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Adrien Blanchet [*], José A. Carrillo [†] & Philippe Laurençot [‡]

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Abstract

This paper is devoted to the analysis of non-negative solutions for a generalisation of the classical parabolic-elliptic Patlak-Keller-Segel system with $d \geq 3$ and porous medium-like non-linear diffusion. Here, the non-linear diffusion is chosen in such a way that its scaling and the one of the Poisson term coincide. We exhibit that the qualitative behaviour of solutions is decided by the initial mass of the system. Actually, there is a sharp critical mass M_c such that if $M \in (0, M_c]$ solutions exist globally in time, whereas there are blowing-up solutions otherwise. We also show the existence of self-similar solutions for $M \in (0, M_c)$. While characterising the eventual infinite time blowing-up profile for $M = M_c$, we observe that the long time asymptotics are much more complicated than in the classical Patlak-Keller-Segel system in dimension two.

1 Introduction

In this work, we analyse qualitative properties of non-negative solutions for the Patlak-Keller-Segel system in dimension $d \geq 3$ with homogeneous non-linear diffusion given by

$$\left\{ \begin{array}{rll} \dfrac{\partial u}{\partial t}(t,x) & = \operatorname{div}\left[\nabla u^m(t,x) - u(t,x)\nabla\phi(t,x)\right] & t>0\,,\ x\in\mathbb{R}^d\,, \\ -\Delta\phi(t,x) & = u(t,x)\,, & t>0\,,\ x\in\mathbb{R}^d\,, \\ u(0,x) & = u_0(x) & x\in\mathbb{R}^d\,. \end{array} \right. \tag{1.1}$$

Initial data will be assumed throughout this paper to verify

$$u_0 \in \mathrm{L}^1(\mathbb{R}^d;(1+|x|^2)\ \mathrm{d}x) \cap \mathrm{L}^\infty(\mathbb{R}^d), \quad \nabla u_0^m \in \mathrm{L}^2(\mathbb{R}^d) \quad \text{and} \quad u_0 \geq 0\,. \tag{1.2}$$

A fundamental property of the solutions to (1.1) is the formal conservation of the total mass of the system

$$M := \int_{\mathbb{R}^d} u_0(x)\ \mathrm{d}x = \int_{\mathbb{R}^d} u(t,x)\ \mathrm{d}x \quad \text{for } t \geq 0\,.$$

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As the solution to the Poisson equation $-\Delta\phi = u$ is given up to an harmonic function, we choose the one given by $\phi = \mathcal{K} * u$ with

$$\mathcal{K}(x) = c_d \, \frac{1}{|x|^{d-2}} \quad \text{and} \quad c_d := \frac{1}{(d-2)\sigma_d}$$

where $\sigma_d := 2\,\pi^{d/2}/\Gamma(d/2)$ is the surface area of the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. This system has been proposed as a model for chemotaxis-driven cell movement or in the study of large ensemble of gravitationally interacting particles, see [17, 12, 4] and the literature therein.

We will concentrate on a particular choice of the non-linear diffusion exponent m in any dimension characterised for producing an exact balance in the scaling of diffusion and potential drift in equation (1.1). To this end we use the by-now classical scaling leading to the nonlinear Fokker-Planck equation for porous media as in [10], that is, let us define ρ by $\rho(s,y) := \mathrm{e}^{dt} u\left(\beta(t), \mathrm{e}^t x\right)$ and $c := \mathcal{K} * \rho$ with β strictly increasing to be chosen. Then, it is straightforward to check that

$$\begin{cases} \dfrac{\partial \rho}{\partial s}(s,y) &= \operatorname{div}\left[y\rho(s,y) + \beta'(t)\left\{\mathrm{e}^{-\lambda t}\nabla\rho^m(s,y) - \mathrm{e}^{-dt}\rho(s,y)\nabla c(s,y)\right\}\right] \qquad s>0\,,\ y\in\mathbb{R}^d\,, \\ -\Delta c(s,y) &= \rho(s,y)\,, \qquad\qquad s>0\,,\ y\in\mathbb{R}^d\,, \\ \rho(0,y) &= u_0(y) \geq 0 \qquad\qquad y\in\mathbb{R}^d\,, \end{cases}$$

with $\lambda = d(m-1)+2$. From this scaling, the only possible choice of m leading to a compensation effect between diffusion and concentration is given by $\lambda = d$ or equivalently

$$m_d := \frac{2\,(d-1)}{d}\,.$$

In that case, $\beta'(t) = \mathrm{e}^{dt}$ determines the change of variables and the final scaled equation reads:

$$\begin{cases} \dfrac{\partial \rho}{\partial s}(s,y) &= \operatorname{div}\left[y\rho(s,y) + \nabla\rho^{m_d}(s,y) - \rho(s,y)\nabla c(s,y)\right] \qquad s>0\,,\ y\in\mathbb{R}^d\,, \\ -\Delta c(s,y) &= \rho(s,y)\,, \qquad\qquad s>0\,,\ y\in\mathbb{R}^d\,, \\ \rho(0,y) &= u_0(y) \geq 0 \qquad\qquad y\in\mathbb{R}^d\,, \end{cases} \tag{1.3}$$

Note that the case $d = 2$ and $m_2 = 1$ corresponds to the Patlak-Keller-Segel system or to the classical Smoluchowski-Poisson system in two dimensions with linear diffusion [29, 19]. In this case, a simple dichotomy result have been shown in [14, 4] improving over previous results in [18, 26], namely, the behaviour of the solutions is just determined by the initial mass of the system. More precisely, there exists a critical value of the mass $M_c := 8\pi$ such that if $0 < M < M_c$ (sub-critical case) the solutions exist globally and if $M > M_c$ (super-critical case) the solutions blow up in finite time. Moreover, in the sub-critical case solutions behave self-similarly as $t \to \infty$ [2, 4]. Finally, the critical case $M = M_c$ was studied in [3] showing that solutions exist globally and blow up as a Dirac mass at the centre of mass as $t \to \infty$. Solutions have to be understood as free energy solutions, concept that we will specify below.

In this work, we will show that a similar situation to the classical PKS system in $d = 2$, although with some important differences, happens for the critical variant of the PKS model in any dimension $d \geq 3$ reading as:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &= \operatorname{div}\left[\nabla u^{m_d}(t,x) - u(t,x)\nabla(\mathcal{K} * u)(t,x)\right] \qquad t>0\,,\ x\in\mathbb{R}^d\,, \\ u(0,x) &= u_0(x) \geq 0 \qquad\qquad x\in\mathbb{R}^d\,. \end{cases} \tag{1.4}$$

We will simply denote by m the critical exponent

$$m := m_d = \frac{2\,(d-1)}{d} \in (1,2)\,,$$

as long as $d \geq 3$, in the rest of the paper for notational convenience. The main tool for the analysis of this equation is the following *free energy* functional:

$$\begin{aligned} t \mapsto \mathcal{F}[u(t)] :&= \int_{\mathbb{R}^d} \frac{u^m(t,x)}{m-1} - \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{K}(x-y)\, u(t,x)\, u(t,y)\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_{\mathbb{R}^d} \frac{u^m(t,x)}{m-1} - \frac{c_d}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^{d-2}}\, u(t,x)\, u(t,y)\, \mathrm{d}x\, \mathrm{d}y \end{aligned}$$

which is related to its time derivative, *the Fisher information*, in the following way: given a smooth positive fast-decaying solution to (1.4), then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[u(t)] = -\int_{\mathbb{R}^d} u(t,x) \left| \nabla \left(\frac{m}{m-1} u^{m-1}(t,x) - \phi(t,x) \right) \right|^2 \mathrm{d}x\,. \tag{1.5}$$

We will give a precise sense to this entropy/entropy-dissipation relation below.

The system (1.4) can formally be considered a particular instance of the general family of PDEs studied in [8, 1, 9]. The free energy functional $\mathcal{F}$ structurally belongs to the general class of free energies for interacting particles introduced in [25, 8, 9]. The functionals treated in those references are of the general form:

$$\mathcal{E}[n] := \int_{\mathbb{R}^d} U[n(x)]\, \mathrm{d}x + \int_{\mathbb{R}^d} n(x)\, V(x)\, \mathrm{d}x + \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} W(x-y)\, n(x)\, n(y)\, \mathrm{d}x\, \mathrm{d}y$$

under the basic assumptions $U : \mathbb{R}^+ \to \mathbb{R}$ is a density of internal energy, $V : \mathbb{R}^d \to \mathbb{R}$ is a convex smooth confinement potential and $W : \mathbb{R}^d \to \mathbb{R}$ is a symmetric convex smooth interaction potential. The internal energy U should satisfy the following dilation condition, introduced in McCann [25]

$$\lambda \longmapsto \lambda^d U(\lambda^{-d}) \qquad \text{is convex non-increasing on } \mathbb{R}^+.$$

In our case, the interaction potential is singular and the key tool of displacement convexity of the functional fails, making the theory in the previous references not useful for our purposes. Nevertheless, the free energy functional plays a central role for this problem as we shall see below. Before proceeding further, let us state the notion of solutions we will deal with in the rest:

Definition 1.1 (Weak and free energy solution) *Let u_0 be an initial condition satisfying* (1.2) *and $T \in (0, \infty]$.*

(i) A weak solution *to* (1.4) *on $[0,T)$ with initial condition u_0 is a non-negative function $u \in C([0,T); \mathrm{L}^1(\mathbb{R}^d))$ such that $u \in \mathrm{L}^\infty((0,t) \times \mathbb{R}^d))$, $u^m \in \mathrm{L}^2(0,t; H^1(\mathbb{R}^d))$ for each $t \in [0,T)$ and*

$$\begin{aligned} \int_{\mathbb{R}^d} u_0(x)\, \psi(0,x)\, \mathrm{d}x = \int_0^T \int_{\mathbb{R}^d} [\nabla u^m(t,x) - u(t,x) \nabla \phi(t,x)] \cdot \nabla \psi(t,x)\, \mathrm{d}x\, \mathrm{d}t \\ - \int_0^T \int_{\mathbb{R}^d} u(t,x)\, \partial_t \psi(t,x)\, \mathrm{d}x\, \mathrm{d}t \end{aligned} \tag{1.6}$$

for any test function $\psi \in \mathcal{D}([0,T) \times \mathbb{R}^d)$ with $\phi = \mathcal{K} * u$.

(ii) A free energy solution *to* (1.4) *on* $[0,T)$ *with initial condition* u_0 *is a weak solution to* (1.4) *on* $[0,T)$ *with initial condition* u_0 *satisfying additionally:* $u^{(2m-1)/2} \in \mathrm{L}^2(0,t;H^1(\mathbb{R}^d))$ *and*

$$\mathcal{F}[u(t)] + \int_0^t \int_{\mathbb{R}^d} \left| \left(\frac{2m}{2m-1} \nabla u^{(2m-1)/2}(s,x) - u^{1/2}(s,x) \nabla \phi(s,x) \right) \right|^2 \, \mathrm{d}x \, \mathrm{d}s \le \mathcal{F}[u_0] \quad (1.7)$$

for all $t \in (0,T)$ *with* $\phi = \mathcal{K} * u$.

In (1.7), we cannot write the Fisher information factorised by u as in (1.5) because of the lack of regularity of u. We note that both (1.6) and (1.7) are meaningful. Indeed, the regularity required for u implies that the solution $\phi = \mathcal{K} * u$ to the Poisson equation satisfies $\phi \in \mathrm{L}^\infty(0,t;H^1(\mathbb{R}^d))$ for all $t \in (0,T)$. In addition, it follows from (1.6) by classical approximation arguments that

$$\|u(t)\|_1 = \int_{\mathbb{R}^d} u(t,x) \, \mathrm{d}x = \int_{\mathbb{R}^d} u_0(x) \, \mathrm{d}x = \|u_0\|_1 = M \quad \text{for} \quad t \in [0,T) \, . \quad (1.8)$$

Let us point out that the existence of free energy solutions for a related problem was essentially obtained in [30, 31, 27] where the Poisson equation is replaced by $-\Delta\phi = u - \phi$. There, the authors also show that the mass is the suitable quantity for (1.4) allowing for a dichotomy. Precisely, the author shows that there exist two masses $0 < M_1 < M_2$ such that if $0 < M < M_1$ the solutions exist globally in time, while for $M > M_2$ there are solutions blowing up in finite time. The values of these masses, are related to the sharp constants of the Sobolev inequality.

Here, we will make a fundamental use of a *variant to the Hardy-Littlewood-Sobolev (VHLS) inequality*, see Lemma 3.2: for all $h \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$, there exists an optimal constant C_* such that

$$\left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{h(x)\, h(y)}{|x-y|^{d-2}} \, \mathrm{d}x \, \mathrm{d}y \right| \le C_* \, \|h\|_m^m \, \|h\|_1^{2/d} \, .$$

This inequality will play the same role as the logarithmic HLS inequality proved in [6] for the classical PKS system in $d = 2$ [14, 4, 3]. The VHLS inequality and the identification of the equality cases allow us to give the first main result of this work, namely, the following sharp critical mass

$$M_c := \left[\frac{2}{(m-1)\, C_* \, c_d} \right]^{d/2}$$

for equation (1.4). More precisely, we will show that free energy solutions *exist globally for* $M \in (0, M_c]$ while there are finite time blowing-up solutions otherwise. However, the long time asymptotics of the solutions is much more complicated compared to the classical PKS system in two dimensions. The main results of this work and the open problems related to large times asymptotics can be summarised as follows:

- Sub-critical case: $0 < M < M_c$, solutions exist globally in time and there exists a radially symmetric compactly supported self-similar solution, although we are not able to show that it attracts all global solutions. See Proposition 4.3, Theorem 5.2 and Corollary 5.7.
- Critical case: $M = M_c$, solutions exist globally in time, see Proposition 4.6. There are infinitely many compactly supported stationary solutions. The second moment of solutions is non-decreasing in time, with two possibilities we cannot exclude: either is uniformly bounded in time or diverges. Moreover, the L^m-norm of the solution could be divergent as $t \to \infty$ or a diverging sequence of times could exist with bounded L^m-norm. However, we show a striking difference with respect to the classical PKS system in two dimensions [3], namely, the existence of global in time solutions not blowing-up in infinite time. We will comment further on these issues in Section 4.2.3.
- Super-critical case: $M > M_c$, we prove that there exist solutions, corresponding to initial data with negative free energy, blowing up in finite time, see Proposition 4.2. However, we cannot exclude the possibility that solutions with positive free energy may be global in time.

The results are organised as follows. Section 2 shows a key maximal time of existence criterion for free energy solutions of equation (1.4). This criterion improves over the results in [30, 31] since it is only based on the boundedness or unboundedness in time of the L^m-norm of the solutions and it has to be compared to a similar criterion based on the logarithmic entropy in the classical PKS system in two dimensions obtained in [3]. Section 3 is devoted to the variational study of the minimisation of the free energy functional over the set of densities with a fixed mass. With that aim the proof of the VHLS inequality and the identification of the equality cases are performed. Section 4 uses this variational information to show the above main results concerning the dichotomy, the global existence for $M < M_c$ and the characterisation by concentration-compactness techniques of the nature of the possible blow-up in the critical case leading to the global existence for this critical value. Finally, the last section is devoted to the study of the free energy functional in self-similar variables and the proof of the existence of self-similar solutions in the sub-critical case.

2 Existence criterion

As in [30, 31], we consider the regularised problem

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t}(t,x) &= \operatorname{div}\left[\nabla\left(f_\varepsilon \circ u_\varepsilon\right)(t,x) - u_\varepsilon(t,x)\nabla\phi_\varepsilon(t,x)\right] \qquad t>0\,,\ x\in\mathbb{R}^d\,, \\ \phi_\varepsilon(t,x) &= \mathcal{K} * u_\varepsilon(t,x)\,, \qquad t>0\,,\ x\in\mathbb{R}^d\,, \\ u_\varepsilon(0,x) &= u_0^\varepsilon \geq 0 \qquad x\in\mathbb{R}^d\,, \end{cases} \tag{2.1}$$

where $f_\varepsilon : [0,\infty) \longrightarrow \mathbb{R}$ is given by $f_\varepsilon(u) := (u+\varepsilon)^m - \varepsilon^m$. Here, u_0^ε is the convolution of u_0 with a sequence of mollifiers and $\|u_0^\varepsilon\|_1 = \|u_0\|_1 = M$ in particular. This regularised problem has global in time smooth solutions. This approximation has been proved to be convergent. More precisely, the result in [31, Section 4] asserts that if we assume that

$$\sup_{0<t<T} \|u_\varepsilon(t)\|_\infty \leq \kappa \tag{2.2}$$

where κ is independent of $\varepsilon > 0$, then there exists a sub-sequence $\varepsilon_n \to 0$, such that

$$u_{\varepsilon_n} \to u \qquad \text{strongly} \qquad \text{in } C([0,T],\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)) \text{ and a.e. in } (0,T)\times\mathbb{R}^d, \tag{2.3}$$

$$\nabla u^m_{\varepsilon_n} \rightharpoonup \nabla u^m \qquad \text{weakly-*} \qquad \text{in } \mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^d)), \tag{2.4}$$

$$\phi_{\varepsilon_n}(t) \to \phi(t) \qquad \text{strongly} \qquad \text{in } \mathrm{L}^r_{\mathrm{loc}}(\mathbb{R}^d) \text{ a.e. in } (0,T), \tag{2.5}$$

$$\nabla\phi_{\varepsilon_n}(t) \to \nabla\phi(t) \qquad \text{strongly} \qquad \text{in } \mathrm{L}^r_{\mathrm{loc}}(\mathbb{R}^d) \text{ a.e. in } (0,T), \tag{2.6}$$

for any $p \in (1,\infty)$ and $r \in (1,\infty]$, and u is a weak solution to (1.4) on $[0,T)$ with $\phi = \mathcal{K} * u$. Moreover, the free energy [30, Proposition 6.1] satisfies $\mathcal{F}[u(t)] \leq \mathcal{F}[u_0]$ for a.e. $t \geq 0$. However, a detailed analysis of the proof in [30, Proposition 6.1] shows that the weak solution is in fact a free energy solution.

Proposition 2.1 (Existence of free energy solutions) *Under assumption* (1.2) *on the initial data and* (2.2) *on the approximation sequence, there exists a free energy solution to* (1.4) *in* $[0,T)$.

Proof. The only remaining points not covered by the results in [30, 31] are the lower semi-continuity of the free energy dissipation and the fact that $u^{(2m-1)/2}$ belongs to $L^2(0,t;H^1(\mathbb{R}^d))$

for $t \in [0,T)$. The latter will actually be shown in the proof of Lemma 2.3, see (2.9) below. Concerning the former, a careful reading of the proof of [30, Proposition 6.1] gives that

$$\mathcal{F}_{\varepsilon,l}[u_\varepsilon(t)]+\frac{3}{4}\int_0^t\int_{\mathbb{R}^d}[u_\varepsilon(s,x)+\varepsilon]\left|\nabla\left(\frac{m}{m-1}[u_\varepsilon(s,x)+\varepsilon]^{m-1}-\phi_\varepsilon(s,x)\right)\right|^2\psi_l(x)\,\mathrm{d}x\,\mathrm{d}t\le\mathcal{F}_{\varepsilon,l}[u_0]$$

for a.e. $t\in(0,T)$ where ψ_l is a standard cut-off function in $\mathbb{R}^d$ for any $l\in\mathbb{N}$ and

$$\mathcal{F}_{\varepsilon,l}[u_\varepsilon(t)]=\int_{\mathbb{R}^d}\frac{[u_\varepsilon(t,x)+\varepsilon]^m}{m-1}\psi_l(x)\,\mathrm{d}x-\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\mathcal{K}(x-y)\,u_\varepsilon(t,x)\,u_\varepsilon(t,y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

In this regularised setting, we can write that

$$(u_\varepsilon+\varepsilon)\left|\nabla\left[\frac{m}{m-1}(u_\varepsilon+\varepsilon)^{m-1}-\phi_\varepsilon\right]\right|^2=\left|\frac{2m}{2m-1}\,\nabla(u_\varepsilon+\varepsilon)^{(2m-1)/2}-(u_\varepsilon+\varepsilon)^{1/2}\nabla\phi_\varepsilon\right|^2\,.$$

As proved in [30], we have $\mathcal{F}_{\varepsilon_n,l}[u_{\varepsilon_n}(t)]\to\mathcal{F}[u(t)]$ as $\varepsilon_n\to0$ and $l\to\infty$. In addition, it is straightforward from the convergence properties (2.3)-(2.6) above to pass to the limit as $\varepsilon_n\to0$ in the free energy dissipation functional with the help of a lower semi-continuity argument. We leave the details to the reader, see *e.g.* [28] or [7, Lemma 10]. Hence, passing to the limit as $l\to\infty$, then u is a free energy solution as it satisfies the free energy inequality (1.7). □

Remark 2.2 *The free energy inequality* (1.7) *can be obtained with constant* $3/4$ *multiplying the entropy dissipation directly from* [30, Proposition 6.1] *and the procedure above. This is a technical issue that can be improved to constant* 1 *by redoing the proof in* [30, Proposition 6.1] *treating more carefully the free energy dissipation term. In fact, the proof in* [30, Proposition 6.1] *shows that you can choose the constant as close to* 1 *as you want.*

We are now ready to characterise the maximal time of existence by showing the local in time boundedness of the L^m-norm independently of the approximation parameter $\varepsilon>0$ and how this estimate implies the local in time L^∞-estimate (2.2).

Lemma 2.3 (From uniform integrability to L^∞-bounds) *For any* $\eta>0$ *there exists* $\tau_\eta>0$ *depending only on* d, M, *and* η *such that, if*

$$\sup_{\varepsilon\in(0,1)}\|u_\varepsilon(t^*)\|_m\le\eta$$

for some $t^*\in[0,\infty)$, *then*

(i) the family $(u_\varepsilon)_\varepsilon$ *is bounded in* $\mathrm{L}^\infty(t^*,t^*+\tau_\eta;\mathrm{L}^m(\mathbb{R}^d))$.

(ii) Moreover, if $(u_\varepsilon(t^*))_\varepsilon$ *is also bounded in* $\mathrm{L}^p(\mathbb{R}^d)$ *for some* $p\in(m,\infty]$, *then* $(u_\varepsilon)_\varepsilon$ *is bounded in* $\mathrm{L}^\infty(t^*,t^*+\tau_\eta;\mathrm{L}^p(\mathbb{R}^d))$.

Proof. To prove this result we need to refine the argument already used in the two-dimensional situation $d=2$ with linear diffusion $m=1$ in [4, 3]. We follow a procedure analogous to the ones in [20, 5, 30, 13].

Step 1 - L^m-estimates: By (2.1) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon\|_m^m&=-m(m-1)\int_{\mathbb{R}^d}u_\varepsilon^{m-2}\,\nabla u_\varepsilon\cdot\left(m(u_\varepsilon+\varepsilon)^{m-1}\,\nabla u_\varepsilon-u_\varepsilon\,\nabla\phi_\varepsilon\right)\,\mathrm{d}x\\&\le-\frac{4m^2(m-1)}{(2m-1)^2}\left\|\nabla u_\varepsilon^{(2m-1)/2}\right\|_2^2-(m-1)\int_{\mathbb{R}^d}u_\varepsilon^m\,\Delta\phi_\varepsilon\,\mathrm{d}x\\&=-\frac{4m^2(m-1)}{(2m-1)^2}\left\|\nabla u_\varepsilon^{(2m-1)/2}\right\|_2^2+(m-1)\,\|u_\varepsilon\|_{m+1}^{m+1}\,.\end{aligned}$$

As

$$1 < \frac{2m}{2m-1} < \frac{2(m+1)}{2m-1} < \frac{2d}{d-2}\,,$$

we have the following Gagliardo-Nirenberg-Sobolev inequality: there exists a positive constant C such that

$$\|w\|_{2(m+1)/(2m-1)} \le C\,\|\nabla w\|_2^{[(2m-1)d]/[(m+1)(2m+d-2)]}\,\|w\|_{2m/(2m-1)}^{2m^2/[(m+1)(2m+d-2)]}$$

which we apply with $w = u_\varepsilon^{(2m-1)/2}$ to obtain

$$\|u_\varepsilon\|_{m+1}^{(2m-1)/2} \le C\,\left\|\nabla u_\varepsilon^{(2m-1)/2}\right\|_2^{[(2m-1)d]/[(m+1)(2m+d-2)]}\,\|u_\varepsilon\|_m^{m^2(2m-1)/[(m+1)(2m+d-2)]}\,.$$

It leads to

$$\begin{aligned}\|u_\varepsilon\|_{m+1}^{m+1} &\le C\,\left\|\nabla u_\varepsilon^{(2m-1)/2}\right\|_2^{2d/(2m+d-2)}\,\|u_\varepsilon\|_m^{2m^2/(2m+d-2)}\\ &\le \frac{2m^2}{(2m-1)^2}\,\left\|\nabla u_\varepsilon^{(2m-1)/2}\right\|_2^2 + C\,\|u_\varepsilon\|_m^{m^2/(m-1)}\,.\end{aligned}$$

We thus end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon\|_m^m + \frac{2m^2(m-1)}{(2m-1)^2}\,\left\|\nabla u_\varepsilon^{(2m-1)/2}\right\|_2^2 \le (m-1)\,C\,\|u_\varepsilon\|_m^{m^2/(m-1)}\,. \tag{2.7}$$

In particular, for any $t_2 \ge t_1 \ge 0$

$$\|u_\varepsilon(t_2)\|_m^m \le \left[\|u_\varepsilon(t_1)\|_m^{-m/(m-1)} - C\,(t_2-t_1)\right]^{-(m-1)} \tag{2.8}$$

Taking $t_1 = t^*$ we deduce from (2.8) that

$$\|u_\varepsilon(t)\|_m^m \le \left(\eta^{-m/(m-1)} - C\,(t-t^*)\right)^{-(m-1)} \qquad \text{for } t \in [t^*, t^*+2\tau_\eta)$$

with $\tau_\eta = 1/\left(2C\eta^{m/(m-1)}\right)$. Consequently, $\|u_\varepsilon(t)\|_m^m \le (C\tau_\eta)^{-(m-1)}$ for $t \in [t^*, t^*+\tau_\eta]$ and the proof of the first assertion of Lemma 2.3 is complete. In addition, coming back to (2.7), we further deduce that

$$\int_{t^*}^{t^*+\tau_\eta} \left\|\nabla u_\varepsilon^{(2m-1)/2}\right\|_2^2 \le C(t^*,\eta)\,. \tag{2.9}$$

Step 2 - L^p-estimates, $p \in (m,\infty)$: For $t \in [t^*, t^*+\tau_\eta]$, $K \ge 1$, and $p > m$, we infer from (2.1) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\|(u_\varepsilon(t)-K)_+\|_p^p \le & -m\,p\,(p-1)\int_{\mathbb{R}^d}(u_\varepsilon(t)-K)_+^{p-2}\,(u_\varepsilon+\varepsilon)^{m-1}\,|\nabla u_\varepsilon|^2\,\mathrm{d}x\\ & + p\,(p-1)\int_{\mathbb{R}^d}\left[(u_\varepsilon(t)-K)_+^{p-1} + K\,(u_\varepsilon(t)-K)_+^{p-2}\right]\,\nabla u_\varepsilon\cdot\nabla\phi_\varepsilon\,\mathrm{d}x\\ \le & -m\,p\,(p-1)\int_{\mathbb{R}^d}(u_\varepsilon(t)-K)_+^{m+p-3}\,|\nabla u_\varepsilon|^2\,\mathrm{d}x\\ & -\int_{\mathbb{R}^d}\left[(p-1)\,(u_\varepsilon(t)-K)_+^p + p\,K\,(u_\varepsilon(t)-K)_+^{p-1}\right]\,\Delta\phi_\varepsilon\,\mathrm{d}x\\ \le & -\frac{4\,m\,p\,(p-1)}{(m+p-1)^2}\left\|\nabla\left[(u_\varepsilon(t)-K)_+^{(m+p-1)/2}\right]\right\|_2^2 + \mathrm{(I)}\end{aligned}$$

with

$$\text{(I)} := p\,K^2\,\|(u_\varepsilon(t)-K)_+\|_{p-1}^{p-1} + (2p-1)\,K\,\|(u_\varepsilon(t)-K)_+\|_p^p + (p-1)\,\|(u_\varepsilon(t)-K)_+\|_{p+1}^{p+1}\,.$$

We now use the following interpolation inequality

$$\|w\|_{p+1}^{p+1} \le C(p)\,\left\|\nabla\left(w^{(m+p-1)/2}\right)\right\|_2^2\,\|w\|_1^{2/d}$$

which is a consequence of the Gagliardo-Nirenberg-Sobolev and Hölder inequalities (see, e.g., [31, Lemma 3.2]) to obtain

$$\begin{aligned}\text{(I)} &\le (p-1)\,K^2\,\|(u_\varepsilon(t)-K)_+\|_p^p + K^2\,|\{x\,:\,u_\varepsilon(t,x)\ge K\}| + (2p-1)\,K\,\|(u_\varepsilon(t)-K)_+\|_p^p \\ &+ C(p)\,\left\|\nabla\left[(u_\varepsilon(t)-K)_+^{(m+p-1)/2}\right]\right\|_2^2\,\|(u_\varepsilon(t)-K)_+\|_1^{2/d}\,.\end{aligned}$$

Noting that

$$\|(u_\varepsilon(t)-K)_+\|_1 \le \|u_\varepsilon(t)\|_m\,\left(\frac{\|u_\varepsilon(t)\|_1}{K}\right)^{(m-1)/m}$$

and recalling that $\|u_\varepsilon(t)\|_1 = M$ we conclude that

$$\begin{aligned}\text{(I)} &\le C(p)\,\frac{\|u_\varepsilon(t)\|_m^{2/d}}{K^{2(m-1)/m\,d}}\,\left\|\nabla\left[(u_\varepsilon(t)-K)_+^{(m+p-1)/2}\right]\right\|_2^2 \\ &+ K\,[2p-1+(p-1)\,K]\,\|(u_\varepsilon(t)-K)_+\|_p^p + K\,M\,.\end{aligned}$$

By Step 1, we may choose $K = K_*$ large enough such that

$$C\,\frac{\|u_\varepsilon(t)\|_m^{2/d}}{K_*^{2(m-1)/m\,d}} \le \frac{4\,m\,p\,(p-1)}{(m+p-1)^2}$$

for all $t\in[t^*,t^*+\tau_\eta]$ and $\varepsilon\in(0,1)$, hence

$$\text{(I)} \le \frac{4\,m\,p\,(p-1)}{(m+p-1)^2}\,\left\|\nabla\left[(u_\varepsilon(t)-K_*)_+^{(m+p-1)/2}\right]\right\|_2^2 + C(p,t^*,\eta)\,\left[1+\|(u_\varepsilon(t)-K_*)_+\|_p^p\right]\,.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|(u_\varepsilon(t)-K_*)_+\|_p^p \le C(p,t^*,\eta)\,\left[1+\|(u_\varepsilon(t)-K_*)_+\|_p^p\right]\,,$$

so that

$$\|(u_\varepsilon(t)-K_*)_+\|_p^p \le C(p,t^*,\eta) \quad \text{for} \quad t\in[t^*,t^*+\tau_\eta] \quad \text{and} \quad \varepsilon\in(0,1)\,.$$

As

$$\|u_\varepsilon(t)\|_p^p \le C(p)\,\left(K_*^{p-m}\,\|u_\varepsilon(t)\|_m^m + \|(u_\varepsilon(t)-K_*)_+\|_p^p\right)\,,$$

the previous inequality and Step 1 warrant that

$$\|u_\varepsilon(t)\|_p \le C(p,t^*,\eta) \quad \text{for} \quad t\in[t^*,t^*+\tau_\eta] \quad \text{and} \quad \varepsilon\in(0,1)\,.$$

Step 3 - L^∞-estimates: As a direct consequence of Step 2 with $p = d+1$ and Morrey's embedding theorem $(\nabla\phi_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^\infty((t^*,t^*+\tau_\eta)\times\mathbb{R}^d;\mathbb{R}^d)$. This property in turn implies that $(u_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^\infty((t^*,t^*+\tau_\eta)\times\mathbb{R}^d)$ and we refer to [5, Lemma 3.2] and [20] for a proof (see also [31, Section 5] and [30, Theorem 1.2] for alternative arguments). □

As a consequence of the previous lemma, we are able to construct a free energy solution defined on a maximal existence time.

Theorem 2.4 (Maximal free energy solution) *Under assumption* (1.2) *on the initial condition there are* $T_\omega \in (0,\infty]$ *and a free energy solution* u *to* (1.4) *on* $[0,T_\omega)$ *with the following alternative: Either* $T_\omega = \infty$ *or* $T_\omega < \infty$ *and* $\|u(t)\|_m \to \infty$ *as* $t \nearrow T_\omega$. *Furthermore there exists a positive constant* C_0 *depending only on* d *such that* u *satisfies*

$$\|u(t_2)\|_m^m \le \left(\|u(t_1)\|_m^{-m/(m-1)} - C_0\,(t_2 - t_1) \right)^{-(m-1)} \tag{2.10}$$

for $t_1 \in [0,T_\omega)$ *and* $t_2 \in (t_1, T_\omega)$.

Proof. We put $\xi_p(t) = \sup_{\varepsilon\in(0,1)} \|u_\varepsilon(t)\|_p \in (0,\infty]$ for $t \ge 0$ and $p \in [m,\infty]$ and

$$T_1 = \sup\left\{ T > 0 \,:\, \xi_m \in \mathrm{L}^\infty(0,T) \right\} .$$

Clearly the definition of the sequence $(u_0^\varepsilon)_\varepsilon$ and (1.2) ensure that $\xi_p(0)$ is finite for all $p \in [m,\infty]$. By Lemma 2.3 there exists $t_1 > 0$ such that ξ_p is bounded on $[0,t_1]$ for all $p \in [m,\infty]$. Then (2.2) is fulfilled for $T = t_1$ and there is a free energy solution to (1.4) on $[0,t_1)$ by Proposition 2.1 and (2.9). This ensures in particular that $T_1 \ge t_1 > 0$.

We next claim that

$$\xi_\infty \in \mathrm{L}^\infty(0,T) \quad \text{for any } T \in [0,T_1) . \tag{2.11}$$

Indeed, consider $T_1^\infty = \sup\{T \in (0,T_1) \,:\, \xi_\infty \in \mathrm{L}^\infty(0,T)\}$ and assume for contradiction that $T_1^\infty < T_1$. Then ξ_m belongs to $\mathrm{L}^\infty(0,T_1^\infty)$ and we put $\eta = \|\xi_m\|_{\mathrm{L}^\infty(0,T_1^\infty)}$ and $t^* = T_1^\infty - (\tau_\eta/2)$, τ_η being defined in Lemma 2.3. As $\xi_m(t^*) \le \eta$ and $\xi_\infty(t^*)$ is finite we may apply Lemma 2.3 to deduce that both ξ_m and ξ_∞ belong to $\mathrm{L}^\infty(t^*, t^* + \tau_\eta)$, the latter property contradicting the definition of T_1^∞ as $t^* + \tau_\eta = T_1^\infty + (\tau_\eta/2)$.

Now, thanks to (2.11), (2.2) is fulfilled for any $T \in [0,T_1)$ and the existence of a free energy solution u to (1.4) on $[0,T_1)$ follows from Proposition 2.1 and (2.9). Moreover, either $T_1 = \infty$ or $T_1 < \infty$ and $\|u(t)\|_m \to \infty$ as $t \nearrow T_1$, and the proof of Theorem 2.4 is complete with $T_\omega = T_1$. Or $T_1 < \infty$ and

$$\liminf_{t\to T_1} \|u(t)\|_m < \infty .$$

In that case, there are $\eta > 0$ and an increasing sequence of positive real numbers $(s_j)_{j\ge1}$ such that $s_j \to T_1$ as $j \to \infty$ and $\|u(s_j)\|_m \le \eta$. Fix $j_0 \ge 1$ such that $s_{j_0} \ge T_1 - (\tau_\eta/2)$ with τ_η defined in Lemma 2.3 and put $\tilde{u}_0 = u(s_{j_0})$; According to Definition 1.1 and (2.4) $\tilde{u}_0$ fulfils (1.2) and we may proceed as above to obtain a free energy solution $\tilde{u}$ to (1.4) on $[0,T_2)$ for some $T_2 \ge \tau_\eta$. Setting $\bar{u}(t) = u(t)$ for $t \in [0,s_{j_0}]$ and $\bar{u}(t) = \tilde{u}(t - s_{j_0})$ for $t \in [s_{j_0}, s_{j_0} + T_2)$ we first note that $\bar{u}$ is a free energy solution to (1.4) on $[0, s_{j_0} + T_2)$ and a true extension of u as $s_{j_0} + T_2 \ge T_1 - (\tau_\eta/2) + \tau_\eta \ge T_1 + (\tau_\eta/2)$. We then iterate this construction as long as the alternative stated in Theorem 2.4 is not fulfilled to complete the proof.

Thanks to the regularity of weak solutions we may next proceed as in the proof of (2.8) to deduce (2.10). □

Corollary 2.5 (Lower bound on the blow-up time) *Let* u *be a free energy solution to* (1.4) *on* $[0,T_\omega)$ *with an initial condition* u_0 *satisfying* (1.2). *If* T_ω *is finite, then*

$$\|u(t)\|_m \ge \left[C_0\,(T_\omega - t) \right]^{-(m-1)/m} ,$$

where C_0 *is defined in Theorem* 2.4.

Proof. Let $t \in (0,T_\omega)$ and $t_2 \in (t,T_\omega)$. By (2.10), we have

$$\|u(t_2)\|_m^{-m/(m-1)} \ge \|u(t)\|_m^{-m/(m-1)} - C_0\,(t_2 - t) .$$

Letting t_2 going to T_ω gives

$$0 \ge \|u(t)\|_m^{-m/(m-1)} - C_0\,(T_\omega - t) ,$$

hence the expected result. □

3 The free energy functional $\mathcal{F}$

As we have just seen in the existence proof, the existence time of a free energy solution to (1.4) heavily depends on the behaviour of its L^m-norm. As the free energy $\mathcal{F}$ involves the L^m-norm, the information given $\mathcal{F}$ will be of paramount importance. Let us then proceed to a deeper study of this functional.

Lemma 3.1 (Scaling properties of the free energy) *Given $h \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$, let us define $h_\lambda(x) := \lambda^d h(\lambda\, x)$, then*

$$\mathcal{F}[h_\lambda] = \lambda^{d-2}\mathcal{F}[h] \qquad \text{for all } \lambda \in (0,\infty)\,.$$

Proof. We have

$$\begin{aligned}\mathcal{F}[h_\lambda] &= \frac{1}{m-1}\int_{\mathbb{R}^d} \lambda^{2d-2} h(\lambda\, x)^m \,\mathrm{d}x - \frac{c_d}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \lambda^{2d}\frac{h(\lambda\, x)\, h(\lambda\, y)}{|x-y|^{d-2}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{\lambda^{d-2}}{m-1}\int_{\mathbb{R}^d} h(x)^m \,\mathrm{d}x - \frac{c_d\,\lambda^{d-2}}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{h(x)\, h(y)}{|x-y|^{d-2}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \lambda^{d-2}\mathcal{F}[h]\,,\end{aligned}$$

giving the announced scaling property. □

We next establish a *variant to the Hardy-Littlewood-Sobolev (VHLS) inequality*:

Lemma 3.2 (VHLS inequality) *For $h \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$ we put*

$$\mathcal{W}(h) := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{h(x)\, h(y)}{|x-y|^{d-2}}\,\mathrm{d}x\,\mathrm{d}y\,.$$

Then

$$C_* := \sup_{h\neq 0}\left\{\frac{\mathcal{W}(h)}{\|h\|_1^{2/d}\,\|h\|_m^m}\,,\ h \in \mathrm{L}^1(\mathbb{R}^d)\cap\mathrm{L}^m(\mathbb{R}^d)\right\} < \infty\,. \tag{3.1}$$

First recall the Hardy-Littlewood-Sobolev (HLS) inequality, see [22, Theorem 4.3], hich states that if

$$\frac{1}{p}+\frac{1}{q}+\frac{\lambda}{d} = 2 \quad \text{and} \quad 0<\lambda<d\,,$$

then for all $f \in \mathrm{L}^p(\mathbb{R}^d)$, $g \in \mathrm{L}^q(\mathbb{R}^d)$, there exists a sharp positive constant $C_{\mathrm{HLS}} > 0$, given by [21], which only depends on p, q and λ such that

$$\left|\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f(x)\, g(y)}{|x-y|^{\lambda}}\,\mathrm{d}x\,\mathrm{d}y\right| \le C_{\mathrm{HLS}}\,\|f\|_p\,\|g\|_q\,. \tag{3.2}$$

Proof of Lemma 3.2. Consider $h \in \mathrm{L}^1(\mathbb{R}^d)\cap\mathrm{L}^m(\mathbb{R}^d)$. Applying the HLS inequality (3.2) with $p = q = 2d/(d+2)$ and $\lambda = d-2$, and then the Hölder inequality with $1 < p = 2d/(d+2) < m$, we obtain

$$|\mathcal{W}(h)| = \left|\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{h(x)\, h(y)}{|x-y|^{d-2}}\,\mathrm{d}x\,\mathrm{d}y\right| \le C_{\mathrm{HLS}}\,\|h\|_p^2 \le C_{\mathrm{HLS}}\,\|h\|_1^{2/d}\,\|h\|_m^m\,.$$

Consequently, C_* is finite and bounded from above by C_{HLS}. □

We next turn to the existence of maximisers for the VHLS inequality which can be proved by similar arguments as for the classical HLS inequality in [21, Theorem 2.5].

Lemma 3.3 (Extremals of the VHLS inequality) *There exists a non-negative, radially symmetric and non-increasing function $P_* \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$ such that $\mathcal{W}(P_*) = C_*$ with $\|P_*\|_1 = \|P_*\|_m = 1$.*

Proof. Define

$$\Lambda(h) := \frac{\mathcal{W}(h)}{\|h\|_1^{2/d}\,\|h\|_m^m} \quad \text{for } h \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)\ ,$$

and consider a maximising sequence $(p_j)_j$ in $\mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$, that is

$$\lim_{j\to\infty} \Lambda(p_j) = C_*\ . \tag{3.3}$$

Step 1 - We first prove that we may assume that p_j is a non-negative, radially symmetric, non-increasing function such that $\|p_j\|_1 = \|p_j\|_m = 1$ for any $j \ge 0$. Indeed, $\Lambda(p_j) \le \Lambda(|p_j|)$ so that $(|p_j|)_j$ is also a maximising sequence. Next, let us introduce $\tilde{p}_j(x) := \lambda_j |p_j(\mu_j x)|$ with $\mu_j := (\|p_j\|_1/\|p_j\|_m)^{m/[d(m-1)]}$ and $\lambda_j := \mu_j^d/\|p_j\|_1$. A direct computation shows that $\Lambda(\tilde{p}_j) = \Lambda(|p_j|)$ and $\|\tilde{p}_j\|_1 = \|\tilde{p}_j\|_m = 1$. Finally, denoting by p_j^* the symmetric decreasing rearrangement of $\tilde{p}_j$, we infer from the Riesz rearrangement properties [21, Lemma 2.1] that

$$\Lambda(p_j^*) = \mathcal{W}(p_j^*) \ge \mathcal{W}(\tilde{p}_j) = \Lambda(\tilde{p}_j) = \Lambda(|p_j|)\ .$$

Consequently, $\left(p_j^*\right)_j$ is also a maximising sequence and the first step is proved.

Step 2 - Let us now prove that the supremum is achieved. For $k \in \{1, m\}$, the monotonicity and the non-negativity of p_j imply that

$$1 = \|p_j\|_k^k = d\,|B(0,1)| \int_0^\infty r^{d-1} p_j^k(r)\ \mathrm{d}r \ge d\,|B(0,1)|\,p_j^k(R) \int_0^R r^{d-1}\,\mathrm{d}r \ge |B(0,1)|\,R^d\,p_j^k(R)\ .$$

So that

$$0 \le p_j(R) \le b(R) := C_1\,\inf\{R^{-d/m};\, R^{-d}\} \quad \text{for } \ R > 0\ . \tag{3.4}$$

Now, we use once more the monotonicity of the p_j's and their boundedness in (R,∞) for any $R > 0$ to deduce from Helly's theorem that there are a sub-sequence of $(p_j)_j$ (not relabelled) and a non-negative and non-increasing function P_* such that $(p_j)_j$ converges to P_* point-wisely. In addition, as $1 < 2d/(d+2) < m$, $x \mapsto b(|x|)$ belongs to $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$ while the HLS inequality (3.2) warrants that

$$(x,y) \mapsto b(|x|)\,b(|y|)\,|x-y|^{-(d-2)} \in \mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)\ .$$

Together with (3.4) and the point-wise convergence of $(p_j)_j$, this implies that

$$\lim_{j\to\infty} \mathcal{W}(p_j) = \mathcal{W}(P_*)$$

by the Lebesgue dominated convergence theorem. Consequently, $\mathcal{W}(P_*) = C_*$ and thus $P_* \neq 0$. In addition, the point-wise convergence of $(p_j)_j$ and Fatou's lemma ensure $\|P_*\|_1 \le 1$ and $\|P_*\|_m \le 1$. Therefore $\Lambda(P_*) \ge C_*$ and using (3.3) we conclude that $\Lambda(P_*) = C_*$. This in turn implies that $\|P_*\|_1 = \|P_*\|_m = 1$. □

We are now in a position to begin the study of the free energy functional $\mathcal{F}$. To this end, let us define the *critical mass* M_c by

$$M_c := \left[\frac{2}{(m-1)\,C_*\,c_d}\right]^{d/2}\ . \tag{3.5}$$

Next, for $M > 0$, we put

$$\mu_M := \inf_{h \in \mathcal{Y}_M} \mathcal{F}[h] \quad \text{where} \quad \mathcal{Y}_M := \{h \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d) \ : \ \|h\|_1 = M\} \ ,$$

and first identify the values of μ_M as a function of $M > 0$.

Proposition 3.4 (Infimum of the free energy) *We have*

$$\mu_M = \begin{cases} 0 & \text{if } M \in (0, M_c], \\ -\infty & \text{if } M > M_c. \end{cases} \tag{3.6}$$

Moreover,

$$\frac{C_* \, c_d}{2} \, \left(M_c^{2/d} - M^{2/d} \right) \, \|h\|_m^m \le \mathcal{F}[h] \le \frac{C_* \, c_d}{2} \, \left(M_c^{2/d} + M^{2/d} \right) \, \|h\|_m^m \tag{3.7}$$

for $h \in \mathcal{Y}_M$. Furthermore, the infimum μ_M is not achieved if $M < M_c$ while there exists one minimiser of $\mathcal{F}$ in $\mathcal{Y}_{M_c}$.

Proof. Consider $h \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$. By the VHLS inequality (3.1),

$$\mathcal{F}[h] \ge \left(\frac{1}{m-1} - \frac{C_* \, c_d}{2} M^{2/d} \right) \|h\|_m^m \ge \frac{C_* \, c_d}{2} \left(M_c^{2/d} - M^{2/d} \right) \|h\|_m^m \ ,$$

and

$$\mathcal{F}[h] \le \left(\frac{1}{m-1} + \frac{C_* \, c_d}{2} M^{2/d} \right) \|h\|_m^m \le \frac{C_* \, c_d}{2} \left(M_c^{2/d} + M^{2/d} \right) \|h\|_m^m \ ,$$

hence (3.7).

Case $M \le M_c$ **-** By (3.7), $\mathcal{F}$ is non-negative, so that $\mu_M \ge 0$. Choosing

$$h_*(t,x) = \frac{M}{(2\,\pi\,t)^{d/2}} \, \mathrm{e}^{-|x|^2/(4t)} \ ,$$

then

$$\|h_*(t)\|_1 = M \quad \text{and} \quad \|h_*(t)\|_m^m = O\left(t^{-d(m-1)/2} \right) \ .$$

Therefore $h_*(t)$ belongs to $\mathcal{Y}_M$ for each $t > 0$ and it follows from (3.7) that $\mathcal{F}[h_*(t)] \to 0$ as $t \to \infty$. The infimum μ_M of $\mathcal{F}$ on $\mathcal{Y}_M$ is thus non-positive, hence $\mu_M = 0$.

Finally, in the case $M < M_c$, $\mu_M = 0$ and (3.7) imply that the infimum of $\mathcal{F}$ in $\mathcal{Y}_M$ is not achieved. If $M = M_c$ and $p \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$ satisfies $\mathcal{W}(p) = C_* \, \|p\|_m^m \, \|p\|_1^{2/d}$ (such a function exists by Lemma 3.3), then

$$\tilde{p}(x) := M_c^{-d/(d-2)} p\left(x M_c^{-m/(d-2)} \right)$$

belongs to $\mathcal{Y}_{M_c}$ with $\|\tilde{p}\|_m = 1$ and $\mathcal{W}(\tilde{p}) = C_* \, M_c^{2/d}$. Therefore, $\mathcal{F}[\tilde{p}] = 0$ and we have thus proved that suitably rescaled extremals of the VHLS inequality (3.1) are minimisers for $\mathcal{F}$ in $\mathcal{Y}_{M_c}$.

Case $M > M_c$ **-** This part of the proof is based on arguments in [32]. Fix $\theta \in \big((M_c/M)^{2/d}, 1 \big)$. By the VHLS inequality (3.1), there exists a non-zero function $h^* \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$, such that

$$\theta \, C_* \le \frac{|\mathcal{W}(h^*)|}{\|h^*\|_m^m \|h^*\|_1^{2/d}} \le C_* \ . \tag{3.8}$$

Since $|\mathcal{W}(h^*)| \le \mathcal{W}(|h^*|)$ we may assume without loss of generality that h^* is non-negative. Let $\lambda > 0$ and consider the function $h_\lambda(x) := \lambda^d h^* \left(\lambda \|h^*\|_1^{1/d} M^{-1/d} x\right)$. Then, $h_\lambda \in \mathcal{Y}_M$ and it follows from the definition of M_c and (3.8) that

$$\begin{aligned}
\mathcal{F}[h_\lambda] &= \lambda^{d-2} \left[\frac{M}{(m-1)\|h^*\|_1} \|h^*\|_m^m - \frac{c_d}{2} \left(\frac{M}{\|h^*\|_1} \right)^{(d+2)/d} \mathcal{W}(h^*) \right] \\
&\le \lambda^{d-2} \left[\frac{M}{(m-1)\|h^*\|_1} \|h^*\|_m^m - \frac{c_d}{2} \left(\frac{M}{\|h^*\|_1} \right)^{(d+2)/d} \theta\, C_* \|h^*\|_m^m \|h^*\|_1^{2/d} \right] \\
&= \lambda^{d-2} \left(\frac{M}{\|h^*\|_1} \right)^{(d+2)/d} \frac{\|h^*\|_m^m}{m-1} \left[\left(\frac{M_c}{M} \right)^{2/d} - \theta \right] .
\end{aligned}$$

Owing to the choice of θ we may let λ go to infinity to obtain that $\mu_M = -\infty$, thus completing the proof. □

Let us now describe the set of minimisers of $\mathcal{F}$ in $\mathcal{Y}_{M_c}$.

Proposition 3.5 (Identification of the minimisers) *Let ζ be the unique positive radial classical solution to*

$$\Delta\zeta + \frac{m-1}{m} \zeta^{1/(m-1)} = 0 \quad \text{in} \quad B(0,1) \quad \text{with} \quad \zeta = 0 \quad \text{on} \quad \partial B(0,1) .$$

If V is a minimiser of $\mathcal{F}$ in $\mathcal{Y}_{M_c}$ there are $R > 0$ and $z \in \mathbb{R}^d$ such that

$$V(x) = \begin{cases} \dfrac{1}{R^d} \left[\zeta \left(\dfrac{x-z}{R} \right) \right]^{d/(d-2)} & \text{if } x \in B(z,R), \\ 0 & \text{if } x \in \mathbb{R}^d \setminus B(z,R). \end{cases}$$

Proof. We have already shown in Proposition 3.4 that the function $\mathcal{F}$ has at least a minimiser in $\mathcal{Y}_{M_c}$. Let V be a minimiser of $\mathcal{F}$ in $\mathcal{Y}_{M_c}$, and define $\tilde{V}(x) := \|V\|_m^{-m/(m-1)} V\left(x \|V\|_m^{-m/(d(m-1))}\right)$ for $x \in \mathbb{R}^d$. We have $\|\tilde{V}\|_1 = M_c$, $\|\tilde{V}\|_m = 1$ and $\mathcal{F}[\tilde{V}] = 0$, so that $\tilde{V}$ is also a minimiser of $\mathcal{F}$ in $\mathcal{Y}_{M_c}$. We next denote by W the symmetric rearrangement of $\tilde{V}$. Then $\|W\|_1 = M_c$, $\|W\|_m = \|\tilde{V}\|_m = 1$ and $\mathcal{W}(W) \ge |\mathcal{W}(\tilde{V})|$ by the Riesz rearrangement properties [21, Lemma 2.1]. Therefore, $\mathcal{F}[W] \le \mathcal{F}[\tilde{V}] = 0$ and thus $\mathcal{F}[W] = 0$ since $W \in \mathcal{Y}_{M_c}$. This in turn implies that $\mathcal{W}(W) = |\mathcal{W}(\tilde{V})|$. Again by [21, Lemma 2.1] there is $y \in \mathbb{R}^d$ such that $\tilde{V}(x) = W(x-y)$ for $y \in \mathbb{R}^d$.

We next derive the Euler-Lagrange equation solved by W and first point out that a difficulty arises from the non-differentiability of the L^1-norm. Nevertheless, we introduce $\Sigma_0 := \{x \in \mathbb{R}^d : W(x) = 0\}$, $\Sigma_+ := \{x \in \mathbb{R}^d : W(x) > 0\}$ and consider $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and $\varepsilon > 0$. The perturbation $M_c \|W + \varepsilon\varphi\|_1^{-1}(W + \varepsilon\varphi)$ belongs to $\mathcal{Y}_{M_c}$ and is such that

$$\mathcal{F}\left[\frac{M_c}{\|W + \varepsilon\varphi\|_1} (W + \varepsilon\varphi) \right] \ge \mathcal{F}[W] \ge 0 .$$

After a few computations that we omit here we may let $\varepsilon \to 0$, and conclude that

$$\begin{aligned}
&2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{W(x)\,\varphi(y)}{|x-y|^{d-2}} \,\mathrm{d}y \,\mathrm{d}x \\
&\qquad \le C_*\, M_c^{2/d}\, m \int_{\mathbb{R}^d} W^{m-1}(x)\varphi(x) \,\mathrm{d}x + \frac{2}{d}\, C_*\, M_c^{(2-d)/d} \left(\int_{\Sigma_+} \varphi(x) \,\mathrm{d}x + \int_{\Sigma_0} |\varphi(x)| \,\mathrm{d}x \right) .
\end{aligned}$$

Using the definition of M_c and $\mathcal{K}$, the above formula also reads

$$\int_{\mathbb{R}^d} \left(\frac{m}{m-1} W^{m-1} - \mathcal{K} * W + \frac{2-m}{m-1} \frac{1}{M_c} \right) \varphi \, \mathrm{d}x \ge \frac{2-m}{m-1} \frac{1}{M_c} \int_{\Sigma_0} (\varphi - |\varphi|) \, \mathrm{d}x \tag{3.9}$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. On the one hand, the right-hand side of (3.9) vanishes for any non-negative $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, so that

$$\frac{m}{m-1} W^{m-1} - \mathcal{K} * W + \frac{2-m}{m-1} \frac{1}{M_c} \ge 0 \quad \text{a.e. in} \quad \mathbb{R}^d \, .$$

Therefore, for almost every $x \in \Sigma_0$, we have $0 \ge \mathcal{K} * W(x) - (2-m)/[(m-1)M_c]$ so that

$$\frac{m}{m-1} W^{m-1}(x) = 0 = \left(\mathcal{K} * W(x) - \frac{2-m}{m-1} \frac{1}{M_c} \right)_+ \quad \text{for almost every} \quad x \in \Sigma_0 \, . \tag{3.10}$$

On the other hand, if $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, a standard approximation argument allows us to take $\varphi = \mathbf{1}_{\Sigma_+} \psi$ in (3.9) and deduce that

$$\int_{\Sigma_+} \left(\frac{m}{m-1} W^{m-1} - \mathcal{K} * W + \frac{2-m}{m-1} \frac{1}{M_c} \right) \psi \, \mathrm{d}x \ge 0 \, .$$

This inequality being also valid for $-\psi$, we conclude that the left-hand side of the above inequality vanishes for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, whence

$$\frac{m}{m-1} W^{m-1} = \mathcal{K} * W - \frac{2-m}{m-1} \frac{1}{M_c} \quad \text{a.e. in} \quad \Sigma_+ \, . \tag{3.11}$$

Combining (3.10) and (3.11) gives

$$\frac{m}{m-1} W^{m-1} = \left(\mathcal{K} * W - \frac{2-m}{m-1} \frac{1}{M_c} \right)_+ \quad \text{a.e. in} \quad \mathbb{R}^d \, .$$

Now, since W is radially symmetric and non-increasing there exists $\rho \in (0, \infty]$ such that

$$\Sigma_+ \subset B(0,\rho) \ \text{ and } \ \Sigma_0 \subset \mathbb{R}^d \setminus B(0,\rho) \, ,$$

and we infer from (3.11) that

$$\frac{m}{m-1} W^{m-1} = \mathcal{K} * W - \frac{2-m}{m-1} \frac{1}{M_c} \quad \text{for a.e. } x \in B(0,\rho) \, . \tag{3.12}$$

Since $W \in \mathrm{L}^r(\mathbb{R}^d)$ for each $r \in (1,m]$ it follows from the HLS inequality (3.2) that $\mathcal{K} * W \in \mathrm{L}^r(\mathbb{R}^d)$ for each $r \in \left(d/(d-2), m/(m-1)^2 \right]$, see [22, Theorem 10.2]. In particular, $\mathcal{K} * W$ and W^{m-1} both belong to $\mathrm{L}^{m/(m-1)}(\mathbb{R}^d)$. This property and (3.12) then exclude that $\rho = \infty$ as $M_c > 0$. Therefore $\rho < \infty$ and

$$\frac{m}{m-1} W^{m-1}(x) = \begin{cases} \mathcal{K} * W(x) - \dfrac{2-m}{m-1} \dfrac{1}{M_c} & \text{if } |x| < \rho \, , \\ 0 & \text{if } |x| > \rho \, . \end{cases}$$

Since $\mathcal{K} * W \in \mathrm{L}^{m/(m-1)^2}(\mathbb{R}^d)$, the above inequality allows us to conclude that $W \in \mathrm{L}^{m/(m-1)}(\mathbb{R}^d)$. We now improve the regularity of W by classical elliptic estimates. Introduce $\theta := W^{m-1}$ and note that

$$\frac{m}{m-1} \theta(x) = \int_{\mathbb{R}^d} \mathcal{K}(x-y) \, W(y) \, \mathrm{d}y + \frac{m-1}{m-2} \frac{1}{M_c}$$

for $x \in B(0,\rho)$ and $W \in \mathrm{L}^{m/(m-1)}(\mathbb{R}^d)$. By [16, Theorem 9.9], we have $\theta \in W^{2,m/(m-1)}(B(0,\rho))$. A bootstrap argument then ensures that θ and W both belong to $W^{2,r}(B(0,\rho))$ for every $r \in (1,\infty)$. It then follows from [16, Lemma 4.2] that $\theta \in \mathcal{C}^2(B(0,\rho))$ with $-\Delta\theta = (m-1)\theta^{m/(m-1)}/m$ in $B(0,\rho)$ while [16, Lemma 4.1] warrants that $\theta \in \mathcal{C}^1(\mathbb{R}^d)$. Then $\theta(x) = 0$ if $|x| = \rho$ and θ is thus a classical solution to $-\Delta\theta = (m-1)\theta^{m/(m-1)}/m$ in $B(0,\rho)$ with $\theta = 0$ on $\partial B(0,\rho)$. By [15, Lemma 2.3], there is a unique positive solution to this problem. In fact, a simple scaling argument shows that

$$\theta(x) = \rho^{2(m-1)/(m-2)}\zeta\left(\frac{x}{\rho}\right) \quad \text{for } x \text{ in } B(0,\rho)$$

and then

$$W(x) = \frac{1}{\rho^d}\left[\zeta\left(\frac{x}{\rho}\right)\right]^{d/(d-2)} \quad \text{for } x \text{ in } B(0,\rho)\,.$$

Coming back to V, we have

$$V(x) = \begin{cases} \lambda^d W(\lambda\, x - y) = 0 & \text{if } x \in \mathbb{R}^d \setminus B\left(\frac{y}{\lambda}, \frac{\rho}{\lambda}\right)\,, \\ \left(\frac{\lambda}{\rho}\right)^d \left[\zeta\left(\left(x - \frac{y}{\lambda}\right)\left(\frac{\rho}{\lambda}\right)^{-1}\right)\right]^{d/(d-2)} & \text{if } x \in B\left(\frac{y}{\lambda}, \frac{\rho}{\lambda}\right)\,, \end{cases}$$

which is the desired result with $R = \rho/\lambda$ and $z = y/\lambda$. □

Remark 3.6 *As a consequence of the identification of the minimisers given in Proposition* 3.5*,* $C_* < C_{\mathrm{HLS}}$*. Otherwise any minimiser* V *of* $\mathcal{F}$ *is* $\mathcal{Y}_{M_c}$ *would also be an extremum for the HLS inequality* (3.2) *and thus be equal to*

$$V(x) = \frac{a}{(1+|x|^2)^{(d+2)/2}}\,,$$

for some $a > 0$*, see* [21, Theorem 3.1]*. This contradicts Proposition* 3.5*.*

Lemma 3.7 (Unboundedness of $\mathcal{F}$) *For each* $M > 0$ *we have*

$$\sup_{h \in \mathcal{Y}_M} \mathcal{F}[h] = +\infty\,. \tag{3.13}$$

If $M \in (0, M_c)$ the claim (3.13) is actually a straightforward consequence of (3.7).

Proof. Let $M > 0$ and assume for contradiction that

$$A := \sup_{h \in \mathcal{Y}_M} \mathcal{F}[h] < \infty\,.$$

Consider $h \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$ and define $h_\lambda(x) := M\lambda^d h(\lambda x)/\|h\|_1$ for $x \in \mathbb{R}^d$ and $\lambda > 0$. Then $\|h_\lambda\|_1 = M$ so that $h_\lambda \in \mathcal{Y}_M$ with

$$\|h_\lambda\|_m^m = \lambda^{d-2}\left(\frac{M}{\|h\|_1}\right)^m \|h\|_m^m \quad \text{and} \quad \mathcal{W}(h_\lambda) = \lambda^{d-2}\left(\frac{M}{\|h\|_1}\right)^2 \mathcal{W}(h)\,.$$

Since $h_\lambda \in \mathcal{Y}_M$ we have $\mathcal{F}[h_\lambda] \le A$, hence

$$\|h_\lambda\|_m^m \le (m-1)\left(A + \frac{c_d}{2}\,\mathcal{W}(h_\lambda)\right)$$

from which we deduce

$$\|h\|_m^m \le (m-1)\left(A\,\lambda^{2-d}\,\left(\frac{\|h\|_1}{M}\right)^m + \frac{c_d}{2}\,\left(\frac{M}{\|h\|_1}\right)^{2/d}\,\mathcal{W}(h)\right)$$
$$\le (m-1)A\,\lambda^{2-d}\,\left(\frac{\|h\|_1}{M}\right)^m + \frac{1}{C_* M_c^{2/d}}\,\left(\frac{M}{\|h\|_1}\right)^{2/d}\,\mathcal{W}(h)\,.$$

This inequality being valid for all $\lambda > 0$ we let $\lambda \to \infty$ and use the HLS inequality (3.2) to obtain

$$\|h\|_m^m \le \frac{1}{C_* M_c^{2/d}}\,\left(\frac{M}{\|h\|_1}\right)^{2/d}\,\mathcal{W}(h) \le \left(\frac{M}{M_c}\right)^{2/d}\,\frac{C_{\rm HLS}}{C_*}\,\frac{\|h\|_{2d/(d+2)}^2}{\|h\|_1^{2/d}}\,.$$

Consequently,

$$\|h\|_m^m\,\|h\|_1^{2/d} \le \left(\frac{M}{M_c}\right)^{2/d}\,\frac{C_{\rm HLS}}{C_*}\,\|h\|_{2d/(d+2)}^2 \tag{3.14}$$

for all $h \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$.

Now, as $2d/(d+2) < m$, we may choose $\gamma \in ((d+2)/d, d/m)$ and put $b_\delta(x) := (\delta + |x|)^{-\gamma}\,\mathbb{1}_{B(0,1)}(x)$ for $x \in \mathbb{R}^d$ and $\delta \in [0,1]$. Clearly b_δ belongs to $\mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$ with $\|b_\delta\|_1 \ge \|b_1\|_1 > 0$ and $\|b_\delta\|_{2d/(d+2)} \le \|b_0\|_{2d/(d+2)} < \infty$ for each $\delta \in (0,1]$. These properties and (3.14) readily imply that $(b_\delta)_{\delta\in(0,1]}$ is bounded in $\mathrm{L}^m(\mathbb{R}^d)$ which is clearly not true according to the choice of γ. Therefore A cannot be finite and Lemma 3.7 is proved. □

4 Critical threshold

It turns out that the critical mass M_c arising in the study of the free energy functional and defined in (3.5) plays also an important role in the dynamics of (1.4). In the next sections we will distinguish the three cases $M > M_c$ (*super-critical case*), $M < M_c$ (*sub-critical case*), and $M = M_c$ (*critical case*), M denoting the $L^1(\mathbb{R}^d)$-norm of the initial condition u_0.

4.1 Finite time blow-up in the super-critical case

We start with the case $M > M_c$ in which we use the standard argument relying on the evolution of the second moment of solutions as originally done in [18] for the PKS system corresponding to $d = 2$ and $m = 1$.

Lemma 4.1 (Virial identity) *Under assumption* (1.2)*, let u be a free energy solution to* (1.4) *on $[0,T)$ with initial condition u_0 for some $T \in (0,\infty]$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |x|^2\,u(t,x)\,\mathrm{d}x = 2\,(d-2)\,\mathcal{F}[u(t)]\,, \quad t \in [0,T)\,.$$

Proof. Here, we show the formal computation leading to this property, the passing to the limit from the approximated problem (2.1) can be done by adapting the arguments in [30, Lemma 6.2] and [4, Lemma 2.1] without any further complication. By integration by parts in (1.4) and symmetrising the second term, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |x|^2\,u(t,x)\,\mathrm{d}x &= 2\,d\int_{\mathbb{R}^d} u^m(t,x)\,\mathrm{d}x + 2\iint_{\mathbb{R}^d\times\mathbb{R}^d}[x\cdot\nabla\mathcal{K}(x-y)]\,u(t,x)\,u(t,y)\,\mathrm{d}y\,\mathrm{d}x\\ &= 2\,d\int_{\mathbb{R}^d} u^m(t,x)\,\mathrm{d}x + \iint_{\mathbb{R}^d\times\mathbb{R}^d}[(x-y)\cdot\nabla\mathcal{K}(x-y)]\,u(t,x)\,u(t,y)\,\mathrm{d}y\,\mathrm{d}x\\ &= 2\,(d-2)\,\mathcal{F}[u(t)]\,,\end{aligned}$$

giving the desired identity. □

Let us mention that a similar argument can be found in [30, Lemma 6.2] and [31] in the present situation where the Poisson equation is substituted by $-\Delta\phi = u - \phi$. The previous evolution for the second moment is simpler in our case than the one in [31] and resembles that arising in the study of critical nonlinear Schrödinger equations [11].

Let us also emphasise that this second moment evolution is more complicated than in the classical PKS system corresponding to $d = 2$ and $m = 1$ where the time derivative of the second moment is a constant.

An easy consequence of the previous lemma is the following blow-up result.

Proposition 4.2 (Blowing-up solutions) *If $M > M_c$, then there are initial data u_0 satisfying* (1.2) *with $\|u_0\|_1 = M$ and negative free energy $\mathcal{F}[u_0]$. Moreover, if u_0 is such an initial condition and u denotes a free energy solution to* (1.4) *on $[0, T_\omega)$ with initial condition u_0, then $T_\omega < \infty$ and the L^m-norm of u blows up in finite time.*

Proof. The proof is based on the idea of Weinstein [32]. By the identification of the minimisers for the critical mass given in Proposition 3.5, $\tilde{u} := \zeta^{d/(d-2)}$ satisfies (1.2) as well as $\|\tilde{u}\|_1 = M_c$ and $\mathcal{F}[\tilde{u}] = 0$. For $M > M_c$, the initial condition $u_0 = (M/M_c)\tilde{u}$ also satisfies (1.2) with $\|u_0\|_1 = M$ and

$$\begin{aligned}\mathcal{F}[u_0] &= \frac{1}{m-1}\left(\frac{M}{M_c}\right)^m \|\tilde{u}\|_m^m - \left(\frac{M}{M_c}\right)^2 \frac{c_d}{2}\mathcal{W}(\tilde{u}) \\ &= \frac{1}{m-1}\left[\left(\frac{M}{M_c}\right)^m - \left(\frac{M}{M_c}\right)^2\right] \|\tilde{u}\|_m^m \,,\end{aligned}$$

is negative as $M > M_c$ and $m < 2$.

Consider next an initial condition u_0 satisfying (1.2) as well as $\|u_0\|_1 > M_c$ and $\mathcal{F}[u_0] < 0$. Denoting by u a corresponding free energy solution to (1.4) on $[0, T)$, we infer from the time monotonicity of $\mathcal{F}$ and Lemma 4.1 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |x|^2\, u(t,x)\, \mathrm{d}x = 2\,(d-2)\,\mathcal{F}[u(t)] \le 2\,(d-2)\,\mathcal{F}[u_0] < 0 \,.$$

This implies that the second moment of $u(t)$ will become negative after some time and contradicts the non-negativity of u. Therefore, T_ω is finite and $\|u\|_m$ blows up in finite time. □

4.2 Global existence

Proposition 4.3 (Global existence in the subcritical case) *Under assumption* (1.2)*, there exists a free energy solution to* (1.4) *in $[0, \infty)$ with initial condition u_0.*

Proof. By Theorem 2.4 there are T_ω and a free energy solution to (1.4) in $[0, T_\omega)$ with initial condition u_0. We then infer from (1.7), (1.8), and (3.7) that $u(t)$ belongs to $\mathcal{Y}_M$ for all $t \in [0, T_\omega)$ and

$$\frac{C_*\, c_d}{2}\,\left(M_c^{2/d} - M^{2/d}\right)\,\|u(t)\|_m^m \le \mathcal{F}[u(t)] \le \mathcal{F}[u_0] \,.$$

As $M < M_c$, we deduce from the previous inequality that u lies in $\mathrm{L}^\infty(0, \min\{T, T_\omega\}; \mathrm{L}^m(\mathbb{R}^d))$ for every $T > 0$ which implies that $T_\omega = \infty$ by Theorem 2.4. □

Let us now discuss the critical case.

4.2.1 How would it blow-up?

Proposition 4.4 (Nature of the blow-up) *Let u_0 be an initial condition satisfying (1.2) with $\|u_0\|_1 = M_c$ and consider a free energy solution u to (1.4) on $[0, T_\omega)$ with initial condition u_0 and $T_\omega \in (0, \infty]$ and such that $\|u(t)\|_m \to \infty$ as $t \nearrow T_\omega$. If $(t_k)_k$ is a sequence of positive real numbers such that $t_k \to T_\omega$ as $k \to \infty$, there are a sub-sequence $(t_{k_j})_j$ of $(t_k)_k$ and a sequence $(x_j)_j$ in $\mathbb{R}^d$ such that*

$$\lim_{j\to\infty} \int_{\mathbb{R}^d} \left| u(t_{k_j}, x + x_j) - \frac{1}{\lambda_{k_j}^d} V\left(\frac{x}{\lambda_{k_j}}\right) \right| \, \mathrm{d}x = 0 \, ,$$

where $\lambda_k := \|u(t_k)\|_m^{-m/(d-2)}$ and V is the unique radially symmetric minimiser of $\mathcal{F}$ in $\mathcal{Y}_{M_c}$ such that $\|V\|_m = 1$. Assume further that

$$\mathcal{M}_2 := \sup_{t\in[0,T)} \int_{\mathbb{R}^d} |x|^2 \, u(t,x) \, \mathrm{d}x < \infty \, ,$$

then

$$\lim_{j\to\infty} x_j = \bar{x} \quad \textit{where} \quad \bar{x} := \frac{1}{M_c} \int_{\mathbb{R}^d} x \, u_0(x) \, \mathrm{d}x \, . \tag{4.1}$$

Since $\mu_{M_1+M_2} = \mu_{M_1} + \mu_{M_2}$ for $M_1 \le M_c$ and $M_2 \le M_c$, the concentration compactness result as stated by P.-L. Lions [24] does not seem to apply directly. However, we follow the approach of M. Weinstein [33] to prove that the conclusion still holds true.

Proof. We set $v_k(x) := \lambda_k^d \, u(t_k, \lambda_k \, x)$ and aim at proving that $(v_k)_k$ converges strongly in $\mathrm{L}^1(\mathbb{R}^d)$. For this purpose we employ in Step 1 the concentration-compactness principle [24, Theorem II.1] to show that $(v_k)_k$ is tight up to translations. We argue in Step 2 as in [33, Theorem 1] to establish that $(v_k)_k$ has a limit in $\mathrm{L}^1(\mathbb{R}^d)$ and identify the limit. In the last step we use the additional bound on the second moment to show that the dynamics does not escape at infinity.
Step 1 - Tightness. Obviously,

$$\|v_k\|_1 = M_c \quad \text{and} \quad \|v_k\|_m = 1 \quad \text{for} \quad k \ge 1 \, . \tag{4.2}$$

The concentration-compactness principle [24] implies that there exists a sub-sequence (not relabelled) satisfying one of the three following properties:

(Compactness) There exists a sequence $(a_k)_k$ in $\mathbb{R}^d$ such that $(v_k(\cdot + a_k))_k \in \mathbb{R}^d$ is tight, that is, for each $\varepsilon > 0$ there is $R_\varepsilon > 0$ such that

$$\int_{B(a_k, R_\varepsilon)} v_k(x) \, \mathrm{d}x \ge M_c - \varepsilon \, . \tag{4.3}$$

(Vanishing) For all $R \ge 0$

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^d} \int_{B(y,R)} v_k(x) \, \mathrm{d}x = 0 \, . \tag{4.4}$$

(Dichotomy) There exists $\mu \in (0, M_c)$ such that for all $\varepsilon > 0$, there exist $k_0 \ge 1$ and three sequences of non-negative, integrable and compactly supported functions $(y_k^\varepsilon)_k$, $(z_k^\varepsilon)_k$, and $(w_k^\varepsilon)_k$ satisfying $v_k = w_k^\varepsilon + y_k^\varepsilon + z_k^\varepsilon$,

$$\begin{cases} \left| \|y_k^\varepsilon\|_1 - \mu \right| \le \varepsilon \, , \left| \|z_k^\varepsilon\|_1 - (M_c - \mu) \right| \le \varepsilon \, , \|w_k^\varepsilon\|_1 \le \varepsilon \, , \\ \lim\limits_{k\to\infty} \mathrm{dist} \, (\mathrm{supp} \, y_k^\varepsilon, \mathrm{supp} \, z_k^\varepsilon) = \infty \, , \end{cases} \tag{4.5}$$

for any $k \ge k_0$.

As usual we shall rule out the possible occurrence of vanishing and dichotomy. To this end we argue as in [24, Theorem II.1]. Let us first notice that by the scaling and non-negativity properties of the free energy, (2.10) and (3.6), $\mathcal{F}[u(t_k)] \in [0, \mathcal{F}[u_0]]$ and

$$\lim_{k\to\infty} \mathcal{F}[v_k] = \lim_{k\to\infty} \|u(t_k)\|_m^{-m}\, \mathcal{F}[u(t_k)] = 0 \; . \tag{4.6}$$

Consequently, since $\|v_k\|_m = 1$ by the definition of λ_k, we have

$$\lim_{k\to\infty} \mathcal{W}(v_k) = \lim_{k\to\infty} \frac{2}{c_d} \left(\frac{1}{m-1} \|v_k\|_m^m - \mathcal{F}[v_k] \right) = \frac{2}{c_d\,(m-1)} > 0 \; . \tag{4.7}$$

• Let us first show that vanishing does not take place and argue by contradiction. We split the non-local term $\mathcal{W}(v_k)$ in three parts. If $|x-y|$ is small, we control the corresponding term by the bound in $\mathrm{L}^1 \cap \mathrm{L}^m$ of v_k. If $|x-y|$ is large the corresponding term is controlled by the L^1-bound of v_k. And the remaining term converges to zero if we assume that vanishing occurs which contradicts (4.7). Indeed, if $q \in ((d-1)/(d-2), d/(d-2))$ and $R > 0$, it follows from the Hölder and Young inequalities that

$$\begin{aligned}
\mathcal{W}(v_k) &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{v_k(x)\,v_k(y)}{|x-y|^{d-2}} \mathbb{1}_{[0,1/R]}(|x-y|)\,\mathrm{d}y\,\mathrm{d}x + \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{v_k(x)\,v_k(y)}{|x-y|^{d-2}} \mathbb{1}_{(1/R,R)}(|x-y|)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad + \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{v_k(x)\,v_k(y)}{|x-y|^{d-2}} \mathbb{1}_{[R,\infty)}(|x-y|)\,\mathrm{d}y\,\mathrm{d}x \\
&\le \|v_k\|_{2q/(2q-1)}^2 \left(\int_{\mathbb{R}^d} |x|^{-q(d-2)} \mathbb{1}_{[0,1/R]}(|x|)\,\mathrm{d}x \right)^{1/q} + R^{d-2} \int_{\mathbb{R}^d} v_k(x) \int_{B(x,R)} v_k(y)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad + \frac{1}{R^{d-2}} \left(\int_{\mathbb{R}^d} v_k(x)\,\mathrm{d}x \right)^2 \\
&\le \|v_k\|_m^{m/[q(m-1)]}\, \|v_k\|_1^{[(2q-1)/q]-[d/(q(d-2))]} \left(d\,\sigma_d \int_0^{1/R} r^{d-1-q(d-2)}\,\mathrm{d}r \right)^{1/q} \\
&\qquad + R^{d-2} M_c \sup_{x\in\mathbb{R}^d} \int_{B(x,R)} v_k(y)\,\mathrm{d}y\,\mathrm{d}x + \frac{M_c^2}{R^{d-2}} \\
&\le C(q)\, \frac{1}{R^{[d-q(d-2)]/q}} + R^{d-2} M_c \sup_{x\in\mathbb{R}^d} \int_{B(x,R)} v_k(y)\,\mathrm{d}y\,\mathrm{d}x + \frac{M_c^2}{R^{d-2}}
\end{aligned}$$

We let $k \to \infty$ in the above inequality and use the vanishing assumption (4.4) to obtain that

$$\limsup_{k\to\infty} \mathcal{W}(v_k) \le C(q) \left(R^{2-d} + R^{-(d-q(d-2))/q} \right) \; .$$

We next let R to infinity to conclude that $\mathcal{W}(v_k)$ converges to zero as $k \to \infty$ which contradicts (4.7).

• Let us next assume for contradiction that dichotomy takes place. We have

$$\mathcal{W}(v_k) - \mathcal{W}(y_k^\varepsilon) - \mathcal{W}(z_k^\varepsilon) = -\mathcal{W}(w_k^\varepsilon) + \mathrm{I}_1 + \mathrm{I}_2 \; ,$$

where

$$\mathrm{I}_1 := 2 \iint_{\mathbb{R}^d\times\mathbb{R}^d} y_k^\varepsilon(x)\, z_k^\varepsilon(y)\, |x-y|^{2-d}\,\mathrm{d}x\,\mathrm{d}y \quad \text{and} \quad \mathrm{I}_2 := 2 \iint_{\mathbb{R}^d\times\mathbb{R}^d} v_k(x)\, w_k^\varepsilon(y)\, |x-y|^{2-d}\,\mathrm{d}x\,\mathrm{d}y \; .$$

On the one hand, setting $d_k^\varepsilon := \mathrm{dist}(\mathrm{supp}\, y_k^\varepsilon, \mathrm{supp}\, z_k^\varepsilon)$, we have

$$|\mathrm{I}_1| \le \iint_{\mathbb{R}^d\times\mathbb{R}^d} y_k^\varepsilon(x)\, z_k^\varepsilon(y)\, \mathbb{1}_{(0,d_k^\varepsilon)}(|x-y|)\, |x-y|^{2-d}\, \mathrm{d}x\, \mathrm{d}y + \iint_{\mathbb{R}^d\times\mathbb{R}^d} y_k^\varepsilon(x)\, z_k^\varepsilon(y)\, \mathbb{1}_{[d_k^\varepsilon,\infty)}(|x-y|)\, |x-y|^{2-d}\, \mathrm{d}x\, \mathrm{d}y\,.$$

Thanks to the definition of d_k^ε the first integral vanishes and we arrive at

$$|\mathrm{I}_1| \le M_c^2\, (d_k^\varepsilon)^{2-d}\,.$$

On the other hand it follows from (4.2), (4.5), the HLS inequality (3.2) applied to $f = v_k$, $g = w_k^\varepsilon$, $\lambda = d-2$ and $p = q = 2d/(d+2)$, and the Hölder inequality with $1 < 2d/(d+2) < m$ that

$$\begin{aligned}|\mathrm{I}_2| &\le C_{\mathrm{HLS}}\, \|v_k\|_{2d/(d+2)}\, \|w_k^\varepsilon\|_{2d/(d+2)} \le C_{\mathrm{HLS}}\, \|v_k\|_m^{m/2}\, \|v_k\|_1^{1/d}\, \|w_k^\varepsilon\|_m^{m/2}\, \|w_k^\varepsilon\|_1^{1/d} \\ &\le C_{\mathrm{HLS}}\, M_c^{1/d}\, \|w_k^\varepsilon\|_m^{m/2}\, \varepsilon^{1/d}\,,\end{aligned}$$

and $0 \le w_k^\varepsilon \le v_k$ and (4.2) imply that $\|w_k^\varepsilon\|_m \le 1$. Similarly by the variant of the HLS inequality (3.1), we obtain

$$|\mathcal{W}(w_k^\varepsilon)| \le C_*\, \|w_k^\varepsilon\|_m^m\, \varepsilon^{2/d} \le C_*\, \varepsilon^{2/d}\,.$$

Combining these estimates, we have thus shown that, given $\varepsilon \in (0,1)$, there exists $k_\varepsilon \ge \varepsilon^{-1}$ such that

$$|\mathcal{W}(v_{k_\varepsilon}) - \mathcal{W}(y_{k_\varepsilon}^\varepsilon) - \mathcal{W}(z_{k_\varepsilon}^\varepsilon)| \le \varepsilon^{1/d}\,. \tag{4.8}$$

Since $w_{k_\varepsilon}^\varepsilon$ is non-negative and the supports of $y_{k_\varepsilon}^\varepsilon$ and $z_{k_\varepsilon}^\varepsilon$ are disjoint we have

$$\|y_{k_\varepsilon}^\varepsilon + z_{k_\varepsilon}^\varepsilon + w_{k_\varepsilon}^\varepsilon\|_m^m \ge \|y_{k_\varepsilon}^\varepsilon + z_{k_\varepsilon}^\varepsilon\|_m^m = \|y_{k_\varepsilon}^\varepsilon\|_m^m + \|z_{k_\varepsilon}^\varepsilon\|_m^m\,,$$

and we deduce from (4.8) that

$$\begin{aligned}\mathcal{F}[v_{k_\varepsilon}] &= \frac{1}{m-1}\|y_{k_\varepsilon}^\varepsilon + z_{k_\varepsilon}^\varepsilon + w_{k_\varepsilon}^\varepsilon\|_m^m - \frac{c_d}{2}\mathcal{W}(v_{k_\varepsilon}) \\ &\ge \frac{1}{m-1}\left(\|y_{k_\varepsilon}^\varepsilon\|_m^m + \|z_{k_\varepsilon}^\varepsilon\|_m^m\right) - \frac{c_d}{2}\mathcal{W}(y_{k_\varepsilon}^\varepsilon) - \frac{c_d}{2}\mathcal{W}(z_{k_\varepsilon}^\varepsilon) - \frac{c_d}{2}\,\varepsilon^{1/d} \\ &\ge \mathcal{F}[y_{k_\varepsilon}^\varepsilon] + \mathcal{F}[z_{k_\varepsilon}^\varepsilon] - \frac{c_d}{2}\,\varepsilon^{1/d}\,.\end{aligned}$$

The above inequality, (4.5), (4.6), and the non-negativity of $\mathcal{F}$ for functions with L^1-norm lower or equal to M_c then entail that

$$\lim_{\varepsilon\to 0} \mathcal{F}[y_{k_\varepsilon}^\varepsilon] = \lim_{\varepsilon\to 0} \mathcal{F}[z_{k_\varepsilon}^\varepsilon] = 0\,. \tag{4.9}$$

Now, (3.7) and (4.5) imply

$$0 = \lim_{\varepsilon\to 0} \mathcal{F}[y_{k_\varepsilon}^\varepsilon] \ge \lim_{\varepsilon\to 0} \frac{C_*\, c_d}{2}\left(M_c^{2/d} - \|y_{k_\varepsilon}^\varepsilon\|_1^{2/d}\right)\, \|y_{k_\varepsilon}^\varepsilon\|_m^m \ge \frac{C_*\, c_d}{2}\left(M_c^{2/d} - \mu^{2/d}\right)\, \lim_{\varepsilon\to 0} \|y_{k_\varepsilon}^\varepsilon\|_m^m\,,$$

and a similar inequality for $z_{k_\varepsilon}^\varepsilon$ (with $M_c - \mu$ instead of μ), hence

$$\lim_{\varepsilon\to 0} \|y_{k_\varepsilon}^\varepsilon\|_m^m = \lim_{\varepsilon\to 0} \|z_{k_\varepsilon}^\varepsilon\|_m^m = 0\,. \tag{4.10}$$

Combining (4.9) and (4.10) gives

$$\lim_{\varepsilon\to 0} \mathcal{W}(y_{k_\varepsilon}^\varepsilon) = \lim_{\varepsilon\to 0} \mathcal{W}(z_{k_\varepsilon}^\varepsilon) = 0\,,$$

which, together with (4.8), implies that $(\mathcal{W}(v_{k_\varepsilon}))_\varepsilon$ goes to 0 as ε goes to infinity and contradicts (4.7).

Having excluded the vanishing and dichotomy phenomena we thus conclude that there exists a sequence $(a_k)_k$ in $\mathbb{R}^d$ such that $(v_k(\cdot + a_k))_k$ is tight, that is, satisfies (4.3).

Step 2 - Compactness in L^m. We now aim at showing that a sub-sequence of $(v_k(\cdot + a_k))_k$ converges in $\mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$ towards a minimiser of $\mathcal{F}$ in $\mathcal{Y}_{M_c}$. We set $V_k(x) := v_k(x + a_k)$ for $x \in \mathbb{R}^d$ and $k \geq 1$. By virtue of (4.2) we may assume (after possibly extracting a sub-sequence) that there is a non-negative $V_\infty \in \mathrm{L}^m(\mathbb{R}^d)$ such that

$$V_k \rightharpoonup V_\infty \quad \text{weakly in } \mathrm{L}^m(\mathbb{R}^d) \,. \tag{4.11}$$

By (4.2), (4.3), and (4.11) we have V_∞ is non-negative with $\|V_\infty\|_1 = M_c$ and $\|V_\infty\|_m \leq 1$.

To prove the convergence of $\mathcal{W}(V_k)$ to $\mathcal{W}(V_\infty)$, we proceed as in Step 1 and split $\mathbb{R}^d \times \mathbb{R}^d$ in three parts. If $q \in ((d-1)/(d-2), d/(d-2))$ we have

$$\begin{aligned} |\mathcal{W}(V_k) - \mathcal{W}(V_\infty)| \leq \frac{2\,M_c^2}{R^{d-2}} + C(q)\,R^{[q(2-d)+d]/q} \\ + \left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} [V_k(x)\,V_k(y) - V_\infty(x)\,V_\infty(y)]\; \frac{\mathbb{1}_{(1/R,R)}(|x-y|)}{|x-y|^{d-2}}\; \mathrm{d}y\,\mathrm{d}x \right| \,. \end{aligned}$$

Since $x \mapsto \mathbb{1}_{(1/R,R)}(|x|)|x|^{2-d} \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^\infty(\mathbb{R}^d)$, the weak convergence (4.11) ensures that $(x,y) \mapsto V_k(x)\,V_k(y)$ converges weakly toward $(x,y) \mapsto V_\infty(x)\,V_\infty(y)$ in $\mathrm{L}^m(\mathbb{R}^d \times \mathbb{R}^d)$ so that the last term of the right-hand side converges to zero as $k \to \infty$. Therefore

$$\limsup_{k\to\infty} |\mathcal{W}(V_k) - \mathcal{W}(V_\infty)| \leq C(q) \left(R^{2-d} + R^{-[d-q(d-2)]/q} \right) \,.$$

We then let $R \to \infty$ to obtain

$$\lim_{k\to\infty} \mathcal{W}(V_k) = \mathcal{W}(V_\infty) \,.$$

Owing to the lower semi-continuity of the L^m-norm and (4.6) we have

$$\mathcal{F}[V_\infty] \leq \frac{1}{m-1} \liminf_{k\to\infty} \|V_k\|_m^m - \frac{c_d}{2} \lim_{k\to\infty} \mathcal{W}(V_k) \leq \lim_{k\to\infty} \mathcal{F}[V_k] = 0 \,,$$

while Proposition 3.4 warrants that $\mathcal{F}[V_\infty] \geq 0$ as $V_\infty \in \mathcal{Y}_{M_c}$. Consequently, $\mathcal{F}[V_\infty] = 0$ and the strong convergence of $(V_k)_k$ to V_∞ in $\mathrm{L}^m(\mathbb{R}^d)$ readily follows: indeed,

$$\frac{1}{m-1}\|V_\infty\|_m^m = \mathcal{F}[V_\infty] + \frac{c_d}{2}\mathcal{W}(V_\infty) = \lim_{k\to\infty} \left(\mathcal{F}[V_k] + \frac{c_d}{2}\mathcal{W}(V_k) \right) = \frac{1}{m-1} \lim_{k\to\infty} \|V_k\|_m^m \,.$$

We have thus shown that V_∞ is a minimiser of $\mathcal{F}$ in $\mathcal{Y}_{M_c}$ with the additional property $\|V_\infty\|_m = 1$. Furthermore, according to the characterisation of the minimisers given in Proposition 3.5, there exists $y_0 \in \mathbb{R}^d$ such that $V_\infty(\cdot + y_0) =: V$ is the unique radially symmetric minimiser of $\mathcal{F}$ in $\mathcal{Y}_{M_c}$ with $\|V\|_m = 1$. Coming back to the original variables we have proved that $\left(x \mapsto \lambda_k^d\, u(t_k, \lambda_k\,(x + a_k + y_0))\right)_k$ converges to V in $\mathrm{L}^1(\mathbb{R}^d)$ and $\mathrm{L}^m(\mathbb{R}^d)$. Setting $x_k = \lambda_k\,(a_k + y_0)$ gives

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} \left| u(t_k, x + x_k) - \frac{1}{\lambda_k^d}\, V\left(\frac{x}{\lambda_k}\right) \right| \; \mathrm{d}x = 0 \,, \tag{4.12}$$

and thus the first assertion of Proposition 4.4.

Step 3 - Convergence of $(x_k)_k$. We first note that

$$\int_{\mathbb{R}^d} x\, u(t,x) \; \mathrm{d}x = \int_{\mathbb{R}^d} x\, u_0(x) \; \mathrm{d}x$$

for $t \in [0, T_\omega)$ so that we have also

$$\bar{x} = \frac{1}{M_c} \int_{\mathbb{R}^d} x\, u(t,x)\ \mathrm{d}x$$

for $t \in [0, T_\omega)$ by (1.8). Next, for $\varepsilon \in (0,1)$, we have

$$\begin{aligned}
\left| (\bar{x} - x_k) \int_{B(x_k,\varepsilon)} u(t_k,x)\ \mathrm{d}x \right| &\le \left| \int_{B(x_k,\varepsilon)} (\bar{x} - x)\, u(t_k,x)\ \mathrm{d}x \right| + \left| \int_{B(x_k,\varepsilon)} (x - x_k)\, u(t_k,x)\ \mathrm{d}x \right| \\
&\le \int_{\{|x-x_k| \ge \varepsilon\}} |\bar{x} - x|\, u(t_k,x)\ \mathrm{d}x + \varepsilon\, M_c \\
&\le \int_{\{|x-x_k| \ge \varepsilon\,,\, |x-\bar{x}| \le 1/\varepsilon\}} |\bar{x} - x|\, u(t_k,x)\ \mathrm{d}x \\
&+ \int_{\{|x-x_k| \ge \varepsilon\,,\, |x-\bar{x}| > 1/\varepsilon\}} |\bar{x} - x|\, u(t_k,x)\ \mathrm{d}x + \varepsilon\, M_c \\
&\le \frac{1}{\varepsilon} \int_{\{|x-x_k| \ge \varepsilon\}} u(t_k,x)\ \mathrm{d}x + \varepsilon\, (\mathcal{M}_2 + M_c) \\
&\le \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \left| u(t_k, y + x_k) - \frac{1}{\lambda_k^d}\, V\left(\frac{y}{\lambda_k}\right) \right|\ \mathrm{d}y \\
&+ \frac{1}{\varepsilon} \int_{\{|x| \ge \varepsilon/\lambda_k\}} V(y)\ \mathrm{d}y + \varepsilon\, (\mathcal{M}_2 + M_c)\ .
\end{aligned}$$

Since $\lambda_k \to 0$ as $k \to \infty$ we infer from (4.12) and the integrability of V that

$$\limsup_{k\to\infty} \left| (\bar{x} - x_k) \int_{B(x_k,\varepsilon)} u(t_k,x)\ \mathrm{d}x \right| \le \varepsilon\, (\mathcal{M}_2 + M_c)\ .$$

Using once more (4.12) we readily deduce that

$$\lim_{k\to\infty} \int_{B(x_k,\varepsilon)} u(t_k,x)\ \mathrm{d}x = \int_{\mathbb{R}^d} V(x)\ \mathrm{d}x = M_c\ .$$

Combining the previous two limits gives

$$M_c \limsup_{k\to\infty} |\bar{x} - x_k| \le \varepsilon\, (\mathcal{M}_2 + M_c)\ ,$$

whence the last assertion of Proposition 4.4 by letting $\varepsilon \to 0$. □

For radially symmetric solutions we can remove the additional assumption on the second moment.

Corollary 4.5 (Radially symmetric blow-up) *Let u_0 be a radially symmetric initial condition satisfying (1.2) with $\|u_0\|_1 = M_c$ and consider a radially symmetric free energy solution u to (1.4) on $[0, T_\omega)$ with initial condition u_0 and $T_\omega \in (0, \infty]$ and such that $\|u(t)\|_m \to \infty$ as $t \nearrow T_\omega$. If $(t_k)_k$ is a sequence of positive real numbers such that $t_k \to T_\omega$ as $k \to \infty$, there is a sub-sequence $(t_{k_j})_j$ of $(t_k)_k$ such that*

$$\lim_{j\to\infty} \int_{\mathbb{R}^d} \left| u(t_{k_j},x) - \frac{1}{\lambda_{k_j}^d}\, V\left(\frac{x}{\lambda_{k_j}}\right) \right|\ \mathrm{d}x = 0\ ,$$

where $\lambda_k := \|u(t_k)\|_m^{-m/(d-2)}$ and V is the unique radially symmetric minimiser of $\mathcal{F}$ in $\mathcal{Y}_{M_c}$ such that $\|V\|_m = 1$.

Proof. The only modification of the proof of Proposition 4.4 is to show that we can choose $a_k = 0$ for all k at the end of Step 1. Indeed, we claim that if $\varepsilon \in (0, M_c/4)$ we have $|a_k| \le R_\varepsilon$, where a_k and R_ε are defined in (4.3). Otherwise $B(a_k, R_\varepsilon)$ and $B(-a_k, R_\varepsilon)$ are disjoint and the radial symmetry of v_k and (4.3) imply that

$$M_c \ge \int_{B(a_k,R_\varepsilon)\cup B(-a_k,R_\varepsilon)} v_k(x)\,\mathrm{d}x = 2\int_{B(a_k,R_\varepsilon)} v_k(x)\,\mathrm{d}x \ge 2\,(M_c-\varepsilon) \ge \frac{3\,M_c}{2}$$

and a contradiction. Therefore $B(a_k, R_\varepsilon) \subset B(0, 2\,R_\varepsilon)$ and thus

$$\int_{B(0,2\,R_\varepsilon)} v_k(x)\,\mathrm{d}x \ge M_c - \varepsilon$$

by (4.3). □

4.2.2 When would it blow-up?

Proposition 4.6 (Global existence in the critical case) *Let u_0 be an initial condition satisfying (1.2) with $\|u_0\|_1 = M_c$ and consider a free energy solution u to (1.4) on $[0, T_\omega)$ with initial condition u_0 and $T_\omega \in (0, \infty]$. Then $T_\omega = \infty$.*

The proof of this proposition relies on Proposition 4.4 and the following control of the behaviour of free energy solutions for large x:

Lemma 4.7 (Control of the tail) *Consider a free energy solution u to (1.4) on $[0, T_\omega)$ with initial condition u_0 satisfying (1.2) and $T_\omega \in (0, \infty]$. If $t \longmapsto \mathcal{F}[u(t)]$ is bounded from below in $[0, T)$ for some $T \le T_\omega$ then*

$$\lim_{R\to\infty} \sup_{t\in[0,T)} \int_{\{|x|>R\}} |x|^2\,u(t,x)\,\mathrm{d}x = 0\,.$$

Proof. Consider a non-decreasing function $\xi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\xi(r) = 0$ for $|r| \le 1$ and $\xi(r) = 1$ for $|r| \ge 2$ and define

$$\Phi_R(r) = r\,\xi^4\left(\frac{r}{R}\right) \quad \text{for} \quad r \in \mathbb{R} \quad \text{and} \quad R > 0\,.$$

The support of Φ_R is included in $\mathbb{R}^d \setminus B(0, R)$ and, introducing

$$\mathcal{I}_R(t) := \int_{\{|x|>R\}} \left|\left(\frac{2\,m}{2\,m-1}\,\nabla u^{(2m-1)/2} - u^{1/2}\,\nabla\phi\right)(t,x)\right|^2\,\mathrm{d}x\,,$$

we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \Phi_R\left(|x|^2\right)\,u(t,x)\,\mathrm{d}x \\
&\qquad = -\int_{\mathbb{R}^d} 2\,x\,\Phi_R'\left(|x|^2\right)\,\left(\frac{2\,m}{2\,m-1}\,\nabla u^{(2m-1)/2} - u^{1/2}\,\nabla\phi\right)(t,x)\,\mathrm{d}x \\
&\qquad \le 2\left(\int_{\mathbb{R}^d} |x|^2\,\left|\Phi_R'\left(|x|^2\right)\right|^2\,u(t,x)\,\mathrm{d}x\right)^{1/2}\,\mathcal{I}_R(t)^{1/2}\,.
\end{aligned}$$

By the definition of Φ_R, we have

$$\begin{aligned}
|\Phi_R'(r)|^2 &\le \left|\xi^4\left(\frac{r}{R}\right) + 4\,\frac{r}{R}\,\xi^3\left(\frac{r}{R}\right)\,\xi'\left(\frac{r}{R}\right)\right|^2 \le 2\,\xi^8\left(\frac{r}{R}\right) + 32\,\left(\frac{r}{R}\right)^2\,\xi^6\left(\frac{r}{R}\right)\,\left|\xi'\left(\frac{r}{R}\right)\right|^2 \\
&\le \xi^4\left(\frac{r}{R}\right)\,\left(2 + 32\,\sup_{z\in\mathbb{R}}\left|z\,\xi'(z)\right|^2\right)\,,
\end{aligned}$$

so that $r\,|\Phi_R'(r)|^2 \le C\,\Phi_R(r)$ for $r \in \mathbb{R}$. Therefore, for any $t \in [0,T)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \Phi_R\left(|x|^2\right)\,u(t,x)\,\mathrm{d}x \le C\,\left(\int_{\mathbb{R}^d} \Phi_R\left(|x|^2\right)\,u(t,x)\,\mathrm{d}x\right)^{1/2} \mathcal{I}_R(t)^{1/2}\,,$$

hence

$$\left(\int_{\mathbb{R}^d} \Phi_R\left(|x|^2\right)\,u(t,x)\,\mathrm{d}x\right)^{1/2} \le \left(\int_{\mathbb{R}^d} \Phi_R\left(|x|^2\right)\,u_0(x)\,\mathrm{d}x\right)^{1/2} + \frac{C}{2}\int_0^t \mathcal{I}_R^{1/2}(s)\,\mathrm{d}s\,.$$

Now, since $\mathcal{F}[u(t)]$ is bounded from below in $[0,T)$, we have

$$\int_0^T\int_{\mathbb{R}^d}\left|\left(\frac{2\,m}{2\,m-1}\,\nabla u^{(2m-1)/2} - u^{1/2}\,\nabla\phi\right)(s,x)\right|^2\,\mathrm{d}x\,\mathrm{d}s \le \sup_{t\in[0,T)}\left\{\mathcal{F}[u_0]-\mathcal{F}[u(t)]\right\} < \infty\,,$$

so that

$$\lim_{R\to\infty}\int_0^T \mathcal{I}_R^{1/2}(s)\,\mathrm{d}s = 0$$

by the Lebesgue dominated convergence theorem. Therefore,

$$\limsup_{R\to\infty}\,\sup_{t\in[0,T)}\int_{\mathbb{R}^d}\Phi_R\left(|x|^2\right)\,u(t,x)\,\mathrm{d}x = 0\,,$$

from which the lemma follows. □

Proof of Proposition 4.6. Assume for contradiction that T_ω is finite and let $(t_k)_k$ be a sequence of positive real numbers such that $t_k \to T_\omega$ as $k \to \infty$. Observe that Theorem 2.4 entails that $\|u(t)\|_m \to \infty$ as $t \to T_\omega$. On the one hand we infer from the nature of the blow-up given in Proposition 4.4 that there are a sub-sequence of $(t_k)_k$ (not relabelled) and a sequence $(x_k)_k$ in $\mathbb{R}^d$ such that

$$\lim_{k\to\infty} x_k = \bar{x} := \frac{1}{M_c}\,\int_{\mathbb{R}^d} x\,u_0(x)\;\mathrm{d}x\,, \tag{4.13}$$

$$\lim_{k\to\infty}\int_{\mathbb{R}^d}\left|u(t_k,x+x_k) - \frac{1}{\lambda_k^d}\,V\left(\frac{x}{\lambda_k}\right)\right|\;\mathrm{d}x = 0 \tag{4.14}$$

with $\lambda_k := \|u(t_k)\|_m^{-m/(d-2)}$. On the other hand it follows from Proposition 3.4 and Lemma 4.1 that $\mathcal{F}[u(t)] \ge 0$ for $t \in [0,T_\omega)$ so that

$$\begin{aligned}\int_{\mathbb{R}^d}|x-\bar{x}|^2\,u(t,x)\;\mathrm{d}x &= \int_{\mathbb{R}^d}|x-\bar{x}|^2\,u_0(x)\;\mathrm{d}x + 2(d-2)\int_0^t\mathcal{F}[u(s)]\;\mathrm{d}s\\ &\ge \int_{\mathbb{R}^d}|x-\bar{x}|^2\,u_0(x)\;\mathrm{d}x > 0\,,\end{aligned} \tag{4.15}$$

and Lemma 4.7 may be applied to obtain

$$\lim_{R\to\infty}\,\sup_{t\in[0,T_\omega)}\int_{\{|x|>R\}}|x|^2\,u(t,x)\,\mathrm{d}x = 0\,. \tag{4.16}$$

Now, for $k \ge 1$ and $R \ge |\bar{x}|$ we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |x-\bar{x}|^2\, u(t_k,x)\ \mathrm{d}x \le 2 \int_{\{|x-\bar{x}|\ge 2R\}} \left(|x|^2+|\bar{x}|^2\right)\ u(t_k,x)\ \mathrm{d}x \\
+ \int_{\{|x-\bar{x}|<2R\}} |x-\bar{x}|^2 \left[u(t,x) - \frac{1}{\lambda_k^d}\, V\left(\frac{x-x_k}{\lambda_k}\right)\right]\ \mathrm{d}x \\
+ \int_{\{|x-\bar{x}|<2R\}} \frac{|x-\bar{x}|^2}{\lambda_k^d}\, V\left(\frac{x-x_k}{\lambda_k}\right)\ \mathrm{d}x \\
\le 4 \int_{\{|x|\ge R\}} |x|^2\, u(t_k,x)\ \mathrm{d}x + R^2 \int_{\mathbb{R}^d} \left|u(t,x) - \frac{1}{\lambda_k^d}\, V\left(\frac{x-x_k}{\lambda_k}\right)\right|\ \mathrm{d}x \\
+ \int_{\mathbb{R}^d} |\lambda_k\, x + x_k - \bar{x}|^2\ V(x)\ \mathrm{d}x \\
\le 4 \sup_{t\in[0,T_\omega)} \int_{\{|x|\ge R\}} |x|^2\, u(t,x)\ \mathrm{d}x + R^2 \int_{\mathbb{R}^d} \left|u(t,x) - \frac{1}{\lambda_k^d}\, V\left(\frac{x-x_k}{\lambda_k}\right)\right|\ \mathrm{d}x \\
+ 2\ |x_k-\bar{x}|^2\ M_c + 2\,\lambda_k^2 \int_{\mathbb{R}^d} |x|^2\, V(x)\ \mathrm{d}x\ .
\end{aligned}$$

Owing to (4.13), (4.14), and the convergence of $(\lambda_k)_k$ to zero we may let $k \to \infty$ in the previous inequality to obtain

$$\limsup_{k\to\infty} \int_{\mathbb{R}^d} |x-\bar{x}|^2\, u(t_k,x)\ \mathrm{d}x \le 4 \sup_{t\in[0,T_\omega)} \int_{\{|x|\ge R\}} |x|^2\, u(t,x)\ \mathrm{d}x\ .$$

We next pass to the limit as $R \to \infty$ with the help of (4.16) to conclude that

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} |x-\bar{x}|^2\, u(t_k,x)\ \mathrm{d}x = 0\ ,$$

which contradicts (4.15). □

4.2.3 Does it blow-up?

Let us first note that Proposition 4.4 allows us to describe the nature of the blow-up when it occurs. We define the two following statements:

$$\text{There exists } (t_k)_k \nearrow \infty \text{ such that } (\|u(t_k)\|_m)_k \text{ is bounded} \tag{S1}$$

$$\mathcal{M}_2^\infty := \lim_{t\to\infty} \int_{\mathbb{R}^d} |x|^2 u(t,x)\ \mathrm{d}x < \infty \tag{S2}$$

- If [not (S1)] and (S2): By Proposition 4.4, the solution blows up as a Dirac mass at the centre of mass as t goes to infinity. Moreover, the blow-up profile is described by the minimisers of $\mathcal{F}$ for the critical mass.
- If (S1) and (S2): By the virial identity Lemma 4.1, $\mathcal{F}[u(t_k)] \to 0$ so that $(u(t_k))_k$ is a minimising sequence for $\mathcal{F}$ in $\mathcal{Y}_{M_c}$. We expect that it converges to the minimiser of $\mathcal{F}$ in $\mathcal{Y}_{M_c}$ with centre of mass $\bar{x}$ defined in (4.1) and second moment $\mathcal{M}_2^\infty$.
- If [not (S1)] and [not (S2)]: By Proposition 4.4, the solution blows up as a Dirac mass. However, we cannot prevent the escape at infinity of the Dirac mass.
- If (S1) and [not (S2)]: No precise information can be deduced in this case. We cannot even rule out the possibility of the existence of another sequence of times for which the L^m-norm diverges.

In the radially symmetric case, if the initial condition is less concentrated than one of the stationary solutions, then we strongly believe that such a property remains true for all times, thus excluding the formation of a Dirac mass. According to the above discussion this prevents the blow-up of the L^m-norm in infinite time and give an example where (S1) and (S2) hold true. This is in sharp contrast with the two-dimensional PKS case where infinite time blow-up always occurs, see [2, 3].

5 Sub-critical self-similar solutions

The aim of this section is to prove the existence of self-similar solution by variational techniques. Actually, it is equivalent to show the existence of minimisers for the free energy $\mathcal{G}$ associated to the rescaled problem (1.3) given by

$$\mathcal{G}[h] := \mathcal{F}[h] + \frac{1}{2} M_2[h] \quad \text{with} \quad M_2[h] := \int_{\mathbb{R}^d} |x|^2 \, |h(x)| \, \mathrm{d}x$$

for $h \in \mathrm{L}^1(\mathbb{R}^d; (1+x^2)\,\mathrm{d}x) \cap \mathrm{L}^m(\mathbb{R}^d)$. For $M > 0$, we define

$$\nu_M := \inf\{\mathcal{G}[h] \,:\, h \in \mathcal{Z}_M\} \quad \text{with} \quad \mathcal{Z}_M := \{h \in \mathcal{Y}_M \,:\, M_2[h] < \infty\} \,.$$

We first establish the following analogue of Proposition 3.4.

Proposition 5.1 (Infimum of the rescaled free energy) *For $M > 0$ and $h \in \mathcal{Z}_M$ we have*

$$\mathcal{G}[h] \ge \frac{C_* \, c_d}{2} \left(M_c^{2/d} - M^{2/d} \right) \|h\|_m^m + \frac{1}{2} M_2[h] \,. \tag{5.1}$$

In addition,

$$\begin{cases} \nu_M > 0 & \text{if } M < M_c \,, \\ \nu_{M_c} = 0 \,, & \\ \nu_M = -\infty & \text{if } M > M_c \,. \end{cases}$$

Proof. The inequality (5.1) readily follows from (3.7) and the definition of $\mathcal{G}$. Consider next $M \ge M_c$ and put

$$h_R(x) := \begin{cases} \dfrac{M}{M_c} \dfrac{1}{R^d} \zeta^{d/(d-2)} \left(\dfrac{x}{R^d} \right) & \text{if } x \in B(0,R) \,, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus B(0,R) \,, \end{cases}$$

where the function ζ is defined in Proposition 3.5 and $R > 0$. We compute $\mathcal{G}[h_R]$ and use the property $\mathcal{F}[\zeta^{d/(d-2)}] = 0$ to obtain

$$\nu_M \le \mathcal{G}[h_R] = \left(\frac{M}{M_c} \right)^2 R^{2-d} \left[\frac{M_c}{2M} R^d M_2 \left[\zeta^{d/(d-2)} \right] - \left(1 - \left(\frac{M_c}{M} \right)^{2-m} \right) \frac{\|\zeta^{d/(d-2)}\|_m^m}{m-1} \right] \,.$$

Now, either $M > M_c$ and the right-hand side of the above inequality diverges to $-\infty$ as $R \to 0$ since $d > 2$ and $m < 2$. Consequently $\nu_m = -\infty$ in that case. Or $M = M_c$ and we may let $R \to 0$ in the above inequality to obtain that $\nu_{M_c} \le 0$. Since $\mathcal{G}$ is non-negative by Proposition 3.4, we conclude that $\nu_{M_c} = 0$.

Finally, assume for contradiction that $\nu_M = 0$ for some $M < M_c$ and let $(h_k)_k$ be a minimising sequence for $\mathcal{G}$ in $\mathcal{Z}_M$. Since $\mathcal{G}[h_k] \ge \mathcal{G}[|h_k|]$, $(|h_k|)_k$ is also a minimising sequence for $\mathcal{G}$ in $\mathcal{Z}_M$ and we infer from (5.1) that

$$\lim_{k \to \infty} \left(\|h_k\|_m + M_2[h_k] \right) = 0 \,.$$

By Vitali's theorem $(|h_k|)_k$ converges towards zero in $\mathrm{L}^1(\mathbb{R}^d)$ which contradicts the fact that $\|h_k\|_{\mathrm{L}^1} = M$ for all $k \ge 1$. Therefore $\nu_M \neq 0$ and the non-negativity of $\mathcal{G}$ in $\mathcal{Z}_M$ entails that $\nu_M > 0$. □

We next identify the minimisers of $\mathcal{G}$ in $\mathcal{Z}_M$ for $M \in (0, M_c)$.

Theorem 5.2 (Identification of minimisers) *If $M \in (0, M_c)$ there is a unique minimiser W_M of $\mathcal{G}$ in $\mathcal{Z}_M$. In addition, W_M is non-negative radially symmetric and non-increasing and there is a unique $\varrho_M > 0$ such that $W_M(x) = 0$ for $|x| \ge \varrho_M$ and $\xi_M := W_M^{m-1}$ solves*

$$\Delta \xi_M + \frac{m-1}{m} \left(\xi_M^{1/(m-1)} + d \right) = 0 \quad in \quad B(0, \varrho_M) \quad with \quad \xi_M = 0 \quad on \quad \partial B(0, \varrho_M) \, .$$

Several steps are required to perform the proof of Theorem 5.2 which borrows several arguments from [21, 23]. We first establish the existence of minimisers of $\mathcal{G}$ in $\mathcal{Z}_M$ for $M \in (0, M_c)$.

Lemma 5.3 (Existence of minimisers) *Consider $M \in (0, M_c)$. The functional $\mathcal{G}$ has at least a minimiser in $\mathcal{Z}_M$. In addition, every minimiser of $\mathcal{G}$ in $\mathcal{Z}_M$ is non-negative radially symmetric and non-increasing.*

Proof. We first recall that, if $h \in \mathrm{L}^1(\mathbb{R}^d; (1 + |x|^2) \, \mathrm{d}x)$ and h^* denotes its symmetric decreasing rearrangement, then $M_2[h^*] \le M_2[h]$. Thanks to this property, we may next argue as in the proof of Lemma 3.3 to conclude that there is at least a minimiser of $\mathcal{G}$ in $\mathcal{Z}_M$.

Next, let W be a minimiser of $\mathcal{G}$ in $\mathcal{Z}_M$ and denote by W^* its symmetric decreasing rearrangement. As

$$\|W^*\|_1 = \|W\|_1 \, , \quad \|W^*\|_m = \|W\|_m \, , \quad \text{and} \quad M_2[W^*] \le M_2[W] \, ,$$

W^* belongs to $\mathcal{Z}_M$. In addition, by Riesz's rearrangement inequality [21, Lemma 2.1], $\mathcal{W}(W) \le \mathcal{W}(W^*)$. Consequently, $\nu_M = \mathcal{G}[W] \ge \mathcal{G}[W^*]$ and W^* is also a minimiser of $\mathcal{G}$ in $\mathcal{Z}_M$. This last property entails that

$$M_2[W^*] = M_2[W] \quad \text{and} \quad \mathcal{W}(W^*) = \mathcal{W}(W) \, .$$

Using once more [21, Lemma 2.1 (ii)] we deduce from $\mathcal{W}(W^*) = \mathcal{W}(W)$ that there is $y \in \mathbb{R}^d$ such that $W(x) = W^*(x + y)$ for $x \in \mathbb{R}^d$. Then $M_2[W^*] = M_2[W]$ implies that $y = 0$, which completes the proof. □

We are thus left with the uniqueness issue to complete the proof of Theorem 5.2. To this end we adapt the proof in [23, Section IV.B] and first proceed as in the proof of Proposition 3.5 to identify the Euler-Lagrange equation satisfied by the minimisers of $\mathcal{G}$ in $\mathcal{Z}_M$.

Lemma 5.4 *Consider $M \in (0, M_c)$ and let W be a minimiser of $\mathcal{G}$ in $\mathcal{Z}_M$. Then there is $\varrho > 0$ such that $W(x) = 0$ if $|x| \ge \varrho$ and $\xi := W^{m-1}$ is a non-negative radially symmetric and non-increasing classical solution to*

$$\Delta \xi + \frac{m-1}{m} \left(\xi^{1/(m-1)} + d \right) = 0 \quad in \quad B(0, \varrho) \quad with \quad \xi = 0 \quad on \quad \partial B(0, \varrho) \, .$$

In addition,

$$\frac{m}{m-1} \, W^{m-1} = \left(\mathcal{K} * W - \frac{|x|^2}{2} + \frac{1}{2} + \frac{m}{m-1} \, M^{m-1} - \frac{c_d}{M} \, \mathcal{W}(W) \right)_+ \qquad a.e. \ in \ \mathbb{R}^d \, . \tag{5.2}$$

Additional properties of minimisers of $\mathcal{G}$ in $\mathcal{Z}_M$ can be deduced from Lemma 5.4.

Lemma 5.5 *Consider $M \in (0, M_c)$ and let W be a minimiser of $\mathcal{G}$ in $\mathcal{Z}_M$. Then*

$$M_2[W] = (d-2)\,\mathcal{F}[W] = 2(m-1)\,\nu_M\,, \tag{5.3}$$

$$\frac{2m}{m-1}\,\|W\|_m^m + M_2[W] = \frac{2m}{m-1}\,M^m + M\,. \tag{5.4}$$

Proof. We proceed as in [23, Lemma 6]. By Lemma 5.4 we have

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{d-1}\,\frac{\mathrm{d}\xi}{\mathrm{d}r}(r)\right) = \frac{m-1}{m}\,\left(r^{d-1}\,W(r) + d\,r^{d-1}\right) \quad \text{for} \quad r \in (0,\varrho)\,,$$

where ϱ denotes the radius of the support of W and $\xi := W^{m-1}$. Introducing

$$Q(r) := \int_{B(0,r)} W(x)\;\mathrm{d}x = \sigma_d\,\int_0^r W(z)\,z^{d-1}\;\mathrm{d}z \quad \text{for} \quad r \in (0,\varrho)\,,$$

we integrate the previous differential equation to obtain

$$-m\,r^{d-1}\,W(r)^{m-2}\,\frac{\mathrm{d}W}{\mathrm{d}r}(r) = \frac{Q(r)}{\sigma_d} + r^d \quad \text{for} \quad r \in (0,\varrho)\,.$$

Multiplying the above identity by $\sigma_d\,r\,W(r)$ and integrating over $(0,\infty)$ then lead us to the formula

$$d\,\|W\|_m^m = \int_0^\infty r\,Q(r)\,W(r)\;\mathrm{d}r + M_2[W]\,.$$

As

$$2\sigma_d\,\int_0^\infty r\,Q(r)\,W(r)\;\mathrm{d}r = \mathcal{W}(W)$$

by Newton's theorem [22, Theorem 9.7], we end up with the identity $(d-2)\,\mathcal{F}[W] = M_2[W]$ and (5.3) follows by the definition of ν_M and $\mathcal{G}$. We next multiply (5.2) by $2\,W$ and integrate over $\mathbb{R}^d$ to obtain (5.4). □

We next prove the following comparison result.

Lemma 5.6 *Consider $M_1 \in (0, M_c)$ and $M_2 \in (0, M_c)$. For $i = 1, 2$ let W_i be a minimiser of $\mathcal{G}$ in $\mathcal{Z}_{M_i}$ and denote by ϱ_i the radius of its support (which is finite according to Lemma 5.4). If $W_1(0) > W_2(0)$ then $Q_1(r) > Q_2(r)$ for $r \in (0, \max\{\varrho_1, \varrho_2\})$ where*

$$Q_i(r) := \int_{B(0,r)} W_i(x)\;\mathrm{d}x \quad \textit{for} \quad r \in (0, \max\{\varrho_1, \varrho_2\}) \quad \textit{and} \quad i = 1, 2\,.$$

Owing to Lemma 5.3 and Lemma 5.4, the proof of Lemma 5.6 is similar to that of [23, Lemma 10] to which we refer.

Proof of Theorem 5.2. Consider $M \in (0, M_c)$ and assume for contradiction that $\mathcal{G}$ has two minimisers W_1 and W_2 in $\mathcal{Z}_M$ with $W_1(0) > W_2(0)$. Denoting by ϱ_i the radius of the support of W_i and introducing

$$Q_i(r) := \int_{B(0,r)} W_i(x)\;\mathrm{d}x$$

for $r \in [0, \max\{\varrho_1, \varrho_2\}]$ and $i = 1, 2$, we infer from Lemma 5.6 that $Q_1(r) > Q_2(r)$ for all $r \in (0, \max\{\varrho_1, \varrho_2\})$. Then $\varrho_1 \le \varrho_2$ and (5.3) warrants that

$$2\,(m-1)\,\nu_M = \sigma_d\,\int_0^\infty r^2\,\frac{\mathrm{d}}{\mathrm{d}r}\,(Q_i - M)\,(r)\;\mathrm{d}r = \int_0^\infty 2\,r\;(M - Q_i(r))\;\mathrm{d}r$$

for $i = 1, 2$. Consequently,

$$\int_0^{\varrho_2} 2\,r\,\left(Q_1 - Q_2\right)(r)\;\mathrm{d}r = 0\;,$$

which implies that $\varrho_1 = \varrho_2$ and $Q_1 = Q_2$, hence a contradiction. □

Corollary 5.7 *If $M \in (0, M_c)$ there exists a self-similar solution U_M to (1.4) given by*

$$U_M(t,x) = \frac{1}{1+dt} W_M\left(\frac{x}{(1+dt)^{1/d}}\right)\;,$$

where W_M is the unique minimiser of $\mathcal{G}$ in $\mathcal{Z}_M$ given in Theorem 5.2.

Remark 5.8 *Given $M \in (0, M_c)$, we expect that this self-similar solution attracts the dynamics of (1.4) for large times. Although we can prove that the ω-limit set of the rescaled equation (1.3) consists of stationary solutions, we are yet lacking a uniqueness result to identify them as W_M.*

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