Y \in \mathcal{L}$,
- $\bar{\alpha}([X]Y) = [\bar{\alpha}(X)]\bar{\alpha}(Y)$ for all $X, Y \in \mathcal{L}$.

Proposition $[X_1|\dots|X_n] \in \mathcal{L}$ is true for the model (E, α) , if there is $i \in \{1 : n\}$ such that $\bar{\alpha}(X_i) = \Omega$. Then it is written $E, \alpha \Vdash [X_1|\dots|X_n]$.

Definition 73 (model). Let E be a Bayesian algebra and α an assignment mapping. (E, α) is a model of DBL if $\vdash [\Gamma] \implies E, \alpha \Vdash [\Gamma]$ for all $[\Gamma] \in \mathcal{L}$. It is a complete model if $\vdash [\Gamma] \iff E, \alpha \Vdash [\Gamma]$ for all $[\Gamma] \in \mathcal{L}$.

Property 74 (model). Let $B_{oole}^{\mathcal{P}}$ be the free Boolean algebra generated by \mathcal{P} ; $B_{oole}^{\mathcal{P}}$ is tangible. Let $B_{aye}^{\mathcal{P}}$ be a Bayesian extension of $B_{oole}^{\mathcal{P}}$ and let $\mu^{\mathcal{P}} : B_{oole}^{\mathcal{P}} \rightarrow B_{aye}^{\mathcal{P}}$ be an injective morphism. Let $\alpha_{\mathcal{P}} : \mathcal{P} \rightarrow B_{oole}^{\mathcal{P}}$ be defined by $\alpha_{\mathcal{P}}(p) = \mu(p)$ for any $p \in \mathcal{P}$. Then $(B_{aye}^{\mathcal{P}}, \alpha_{\mathcal{P}})$ is a model of DBL.

Proof. Since $B_{aye}^{\mathcal{P}}$ is Boolean algebra, it is known that $\bar{\alpha}_{\mathcal{P}}(X) = \Omega$ for any proposition $X \in \mathcal{L}$ which is a classical axiom. It is deduced from the Boolean structure the following algebraic equivalent of *Modus Ponens*:

$$\left. \begin{array}{l} \bar{\alpha}_{\mathcal{P}}(X) = \Omega \\ \text{or } \exists i \in \{1 : n\}, \bar{\alpha}_{\mathcal{P}}(Z_i) = \Omega \\ \bar{\alpha}_{\mathcal{P}}(Y) \cup \sim \bar{\alpha}_{\mathcal{P}}(X) = \Omega \\ \text{or } \exists i \in \{1 : m\}, \bar{\alpha}_{\mathcal{P}}(Z_{i+n}) = \Omega \end{array} \right\} \Rightarrow [\bar{\alpha}_{\mathcal{P}}(Y) = \Omega \text{ or } \exists i \in \{1 : n + m\}, \bar{\alpha}_{\mathcal{P}}(Z_i) = \Omega].$$

It is also easy to check that the meta-rules $\mu\mathbf{P}$, $\mu\mathbf{C}$ and $\mu\mathbf{W}$ are implied by the definition of \Vdash . Now, let consider the Bayesian rules and axioms.

The Bayesian characteristic **Def** is exactly an algebraic equivalent of rule **Cd**. It is deduced from the Bayesian characteristic **Bool** that:

$$\left\{ \begin{array}{l} [\bar{\alpha}_{\mathcal{P}}(X)](\sim \bar{\alpha}_{\mathcal{P}}(Y) \cup \bar{\alpha}_{\mathcal{P}}(Y)) \subset \sim [\bar{\alpha}_{\mathcal{P}}(X)]\bar{\alpha}_{\mathcal{P}}(Y) \cup [\bar{\alpha}_{\mathcal{P}}(X)]\bar{\alpha}_{\mathcal{P}}(Z), \\ [\bar{\alpha}_{\mathcal{P}}(X)] \sim \bar{\alpha}_{\mathcal{P}}(Y) = \sim [\bar{\alpha}_{\mathcal{P}}(X)]\bar{\alpha}_{\mathcal{P}}(Y), \end{array} \right.$$

which implies $\bar{\alpha}_{\mathcal{P}}(X) = \Omega$ for any instance of axiom **K** or **Neg**. From **Ind** it is deduced $[\bar{\alpha}_{\mathcal{P}}(X)]\bar{\alpha}_{\mathcal{P}}(Y) = \Omega \Rightarrow [\sim \bar{\alpha}_{\mathcal{P}}(X)]\bar{\alpha}_{\mathcal{P}}(Y)$ which implies an algebraic equivalent of rule **Ind**. At last, **Inf** implies:

$$\begin{aligned} [\bar{\alpha}_{\mathcal{P}}(X)]\bar{\alpha}_{\mathcal{P}}(Y) &\subset \sim \bar{\alpha}_{\mathcal{P}}(X) \cup (\bar{\alpha}_{\mathcal{P}}(X) \cap [\bar{\alpha}_{\mathcal{P}}(X)]\bar{\alpha}_{\mathcal{P}}(Y)) \\ &= \sim \bar{\alpha}_{\mathcal{P}}(X) \cup (\bar{\alpha}_{\mathcal{P}}(X) \cap \bar{\alpha}_{\mathcal{P}}(Y)) = \sim \bar{\alpha}_{\mathcal{P}}(X) \cup \bar{\alpha}_{\mathcal{P}}(Y), \end{aligned}$$

which implies $\bar{\alpha}_{\mathcal{P}}(X) = \Omega$ for any instance of axiom **Cd**⁻¹.

By considering these algebraic equivalents of rules of axioms of DBL, it is implied by induction that $\vdash [\Gamma]$ implies $B_{ayes}^{\mathcal{P}}, \bar{\alpha}_{\mathcal{P}} \Vdash [\Gamma]$ for any $[\Gamma] \in \mathcal{L}$. \square

Consequence. First at all, property 74 implies that DBL is not degenerated, nor reduced to the classical logic. It also implies that DBL is compatible with any probability distribution defined on classical propositions — by means of the free Bayesian algebra $B_{oole}^{\mathcal{P}}$. The other important point with model $(B_{aye}^{\mathcal{P}}, \alpha_{\mathcal{P}})$ is that it contains the free Boolean algebra generated by \mathcal{P} by means of the injective morphism μ . It is known that the free Boolean algebra generated by \mathcal{P} is a complete model for classical logic [10]. Thus, it is deduced the following property.

Property 75. Let $[X] \in \mathcal{L}_C$. Then $B_{ayes}^{\mathcal{P}}, \bar{\alpha}_{\mathcal{P}} \Vdash [X]$ implies $\vdash_C [X]$.

As a corollary, the following theorem is deduced, which means that any classical proposition, which is proved in DBL, is provable in classical logic.

Theorem 3. *Let $\lfloor X \rfloor \in \mathcal{L}_C$. Then $\vdash \lfloor X \rfloor$ implies $\vdash_C \lfloor X \rfloor$.*

While $(B_{\text{ayes}}^{\mathcal{P}}, \bar{\alpha}_{\mathcal{P}})$ is a complete model for classical logic, it has not been proved that it is a complete model for DBL. This result is foreseeable however for the algebra defined in section 2, since it is based on a free construction. The proof of this assertion is however outside the scope of this introductory definition of DBL.

3.4 Theorems of DBL

In this section, some noteworthy theorems of DBL are derived. In particular, this presentation is followed by a discussion about the mathematical meaning of the conditional operator.

3.5 Theorems

Subsequently, W, X, Y, Z are any propositions of \mathcal{L} . The proofs of the subsequent properties are done in appendix A, except for the first which is given as an example.

Definition 76 (independence). *Y is independent of X if $\vdash \lfloor [X]Y \leftrightarrow Y \rfloor$.*

Property 77 (full universe). *If $\vdash \lfloor \Gamma | X \rfloor$ then $\vdash \lfloor \Gamma | [X]Y \leftrightarrow Y \rfloor$. In particular $\vdash \lfloor \Gamma | \top | Y \leftrightarrow Y \rfloor$.*

Proof. From \mathbf{Cd}^{-1} , it is deduced $\vdash \lfloor [X]Y \rightarrow (X \rightarrow Y) \rfloor$ and $\vdash \lfloor [X]\neg Y \rightarrow (X \rightarrow \neg Y) \rfloor$. As a consequence, $\vdash \lfloor X \rightarrow ([X]Y \rightarrow Y) \rfloor$ and $\vdash \lfloor X \rightarrow ([X]\neg Y \rightarrow \neg Y) \rfloor$. By applying **Neg**, it comes $\vdash \lfloor X \rightarrow ([X]Y \leftrightarrow Y) \rfloor$. Then $\vdash \lfloor \Gamma | [X]Y \leftrightarrow Y \rfloor$ follows from MP. \square

Property 78 (empty universe). *If $\vdash \lfloor \Gamma | \neg X \rfloor$ then $\vdash \lfloor \Gamma | [X]Y \leftrightarrow Y \rfloor$. In particular $\vdash \lfloor \Gamma | \perp | Y \leftrightarrow Y \rfloor$.*

Property 79 (contradiction and tautology). *If $\vdash \lfloor \Gamma | Y \rfloor$ then $\vdash \lfloor \Gamma | [X]Y \rfloor$. In particular $\vdash \lfloor \Gamma | [X]\top \rfloor$. If $\vdash \lfloor \Gamma | \neg Y \rfloor$ then $\vdash \lfloor \Gamma | \neg [X]Y \rfloor$. In particular $\vdash \lfloor \Gamma | \neg [X]\perp \rfloor$.*

Interpretation of properties 77 to 79: the tautology and the contradiction are independent of all propositions of \mathcal{L} , and all propositions of \mathcal{L} are independent of the tautology and of the contradiction.

Property 80 (sub-universes are classical). $\vdash \lfloor [X](Y \rightarrow Z) \leftrightarrow ([X]Y \rightarrow [X]Z) \rfloor$ and $\vdash \lfloor [X]\neg Y \leftrightarrow \neg [X]Y \rfloor$.

Corollary 81. $\vdash \lfloor [X](Y \wedge Z) \leftrightarrow ([X]Y \wedge [X]Z) \rfloor$, $\vdash \lfloor [X](Y \vee Z) \leftrightarrow ([X]Y \vee [X]Z) \rfloor$ and $\vdash \lfloor [X](Y \leftrightarrow Z) \leftrightarrow ([X]Y \leftrightarrow [X]Z) \rfloor$.

Corollary 82 (right equivalence). *If $\vdash \lfloor \Gamma | Y \leftrightarrow Z \rfloor$ then $\vdash \lfloor \Gamma | [X]Y \leftrightarrow [X]Z \rfloor$.*

Interpretation: the conditional modalities $[X]$ are morphisms for the classical operators. In other words, the subuniverse of $[X]$ has a classical structure.

Property 83 (inference). $\vdash [(X \wedge [X]Y) \leftrightarrow (X \wedge Y)]$.

Interpretation: The operator $[]$ works like an inference.

Property 84 (introspection). $\vdash [\neg X|[X]X]$

Proof is immediate from **Cd**.

Interpretation: a non-empty proposition sees itself as ever true.

Property 85 (independence with hypothesis). $\vdash [[X][X]Y \leftrightarrow [X]Y]$.

Interpretation: a conditional proposition $[X]Y$ is independent with its hypothesis X . In other words, the sub-universe of a proposition is independent with this proposition.

Property 86 (equivalence). *If $\vdash [\Gamma|W \leftrightarrow X]$ and $\vdash [\Gamma|Y \leftrightarrow Z]$ then $\vdash [\Gamma|[W]Y \leftrightarrow [X]Z]$.*

Property 87 (independence).

$$\text{If } \vdash [\Gamma|[X]X \leftrightarrow X] \text{ then } \vdash [\Gamma|\neg X|X], \quad (7)$$

$$\text{If } \vdash [\Gamma|[X]Y \leftrightarrow Y] \text{ and } \vdash [\Gamma|X \rightarrow Y] \text{ then } \vdash [\Gamma|\neg X|Y], \quad (8)$$

$$\text{If } \vdash [\Gamma|[X]Y \leftrightarrow Y] \text{ and } \vdash [\Gamma|X \vee Y] \text{ then } \vdash [\Gamma|X|Y], \quad (9)$$

$$\begin{aligned} \text{If } \vdash [\Gamma|(X \wedge Y) \rightarrow (X \wedge Z)], \vdash [\Gamma|[X]Y \leftrightarrow Y] \\ \text{and } \vdash [\Gamma|[X]Z \leftrightarrow Z] \text{ then } \vdash [\Gamma|\neg X|Y \rightarrow Z]. \end{aligned} \quad (10)$$

Interpretation: these properties illustrate the consequence of independence in term of proof factorization.

3.6 Logical independence

In definition 76, a notion of independence has been defined within DBL by means of the conditional inference: Y is independent of X if $\vdash [[X]Y \leftrightarrow Y]$. This independence relation is of course inspired by the probabilistic independence, $P(Y|X) = P(Y)$. It is noticed however that the independence relation of DBL is not necessary symmetric, unlike the probabilistic independence.

In mathematical logic, the notion of independence refers to the impossibility to infer or refute a proposition from a set of propositions. The relation (8), which is a direct consequence of rule **Cd** is a good illustration of the link between the independence relation of DBL and the logical independence:

$$\vdash [[X]Y \leftrightarrow Y] \text{ and } \vdash [X \rightarrow Y] \text{ imply } \vdash [\neg X|Y].$$

This relation is interpreted as follows: *if Y is independent of X and X infers Y , then the inference is trivial – i.e X is a contradiction or Y is a tautology.*

On the other hand, the relation (8) implies immediately the following:

$$\vdash [[X]Y \leftrightarrow Y] \text{ and } \vdash [\neg(X \wedge Y)] \text{ then } \vdash [\neg X|\neg Y], \quad (11)$$

which is interpreted as: *if Y is independent of X and X contradicts Y , then the contradiction is trivial* – i.e *X is a contradiction or Y is a contradiction*. Given these findings, it follows that the independence relation, as defined in definition **Cd**, may be considered as a logical independence. An interesting point is that DBL makes possible the manipulation of the concept of logical independence as a relation within the logic itself. Owing to the extensions theorems, DBL is also a link between the notions of probabilistic conditionals/independence and logical conditionals/independence.

4 Conclusion

In the first part of this paper, a new algebraic structure has been introduced, extending the Boolean algebra with an operator for the algebraic representation of the Bayesian inference. It has been shown that it is possible to construct such extension for any Boolean algebra, which admits a strictly positive finite measure. This construction is such that any probability defined on a Boolean algebra may be extended to this extension in compliance with the definition of the conditional probability. As a corollary, this result complements the triviality of Lewis, by providing a positive answer to the definition of an algebraic conditional operator by means of an extension of the space of event.

In a second part of this paper, this algebraic extension has been applied to a model-based definition of a bivalent Bayesian extension of classical logic. It has been shown that this logic implements intrinsically a relation of *logical independence*. Various elementary properties have been derived.

This work addressed the delicate issue of Lewis' triviality by complementing it positively. However, it introduced new questions. First at all, is it possible to generalize our construction to any Boolean algebra? Our work is limited to the case of Boolean algebras. Is it possible to construct a similar algebraic extension of a measurable space? The author surmises that it is possible.

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A Proofs of DBL theorems

For the sake of simplicity, the rules $\mu\mathbf{P}$, $\mu\mathbf{C}$ and $\mu\mathbf{W}$ are implicitly used in the subsequent proofs.

Proof of property 78. If $\vdash [\Gamma|\neg X]$ then $\vdash [\Gamma|[X]Y \leftrightarrow Y]$.

Proof. Assume $\vdash [\Gamma|\neg X]$. From property 77, it is deduced $\vdash [\Gamma|[\neg X]Y \leftrightarrow Y]$. From classical theorem $\vdash [X \leftrightarrow \neg\neg X]$ and **Ind**, it is then deduced $\vdash [\Gamma|[X]Y \leftrightarrow Y]$. \square

Proof of property 79. If $\vdash [\Gamma|Y]$ then $\vdash [\Gamma|[X]Y]$. If $\vdash [\Gamma|\neg Y]$ then $\vdash [\Gamma|\neg[X]Y]$.

Proof. Assume $\vdash [\Gamma|Y]$. Then $\vdash [\Gamma|[X]Y]$, it is deduced $\vdash [\Gamma|Y]$ then $\vdash [\Gamma|X \rightarrow Y]$. By applying **Cd**, it comes $\vdash [\Gamma|\neg X|[X]Y]$. From property 78, it is deduced $\vdash [\Gamma|[X]Y \leftrightarrow Y|[X]Y]$. By applying MP with $\vdash [\Gamma|Y]$, it is deduced $\vdash [\Gamma|[X]Y|[X]Y]$ and thus $\vdash [\Gamma|[X]Y]$.

Now assume $\vdash [\Gamma|\neg Y]$. It is similarly deduced $\vdash [\Gamma|[X]\neg Y]$, and then $\vdash [\Gamma|\neg[X]Y]$ by **Neg+MP**. \square

Proof of property 80. $\vdash [[X](Y \rightarrow Z) \leftrightarrow ([X]Y \rightarrow [X]Z)]$ and $\vdash [[X]\neg Y \leftrightarrow \neg[X]Y]$.

Proof. This property is almost obtained from **K** and **Neg**. However, it is necessary to prove $\vdash [([X]Y \rightarrow [X]Z) \rightarrow [X](Y \rightarrow Z)]$. From $\vdash [\neg Y \rightarrow (Y \rightarrow Z)]$ and $\vdash [Z \rightarrow (Y \rightarrow Z)]$, it is deduced $\vdash [[X](\neg Y \rightarrow (Y \rightarrow Z))]$ and $\vdash [[X](Z \rightarrow (Y \rightarrow Z))]$, by applying property 79. By applying **K+MP**, it comes $\vdash [[X]\neg Y \rightarrow [X](Y \rightarrow Z)]$ and $\vdash [[X]Z \rightarrow [X](Y \rightarrow Z)]$. By applying **Neg+MP**, it is also deduced $\vdash [\neg[X]Y \rightarrow [X](Y \rightarrow Z)]$. As a consequence, $\vdash [([X]Y \rightarrow [X]Z) \rightarrow [X](Y \rightarrow Z)]$. \square

Proof of corollary 82. If $\vdash [\Gamma|Y \leftrightarrow Z]$ then $\vdash [\Gamma|[X]Y \leftrightarrow [X]Z]$.

Proof. From $\vdash [\Gamma|Y \leftrightarrow Z]$, it is deduced $\vdash [\Gamma|[X](Y \leftrightarrow Z)]$ by property 79. Then it is deduced $\vdash [\Gamma|[X]Y \leftrightarrow [X]Z]$ from corollary 81. \square

Proof of property 83. $\vdash [(X \wedge [X]Y) \leftrightarrow (X \wedge Y)]$.

Proof. From **Cd⁻¹**, it is deduced $\vdash [[X]\neg Y \rightarrow (X \rightarrow \neg Y)]$. Then it is deduced $\vdash [\neg(X \rightarrow \neg Y) \rightarrow \neg[X]\neg Y]$. Then $\vdash [(X \wedge Y) \rightarrow [X]Y]$ by applying **Neg**, and finally $\vdash [(X \wedge Y) \rightarrow (X \wedge [X]Y)]$ by applying **Neg**. Conversely, $\vdash [[X]Y \rightarrow (X \rightarrow Y)]$ is deduced from **Cd⁻¹**, and then $\vdash [(X \wedge [X]Y) \rightarrow (X \wedge Y)]$. \square

Proof of property 85. $\vdash \llbracket [X][X]Y \leftrightarrow [X]Y \rrbracket$.

Proof. By properties 83 and 82, it is deduced $\vdash \llbracket [X](X \wedge [X]Y) \leftrightarrow [X](X \wedge Y) \rrbracket$. By applying property 81, it comes $\vdash \llbracket ([X]X \wedge [X][X]Y) \leftrightarrow ([X]X \wedge [X]Y) \rrbracket$. As a consequence, $\vdash \llbracket [X]X \rightarrow ([X][X]Y \leftrightarrow [X]Y) \rrbracket$. Then, by applying property 84 and MP, it comes $\vdash \llbracket \neg X \mid [X][X]Y \leftrightarrow [X]Y \rrbracket$. Now by applying property 78, it is deduced $\vdash \llbracket [X][X]Y \leftrightarrow [X]Y \mid [X][X]Y \leftrightarrow [X]Y \rrbracket$ and the result follows from $\mu\mathbf{W}$. \square

Proof of property 86. If $\vdash \llbracket \Gamma \mid W \leftrightarrow X \rrbracket$ and $\vdash \llbracket \Gamma \mid Y \leftrightarrow Z \rrbracket$ then $\vdash \llbracket \Gamma \mid [W]Y \leftrightarrow [X]Z \rrbracket$.

Proof. It is sufficient to prove $\vdash \llbracket \Gamma \mid W \leftrightarrow X \rrbracket \Rightarrow \vdash \llbracket \Gamma \mid [W]Y \leftrightarrow [X]Y \rrbracket$. Assume $\vdash \llbracket \Gamma \mid W \leftrightarrow X \rrbracket$. Then $\vdash \llbracket \Gamma \mid \neg W \leftrightarrow \neg X \rrbracket$. Now, $\vdash \llbracket [X][X]Y \leftrightarrow [X]Y \rrbracket$ by property 85. Applying **Ind**, it is deduced $\vdash \llbracket \Gamma \mid \neg W \mid [X]Y \leftrightarrow [X]Y \rrbracket$. Now $\vdash \llbracket \Gamma \mid W \leftrightarrow \neg \neg W \rrbracket$. Applying **Ind** again, it is deduced $\vdash \llbracket \Gamma \mid [W][X]Y \leftrightarrow [X]Y \rrbracket$.

Now, property 84 implies $\vdash \llbracket \neg W \mid [W]W \rrbracket$. It is thus deduced $\vdash \llbracket \Gamma \mid \neg W \mid ([W]W \wedge [W][X]Y) \leftrightarrow [X]Y \rrbracket$. Since $\vdash \llbracket (W \wedge [X]Y) \leftrightarrow (W \wedge Y) \rrbracket$, it is deduced $\vdash \llbracket \Gamma \mid \neg W \mid [W](W \wedge Y) \leftrightarrow [X]Y \rrbracket$ and finally $\vdash \llbracket \Gamma \mid \neg W \mid [W]Y \leftrightarrow [X]Y \rrbracket$. The result is concluded by applying property 78. \square

Proof of property 87.

If $\vdash \llbracket \Gamma \mid [X]X \leftrightarrow X \rrbracket$ then $\vdash \llbracket \Gamma \mid \neg X \mid X \rrbracket$.

Proof. A direct consequence of property 84. \square

If $\vdash \llbracket \Gamma \mid [X]Y \leftrightarrow Y \rrbracket$ and $\vdash \llbracket \Gamma \mid X \rightarrow Y \rrbracket$ then $\vdash \llbracket \Gamma \mid \neg X \mid Y \rrbracket$.

Proof. A consequence of **Cd**. \square

If $\vdash \llbracket \Gamma \mid [X]Y \leftrightarrow Y \rrbracket$ and $\vdash \llbracket \Gamma \mid X \vee Y \rrbracket$ then $\vdash \llbracket \Gamma \mid X \mid Y \rrbracket$.

Proof. Assume $\vdash \llbracket \Gamma \mid [X]Y \leftrightarrow Y \rrbracket$ and $\vdash \llbracket \Gamma \mid X \vee Y \rrbracket$. By **Ind**, it is deduced $\vdash \llbracket \Gamma \mid \neg X \mid Y \leftrightarrow Y \rrbracket$. Since $X \vee Y = (\neg X \rightarrow Y)$, the result is obtained by applying previous case. \square

If $\vdash \llbracket \Gamma \mid (X \wedge Y) \rightarrow (X \wedge Z) \rrbracket$, $\vdash \llbracket \Gamma \mid [X]Y \leftrightarrow Y \rrbracket$ and $\vdash \llbracket \Gamma \mid [X]Z \leftrightarrow Z \rrbracket$ then $\vdash \llbracket \Gamma \mid \neg X \mid Y \rightarrow Z \rrbracket$.

Proof. It is first derived $\vdash \llbracket \neg X \mid (Y \rightarrow Z) \leftrightarrow (Y \rightarrow Z) \rrbracket$ by application of **Ind** and property 80. It is noticed $\vdash \llbracket ((X \wedge Y) \rightarrow (X \wedge Z)) \leftrightarrow (\neg X \vee (Y \rightarrow Z)) \rrbracket$. Then, the result is deduced from previous case. \square