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EFFICIENT PRIME COUNTING AND THE CHEBYSHEV PRIMES

MICHEL PLANAT AND PATRICK SOLÉ

ABSTRACT. The function $\epsilon(x) = \text{li}(x) - \pi(x)$ is known to be positive up to the (very large) Skewes' number. Besides, according to Robin's work, the functions $\epsilon_\theta(x) = \text{li}[\theta(x)] - \pi(x)$ and $\epsilon_\psi(x) = \text{li}[\psi(x)] - \pi(x)$ are positive if and only if Riemann hypothesis (RH) holds (the first and the second Chebyshev function are $\theta(x) = \sum_{p \leq x} \log p$ and $\psi(x) = \sum_{n=1}^x \Lambda(n)$, respectively, $\text{li}(x)$ is the logarithmic integral, $\mu(n)$ and $\Lambda(n)$ are the Möbius and the Von Mangoldt functions). Negative jumps in the above functions ϵ , ϵ_θ and ϵ_ψ may potentially occur only at $x + 1 \in \mathcal{P}$ (the set of primes). One denotes $j_p = \text{li}(p) - \text{li}(p - 1)$ and one investigates the jumps j_p , $j_{\theta(p)}$ and $j_{\psi(p)}$. In particular, $j_p < 1$, and $j_{\theta(p)} > 1$ for $p < 10^{11}$. Besides, $j_{\psi(p)} < 1$ for any odd $p \in \text{Ch}$, an infinite set of so-called *Chebyshev primes* with partial list $\{109, 113, 139, 181, 197, 199, 241, 271, 281, 283, 293, 313, 317, 443, 449, 461, 463, \dots\}$.

We establish a few properties of the set Ch and give accurate approximations of the jump $j_{\psi(p)}$. Finally, we introduce a *good* prime counting function $S_N(x) = \sum_{n=1}^N \frac{\mu(n)}{n} \text{li}[\psi(x)^{1/n}]$, that is found to be about 5 times better if $x < 10^5$ and 15 times better if $x < 10^6$ than the standard Riemann prime counting function.

INTRODUCTION

Let us introduce the first and the second Chebyshev function $\theta(x) = \sum_{p \leq x} \log p$ (where $p \in \mathcal{P}$: the set of prime numbers) and $\psi(x) = \sum_{n=1}^x \Lambda(n)$, the logarithmic integral $\text{li}(x)$, the Möbius function $\mu(n)$ and the Von Mangoldt function $\Lambda(n)$ [1, 3]. The number of primes up to x is denoted $\pi(x)$. Indeed, $\theta(x)$ and $\psi(x)$ are the product of all primes up to x , and the logarithm of the least common multiple of all positive integers up to x , respectively.

It has been known for a long time that $\theta(x)$ and $\psi(x)$ are of order x (see [3], p. 341). There also exists an explicit formula, due to Von Mangoldt, relating $\psi(x)$ to the non-trivial zeros ρ of the Riemann zeta function $\zeta(s)$ [1]

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - x^{-2}).$$

Under RH, $|\theta(x) - x| \sim |\psi(x) - x| \sim O(x^{1/2+\epsilon_0})$, for every $\epsilon_0 > 0$.

The function $\epsilon(x) = \text{li}(x) - \pi(x)$ is known to be positive up to the (very large) Skewes' number [2]. In this paper, we are first interested in the jumps (they occur at primes p) in the function $\epsilon_{\theta(x)} = \text{li}[\theta(x)] - \pi(x)$. Following Robin's work on the relation between $\epsilon_{\theta(x)}$ and RH (Theorem 1.1), this allows us to derive a new

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statement (Theorem 1.7) about the jumps of $\text{li}[\theta(p)]$ and Littlewood's oscillation theorem.

Then, we study the refined function $\epsilon_{\psi(x)} = \text{li}[\psi(x)] - \pi(x)$ and we observe that the sign of the jumps of $\text{li}[\psi(p)]$ is controlled by an infinite sequence of primes that we call the *Chebyshev primes* Ch_n (see proposition 1.9). The primes Ch_n (and the generalized primes $\text{Ch}_n^{(l)}$) are also obtained by using an accurate calculation of the jumps of $\text{li}[\psi(p)]$, as in conjecture 1.12 (and of the jumps of the function $\text{li}[\psi(p^l)]$, as in conjecture 1.15). There exists a potential link between the non-trivial zeros ρ of $\zeta(s)$ and the position of the $\text{Ch}_n^{(l)}$'s but we are still unable to display it at this stage of the work. We only provide numerical results about the distribution of the Ch_n 's and conjecture that the function $\text{Ch}_n - p_{2n}$ has infinitely many zeros.

Finally, we introduce a new prime counting function $R(x) = \sum_{n>1} \frac{\mu(n)}{n} \text{li}(x^{1/n})$, better than the standard Riemann's one, even with three terms in the expansion.

1. SELECTED RESULTS ABOUT THE FUNCTIONS ϵ , ϵ_θ , ϵ_ψ

Let p_n be the n -th prime number and $j(p_n) = \text{li}(p_n) - \text{li}(p_n - 1)$ be the jump in the logarithmic integral at p_n . For any $n > 2$ one numerically observes that $j_{p_n} < 1$. This statement is not useful for the rest of the paper. But it is enough to observe that $j_5 = 0.667\dots$ and that the sequence j_{p_n} is strictly decreasing.

The next two subsections deal with the jumps in the function $\text{li}[\theta(x)]$ and $\text{li}[\psi(x)]$.

1.1. The jumps in the function $\text{li}[\theta(x)]$.

Theorem 1.1. (*Robin*). *The statement $\epsilon_{\theta(x)} = \text{li}[\theta(x)] - \pi(x) > 0$ is equivalent to RH [4, 5].*

Corollary 1.2. (related to Robin [4]). *The statement $\epsilon_{\psi(x)} = \text{li}[\psi(x)] - \pi(x) > 0$ is equivalent to RH.*

Proof. If RH is true then, using the fact $\psi(x) > \theta(x)$ and that $\text{li}(x)$ is a strictly growing function when $x > 1$, this follows from theorem 1 in Robin [4]. If RH is false, Lemma 2 in Robin ensures the violation of the inequality. \square

Using the fact that $\theta(p_{n+1} - 1) = \theta(p_n)$, define the jump of index n as

$$J_n = j_{\theta(p_n)} = \text{li}[\theta(p_{n+1})] - \text{li}[\theta(p_n)] = \int_{\theta(p_n)}^{\theta(p_{n+1})} \frac{dt}{\log t}.$$

Proposition 1.3. *If $p_{n+1} < 10^{11}$, then $J_n = j_{\theta(p_n)} > 1$.*

Proof. The integral definition of the jump yields

$$J_n \geq \frac{\theta(p_{n+1}) - \theta(p_n)}{\log \theta(p_{n+1})} = \frac{\log p_{n+1}}{\log \theta(p_{n+1})}.$$

The result now follows after observing that by [9, Theorem18], we have $\theta(x) < x$ for $x < 10^8$, and by using the note added in proof of [10] that establishes that $\theta(x) < x$ for $x < 10^{11}$. \square

By seeing this result it would be natural to make the

Conjecture 1.4. $\forall n \geq 1$ we have $J_n > 1$.

However, building on Littlewood's oscillation theorem for θ we can prove that J_n oscillates about 1 with a small amplitude. Let us recall the Littlewood's oscillation theorem [8, Theorem 6.3, p.200],[7, Theorem 34]

$$\theta(x) - x = \Omega_{\pm}(x^{1/2} \log_3 x), \text{ when } x \rightarrow \infty,$$

where $\log_3 x = \log \log \log x$. The omega notations means that there are infinitely many numbers x , and constants C_+ and C_- , satisfying

$$\theta(x) \leq x - C_- \sqrt{x} \log_3 x \text{ or } \theta(x) \geq x + C_+ \sqrt{x} \log_3 x.$$

We now prepare the proof of the invalidity of conjecture (1.4) by writing two lemmas.

Lemma 1.5. *For $n \geq 1$, we have the bounds*

$$\frac{\log p_{n+1}}{\log \theta(p_{n+1})} \leq J_n \leq \frac{\log p_{n+1}}{\log \theta(p_n)}.$$

Proof. This is straightforward from the integral definition of the jump. \square

Lemma 1.6. *For n large, we have*

$$\theta(p_{n+1}) = p_{n+1} + \Omega_{\pm}(\sqrt{p_{n+1}} \log_3 p_{n+1}).$$

Proof. We know that by [8, Theorem 6.3, p.200], we have for $x > 0$ and large

$$\theta(x) - x = \Omega_{\pm}(\sqrt{x} \log_3 x).$$

The result follows by considering the primes closest to x . \square

We can now state and prove the main result of this section.

Theorem 1.7. *For n large we have*

$$J_n = 1 + \Omega_{\pm}\left(\frac{\log_3 p_{n+1}}{\sqrt{p_{n+1}} \log p_{n+1}}\right).$$

Proof. By lemma 1.6 we know there is a constant C_- such that for infinitely many n 's we have

$$\theta(p_{n+1}) < p_{n+1} - C_- \sqrt{p_{n+1}} \log_3 p_{n+1}.$$

By combining with the first inequality of lemma 1.5 and writing

$$\log p_{n+1} = \log p_n + \log\left(1 - C_- \frac{\log_3 p_{n+1}}{\sqrt{p_{n+1}}}\right),$$

the minus part of the statement follows after some standard asymptotics. To prove the plus part write $\theta(p_n) = \theta(p_{n+1}) - \log p_{n+1}$, and proceed as before. \square

1.2. The jumps in the function $\text{li}[\psi(x)]$ and the Chebyshev primes.

Definition 1.8. Let $p \in \mathcal{P}$ be a odd prime number and the function $j_{\psi(p)} = \text{li}[\psi(p)] - \text{li}[\psi(p-1)]$. The primes p such that $j_{\psi(p)} < 1$ are called here *Chebyshev primes* Ch_n ¹. In increasing order, they are

{109, 113, 139, 181, 197, 199, 241, 271, 281, 283, 293, 313, 317, 443, 449, 461, 463, ...}.

¹Our terminology should not be confused with that used in [6] where the *Tchebychev* primes are primes of the form $4n2^m + 1$, with $m > 0$ and n an odd prime. We used the Russian spelling Chebyshev to distinguish both meanings.

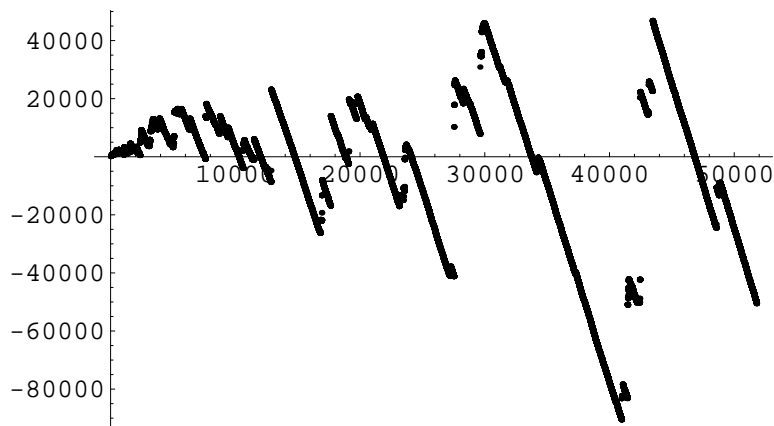


FIGURE 1. A plot of the function $\text{Ch}_n - p_{2n}$ up to the 10^5 -th prime

The number of Chebyshev primes less than 10^n , ($n = 1, 2, \dots$) is the sequence $\{0, 0, 42, 516, 4498, 41423 \dots\}$. This sequence suggests the density $\frac{1}{2}\pi(x)$ for the Chebyshev primes. The largest gaps between the Chebyshev primes are

$\{4, 26, 42, 126, 146, 162, 176, 470, 542, 1370, 1516, 4412, 8196, 14928, 27542, 30974, 51588, 62906 \dots\}$,

and they occur at

$\{109, 113, 139, 317, 887, 1327, 1913, 3089, 8297, 11177, 29761, 45707, 113383, 164893, 291377, 401417, 638371, 1045841 \dots\}$.

Proposition 1.9. There are infinitely many Chebyshev primes Ch_n .

Proof. Let us define the jump of index n as

$$K_n = \text{li}[\psi(p_{n+1})] - \text{li}[\psi(p_{n+1} - 1)] = \int_{\psi(p_{n+1}-1)}^{\psi(p_{n+1})} \frac{dt}{\log t}.$$

We need to check that, for infinitely many values of n , we have $K_n < 1$. The jump K_n is bounded as

$$K_n \leq \frac{\psi(p_{n+1}) - \psi(p_{n+1} - 1)}{\log[\psi(p_{n+1} - 1)]}.$$

By definition, one has $\psi(p_{n+1}) - \psi(p_{n+1} - 1) = \log p_{n+1}$. Therefore,

$$K_n \leq \frac{\log p_{n+1}}{\log[\psi(p_{n+1} - 1)]}.$$

Invoking once again the Littlewood's oscillation theorem [8, Theorem 6.3, p.200], we see that there exists a constant C , such that for infinitely many values of n , one gets

$$\psi(p_{n+1} - 1) > p_{n+1} - 1 + C\sqrt{p_{n+1} - 1}.$$

The \log_3 factor is not needed for the ψ function. The result follows. \square

One observes that the sequence Ch_n oscillates around p_{2n} and the largest deviations from p_{2n} seem to be unbounded at large n . This behaviour is illustrated in Fig 1. Based on this numerical results, we are led to the conjecture

Conjecture 1.10. The function $\text{Ch}_n - p_{2n}$ possesses infinitely many zeros.

Comment 1.11. The first eleven zeros of $\text{Ch}_n - p_{2n}$ occur at the indices $\{510, 10271, 11259, 11987, 14730, 18772, 18884, 21845, 24083, 33723, 46789\}$ where the corresponding Chebyshev primes are $\{164051, 231299, 255919, 274177, 343517, 447827, 450451, 528167, 587519, 847607, 1209469\}$.

At the moment, we do not have easy analytical arguments to prove/disprove the conjecture 1.10. The position and the distribution of the Chebyshev primes depend on the non-trivial zeros ρ of $\zeta(s)$ through the Von Mangoldt formula for the Chebyshev function $\psi(x)$. But the explicit link is out of our reach at the moment.

Conjecture 1.12. The jump at primes p_n of the function $\text{li}[\psi(x)]$ may be written as $K_{n-1} = \tilde{K}_{n-1} + O(1/p_n^2)$, with $\tilde{K}_{n-1} = \frac{\log p_n}{\log[(\psi(p_n) + \psi(p_{n-1}))/2]}$. In particular, the sign of $\tilde{K}_{n-1} - 1$ is that of $K_n - 1$.

Comment 1.13. The jump of index $n - 1$ (at the prime number p_n) is

$$K_{n-1} = \int_{\psi(p_{n-1})}^{\psi(p_n)} \frac{dt}{\log t}.$$

There exists a real c_n depending of the index n , with $\psi(p_{n-1}) < c_n < \psi(p_n)$ such that

$$K_{n-1} = \frac{\psi(p_n) - \psi(p_{n-1})}{\log c_n} = \frac{\log p_n}{\log c_n}.$$

Using the known locations of the Chebyshev primes of low index, it is straightforward to check that the real c_n reads

$$c_n \sim \frac{1}{2} [\psi(p_n) + \psi(p_{n-1})].$$

This numerical calculations support our conjecture (1.12) that the Chebyshev primes may be derived from \tilde{K}_{n-1} instead of K_{n-1} .

1.3. The generalized Chebyshev primes.

Definition 1.14. Let $p \in \mathcal{P}$ be a odd prime number and the function $j_{\psi(p^l)} = \text{li}[\psi(p^l)] - \text{li}[\psi(p^l - 1)]$, $l \geq 1$. The primes p such that $j_{\psi(p^l)} < 1/l$ are called here *generalized Chebyshev primes* $\text{Ch}_n^{(l)}$.

A short list of second order Chebyshev primes is as follows

$\{17, 29, 41, 53, 61, 71, 83, 89, 101, 103, 113, 128, 137, 149, 151, 157, 193, 211, 239, 241, \dots\}$.

A short list of third order Chebyshev primes is as follows

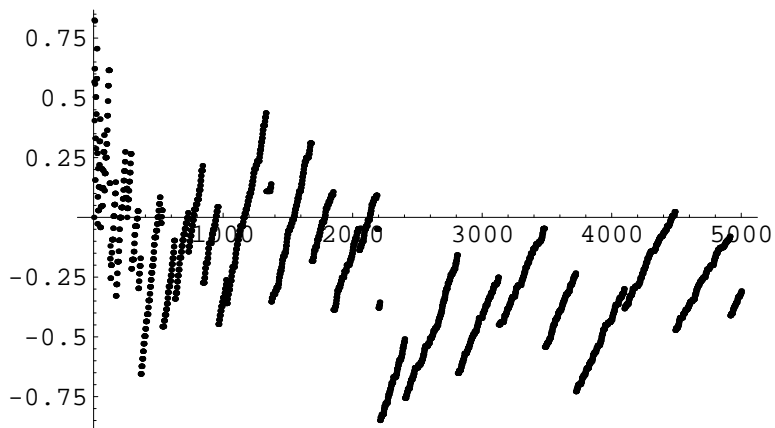
$\{11, 19, 29, 61, 71, 97, 101, 107, 109, 113, 127, 131, 149, 151, 173, 181, 191, 193, 197, 199, \dots\}$.

A short list of fourth order Chebyshev primes is as follows

$\{5, 7, 17, 19, 31, 37, 41, 43, 53, 59, 67, \dots\}$.

Conjecture 1.15. The jump at power of primes p_n^l of the function $\text{li}[\psi(x)]$ may be written as $K_{n-1}^{(l)} = \tilde{K}_{n-1}^{(l)} + O(1/p_n^{2l})$, with $\tilde{K}_{n-1}^{(l)} = \frac{\log p_n}{\log[(\psi(p_n^l) + \psi(p_n^l - 1))/2]}$. In particular the sign of $\tilde{K}_{n-1}^{(l)} - 1$ is that of $K_{n-1}^{(l)} - 1$.

Comment 1.16. Our comment is similar to the comment given in the context of proposition (1.12) but refers to the generalized Chebyshev primes $\text{Ch}_n^{(l)}$. To summarize, the jump of the function $\text{li}[\psi(x)]$ is accurately defined by a *generalized Mangoldt function* Λ_n^{new} that is $\tilde{K}_{n-1}^{(l)}$ if $n = p^l$ and 0 otherwise, with $\tilde{K}_{n-1}^{(l)}$ as

FIGURE 2. A plot of the function $\eta_3(x)$.

defined in the present proposition. The sign of the function $\tilde{K}_{n-1}^{(l)} - 1/l$ determines the position of the generalized Chebyshev primes.

2. AN EFFICIENT PRIME COUNTING FUNCTION

In this section, one finds that the Riemann prime counting function [11] $R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \text{li}(x^{1/n}) \sim \pi(x)$ may be much improved by replacing it by $R[\psi(x)]$. One denotes $\eta_N(x) = \sum_{n=1}^N \frac{\mu(n)}{n} \text{li}[\psi(x)^{1/n}] - \pi(x)$, $N \geq 1$, the offset in the new prime counting function. Indeed, $\eta_1(x) = \epsilon_{\psi(x)}$.

By definition, the negative jumps in the function $\eta_N(x)$ may only occur at $x+1 \in \mathcal{P}$. For $N = 1$, they occur at primes $p \in \text{Ch}$ (the Chebyshev primes: see definition 1.8). For $N > 1$, negative jumps are numerically found to occur at all $x+1 \in \mathcal{P}$ with an amplitude decreasing to zero. We are led to the conjecture

Conjecture 2.1. Let $\eta_N(x) = \sum_{n=1}^N \frac{\mu(n)}{n} \text{li}[\psi(x)^{1/n}] - \pi(x)$, $N > 1$. Negative jumps of the function $\eta_N(x)$ occur at all primes $x+1 \in \mathcal{P}$ and $\lim_{p \rightarrow \infty} [\eta_N(p) - \eta_N(p-1)] = 0$.

More generally, the jumps of $\eta_N(x)$ at power of primes $x+1 = p^l$, $l > 1$ are described by the following

Conjecture 2.2. Let $\eta_N(x) = \sum_{n=1}^N \frac{\mu(n)}{n} \text{li}[\psi(x)^{1/n}] - \pi(x)$, $N > 1$. Positive jumps of the function $\eta_N(x)$ occur at all power of primes $x+1 = p^l$, $p \in \mathcal{P}$ and $l > 1$. Moreover, the jumps are such that $\eta_N(p^l) - \eta_N(p^l - 1) - 1/l > 0$ and $\lim_{p \rightarrow \infty} \eta_N(p^l) - \eta_N(p^l - 1) - 1/l = 0$

A sketch of the function $\eta_3(n)$ (for $2 < n < 1500$) is given in Fig. 2. One easily detects the large positive jumps at $n = p^2$ ($p \in \mathcal{P}$), the intermediate positive jumps at $n = p^l$ ($l > 2$), and the (very small) negative jumps at primes p . This plot can be compared to that of the function $R(n) - \pi(n)$ displayed in [11].

Comment 2.3. The arithmetical structure of $\eta_N(x)$ just described leads to $|\eta_N(x)| < \eta_{\max}$ when $N \geq 3$. Table 1 represents the maximum value η_{\max} that is reached

and the position x_{\max} of the extremum, for several small values of N and $x < 10^5$. Thus, the function $\sum_{n=1}^N \frac{\mu(n)}{n} \text{li}[\psi(x)^{1/n}]$ is a good prime counting function with only a few terms in the summation. This is about a fivefold improvement of the accuracy obtained with the standard Riemann prime counting function $R(x)$ (in the range $x < 10^4$) and an even better improvement when $x > 10^4$, already with three terms in the expansion.

It is known that $R(x)$ converges for any x and may also be written as the Gram series [11] $R(x) = 1 + \sum_{k=1}^{\infty} \frac{(\log x)^k}{k!k\zeta(k+1)}$. A similar formula is not established here.

TABLE 1. Upper part of the table: maximum error η_{\max} in the new prime counting function for $x < 10^4$ (left hand part) in comparison to the maximum error using the Riemann prime counting function (right hand part). Lower part of the table: as above in the range $x < 10^5$.

N	x_{\max}	η_{\max}	x_{\max}	$(R - \pi)_{\max}$
3	6889	1.118	7450	6.174
4	6889	1.118	7450	6.174
5	1330	-1.061	9859	-5.506
6	7	-0.862	7450	5.879
7	1330	-0.936	9949	-5.609
10	7	-0.884	7450	5.661
50	1330	-0.885	9949	-5.557
3	80090	1.840	87888	15.304
10	49727	-1.158	59797	-15.729

CONCLUSION

This (preliminary) work sheds light on the structure and the distribution of the generalized Chebyshev primes Ch_n^l arising from the jumps of the function $\text{li}[\psi(x)]$. It is inspired by Robin's work [4] relating the sign of the functions $\epsilon_{\theta(x)}$ and $\epsilon_{\psi(x)}$ to RH [4]. Our most puzzling observation is that the non-trivial zeros ρ of the Riemann zeta function are mirrored in the (generalized) Chebyshev primes, whose existence at infinity crucially depends on the Littlewood's oscillation theorem. In addition, a new accurate prime counting function, based on $\text{li}[\psi(x)]$ has been proposed. Future work should concentrate on an effective analytic map between the the zeros ρ and the sequence Ch_n .

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[109, 113, 139, 181, 197, 199, 241, 271, 281, 283, 293, 313, 317, 443, 449, 461, 463, 467, 479, 491, 503, 509, 523, 619, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 761, 769, 773, 829, 859, 863, 883, 887, 1033, 1039, 1049, 1051, 1061, 1063, 1069, 1091, 1093, 1097, 1103, 1109, 1117, 1123, 1129, 1153, 1231, 1237, 1301, 1303, 1307, 1319, 1321, 1327, 1489, 1493, 1499, 1511, 1571, 1579, 1583, 1601, 1607, 1609, 1613, 1619, 1621, 1627, 1637, 1657, 1663, 1667, 1669, 1693, 1697, 1699, 1709, 1721, 1723, 1733, 1741, 1747, 1753, 1759, 1783, 1787, 1789, 1801, 1811, 1877, 1879, 1889, 1907, 1913, 2089, 2113, 2141, 2143, 2153, 2161, 2297, 2311, 2351, 2357, 2381, 2383, 2389, 2393, 2399, 2411, 2417, 2423, 2437, 2441, 2447, 2459, 2467, 2473, 2477, 2557, 2711, 2713, 2719, 2729, 2731, 2741, 2749, 2753, 2767, 2777, 2789, 2791, 2797, 2801, 2803, 2819, 2833, 2837, 2843, 2851, 2857, 2861, 2879, 2887, 2897, 2903, 2909, 2917, 2927, 2939, 2953, 2957, 2963, 2969, 2971, 3001, 3011, 3019, 3023, 3037, 3041, 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