CHEN–RUAN ORBIFOLD COHOMOLOGY OF THE BIANCHI GROUPS

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Abstract. We give formulae for the Chen–Ruan orbifold cohomology for the orbifolds given by a Bianchi group acting on a model for its classifying space for proper actions: complex hyperbolic space.
The Bianchi groups are the arithmetic groups $\mathrm{PSL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers in an imaginary quadratic number field. The underlying real orbifolds which help us in our study, have applications in Physics.

1. Introduction

Recently motivated by string theory in theoretical physics, a stringy geometry and topology of orbifolds has been introduced in mathematics [1]. Its essential innovations consist of Chen–Ruan orbifold cohomology [7], [8] and orbifold K–theory. Ruan's cohomological crepant resolution conjecture [7] associates Chen–Ruan orbifold cohomology with the ordinary cohomology of a resolution of the singularities of the given orbifold. In three complex dimensions and outside the global quotient case, where the conjecture is completely open, we are going to calculate the Chen–Ruan cohomology of an infinite family of orbifolds in this article; and indicate in section 6 how to obtain the cohomology of their resolution of singularities.

Denote by $\mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer, an imaginary quadratic number field, and by $\mathcal{O}_{-m}$ its ring of integers. The *Bianchi groups* are the projective special linear groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$. The Bianchi groups may be considered as a key to the study of a larger class of groups, the *Kleinian* groups, which date back to work of Henri Poincaré [20]. In fact, each non-co-compact arithmetic Kleinian group is commensurable with some Bianchi group [15]. A wealth of information on the Bianchi groups can be found in the monographs [9,11,15]. These groups act in a natural way on hyperbolic three-space, which is isomorphic to the symmetric space associated to them. This yields orbifolds that are studied in Cosmology [4].

The orbifold structure obtained by our group action is determined by a fundamental domain and its stabilisers and identifications. The computation of this information has been implemented for all Bianchi groups [22].

The vector space structure of Chen–Ruan orbifold cohomology. Let Γ be a discrete group acting *properly*, i.e. with finite stabilisers, by diffeomorphisms on a manifold Y. For any element $g \in \Gamma$, denote by $C_\Gamma(g)$ the centraliser of g in Γ. Denote by Y^g the subset of Y consisting of the fixed points of g.

Definition 1. *Let $T \subset \Gamma$ be a set of representatives of the conjugacy classes of elements of finite order in Γ. Then we set*

$$\mathrm{H}^*_{orb}(Y//\Gamma) := \bigoplus_{g\in T} \mathrm{H}^*\left(Y^g/C_\Gamma(g);\mathbb{Q}\right).$$

Date: September 11, 2012.
2000 *Mathematics Subject Classification.* 55N32, Orbifold cohomology.
Funded by the Irish Research Council.

It can be checked that this definition gives the vector space structure of the orbifold cohomology defined by Chen and Ruan [7], if we forget the grading of the latter. We can verify this analogously to the case where Γ is a finite group, treated by Fantechi and Göttsche [10]. The additional argument needed when considering some element g in Γ of infinite order, is the following. As the action of Γ on Y is proper, g does not admit any fixed point in Y. Thus, $\mathrm{H}^*\left(Y^g/C_\Gamma(g);\mathbb{Q}\right) = \mathrm{H}^*\left(\emptyset;\mathbb{Q}\right) = 0$.

Statement of the results. We complexify our orbifolds by complexifying the real hyperbolic three-space. We obtain orbifolds given by the induced action of the Bianchi groups on complex hyperbolic three-space. Then we compute the Chen–Ruan Orbifold Cohomology for these complex orbifolds. We can determine its product structure with theorem 6.

As a result of theorems 13 and 14, we can express the vector space structure of the orbifold cohomology in terms of the numbers of conjugacy classes of finite subgroups and the cohomology of the quotient space, as follows.

Except for the Gaußian and Eisenstein integers, which have to be treated separately, all the rings of integers of imaginary quadratic number fields admit as only units $\{\pm 1\}$. In the latter case, we call $\mathrm{PSL}_2(\mathcal{O})$ a Bianchi group with units $\{\pm 1\}$.

Corollary 2. *Let $\Gamma := PSL_2(\mathcal{O}_{-m})$ be any Bianchi group with units $\{\pm 1\}$. Denote by λ_{2n} the number of conjugacy classes of subgroups of type $\mathbb{Z}/_{n\mathbb{Z}}$ in Γ. Denote by λ_{2n}^* the number of conjugacy classes of those of them which are contained in a subgroup of type $\mathcal{D}_n$ in Γ. Then,*

$$\mathrm{H}^d_{orb}\left(\mathcal{H}_\mathbb{C}//PSL_2(\mathcal{O}_{-m})\right) \cong \mathrm{H}^d\left(\mathcal{H}_\mathbb{C}/_{PSL_2(\mathcal{O}_{-m})};\ \mathbb{Q}\right) \oplus \begin{cases} \mathbb{Q}^{\lambda_4+2\lambda_6-\lambda_6^*}, & d=2,\\ \mathbb{Q}^{\lambda_4-\lambda_4^*+2\lambda_6-\lambda_6^*}, & d=3,\\ 0, & \text{otherwise.}\end{cases}$$

The (co)homology of the quotient space $\mathcal{H}_\mathbb{R}/_\Gamma$ has been has been computed numerically for a large scope of Bianchi groups [30], [26], [24]; and bounds for its Betti numbers have been given in [13]. Krämer [14] has determined number-theoretic formulae for the numbers λ_{2n} and λ_{2n}^* of conjugacy classes of finite subgroups in the Bianchi groups. Krämer's formulae have been evaluated for hundreds of thousands of Bianchi groups [23], and these values are matching with the ones from the orbifold structure computations with [22] in the cases where the latter are available.

Acknowledgements. I would like to thank Martin Deraux, and very specially Fabio Perroni, for helpful and motivating discussions, which initiated the present work.
Warmest thanks go to Alessandro Chiodo and Yongbin Ruan for explanations on the cohomological crepant resolution conjecture, to José Bertin, Dmitry Kerner, Christian Lehn and Sergei Yakovenko for answering questions on resolutions of singularities, to John Ratcliffe for answers on orbifold topology, and to Graham Ellis for support and encouragement.

Organisation of the paper. In section 2, we work out the product structure of Chen-Ruan orbifold cohomology for arbitrary groups of hyperbolic motions. Then we specialise to the Bianchi groups, and work out the information that we need about the conjugacy classes of their finite order elements in section 3. We use this in section 4 to state the main results of the paper, which imply corollary 2. We illustrate them with the computation of some explicit examples in section 5. Then we outline in section 6 how our results can be used to check the cohomological crepant resolution conjecture. Finally, the appendix A provides an alternative approach to compute subgroups in the centralisers relevant for the determination of the twisted sectors that are issue of the statements in 4; so it can be used for a check.

2. The orbifold cohomology product

In order to equip the orbifold cohomology vector space with the Chen–Ruan product structure, we need an almost complex orbifold structure on $Y//\Gamma$.
Let Y be a complex manifold of dimension D with a proper action of a discrete group Γ by diffeomorphisms, the differentials of which are holomorphic. For any $g \in \Gamma$ and $y \in Y^g$, we consider the eigenvalues $\lambda_1, \dots, \lambda_D$ of the action of g on the tangent space $\mathrm{T}_y Y$. As the action of g on $\mathrm{T}_y Y$ is complex linear, its eigenvalues are roots of unity.

Definition 3. *Write $\lambda_j = e^{2\pi i r_j}$, where r_j is a rational number in the interval $[0, 1[$. The* degree shifting number *of g in y is the rational number* $\operatorname{shift}(g, y) := \sum_{j=1}^{D} r_j$.

We see in [10] that the degree shifting number agrees with the one defined by Chen and Ruan. It is also called the fermionic shift number in [32]. The degree shifting number of an element g is constant on a connected component of its fixed point set Y^g. For the groups under our consideration, Y^g is connected, so we can omit the argument y. Details for this and the explicit value of the degree shifting number are given in lemma 5. Then we can define the graded vector space structure of the orbifold cohomology as

$$\mathrm{H}^d_{orb}(Y//\Gamma) := \bigoplus_{g \in T} \mathrm{H}^{d - 2\operatorname{shift}(g)}\left(Y^g / C_\Gamma(g); \mathbb{Q}\right). \tag{1}$$

Denote by g, h two elements of finite order in Γ, and by $Y^{g,h}$ their common fixed point set. Chen and Ruan construct a certain vector bundle on $Y^{g,h}$ we call the *obstruction bundle*. We denote by $c(g, h)$ its top Chern class. In our cases, $Y^{g,h}$ is a connected manifold. In the general case, the fibre dimension of the obstruction bundle can vary between the connected components of $Y^{g,h}$, and $c(g, h)$ is the cohomology class restricting to the top Chern class of the obstruction bundle on each connected component. The obstruction bundle is at the heart of the construction [7] of the Chen–Ruan orbifold cohomology product. In [10], this product, when applied to a cohomology class associated to Y^g and one associated to Y^h, is described as a push-forward of the cup product of these classes restricted to $Y^{g,h}$ and multiplied by $c(g, h)$. The following statement is made for global quotient orbifolds, but it is a local property, so we can apply it in our proper actions case.

Lemma 4 (Fantechi–Göttsche)**.** *Let $Y^{g,h}$ be connected. Then the obstruction bundle on it is a vector bundle of fibre dimension*

$$\operatorname{shift}(g) + \operatorname{shift}(h) - \operatorname{shift}(gh) - \operatorname{codim}_{\mathbb{C}}\left(Y^{g,h} \subset Y^{gh}\right).$$

In [10], a proof is given in the more general setting that $Y^{g,h}$ needs not be connected.
Examples where the product structure is worked out in the non-global quotient case, are for instance given in [7, 5.3] and [6].

2.1. Groups of hyperbolic motions.

A class of examples with complex structures admitting the grading (1) is given by the discrete subgroups Γ of the orientation preserving isometry group $\mathrm{PSL}_2(\mathbb{C})$ of real hyperbolic 3-space $\mathcal{H}^3_{\mathbb{R}}$. The Lobachevsky model of $\mathcal{H}^3_{\mathbb{R}}$ gives a natural identification of the orientation preserving isometries of $\mathcal{H}^3_{\mathbb{R}}$ with matrices in $\mathrm{PSO}(3, 1)$. By the subgroup inclusion $\mathrm{PSO}(3, 1) \hookrightarrow \mathrm{PSU}(3, 1)$, these matrices specify isometries of the complex hyperbolic space $\mathcal{H}^3_{\mathbb{C}}$.

Lemma 5. *The degree shifting number of any non-trivial rotation of $\mathcal{H}^3_{\mathbb{C}}$ on its fixed points set is* 1.

Proof. For any rotation $\hat{\theta}$ of angle θ around a geodesic line in $\mathcal{H}^3_{\mathbb{R}}$, there is a basis for the construction of the Lobachevsky model such that the matrix of $\hat{\theta}$ takes the shape

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{PSO}(3,1).$$

This matrix, considered as an element of $\mathrm{PSU}(3,1)$, performs a rotation of angle θ around the "complexified geodesic line" with respect to the inclusion $\mathcal{H}^3_{\mathbb{R}} \hookrightarrow \mathcal{H}^3_{\mathbb{C}}$. The fixed points of this rotation are exactly the points p lying on this complexified geodesic line, and the action on their tangent space $\mathrm{T}_p\mathcal{H}^3_{\mathbb{C}} \cong \mathbb{C}^3$ is again a rotation of angle θ. Hence we can choose a basis of this tangent space such that this rotation is expressed by the matrix

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}_3(\mathbb{C}).$$

Therefore the degree shifting number of the rotation $\hat{\theta}$ at p is 1. □

Theorem 6. *Let* Γ *be a group generated by translations and rotations of* $\mathcal{H}^3_{\mathbb{C}}$*. Then all obstruction bundles of the orbifold* $\mathcal{H}^3_{\mathbb{C}}//\Gamma$ *are of fibre dimension zero.*

Proof. Non-trivial obstruction bundles can only appear for two elements of Γ with common fixed points, and such that one of these is not a power of the other one. The translations of $\mathcal{H}^3_{\mathbb{C}}$ have their fixed point on the boundary and not in $\mathcal{H}^3_{\mathbb{C}}$. So let b and c be non-trivial hyperbolic rotations around distinct axes intersecting in the point $p \in \mathcal{H}^3_{\mathbb{C}}$. Then bc is again a hyperbolic rotation around a third distinct axis passing through p. Obviously, these rotation axes constitute the fixed point sets Y^b, Y^c and Y^{bc}. Hence the only fixed point of the group generated by b and c is p. Now lemma 4 yields the following fibre dimension for the obstruction bundle on $Y^{b,c}$:

$$\mathrm{shift}(b) + \mathrm{shift}(c) - \mathrm{shift}(bc) - \mathrm{codim}_{\mathbb{C}}\left(Y^{b,c} \subset Y^{bc}\right).$$

After computing degree shifting numbers using lemma 5, we see that this fibre dimension is zero. □

Hence the obstruction bundle is trivial, and its top Chern class is the neutral element of the cohomological cup product. By Fantechi–Göttsche's description, the Chen–Ruan orbifold cohomology product is then a push-forward of the cup product of the cohomology classes restricted to the intersection of the fixed points sets.

3. The conjugacy classes of finite order elements in the Bianchi groups

Let $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m})$ be a Bianchi group. In 1892, Luigi Bianchi [5] computed fundamental domains for some of these groups. Such a fundamental domain has the shape of a hyperbolic polyhedron (up to a missing vertex at certain cusps, which represent the ideal classes of $\mathcal{O}_{-m}$), so we will call it the *Bianchi fundamental polyhedron.*

It is well-known [12] that any element of Γ fixing a point inside real hyperbolic 3-space $\mathcal{H}^3_{\mathbb{R}}$ acts as a rotation of finite order. Hence the induced action of Γ on complex hyperbolic 3-space $\mathcal{H}^3_{\mathbb{C}}$ is proper. For the remainder of this section, as well as Theorems 13 and 14, we will reduce all our considerations to the action on real hyperbolic 3-space $\mathcal{H}^3_{\mathbb{R}}$.

Let Z be the refined cellular complex obtained from the action of Γ on hyperbolic 3-space as described in [21], namely we subdivide $\mathcal{H}^3_{\mathbb{R}}$ until the stabiliser in Γ of any cell σ fixes σ point-wise. We achieve this by computing Bianchi's fundamental polyhedron for the action of Γ, taking as preliminary set of 2-cells its facets lying on the Euclidean hemispheres and vertical planes of the upper-half space model for $\mathcal{H}^3_{\mathbb{R}}$, and then subdividing along the rotation axes of the elements of Γ. Let ℓ be a prime number.

Definition 7. *The* ℓ–torsion sub-complex *is the sub-complex of Z consisting of all the cells which have stabilisers in* Γ *containing elements of order* ℓ*.*

For ℓ being one of the two occurring primes 2 and 3, the orbit space of this sub-complex is a finite graph, because the cells of dimension greater than 1 are trivially stabilised in the refined cellular complex. We reduce this sub-complex with the following procedure, motivated in [23].

Condition A. In the ℓ–torsion sub-complex, let σ be a cell of dimension $n-1$, lying in the boundary of precisely two n–cells τ_1 and τ_2, the latter cells representing two different orbits. Assume further that no higher-dimensional cells of the ℓ–torsion sub-complex touch σ; and that the n–cell stabilisers admit an isomorphism $\Gamma_{\tau_1} \cong \Gamma_{\tau_2}$.

Where this condition is fulfilled in the ℓ–torsion sub-complex, we merge the cells τ_1 and τ_2 along σ and do so for their entire orbits, if and only if they meet the following additional condition. We use Zassenhaus's notion for a finite group to be *ℓ–normal*, if the centre of one of its Sylow ℓ–subgroups is the centre of every Sylow ℓ–subgroup in which it is contained.

Condition B. Either $\Gamma_{\tau_1} \cong \Gamma_\sigma$,
or Γ_σ is ℓ–normal and $\Gamma_{\tau_1} \cong \mathrm{N}_{\Gamma_\sigma}(\mathrm{center}(\mathrm{Sylow}_\ell(\Gamma_\sigma)))$.

Here, we write $\mathrm{N}_{\Gamma_\sigma}$ for taking the normaliser in Γ_σ and Sylow_ℓ for picking an arbitrary Sylow ℓ–subgroup. This is well defined because all Sylow ℓ–subgroups are conjugate.

The reduced ℓ–torsion sub-complex is the Γ–complex obtained by orbit-wise merging two n–cells of the ℓ–torsion sub-complex satisfying conditions A and B.

We use the following classification of Felix Klein [12].

Lemma 8 (Klein)**.** *The finite subgroups in* $\mathrm{PSL}_2(\mathcal{O})$ *are exclusively of isomorphism types the cyclic groups of orders* 1, 2 *and* 3*, the dihedral groups* $\mathcal{D}_2$ *and* $\mathcal{D}_3$ *(isomorphic to the Klein four-group* $\mathbb{Z}/2 \times \mathbb{Z}/2$*, respectively the symmetric group on three symbols) and the alternating group* $\mathcal{A}_4$*.*

Now we investigate the associated normaliser groups. Straight-forward verification using the multiplication tables of the implied finite groups yields the following.

Lemma 9. *Let G be a finite subgroup of* $\mathrm{PSL}_2(\mathcal{O}_{-m})$*. Then the type of the normaliser of any subgroup of type $\mathbb{Z}/\ell$ in G is given as follows for $\ell = 2$ and $\ell = 3$, where we print only cases with existing subgroup of type $\mathbb{Z}/\ell$.*

Isomorphism type of G	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{A}_4$
normaliser of $\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathcal{D}_2$	$\mathbb{Z}/2$	$\mathcal{D}_2$
normaliser of $\mathbb{Z}/3$			$\mathbb{Z}/3$		$\mathcal{D}_3$	$\mathbb{Z}/3$.

Lemma 10 ([23])**.** *Let $v \in \mathcal{H}^3_{\mathbb{R}}$ be a vertex with stabiliser in Γ of type $\mathcal{D}_2$ or $\mathcal{A}_4$. Let γ in Γ be a rotation of order 2 around an edge e adjacent to v. Then the centraliser $C_\Gamma(\gamma)$ reflects $\mathcal{H}^\gamma$ — which is the geodesic line through e — onto itself at v.*

Let α be any torsion element in Γ. We construct a *chain of edges for* α as follows. Consider the edge of the reduced torsion sub-complex to which the edge stabilised by α belongs. By [23, corollary 22], we can represent our edge of the reduced torsion sub-complex by a chain of edges on the intersection of one geodesic line with a strict fundamental domain for Γ in $\mathcal{H}$. This chain is connected in the orbit space. Now, α is conjugate to an element $\gamma\alpha\gamma^{-1}$ of the stabiliser of one of the edges in the chain. By [23, lemma 24], for any elements α and γ of $\mathrm{PSL}_2(\mathbb{C})$, the fixed point set in $\mathcal{H}$ of α is identified by γ with the fixed point set of $\gamma\alpha\gamma^{-1}$. Hence, the element $\gamma^{-1} \in \Gamma$ maps the mentioned geodesic line to the rotation axis of α. The image under γ^{-1} of the chain of edges under consideration is the desired chain for α. So the chain of edges for α exists and is unique up to translation on the rotation axis of α.

Furthermore, the following easy-to-check statement will be useful for our orbifold cohomology computations.

Lemma 11. *There is only one conjugacy class of elements of order 2 in $\mathcal{D}_3$ as well as in $\mathcal{A}_4$. In $\mathcal{D}_3$, there is also only one conjugacy class of elements of order 3, whilst in $\mathcal{A}_4$ there is an element γ such that γ and γ^2 represent the two conjugacy classes of elements of order 3.*

Proof. In cycle type notation, we can explicitly establish the multiplication tables of $\mathcal{D}_3$ and $\mathcal{A}_4$, and compute the conjugacy classes. $\square$

Corollary 12 (Corollary to lemma 11)**.** *Let γ be an element of order 3 in a Bianchi group Γ with units $\{\pm 1\}$. Then, γ is conjugate in Γ to its square γ^2 if and only if there exists a group $G \cong \mathcal{D}_3$ with $\langle \gamma \rangle \subsetneq G \subsetneq \Gamma$.*

4. The orbifold cohomology of the Bianchi groups

Our main results on the vector space structure of the Chen–Ruan orbifold cohomology of the Bianchi groups are the following.

Theorem 13. *For any element γ of order 3 in a Bianchi group Γ with units $\{\pm 1\}$, the quotient space $\mathcal{H}^\gamma/_{C_\Gamma(\gamma)}$ of the rotation axis modulo the centraliser of γ is homeomorphic to a circle.*

Proof. By [23, lemma 27], for any non-trivial torsion element α in a Bianchi group Γ, the Γ–image of the chain of edges for α contains the rotation axis of α. Therefore, the Γ–image of the chain of edges for γ contains the rotation axis $\mathcal{H}^\gamma$. Now we can observe two cases.

- First, assume that the rotation axis of γ does not contain any vertex of stabiliser type $\mathcal{D}_3$ (from [23], we know that this gives us a circle as a path component in the quotient of the 3–torsion sub-complex). Assume that there exists a reflection of $\mathcal{H}^\gamma$ onto itself by an element of Γ. Such a reflection would fix a point on $\mathcal{H}^\gamma$. Then the normaliser of $\langle\gamma\rangle$ in the stabiliser of this point would contain the reflection. This way, lemma 9 tells us that this stabiliser is of type $\mathcal{D}_3$, which we have excluded. Thus, there can be no reflection of $\mathcal{H}^\gamma$ onto itself by an element of Γ.
  As Γ acts by CAT(0) isometries, every element $g \in \Gamma$ sending an edge of the chain for γ to an edge on $\mathcal{H}^\gamma$ outside the fundamental domain, can then only perform a translation on $\mathcal{H}^\gamma$. A translation along the rotation axis of γ commutes with γ, so $g \in C_\Gamma(\gamma)$. Hence the quotient space $\mathcal{H}^\gamma/_{C_\Gamma(\gamma)}$ is homeomorphic to a circle.
- If $\mathcal{H}^\gamma$ contains a point with stabiliser in Γ of type $\mathcal{D}_3$, then there are exactly two Γ–orbits of such points. The elements of order 2 do not commute with the elements of order 3 in $\mathcal{D}_3$, so the centraliser of γ does not contain the former ones. Hence, $C_\Gamma(\gamma)$ does not contain any reflection of $\mathcal{H}^\gamma$ onto itself. Denote by α and β elements of order 2 of each of the stabilisers of the two endpoints of a chain of edges for γ. Then $\alpha\beta$ performs a translation on $\mathcal{H}^\gamma$ and hence commutes with γ. A fundamental domain for the action of $\langle\alpha\beta\rangle$ on $\mathcal{H}^\gamma$ is given by the chain of edges for γ united with its reflection through one of its endpoints. As no such reflection belongs to the centraliser of γ and the latter endpoint is the only one on its Γ–orbit in this fundamental domain, the quotient $\mathcal{H}^\gamma/_{C_\Gamma(\gamma)}$ matches with the quotient $\mathcal{H}^\gamma/_{\langle\alpha\beta\rangle}$, which is homeomorphic to a circle.

□

Theorem 14. *Let γ be an element of order 2 in a Bianchi group Γ with units $\{\pm 1\}$. Then, the homeomorphism type of the quotient space $\mathcal{H}^\gamma/_{C_\Gamma(\gamma)}$ is*

- an edge without identifications, if $\langle\gamma\rangle$ is contained in a subgroup of type $\mathcal{D}_2$ inside Γ and
- a circle, otherwise.

Proof. By [23, lemma 26], for any 2–torsion element α in Γ, the chain of edges for α is a fundamental domain for the centraliser of α on the rotation axis of α. Thus, the chain of edges for γ is a fundamental domain for $C_\Gamma(\gamma)$ on $\mathcal{H}^\gamma$. Again, we have two cases.

- If $\langle\gamma\rangle$ is contained in a subgroup of type $\mathcal{D}_2$ inside Γ, then any chain of edges for γ admits endpoints of stabiliser types $\mathcal{D}_2$ or $\mathcal{A}_4$. As $\mathcal{D}_2$ is an Abelian group and the reflections in $\mathcal{A}_4$ are contained in the normal subgroup $\mathcal{D}_2$, the reflections in these endpoint stabilisers commute with γ, so the quotient space $\mathcal{H}^\gamma/_{C_\Gamma(\gamma)}$ can be identified with a chain of edges for γ. By [23, corollary 22], we know that any edge of the reduced torsion sub-complex can be represented by a chain of edges on the intersection of one geodesic line with a strict fundamental domain for Γ in $\mathcal{H}$. Hence, this chain of edges for γ represents a reduced edge in the 3–torsion sub-complex with distinct endpoints, so especially there is no identification on this chain by Γ. So, the homeomorphism type of $\mathcal{H}^\gamma/_{C_\Gamma(\gamma)}$ is an edge without identifications.

- $\circ\!\!\bullet$ The other case is analogous to the first case of the proof of theorem 13, the rôle of $\mathcal{D}_3$ being played by $\mathcal{D}_2$ and $\mathcal{A}_4$.

$\square$

Denote by λ_{2n} the number of conjugacy classes of subgroups of type $\mathbb{Z}/_{n\mathbb{Z}}$ in a Bianchi group $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$ with units $\{\pm 1\}$. Denote by λ_{2n}^* the number of conjugacy classes of those of them which are contained in a subgroup of type $\mathcal{D}_n$ in Γ. By corollary 12, there are $2\lambda_6 - \lambda_6^*$ conjugacy classes of elements of order 3. As a result of theorems 13 and 14, the vector space structure of the orbifold cohomology of $\mathcal{H}_\mathbb{R}//\Gamma$ is given as

$$\mathrm{H}^\bullet_{orb}(\mathcal{H}_\mathbb{R}//\Gamma) \cong \mathrm{H}^\bullet\left(\mathcal{H}_\mathbb{R}/_\Gamma;\mathbb{Q}\right) \bigoplus^{\lambda_4^*} \mathrm{H}^\bullet\left(\bullet\!\!-\!\!\bullet;\mathbb{Q}\right) \bigoplus^{(\lambda_4-\lambda_4^*)} \mathrm{H}^\bullet\left(\circ\!\!\bullet;\mathbb{Q}\right) \bigoplus^{(2\lambda_6-\lambda_6^*)} \mathrm{H}^\bullet\left(\circ\!\!\bullet;\mathbb{Q}\right).$$

The (co)homology of the quotient space $\mathcal{H}_\mathbb{R}/_\Gamma$ has been has been computed numerically for a large scope of Bianchi groups [30], [26], [24]; and bounds for its Betti numbers have been given in [13]. Krämer [14] has determined number-theoretic formulae for the numbers λ_{2n} and λ_{2n}^* of conjugacy classes of finite subgroups in the Bianchi groups. Krämer's formulae have been evaluated for hundreds of thousands of Bianchi groups [23], and these values are matching with the ones from the orbifold structure computations with [22] in the cases where the latter are available.

When we pass to the complexified orbifold $\mathcal{H}_\mathbb{C}//\Gamma$, the real line that is the rotation axis in $\mathcal{H}_\mathbb{R}$ of an element of finite order, becomes a complex line. However, the centraliser still acts in the same way by reflections and translations. So, the interval $\bullet\!\!-\!\!\bullet$ as a quotient of the real line yields a stripe $\bullet\!\!-\!\!\bullet \times \mathbb{R}$ as a quotient of the complex line. And the circle $\circ\!\!\bullet$ as a quotient of the real line yields a cylinder $\circ\!\!\bullet \times \mathbb{R}$ as a quotient of the complex line. Therefore, using the degree shifting numbers of lemma 5, we obtain the result of corollary 2,

$$\mathrm{H}^d_{orb}\left(\mathcal{H}_\mathbb{C}//\mathrm{PSL}_2(\mathcal{O}_{-m})\right) \cong \mathrm{H}^d\left(\mathcal{H}_\mathbb{C}/_{\mathrm{PSL}_2(\mathcal{O}_{-m})};\,\mathbb{Q}\right) \oplus \begin{cases} \mathbb{Q}^{\lambda_4+2\lambda_6-\lambda_6^*}, & d=2,\\ \mathbb{Q}^{\lambda_4-\lambda_4^*+2\lambda_6-\lambda_6^*}, & d=3,\\ 0, & \text{otherwise.}\end{cases}$$

As we can calculate the Bredon homology $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma;R_\mathbb{C})$ of the Bianchi groups with coefficients in the complex representation ring functor $R_\mathbb{C}$, the following lemma provides a check on our computations.

Lemma 15 (Mislin [17])**.** *Let Γ be an arbitrary group and write $\mathrm{FC}(\Gamma)$ for the set of conjugacy classes of elements of finite order in Γ. Then there is an isomorphism*

$$\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma;R_\mathbb{C}) \otimes_\mathbb{Z} \mathbb{C} \cong \mathbb{C}[\mathrm{FC}(\Gamma)].$$

5. Sample orbifold cohomology computations for the Bianchi groups

We will carry out our computations in the upper-half space model

$$\{x + iy + rj \in \mathbb{C} \oplus \mathbb{R}j \mid r > 0\}$$

for $\mathcal{H}^3_{\mathbb{R}}$. Details on how to compute Chen–Ruan orbifold cohomology can be found in [19].

The case $\Gamma = \mathbf{PSL}_2(\mathbb{Z}[\sqrt{-2}\,])$. Let $\omega := \sqrt{-2}$. A fundamental domain for $\Gamma := \mathrm{PSL}_2(\mathbb{Z}[\omega])$ in real hyperbolic 3-space has been found by Luigi Bianchi [5]. We can obtain it by taking the geodesic convex envelope of of its lower boundary (half of which is depicted in figure 1) and the vertex ∞, and then removing the vertex ∞, making it non-compact. The other half of the lower boundary consists of one isometric Γ–image of each of the depicted 2-cells (in fact, the depicted 2-cells are a fundamental domain for a Γ–equivariant retract of $\mathcal{H}$, which is described in [25]). The coordinates of the vertices of figure 1 in the upper-half space model are $(1) = j$, $(1)' = \omega + j$, $(2) = \frac{1}{2}\omega + \sqrt{\frac{1}{2}}j$, $(7) = \frac{1}{2} + \sqrt{\frac{3}{4}}j$, $(7)' = \frac{1}{2} + \omega + \sqrt{\frac{3}{4}}j$, $(8) = \frac{1}{2} + \frac{1}{2}\omega + \sqrt{\frac{1}{4}}j$.

The 2-torsion sub-complex (- -) and the 3-torsion sub-complex (·····) are coloured in the figure. The set of representatives of conjugacy classes can be chosen

$$T = \{\mathrm{Id},\, \alpha,\, \gamma,\, \beta,\, \beta^2\},$$

with $\alpha = \pm\begin{pmatrix} 1 & \omega \\ \omega & -1 \end{pmatrix}$, $\beta = \pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\gamma = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so α and γ are of order 2, and β is of order 3. Using lemma 15 and with the help of our Bredon homology computations, we check the cardinality of T. The fixed point sets are then the following subsets of complex hyperbolic space $\mathcal{H} := \mathcal{H}^3_{\mathbb{C}}$:
$\mathcal{H}^{\mathrm{Id}} = \mathcal{H}$,
$\mathcal{H}^{\alpha}$ = the complex geodesic line through (2) and (8),
$\mathcal{H}^{\gamma}$ = the complex geodesic line through (1) and (2),
$\mathcal{H}^{\beta} = \mathcal{H}^{\beta^2}$ = the complex geodesic line through (7) and (8).

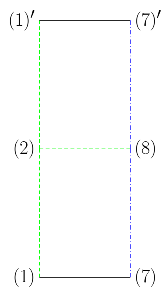

Figure 1. Fundamental domain in the case $m = 2$.

The matrix $g = \pm\begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix}$ performs a translation preserving the j-coordinate and sends the edge (1)(7) onto the edge (1)′(7)′, so the orbit space $\mathcal{H}_{\mathbb{R}}/_{\Gamma}$ is homotopy equivalent to a circle. Consider the real geodesic line $\mathcal{H}^{\gamma}_{\mathbb{R}}$ on the unit circle of real part zero. The edge $g^{-1} \cdot ((2)(1)') = \left(g^{-1}(2)\right)(1)$ lies on $\mathcal{H}^{\gamma}_{\mathbb{R}}$ and is not Γ–equivalent to the edge (1)(2). Because of lemma 10, the centraliser $C_\Gamma(\gamma)$ reflects the line $\mathcal{H}^{\gamma}_{\mathbb{R}}$ onto itself at (2), and again at $g^{-1}(2)$. Furthermore, none of the four elements of Γ sending (2) to $g^{-1}(2)$ belongs to $C_\Gamma(\gamma)$. Hence the quotient space $\mathcal{H}^{\gamma}_{\mathbb{R}}/_{C_\Gamma(\gamma)}$ consists of a contractible segment of two adjacent edges. Thus $\mathrm{H}^{d-2}\left(\mathcal{H}^{\gamma}_{\mathbb{C}}/_{C_\Gamma(\gamma)}; \mathbb{Q}\right) \cong \begin{cases} \mathbb{Q}, & d = 2 \\ 0 & \text{else} \end{cases}$ is contributed to the orbifold cohomology.

Next, consider the real geodesic line $\mathcal{H}^{\beta}_{\mathbb{R}}$ on the circle of constant real coordinate $\frac{1}{2}$, of centre $\frac{1}{2}$ and radius $\sqrt{\frac{3}{4}}$. The edge $g^{-1} \cdot ((8)(7)') = \left(g^{-1}(8)\right)(7)$ lies on $\mathcal{H}^{\beta}_{\mathbb{R}}$ and is not Γ–equivalent to the edge (7)(8). The centraliser of β contains the matrix $V := \pm\begin{pmatrix} 21-\omega & \\ \omega-1 & 1+\omega \end{pmatrix}$ of infinite order, which sends the edge $\left(g^{-1}(8)\right)(7)$ to $(8)z$ with $z = \frac{1}{2} + \frac{3}{5}\omega + \sqrt{\frac{3}{100}}j$. We conclude that the translation action of the group $\langle V\rangle$ on the line $\mathcal{H}^{\beta}_{\mathbb{R}}$ is transitive, with quotient space represented by the circle $\left(g^{-1}(8)\right)(7) \cup (7)(8)$, first and last vertex identified. Thus

$\mathrm{H}^{d-2}\left(\mathcal{H}_{\mathbb{C}}^{\beta}/_{C_\Gamma(\beta)};\mathbb{Q}\right) \cong \mathrm{H}^{d-2}\left(\mathcal{H}_{\mathbb{C}}^{\beta^2}/_{C_\Gamma(\beta^2)};\mathbb{Q}\right) \cong \begin{cases}\mathbb{Q}, & d=2,3\\ 0 & \text{else}\end{cases}$ is contributed to the orbifold cohomology.

Because of lemma 10, the centraliser $C_\Gamma(\alpha)$ reflects the line $\mathcal{H}_{\mathbb{R}}^{\alpha}$ onto itself at (2), and again at (8). So, the quotient space $\mathcal{H}_{\mathbb{R}}^{\alpha}/_{C_\Gamma(\alpha)}$ is represented by the single contractible edge (2)(8). This yields that $\mathrm{H}^{d-2}\left(\mathcal{H}_{\mathbb{C}}^{\alpha}/_{C_\Gamma(\alpha)};\mathbb{Q}\right) \cong \begin{cases}\mathbb{Q}, & d=2\\ 0 & \text{else}\end{cases}$ is contributed to the orbifold cohomology.

Summing up over T, we obtain

$$\mathrm{H}^d_{orb}\left(\mathcal{H}_{\mathbb{C}}//\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-2}\,])\right) \cong \mathrm{H}^d\left(\mathcal{H}_{\mathbb{C}}/_{\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-2}\,])};\,\mathbb{Q}\right) \oplus \begin{cases}\mathbb{Q}^4, & d=2,\\ \mathbb{Q}^2, & d=3,\\ 0, & \text{otherwise.}\end{cases}$$

The case $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-11})$.

Let $\mathcal{O}_{-11}$ be the ring of integers in $\mathbb{Q}(\sqrt{-11}\,)$. Then $\mathcal{O}_{-11} = \mathbb{Z}[\omega]$ with $\omega = \frac{-1+\sqrt{-11}}{2}$.

A fundamental domain for $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-11})$ in real hyperbolic 3-space has been found by Luigi Bianchi [5]. Half of its lower boundary given in figure 2. The coordinates of the vertices of figure 2 in the upper-half space model are $(3) = j$, $(3)' = 1+\omega+j$, $(6) = \frac{1}{2}+\sqrt{\frac{3}{4}}j$, $(6)' = \frac{1}{2}+\omega+\sqrt{\frac{3}{4}}j$, $(8) = \frac{3}{11}+\frac{3}{11}\omega+\sqrt{\frac{2}{11}}j$, $(9) = \frac{8}{11}+\frac{5}{11}\omega+\sqrt{\frac{2}{11}}j$. The set of representatives of conjugacy classes can be chosen

$$T = \{\mathrm{Id},\ \gamma,\ \beta,\ \beta^2\},$$

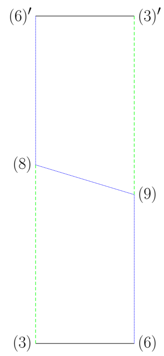

FIGURE 2. Fundamental domain in the case $m = 11$.

with $\beta = \pm\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$ and $\gamma = \pm\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, so γ is of order 2, and β is of order 3. Using lemma 15 and with the help of our Bredon homology computations, we check the cardinality of T. That we have one less conjugacy class of finite order elements than in the case $\mathcal{O}_{-2}$, comes from the fact that by lemma 11, there is only one conjugacy class of order–2–elements in $\mathcal{A}_4$.

The fixed point sets are then the following subsets of complex hyperbolic space $\mathcal{H} := \mathcal{H}^3_{\mathbb{C}}$:
$\mathcal{H}^{\mathrm{Id}} = \mathcal{H}$,
$\mathcal{H}^{\gamma}$ = the complex geodesic line through (3) and (8),
$\mathcal{H}^{\beta} = \mathcal{H}^{\beta^2}$ = the complex geodesic line through (6) and (9).

The 2–torsion sub-complex is of homeomorphism type •—• and the 3–torsion sub-complex is of homeomorphism type ○•. Therefore, we obtain

$$\mathrm{H}^d_{orb}\left(\mathcal{H}_{\mathbb{C}}//\mathrm{PSL}_2(\mathcal{O}_{-11})\right) \cong \mathrm{H}^d_{orb}\left(\mathcal{H}_{\mathbb{C}}/_{\mathrm{PSL}_2(\mathcal{O}_{-11})};\mathbb{Q}\right) \oplus \begin{cases}\mathbb{Q}^{1+2}, & d=2,\\ \mathbb{Q}^2, & d=3,\\ 0, & \text{otherwise.}\end{cases}$$

The case $\Gamma = \mathbf{PSL_2}(\mathcal{O}_{-191})$.

Let $\mathcal{O}_{-191}$ be the ring of integers in $\mathbb{Q}(\sqrt{-191}\,)$. Again, the set of representatives of conjugacy classes can be chosen

$$T = \{\mathrm{Id},\, \gamma,\, \beta,\, \beta^2\},$$

with $\beta = \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\gamma = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so γ is of order 2, and β is of order 3. Both the 2– and the 3–torsion sub-complexes are of homeomorphism type . Then,

$$\mathrm{H}^d_{orb}\left(\mathcal{H}_\mathbb{C}//\mathrm{PSL}_2(\mathcal{O}_{-191})\right)$$
$$\cong \mathrm{H}^d_{orb}\left(\mathcal{H}_\mathbb{C}/_{\mathrm{PSL}_2(\mathcal{O}_{-191})};\, \mathbb{Q}\right) \oplus \begin{cases} \mathbb{Q}^{1+2}, & d = 2, \\ \mathbb{Q}^{1+2}, & d = 3, \\ 0, & \text{otherwise.} \end{cases}$$

We conclude this section with the following explanation why in our fundamental domain diagrams, there occurs only one representative per torsion-stabilised edge.

***Remark* 16.** Let e be a non-trivially stabilised edge in the fundamental domain for the refined cell complex. Then the fundamental domain for the 2–dimensional retract can be chosen such that it contains e as the only edge on its orbit.

Sketch of proof. Observe that the inner dihedral angle $\frac{2\pi}{q}$ of the Bianchi fundamental polyhedron is $\frac{2\pi}{\ell}$ or $\frac{\pi}{\ell}$ at its edges admitting a rotation of order ℓ from the Bianchi group. We can verify this in the vertical half-plane where the action of $\mathrm{PSL}_2(\mathbb{Z})$ is embedded into the action of the Bianchi group, for the generators of orders $\ell = 2$ and $\ell = 3$ of $\mathrm{PSL}_2(\mathbb{Z})$ which fix edges orthogonal to the vertical half-plane. These angles are transported to all edges stabilised by Bianchi group elements conjugate to these two rotations. Poincaré [20] partitions the edges of the Bianchi fundamental polyhedron into cycles, consisting of the edges on the same orbit, of length $\frac{q}{\ell} = 1$ or 2. In the case of length 2, Poincaré's description implies that each of the two 2–cells separated by the first edge of the cycle, is respectively on the same orbit as one of the 2–cells separated by the second edge of the cycle. As the fundamental domain for the 2–dimensional retract is strict with respect to the 2–cells, it can be chosen such that it contains e as the only edge on its orbit. □

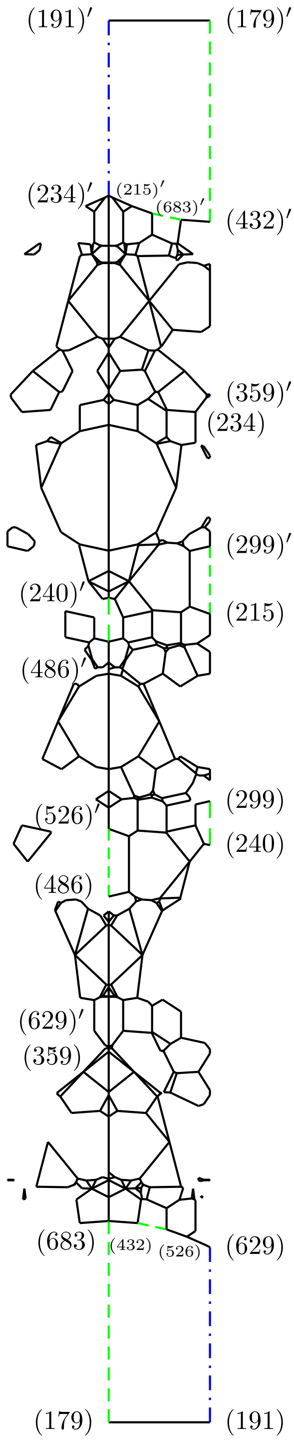

FIGURE 3. Fundamental domain in the case $m = 191$. The coordinates of the vertices can be displayed by [22].

6. Cohomology of the crepant resolution of the singularities

Perroni [19] has shown that the cohomological crepant resolution conjecture cannot hold in the original form; and has suggested some little modifications that, together with the sign modification already introduced by Fantechi and Göttsche [10] make the conjecture robust enough to support all the known examples. We now give a very coarse sketch of how the conjectured association between the Chen–Ruan orbifold cohomology and the ordinary cohomology of a crepant resolution of the singularities of the quotient spaces could possibly be checked for the Bianchi groups.

(1) Mennicke and Grunewald [16] provide a structure as a topological manifold for the quotient of real hyperbolic space by a Bianchi group. In order to get such a structure also for complexified hyperbolic space, we could take Armstrong's triangulations [2], [3] and follow Woodruff's way of studying angular sectors [31] in order to check Seifert and Threlfall's conditions [27] for the triangulated orbit space to be a topological manifold.
(2) By the theorems of Nash [18] and Tognoli [29], compact real manifolds are diffeomorphic to classical real algebraic varieties. So there exist algebraic varieties that are homeomorphic to the quotient of compactified complex hyperbolic space by the Bianchi groups.
(3) We want to show that the homology of the twisted sectors equals the homology of the corresponding ADE singularities of the variety once they are replaced by a crepant resolution.
(4) We might be able to assemble the local evidence for Ruan's conjecture produced in step (3) together to the whole orbifold, using the results of [19].

The crucial part is step (3), on which we give some details now.

Let us consider for instance the singularity obtained by locally passing to the quotient of the action of a group with two elements which is contained in a group of type $\mathcal{D}_2$ inside Γ. This singularity is a product of a complex line segment and a singularity of type A_1. We can think of the A_1-singularity as a cone, and we obtain a crepant resolution for it by taking a complex cylinder surface and pinching a diametral circle off into a single point. On cohomology, the complex cylinder contributes a generator to H^2, and the Chen–Ruan cohomology has an isomorphic contribution $\mathrm{H}^\bullet\left(\bullet\!\!-\!\!\bullet; \mathbb{Q}\right)$. This occurs λ_4^* times, where we denote by λ_{2n} the number of conjugacy classes of subgroups of type $\mathbb{Z}/_{n\mathbb{Z}}$ in Γ; and by λ_{2n}^* the number of conjugacy classes of those of them which are contained in a subgroup of type $\mathcal{D}_n$ in Γ. Furthermore, we obtain the following contributions.

Multiplicity	λ_4^*	$\lambda_4 - \lambda_4^*$	λ_6^*	$\lambda_6 - \lambda_6^*$
Orbifold cohomology contributed	$\mathrm{H}^\bullet\left(\bullet\!\!-\!\!\bullet; \mathbb{Q}\right)$	$\mathrm{H}^\bullet\left(\circlearrowleft; \mathbb{Q}\right)$	$\mathrm{H}^\bullet\left(\circlearrowleft; \mathbb{Q}\right)$	$\left(\mathrm{H}^\bullet\left(\circlearrowleft; \mathbb{Q}\right)\right)^2$
Quotient singularity	$\mathrm{A}_1 \times (\bullet\!\!-\!\!\bullet \times \mathbb{R})$	$\mathrm{A}_1 \times (\circlearrowleft \times \mathbb{R})$	$\mathrm{A}_2 \times (\bullet\!\!-\!\!\bullet \times \mathbb{R})$	$\mathrm{A}_2 \times (\circlearrowleft \times \mathbb{R})$

Concerning the 3-torsion, a small smoothing deformation of the singularity of type A_2 creates in a neighbourhood of the singular point two cocycles in H^2, i.e. the cohomology group H^2 of the crepant resolution grows by $\mathbb{Q}^2$ compared with the quotient space containing the singularity. Recall also that the conjugacy classes of subgroups with three elements which are not contained in a subgroup of type $\mathcal{D}_3$ in Γ have their generator not conjugate to its square, so to contribute $\left(\mathrm{H}^\bullet\left(\circlearrowleft; \mathbb{Q}\right)\right)^2$ to the Chen–Ruan orbifold cohomology.

Appendix A. Computation of subgroups in the centralisers

We can check our computations using the following algorithm for the computation of subgroups in the centralisers. We start by computing a subgroup in the centraliser of β, for β running through the representatives of conjugacy classes of elements of finite order in Γ. For an arbitrary matrix $\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \beta$, the centraliser elements are of the form $\begin{pmatrix} a & b \\ \frac{g}{f}b & a + b\frac{h-e}{f} \end{pmatrix}$, and the determinant 1 equation splits as follows into real and imaginary part. Assume that m is congruent to 1 or 2 $\mod 4$, so the ring of integers of $\mathbb{Q}(\sqrt{-m})$ is given by $\mathbb{Z}[\sqrt{-m}\,]$. Let $\omega := \sqrt{-m}$, and write $a =: j + k\omega$, $b =: \ell + n\omega$ with $j,\ k,\ \ell,\ n \in \mathbb{Z}$, and set $\frac{g}{f} =: R + \omega J$ as well as $\frac{h-e}{f} =: \rho + \omega\iota$. Then

$$(\mathrm{Re}): \qquad j^2 - mk^2 + (j\ell - mkn)\rho - (jn + k\ell)m\iota + (-l^2 + mn^2)R + 2mJln = 1,$$

$$(\mathrm{Im}): \qquad 2jk + (jn - k\ell)\rho + (j\ell - mkn)\iota - 2R\ell n - J\ell^2 + Jmn^2 = 0.$$

In the special case of $\beta = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, a matrix which is contained in all the Bianchi groups via the inclusion $\mathrm{PSL}_2(\mathbb{Z}) < \Gamma$, these equations reduce to

$$(\mathrm{Re}): \qquad j^2 - mk^2 + \ell^2 - mn^2 - j\ell + mkn = 1,$$

$$(\mathrm{Im}): \qquad (2k - n)j + 2\ell n - k\ell = 0.$$

First case: $n = 2k$. The equation (Im) yields $k\ell = 0$. If $k = 0$, we obtain the matrix group $\langle\beta\rangle \cong \mathbb{Z}/3$ generated by our centralised matrix β.
Otherwise $\ell = 0$, and equation (Re) gives

$$j^2 = 3mk^2 + 1. \tag{2}$$

Second case: $n \neq 2k$. This means that we can transform the equation (Im) into

$$j = \ell\frac{k - 2n}{2k - n},$$

which we insert into the equation (Re):

$$\ell^2\left(\frac{k - 2n}{2k - n}\right)^2 - mk^2 + \ell^2 - mn^2 - \ell^2\frac{k - 2n}{2k - n} + mkn = 1.$$

We solve for ℓ^2 and find

$$\ell^2 = \frac{mk^2 + mn^2 - mkn + 1}{1 + \frac{k-2n}{2k-n}\left(\frac{k-2n}{2k-n} - 1\right)}. \tag{3}$$

As m is fixed, we can numerically compute the integer solutions of equations (2) and (3), up to a chosen bound for the absolute value of k, which shall also bound the absolute value of n in the second equation. We obtain a subset S of $C_\Gamma(\beta)$ that contains all the matrices in $C_\Gamma(\beta)$ the entries of which have absolute value at most the chosen bound. Then we calculate the group $\langle S\rangle$ generated by S. Once the quotient space $\mathcal{H}^\beta/\langle S\rangle$ contains no more than one representative for any Γ-orbit of cells, we know that we have obtained the quotient space $\mathcal{H}^\beta/_{C_\Gamma(\beta)}$, because of the subgroup inclusions $\langle S\rangle < C_\Gamma(\beta) < \Gamma$.

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