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# Non-diffusive large time behaviour for a degenerate viscous Hamilton-Jacobi equation

Philippe Laurençot\*

March 10, 2018

## Abstract

The convergence to non-diffusive self-similar solutions is investigated for non-negative solutions to the Cauchy problem  $\partial_t u = \Delta_p u + |\nabla u|^q$  when the initial data converge to zero at infinity. Sufficient conditions on the exponents  $p > 2$  and  $q > 1$  are given that guarantee that the diffusion becomes negligible for large times and the  $L^\infty$ -norm of  $u(t)$  converges to a positive value as  $t \rightarrow \infty$ .

## 1 Introduction

The quasilinear degenerate parabolic equation

$$(1.1) \quad \partial_t u = \Delta_p u + |\nabla u|^q, \quad (t, x) \in Q_\infty := (0, \infty) \times \mathbb{R}^N,$$

includes two competing mechanisms acting on the space variable  $x$ , a degenerate diffusion  $\Delta_p u$  involving the  $p$ -Laplacian operator defined by

$$\Delta_p u := \operatorname{div} (|\nabla u|^{p-2} \nabla u), \quad p > 2,$$

and a source term  $|\nabla u|^q$ ,  $q > 1$ , depending solely on the gradient of  $u$ . The aim of this work is to identify a range of the parameters  $p$  and  $q$  for which the large time behaviour of non-negative solutions to (1.1) is dominated by the source term. More precisely, we consider the Cauchy problem and supplement (1.1) with the initial condition

$$(1.2) \quad u(0) = u_0 \geq 0, \quad x \in \mathbb{R}^N.$$

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Throughout the paper, the initial condition  $u_0$  is assumed to fulfill

$$(1.3) \quad u_0 \in \mathcal{C}_0(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N), \quad u_0 \geq 0, \quad u_0 \not\equiv 0,$$

where

$$\mathcal{C}_0(\mathbb{R}^N) := \left\{ w \in \mathcal{BC}(\mathbb{R}^N) : \lim_{R \rightarrow \infty} \sup_{\{|x| \geq R\}} \{|w(x)|\} = 0 \right\},$$

and  $\mathcal{BC}(\mathbb{R}^N) := \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ .

For such an initial condition, the Cauchy problem (1.1), (1.2) has a unique non-negative (viscosity) solution  $u \in \mathcal{BC}([0, \infty) \times \mathbb{R}^N)$  (see Proposition 2.1 below). Moreover,  $t \mapsto \|u(t)\|_\infty$  is a non-increasing function and has a limit  $M_\infty \in [0, \|u_0\|_\infty]$  as  $t \rightarrow \infty$ . Our main result is then the following:

**Theorem 1.1** *Assume that  $p > 2$  and  $q \in (1, p)$ . Consider a non-negative function  $u_0$  satisfying (1.3) and let  $u$  be the corresponding (viscosity) solution to (1.1), (1.2). Assume further that*

$$(1.4) \quad M_\infty := \lim_{t \rightarrow \infty} \|u(t)\|_\infty > 0.$$

Then

$$(1.5) \quad \lim_{t \rightarrow \infty} \|u(t) - h_\infty(t)\|_\infty = 0,$$

where  $h_\infty$  is given by

$$(1.6) \quad h_\infty(t, x) := H_\infty\left(\frac{x}{t^{1/q}}\right) \quad \text{and} \quad H_\infty(x) := (M_\infty - \gamma_q |x|^{q/(q-1)})_+$$

for  $(t, x) \in Q_\infty$  and  $\gamma_q := (q-1) q^{-q/(q-1)}$ .

Here and below,  $r_+ := \max\{r, 0\}$  denotes the positive part of the real number  $r$ .

The convergence (1.5) clearly indicates that the large time behaviour of non-negative solutions to (1.1), (1.2) fulfilling the condition (1.4) is governed by the gradient source term. Indeed,  $h_\infty$  is actually a self-similar solution to the Hamilton-Jacobi equation

$$(1.7) \quad \partial_t h = |\nabla h|^q, \quad (t, x) \in Q_\infty,$$

and an alternative formula for  $h_\infty$  reads

$$(1.8) \quad h_\infty(t, x) = \sup_{y \in \mathbb{R}^N} \left\{ M_\infty \mathbf{1}_{\{0\}}(y) - \gamma_q \frac{|x - y|^{q/(q-1)}}{t^{1/(q-1)}} \right\}$$

for  $(t, x) \in [0, \infty) \times \mathbb{R}^N$ ,  $\mathbf{1}_{\{0\}}$  denoting the indicator function of the singleton set  $\{0\}$ . The formula (1.8) is the well-known Hopf-Lax-Oleinik representation formula for viscosity solutions to (1.7) (see, e.g., [10, Chapter 3]) and  $h_\infty$  turns out to be the unique viscosity solution in  $\mathcal{BUC}(Q_\infty)$  to (1.7) with the bounded and upper semicontinuous initial condition  $h_\infty(0, x) = \mathbf{1}_{\{0\}}(x)$  for  $x \in \mathbb{R}^N$  [24].

**Remark 1.2** *The convergence (1.5) also holds true for the viscosity solution to the Hamilton-Jacobi equation (1.7) with a non-negative initial condition  $u_0 \in \mathcal{C}_0(\mathbb{R}^N)$  but with  $\|u_0\|_\infty$  instead of  $M_\infty$  in the formula (1.6) giving  $H_\infty$ . For (1.1), (1.2), the constant  $M_\infty$  takes into account that, though negligible for large times, the diffusion erodes the supremum of  $u$  during the time evolution.*

For  $p = 2$ , Theorem 1.1 is also valid and is proved in [7], the proof relying on a rescaling technique: The crucial step is then to identify the possible limits of the rescaled sequence and this is done by an extensive use of the Hopf-Lax-Oleinik representation formula. The proof we perform here is of a completely different nature and relies on the relaxed half-limits method introduced in [3]. A similar approach has been used in [22] and [23] to investigate the large time behaviour of solutions to first-order Hamilton-Jacobi equations  $\partial_t w + H(x, \nabla w) = 0$  in  $Q_\infty$ . It has also been used in [20] to study the convergence to non-diffusive localized self-similar patterns for non-negative and compactly supported solutions to  $\partial_t w - \Delta_p w + |\nabla w|^q = 0$  in  $Q_\infty$  when  $p > 2$  and  $q \in (1, p - 1)$ .

In order to apply Theorem 1.1, one should check whether there are non-negative solutions to (1.1), (1.2) for which (1.4) holds true. The next result provides sufficient conditions for (1.4) to be fulfilled.

**Theorem 1.3** *Assume that  $p > 2$  and  $q > 1$ . Consider a non-negative function  $u_0$  satisfying (1.3) and let  $u$  be the corresponding solution to (1.1), (1.2). Introducing*

$$(1.9) \quad q_\star := p - \frac{N}{N+1},$$

*then  $u$  fulfills (1.4) if*

- (a) *either  $q \in (1, q_\star]$ ,*
- (b) *or  $q \in (q_\star, p)$ ,  $u_0 \in W^{2,\infty}(\mathbb{R}^N)$ , and*

$$(1.10) \quad \|u_0\|_\infty > \kappa_0 \left| \inf_{y \in \mathbb{R}^N} \{ \Delta_p u_0(y) \} \right|^{(p-q)/q}.$$

*for some  $\kappa_0 > 0$  which depends only on  $N$ ,  $p$ , and  $q$ .*

A similar result is already available for  $p = 2$  and has been established in [7, 12]. It is also shown in [7, 12, 18] that, still for  $p = 2$ ,  $\|u(t)\|_\infty \rightarrow 0$  as  $t \rightarrow \infty$  if either  $q \geq p$  or  $q \in (q_\star, p)$  and a suitable norm of  $u_0$  is small enough. A similar result is likely to be true also for  $p > 2$ , so that different large time behaviours are only to be found in the intermediate range  $q \in (q_\star, p)$ .

The proof of Theorem 1.3 for  $q \in (p - 1, p)$  and  $p > 2$  borrows some steps from the case  $p = 2$ . However, it relies on semiconvexity estimates for solutions to (1.1), (1.2) which seem to be new for  $p > 2$  and  $q \in (1, p)$  and are stated now.

**Proposition 1.4** *Assume that  $p > 2$  and  $q \in (1, p]$ . Let  $u$  be the viscosity solution to (1.1), (1.2) with initial condition  $u_0 \in \mathcal{BUC}(\mathbb{R}^N)$  (that is,  $u_0 \in \mathcal{BC}(\mathbb{R}^N)$  and is uniformly continuous in  $\mathbb{R}^N$ ). Then  $\nabla u(t)$  belongs to  $L^\infty(\mathbb{R}^N)$  for each  $t > 0$  and there is  $\kappa_1 > 0$  depending only on  $N$ ,  $p$ , and  $q$  such that*

$$(1.11) \quad \Delta_p u(t, x) \geq -\kappa_1 \|u(s)\|_\infty^{(p-q)/q} (t-s)^{-p/q}, \quad t > s \geq 0,$$

*in the sense of distributions. In addition, if  $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ , there holds*

$$(1.12) \quad \Delta_p u(t, x) \geq -\frac{N(p-1)}{q(q-1)} \frac{\|\nabla u_0\|_\infty^{p-q}}{t}$$

*for  $t > 0$  in the sense of distributions.*

The proof of Proposition 1.4 relies on the comparison principle combined with a gradient estimate established in [6].

Similar semiconvexity estimates for solutions to (1.1), (1.2) have already been obtained in [14] and [21, Lemma 5.1] for  $p = q = 2$ , in [7, Proposition 3.2] for  $p = 2$  and  $q \in (1, 2]$ , and in [9, Theorem 1] for  $p = q > 2$ . We extend these results to the range  $p > 2$  and  $q \in (1, p]$ . As we shall see below, the estimate (1.11) plays an important role in the proof of Theorem 1.3 and is also helpful to construct a subsolution in the proof of Theorem 1.1.

Let us finally emphasize that the validity of Proposition 1.4 is not restricted to non-negative solutions and that the solutions to the Hamilton-Jacobi equation (1.7) also enjoy the semiconvexity estimates (1.11) and (1.12). These two estimates thus stem from the reaction term  $|\nabla u|^q$  and not from the diffusion.

In the next section, we recall the well-posedness of (1.1), (1.2) in  $\mathcal{BUC}(\mathbb{R}^N)$ , as well as some properties of the solutions established in [6]. We also show the finite speed of propagation of the support for non-negative compactly supported initial data. Section 3 is devoted to the proof of the semiconvexity estimates (Proposition 1.4) and Section 4 to that of Theorem 1.1. Theorem 1.3 is shown in the last section, its proof combining arguments of [7, 12, 19] used to established analogous results when  $p = 2$ .

Throughout the paper,  $C$  and  $C_i$ ,  $i \geq 1$ , denote positive constants depending only on  $p$ ,  $q$ , and  $N$ . Dependence upon additional parameters will be indicated explicitly. Also,  $\mathcal{M}_N(\mathbb{R})$  denotes the space of real-valued  $N \times N$  matrices and  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  if  $i \neq j$ ,  $1 \leq i, j \leq N$ . Given a matrix  $A = (a_{ij}) \in \mathcal{M}_N(\mathbb{R})$ ,  $tr(A)$  denotes its trace and is given by  $tr(A) := \sum a_{ii}$ .

## 2 Preliminary results

Let us first recall the well-posedness (in the framework of viscosity solutions) of (1.1), (1.2), together with some properties of the solutions established in [6].

**Proposition 2.1** Consider a non-negative initial condition  $u_0 \in \mathcal{BUC}(\mathbb{R}^N)$ . There is a unique non-negative viscosity solution  $u \in \mathcal{BC}([0, \infty) \times \mathbb{R}^N)$  to (1.1), (1.2) such that

$$(2.1) \quad 0 \leq u(t, x) \leq \|u_0\|_\infty, \quad (t, x) \in Q_\infty,$$

$$(2.2) \quad \|\nabla u(t)\|_\infty \leq \min \left\{ C_1 \|u(s)\|_\infty^{1/q} (t-s)^{-1/q}, \|\nabla u(s)\|_\infty \right\},$$

and

$$(2.3) \quad \int_{\mathbb{R}^N} (u(t, x) - u(s, x)) \vartheta(x) dx + \int_s^t \int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \cdot \nabla \vartheta - |\nabla u|^q \vartheta) dx d\tau = 0$$

for  $t > s \geq 0$  and  $\vartheta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ . In addition,  $t \mapsto \|u(t)\|_\infty$  is a non-increasing function.

**Proof.** We put  $\tilde{u}_0 := \|u_0\|_\infty - u_0$ . As  $\tilde{u}_0$  is a non-negative function in  $\mathcal{BUC}(\mathbb{R}^N)$ , it follows from [6, Theorem 1.1] that there is a unique non-negative viscosity solution  $\tilde{u}$  to

$$(2.4) \quad \partial_t \tilde{u} - \Delta_p \tilde{u} + |\nabla \tilde{u}|^q = 0, \quad (t, x) \in Q_\infty,$$

with initial condition  $\tilde{u}(0, x) = \tilde{u}_0(x)$  for  $x \in \mathbb{R}^N$ . It also satisfies  $0 \leq \tilde{u}(t, x) \leq \|u_0\|_\infty$  and

$$\int_{\mathbb{R}^N} (\tilde{u}(t, x) - \tilde{u}(s, x)) \vartheta(x) dx + \int_s^t \int_{\mathbb{R}^N} (|\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla \vartheta + |\nabla \tilde{u}|^q \vartheta) dx d\tau = 0$$

for  $t > s \geq 0$ ,  $x \in \mathbb{R}^N$ , and  $\vartheta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ . In addition,  $\nabla \tilde{u}(t)$  belongs to  $L^\infty(\mathbb{R}^N)$  for each  $t > 0$  and

$$\|\nabla \tilde{u}(t)\|_\infty \leq C_1 \|\tilde{u}_0\|_\infty^{1/q} t^{-1/q}$$

by [6, Lemma 4.1]. Setting  $u := \|u_0\|_\infty - \tilde{u}$ , we readily deduce from the properties of  $\tilde{u}$  that  $u$  is a non-negative viscosity solution to (1.1), (1.2) satisfying (2.1) and (2.3). Also,  $\nabla u(t)$  belongs to  $L^\infty(\mathbb{R}^N)$  for each  $t > 0$ . The uniqueness and the time monotonicity of  $\|u\|_\infty$  then both follow from the comparison principle, see [8] or [11, Theorem 2.1]. Finally, given  $s \geq 0$ ,  $(t, x) \mapsto \|u(s)\|_\infty - u(t+s, x)$  is the unique non-negative viscosity solution to the Cauchy problem (2.4) with initial condition  $x \mapsto \|u(s)\|_\infty - u(s, x)$  and we infer from [6, Lemma 4.1] that

$$\|\nabla u(t+s)\|_\infty \leq C_1 \|\|u(s)\|_\infty - u(s)\|_\infty^{1/q} t^{-1/q} \leq C_1 \|u(s)\|_\infty^{1/q} t^{-1/q}$$

for  $t > 0$ , whence (2.2). □

We next turn to the propagation of the support of non-negative solutions to (1.1), (1.2) with non-negative compactly supported initial data.

**Proposition 2.2** Consider a non-negative solution  $u$  to (1.1), (1.2) with an initial condition  $u_0$  satisfying (1.3). Assume further that  $u_0$  is compactly supported in a ball  $B(0, R_0)$  of  $\mathbb{R}^N$  for some  $R_0 > 0$ . Then  $u(t)$  is compactly supported for each  $t \geq 0$ .

**Proof.** We argue by comparison with travelling wave solutions. By [13, Application 9.4], there is a travelling wave solution  $w$  to the convection-diffusion equation

$$(2.5) \quad \partial_t w - \partial_1^2 (w^{p-1}) + \partial_1 (w^q) = 0, \quad (t, x_1) \in (0, \infty) \times \mathbb{R},$$

with wave speed unity. It is given by  $w(t, x_1) = f(x_1 - t)$  for  $(t, x_1) \in (0, \infty) \times \mathbb{R}$ , the function  $f$  being implicitly defined by

$$(p-1) \int_0^{f(y)} \frac{z^{p-3}}{1-z^{q-1}} dz = (-y)_+, \quad y \in \mathbb{R}.$$

In particular,  $f$  satisfies  $f(y) = 0$  if  $y > 0$  and  $f(y) \rightarrow 1$  as  $y \rightarrow -\infty$ . Introducing

$$F(y) := \int_y^\infty f(z) dz, \quad y \in \mathbb{R},$$

the properties of  $f$  ensure that  $F$  is a decreasing function on  $(-\infty, 0)$  with  $F(y) = 0$  if  $y > 0$ ,  $F(y) \leq |y|$  if  $y < 0$ , and  $F(y) \rightarrow \infty$  as  $y \rightarrow -\infty$ . There is therefore a unique  $\mu \in (-\infty, 0)$  such that  $F(R_0 + \mu) = \|u_0\|_\infty$ . In addition, it readily follows from (2.5) and the invariance by translation of (1.1) that  $W_\mu(t, x) := F(x_1 + \mu - t)$  is a travelling wave solution to (1.1). Now,  $u$  and  $W_\mu$  are both solutions to (1.1) in  $(0, \infty) \times H_+$ , the half-space  $H_+$  being defined by  $H_+ := \{x \in \mathbb{R}^N : x_1 > R_0\}$ . Owing to the monotonicity of  $F$ , the bound  $0 \leq f \leq 1$ , and (2.1), we have also

$$u_0(x) - W_\mu(0, y) = 0 - W_\mu(0, y) \leq W_\mu(0, x) - W_\mu(0, y) \leq |x - y|$$

for  $x \in H_+$  and  $y \in H_+$ ,

$$\begin{aligned} u(t, x) - W_\mu(t, y) &\leq \|u_0\|_\infty - W_\mu(t, y) \\ &\leq F(R_0 + \mu - t) - W_\mu(t, y) = W_\mu(t, x) - W_\mu(t, y) \leq |x - y| \end{aligned}$$

for  $t > 0$ ,  $x \in \partial H_+$ ,  $y \in H_+$ , and

$$u(t, x) - W_\mu(t, y) \leq \|u_0\|_\infty - F(R_0 + \mu - t) \leq 0$$

for  $t > 0$ ,  $x \in H_+$ ,  $y \in \partial H_+$ . We are then in a position to use the comparison principle stated in [11, Theorem 2.1] to conclude that  $u(t, x) \leq W_\mu(t, x)$  for  $(t, x) \in (0, \infty) \times H_+$ . Consequently,  $u(t, x) \leq F(x_1 + \mu - t) = 0$  if  $t \geq 0$  and  $x_1 > \max\{R_0, t - \mu\}$ , and the rotational invariance of (1.1) allows us to conclude that  $u(t, x) = 0$  for  $t \geq 0$  and  $|x| > \max\{R_0, t - \mu\}$ .  $\square$

We finally recall the convergence to self-similar solutions for non-negative and compactly supported solutions to the  $p$ -Laplacian equation [17]

$$(2.6) \quad \partial_t \varphi = \Delta_p \varphi, \quad (t, x) \in Q_\infty.$$

**Proposition 2.3** *Let  $\varphi_0$  be a non-negative and compactly supported function in  $L^1(\mathbb{R}^N)$  and  $\varphi$  denote the unique weak solution to (2.6) with initial condition  $\varphi_0$ . Then*

$$(2.7) \quad \lim_{t \rightarrow \infty} t^{(N(r-1))/(r(N(p-2)+p))} \|\varphi(t) - \mathcal{B}_{\|\varphi_0\|_1}(t)\|_r = 0 \quad \text{for } r \in [1, \infty],$$

where  $\mathcal{B}_L$  denotes the Barenblatt solution to (2.6) given by

$$\begin{aligned} \mathcal{B}_L(t, x) &:= t^{-N/(N(p-2)+p)} b_L(xt^{-1/(N(p-2)+p)}), \\ b_L(x) &:= (C_2 L^{(p(p-2))/(p-1)(N(p-2)+p)} - C_3 |x|^{p/(p-1)})_+^{(p-1)/(p-2)} \end{aligned}$$

for  $(t, x) \in Q_\infty$  and  $L > 0$ .

The convergence (2.7) is proved in [17, Theorem 2] for  $r = \infty$ . As  $\varphi_0$  is compactly supported, so is  $\varphi(t)$  for each  $t > 0$  and the support of  $\varphi(t)$  is included in  $B(0, C_4(\varphi_0) t^{1/(N(p-2)+p)})$  for  $t \geq 1$  [17, Proposition 2.2]. Combining this property with [17, Theorem 2] readily provide the convergence (2.7) for all  $r \in [1, \infty)$ .

### 3 Semiconvexity

In this section, we prove the semiconvexity estimates (1.11) and (1.12). To this end, we would like to derive an equation for  $\Delta_p u$  to which we could apply the comparison principle. The poor regularity of  $u$  however does not allow to perform directly such a computation and an approximation procedure is needed. As a first step, we report the following result:

**Lemma 3.1** *Let  $a$  and  $b$  be two non-negative function in  $C^\infty([0, \infty))$  satisfying*

$$(3.1) \quad a(r) > 0, \quad a'(r) > 0, \quad a'(r) b'(r) - a(r) b''(r) > 0,$$

$$(3.2) \quad c(r) := 2 \left( \frac{b'}{a} \right)'(r) + \frac{4r (a b'' - a' b')(r)}{a^2(r) + 2r a(r) a'(r)} \geq 0.$$

Consider a classical solution  $v$  to

$$(3.3) \quad \partial_t v - \operatorname{div} (a(|\nabla v|^2) \nabla v) = b(|\nabla v|^2), \quad (t, x) \in Q_\infty,$$

and put

$$w := \operatorname{div} (a(|\nabla v|^2) \nabla v) \quad \text{and} \quad z_i := a(|\nabla v|^2) \partial_i v$$

for  $i \in \{1, \dots, N\}$ . Then

$$(3.4) \quad \partial_t w - \mathcal{L}w - \mathcal{V} \cdot \nabla w - \frac{c(|\nabla v|^2)}{N} w^2 \geq 0 \quad \text{in } Q_\infty,$$

where

$$\mathcal{L}w := \sum_{i,j} \partial_i (a(|\nabla v|^2) E_{ij} \partial_j w), \quad \mathcal{V} := 2 b'(|\nabla v|^2) \nabla v,$$

$$E_{ij} := \delta_{ij} + 2 \frac{a'}{a} (|\nabla v|^2) \partial_i v \partial_j v, \quad 1 \leq i, j \leq N.$$



The proof of Lemma 3.1 borrows some steps from the proof of [9, Theorem 1] for  $p = q > 2$  but requires additional arguments to handle the term coming from the fact that  $q \neq p$ . In particular, we recall the following elementary result which will be helpful to estimate this term.

**Lemma 3.2** *Let  $A$  and  $B$  be two symmetric matrices in  $\mathcal{M}_N(\mathbb{R})$  and put  $M := ABA$ . Then  $M$  is a symmetric matrix in  $\mathcal{M}_N(\mathbb{R})$  and*

$$(3.5) \quad |MX|^2 \leq \text{tr}(M^2) |X|^2 \quad \text{for } X \in \mathbb{R}^N.$$

**Proof of Lemma 3.1.** We first note that

$$(3.6) \quad \partial_j z_i = a(|\nabla v|^2) \sum_k E_{ik} \partial_k \partial_j v,$$

$$(3.7) \quad \partial_t z_i = a(|\nabla v|^2) \sum_k E_{ik} \partial_k \partial_t v,$$

for  $1 \leq i, j \leq N$ . According to the definition of  $w$ , we infer from (3.3), (3.6), and (3.7) that

$$\begin{aligned} \partial_t w &= \sum_{i,k} \partial_i (a(|\nabla v|^2) E_{ik} \partial_k \partial_t v) \\ &= \sum_{i,k} \partial_i (a(|\nabla v|^2) E_{ik} \partial_k (w + b(|\nabla v|^2))) \\ &= \mathcal{L}w + 2 \sum_{i,k} \partial_i \left( (ab') (|\nabla v|^2) E_{ik} \sum_j \partial_j v \partial_j \partial_k v \right) \\ &= \mathcal{L}w + 2 \sum_{i,j} \partial_i \left( (ab') (|\nabla v|^2) \partial_j v \sum_k E_{ik} \partial_k \partial_j v \right) \\ &= \mathcal{L}w + 2 \sum_{i,j} \partial_i \left( \left( \frac{b'}{a} \right) (|\nabla v|^2) z_j \partial_j z_i \right) \\ &= \mathcal{L}w + 4 \sum_{i,j} \left( \frac{b'}{a} \right)' (|\nabla v|^2) \sum_k \partial_k v \partial_k \partial_i v z_j \partial_j z_i \\ &\quad + 2 \sum_{i,j} \left( \frac{b'}{a} \right) (|\nabla v|^2) \partial_i z_j \partial_j z_i + 2 \sum_{i,j} \left( \frac{b'}{a} \right) (|\nabla v|^2) z_j \partial_j \partial_i z_i. \end{aligned}$$

Since  $w = \sum \partial_i z_i$ , the last term of the right-hand side of the above inequality is equal to  $\mathcal{V} \cdot \nabla w$  and

$$(3.8) \quad \begin{aligned} \partial_t w &= \mathcal{L}w + \mathcal{V} \cdot \nabla w + 4 \left[ a \left( \frac{b'}{a} \right)' \right] (|\nabla v|^2) \sum_{i,j,k} \partial_j v \partial_k v \partial_k \partial_i v \partial_j z_i \\ &\quad + 2 \left( \frac{b'}{a} \right) (|\nabla v|^2) \sum_{i,j} \partial_i z_j \partial_j z_i. \end{aligned}$$

On the one hand, introducing the matrix  $\mathcal{E} := (E_{ij})$  and the Hessian matrix  $D^2v = (\partial_i \partial_j v)$  of  $v$ , we infer from (3.6) that

$$\begin{aligned}
\sum_{i,j} \partial_i z_j \partial_j z_i &= a^2 (|\nabla v|^2) \sum_{i,j,k,l} E_{ik} \partial_k \partial_j v E_{jl} \partial_l \partial_i v \\
&= a^2 (|\nabla v|^2) \sum_{i,j} (\mathcal{E} D^2v)_{ij} (\mathcal{E} D^2v)_{ji} \\
(3.9) \quad \sum_{i,j} \partial_i z_j \partial_j z_i &= a^2 (|\nabla v|^2) \operatorname{tr} \left( (\mathcal{E} D^2v)^2 \right).
\end{aligned}$$

On the other hand, using once more (3.6), we obtain

$$\begin{aligned}
\sum_{i,j,k} \partial_j v \partial_k v \partial_k \partial_i v \partial_j z_i &= a (|\nabla v|^2) \sum_{i,j,k,l} \partial_j v \partial_k v \partial_k \partial_i v E_{il} \partial_l \partial_j v \\
&= a (|\nabla v|^2) \sum_{i,l} \left( \sum_k \partial_i \partial_k v \partial_k v \right) E_{il} \left( \sum_j \partial_l \partial_j v \partial_j v \right) \\
(3.10) \quad \sum_{i,j,k} \partial_j v \partial_k v \partial_k \partial_i v \partial_j z_i &= a (|\nabla v|^2) \langle D^2v \nabla v, (\mathcal{E} D^2v) \nabla v \rangle.
\end{aligned}$$

Inserting (3.9) and (3.10) in (3.8), we end up with

$$\begin{aligned}
(3.11) \quad \partial_t w &= \mathcal{L}w + \mathcal{V} \cdot \nabla w + 2 (ab') (|\nabla v|^2) \operatorname{tr} \left( (\mathcal{E} D^2v)^2 \right) \\
&\quad + 4 (a b'' - a' b') (|\nabla v|^2) \langle D^2v \nabla v, (\mathcal{E} D^2v) \nabla v \rangle.
\end{aligned}$$

We next observe that

$$(3.12) \quad \mathcal{E} \nabla v = \left( 1 + 2 |\nabla v|^2 \left( \frac{a'}{a} \right) (|\nabla v|^2) \right) \nabla v$$

and that, for  $X \in \mathbb{R}^N$ ,

$$\langle \mathcal{E} X, X \rangle = |X|^2 + 2 \left( \frac{a'}{a} \right) (|\nabla v|^2) \langle X, \nabla v \rangle^2 \geq |X|^2$$

as  $a$  and  $a'$  are both positive by (3.1). Consequently,  $\mathcal{E}$  is a positive definite symmetric matrix in  $\mathcal{M}_N(\mathbb{R})$  and there exists a positive definite matrix  $\mathcal{E}_{1/2}$  such that  $\mathcal{E}_{1/2}^2 = \mathcal{E}$ . We then infer from the definition of  $\mathcal{E}_{1/2}$ , (3.12), and Lemma 3.2 (with  $A = \mathcal{E}_{1/2}$ ,  $B = D^2v$  and

$X = \mathcal{E}_{1/2}^{-1} \nabla v$ ) that

$$\begin{aligned}
\langle D^2 v \nabla v, (\mathcal{E} D^2 v) \nabla v \rangle &= |(\mathcal{E}_{1/2} D^2 v) \nabla v|^2 \\
&= \left| \left( (\mathcal{E}_{1/2} D^2 v \mathcal{E}_{1/2}) \mathcal{E}_{1/2}^{-1} \right) \nabla v \right|^2 \\
&\leq \operatorname{tr} (\mathcal{E}_{1/2} D^2 v \mathcal{E}_{1/2} \mathcal{E}_{1/2} D^2 v \mathcal{E}_{1/2}) \langle \mathcal{E}_{1/2}^{-1} \nabla v, \mathcal{E}_{1/2}^{-1} \nabla v \rangle \\
&\leq \operatorname{tr} \left( (\mathcal{E} D^2 v)^2 \right) \langle \nabla v, \mathcal{E}^{-1} \nabla v \rangle \\
&\leq \operatorname{tr} \left( (\mathcal{E} D^2 v)^2 \right) |\nabla v|^2 \left( 1 + 2 |\nabla v|^2 \left( \frac{a'}{a} \right) (|\nabla v|^2) \right)^{-1}.
\end{aligned}$$

Owing to the non-positivity (3.1) of  $a b'' - a' b'$ , we deduce from (3.11) and the above inequality that

$$\partial_t w \geq \mathcal{L}w + \mathcal{V} \cdot \nabla w + (a^2 c) (|\nabla v|^2) \operatorname{tr} \left( (\mathcal{E} D^2 v)^2 \right),$$

the function  $c$  being defined in (3.2). We finally use the inequality

$$\operatorname{tr} (A^2) \geq \frac{1}{N} \operatorname{tr} (A)^2, \quad A \in \mathcal{M}_N(\mathbb{R}),$$

the identity

$$w = \sum_i \partial_i z_i = a (|\nabla v|^2) \operatorname{tr} (\mathcal{E} D^2 v),$$

and the non-negativity (3.2) of  $c$  to conclude that

$$\begin{aligned}
\partial_t w &\geq \mathcal{L}w + \mathcal{V} \cdot \nabla w + \frac{1}{N} (a^2 c) (|\nabla v|^2) \operatorname{tr} (\mathcal{E} D^2 v)^2 \\
&\geq \mathcal{L}w + \mathcal{V} \cdot \nabla w + \frac{c (|\nabla v|^2)}{N} w^2,
\end{aligned}$$

and complete the proof.  $\square$

**Proof of Proposition 1.4.** To be able to use Lemma 3.1, we shall first construct a suitable approximation of (1.1), (1.2). Such a construction has already been performed in [6] for similar purposes and we recall it now. Given  $u_0$  satisfying (1.3), there is a sequence of functions  $(u_{0,k})_{k \geq 1}$  such that, for each integer  $k \geq 1$ ,  $u_{0,k} \in \mathcal{BC}^\infty(\mathbb{R}^N)$ ,  $u_0 \leq u_{0,k+1} \leq u_{0,k}$ , and  $(u_{0,k}, \nabla u_{0,k})_k$  converge towards  $(u_0, \nabla u_0)$  uniformly on every compact subset of  $\mathbb{R}^N$  as  $k \rightarrow \infty$ . Next, for  $\varepsilon \in (0, 1)$  and  $r \geq 0$ , we set

$$a_\varepsilon(r) := (r + \varepsilon^2)^{(p-2)/2} \quad \text{and} \quad b_\varepsilon(r) := (r + \varepsilon^2)^{q/2} - \varepsilon^q.$$

Then the Cauchy problem

$$(3.13) \quad \partial_t u_{k,\varepsilon} = \operatorname{div} (a_\varepsilon (|\nabla u_{k,\varepsilon}|^2) \nabla u_{k,\varepsilon}) + b_\varepsilon (|\nabla u_{k,\varepsilon}|^2), \quad (t, x) \in Q_\infty,$$

$$(3.14) \quad u_{k,\varepsilon}(0) = u_{0,k} + \varepsilon^\nu, \quad x \in \mathbb{R}^N,$$

has a unique classical solution  $u_{k,\varepsilon}$ , the parameter  $\nu > 0$  depending  $p, q$ , and  $N$  and being appropriately chosen. Furthermore,

$$(3.15) \quad \|\nabla u_{k,\varepsilon}(t)\|_\infty \leq \|\nabla u_{0,k}\|_\infty, \quad t \geq 0,$$

$$(3.16) \quad \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} u_{k,\varepsilon}(t, x) = u(t, x),$$

the latter convergence being uniform on every compact subset of  $[0, \infty) \times \mathbb{R}^N$ , see [6, Section 3] (after performing the same change of unknown function as in the proof of Proposition 2.1).

Introducing

$$c_\varepsilon(r) = 2 \left( \frac{b'_\varepsilon}{a_\varepsilon} \right) (r) + \frac{4r (a_\varepsilon b''_\varepsilon - a'_\varepsilon b'_\varepsilon)(r)}{a_\varepsilon^2(r) + 2r a_\varepsilon(r) a'_\varepsilon(r)}, \quad r \geq 0,$$

let us check that  $a_\varepsilon$  and  $b_\varepsilon$  fulfill the conditions (3.1) and (3.2). Clearly,  $a_\varepsilon > 0$  and  $a'_\varepsilon > 0$  as  $p > 2$ . Next, since  $1 < q \leq p$ ,

$$\begin{aligned} (a'_\varepsilon b'_\varepsilon - a_\varepsilon b''_\varepsilon)(r) &= \frac{q(p-q)}{4} (r + \varepsilon^2)^{(p+q-6)/2} \geq 0, \\ c_\varepsilon(r) &= q \frac{r(q-1) + \varepsilon^2}{r(p-1) + \varepsilon^2} (r + \varepsilon^2)^{(q-p)/2} \geq 0. \end{aligned}$$

We may then apply Lemma 3.1 to deduce that  $w_{k,\varepsilon} := \operatorname{div} (a_\varepsilon (|\nabla u_{k,\varepsilon}|^2) \nabla u_{k,\varepsilon})$  satisfies

$$\partial_t w_{k,\varepsilon} - \mathcal{L}_{k,\varepsilon} w_{k,\varepsilon} - \mathcal{V}_{k,\varepsilon} \cdot \nabla w_{k,\varepsilon} - \frac{c_\varepsilon (|\nabla u_{k,\varepsilon}|^2)}{N} w_{k,\varepsilon}^2 \geq 0$$

in  $Q_\infty$ . Observe next that the condition  $1 < q \leq p$  implies that  $c_\varepsilon$  is a non-increasing function. It then follows from (3.15) that  $c_\varepsilon (|\nabla u_{k,\varepsilon}|^2) \geq c_\varepsilon (\|\nabla u_{0,k}\|_\infty^2)$  and we end up with

$$(3.17) \quad \partial_t w_{k,\varepsilon} - \mathcal{L}_{k,\varepsilon} w_{k,\varepsilon} - \mathcal{V}_{k,\varepsilon} \cdot \nabla w_{k,\varepsilon} - \frac{c_\varepsilon (\|\nabla u_{0,k}\|_\infty^2)}{N} w_{k,\varepsilon}^2 \geq 0$$

in  $Q_\infty$ . Clearly,  $t \mapsto -N / (c_\varepsilon (\|\nabla u_{0,k}\|_\infty^2) t)$  is a subsolution to (3.17) and the comparison principle warrants that

$$(3.18) \quad w_{k,\varepsilon}(t, x) \geq -\frac{N}{c_\varepsilon (\|\nabla u_{0,k}\|_\infty^2) t}, \quad (t, x) \in Q_\infty.$$

Letting  $\varepsilon \rightarrow 0$  and  $k \rightarrow \infty$  in the previous inequality with the help of (3.16) gives (1.12).

Next, since (1.1) is autonomous, we infer from (2.2) (with  $s = 0$ ) and (1.12) that

$$\begin{aligned} \Delta_p u(t, x) &\geq -\frac{2N(p-1)}{q(q-1)} \frac{\|\nabla u(t/2)\|_\infty^{p-q}}{t} \\ &\geq -\frac{2^{p/q} N(p-1)}{q(q-1)} C_1^{p-q} \|u_0\|_\infty^{(p-q)/q} t^{-p/q}, \end{aligned}$$

whence (1.11) for  $s = 0$ . To prove the general case  $s \in (0, t)$ , we use again the fact that (1.1) is autonomous.  $\square$

We have a similar result when  $u_0$  is more regular.

**Corollary 3.3** *Assume that  $p > 2$  and  $q \in (1, p]$ . Let  $u$  be the solution to (1.1), (1.2) with an initial condition  $u_0$  satisfying  $u_0 \in W^{2,\infty}(\mathbb{R}^N)$  in addition to (1.3). Then*

$$(3.19) \quad \Delta_p u(t, x) \geq - \left| \inf_{y \in \mathbb{R}^N} \Delta_p u_0(y) \right|$$

*in the sense of distributions.*

**Proof.** Keeping the notations introduced in the proof of Proposition 1.4, we readily infer from (3.17) and the comparison principle that

$$(3.20) \quad w_{k,\varepsilon}(t, x) \geq - \left| \inf_{y \in \mathbb{R}^N} \Delta_p u_{0,k}(y) \right|, \quad (t, x) \in Q_\infty.$$

Owing to the regularity of  $u_0$ , it is possible to construct the sequence  $(u_{0,k})_k$  such that it satisfies

$$\lim_{k \rightarrow \infty} \inf_{y \in \mathbb{R}^N} \Delta_p u_{0,k}(y) = \inf_{y \in \mathbb{R}^N} \Delta_p u_0(y).$$

We may then pass to the limit first as  $\varepsilon \rightarrow 0$  and then as  $k \rightarrow \infty$  in (3.20) and use (3.16) and the above convergence to complete the proof.  $\square$

Another useful consequence of the semiconvexity estimates derived in Proposition 1.4 is that the solution  $u$  to (1.1), (1.2) is a supersolution to a first-order Hamilton-Jacobi equation.

**Corollary 3.4** *Consider an initial condition  $u_0$  satisfying (1.3). Setting  $F(t, \xi_0, \xi) := \xi_0 - |\xi|^q + \kappa_1 \|u_0\|_\infty^{(p-q)/q} t^{-p/q}$  for  $t \in (0, \infty)$ ,  $\xi_0 \in \mathbb{R}$ , and  $\xi \in \mathbb{R}^N$  (recall that  $\kappa_1$  is defined in (1.11)), the solution  $u$  to (1.1), (1.2) is a supersolution to  $F(t, \partial_t w, \nabla w) = 0$  in  $Q_\infty$ .*

**Proof.** We still use the notations introduced in the proof of Proposition 1.4. As  $w_{k,\varepsilon} = \operatorname{div} (a_\varepsilon (|\nabla u_{k,\varepsilon}|^2) \nabla u_{k,\varepsilon})$ , we infer from (3.13) and (3.18) that

$$\partial_t u_{k,\varepsilon} - b_\varepsilon (|\nabla u_{k,\varepsilon}|^2) \geq - \frac{N}{c_\varepsilon (\|\nabla u_{0,k}\|_\infty^2) t}$$

in  $Q_\infty$ . We then use (3.16) and the stability of viscosity solutions [1, 2, 8] to pass to the limit as  $\varepsilon \rightarrow 0$  and  $k \rightarrow \infty$  in the previous inequality and conclude that  $u$  is a supersolution to

$$\partial_t w - |\nabla w|^q + \frac{N(p-1)}{q(q-1)} \frac{\|\nabla u_0\|_\infty^{p-q}}{t} = 0 \quad \text{in } Q_\infty.$$

Now, fix  $T \geq 0$ . As (1.1) is an autonomous equation, the function  $(t, x) \mapsto u(t+T, x)$  is the solution to (1.1) with initial condition  $u(T)$  and the above analysis allows us to conclude that  $u$  is a supersolution to

$$\partial_t w - |\nabla w|^q + \frac{N(p-1)}{q(q-1)} \frac{\|\nabla u(T)\|_\infty^{p-q}}{t-T} = 0 \quad \text{in } (T, \infty) \times \mathbb{R}^N.$$

We then use (2.2) (with  $T = t/2$ ) to complete the proof.  $\square$

## 4 Convergence to self-similarity

We change the variables and the unknown function so that the convergence (1.5) is transformed to the convergence towards a steady state. More precisely, we introduce the self-similar (or scaling) variables

$$\tau = \frac{1}{q} \log(1+t), \quad y = \frac{x}{(1+t)^{1/q}},$$

and the new unknown function  $v$  defined by

$$(4.1) \quad u(t, x) = v\left(\frac{\log(1+t)}{q}, \frac{x}{(1+t)^{1/q}}\right), \quad (t, x) \in [0, \infty) \times \mathbb{R}^N.$$

Equivalently,  $v(\tau, y) = u(e^{q\tau} - 1, ye^\tau)$  for  $(\tau, y) \in [0, \infty) \times \mathbb{R}^N$  and it follows from (1.1), (1.2) that  $v$  solves

$$(4.2) \quad \partial_\tau v = y \cdot \nabla v + q |\nabla v|^q + q e^{-(p-q)\tau} \Delta_p v, \quad (\tau, y) \in Q_\infty,$$

$$(4.3) \quad v(0) = u_0, \quad y \in \mathbb{R}^N.$$

We also infer from (2.1) and (2.2) that there is a positive constant  $C_5(u_0)$  depending only on  $N, p, q$ , and  $u_0$  such that

$$(4.4) \quad \|v(\tau)\|_\infty + \|\nabla v(\tau)\|_\infty \leq C_5(u_0), \quad \tau \geq 0,$$

while (1.4) reads

$$(4.5) \quad \lim_{\tau \rightarrow \infty} \|v(\tau)\|_\infty = M_\infty > 0.$$

Formally, since  $p > q$ , the diffusion term vanishes in the large time limit and we expect the large time behaviour of the solution  $v$  to (4.2), (4.3) to look like that of the solutions to the first-order Hamilton-Jacobi equation

$$(4.6) \quad \partial_\tau w - y \cdot \nabla w - q |\nabla w|^q = 0 \quad \text{in } Q_\infty.$$

Now, to investigate the large time behaviour of first-order Hamilton-Jacobi equations, an efficient approach has been developed in [22, 23] which relies on the relaxed half-limits method introduced in [3]. More precisely, for  $(\tau, y) \in Q_\infty$ , we define the relaxed half-limits  $v_*$  and  $v^*$  by

$$(4.7) \quad v_*(y) := \liminf_{(\sigma, z, \lambda) \rightarrow (\tau, y, \infty)} v(\sigma + \lambda, z) \quad \text{and} \quad v^*(y) := \limsup_{(\sigma, z, \lambda) \rightarrow (\tau, y, \infty)} v(\sigma + \lambda, z).$$

These relaxed half-limits are well-defined thanks to (4.4) and we first note that the right-hand sides of the above definitions indeed do not depend on  $\tau > 0$ . In addition,

$$(4.8) \quad 0 \leq v_*(x) \leq v^*(x) \leq M_\infty \quad \text{for } y \in \mathbb{R}^N$$

by (4.5), while (4.4) and the Rademacher theorem ensure that  $v_*$  and  $v^*$  both belong to  $W^{1,\infty}(\mathbb{R}^N)$ . Finally, by [2, Théorème 4.1] applied to equation (4.2),  $v^*$  and  $v_*$  are viscosity subsolution and supersolution, respectively, to the Hamilton-Jacobi equation

$$(4.9) \quad \mathcal{H}(y, \nabla w) := -y \cdot \nabla w - q |\nabla w|^q = 0 \quad \text{in } \mathbb{R}^N.$$

We now aim at showing that  $v^*$  and  $v_*$  coincide. However, the equation (4.9) has infinitely many solutions as  $y \mapsto (c - \gamma_q |y|^{q/(q-1)})_+$  solves (4.9) for any  $c > 0$ . The information obtained so far on  $v_*$  and  $v^*$  are thus not sufficient and are supplemented by the next two results.

**Lemma 4.1** *Given  $\varepsilon \in (0, 1)$ , there is  $R_\varepsilon > 1/\varepsilon$  such that*

$$(4.10) \quad v(\tau, y) \leq \varepsilon \quad \text{for } \tau \geq 0 \quad \text{and } y \in \mathbb{R}^N \setminus B(0, R_\varepsilon),$$

and  $0 \leq v_*(y) \leq v^*(y) \leq \varepsilon$  for  $y \in \mathbb{R}^N \setminus B(0, R_\varepsilon)$ .

In other words,  $v(\tau)$  belongs to  $\mathcal{C}_0(\mathbb{R}^N)$  for each  $\tau \geq 0$  in a way which is uniform with respect to  $\tau \geq 0$ .

**Proof.** We first construct a supersolution to (4.2) in  $(0, \infty) \times (\mathbb{R}^N \setminus B(0, R))$  for  $R$  large enough. To this end, consider  $R \geq R_c := 1 + (q(2\|u_0\|_\infty)^{q-1} + 3pq(2\|u_0\|_\infty)^{p-2})^{1/q}$  and put  $\Sigma_R(y) = \|u_0\|_\infty R^2 |y|^{-2}$  for  $y \in \mathbb{R}^N \setminus B(0, R)$ . Let  $\mathcal{L}$  be the parabolic operator defined by

$$\mathcal{L}w(\tau, y) := \partial_\tau w(\tau, y) - y \cdot \nabla w(\tau, y) - q |\nabla w(\tau, y)|^q - q e^{-(p-q)\tau} \Delta_p w(\tau, y)$$

for  $(\tau, y) \in Q_\infty$  (so that  $\mathcal{L}v = 0$  by (4.2)). Then, if  $y \in \mathbb{R}^N \setminus B(0, R)$ , we have

$$\begin{aligned} \mathcal{L}\Sigma_R(y) &= 2 \Sigma_R(y) - q \frac{2^q}{|y|^q} \Sigma_R(y)^q + q 2^{p-1} \frac{N+2-3p}{|y|^p} \Sigma_R(y)^{p-1} e^{-(p-q)\tau} \\ &\geq 2 \Sigma_R(y) \left\{ 1 - q (2\|u_0\|_\infty)^{q-1} \frac{R^{2(q-1)}}{|y|^{3q-2}} - 3pq e^{-(p-q)\tau} (2\|u_0\|_\infty)^{p-2} \frac{R^{2(p-2)}}{|y|^{3p-4}} \right\} \\ &\geq 2 \Sigma_R(y) \left\{ 1 - q (2\|u_0\|_\infty)^{q-1} R^{-q} - 3pq e^{-(p-q)\tau} (2\|u_0\|_\infty)^{p-2} R^{-p} \right\} \\ &\geq 0 \end{aligned}$$

by the choice of  $R$ . Consequently,  $\Sigma_R$  is a supersolution to (4.2) in  $(0, \infty) \times (\mathbb{R}^N \setminus B(0, R))$  for  $R \geq R_c$ .

Now, fix  $\varepsilon \in (0, 1)$ . Since  $u_0 \in \mathcal{C}_0(\mathbb{R}^N)$ , there is  $\rho_\varepsilon \geq \max\{R_c, \varepsilon^{-1}\}$  such that  $u_0(y) \leq \varepsilon/2$  if  $|y| \geq \rho_\varepsilon$ . We then infer from the monotonicity of  $\Sigma_R$  and (2.1) that

$$u_0(y) - \frac{\varepsilon}{2} - \Sigma_{\rho_\varepsilon}(z) \leq -\Sigma_{\rho_\varepsilon}(z) \leq 0$$

if  $|y| \geq \rho_\varepsilon$  and  $|z| \geq \rho_\varepsilon$ ,

$$v(\tau, y) - \frac{\varepsilon}{2} - \Sigma_{\rho_\varepsilon}(z) \leq \|u_0\|_\infty - \Sigma_{\rho_\varepsilon}(z) = \Sigma_{\rho_\varepsilon}(y) - \Sigma_{\rho_\varepsilon}(z) \leq \frac{2\|u_0\|_\infty}{\rho_\varepsilon} |y - z|$$

if  $|y| = \rho_\varepsilon$ ,  $|z| \geq \rho_\varepsilon$ , and  $\tau \geq 0$ , and

$$v(\tau, y) - \frac{\varepsilon}{2} - \Sigma_{\rho_\varepsilon}(z) \leq \|u_0\|_\infty - \|u_0\|_\infty \leq 0$$

if  $|y| \geq \rho_\varepsilon$ ,  $|z| = \rho_\varepsilon$ , and  $\tau \geq 0$ . As  $v - \varepsilon/2$  and  $\Sigma_{\rho_\varepsilon}$  are subsolution and supersolution, respectively, to (4.2), the comparison principle [11, Theorem 4.1] warrants that  $v(\tau, y) - \varepsilon/2 \leq \Sigma_{\rho_\varepsilon}(y)$  for  $\tau \geq 0$  and  $|y| \geq \rho_\varepsilon$ . It remains to choose  $R_\varepsilon \geq \rho_\varepsilon$  such that  $\Sigma_{\rho_\varepsilon}(y) \leq \varepsilon/2$  for  $|y| \geq R_\varepsilon$  to complete the proof of (4.10). The last assertion of Lemma 4.1 is then a straightforward consequence of the definition (4.7) and (4.10).  $\square$

We next use the semiconvexity estimate (1.11) (and more precisely its consequence stated in Corollary 3.4) to show that  $v_*$  lies above the profile  $H_\infty$  defined in (1.6).

**Lemma 4.2** *For  $y \in \mathbb{R}^N$ , we have*

$$(4.11) \quad H_\infty(y) \leq v_*(y) \leq v^*(y).$$

**Proof.** For  $\tau \geq 0$ ,  $y \in \mathbb{R}^N$ ,  $\xi_0 \in \mathbb{R}$  and  $\xi \in \mathbb{R}^N$ , we set  $\mathcal{F}(\tau, y, \xi_0, \xi) := \xi_0 - y \cdot \xi - q |\xi|^q + \kappa_2 e^{-(p-q)\tau}$  with  $\kappa_2 := q \kappa_1 \|u_0\|_\infty^{(p-q)/q} (e^q/(e^q - 1))^{p/q}$ , the constant  $\kappa_1$  being defined in (1.11). It then readily follows from Corollary 3.4 that

$$(4.12) \quad v \text{ is a supersolution to } \mathcal{F}(\tau, y, \partial_\tau w, \nabla w) = 0 \text{ in } (1, \infty) \times \mathbb{R}^N.$$

We next fix  $\tau_0 > 1$  and denote by  $V$  the (viscosity) solution to

$$\begin{aligned} \partial_\tau V - y \cdot \nabla V - q |\nabla V|^q &= 0, & (\tau, y) \in (\tau_0, \infty) \times \mathbb{R}^N, \\ V(\tau_0) &= v(\tau_0), & y \in \mathbb{R}^N. \end{aligned}$$

On the one hand, a straightforward computation shows that the function  $\tilde{V}$  defined by

$$\tilde{V}(\tau, y) := V(\tau, y) - \kappa_2 \int_{\tau_0}^\tau e^{-(p-q)s} ds, \quad (\tau, y) \in (\tau_0, \infty) \times \mathbb{R}^N,$$

is the (viscosity) solution to  $\mathcal{F}(\tau, y, \partial_\tau \tilde{V}, \nabla \tilde{V}) = 0$  in  $(\tau_0, \infty) \times \mathbb{R}^N$  with initial condition  $\tilde{V}(\tau_0) = v(\tau_0)$ . Recalling (4.12), we infer from the comparison principle that

$$(4.13) \quad \tilde{V}(\tau, y) \leq v(\tau, y) \text{ for } (\tau, y) \in (\tau_0, \infty) \times \mathbb{R}^N.$$

On the other hand, it follows from Proposition A.1 that

$$\limsup_{\tau \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \left| V(\tau, y) - (\|v(\tau_0)\|_\infty - \gamma_q |y|^{q/(q-1)})_+ \right| = 0.$$

We may then pass to the limit as  $\tau \rightarrow \infty$  in (4.13) and use the definition (4.7) to conclude that

$$(\|v(\tau_0)\|_\infty - \gamma_q |y|^{q/(q-1)})_+ - \kappa_2 \int_{\tau_0}^\infty e^{-(p-q)s} ds \leq v_*(y) \leq v^*(y)$$



for  $y \in \mathbb{R}^N$ . Letting  $\tau_0 \rightarrow \infty$  in the above inequality with the help of (4.5) completes the proof of the lemma.  $\square$

We are now in a position to complete the proof of Theorem 1.1. To this end, fix  $\varepsilon \in (0, 1)$ . Lemma 4.1 ensures that  $v^*(y) \leq \varepsilon$  for  $|y| \geq R_\varepsilon \geq 1/\varepsilon$  while the continuity of  $H_\infty$  implies that there is  $r_\varepsilon \in (0, \varepsilon)$  such that  $H_\infty(y) \geq M_\infty - \varepsilon$  for  $|y| \leq r_\varepsilon$ . Recalling (4.8), we realize that

$$(4.14) \quad \begin{cases} v^*(y) - \varepsilon \leq 0 \leq H_\infty(y) & \text{if } |y| = R_\varepsilon, \\ v^*(y) - \varepsilon \leq M_\infty - \varepsilon \leq H_\infty(y) & \text{if } |y| = r_\varepsilon. \end{cases}$$

Moreover, introducing  $\psi(y) = -\gamma_q |y|^{q/(q-1)}/2$ , we have

$$(4.15) \quad \mathcal{H}(y, \nabla\psi(y)) = \frac{q\gamma_q}{2(q-1)} |y|^{q/(q-1)} \left(1 - \frac{1}{2^{q-1}}\right) > 0 \quad \text{if } r_\varepsilon < |y| < R_\varepsilon,$$

the Hamiltonian  $\mathcal{H}$  being defined in (4.9). Summarizing, we have shown that  $H_\infty$  and  $v^* - \varepsilon$  are supersolution and subsolution, respectively, to (4.9) in  $\Omega_\varepsilon := \{y \in \mathbb{R}^N : r_\varepsilon < |y| < R_\varepsilon\}$  with  $v^* - \varepsilon \leq H_\infty$  on  $\partial\Omega_\varepsilon$  by (4.14). Owing to (4.15) and the concavity of  $\mathcal{H}$  with respect to its second variable, we may apply [15, Theorem 1] to conclude that  $v^* - \varepsilon \leq H_\infty$  in  $\Omega_\varepsilon$ . This property being valid for each  $\varepsilon \in (0, 1)$ , we actually have  $v^* \leq H_\infty$  in  $\mathbb{R}^N$  by passing to the limit as  $\varepsilon \rightarrow 0$  thanks to the properties of  $r_\varepsilon$  and  $R_\varepsilon$ . Recalling (4.11), we have thus established that  $v^* = v_* = H_\infty$  in  $\mathbb{R}^N$ . In particular, the property  $v^* = v_*$  and the definition (4.7) provide the uniform convergence of  $\{v(\tau)\}_{\tau \geq 0}$  towards  $v^* = H_\infty$  on every compact subset of  $\mathbb{R}^N$  as  $\tau \rightarrow \infty$ , see [2, Lemme 4.1] or [1, Lemma V.1.9]. Combining this local convergence with Lemma 4.1 actually gives

$$(4.16) \quad \lim_{\tau \rightarrow \infty} \|v(\tau) - H_\infty\|_\infty = 0.$$

Theorem 1.1 then readily follows after writing the convergence (4.16) in the original variables  $(t, x)$  for the function  $u$  and noticing that  $\|h_\infty(1+t) - h_\infty(t)\|_\infty \rightarrow 0$  as  $t \rightarrow \infty$ .  $\square$

## 5 Limit value of $\|u(t)\|_\infty$

This section is devoted to the proof of Proposition 1.4, for which three cases are to be distinguished and handled differently:  $q \in (1, p-1]$ ,  $q \in (p-1, q_*]$ , and  $q \in (q_*, p)$ .

**Proof of Proposition 1.4:**  $q \in (1, p-1]$ . We proceed as in [19, Proposition 1] (where a similar result is proved for  $p = 2$  and  $q = 1$ ). For  $\alpha > N/2$ ,  $\delta > 0$ , and  $x \in \mathbb{R}^N$ , we set  $\varrho_\delta(x) := (1 + \delta |x|^2)^{-\alpha}$ . Clearly,  $\varrho_\delta \in L^1(\mathbb{R}^N)$  and it follows from (2.3) that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} \varrho_\delta(x) u(t, x) dx &= \int_{\mathbb{R}^N} \left\{ \varrho_\delta(x) |\nabla u(t, x)|^q - |\nabla u(t, x)|^{p-2} \nabla u(t, x) \cdot \nabla \varrho_\delta(x) \right\} dx \\ &\geq \int_{\mathbb{R}^N} \varrho_\delta(x) |\nabla u(t, x)|^q \left( 1 - |\nabla u(t, x)|^{p-1-q} \frac{|\nabla \varrho_\delta(x)|}{\varrho_\delta(x)} \right) dx. \end{aligned}$$

Recalling that  $\|\nabla u(t)\|_\infty \leq \|\nabla u_0\|_\infty$  by (1.3) and (2.2) and noticing that  $|\nabla \varrho_\delta| \leq \alpha \delta^{1/2} \varrho_\delta$ , we further obtain

$$\frac{d}{dt} \int_{\mathbb{R}^N} \varrho_\delta(x) u(t, x) dx \geq \int_{\mathbb{R}^N} \varrho_\delta(x) |\nabla u(t, x)|^q (1 - \alpha \delta^{1/2} \|\nabla u_0\|_\infty^{p-1-q}) dx.$$

Choosing  $\delta = \|\nabla u_0\|_\infty^{2(q+1-p)}/\alpha^2$  and integrating with respect to time give

$$\|u(t)\|_\infty \|\varrho_\delta\|_1 \geq \int_{\mathbb{R}^N} \varrho_\delta(x) u(t, x) dx \geq \int_{\mathbb{R}^N} \varrho_\delta(x) u_0(x) dx > 0.$$

We then pass to the limit as  $t \rightarrow \infty$  to conclude that  $M_\infty > 0$ .  $\square$

We next turn to the case  $q \in (p-1, q_\star]$  which turns out to be more complicated and requires two preparatory results.

**Lemma 5.1** *Assume that  $q \in (1, q_\star]$  and let  $u$  be a non-negative solution to (1.1), (1.2) with a compactly supported initial condition  $u_0$  satisfying (1.3). Then  $u(t) \in L^1(\mathbb{R}^N)$  for each  $t \geq 0$ , the function  $t \mapsto \|u(t)\|_1$  is non-decreasing and*

$$(5.1) \quad \lim_{t \rightarrow \infty} \|u(t)\|_1 = \infty.$$

**Proof.** For every  $t \geq 0$ ,  $u(t)$  is bounded and compactly supported by (2.1) and Proposition 2.2, and is thus in  $L^1(\mathbb{R}^N)$ . The time monotonicity of the  $L^1$ -norm of  $u$  then readily follows from (2.3) with  $\vartheta = 1$ , a valid choice in this particular case as  $u(t)$  is compactly supported. It further follows from (2.3) with  $\vartheta = 1$  that

$$(5.2) \quad \|u(t)\|_1 \geq \|u(T)\|_1 + \int_T^t \|\nabla u(s)\|_q^q ds \quad \text{for } t > T \geq 0.$$

Consider next  $T > 0$  and  $t > T$ . Recalling the Gagliardo-Nirenberg inequality

$$(5.3) \quad \|w\|_q \leq C_6 \|\nabla w\|_q^{N(q-1)/(N(q-1)+q)} \|w\|_1^{q/(N(q-1)+q)}, \quad w \in W^{1,q}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N),$$

we infer from (5.2), (5.3), and the time monotonicity of the  $L^1$ -norm of  $u$  that

$$\begin{aligned} \|u(t)\|_1^{1+(q^2/N(q-1))} &\geq \|u(t)\|_1^{q^2/N(q-1)} \left( \|u(T)\|_1 + \int_T^t \|\nabla u(s)\|_q^q ds \right) \\ &\geq \int_T^t \|u(s)\|_1^{q^2/N(q-1)} \|\nabla u(s)\|_q^q ds \\ &\geq C_7 \int_T^t (\|u(s)\|_q^q)^{(N(q-1)+q)/N(q-1)} ds. \end{aligned}$$

If  $\varphi$  denotes the solution to the  $p$ -Laplacian equation  $\partial_t \varphi - \Delta_p \varphi = 0$  in  $Q_\infty$  with initial condition  $\varphi(0) = u_0$ , the comparison principle readily implies that

$$(5.4) \quad \varphi(t, x) \leq u(t, x), \quad (t, x) \in Q_\infty.$$

Inserting this estimate in the previous lower bound for  $\|u(t)\|_1$ , we end up with

$$(5.5) \quad \|u(t)\|_1^{1+(q^2/N(q-1))} \geq C_7 \int_T^t (\|\varphi(s)\|_q^q)^{(N(q-1)+q)/N(q-1)} ds.$$

Now, by Proposition 2.3 we have

$$\lim_{s \rightarrow \infty} s^{N(q-1)/(N(p-2)+p)} \|\varphi(s) - \mathcal{B}_{\|u_0\|_1}(s)\|_q^q = 0$$

and

$$\|\mathcal{B}_{\|u_0\|_1}(s)\|_q^q = C_8 s^{-N(q-1)/(N(p-2)+p)},$$

so that

$$\begin{aligned} \|\varphi(s)\|_q^q &\geq \left( \|\mathcal{B}_{\|u_0\|_1}(s)\|_q - \|\varphi(s) - \mathcal{B}_{\|u_0\|_1}(s)\|_q \right)^q \\ &\geq s^{-N(q-1)/(N(p-2)+p)} \left( C_8 - \|\varphi(s) - \mathcal{B}_{\|u_0\|_1}(s)\|_q \right)^q \\ &\geq \left( \frac{C_8}{2} \right)^q s^{-N(q-1)/(N(p-2)+p)} \end{aligned}$$

for  $s \geq T$ , provided  $T$  is chosen sufficiently large. Inserting this estimate in (5.5) gives

$$\begin{aligned} \|u(t)\|_1^{1+(q^2/N(q-1))} &\geq C_9 \int_T^t s^{-(N(q-1)+q)/(N(p-2)+p)} ds \\ &\geq C_{10} \begin{cases} t^{(N+1)(q_*-q)/(N(p-2)+p)} - T^{(N+1)(q_*-q)/(N(p-2)+p)} & \text{if } q \in (1, q_*), \\ \log(t/T) & \text{if } q = q_*. \end{cases} \end{aligned}$$

We then let  $t \rightarrow \infty$  to obtain the claimed result.  $\square$

We next argue as in [12, Lemma 14] (for  $p = 2$ ) to show that, if  $q \in (p-1, p)$  and  $M_\infty = 0$ , then the  $L^\infty$ -norm of  $u(t)$  decays faster than an explicit rate.

**Lemma 5.2** *Assume that  $q \in (p-1, p)$  and let  $u$  be a non-negative solution to (1.1), (1.2) with an initial condition  $u_0$  satisfying (1.3). If  $M_\infty = 0$  in (1.4), then*

$$(5.6) \quad \|u(t)\|_\infty \leq C_{11} t^{-(p-q)/(2q-p)} \quad \text{for } t > 0.$$

Observe that the assumptions  $p > 2$  and  $q \in (p-1, p)$  imply that  $2q > p$  and  $(p-q)/(2q-p) > 0$ .

**Proof.** Consider a non-negative function  $\eta \in \mathcal{C}^\infty(\mathbb{R}^N)$  with compact support in  $B(0, 1)$  and  $\|\eta\|_1 = 1$ . We then define a sequence of mollifiers  $(\eta_\delta)_\delta$  by  $\eta_\delta(x) := \eta(x/\delta)/\delta^N$  for  $x \in \mathbb{R}^N$

and  $\delta \in (0, 1)$ . For  $(t, x_0) \in Q_\infty$  and  $T > t$ , we take  $\vartheta(x) = \eta_\delta(x - x_0)$  in (2.3) and infer from (1.11) (with  $s = t/2$ ) that

$$\begin{aligned}
\|u(T)\|_\infty &\geq \int_{\mathbb{R}^N} u(T, x) \eta_\delta(x - x_0) dx \\
&\geq \int_{\mathbb{R}^N} u(t, x) \eta_\delta(x - x_0) dx - \int_t^T \int_{\mathbb{R}^N} |\nabla u(s, x)|^{p-2} \nabla u(s, x) \cdot \nabla \eta_\delta(t, x - x_0) dx ds \\
&\geq \int_{\mathbb{R}^N} u(t, x) \eta_\delta(x - x_0) dx - 2^{p/q} \kappa_1 \left\| u\left(\frac{t}{2}\right) \right\|_\infty^{(p-q)/q} \int_t^T (2s - t)^{-p/q} ds \\
&\geq \int_{\mathbb{R}^N} u(t, x) \eta_\delta(x - x_0) dx - C_{12} \left\| u\left(\frac{t}{2}\right) \right\|_\infty^{(p-q)/q} (t^{(q-p)/q} - T^{(q-p)/q}).
\end{aligned}$$

Owing to the continuity of  $u$ , we may pass to the limit as  $\delta \rightarrow 0$  in the above inequality and deduce that

$$\|u(T)\|_\infty \geq u(t, x_0) - C_{12} \left\| u\left(\frac{t}{2}\right) \right\|_\infty^{(p-q)/q} (t^{(q-p)/q} - T^{(q-p)/q}).$$

But the above inequality is valid for all  $x_0 \in \mathbb{R}^N$  and we thus end up with

$$\|u(T)\|_\infty \geq \|u(t)\|_\infty - C_{12} \left\| u\left(\frac{t}{2}\right) \right\|_\infty^{(p-q)/q} (t^{(q-p)/q} - T^{(q-p)/q}).$$

Finally, as  $q < p$ , we may let  $T \rightarrow \infty$  in the previous inequality and use the assumption  $M_\infty = 0$  to conclude that

$$\|u(t)\|_\infty \leq C_{12} \left\| u\left(\frac{t}{2}\right) \right\|_\infty^{(p-q)/q} t^{(q-p)/q},$$

or, equivalently, as  $2q > p$ ,

$$t^{(p-q)/(2q-p)} \|u(t)\|_\infty \leq C_{13} \left\{ \left(\frac{t}{2}\right)^{(p-q)/(2q-p)} \left\| u\left(\frac{t}{2}\right) \right\|_\infty \right\}^{(p-q)/q}$$

for  $t \geq 0$ . Introducing

$$A(t) := \sup_{s \in (0, t)} \{s^{(p-q)/(2q-p)} \|u(s)\|_\infty\} \in [0, \infty), \quad t \geq 0,$$

we deduce from the previous inequality that  $A(t) \leq C_{13} A(t)^{(p-q)/q}$ , whence  $A(t) \leq C_{13}^{q/(2q-p)}$  for  $t \geq 0$ . This bound being valid for each  $t > 0$ , the proof of (5.6) is complete.  $\square$

**Proof of Proposition 1.4:**  $q \in (p - 1, q_\star]$ .

*Step 1:* We first consider a compactly supported initial condition  $u_0$  satisfying (1.3) and assume for contradiction that  $M_\infty = 0$ . On the one hand, according to Lemma 5.2 and the assumption  $q \leq q_\star$ , there holds

$$(5.7) \quad \limsup_{t \rightarrow \infty} t^{N/(N(p-2)+p)} \|u(t)\|_\infty \leq C_{11} t^{(p(N+1)(q-q_\star))/((2q-p)(N(p-2)+p))} \leq C_{11}.$$

On the other hand, fix  $t_0 > 0$  and let  $\varphi$  be the solution to the  $p$ -Laplacian equation  $\partial_t \varphi - \Delta_p \varphi = 0$  in  $Q_\infty$  with initial condition  $\varphi(0) = u(t_0)$ . As  $u_0$  is compactly supported, so is  $u(t_0)$  by Proposition 2.2 and  $u(t_0)$  thus belongs to  $L^1(\mathbb{R}^N)$ . Moreover, the comparison principle warrants that  $u(t, x) \geq \varphi(t - t_0, x)$  for  $(t, x) \in [t_0, \infty) \times \mathbb{R}^N$ . We then infer from the above properties and Proposition 2.3 that, for  $t > t_0$ ,

$$\begin{aligned} t^{N/(N(p-2)+p)} \|u(t)\|_\infty &\geq (t - t_0)^{N/(N(p-2)+p)} \|u(t)\|_\infty \\ &\geq (t - t_0)^{N/(N(p-2)+p)} \|\varphi(t - t_0)\|_\infty \\ &\geq (t - t_0)^{N/(N(p-2)+p)} \|\mathcal{B}_{\|u(t_0)\|_1}(t - t_0)\|_\infty \\ &\quad - (t - t_0)^{N/(N(p-2)+p)} \|\mathcal{B}_{\|u(t_0)\|_1}(t - t_0) - \varphi(t - t_0)\|_\infty \\ &\geq C_{14} \|u(t_0)\|_1^{p/(N(p-2)+p)} \\ &\quad - (t - t_0)^{N/(N(p-2)+p)} \|\mathcal{B}_{\|u(t_0)\|_1}(t - t_0) - \varphi(t - t_0)\|_\infty. \end{aligned}$$

Using once more Proposition 2.3, we may pass to the limit as  $t \rightarrow \infty$  in the previous inequality to obtain

$$(5.8) \quad \liminf_{t \rightarrow \infty} t^{N/(N(p-2)+p)} \|u(t)\|_\infty \geq C_{14} \|u(t_0)\|_1^{p/(N(p-2)+p)}.$$

Combining (5.7) and (5.8) yields  $\|u(t_0)\|_1 \leq C_{15}$  for all  $t_0 > 0$  which contradicts Lemma 5.1. Therefore,  $M_\infty > 0$ .

*Step 2:* Now, if  $u_0$  is an arbitrary initial condition satisfying (1.3), there clearly exists a compactly supported initial condition  $\tilde{u}_0$  satisfying (1.3) and such that  $u_0 \geq \tilde{u}_0$  in  $\mathbb{R}^N$ . Introducing the solution  $\tilde{u}$  to (1.1) with initial condition  $\tilde{u}_0$ , the comparison principle entails that  $u \geq \tilde{u}$  in  $Q_\infty$ , hence

$$M_\infty \geq \lim_{t \rightarrow \infty} \|\tilde{u}(t)\|_\infty.$$

The first step of the proof ensures that the right-hand side of the above inequality is positive which completes the proof.  $\square$

It remains to investigate the case  $q \in (q_\star, p)$ , for which we adapt the proof of [7, Theorem 2.4(b)].

**Proof of Proposition 1.4:**  $q \in (q_\star, p)$ . We put

$$m_0 := \left| \inf_{y \in \mathbb{R}^N} \Delta_p u_0(y) \right|.$$

As in the proof of Lemma 5.1, let  $\eta \in C^\infty(\mathbb{R}^N)$  be a non-negative function with compact support in  $B(0,1)$  and  $\|\eta\|_1 = 1$ , and define a sequence of mollifiers  $(\eta_\delta)_\delta$  by  $\eta_\delta(x) := \eta(x/\delta)/\delta^N$  for  $x \in \mathbb{R}^N$  and  $\delta \in (0,1)$ . For  $(t, x_0) \in Q_\infty$  and  $T \in (0, t)$ , we take  $\vartheta(x) = \eta_\delta(x - x_0)$  in (2.3) and infer from (1.11) (with  $s = 0$ ) and Corollary 3.3 that

$$\begin{aligned} \|u(t)\|_\infty &\geq \int_{\mathbb{R}^N} u(t, x) \eta_\delta(x - x_0) dx \\ &\geq \int_{\mathbb{R}^N} u_0(x) \eta_\delta(x - x_0) dx - \int_0^t \int_{\mathbb{R}^N} |\nabla u(s, x)|^{p-2} \nabla u(s, x) \cdot \nabla \eta_\delta(t, x - x_0) dx ds \\ &\geq \int_{\mathbb{R}^N} u_0(x) \eta_\delta(x - x_0) dx - \int_0^T m_0 ds - \kappa_1 \|u_0\|_\infty^{(p-q)/q} \int_T^t s^{-p/q} ds \\ &\geq \int_{\mathbb{R}^N} u_0(x) \eta_\delta(x - x_0) dx - T m_0 - C_{16} \|u_0\|_\infty^{(p-q)/q} (T^{(q-p)/q} - t^{(q-p)/q}) . \end{aligned}$$

Owing to the continuity of  $u_0$ , we may pass to the limit as  $\delta \rightarrow 0$  in the above inequality and deduce that

$$\|u(t)\|_\infty \geq u_0(x_0) - T m_0 - C_{16} \|u_0\|_\infty^{(p-q)/q} (T^{(q-p)/q} - t^{(q-p)/q}) .$$

Since  $q < p$ , we may let  $t \rightarrow \infty$  in the above inequality and take the supremum with respect to  $x_0$  to conclude that

$$M_\infty \geq \|u_0\|_\infty - T m_0 - C_{16} \|u_0\|_\infty^{(p-q)/q} T^{(q-p)/q} .$$

Next, for  $\beta \in (0, 1)$ , the choice  $T = \|u_0\|_\infty^{(p-q)/p} (\beta + m_0)^{-q/p}$  in the previous inequality yields

$$M_\infty \geq \|u_0\|_\infty^{(p-q)/p} (\|u_0\|_\infty^{q/p} - (1 + C_{16}) (\beta + m_0)^{(p-q)/p}) .$$

This inequality being valid for every  $\beta \in (0, 1)$ , we conclude that

$$M_\infty \geq \|u_0\|_\infty^{(p-q)/p} \left( \|u_0\|_\infty^{q/p} - (1 + C_{16}) m_0^{(p-q)/p} \right) > 0$$

as soon as (1.10) is fulfilled with  $\kappa_0 = (1 + C_{16})^{p/q}$ .  $\square$

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## A Convergence for the Hamilton-Jacobi equation (4.6)

In this section, we study the large behaviour of non-negative solutions to the Hamilton-Jacobi equation (4.6) with initial data in  $\mathcal{C}_0(\mathbb{R}^N)$  and show their convergence to a steady state uniquely determined by the  $L^\infty$ -norm of the initial data. Though the large time behaviour of solutions to first-order Hamilton-Jacobi equations has received considerable attention in recent years (see [4, 5, 16, 22, 23] and the references therein), the particular case of (4.6) does not seem to have been investigated in the literature. We thus provide a simple proof relying on the Hopf-Lax-Oleinik formula.

**Proposition A.1** *Let  $q > 1$  and consider a non-negative function  $h_0 \in \mathcal{C}_0(\mathbb{R}^N)$ . Let  $h$  be the unique viscosity solution to the Cauchy problem*

$$(A.1) \quad \partial_\tau h - y \cdot \nabla h - q |\nabla h|^q = 0, \quad (\tau, y) \in (0, \infty) \times \mathbb{R}^N,$$

$$(A.2) \quad h(0) = h_0, \quad y \in \mathbb{R}^N.$$

Then

$$(A.3) \quad \lim_{\tau \rightarrow \infty} \|h(\tau) - h_s\|_\infty = 0$$

with

$$h_s(y) := \left( \|h_0\|_\infty - \gamma_q |y|^{q/(q-1)} \right)_+, \quad y \in \mathbb{R}^N,$$

the constant  $\gamma_q = (q-1) q^{-q/(q-1)}$  being defined in Theorem 1.1.

Thanks to the concavity of the Hamiltonian  $\mathcal{H}(y, \xi) = -y \cdot \xi - q |\xi|^q$ ,  $(y, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$ , with respect to its second variable, the Hopf-Lax-Oleinik formula provides a representation formula for the solution  $h$  to (A.1), (A.2) which can be used to prove (A.3).

**Proof.** We first recall that  $h$  is given by the Hopf-Lax-Oleinik formula

$$h(\tau, y) = \sup_{z \in \mathbb{R}^N} \left\{ h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \right\}$$

for  $(\tau, y) \in [0, \infty) \times \mathbb{R}^N$ , see, e.g., [10, Chapter 3]. Since  $h(\tau, y) \geq h_0(ye^\tau) \geq 0$ , we have in fact

$$h(\tau, y) = \sup_{z \in \mathbb{R}^N} \left\{ \left( h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \right)_+ \right\}$$

for  $(\tau, y) \in [0, \infty) \times \mathbb{R}^N$ .

Consider now  $\beta \in (0, 1)$ . As  $h_0 \in \mathcal{C}_0(\mathbb{R}^N)$ , there is  $R_\beta > (\|h_0\|_\infty / \gamma_q)^{(q-1)/q}$  such that

$$(A.4) \quad h_0(z) \leq \beta \quad \text{for} \quad |z| \geq R_\beta.$$

On the one hand, if  $(\tau, y) \in [\log R_\beta, \infty) \times \mathbb{R}^N$  and  $z \in \mathbb{R}^N$ , we have either  $|z| \geq R_\beta$  and

$$\begin{aligned} & \left| \left( h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \right)_+ - \left( h_0(z) - \gamma_q |y|^{q/(q-1)} \right)_+ \right| \\ & \leq \left( h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \right)_+ + \left( h_0(z) - \gamma_q |y|^{q/(q-1)} \right)_+ \\ & \leq 2\beta \end{aligned}$$

by (A.4) or  $z \in B(0, R_\beta)$  and

$$\begin{aligned}
& \left| \left( h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \right)_+ - \left( h_0(z) - \gamma_q |y|^{q/(q-1)} \right)_+ \right| \\
& \leq \gamma_q |y - z e^{-\tau}|^{q/(q-1)} \left\{ (1 - e^{-q\tau})^{-1/(q-1)} - 1 \right\} + \gamma_q \left| |y - z e^{-\tau}|^{q/(q-1)} - |y|^{q/(q-1)} \right| \\
& \leq \gamma_q (|y| + R_\beta e^{-\tau})^{q/(q-1)} \left\{ (1 - e^{-q\tau})^{-1/(q-1)} - 1 \right\} + \frac{q \gamma_q}{q-1} (|y| + |z| e^{-\tau})^{1/(q-1)} |z| e^{-\tau} \\
& \leq \gamma_q (|y| + 1)^{1/(q-1)} \left\{ \frac{q}{q-1} + |y| + 1 \right\} \left\{ (1 - e^{-q\tau})^{-1/(q-1)} - 1 + R_\beta e^{-\tau} \right\}
\end{aligned}$$

as  $\tau \geq \log R_\beta$ . Combining the above two estimates gives

$$\begin{aligned}
& \left| h(\tau, y) - \sup_{z \in \mathbb{R}^N} \left\{ \left( h_0(z) - \gamma_q |y|^{q/(q-1)} \right)_+ \right\} \right| \\
& \leq C(q) (|y| + 1)^{q/(q-1)} \left\{ (1 - e^{-q\tau})^{-1/(q-1)} - 1 + R_\beta e^{-\tau} \right\} + 2\beta,
\end{aligned}$$

whence

$$(A.5) \quad |h(\tau, y) - h_s(y)| \leq C(q) (|y| + 1)^{q/(q-1)} \left\{ (1 - e^{-q\tau})^{-1/(q-1)} - 1 + R_\beta e^{-\tau} \right\} + 2\beta$$

for  $(\tau, y) \in [\log R_\beta, \infty) \times \mathbb{R}^N$ . On the other hand, if  $\tau \geq \log(R_\beta)$ ,  $|y| \geq Y := 1 + (\|h_0\|_\infty / \gamma_q)^{(q-1)/q}$  and  $z \in \mathbb{R}^N$ , we have either  $|y - z e^{-\tau}| \geq Y - 1$  and

$$\begin{aligned}
& h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \\
& \leq (1 - e^{-q\tau})^{-1/(q-1)} \left\{ \|h_0\|_\infty (1 - e^{-q\tau})^{1/(q-1)} - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} \right\} \\
& \leq (1 - e^{-q\tau})^{-1/(q-1)} \left\{ \|h_0\|_\infty - \gamma_q (Y - 1)^{q/(q-1)} \right\} \\
& \leq 0,
\end{aligned}$$

or  $|y - z e^{-\tau}| < Y - 1$  and

$$|z| \geq |y e^\tau| - |z - y e^\tau| \geq Y e^\tau - (Y - 1) e^\tau = e^\tau \geq R_\beta,$$

so that

$$h_0(z) - \gamma_q |y - z e^{-\tau}|^{q/(q-1)} (1 - e^{-q\tau})^{-1/(q-1)} \leq \beta$$

by (A.4). Therefore,

$$(A.6) \quad h(\tau, y) \leq \beta \quad \text{for} \quad (\tau, y) \in [\log R_\beta, \infty) \times (\mathbb{R}^N \setminus B(0, Y)).$$

The claim (A.3) then easily follows from (A.5) and (A.6).  $\square$



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