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Fabio Perroni, Alexander D. Rahm

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On the Cohomological Crepant Resolution Conjecture for the complexified Bianchi orbifolds

FABIO PERRONI AND ALEXANDER D. RAHM

We give formulae for the Chen–Ruan orbifold cohomology for the orbifolds given by a Bianchi group acting on complex hyperbolic 3-space.

The Bianchi groups are the arithmetic groups $\mathrm{PSL}_2(\mathcal{O})$, where \mathcal{O} is the ring of integers in an imaginary quadratic number field. The underlying real orbifolds which help us in our study, given by the action of a Bianchi group on real hyperbolic 3-space (which is a model for its classifying space for proper actions), have applications in physics.

We then prove that, for any such orbifold, its Chen–Ruan orbifold cohomology ring is isomorphic to the usual cohomology ring of any crepant resolution of its coarse moduli space. By vanishing of the quantum corrections, we show that this result fits in with Ruan’s Cohomological Crepant Resolution Conjecture.

[55N32, Orbifold cohomology](#)

1 Introduction

Recently motivated by string theory in theoretical physics, a stringy topology of orbifolds has been introduced in mathematics [1]. Its essential innovations consist of Chen–Ruan orbifold cohomology [9], [10] and orbifold K –theory. They are of interest as topological quantum field theories [19]. Ruan’s cohomological crepant resolution conjecture [38] associates Chen–Ruan orbifold cohomology with the ordinary cohomology of a resolution of the singularities of the coarse moduli space of the given orbifold. We place ourselves where the conjecture is still open: in three complex dimensions and outside the global quotient case. There, we are going to calculate the Chen–Ruan cohomology of an infinite family of orbifolds in this article; and prove in Section 6 that it is isomorphic as a ring to the cohomology of their crepant resolution of singularities.

Denote by $\mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer, an imaginary quadratic number field, and by \mathcal{O}_{-m} its ring of integers. The *Bianchi groups* are the projective special linear groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$. The Bianchi groups may be considered as a key to the study of a larger class of groups, the *Kleinian groups*, which date back to work of Henri Poincaré [30]. In fact, each non-co-compact arithmetic Kleinian group is commensurable with some Bianchi group [24]. A wealth of information on the Bianchi groups can be found in the monographs [13, 15, 24]. These groups act in a natural way on hyperbolic three-space, which is isomorphic to the symmetric space associated to them. This yields orbifolds that are studied in cosmology [4].

The orbifold structure obtained by our group action is determined by a fundamental domain and its stabilizers and identifications. The computation of this information has been implemented for all Bianchi groups [33].

The vector space structure of Chen–Ruan orbifold cohomology

Let Γ be a discrete group acting *properly discontinuously*, hence with finite stabilizers, by bi-holomorphisms on a complex manifold Y . For any element $g \in \Gamma$, denote by $C_\Gamma(g)$ the centralizer of g in Γ . Denote by Y^g the subspace of Y consisting of the fixed points of g .

Definition 1 ([9]) *Let $T \subset \Gamma$ be a set of representatives of the conjugacy classes of elements of finite order in Γ . Then the Chen–Ruan orbifold cohomology vector space of $[Y/\Gamma]$ is:*

$$H_{\text{CR}}^*([Y/\Gamma]) := \bigoplus_{g \in T} H^*(Y^g/C_\Gamma(g); \mathbb{Q}).$$

The grading on this vector space is reviewed in (1).

This definition is slightly different from, but equivalent to, the original one in [9]. We can verify this fact using arguments analogous to those used by Fantechi and Göttsche [14] in the case of a finite group Γ acting on Y . The additional argument needed when considering some element g in Γ of infinite order is the following. As the action of Γ on Y is properly discontinuous, g does not admit any fixed point in Y . Thus, $H^*(Y^g/C_\Gamma(g); \mathbb{Q}) = H^*(\emptyset; \mathbb{Q}) = 0$. For another proof, see [1].

Statement of the results

We complexify our orbifolds by complexifying the real hyperbolic three-space (Construction 6). We obtain orbifolds given by the induced action of the Bianchi groups on complex hyperbolic three-space $\mathcal{H}_{\mathbb{C}}^3$. Then we compute the Chen–Ruan Orbifold Cohomology for these complex orbifolds $[\mathcal{H}_{\mathbb{C}}^3/\Gamma]$. We can determine its product structure with Theorem 10.

As a result of Theorems 17 and 18, we can express the vector space structure of the orbifold cohomology in terms of the numbers of conjugacy classes of finite subgroups and the cohomology of the quotient space, as follows.

Except for the Gaussian and Eisenstein integers, which we have treated separately (Sections 8.2 and 8.3), all the rings of integers of imaginary quadratic number fields admit as only units $\{\pm 1\}$. In the latter case, we call $\text{PSL}_2(\mathcal{O})$ a *Bianchi group with units $\{\pm 1\}$* .

As a corollary to Theorems 17 and 18, which we are going to prove in Section 5, and to Theorem 16, we obtain:

Corollary 2 *Let Γ be a finite index subgroup in a Bianchi group with units $\{\pm 1\}$. Denote by λ_{2n} the number of conjugacy classes of cyclic subgroups of order n in Γ . Denote by λ_{2n}^* the number of conjugacy classes of those of them which are contained in a dihedral subgroup of order $2n$ in Γ . Then,*

$$H_{\text{CR}}^d([\mathcal{H}_{\mathbb{C}}^3/\Gamma]) \cong H^d(\mathcal{H}_{\mathbb{R}}^3/\Gamma; \mathbb{Q}) \oplus \begin{cases} \mathbb{Q}^{\lambda_4+2\lambda_6-\lambda_6^*}, & d = 2, \\ \mathbb{Q}^{\lambda_4-\lambda_4^*+2\lambda_6-\lambda_6^*}, & d = 3, \\ 0, & \text{otherwise.} \end{cases}$$

Together with the example computations for the Gaussian and Eisensteinian cases (Sections 8.2 and 8.3), we obtain $H_{\text{CR}}^d([\mathcal{H}_{\mathbb{C}}^3/\Gamma])$ for all Bianchi groups Γ .

The (co)homology of the quotient space $\mathcal{H}_{\mathbb{R}}^3/\Gamma$ has been computed numerically for a large scope of Bianchi groups [44], [39], [36]; and bounds for its Betti numbers have been given in [22]. Krämer [23] has determined number-theoretic formulae for the numbers λ_{2n} and λ_{2n}^* of conjugacy classes of finite subgroups in the Bianchi groups. Krämer's formulae have been evaluated for hundreds of thousands of Bianchi groups [35], and these values are matching with the ones from the orbifold structure computations with [33] in the cases where the latter are available.

The quotient $\mathcal{H}_{\mathbb{C}}^3/\Gamma$ has a crepant resolution, since the stabilizer in Γ of any point of $\mathcal{H}_{\mathbb{C}}^3$ acts on the tangent space at the point with determinant 1 (see Section 3). Then we prove the following result.

Theorem 3 *Let Γ be a Bianchi group and let $[\mathcal{H}_{\mathbb{C}}^3/\Gamma]$ be the corresponding orbifold, with coarse moduli space $\mathcal{H}_{\mathbb{C}}^3/\Gamma$. Let $f: Y \rightarrow \mathcal{H}_{\mathbb{C}}^3/\Gamma$ be a crepant resolution of $\mathcal{H}_{\mathbb{C}}^3/\Gamma$. Then there is an isomorphism as graded \mathbb{Q} -algebras between the Chen–Ruan cohomology ring of $[\mathcal{H}_{\mathbb{C}}^3/\Gamma]$ and the singular cohomology ring of Y :*

$$(H_{\text{CR}}^*([\mathcal{H}_{\mathbb{C}}^3/\Gamma]), \cup_{\text{CR}}) \cong (H^*(Y), \cup).$$

The proof of this theorem, which we shall give in Section 6, uses the McKay correspondence and our computations of the Chen–Ruan orbifold cohomology of Bianchi orbifolds. In Section 7 we compare this result with Ruan's Cohomological Crepant Resolution Conjecture ([38], [12]). Even though $\mathcal{H}_{\mathbb{C}}^3/\Gamma$ and Y are not projective varieties, hence Ruan's conjecture does not apply directly, our results confirm the validity of this conjecture.

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Organization of the paper

In Section 2, we work out the product structure of Chen–Ruan orbifold cohomology for arbitrary groups of hyperbolic motions. Then we specialize to the Bianchi groups, and work out the information that we

need about the conjugacy classes of their finite order elements in Section 3. We use this in Section 5 to develop the dimension formulas stated in Corollary 2. What reduces our considerations on complex orbifolds here to the easier case of real orbifolds, is the spine described in Section 4. In Section 6, we prove the Chen–Ruan ring isomorphism stated in Theorem 3. In Section 7, we show how the vanishing of the quantum corrections makes this ring isomorphism fit in with Ruan’s Cohomological Crepant Resolution Conjecture. We illustrate our study with the computation of some explicit examples in Section 8.

Notation

We use the following symbols for the finite subgroups in $\mathrm{PSL}_2(\mathcal{O})$: for cyclic groups of order n , we write \mathbb{Z}/n ; for the Klein four-group $\mathbb{Z}/2 \times \mathbb{Z}/2$, we write \mathcal{D}_2 ; for the dihedral group with six elements, we write \mathcal{D}_3 ; and for the alternating group on four symbols, we write \mathcal{A}_4 .

2 The orbifold cohomology product

In order to equip the orbifold cohomology vector space with the Chen–Ruan product structure, we use the orbifold complex structure on $[Y/\Gamma]$.

Let Y be a complex manifold of dimension D with a properly discontinuous action of a discrete group Γ by bi-holomorphisms. For any $g \in \Gamma$ and $y \in Y^g$, we consider the eigenvalues $\lambda_1, \dots, \lambda_D$ of the action of g on the tangent space $T_y Y$. As the action of g on $T_y Y$ is complex linear, its eigenvalues are roots of unity.

Definition 4 Write $\lambda_j = e^{2\pi i r_j}$, where r_j is a rational number in the interval $[0, 1[$. The degree shifting number of g in y is the rational number $\mathrm{shift}(g, y) := \sum_{j=1}^D r_j$.

We see in [14] that the degree shifting number agrees with the one defined by Chen and Ruan. It is also called the fermionic shift number in [46]. The degree shifting number of an element g is constant on a connected component of its fixed point set Y^g . For the groups under our consideration, Y^g is connected, so we can omit the argument y . Details for this and the explicit value of the degree shifting number are given in Lemma 9. Then we can define the graded vector space structure of the Chen–Ruan orbifold cohomology as

$$(1) \quad \mathrm{H}_{\mathrm{CR}}^d([Y/\Gamma]) := \bigoplus_{g \in T} \mathrm{H}^{d-2 \mathrm{shift}(g)}(Y^g/C_\Gamma(g); \mathbb{Q}).$$

Denote by g, h two elements of finite order in Γ , and by $Y^{g,h}$ their common fixed point set. Chen and Ruan construct a certain vector bundle on $Y^{g,h}$, the *obstruction bundle*. We denote by $c(g, h)$ its top Chern class. In our cases, $Y^{g,h}$ is a connected manifold. In the general case, the fiber dimension of the obstruction bundle can vary between the connected components of $Y^{g,h}$, and $c(g, h)$ is the cohomology class restricting to the top Chern class of the obstruction bundle on each connected component. The obstruction bundle is at the heart of the construction [9] of the Chen–Ruan orbifold cohomology product. In [14], this product, when applied to a cohomology class associated to Y^g and one associated to Y^h , is described as a push-forward of the cup product of these classes restricted to $Y^{g,h}$ and multiplied by

$c(g, h)$.

The following statement is made for global quotient orbifolds, but it is a local property, so we can apply it in our case.

Lemma 5 (Fantechi–Göttsche) *Let $Y^{g,h}$ be connected. Then the obstruction bundle on it is a vector bundle of fiber dimension*

$$\text{shift}(g) + \text{shift}(h) - \text{shift}(gh) - \text{codim}_{\mathbb{C}}(Y^{g,h} \subset Y^{gh}).$$

In [14], a proof is given in the more general setting that $Y^{g,h}$ needs not be connected. Examples where the product structure is worked out in the non-global quotient case, are for instance given in [9, 5.3], [29] and [6].

2.1 Groups of hyperbolic motions

A class of examples with complex structures admitting the grading (1) is given by the discrete subgroups Γ of the orientation preserving isometry group $\text{PSL}_2(\mathbb{C})$ of real hyperbolic 3-space $\mathcal{H}_{\mathbb{R}}^3$. The Kleinian model of $\mathcal{H}_{\mathbb{R}}^3$ gives a natural identification of the orientation preserving isometries of $\mathcal{H}_{\mathbb{R}}^3$ with matrices in $\text{PSO}(3, 1)$. By the subgroup inclusion $\text{PSO}(3, 1) \hookrightarrow \text{PSU}(3, 1)$, these matrices specify isometries of the complex hyperbolic space $\mathcal{H}_{\mathbb{C}}^3$. The details are as follows.

Construction 6 Given an orbifold $[\mathcal{H}_{\mathbb{R}}^3/\Gamma]$, we construct as follows the complexified orbifold $[\mathcal{H}_{\mathbb{C}}^3/\Gamma]$. Recall the Kleinian model for $\mathcal{H}_{\mathbb{R}}^3$ described in [13]: For this, we take a basis $\{f_0, f_1, f_2, f_3\}$ for \mathbb{R}^4 , and rewrite \mathbb{R}^4 as $\tilde{E}_1 := \mathbb{R}f_0 \oplus \mathbb{R}f_1 \oplus \mathbb{R}f_2 \oplus \mathbb{R}f_3$. Then we define the quadratic form q_1 by

$$q_1(y_0f_0 + y_1f_1 + y_2f_2 + y_3f_3) = y_0^2 - y_1^2 - y_2^2 - y_3^2.$$

We consider the real projective 3-space $\mathbb{P}\tilde{E}_1 = (\tilde{E}_1 \setminus \{0\})/\mathbb{R}^*$, where \mathbb{R}^* stands for the multiplicative group $\mathbb{R} \setminus \{0\}$. The set underlying the Kleinian model is then

$$\mathbb{K} := \{[y_0 : y_1 : y_2 : y_3] \in \mathbb{P}\tilde{E}_1 \mid q_1(y_0, y_1, y_2, y_3) > 0\}.$$

Once that \mathbb{K} is equipped with the hyperbolic metric, its group of orientation preserving isometries is $\text{PSO}_4(q_1, \mathbb{R}) =: \text{PSO}(3, 1)$. The isomorphism of \mathbb{K} to the upper-half space model of $\mathcal{H}_{\mathbb{R}}^3$ yields an isomorphism between the groups of orientation preserving isometries, $\text{PSO}(3, 1) \cong \text{PSL}_2(\mathbb{C})$. This is how we include Γ into $\text{PSO}(3, 1)$.

Now we consider complex Euclidean 4-space $\tilde{E}_1 \otimes_{\mathbb{R}} \mathbb{C} := \mathbb{C}f_0 \oplus \mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \mathbb{C}f_3$, complex projective 3-space $\mathbb{P}(\tilde{E}_1 \otimes_{\mathbb{R}} \mathbb{C}) = (\tilde{E}_1 \otimes_{\mathbb{R}} \mathbb{C} \setminus \{0\})/\mathbb{C}^*$, and obtain a model

$$\mathbb{K}_{\mathbb{C}} := \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}(\tilde{E}_1 \otimes_{\mathbb{R}} \mathbb{C}) \mid q_1(|z_0|, |z_1|, |z_2|, |z_3|) > 0\}$$

for complex hyperbolic 3-space $\mathcal{H}_{\mathbb{C}}^3$, where $|z|$ denotes the modulus of the complex number z . The latter model admits $\text{PSU}(3, 1)$ as its group of orientation preserving isometries, with a natural inclusion of $\text{PSO}(3, 1)$.

This is how we obtain our action of Γ on $\mathcal{H}_{\mathbb{C}}^3$. In the remainder of this section we show some properties of this action that will be used in the following.

Lemma 7 *The action of Γ on $\mathcal{H}_{\mathbb{C}}^3$ just defined is properly discontinuous.*

Proof This fact should be well known and can be proved using the existence of Dirichlet fundamental domains for the Γ -action on $\mathcal{H}_{\mathbb{C}}^3$ [18, Section 9.3]. We include here for completeness a self-contained proof which relies on the fact that the Γ -action on $\mathcal{H}_{\mathbb{R}}^3$ is properly discontinuous [13, Theorems 1.2, p. 34, and 1.1 p. 311].

Let $\{\gamma_n\}_{n \geq 1}$ be a sequence of elements of Γ and let $x \in \mathcal{H}_{\mathbb{C}}^3$ be a point, such that $\{\gamma_n \cdot x\}_{n \geq 1}$ is infinite. We show that $\{\gamma_n \cdot x\}_{n \geq 1}$ has no accumulation point in $\mathcal{H}_{\mathbb{C}}^3$. To this aim, assume by contradiction that $x_{\infty} \in \mathcal{H}_{\mathbb{C}}^3$ is an accumulation point for $\{\gamma_n \cdot x\}_{n \geq 1}$. Let $p: \mathcal{H}_{\mathbb{C}}^3 \rightarrow \mathcal{H}_{\mathbb{R}}^3$ be the projection defined in the proof of Theorem 16 and consider $p(x_{\infty})$, $\{p(\gamma_n \cdot x)\}_{n \geq 1}$. Notice that, since p is Γ -equivariant, $p(\gamma_n \cdot x) = \gamma_n \cdot p(x)$, and $\{\gamma_n \cdot p(x)\}_{n \geq 1}$ is infinite, because Γ acts properly discontinuously on $\mathcal{H}_{\mathbb{R}}^3$. It follows that $p(x_{\infty})$ is an accumulation point for $\{\gamma_n \cdot p(x)\}_{n \geq 1}$, hence a contradiction. \square

Lemma 8 *For any $g \in \Gamma$, the natural map $(\mathcal{H}_{\mathbb{C}}^3)^g / C_{\Gamma}(g) \rightarrow \mathcal{H}_{\mathbb{C}}^3 / \Gamma$ induced by the inclusion $(\mathcal{H}_{\mathbb{C}}^3)^g \subset \mathcal{H}_{\mathbb{C}}^3$ is proper.*

Proof The proof is given in two steps, in the first one we show that the map has finite fibre. Since this fact holds true in general, for any discrete group Γ acting properly discontinuously by bi-holomorphisms on a complex manifold M , we prove it in this generality. Let us denote by $f: M^g / C_{\Gamma}(g) \rightarrow M / \Gamma$ the natural map induced by the inclusion $(\mathcal{H}_{\mathbb{C}}^3)^g \subset \mathcal{H}_{\mathbb{C}}^3$. For any $x \in M^g$, let $[x] \in M^g / C_{\Gamma}(g)$ be its equivalence class. Then

$$f^{-1}(f([x])) = \{y \in M^g \mid y \in \Gamma \cdot x\} / C_{\Gamma}(g),$$

where $\Gamma \cdot x$ denotes the orbit of x . Notice that, for any $h \in \Gamma$, if $h \cdot x \in M^g$, then $g \in \text{Stab}(h \cdot x) = h \text{Stab}(x) h^{-1}$, and so there exists a unique $g_h \in \text{Stab}(x)$ such that $h g_h h^{-1} = g$, $g_h = h^{-1} g h$, here for any element y , $\text{Stab}(y)$ denotes its stabilizer. Furthermore, if $h_1, h_2 \in \Gamma$ are such that $h_1^{-1} g h_1 = h_2^{-1} g h_2$, then $g = h_2 h_1^{-1} g h_1 h_2^{-1} = (h_2 h_1^{-1}) g (h_2 h_1^{-1})^{-1}$. Therefore $h_2 h_1^{-1} \in C_{\Gamma}(g)$ and hence $h_2 \in C_{\Gamma}(g) \cdot h_1$. This implies that, if we define

$$\Gamma_{x,g} := \{h \in \Gamma \mid h \cdot x \in M^g\},$$

then the map $f_{x,g}: \Gamma_{x,g} \rightarrow \text{Stab}(x)$, $h \mapsto g_h = h^{-1} g h$, descends to an injective map $\Gamma_{x,g} / C_{\Gamma}(g) \rightarrow \text{Stab}(x)$. The claim now follows from the fact that $\text{Stab}(x)$ is finite and $\Gamma_{x,g} / C_{\Gamma}(g)$ is bijective to $f^{-1}(f([x]))$.

In the second step of the proof, $M = \mathcal{H}_{\mathbb{C}}^3$ and $f: (\mathcal{H}_{\mathbb{C}}^3)^g / C_{\Gamma}(g) \rightarrow \mathcal{H}_{\mathbb{C}}^3 / \Gamma$. Let d be the distance function on $\mathcal{H}_{\mathbb{C}}^3$ induced by the Bergman metric. By restriction d induces a distance function on $(\mathcal{H}_{\mathbb{C}}^3)^g$. Moreover, defining for any $[x], [y] \in \mathcal{H}_{\mathbb{C}}^3 / \Gamma$ (respectively $[x], [y] \in (\mathcal{H}_{\mathbb{C}}^3)^g / C_{\Gamma}(g)$),

$$\tilde{d}([x], [y]) := \text{Inf}\{d(\xi, \eta) \mid \xi \in \Gamma \cdot x, \eta \in \Gamma \cdot y\},$$

we have a distance function on $\mathcal{H}_{\mathbb{C}}^3 / \Gamma$ (on $(\mathcal{H}_{\mathbb{C}}^3)^g / C_{\Gamma}(g)$ respectively, where \tilde{d} is defined accordingly). By elementary topology, for topological spaces with distance functions, a subspace K is compact, if and only if any infinite subset $Z \subset K$ has an accumulation point in K . So, let $K \subset \mathcal{H}_{\mathbb{C}}^3 / \Gamma$ be a compact subspace. To show that $f^{-1}(K)$ is compact, let $Z \subset f^{-1}(K)$ be an infinite subset. Since f has finite fibre, $f(Z)$ is infinite, so it has an accumulation point, say $[x_0] \in K$. Notice that $f^{-1}([x_0]) \neq \emptyset$, since $\text{Im}(f)$

is closed. To see this, let $[x] \notin \text{Im}(f)$. Then $\Gamma \cdot x \cap (\mathcal{H}_{\mathbb{C}}^3)^g = \emptyset$, in other words, for any $y \in \Gamma \cdot x$, $g \notin \text{Stab}(y)$. Since the action is properly discontinuous, any $y \in \Gamma \cdot x$ has a neighborhood U such that $\gamma \cdot U \cap U \neq \emptyset$, if and only if $\gamma \in \text{Stab}(y)$, for any $\gamma \in \Gamma$. In particular the stabilizer of any point in U is contained in $\text{Stab}(y)$, and hence $\Gamma \cdot U \cap (\mathcal{H}_{\mathbb{C}}^3)^g = \emptyset$. So, $\Gamma \cdot U$ gives an open neighbourhood of $[x]$ which has empty intersection with $\text{Im}(f)$. To finish the proof of the lemma, we observe that, if $[x_0] \in K$ is an accumulation point for $f(Z)$, and $f^{-1}([x_0]) \neq \emptyset$, then there exists $[y_0] \in f^{-1}([x_0]) \subset f^{-1}(K)$ which is an accumulation point for Z , since f has finite fibres. \square

Lemma 9 *The degree shifting number of any non-trivial rotation of $\mathcal{H}_{\mathbb{C}}^3$ on its fixed points set is 1.*

Proof For any rotation $\hat{\theta}$ of angle θ around a geodesic line in $\mathcal{H}_{\mathbb{R}}^3$, there is a basis for the construction of the Kleinian model such that the matrix of $\hat{\theta}$ takes the shape

$$\begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{PSO}(3, 1).$$

This matrix, considered as an element of $\text{PSU}(3, 1)$, performs a rotation of angle θ around the “complexified geodesic line” with respect to the inclusion $\mathcal{H}_{\mathbb{R}}^3 \hookrightarrow \mathcal{H}_{\mathbb{C}}^3$. The fixed points of this rotation are exactly the points p lying on this complexified geodesic line, and the action on their tangent space $T_p \mathcal{H}_{\mathbb{C}}^3 \cong \mathbb{C}^3$ is again a rotation of angle θ . Hence we can choose a basis of this tangent space such that this rotation is expressed by the matrix

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}_3(\mathbb{C}).$$

Therefore the degree shifting number of the rotation $\hat{\theta}$ at p is 1. \square

Theorem 10 *Let Γ be a group generated by translations and rotations of $\mathcal{H}_{\mathbb{C}}^3$. Then all obstruction bundles of the orbifold $[\mathcal{H}_{\mathbb{C}}^3/\Gamma]$ are of fiber dimension zero.*

Proof Non-trivial obstruction bundles can only appear for two elements of Γ with common fixed points, and such that one of these is not a power of the other one. The translations of $\mathcal{H}_{\mathbb{C}}^3$ have their fixed point on the boundary and not in $\mathcal{H}_{\mathbb{C}}^3$. So let b and c be non-trivial hyperbolic rotations around distinct axes intersecting in the point $p \in \mathcal{H}_{\mathbb{C}}^3$. Then bc is again a hyperbolic rotation around a third distinct axis passing through p . Obviously, these rotation axes constitute the fixed point sets Y^b , Y^c and Y^{bc} . Hence the only fixed point of the group generated by b and c is p . Now Lemma 5 yields the following fiber dimension for the obstruction bundle on $Y^{b,c}$:

$$\text{shift}(b) + \text{shift}(c) - \text{shift}(bc) - \text{codim}_{\mathbb{C}}(Y^{b,c} \subset Y^{bc}).$$

After computing degree shifting numbers using Lemma 9, we see that this fiber dimension is zero. \square

Hence the obstruction bundle is trivial, and its top Chern class is the neutral element of the cohomological

cup product. By Fantechi–Göttsche’s description, the Chen–Ruan orbifold cohomology product is then a push-forward of the cup product of the cohomology classes restricted to the intersection of the fixed points sets.

3 The centralizers of finite cyclic subgroups in the Bianchi groups

Let Γ be a finite index subgroup in a Bianchi group $\mathrm{PSL}_2(\mathcal{O}_{-m})$. In 1892, Luigi Bianchi [5] computed fundamental domains for some of the full Bianchi groups. Such a fundamental domain has the shape of a hyperbolic polyhedron (up to a missing vertex at certain cusps, which represent the ideal classes of \mathcal{O}_{-m}), so we will call it the *Bianchi fundamental polyhedron*.

It is well-known [21] that any element of Γ fixing a point inside real hyperbolic 3-space $\mathcal{H}_{\mathbb{R}}^3$ acts as a rotation of finite order. For the remainder of this section, as well as Theorems 17 and 18, we will reduce all our considerations to the action on real hyperbolic 3-space $\mathcal{H}_{\mathbb{R}}^3$.

Let Z be the refined cellular complex obtained from the action of Γ on hyperbolic 3-space as described in [32], namely we subdivide $\mathcal{H}_{\mathbb{R}}^3$ until the stabilizer in Γ of any cell σ fixes σ point-wise. We achieve this by computing Bianchi’s fundamental polyhedron for the action of Γ , taking as preliminary set of 2-cells its facets lying on the Euclidean hemispheres and vertical planes of the upper-half space model for $\mathcal{H}_{\mathbb{R}}^3$, and then subdividing along the rotation axes of the elements of Γ . Let ℓ be a prime number.

Definition 11 *The ℓ -torsion sub-complex is the sub-complex of Z consisting of all the cells which have stabilizers in Γ containing elements of order ℓ .*

For ℓ being one of the two occurring primes 2 and 3, the orbit space of this sub-complex is a finite graph, because the cells of dimension greater than 1 are trivially stabilized in the refined cellular complex. We reduce this sub-complex with the following procedure, motivated in [35].

Condition A In the ℓ -torsion sub-complex, let σ be a cell of dimension $n - 1$, lying in the boundary of precisely two n -cells τ_1 and τ_2 , the latter cells representing two different orbits. Assume further that no higher-dimensional cells of the ℓ -torsion sub-complex touch σ ; and that the n -cell stabilizers admit an isomorphism $\Gamma_{\tau_1} \cong \Gamma_{\tau_2}$.

Where this condition is fulfilled in the ℓ -torsion sub-complex, we merge the cells τ_1 and τ_2 along σ and do so for their entire orbits, if and only if they meet the following additional condition. We never merge two cells the interior of which contains two points on the same orbit.

Condition B The inclusion $\Gamma_{\tau_1} \subset \Gamma_{\sigma}$ induces an isomorphism on group homology with \mathbb{Z}/ℓ -coefficients under the trivial action.

The reduced ℓ -torsion sub-complex is the Γ -complex obtained by orbit-wise merging two n -cells of the ℓ -torsion sub-complex satisfying conditions A and B.

We use the following classification of Felix Klein [21].

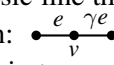
Lemma 12 (Klein) *The finite subgroups in $\mathrm{PSL}_2(\mathcal{O})$ are exclusively of isomorphism types the cyclic groups of orders 1, 2 and 3, the dihedral groups \mathcal{D}_2 and \mathcal{D}_3 (isomorphic to the Klein four-group $\mathbb{Z}/2 \times \mathbb{Z}/2$, respectively to the symmetric group on three symbols) and the alternating group \mathcal{A}_4 .*

Now we investigate the associated normalizer groups. Straight-forward verification using the multiplication tables of the implied finite groups yields the following.

Lemma 13 *Let G be a finite subgroup of $\mathrm{PSL}_2(\mathcal{O}_{-m})$. Then the type of the normalizer of any subgroup of type \mathbb{Z}/ℓ in G is given as follows for $\ell = 2$ and $\ell = 3$, where we print only cases with existing subgroup of type \mathbb{Z}/ℓ .*

Isomorphism type of G	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	\mathcal{D}_2	\mathcal{D}_3	\mathcal{A}_4
normaliser of $\mathbb{Z}/2$		$\mathbb{Z}/2$		\mathcal{D}_2	$\mathbb{Z}/2$	\mathcal{D}_2
normaliser of $\mathbb{Z}/3$			$\mathbb{Z}/3$		\mathcal{D}_3	$\mathbb{Z}/3$.

Lemma 14 *Let $v \in \mathcal{H}_{\mathbb{R}}^3$ be a vertex with stabilizer in Γ of type \mathcal{D}_2 or \mathcal{A}_4 . Let γ in Γ be a rotation of order 2 around an edge e adjacent to v . Then the centralizer $C_{\Gamma}(\gamma)$ reflects \mathcal{H}^{γ} — which is the geodesic line through e — onto itself at v .*

Proof Denote by Γ_v the stabilizer of the vertex v . In the case that Γ_v is of type \mathcal{D}_2 , which is Abelian, it admits two order-2-elements centralizing γ and turning the geodesic line through e onto itself such that the image of e touches v from the side opposite to e (illustration: ). In the case that Γ_v is of type \mathcal{A}_4 , it contains a normal subgroup of type \mathcal{D}_2 that admits again two such elements. \square

Any edge of the reduced torsion sub-complex is obtained by merging a chain of edges on the intersection of one geodesic line with some strict fundamental domain for Γ in \mathcal{H} . We call this chain the *chain of edges associated to α* . It is well defined up to translation along the rotation axis of α .

Lemma 15 *Let α be any 2-torsion element in Γ . Then the chain of edges associated to α is a fundamental domain for the action of the centralizer of α on the rotation axis of α .*

Proof We distinguish the following two cases of how $\langle \alpha \rangle \cong \mathbb{Z}/2$ is included into Γ .

The case \odot . Suppose that there is no subgroup of type \mathcal{D}_2 in Γ which contains $\langle \alpha \rangle$. Then the connected component to which the rotation axis of α passes in the quotient of the 2-torsion subcomplex, is homeomorphic to a circle. We can write $\Gamma_e = \langle \alpha \rangle$ and $\Gamma_{e'} = \langle \gamma \alpha \gamma^{-1} \rangle$. One immediately checks that any fixed point $x \in \mathcal{H}$ of α induces the fixed point $\gamma \cdot x$ of $\gamma \alpha \gamma^{-1}$. As $\mathrm{PSL}_2(\mathbb{C})$ acts by isometries, the whole fixed point set in \mathcal{H} of α is hence identified by γ with the fixed point set of $\gamma \alpha \gamma^{-1}$. This gives us the identification γ^{-1} from e' to an edge on the rotation axis of α , adjacent to e because of the first condition on γ . We repeat this step until we have attached an edge δe on the orbit of the first edge e , with $\delta \in \Gamma$. As δ is an isometry, the whole chain is translated by δ from the start at e to the start at δe . So the group $\langle \delta \rangle$ acts on the rotation axis with fundamental domain our chain of edges. And $\delta \alpha \delta^{-1}$ is again the rotation of order 2 around the axis of α . So, $\delta \alpha \delta^{-1} = \alpha$ and therefore $\langle \delta \rangle < C_{\Gamma}(\alpha)$.

The cases $\bullet \rightarrow \bullet$, $\bullet \rightarrow \bullet \rightarrow \bullet$, $\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$, $\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$, \dots . Suppose that there is a subgroup G of Γ of type $G \cong D_2$ containing $\langle \alpha \rangle$. If there is no further inclusion $G < G' < \Gamma$ with $G' \cong \mathcal{A}_4$, let $G' := G$. Then the chain associated to α can be chosen such that one of its endpoints is stabilized by G' . The other endpoint of this chain must then lie on a different Γ -orbit, and admit as stabilizer a group H' containing $\langle \alpha \rangle$, of type D_2 or \mathcal{A}_4 . By Lemma 14, each G' and H' contain a reflection of the rotation axis of α , centralizing α . These two reflections must differ from one another because they do not fix the chain of edges. So their free product tessellates the rotation axis of α with images of the chain of edges associated to α . \square

4 A spine for the complexified Bianchi orbifolds

In this section we prove the following theorem, which will be used to prove Theorem 3.

Theorem 16 *Let Γ be a Bianchi group, then there is a Γ -equivariant homotopy equivalence between $\mathcal{H}_{\mathbb{C}}^3$ and $\mathcal{H}_{\mathbb{R}}^3$. In particular, $\mathcal{H}_{\mathbb{C}}^3/\Gamma$ is homotopy equivalent to $\mathcal{H}_{\mathbb{R}}^3/\Gamma$.*

Proof We consider the ball model for complex hyperbolic 3-space $\mathcal{H}_{\mathbb{C}}^3$ [18] (which is called the Klein model in [13]). This provides us with a complex structure such that $\mathcal{H}_{\mathbb{R}}^3$ is naturally embedded into $\mathcal{H}_{\mathbb{C}}^3$ as the fixed points set of the complex conjugation. In the other direction, following [18], we define a projection as follows. For any point $z \in \mathcal{H}_{\mathbb{C}}^3$, there is a unique geodesic arc, with respect to the Bergman metric, $\alpha_{z,\bar{z}}$ from z to its complex conjugate \bar{z} (see e.g. [18, Theorem 3.1.11]); and the intersection point $p(z) = \alpha_{z,\bar{z}} \cap \mathcal{H}_{\mathbb{R}}^3$ is equidistant to z and \bar{z} [18, Section 3.3.6]. This defines a projection $p: \mathcal{H}_{\mathbb{C}}^3 \rightarrow \mathcal{H}_{\mathbb{R}}^3$. Notice that p is $\text{PSO}(3, 1)$ -equivariant and hence also Γ -equivariant.

Clearly, the restriction $p|_{\mathcal{H}_{\mathbb{R}}^3}$ is the identity. On the other hand, let

$$H: \mathcal{H}_{\mathbb{C}}^3 \times [0, 1] \rightarrow \mathcal{H}_{\mathbb{C}}^3, \quad H(z, t) = \alpha_{z,\bar{z}}(t\rho(z, p(z)))$$

where ρ is the hyperbolic distance and we have parametrized the geodesic arc such that $\alpha_{z,\bar{z}}(0) = p(z)$ and $\alpha_{z,\bar{z}}(\rho(z, p(z))) = z$. Then H is an homotopy between p and the identity map of $\mathcal{H}_{\mathbb{C}}^3$. Furthermore, since $\text{PSO}(3, 1)$ is a group of isometries of $\mathcal{H}_{\mathbb{C}}^3$, it sends geodesics to geodesics and so, for any $M \in \text{PSO}(3, 1)$,

$$\begin{aligned} H(Mz, t) &= \alpha_{Mz, \overline{Mz}}(t\rho(Mz, p(Mz))) \\ &= M\alpha_{z,\bar{z}}(t\rho(z, p(z))) = MH(z, t). \end{aligned}$$

It follows that H is $\text{PSO}(3, 1)$ -equivariant, in particular it is Γ -equivariant. \square

5 Orbifold cohomology of real Bianchi orbifolds

Our main results on the vector space structure of the Chen–Ruan orbifold cohomology of Bianchi orbifolds are the below two theorems.

Theorem 17 *For any element γ of order 3 in a finite index subgroup Γ in a Bianchi group with units $\{\pm 1\}$, the quotient space $\mathcal{H}^\gamma/C_\Gamma(\gamma)$ of the rotation axis modulo the centralizer of γ is homeomorphic to a circle.*

Proof As γ is a non-trivial torsion element, by [35, lemma 22] the Γ -image of the chain of edges associated to γ contains the rotation axis \mathcal{H}^γ . Now we can observe two cases.

- First, assume that the rotation axis of γ does not contain any vertex of stabilizer type \mathcal{D}_3 (from [35], we know that this gives us a circle as a path component in the quotient of the 3-torsion sub-complex). Assume that there exists a reflection of \mathcal{H}^γ onto itself by an element of Γ . Such a reflection would fix a point on \mathcal{H}^γ . Then the normalizer of $\langle \gamma \rangle$ in the stabilizer of this point would contain the reflection. This way, Lemma 13 yields that this stabilizer is of type \mathcal{D}_3 , which we have excluded. Thus, there can be no reflection of \mathcal{H}^γ onto itself by an element of Γ .

As Γ acts by CAT(0) isometries, every element $g \in \Gamma$ sending an edge of the chain for γ to an edge on \mathcal{H}^γ outside the fundamental domain, can then only perform a translation on \mathcal{H}^γ . A translation along the rotation axis of γ commutes with γ , so $g \in C_\Gamma(\gamma)$. Hence the quotient space $\mathcal{H}^\gamma / C_\Gamma(\gamma)$ is homeomorphic to a circle.

- If \mathcal{H}^γ contains a point with stabilizer in Γ of type \mathcal{D}_3 , then there are exactly two Γ -orbits of such points. The elements of order 2 do not commute with the elements of order 3 in \mathcal{D}_3 , so the centralizer of γ does not contain the former ones. Hence, $C_\Gamma(\gamma)$ does not contain any reflection of \mathcal{H}^γ onto itself. Denote by α and β elements of order 2 of each of the stabilizers of the two endpoints of a chain of edges for γ . Then $\alpha\beta$ performs a translation on \mathcal{H}^γ and hence commutes with γ . A fundamental domain for the action of $\langle \alpha\beta \rangle$ on \mathcal{H}^γ is given by the chain of edges for γ united with its reflection through one of its endpoints. As no such reflection belongs to the centralizer of γ and the latter endpoint is the only one on its Γ -orbit in this fundamental domain, the quotient $\mathcal{H}^\gamma / C_\Gamma(\gamma)$ matches with the quotient $\mathcal{H}^\gamma / \langle \alpha\beta \rangle$, which is homeomorphic to a circle.

□

Theorem 18 *Let γ be an element of order 2 in a Bianchi group Γ with units $\{\pm 1\}$. Then, the homeomorphism type of the quotient space $\mathcal{H}^\gamma / C_\Gamma(\gamma)$ is*

- an edge without identifications, if $\langle \gamma \rangle$ is contained in a subgroup of type \mathcal{D}_2 inside Γ and
- a circle, otherwise.

Proof By Lemma 15, the chain of edges for γ is a fundamental domain for $C_\Gamma(\gamma)$ on the rotation axis \mathcal{H}^γ of γ . Again, we have two cases.

- If $\langle \gamma \rangle$ is contained in a subgroup of type \mathcal{D}_2 inside Γ , then any chain of edges for γ admits endpoints of stabilizer types \mathcal{D}_2 or \mathcal{A}_4 , because we can merge any two adjacent edges on a 2-torsion axis with touching point of stabilizer type $\mathbb{Z}/2$ or \mathcal{D}_3 . As \mathcal{D}_2 is an Abelian group and the reflections in \mathcal{A}_4 are contained in the normal subgroup \mathcal{D}_2 , the reflections in these endpoint stabilizers commute with γ , so the quotient space $\mathcal{H}^\gamma / C_\Gamma(\gamma)$ is represented by a chain of edges for γ . What remains to show, is that there is no element of $C_\Gamma(\gamma)$ identifying the two endpoints of stabilizer type \mathcal{D}_2 (respectively \mathcal{A}_4). Assume that there is an element $g \in C_\Gamma(\gamma)$ carrying out this identification. Any one of the two endpoints, denote it by x , contains in its stabilizer a reflection α of the rotation axis of γ . The other endpoint is then $g \cdot x$ and contains in its stabilizer the conjugate ${}^g\alpha$ by g . Denote by m the point in the middle of $(x, g \cdot x)$, i.e. the point on \mathcal{H}^γ with equal distance to x and to $g \cdot x$. As $\langle {}^g\alpha, \gamma \rangle$ is Abelian, ${}^g\alpha$ is in $C_\Gamma(\gamma)$ and hence (x, m) and $(g \cdot x, m)$ are equivalent

modulo $C_\Gamma(\gamma)$ via the element ${}^g\alpha g$ (illustration: $\begin{array}{ccccccc} & & \langle \alpha, \gamma \rangle & & \langle {}^g\alpha, \gamma \rangle & & \\ & & \bullet & \text{---} & \bullet & \text{---} & \\ & & x & m & gx & gm & \end{array}$). Then the chain of edges for γ does not reach from x to $g \cdot x$. This contradicts our hypotheses, so the homeomorphism type of $\mathcal{H}^\gamma / C_\Gamma(\gamma)$ is an edge without identifications.

- The other case is analogous to the first case of the proof of Theorem 17, the rôle of \mathcal{D}_3 being played by \mathcal{D}_2 and \mathcal{A}_4 .

□

Furthermore, the following easy-to-check statement will be useful for our orbifold cohomology computations.

Remark 19 There is only one conjugacy class of elements of order 2 in \mathcal{D}_3 as well as in \mathcal{A}_4 . In \mathcal{D}_3 , there is also only one conjugacy class of elements of order 3, whilst in \mathcal{A}_4 there is an element γ such that γ and γ^2 represent the two conjugacy classes of elements of order 3.

Proof In cycle type notation, we can explicitly establish the multiplication tables of \mathcal{D}_3 and \mathcal{A}_4 , and compute the conjugacy classes. □

Corollary 20 (Corollary to Remark 19) *Let γ be an element of order 3 in a Bianchi group Γ with units $\{\pm 1\}$. Then, γ is conjugate in Γ to its square γ^2 if and only if there exists a group $G \cong \mathcal{D}_3$ with $\langle \gamma \rangle \subsetneq G \subsetneq \Gamma$.*

Denote by $\lambda_{2\ell}$ the number of conjugacy classes of subgroups of type $\mathbb{Z}/\ell\mathbb{Z}$ in a finite index subgroup Γ in a Bianchi group with units $\{\pm 1\}$. Denote by $\lambda_{2\ell}^*$ the number of conjugacy classes of those of them which are contained in a subgroup of type \mathcal{D}_n in Γ . By Corollary 20, there are $2\lambda_6 - \lambda_6^*$ conjugacy classes of elements of order 3. As a result of Theorems 17 and 18, we have the following isomorphism of vector spaces:

$$\begin{aligned} & \bigoplus_{\gamma \in T} \mathrm{H}^\bullet((\mathcal{H}_{\mathbb{R}})^\gamma / C_\Gamma(\gamma); \mathbb{Q}) \\ & \cong \mathrm{H}^\bullet(\mathcal{H}_{\mathbb{R}}/\Gamma; \mathbb{Q}) \bigoplus^{\lambda_4^*} \mathrm{H}^\bullet(\bullet\text{---}\bullet; \mathbb{Q}) \bigoplus^{(\lambda_4 - \lambda_4^*)} \mathrm{H}^\bullet(\circ; \mathbb{Q}) \bigoplus^{(2\lambda_6 - \lambda_6^*)} \mathrm{H}^\bullet(\circ; \mathbb{Q}), \end{aligned}$$

where $T \subset \Gamma$ is a set of representatives of conjugacy classes of elements of finite order in Γ . The (co)homology of the quotient space $\mathcal{H}_{\mathbb{R}}/\Gamma$ has been computed numerically for a large scope of Bianchi groups [44], [39], [36]; and bounds for its Betti numbers have been given in [22]. Krämer [23] has determined number-theoretic formulae for the numbers $\lambda_{2\ell}$ and $\lambda_{2\ell}^*$ of conjugacy classes of finite subgroups in the full Bianchi groups. Krämer's formulae have been evaluated for hundreds of thousands of Bianchi groups [35], and these values are matching with the ones from the orbifold structure computations with [33] in the cases where the latter are available.

When we pass to the complexified orbifold $[\mathcal{H}_{\mathbb{C}}^3/\Gamma]$, the real line that is the rotation axis in $\mathcal{H}_{\mathbb{R}}$ of an element of finite order, becomes a complex line. However, the centralizer still acts in the same way by reflections and translations. So, the interval $\bullet\text{---}\bullet$ as a quotient of the real line yields a stripe $\bullet\text{---}\bullet \times \mathbb{R}$ as a quotient of the complex line. And the circle \circ as a quotient of the real line yields a cylinder $\circ \times \mathbb{R}$

as a quotient of the complex line. Therefore, using the degree shifting numbers of Lemma 9, we obtain the result of Corollary 2,

$$\mathrm{H}_{\mathrm{CR}}^d([\mathcal{H}_{\mathbb{C}}^3/\Gamma]) \cong \mathrm{H}^d(\mathcal{H}_{\mathbb{C}}/\Gamma; \mathbb{Q}) \oplus \begin{cases} \mathbb{Q}^{\lambda_4+2\lambda_6-\lambda_6^*}, & d = 2, \\ \mathbb{Q}^{\lambda_4-\lambda_4^*+2\lambda_6-\lambda_6^*}, & d = 3, \\ 0, & \text{otherwise.} \end{cases}$$

As we can calculate the Bredon homology $\mathrm{H}_0^{\tilde{\mathrm{in}}}(\Gamma; R_{\mathbb{C}})$ of the Bianchi groups with coefficients in the complex representation ring functor $R_{\mathbb{C}}$, the following lemma provides a check on our computations.

Lemma 21 (Mislin [26]) *Let Γ be an arbitrary group and write $\mathrm{FC}(\Gamma)$ for the set of conjugacy classes of elements of finite order in Γ . Then there is an isomorphism*

$$\mathrm{H}_0^{\tilde{\mathrm{in}}}(\Gamma; R_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}[\mathrm{FC}(\Gamma)].$$

6 The cohomology ring isomorphism

In this section we prove Theorem 3. To this aim we define a linear and grading-preserving function $\Phi: \mathrm{H}_{\mathrm{CR}}^*([\mathcal{H}_{\mathbb{C}}^3/\Gamma]) \rightarrow \mathrm{H}^*(Y)$ as follows. For any cohomology class of the non-twisted sector $\alpha \in \mathrm{H}^d(\mathcal{H}_{\mathbb{C}}^3/\Gamma)$, we define $\Phi(\alpha) := f^*(\alpha) \in \mathrm{H}^d(Y)$. Let now $\alpha_g \in \mathrm{H}^{d-2\mathrm{shift}(g)}([\mathcal{H}_{\mathbb{C}}^3]^g/C_{\Gamma}(g)) \subset \mathrm{H}_{\mathrm{CR}}^d([\mathcal{H}_{\mathbb{C}}^3/\Gamma])$ be a cohomology class of the twisted sector $[(\mathcal{H}_{\mathbb{C}}^3)^g/C_{\Gamma}(g)]$. Let us consider the following Cartesian diagram

$$\begin{array}{ccc} E & \xrightarrow{j} & Y \\ \pi \downarrow & & \downarrow f \\ (\mathcal{H}_{\mathbb{C}}^3)^g/C_{\Gamma}(g) & \xrightarrow{\iota} & \mathcal{H}_{\mathbb{C}}^3/\Gamma, \end{array}$$

where ι is the morphism induced by the inclusion $(\mathcal{H}_{\mathbb{C}}^3)^g \hookrightarrow \mathcal{H}_{\mathbb{C}}^3$. Then we define $\Phi(\alpha_g) := j_*\pi^*(\alpha_g)$, where $\pi^*(\alpha_g) \in \mathrm{H}^{d-2\mathrm{shift}(g)}(E)$ is the usual pull-back, and $j_*: \mathrm{H}^{d-2\mathrm{shift}(g)}(E) \rightarrow \mathrm{H}^{\delta}(Y)$, with $\delta = 2(\dim Y - \dim E - \mathrm{shift}(g)) + d$, is the composition of the following linear maps

$$\mathrm{H}^{d-2\mathrm{shift}(g)}(E) \cong \mathrm{H}_{2\dim E-d+2\mathrm{shift}(g)}^{\mathrm{BM}}(E) \rightarrow \mathrm{H}_{2\dim E-d+2\mathrm{shift}(g)}^{\mathrm{BM}}(Y) \cong \mathrm{H}^{\delta}(Y),$$

where the first and last isomorphisms are given by Poincaré duality with Borel-Moore homology ([17]), the linear map in the middle is the push-forward induced by j (notice that j is proper because ι is so by Lemma 8). From the description of finite subgroups of Γ in Section 3 it follows that g acts, locally around $(\mathcal{H}_{\mathbb{C}}^3)^g$, as a rotation; from Lemma 9 we conclude that $\mathrm{shift}(g) = 1$. Furthermore, $E \subset Y$ is a divisor, hence $\dim Y - \dim E = 1$ and so $\delta = d$.

The linear function Φ so defined is an isomorphism of graded vector spaces. Indeed, from the McKay correspondence ([7], [11]) the graded vector spaces $\mathrm{H}_{\mathrm{CR}}^*([\mathcal{H}_{\mathbb{C}}^3/\Gamma])$ and $\mathrm{H}^*(Y)$ are isomorphic, so it is enough to show that Φ is injective. To this aim, let $\alpha \in \mathrm{H}^d(\mathcal{H}_{\mathbb{C}}^3/\Gamma) \setminus \{0\}$. Then, by Poincaré duality there exists $\beta \in \mathrm{H}_{\mathbb{C}}^{3-d}(\mathcal{H}_{\mathbb{C}}^3/\Gamma)$, such that $\int_{\mathcal{H}_{\mathbb{C}}^3/\Gamma} \alpha \wedge \beta \neq 0$ ([1]), where $\mathrm{H}_{\mathbb{C}}^*$ denotes cohomology with compact support. Since f maps the fundamental class of Y to that of $\mathcal{H}_{\mathbb{C}}^3/\Gamma$ [17], it follows that $\int_Y f^*(\alpha \wedge \beta) \neq 0$,

hence $\Phi(\alpha) = f^*(\alpha) \neq 0$. The fact that also the restriction of Φ to the cohomology of the twisted sectors is injective can be proved as in [29, Proposition 4.9].

To conclude the proof of the theorem, we show that Φ is a ring homomorphism. Notice that on the non-twisted sector Φ respects the cup products because f^* is a ring homomorphism. So let α_g, β_h be cohomology classes of the twisted sectors $(\mathcal{H}_{\mathbb{C}}^3)^g/C_{\Gamma}(g), (\mathcal{H}_{\mathbb{C}}^3)^h/C_{\Gamma}(h)$. Since $\text{shift}(g) = \text{shift}(h) = 1$, the Chen–Ruan degrees $\deg(\alpha_g), \deg(\beta_h)$ are ≥ 2 , hence $\deg(\alpha_g \cup_{\text{CR}} \beta_h) \geq 4$. By Theorem 16 we conclude that $H_{\text{CR}}^d([\mathcal{H}_{\mathbb{C}}^3/\Gamma]) = 0$, if $d \geq 4$, so $\alpha_g \cup_{\text{CR}} \beta_h = 0$. On the other hand, since Φ is grading preserving, $\deg(\Phi(\alpha_g) \cup \Phi(\beta_h)) \geq 4$, so also $\Phi(\alpha_g) \cup \Phi(\beta_h) = 0$ because $H^d(Y) \cong H_{\text{CR}}^d([\mathcal{H}_{\mathbb{C}}^3/\Gamma])$, for any d . Finally, let $\alpha_g \in H^*((\mathcal{H}_{\mathbb{C}}^3)^g/C_{\Gamma}(g))$ and $\beta \in H^*(\mathcal{H}_{\mathbb{C}}^3/\Gamma)$. Then, $\alpha_g \cup_{\text{CR}} \beta = \alpha_g \cup \iota^* \beta \in H^*((\mathcal{H}_{\mathbb{C}}^3)^g/C_{\Gamma}(g))$, so $\Phi(\alpha_g \cup_{\text{CR}} \beta) = J_* \pi^*(\alpha_g \cup \iota^* \beta)$. On the other hand,

$$\begin{aligned} \Phi(\alpha_g) \cup \Phi(\beta) &= J_* \pi^*(\alpha_g) \cup f^*(\beta) \\ &= J_* (\pi^*(\alpha_g) \cup j^*(f^*(\beta))) && \text{(projection formula)} \\ &= J_* (\pi^*(\alpha_g) \cup \pi^*(\iota^*(\beta))) && (f \circ j = \iota \circ \pi) \\ &= J_* \pi^*(\alpha_g \cup \iota^*(\beta)) \\ &= \Phi(\alpha_g \cup_{\text{CR}} \beta). \end{aligned}$$

□

7 Cohomological Crepant Resolution Conjecture for Bianchi orbifolds

In this section we compare the results obtained so far with the Cohomological Crepant Resolution Conjecture of Ruan. We begin by briefly reviewing the statement of this conjecture, referring to [38], [12], and the references therein, for further details.

Let \mathcal{X} be a complex orbifold, and let X be its coarse moduli space. We assume that X is a complex projective variety which has a crepant resolution $f: Y \rightarrow X$. The quantum corrected cohomology ring of $f: Y \rightarrow X$ is a family of rings on the vector space $H^*(Y, \mathbb{C}) = \bigoplus_{d \geq 0} H^d(Y, \mathbb{C})$, whose definition depends on the choice of a basis of $\ker(f_*: H_2(Y, \mathbb{Q}) \rightarrow H_2(X, \mathbb{Q}))$ consisting of homology classes of effective curves β_1, \dots, β_n . One defines the 3-points function

$$(2) \quad (\alpha_1, \alpha_2, \alpha_3)(q_1, \dots, q_n) = \sum_{(k_1, \dots, k_n) \in \mathbb{N}^n} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{\beta}^Y q_1^{k_1} \cdots q_n^{k_n},$$

where $\beta = k_1 \beta_1 + \dots + k_n \beta_n \in H_2(Y, \mathbb{Z})$, and $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{\beta}^Y$ is the Gromov-Witten invariant of Y , of genus 0, of homology class β , with respect to the cohomology classes $\alpha_1, \alpha_2, \alpha_3 \in H^*(Y, \mathbb{C})$. Recall that a compact complex curve $D \subset Y$ of homology class β is called an *exceptional curve* for f . To simplify the discussion we assume that the 3-points function (2) converges in a neighborhood of the origin $(q_1, \dots, q_n) = (0, \dots, 0)$, and then, for any (q_1, \dots, q_n) in this neighborhood, we define a product \star_f on the cohomology of Y as follows: given cohomology classes α_1, α_2 , then $\alpha_1 \star_f \alpha_2$ is the cohomology class which satisfies the following equation:

$$(\alpha_1 \star_f \alpha_2, \alpha_3) = (\alpha_1, \alpha_2, \alpha_3)(q_1, \dots, q_n), \quad \forall \alpha_3 \in H^*(Y, \mathbb{C}),$$

where the pairing $(,)$ to the left hand side is the Poincaré pairing of Y . The product \star_f satisfies the usual properties of the cup product, e.g. it is associative, graded-commutative, and 1 is its neutral element. The

family of rings $(H^*(Y, \mathbb{C}), \star_f)$ is by definition the quantum corrected cohomology ring of $f: Y \rightarrow X$. Notice that, if $(q_1, \dots, q_n) = (0, \dots, 0)$, then \star_f coincides with the usual cup product, as it follows from the fact that $(\alpha_1, \alpha_2, \alpha_3)(0, \dots, 0) = \int_Y \alpha_1 \cup \alpha_2 \cup \alpha_3$. So, the quantum corrected cohomology ring of $f: Y \rightarrow X$ is regarded as a deformation of the usual cohomology ring of Y .

Ruan's Cohomological Crepant Resolution Conjecture predicts that there is an analytic continuation of (2) to a region containing a point $(\bar{q}_1, \dots, \bar{q}_n)$ such that, for $(q_1, \dots, q_n) = (\bar{q}_1, \dots, \bar{q}_n)$, the ring $(H^*(Y, \mathbb{C}), \star_f)$ is isomorphic to the Chen–Ruan orbifold cohomology ring $(H_{\text{CR}}^*(\mathcal{X}), \cup_{\text{CR}})$ of \mathcal{X} .

In the case of a Bianchi orbifold $[\mathcal{H}_{\mathbb{C}}^3/\Gamma]$, the coarse moduli space $\mathcal{H}_{\mathbb{C}}^3/\Gamma$ is not a projective variety [13], and so, for every crepant resolution $f: Y \rightarrow \mathcal{H}_{\mathbb{C}}^3/\Gamma$, Y is not a projective variety. Hence the Gromov-Witten invariants of Y are, in general, not well defined. However, we will see that $[\mathcal{H}_{\mathbb{C}}^3/\Gamma]$ has a Kähler structure, and that one does not expect non-zero quantum corrections coming from exceptional curves for f . This will be motivated by a deformation theoretic argument about the complex structure of Y (let us recall that the Gromov-Witten invariants are invariant under deformations of the complex structure). This allows us to convince ourselves that the quantum corrections vanish. We prove the latter fact in one special case, namely $\Gamma = \text{PSL}_2(\mathcal{O}_{-5})$, while the general case should be feasible with similar arguments. Hence, in accordance with Ruan's conjecture, there should be a ring isomorphism $(H_{\text{CR}}^*([\mathcal{H}_{\mathbb{C}}^3/\Gamma]), \cup_{\text{CR}}) \cong (H^*(Y), \cup)$. This is confirmed by our Theorem 3.

Proposition 22 *Let $[\mathcal{H}_{\mathbb{C}}^3/\Gamma]$ be a Bianchi orbifold. Then the Bergman metric on $\mathcal{H}_{\mathbb{C}}^3$ descends to a Kähler (orbifold) metric on $[\mathcal{H}_{\mathbb{C}}^3/\Gamma]$.*

Proof Let

$$ds^2 = \sum g_{\alpha\bar{\beta}} dz_{\alpha} d\bar{z}_{\beta}$$

be the Bergman metric on $\mathcal{H}_{\mathbb{C}}^3$. By [27, Theorem 8.4, p. 144], ds^2 is invariant under the action of Γ , hence it induces a Kähler metric on the orbifold $[\mathcal{H}_{\mathbb{C}}^3/\Gamma]$. \square

Let now $f: Y \rightarrow \mathcal{H}_{\mathbb{C}}^3/\Gamma$ be a crepant resolution. Let $D \subset Y$ be an exceptional compact complex curve, that is $f_*([D]) = 0$, where $[D]$ is the fundamental class of D . Since $[\mathcal{H}_{\mathbb{C}}^3/\Gamma]$ is Kähler, $f(D)$ is a point, so D is contained in the exceptional divisor of f . In particular, for any homology class $\beta \in \ker(f_*: H_2(Y, \mathbb{Q}) \rightarrow H_2(\mathcal{H}_{\mathbb{C}}^3/\Gamma, \mathbb{Q}))$, and for any stable map $\mu: C \rightarrow Y$, such that $\mu_*([C]) = \beta$, the image of μ is contained in the exceptional divisor of f . Hence it suffices to consider the problem locally in a neighbourhood of the exceptional divisor.

From the results of Sections 3 and 5 we have that the singular locus of $\mathcal{H}_{\mathbb{C}}^3/\Gamma$ is the union of several irreducible components, each of which is isomorphic either to \mathbb{C} or to \mathbb{C}^* . Furthermore, the generic point of each irreducible component of the singular locus is a transverse singularity of type A_n of $\mathcal{H}_{\mathbb{C}}^3/\Gamma$, with $n = 1$ or 2 . Recall that a point p of a variety (an analytic space, respectively) X is a transverse singularity of type A_n , if there is a neighbourhood of p in the analytic topology of X that is isomorphic to a neighbourhood of a singular point of $\{(u, v, w) \in \mathbb{C}^3 \mid w^{n+1} = uv\} \times \mathbb{C}^{d-2}$, where $d = \dim(X)$.

Let us now consider the special case where $\Gamma = \text{PSL}_2(\mathcal{O}_{-5})$ (see Section 8.6). In this case we explain in detail why the quantum corrections to the cohomology ring of Y coming from exceptional curves should vanish. The singular locus of $X = \mathcal{H}_{\mathbb{C}}^3/\Gamma$ has two connected components, $X_{(2)} \cong \mathbb{C}^*$, whose

points are transverse singularities of type A_2 , and $X_{(1)}$, that is the union of three irreducible components, $X'_{(1)}, X''_{(1)}, X'''_{(1)} \cong \mathbb{C}$, that meet in two points P, Q and the complement $X_{(1)} \setminus \{P, Q\}$ is a locus of transverse singularities of type A_1 (see Figure 8 and Section 8.6). The exceptional divisor of $f: Y \rightarrow X$ has two connected components, $E_{(2)}$, that is mapped to $X_{(2)}$ by f , and $E_{(1)}$, such that $f(E_{(1)}) = X_{(1)}$. Furthermore, $E_{(1)}$ has three irreducible components, $E'_{(1)}, E''_{(1)}, E'''_{(1)}$, that are mapped by f to $X'_{(1)}, X''_{(1)}, X'''_{(1)}$, respectively.

Let us consider first the A_2 -singularities $X_{(2)}$. We can find an open neighbourhood U of $X_{(2)}$, such that $U \cong \tilde{U}/(\mathbb{Z}/3\mathbb{Z})$, where \tilde{U} is a complex manifold with an action of $\mathbb{Z}/3\mathbb{Z}$, such that the fixed-points locus $\tilde{U}^{\mathbb{Z}/3\mathbb{Z}}$ is a smooth submanifold of \tilde{U} isomorphic to $X_{(2)}$. Furthermore, up to deformation, we can assume that \tilde{U} is an open neighbourhood of the zero-section of the normal bundle $N_{\tilde{U}^{\mathbb{Z}/3\mathbb{Z}}|\tilde{U}}$ of $\tilde{U}^{\mathbb{Z}/3\mathbb{Z}}$ in \tilde{U} . This can be achieved using the *deformation to the normal cone* of the imbedding $\tilde{U}^{\mathbb{Z}/3\mathbb{Z}} \subset \tilde{U}$ ([17, Chapter 5]). The vector bundle map $N_{\tilde{U}^{\mathbb{Z}/3\mathbb{Z}}|\tilde{U}} \rightarrow \tilde{U}^{\mathbb{Z}/3\mathbb{Z}} \cong X_{(2)}$ induces a morphism $\tilde{U}/(\mathbb{Z}/3\mathbb{Z}) \rightarrow X_{(2)}$ that gives to $U \cong \tilde{U}/(\mathbb{Z}/3\mathbb{Z})$ the structure of a fibration over $X_{(2)}$ with fibres all isomorphic to the surface singularity of type A_2 . The important fact is that this fibration is trivial. To see this, let us recall that the action of $\mathbb{Z}/3\mathbb{Z}$ on the fibres of $N_{\tilde{U}^{\mathbb{Z}/3\mathbb{Z}}|\tilde{U}}$ induces a splitting, $N_{\tilde{U}^{\mathbb{Z}/3\mathbb{Z}}|\tilde{U}} = \mathbb{L} \oplus \mathbb{M}$, where \mathbb{L} and \mathbb{M} are the eigenbundles corresponding to the irreducible characters of the representation of $\mathbb{Z}/3\mathbb{Z}$ on the fibres of $N_{\tilde{U}^{\mathbb{Z}/3\mathbb{Z}}|\tilde{U}}$. In our case \mathbb{L} and \mathbb{M} are trivial line bundles on $X_{(2)}$, as it follows from Theorem 10 and the computation of the Chen–Ruan cohomology ring of orbifolds with transverse singularities of type A_n [29, Theorem 3.12]. Therefore the fibration $N_{\tilde{U}^{\mathbb{Z}/3\mathbb{Z}}|\tilde{U}}/(\mathbb{Z}/3\mathbb{Z}) \rightarrow X_{(2)}$ is trivial, that is, it is isomorphic to the projection to the first factor of $X_{(2)} \times \{(u, v, w) \in \mathbb{C}^3 \mid uv = w^3\}$. Now, using the theory of deformations of rational double points (see [8], [43]), we deform the family $N_{\tilde{U}^{\mathbb{Z}/3\mathbb{Z}}|\tilde{U}}/(\mathbb{Z}/3\mathbb{Z}) \rightarrow X_{(2)}$ to a family of affine smooth surfaces. Finally, consider the neighbourhood $\mathcal{U} := f^{-1}(U)$ of $E_{(2)}$. Taking a simultaneous resolution of the previous deformation of $N_{\tilde{U}^{\mathbb{Z}/3\mathbb{Z}}|\tilde{U}}/(\mathbb{Z}/3\mathbb{Z}) \rightarrow X_{(2)}$, we obtain a deformation of \mathcal{U} to a manifold that does not contain compact complex curves.

Let us now consider the exceptional curves that are contained in $E_{(1)}$. Notice that each component $E'_{(1)}, E''_{(1)}, E'''_{(1)}$, can be seen as the exceptional divisor of a crepant resolution of a transverse singularity of type A_1 . Hence, from our description of the obstruction bundles (Theorem 10) and from [29, Theorem 7.6], it follows that the exceptional curves contained in one of these components do not contribute to the quantum corrected cohomology ring of Y . If $D \subset Y$ is a connected exceptional curve which is contained in more than one component of $E_{(1)}$, then $f(D)$ coincides with P or Q , the points where the components $X'_{(1)}, X''_{(1)}, X'''_{(1)}$ meet together. Near P and Q , X is isomorphic to the singularity $\mathbb{C}^3/\mathcal{D}_2$ (see Section 8.6), where $\mathcal{D}_2 = \langle \xi, \eta \mid \xi^2 = \eta^2 = (\xi\eta)^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We can realize the quotient $\mathbb{C}^3/\mathcal{D}_2$ as $(\mathbb{C}^3/\langle \xi \rangle) / \langle \eta \rangle$, and notice that $\mathbb{C}^3/\langle \xi \rangle \cong \{(u, v, w, z) \in \mathbb{C}^3 \times \mathbb{C} \mid uv = w^2\}$ with the action of $\langle \eta \rangle$ given by $\eta \cdot (u, v, w, z) \mapsto (u, v, -w, -z)$. The semi-universal deformation of $\{(u, v, w) \in \mathbb{C}^3 \mid uv = w^2\}$ is $uv = w^2 + t$, where t is the deformation parameter. Notice that the action of $\langle \eta \rangle$ on $\mathbb{C}^3/\langle \xi \rangle$ extends to $\{(u, v, w, z) \in \mathbb{C}^3 \times \mathbb{C} \mid uv = w^2 + t\}$, for all t , as follows: $\eta \cdot (u, v, w, z) = (u, v, -w, -z)$. Hence $\{(u, v, w, z) \in \mathbb{C}^3 \times \mathbb{C} \mid uv = w^2 + t\}/\langle \eta \rangle$, for $t \in \mathbb{C}$, is a deformation of $\mathbb{C}^3/\mathcal{D}_2$. Notice that, for $t \neq 0$, $\{(u, v, w, z) \in \mathbb{C}^3 \times \mathbb{C} \mid uv = w^2 + t\}/\langle \eta \rangle$ has transverse singularities of type A_1 , and they can be smoothed by a deformation as follows. Taking the invariants of the $\langle \eta \rangle$ -action, we see that

$$\{(u, v, w, z) \in \mathbb{C}^3 \times \mathbb{C} \mid uv = w^2 + t\}/\langle \eta \rangle \cong \{(u, v, \rho, \sigma, \tau) \in \mathbb{C}^5 \mid uv = \rho + t, \rho\sigma = \tau^2\},$$

where $\rho = w^2, \sigma = z^2, \tau = wz$, and so, $\{(u, v, \rho, \sigma, \tau) \in \mathbb{C}^5 \mid uv = \rho + t, \rho\sigma = \tau^2 + s\}$ is a deformation of $\mathbb{C}^3/\mathcal{D}_2$, with deformation parameters t and s . For $t \neq 0$ and $s \neq 0$, the variety $\{(u, v, \rho, \sigma, \tau) \in \mathbb{C}^5 \mid uv = \rho + t, \rho\sigma = \tau^2 + s\}$ is an affine smooth variety, so a simultaneous resolution

of this family yields a deformation of a neighbourhood of $f^{-1}(P)$ ($f^{-1}(Q)$, respectively), such that the generic member of the family is a smooth affine variety, hence it does not contain compact complex curves.

8 Orbifold cohomology computations for sample Bianchi orbifolds

We will carry out our computations in the upper-half space model $\{x + iy + rz \in \mathbb{C} \oplus \mathbb{R}j \mid r > 0\}$ for $\mathcal{H}_{\mathbb{R}}^3$ in three cases. Details on how to compute Chen–Ruan orbifold cohomology can be found in [29]. In the case $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-5}])$, we also compute the cohomology ring structure.

8.1 The case $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-2}])$.

Let $\omega := \sqrt{-2}$. A fundamental domain for $\Gamma := \mathrm{PSL}_2(\mathbb{Z}[\omega])$ in the real hyperbolic 3-space \mathcal{H} has been found by Luigi Bianchi [5].

We can obtain it by taking the geodesic convex envelope of its lower boundary (half of which is depicted in Figure 1) and the vertex ∞ , and then removing the vertex ∞ , making it non-compact. The other half of the lower boundary consists of one isometric Γ -image of each of the depicted 2-cells (in fact, the depicted 2-cells are a fundamental domain for a Γ -equivariant retract of \mathcal{H} , which is described in [37]). The coordinates of the vertices of Figure 1 in the upper-half space model are $(1) = j$, $(1)' = \omega + j$, $(2) = \frac{1}{2}\omega + \sqrt{\frac{1}{2}}j$, $(7) = \frac{1}{2} + \sqrt{\frac{3}{4}}j$, $(7)' = \frac{1}{2} + \omega + \sqrt{\frac{3}{4}}j$, $(8) = \frac{1}{2} + \frac{1}{2}\omega + \frac{1}{2}j$.

The 2-torsion sub-complex (---) and the 3-torsion sub-complex (.....) are colored in the figure. The set of representatives of conjugacy classes can be chosen

$$T = \{\mathrm{Id}, \alpha, \gamma, \beta, \beta^2\},$$

with $\alpha = \pm \begin{pmatrix} 1 & \omega \\ \omega & -1 \end{pmatrix}$, $\beta = \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\gamma = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so α and γ are of order 2, and β is of order 3. Using Lemma 21 and with the help of our Bredon homology computations, we check the cardinality of T . The fixed point sets are then the following subsets of complex hyperbolic space $\mathcal{H} := \mathcal{H}_{\mathbb{C}}^3$:

$$\mathcal{H}^{\mathrm{Id}} = \mathcal{H},$$

$$\mathcal{H}^{\alpha} = \text{the complex geodesic line through } (2) \text{ and } (8),$$

$$\mathcal{H}^{\gamma} = \text{the complex geodesic line through } (1) \text{ and } (2),$$

$$\mathcal{H}^{\beta} = \mathcal{H}^{\beta^2} = \text{the complex geodesic line through } (7) \text{ and } (8).$$

The matrix $g = \pm \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix}$ performs a translation preserving the j -coordinate and sends the edge $(1)(7)$ onto the edge $(1)'(7)'$, so the orbit space $\mathcal{H}_{\mathbb{R}}/\Gamma$ is homotopy equivalent to a circle. Consider the real geodesic line $\mathcal{H}_{\mathbb{R}}^{\gamma}$ on the unit circle of real part zero. The edge $g^{-1} \cdot ((2)(1)') = (g^{-1}(2))(1)$ lies on $\mathcal{H}_{\mathbb{R}}^{\gamma}$ and is not Γ -equivalent to the edge $(1)(2)$. Because of Lemma 14, the centralizer $C_{\Gamma}(\gamma)$ reflects the line $\mathcal{H}_{\mathbb{R}}^{\gamma}$ onto itself at (2) , and again at $g^{-1}(2)$. Furthermore, none of the four elements of Γ sending (2) to

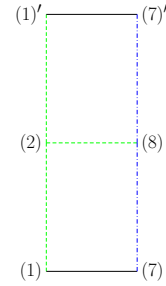


Figure 1: Fundamental domain in the case $m = 2$.

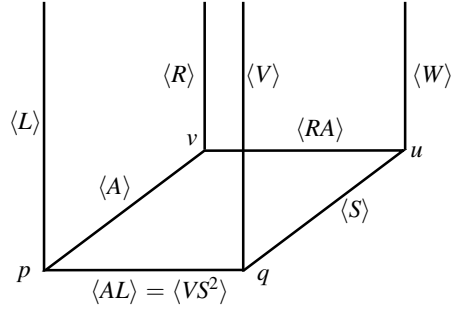


Figure 2: Half of a fundamental domain for the action of $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-1}])$ on \mathcal{H} , open towards the cusp at ∞ . The second half can be obtained as a copy of the open pyramid glued from below to its base square.

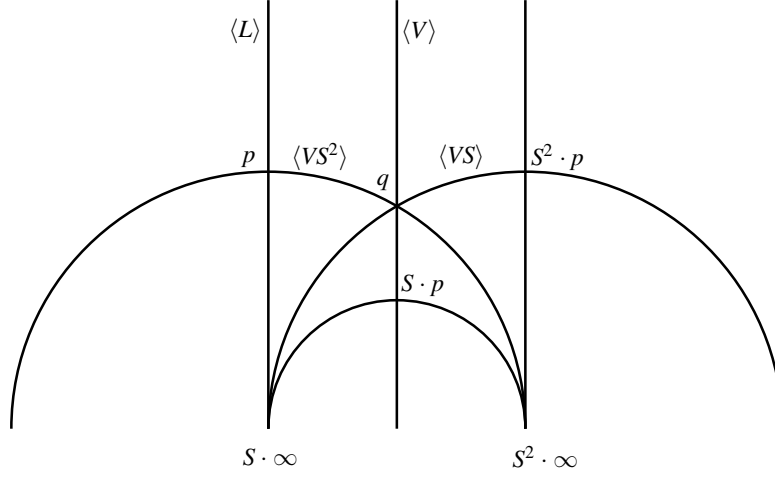
$g^{-1}(2)$ belongs to $C_\Gamma(\gamma)$. Hence the quotient space $\mathcal{H}_\mathbb{R}^\gamma / C_\Gamma(\gamma)$ consists of a contractible segment of two adjacent edges. Thus $H^{d-2}(\mathcal{H}_\mathbb{C}^\gamma / C_\Gamma(\gamma); \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & d=2 \\ 0 & \text{else} \end{cases}$ is contributed to the orbifold cohomology.

Next, consider the real geodesic line $\mathcal{H}_\mathbb{R}^\beta$ on the circle of constant real coordinate $\frac{1}{2}$, of center $\frac{1}{2}$ and radius $\sqrt{\frac{3}{4}}$. The edge $g^{-1} \cdot ((8)(7)') = (g^{-1}(8))(7)$ lies on $\mathcal{H}_\mathbb{R}^\beta$ and is not Γ -equivalent to the edge $(7)(8)$. The centralizer of β contains the matrix $V := \pm \begin{pmatrix} 21-\omega & \\ \omega-1 & 1+\omega \end{pmatrix}$ of infinite order, which sends the edge $(g^{-1}(8))(7)$ to $(8)z$ with $z = \frac{1}{2} + \frac{3}{5}\omega + \sqrt{\frac{3}{100}}j$. We conclude that the translation action of the group $\langle V \rangle$ on the line $\mathcal{H}_\mathbb{R}^\beta$ is transitive, with quotient space represented by the circle $(g^{-1}(8))(7) \cup (7)(8)$, first and last vertex identified. Thus $H^{d-2}(\mathcal{H}_\mathbb{C}^\beta / C_\Gamma(\beta); \mathbb{Q}) \cong H^{d-2}(\mathcal{H}_\mathbb{C}^{\beta^2} / C_\Gamma(\beta^2); \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & d=2,3 \\ 0 & \text{else} \end{cases}$ is contributed to the orbifold cohomology.

Because of Lemma 14, the centralizer $C_\Gamma(\alpha)$ reflects the line $\mathcal{H}_\mathbb{R}^\alpha$ onto itself at (2) , and again at (8) . So, the quotient space $\mathcal{H}_\mathbb{R}^\alpha / C_\Gamma(\alpha)$ is represented by the single contractible edge $(2)(8)$. This yields that $H^{d-2}(\mathcal{H}_\mathbb{C}^\alpha / C_\Gamma(\alpha); \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & d=2 \\ 0 & \text{else} \end{cases}$ is contributed to the orbifold cohomology.

Summing up over T , we obtain

$$H_{orb}^d([\mathcal{H}_\mathbb{C}^3 / \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-2}])]) \cong H^d(\mathcal{H}_\mathbb{C} / \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-2}]); \mathbb{Q}) \oplus \begin{cases} \mathbb{Q}^4, & d=2, \\ \mathbb{Q}^2, & d=3, \\ 0, & \text{otherwise.} \end{cases}$$


 Figure 3: Geodesics fixed by certain finite order elements of $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-1}])$.

8.2 The case $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-1}])$.

Let $i := \sqrt{-1}$. A fundamental domain for the action of $\Gamma := \mathrm{PSL}_2(\mathbb{Z}[i])$ on real hyperbolic 3-space \mathcal{H} has been found by Luigi Bianchi, and the stabilizers have been computed by Flöge [16], whose notation we are going to adopt. It is drawn in Figure 2. Here the vertex stabilizers are

$$\begin{aligned} \Gamma_p &= \langle A, L \rangle \cong \mathcal{D}_2, \quad \Gamma_q = \langle V, S \rangle \cong \mathcal{D}_3, \quad \Gamma_u = \langle W, S \rangle \cong \mathcal{A}_4, \quad \Gamma_v = \langle R, A \rangle \cong \mathcal{D}_3, \\ &\text{where } A = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L = \pm \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad S = \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \\ R &= \pm \begin{pmatrix} -i & 1 \\ 0 & i \end{pmatrix}, \quad V = \pm \begin{pmatrix} -i & -i \\ 0 & i \end{pmatrix} \text{ and } W = \pm \begin{pmatrix} -i & 1-i \\ 0 & i \end{pmatrix}; \\ &1 = A^2 = L^2 = V^2 = R^2 = S^3 = W^2. \end{aligned}$$

The matrices mentioned in Figure 2 (and their square when they are of order 3) constitute a system of representatives modulo Γ of the non-trivial elements of finite order. So we compute the respective quotients of their rotation axis by their centralizer, in order to obtain the CR orbifold cohomology. For the elements of order 3, namely RA and S , Theorem 17 and its proof pass unchanged, so $\mathcal{H}^{RA}/_{C_\Gamma(\langle RA \rangle)} \cong \mathcal{O}$ and $\mathcal{H}^S/_C_\Gamma(\langle S \rangle) \cong \mathcal{O}$.

For the elements of order 2, we study the quotient of their fixed geodesic by their centralizer through Figure 3. Further, we obtain another such figure useful for our purpose by making the following replacements on Figure 3: $q \mapsto v$, $S \mapsto (RA)^2$, $VS^2 \mapsto A$, $V \mapsto R$. The symmetries obtained from combining complex conjugation with the rotation by L ensure that the relabeled figure is isometric to the printed one.

The points p , $S \cdot p$, $S^2 \cdot p$, $(RA)^2 \cdot p$, $R \cdot p$ all have stabilizer type \mathcal{D}_2 , because they are on the orbit of p , and hence the 2-torsion axes passing through them are mirrored by order-2-elements commuting with the rotation around the respective axis. We immediately conclude that $\mathcal{H}^L/_C_\Gamma(\langle L \rangle)$ is represented by the half-open interval $[p, \infty)$. In the stabilizer of q , which is of type \mathcal{D}_3 , apart from the trivial element, only the order 3 element and its square commute with each other. So there are no mirrorings at q in the centralizer of the rotations with axis passing through q . Hence, $\mathcal{H}^V/_C_\Gamma(\langle V \rangle) \cong [S \cdot p, q, \infty)$ and $\mathcal{H}^{VS^2}/_{C_\Gamma(\langle VS^2 \rangle)} \cong [p, q, S^2 \cdot \infty)$.

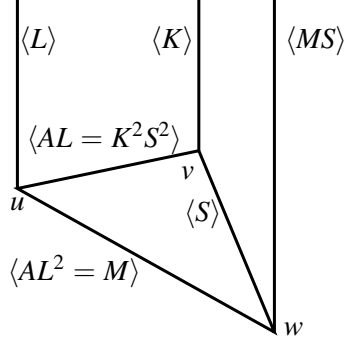


Figure 4: Half of a fundamental domain for the action of $\mathrm{PSL}_2(\mathcal{O}_{-3})$ on \mathcal{H} , open towards the cusp at ∞ . The second half can be obtained as a copy of the open pyramid glued from below to its base square.

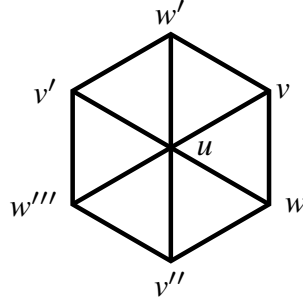


Figure 5: The 2-cells in \mathcal{H} equidistant to the cusps at 0 and ∞ , with no other $\mathrm{PSL}_2(\mathcal{O}_{-3})$ -cusp being closer. The triangle (u, v, w) is the same one as in Figure 4, and the vertex u sits on the middle of the geodesic $(0, \infty)$.

By the above described replacements on Figure 3, we obtain analogously that

$$\mathcal{H}^R /_{C_\Gamma(\langle R \rangle)} \cong [(RA)^2 \cdot p, v, \infty) \text{ and } \mathcal{H}^A /_{C_\Gamma(\langle A \rangle)} \cong [p, v, RA \cdot \infty).$$

In the stabilizer of the point u , there are order-2-elements commuting with W , and therefore

$$\mathcal{H}^W /_{C_\Gamma(\langle W \rangle)} \cong [u, \infty).$$

Summing up, and taking into account that \mathcal{H}/Γ is contractible, we obtain the CR orbifold cohomology

$$H_{orb}^d \left([\mathcal{H}_{\mathbb{C}}^3 / \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-1}])] \right) \cong \begin{cases} \mathbb{Q}, & d = 0, \\ \mathbb{Q}^{10}, & d = 2, \\ \mathbb{Q}^4, & d = 3, \\ 0, & \text{otherwise.} \end{cases}$$

8.3 The case $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-3})$.

Let $\omega := \frac{\sqrt{-3}-1}{2}$. A fundamental domain for the action of $\Gamma := \mathrm{PSL}_2(\mathbb{Z}[\omega])$ on real hyperbolic 3-space \mathcal{H} has been found by Luigi Bianchi, and the stabilizers have been computed by Flöge [16], whose notation we are going to adopt. It is drawn in Figure 4. Here the vertex stabilizers are

$$\begin{aligned} \Gamma_u &= \langle A, L \rangle \cong \mathcal{D}_3, \Gamma_v = \langle K, S \rangle \cong \mathcal{A}_4, \Gamma_w = \langle M, S \rangle \cong \mathcal{A}_4, \\ \text{where } A &= \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, L = \pm \begin{pmatrix} -\omega^2 & 0 \\ 0 & \omega \end{pmatrix}, S = \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, K = \pm \begin{pmatrix} \omega^2 & -\omega \\ 0 & \omega \end{pmatrix}; \\ &1 = A^2 = L^3 = K^3 = S^3 = M^2. \end{aligned}$$

As $\langle S \rangle \cong \mathbb{Z}/3$ and $\Gamma_v \cong \mathcal{A}_4 \cong \Gamma_w$, the latter two vertex stabilizers do neither reflect \mathcal{H}^S , nor do they contribute any element to $C_\Gamma(\langle S \rangle)$. That is why though all cusps are on one Γ -orbit, the centralizer $C_\Gamma(\langle S \rangle) \cong \langle S \rangle$ leaves pointwise fixed \mathcal{H}^S , which is the geodesic line through (v, w) starting at a cusp s in the Γ_v -orbit of ∞ and ending at a cusp e in the Γ_w -orbit of ∞ . By the \mathcal{A}_4 -symmetries in v and w , (s, v) is mapped to (∞, v) and (e, w) is mapped to (∞, w) . Hence there can be no translations of \mathcal{H}^S in Γ , and therefore $\mathcal{H}^S / C_\Gamma(\langle S \rangle) = \mathcal{H}^S$. The \mathcal{A}_4 -symmetries enforce all 3-torsion axes passing through a representative of v or w to admit the same centralizer quotient. Hence also $\mathcal{H}^K / C_\Gamma(\langle K \rangle) = \mathcal{H}^K$ and $\mathcal{H}^{MS} / C_\Gamma(\langle MS \rangle) = \mathcal{H}^{MS}$ are open geodesic lines starting and ending at cusps.

In contrast, \mathcal{H}^L is getting reflected onto itself by Γ_u . But the elements of order 2 in $\Gamma_u \cong \mathcal{D}_3$ do not commute with L , and hence $\mathcal{H}^L = \mathcal{H}^L / C_\Gamma(\langle L \rangle)$ is the geodesic line $(\infty, M \cdot \infty)$ with the vertex u on its middle.

Concerning the 2-torsion axes, \mathcal{H}^M does not get reflected by $\Gamma_u \cong \mathcal{D}_3$. It gets reflected by order-2-elements in Γ_w and $\Gamma_{v'}$ commuting with M (see Figure 5); hence $\mathcal{H}^M / C_\Gamma(\langle M \rangle) \cong \bullet \bullet$. By the \mathcal{D}_3 -symmetry in u , the same happens for \mathcal{H}^{AL} : It gets reflected in v and w''' by centralizing elements and not in u ; therefore $\mathcal{H}^{AL} / C_\Gamma(\langle AL \rangle) \cong \bullet \bullet$.

Summing up, and taking into account that \mathcal{H}/Γ is contractible, we obtain the CR orbifold cohomology

$$H_{orb}^d \left(\left[\mathcal{H}_{\mathbb{C}}^3 / \mathrm{PSL}_2 \left(\mathbb{Z} \left[\frac{\sqrt{-3}-1}{2} \right] \right) \right] \right) \cong \begin{cases} \mathbb{Q}, & d = 0, \\ \mathbb{Q}^{10}, & d = 2, \\ 0, & \text{otherwise.} \end{cases}$$

8.4 The case $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-11})$.

Let \mathcal{O}_{-11} be the ring of integers in $\mathbb{Q}(\sqrt{-11})$.
Then $\mathcal{O}_{-11} = \mathbb{Z}[\omega]$ with $\omega = \frac{-1+\sqrt{-11}}{2}$.

A fundamental domain for $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-11})$ in real hyperbolic 3-space \mathcal{H} has been found by Luigi Bianchi [5]. Half of its lower boundary given in Figure 6. The coordinates of the vertices of Figure 6 in the upper-half space model are $(3) = j$, $(3)' = 1 + \omega + j$, $(6) = \frac{1}{2} + \sqrt{\frac{3}{4}}j$, $(6)' = \frac{1}{2} + \omega + \sqrt{\frac{3}{4}}j$, $(8) = \frac{3}{11} + \frac{3}{11}\omega + \sqrt{\frac{2}{11}}j$, $(9) = \frac{8}{11} + \frac{5}{11}\omega + \sqrt{\frac{2}{11}}j$. The set of representatives of conjugacy classes can be chosen

$$T = \{\mathrm{Id}, \gamma, \beta, \beta^2\},$$

with $\beta = \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\gamma = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

so γ is of order 2, and β is of order 3. Using Lemma 21 and with the help of our Bredon homology computations, we check the cardinality of T . That we have one less conjugacy class of finite order elements than in the case \mathcal{O}_{-2} , comes from the fact that by Remark 19, there is only one conjugacy class of order-2-elements in \mathcal{A}_4 .

The fixed point sets are then the following subsets of complex hyperbolic space $\mathcal{H} := \mathcal{H}_{\mathbb{C}}^3$:

$$\mathcal{H}^{\mathrm{Id}} = \mathcal{H},$$

$$\mathcal{H}^{\gamma} = \text{the complex geodesic line through } (3) \text{ and } (8),$$

$$\mathcal{H}^{\beta} = \mathcal{H}^{\beta^2} = \text{the complex geodesic line through } (6) \text{ and } (9).$$

The 2-torsion sub-complex is of homeomorphism type $\bullet \dashrightarrow \bullet$ and the 3-torsion sub-complex is of homeomorphism type \circ . Therefore, we obtain

$$\mathrm{H}_{\mathrm{CR}}^d([\mathcal{H}_{\mathbb{C}}^3/\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-11}])]) \cong \mathrm{H}^d(\mathcal{H}_{\mathbb{C}}/\mathrm{PSL}_2(\mathcal{O}_{-11}); \mathbb{Q}) \oplus \begin{cases} \mathbb{Q}^{1+2}, & d = 2, \\ \mathbb{Q}^2, & d = 3, \\ 0, & \text{otherwise.} \end{cases}$$

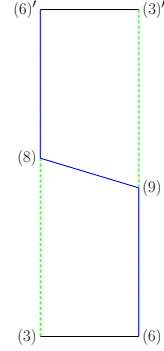


Figure 6: Fundamental domain in the case $m = 11$.

8.5 The case $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-191})$.

Let \mathcal{O}_{-191} be the ring of integers in $\mathbb{Q}(\sqrt{-191})$. Again, the set of representatives of conjugacy classes can be chosen

$$T = \{\mathrm{Id}, \gamma, \beta, \beta^2\},$$

with $\beta = \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\gamma = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so γ is of order 2, and β is of order 3. Both the 2- and the 3-torsion sub-complexes are of homeomorphism type \mathcal{O} . Then,

$$\begin{aligned} & \mathrm{H}_{orb}^d([\mathcal{H}_{\mathbb{C}}^3/\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-191}])]) \\ & \cong \mathrm{H}^d(\mathcal{H}_{\mathbb{C}}/\mathrm{PSL}_2(\mathcal{O}_{-191}); \mathbb{Q}) \oplus \begin{cases} \mathbb{Q}^{1+2}, & d = 2, \\ \mathbb{Q}^{1+2}, & d = 3, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We conclude this section with the following explanation why in our fundamental domain diagrams, there occurs only one representative per torsion-stabilized edge.

Remark 23 Let e be a non-trivially stabilized edge in the fundamental domain for the refined cell complex. Then the fundamental domain for the 2-dimensional retract can be chosen such that it contains e as the only edge on its orbit.

Sketch of proof. Observe that the inner dihedral angle $\frac{2\pi}{q}$ of the Bianchi fundamental polyhedron is $\frac{2\pi}{\ell}$ or $\frac{\pi}{\ell}$ at its edges admitting a rotation of order ℓ from the Bianchi group. We can verify this in the vertical half-plane where the action of $\mathrm{PSL}_2(\mathbb{Z})$ is embedded into the action of the Bianchi group, for the generators of orders $\ell = 2$ and $\ell = 3$ of $\mathrm{PSL}_2(\mathbb{Z})$ which fix edges orthogonal to the vertical half-plane. These angles are transported to all edges stabilized by Bianchi group elements conjugate under $\mathrm{SL}_2(\mathbb{C})$ to these two rotations. Poincaré [30] partitions the edges of the Bianchi fundamental polyhedron into cycles, consisting of the edges on the same orbit, of length $\frac{q}{\ell} = 1$ or 2. In the case of length 2, Poincaré’s description implies that each of the two 2-cells separated by the first edge of the cycle, is respectively on the same orbit as one of the 2-cells separated by the second edge of the cycle. As the fundamental domain for the 2-dimensional retract is strict with respect to the 2-cells, it can be chosen such that it contains e as the only edge on its orbit. \square

Note that we can check our computations using the algorithm of [34, Section 5.3] for the computation of subgroups in the centralizers.

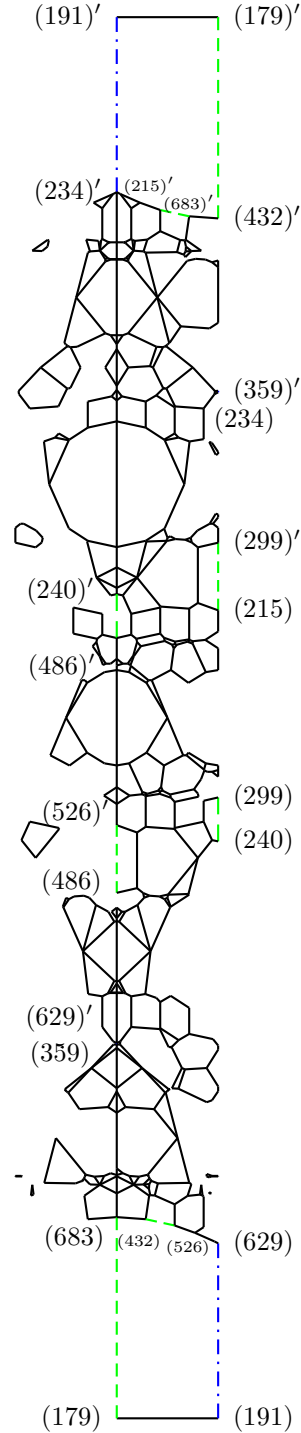


Figure 7: Fundamental domain in the case $m = 191$. The coordinates of the vertices can be displayed by [33].

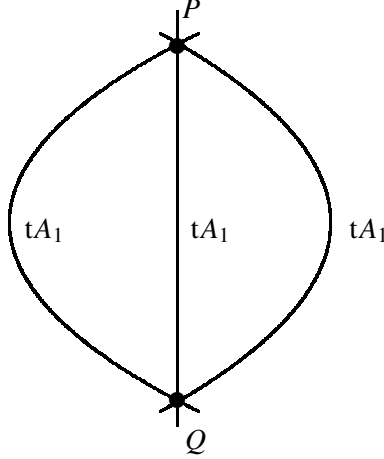


Figure 8: Two singular points P, Q which are analytically isomorphic to the singularity at the origin of $\mathbb{C}^3/\mathcal{D}_2$.

8.6 The case $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-5})$.

We start by analysing the case where

$$\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-5}).$$

In this case the singular locus of X has two connected components. One component is a transverse singularity of type A_2 (we write tA_2). The other component, drawn in Figure 8, contains two singular points P, Q which are analytically isomorphic to the singularity at the origin of $\mathbb{C}^3/\mathcal{D}_2$, where

$$\mathcal{D}_2 = \langle \xi, \eta \mid \xi^2 = \eta^2 = (\xi\eta)^2 = 1 \rangle \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

is the Klein-four-group acting via the standard diagonal representation $\mathcal{D}_2 \rightarrow \mathrm{SL}_3(\mathbb{C})$:

$$\xi \mapsto \mathrm{diag}(-1, -1, 1), \quad \eta \mapsto \mathrm{diag}(-1, 1, -1).$$

The points P, Q are joined by three curves of transverse singularities of type A_1 (tA_1), which correspond in a neighbourhood of P (resp. Q) to the image in $\mathbb{C}^3/\mathcal{D}_2$ of the coordinate axes of \mathbb{C}^3 .

From Corollary 2, we have the following presentation of the Chen–Ruan cohomology:

$$H_{\mathrm{CR}}^d([\mathcal{H}_{\mathbb{C}}^3/\mathrm{PSL}_2(\mathcal{O}_{-5})], \mathbb{Q}) \cong H^d(\mathcal{H}_{\mathbb{C}}^3/\mathrm{PSL}_2(\mathcal{O}_{-5}), \mathbb{Q}) \oplus \begin{cases} \mathbb{Q}^2 \oplus \mathbb{Q}^3 & d = 2 \\ \mathbb{Q}^2 \oplus \{0\} & d = 3 \end{cases}$$

where the first direct summand is the cohomology of the non-twisted sector. The second direct summand $\begin{pmatrix} \mathbb{Q}^2 \\ \mathbb{Q}^2 \end{pmatrix}$ is the cohomology of the 3-torsion twisted sector $\mathcal{X}_{(3)}$ whose coarse moduli space is the connected component of the singular locus of X corresponding to the tA_2 -singularity. Notice that this locus is topologically isomorphic to $S^1 \times \mathbb{R} \cong \mathbb{C}^*$, $\lambda_6 = 1$ and $\lambda_6^* = 0$, where $\lambda_{2n}, \lambda_{2n}^*$ are as defined in Corollary 2. Finally, the third direct summand $\begin{pmatrix} \mathbb{Q}^3 \\ \{0\} \end{pmatrix}$ is the cohomology of the 2-torsion twisted sector

$\mathcal{X}_{(2)}$. This sector has three connected components each one homeomorphic to the strip $[0, 1] \times \mathbb{R}$ and corresponding to the tA_1 -singularities joining the points P and Q in Figure 8. In the coarse moduli space X , these components form the configuration in Figure 8. Here we have $\lambda_4 = \lambda_4^* = 3$.

Now we study the Chen–Ruan cup product \cup_{CR} , verifying first that the ordinary cup product on the non-twisted sector $H^*(\mathcal{H}_{\mathbb{C}}^3/\Gamma, \mathbb{Q})$ vanishes. From the explicit description of the quotient space $\mathcal{H}_{\mathbb{R}}^3/\text{PSL}_2(\mathcal{O}_{-5})$ in [37], we get the picture of the Borel–Serre compactification of $\mathcal{H}_{\mathbb{R}}^3/\text{PSL}_2(\mathcal{O}_{-5})$ drawn in Figure 9. Here, we have expanded the singular cusp at $\frac{\sqrt{-5}+1}{2}$ to a fundamental rectangle (s, s', s'', s''') for the action of the cusp stabilizer $\Gamma_{\frac{\sqrt{-5}+1}{2}}$ on the plane attached by the Borel–Serre bordification. In the same way, we expand the cusp at infinity to a fundamental rectangle $(\infty, \infty', \infty'', \infty''')$ for the action of the cusp stabilizer Γ_{∞} on the plane attached there. This is not visible in our 2-dimensional diagram, but is located above the rectangle (o, o', o'', o''') , where o is of height 1 one above the cusp 0. The fundamental polyhedron for the Γ -action is then spanned by the rectangle $(\infty, \infty', \infty'', \infty''')$ and the polygons of Figure 9. The face identifications of the fundamental polyhedron are

$$\begin{aligned}
(3) \quad & (\infty, o, t, o', \infty') \sim (\infty''', o''', t', o'', \infty''), \\
(4) \quad & (\infty, o, b, u, o''', \infty''') \sim (\infty', o', b', u', o'', \infty''), \\
(5) \quad & (a''', s''', s'', a'', v') \sim (a, s, s', a', v), \\
(6) \quad & (u, a''', s''', s, a, b) \sim (u', a'', s'', s', a', b'), \\
(7) \quad & (o, t, v, a, b) \sim (o', t, v, a', b'), \\
(8) \quad & (o''', t', v', a''', u) \sim (o'', t', v', a'', u').
\end{aligned}$$

Here, we did not respect the orientation of the 2-cells, but have written them in the way in which their vertices are identified.

It is well known that the Borel–Serre compactification of $\mathcal{H}_{\mathbb{R}}^3/\text{PSL}_2(\mathcal{O}_{-m})$ is homotopy equivalent to $\mathcal{H}_{\mathbb{R}}^3/\text{PSL}_2(\mathcal{O}_{-m})$ itself, and it has been worked out in [31] how the boundary is attached in the compactification.

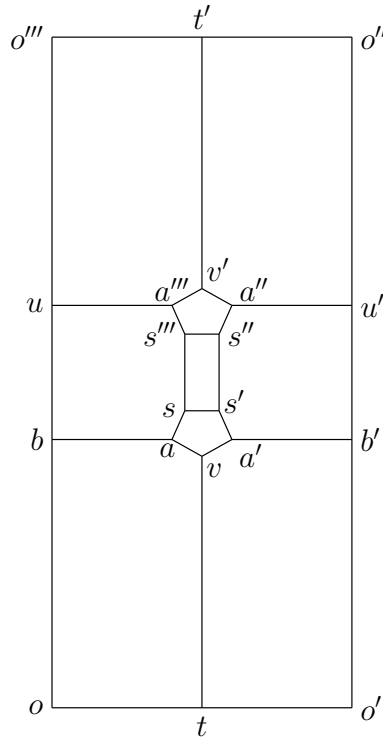
So we can describe the cohomology cocycles of $\mathcal{H}_{\mathbb{R}}^3/\text{PSL}_2(\mathcal{O}_{-5})$ in terms of the above fundamental polyhedron and face identifications. By [18][section 9.3], $\mathcal{H}_{\mathbb{C}}^3$ admits a fundamental polyhedron $P_{\mathbb{C}}$ for Γ with the interior of its top-dimensional facets (called *sides*) being open smooth submanifolds. This yields a Γ -equivariant cell structure on $\mathcal{H}_{\mathbb{C}}^3$. The natural map $\mathcal{H}_{\mathbb{R}}^3 \hookrightarrow \mathcal{H}_{\mathbb{C}}^3 \rightarrow \mathcal{H}_{\mathbb{R}}^3$ induces a map of the sides with respect to the fundamental polyhedron $P_{\mathbb{R}}$ for Γ on $\mathcal{H}_{\mathbb{C}}^3$,

$$\text{sides}(P_{\mathbb{R}}) \hookrightarrow \text{sides}(P_{\mathbb{C}}) \rightarrow \text{sides}(P_{\mathbb{R}}),$$

which respects the side identifications (*side pairings*). All of the side pairings of $P_{\mathbb{C}}$ are detected this way, because they generate the group Γ (see [18][section 9.3]), and so do already the side pairings of $P_{\mathbb{R}}$. Hence there are no additional identifications when complexifying the orbifold, and thus there are no additional cohomology cocycles on $\mathcal{H}_{\mathbb{C}}^3/\text{PSL}_2(\mathcal{O}_{-5})$. Generators for $H^1(\mathcal{H}_{\mathbb{R}}^3/\text{PSL}_2(\mathcal{O}_{-5}), \mathbb{Q})$ are, with reference to the above numbering of the identifications, obtained from

$$(\infty, \infty''') \text{ under (1)} \quad \text{and } (s, s''') \text{ under (3)}.$$

Both (∞, ∞') under (2) and (s, s') under (4) yield trivial cocycles because of the identifications (5) and (6).

Figure 9: Fundamental domain for the Borel-Serre compactification in the case $m = 5$.

For instance using the arc method introduced in [20][section 3.2], we can now check explicitly that the cup product of the two cocycles obtained from (∞, ∞''') under (1) and (s, s''') under (3) vanishes. As further to the two 1-dimensional cocycles, $\mathcal{H}_{\mathbb{R}}^3/\Gamma$ only admits a 0- and a 2-dimensional cocycle, and as there are no further identifications when complexifying, we arrive at the claimed vanishing of the ordinary cup product on the non-twisted sector $H^*(\mathcal{H}_{\mathbb{C}}^3/\Gamma, \mathbb{Q})$.

The cup product of two classes coming from the twisted sectors would be a class in dimension ≥ 4 , where the twisted sectors vanish, and by the above calculation, so does the non-twisted sector.

Therefore, the Chen–Ruan cup product \cup_{CR} is trivial on $[\mathcal{H}_{\mathbb{C}}^3/\text{PSL}_2(\mathcal{O}_{-5})]$.

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University of Trieste, Department of Mathematics and Geosciences, Via A. Valerio 12/1, 34127 Trieste, Italia
 Université du Luxembourg, Mathematics Research Unit, 6, rue Richard Coudenhove-Kalergi, L-1359 Luxembourg
fperroni@units.it, Alexander.Rahm@uni.lu
<http://perroni.dmg.units.it/>, <http://math.uni.lu/~rahm/>