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Monge Extensions of Cooperation and Communication Structures

U. Faigle*, M. Grabisch†, M. Heyne

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Abstract

Cooperation structures without any *a priori* assumptions on the combinatorial structure of feasible coalitions are studied and a general theory for marginal values, cores and convexity is established. The theory is based on the notion of a Monge extension of a general characteristic function, which is equivalent to the Lovász extension in the special situation of a classical cooperative game. It is shown that convexity of a cooperation structure is tantamount to the equality of the associated core and Weber set. Extending Myerson's graph model for game theoretic communication, general communication structures are introduced and it is shown that a notion of supermodularity exists for this class that characterizes convexity and properly extends Shapley's convexity model for classical cooperative games.

Keywords: communication structure, convex game, cooperation structure, Monge extension, Lovász extension, marginal value, ranking, Shapley value, supermodularity, Weber set

AMS Classification: 91A12, 91A40.

1 Introduction

The classical model of cooperative games assumes that every subset of a set N of agents may form a coalition to execute the game. However, many

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situations require a more refined model in which only a restricted collection \mathcal{F} of subsets describes feasible cooperation. In Myerson's [25] communication graph model, for example, only those sets of agents are feasible for communication that induce connected subgraphs. Other examples arise from models where N is (partially) ordered by some dominance or preference relation (*e.g.*, Derks and Gilles [8], Faigle and Kern [14, 15], Gilles *et al.* [17], Grabisch and Lange [18], Hsiao and Raghavan [19]). The latter model was further relaxed and studied by Algaba *et al.* [3], Bilbao *et al.* [2, 5] to combinatorial coalition structures of so-called antimatroids, convex geometries and augmenting systems, and by Lange and Grabisch to regular set systems [22]. All these generalized models for cooperation rely on their particular combinatorial structure for the definition of Shapley-type values, Weber sets and cores. Indeed, it appears difficult to reasonably define a notion of a "marginal value" for cooperation models without special structural properties. Moreover, it seems to be impossible to extend the concept of *supermodular* characteristic functions, and hence of *convex* games, to coalition systems that are not closed under union and intersection.

On the other hand, a natural notion for the core of a general cooperation structure exists as a certain convex set in the Euclidean parameter space \mathbb{R}^N (Faigle [12]), which suggests to study general cooperation from the point of view of real convex analysis. For the classical model, such an approach was indicated by Lovász [23] (see also Algaba *et al.* [4]). It is the purpose of our present investigation to show that Lovász' construction is actually a special case of a quite general construction that is meaningful for arbitrary cooperation structures.

The key in our analysis is the relaxation of the notion of a cooperative game to cooperative *game instances* with given bounds on the activity levels of individual agents. We obtain game instances by a straightforward rule that goes back to Monge [24] and corresponds to the well-known *north-west corner rule* for the construction of feasible solutions for transportation problems (Section 3). Our rule yields the *Monge extension* of the characteristic function v of the underlying cooperation structure to a function $\hat{v} : \mathbb{R}^N \rightarrow \mathbb{R}$. Convexity properties of arbitrary cooperation structures can thus be studied *via* their Monge extensions.

The Monge algorithm furthermore implies a natural ranking notion for agents and thus a framework for marginal vectors, Weber sets and Shapley values (Section 4). In a far-reaching extension of the classical results we find that the Monge extension of a cooperation structure is concave (a.k.a. convex down) if and only if its core and Weber set coincide (Theorem 5.2).

In Section 6, we introduce *communication structures* as a particular class of cooperation structures that are union-closed in a weak sense and hence include Myerson's communication graph model as a special case. We show that a meaningful notion of "supermodularity" exists for this class and characterizes convexity (Theorem 6.2). Hence convex communication structures generalize in particular Shapley's [28] convex cooperative games. Moreover, we show that our general model of convexity implies the notion of convexity introduced by Bilbao and Ordóñez [6] for games on so-called *augmenting systems*, which form a subclass of communication structures.

We always assume that the characteristic function of a cooperation structure describes the *gain* a feasible coalition may achieve. As in the classical case, our cores may equally well be interpreted as arising from associated *cost* games. However, we will not explore the latter model in detail here.

2 Cooperation Structures

Let $N = \{1, \dots, n\}$ be a finite set of *players*. A *cooperation structure* on N is a pair $\Gamma = (\mathcal{F}, v)$, where \mathcal{F} is a family of non-empty subsets of N and $v : \mathcal{F} \rightarrow \mathbb{R}_+$ is a non-negative *valuation* on \mathcal{F} . We refer to a set $F \in \mathcal{F}$ as a *feasible coalition* of Γ . In the case $\mathcal{F} = 2^N \setminus \{\emptyset\}$, i.e., when each non-empty subset of N constitutes a feasible coalition, we say that Γ is a *classical cooperative game*.

REMARK. Strictly speaking, a classical cooperative game may include coalitions F with negative value $v(F) < 0$. Modifying v to a valuation \bar{v} with

$$\bar{v}(F) = v(F) + \kappa \cdot |F| \quad (F \subseteq N),$$

where $\kappa > 0$ is a suitably large constant, however, any classical game is seen to be essentially equivalent to a non-negative classical game.

The next example may serve as a motivation for leaving the classical context. (It will be taken up in Section 6.)

Example 2.1 (Myerson Games [25]) Let $G = (N, E)$ be a graph with node set N and edge set E with the interpretation that $x, y \in N$ may "communicate" if $\{x, y\} \in E$. One is interested in the family \mathcal{F} of those non-empty subsets $F \subseteq N$ that induce a connected subgraph of G and hence allow communication paths among all members of F to be established. $v(F)$ describes the value of the communication within the connected subgraph with node set F .

Throughout the paper we index the coalitions in $\mathcal{F} = \{F_1, \dots, F_m\}$ so that

$$(I_1) \quad F_i \supseteq F_j \implies i < j.$$

In some parts of the paper, we will suppose that v is *monotone* in the sense

$$F \subseteq F' \implies v(F) \leq v(F').$$

If monotonicity holds, we can (and will) assume in addition that the indexing of coalitions also satisfies the property

$$(I_2) \quad v(F_1) \geq \dots \geq v(F_m).$$

2.1 Game Instances with Activity Bounds

Let $c \in \mathbb{R}^N$ be a fixed parameter vector. A *c-feasible game instance* is a parameter vector $y \in \mathbb{R}^{\mathcal{F}}$ such that $y_F \geq 0$ holds for all $F \neq N$ and

$$a_j(y) = \sum_{F \ni j} y_F \leq c_j \quad \text{for all } j \in N.$$

We interpret y_F as the activity level of the coalition $F \in \mathcal{F}$ (i.e., the activity contribution of each $j \in F$ relative to F) in the cooperation effort. So $a_j(y)$ measures the total activity of the player j with respect to y , and the vector c plays the role of an *activity bound*. The *value* of the game instance y is the parameter

$$y(v) = \sum_{F \in \mathcal{F}} y_F v(F).$$

Writing $y_F = y_F^+ - y_F^-$, where $y_F^+ = \max\{0, y_F\}$ and $y_F^- = \max\{0, -y_F\}$, we note

$$y_F = y_F^+ \geq 0 \quad \text{for all } F \neq N.$$

In the case $N \in \mathcal{F}$, we may view $\sigma(y) = y_N^- \cdot v(N)$ as the *setup cost* for the game instance y and the numbers $y_F^+ v(F)$ as the values generated by the coalitions $F \in \mathcal{F}$ at the activity levels y_F^+ . The players j thus respect the activity bounds

$$0 \leq \sum_{F \ni j} y_F^+ \leq c_j + y_N^-.$$

In the following, we will allow for setup costs and therefore assume

- $N \in \mathcal{F}$ (and thus $F_1 = N$ in the listing $\mathcal{F} = \{F_1, \dots, F_m\}$),

unless stated otherwise.

3 Monge Extensions

Assuming $N \in \mathcal{F}$, we turn our attention to the construction of c -feasible game instances y in the context (\mathcal{F}, v) according to a generalized north-west corner rule for transportation problems. We therefore refer to these game instances as being *Monge*. For the description of the algorithm, we use the notation

$$\mathcal{F}(X) := \{F \in \mathcal{F} \mid F \subseteq X\} \quad \text{for any } X \subseteq N.$$

3.1 The Monge Algorithm

We construct sequences μ, π and a vector $y \in \mathbb{R}^m$ as follows for any given $c \in \mathbb{R}^N$. As usual, if μ, μ' are sequences, $\mu\mu'$ denotes the concatenation of the two sequences, and \square denotes the empty sequence.

Monge Algorithm (MA):

- (0) Set $X = N$, $\mu = \square$, $\pi = \square$ and $y_i = 0$ for all $i = 1, \dots, m$.
Set $\gamma_j = c_j$ for all $j = 1, \dots, n$.
- (1) Select the $F_s \in \mathcal{F}(X)$ with the smallest index s and the smallest $p \in F_s$ with $\gamma_p = \min\{\gamma_t \mid t \in F_s\}$.
- (2) Update $\mu \leftarrow [\mu s]$, $\pi \leftarrow [\pi p]$, $y_s \leftarrow \gamma_p$, $X \leftarrow [X \setminus p]$;
Update $\gamma_t \leftarrow [\gamma_t - \gamma_p]$ for all $t \in F_s$.
- (3) If $\mathcal{F}(X) = \emptyset$ then output (μ, π, y) and stop;
Otherwise goto (1).

Let (μ, π, y) be the output of the Monge algorithm and assume $\mu = i_1 \dots i_k$ (with $i_1 = 1$). Setting

$$\mathcal{M} = \mathcal{M}(\mu) := \{M_1, \dots, M_k\} = \{F_{i_1}, \dots, F_{i_k}\} \quad (\text{i.e., } M_s = F_{i_s}),$$

we find

$$\langle v, y \rangle = \sum_{i=1}^m y_i v(F_i) = \sum_{s=1}^k y_{i_s} v(M_s).$$

Notice that the selection rule (1) and the update rule (2) in MA guarantee $y_i \geq 0$ for all $F_i \neq N$. So y yields indeed a game instance. Moreover, we have for all $j \in N$,

$$\sum_{F_i \ni j} y_i \quad \begin{cases} = c_j & \text{if } j \text{ occurs in } \pi \\ \leq c_j & \text{otherwise.} \end{cases}$$

With the interpretation $y_{F_i} = y_i$ for $i = 1, \dots, m$, the Monge algorithm thus generates a c -feasible game instance. The output sequence π of MA is not necessarily a permutation of N , *i.e.*, not every $j \in N$ may occur in π . However, we observe that π is representative for \mathcal{F} in the following sense:

Lemma 3.1 *Let $(\mu, \pi = p_1 \dots p_k, y)$ be the output of the Monge algorithm for some $c \in \mathbb{R}^n$. Then $F \cap \{p_1, \dots, p_k\} \neq \emptyset$ holds for all $F \in \mathcal{F}$.*

◇

Example 3.1 *Let $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{F} = \{12345, 2345, 1345, 124, 234, 345, 12, 35, 2, 5\}$, where "12345" stands for $\{1, 2, 3, 4, 5\}$ etc. (see Figure 1). For any $c \in \mathbb{R}^N$ with $c_4 < c_3 < c_2 < c_1 < c_5$, the algorithm will*

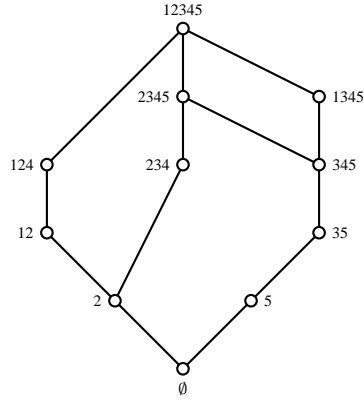


Figure 1: Example of a family of feasible coalitions, ordered by inclusion.

produce the sequences $\mu = (1, 7, 8, 10)$ (corresponding to the coalitions 12345, 12, 35 and 5), $\pi = (4, 2, 3, 5)$ and the vector

$$y = (c_4, 0, 0, 0, 0, c_2 - c_4, c_3 - c_4, c_5 - c_3) \in \mathbb{R}^{\mathcal{F}}.$$

3.1.1 The Greedy Algorithm

If v is monotone and the coalitions are indexed according to the rules (I_1) and (I_2) , the Monge algorithm may be viewed as a *greedy algorithm* for the construction of a game instance: *Sequentially pick a feasible coalition F_s of maximal value $v(F_s)$ and assign to the variable y_s the maximal possible value \tilde{y}_s without violating the individual activity bounds c_j .*

Viewed as a greedy algorithm, the Monge algorithm is also meaningful in the case $N \notin \mathcal{F}$. The output vector \tilde{y} (the so-called "greedy solution") will be feasible for the linear program

$$\max \langle v, y \rangle = \sum_{F \in \mathcal{F}} v(F) y_F \quad \text{s.t.} \quad \sum_{F \ni p} y_F \leq c_p, \quad \forall p \in N.$$

Moreover, \tilde{y} will be nonnegative for any (nonnegative) input $c \geq 0$.

3.1.2 Rankings

The output $\pi = p_1 \dots p_k$ of the Monge algorithm provides a *ranking* of the players of N : *Sequentially pick a representative p of a feasible coalition F_s of maximal possible value $v(F_s)$ and discard the coalitions already represented from further consideration.*

3.2 The Extension Function

Notice that the output (μ, π, y) of the Monge algorithm is uniquely determined by the input $c \in \mathbb{R}^n$, provided the indexing of coalitions in \mathcal{F} is fixed. So MA yields a well-defined function

$$c \in \mathbb{R}^n \mapsto \hat{v}(c) := \langle v, y \rangle \in \mathbb{R}.$$

We call $\hat{v} : \mathbb{R}^n \rightarrow \mathbb{R}$ the *Monge extension* of the valuation $v : \mathcal{F} \rightarrow \mathbb{R}$ and justify the terminology as follows.

Lemma 3.2 $\hat{v}(\mathbf{1}_F) = v(F)$ holds for all $F \in \mathcal{F}$, where $\mathbf{1}_F \in \{0, 1\}^N$ is the incidence vector of $F \subseteq N$ (with components $(\mathbf{1}_F)_j = 1$ if and only if $j \in F$).

Proof. Take $F \in \mathcal{F}$ and consider $c = \mathbf{1}_F$. Since $F \in \mathcal{F}$ and all elements corresponding to zeroes of c are selected first, $M_s = F$ at some step s . So

$$\hat{v}(\mathbf{1}_F) = v(M_s) = v(F)$$

follows by the definition of y . ◇

REMARK. In the case $\mathcal{F} = 2^N \setminus \emptyset$, the Monge extension \hat{v} corresponds to the extension introduced by Lovász [23] for the set function v , which equals the discrete Choquet integral [7] when v is monotone. The authors show in a companion paper [13] how the Choquet integral extends to arbitrary set families \mathcal{F} via the Monge algorithm.

4 Core and Weber Set

Let (\mathcal{F}, v) be a cooperation structure with a monotone valuation v . We define the *core* of $\Gamma = (\mathcal{F}, v)$ as the closed convex set

$$\text{core}(v) := \{x \in \mathbb{R}^N \mid \langle c, x \rangle \geq \hat{v}(c), \forall c \in \mathbb{R}^N\} \subseteq \mathbb{R}^N.$$

We next give a direct characterization of the core which exhibits $\text{core}(v)$ as a non-negative and bounded polyhedron. As usual, we employ the notation $x(S) := \langle \mathbf{1}_S, x \rangle = \sum_{j \in S} x_j$ for any $x \in \mathbb{R}^N$ and $S \subseteq N$.

Theorem 4.1 *Assume $\mathcal{F} \ni N$ and v monotone. Then one has*

$$\text{core}(v) = \{x \in \mathbb{R}_+^N \mid x(N) = v(N), x(F) \geq v(F), \forall F \in \mathcal{F}\}. \quad (1)$$

Proof. Let $\mathbb{P}(v) = \{x \in \mathbb{R}_+^N \mid x(N) = v(N), x(F) \geq v(F), \forall F \in \mathcal{F}\}$ and consider any $x \in \text{core}(v)$. Since v is non-negative by monotonicity, $\hat{v}(c) \geq 0$ holds for every $c \geq 0$. Letting $c = \mathbf{1}_j$ be the j th unit vector in \mathbb{R}^N , we obtain

$$x_j = \langle \mathbf{1}_j, x \rangle \geq \hat{v}(\mathbf{1}_j) \geq 0 \quad \text{for all } j \in N.$$

Moreover, $\hat{v}(\mathbf{1}_N) = v(N)$ and $\hat{v}(-\mathbf{1}_N) = -v(N)$ immediately yields $x(N) = \langle \mathbf{1}_N, x \rangle = v(N)$. In view of $\hat{v}(\mathbf{1}_F) = v(F)$ (Lemma 3.2), we thus conclude $x \in \mathbb{P}(v)$.

To prove the converse, observe that any $z \in \mathbb{P}(v)$ is a feasible solution for the linear program

$$\min_{x \geq 0} \langle c, x \rangle \quad \text{s.t.} \quad x(N) = v(N), x(F) \geq v(F), \forall F \in \mathcal{F}.$$

Let \bar{y} be the output of the Monge algorithm with respect to c . Then \bar{y} is a feasible solution for the dual linear program

$$\max_y \langle v, y \rangle \quad \text{s.t.} \quad \sum_{F \ni j} y_F \leq c_j, \forall j \in N, y_F \geq 0 \forall F \in \mathcal{F} \setminus \{N\}.$$

So $\langle c, z \rangle \geq \langle v, \bar{y} \rangle = \hat{v}(c)$ and hence $z \in \text{core}(v)$ follows from linear programming duality.

◇

REMARK. Theorem 4.1 shows that $\text{core}(v)$ coincides with the notion of the *positive core* for "cooperative games with restricted cooperation" introduced in Faigle [12].

4.1 Marginal Vectors

To study marginal vectors relative to the cooperation structure $\Gamma = (\mathcal{F}, v)$, consider the output $(\mu = i_1 \dots i_k, \pi = p_1 \dots p_k, y)$ of the Monge algorithm with respect to the input c . Note that μ and y can be reconstructed from the knowledge of the ranking sequence $\pi = p_1 \dots p_k$ (given the fixed linear arrangement $\mathcal{F} = \{F_1, \dots, F_m\}$). We let Π denote the collection of all possible ranking sequences.

Recalling the notation $\mathcal{M}(\mu) = \{M_1, \dots, M_k\}$, consider the (π, μ) -incidence matrix $R = [r_{st}] \in \{0, 1\}^{k \times k}$ with the coefficients

$$r_{st} = \begin{cases} 1 & \text{if } p_s \in M_t \\ 0 & \text{otherwise.} \end{cases}$$

R is (lower) triangular with diagonal elements $r_{ss} = 1$ and hence invertible. Let \bar{y} (resp. \bar{v}) and \bar{c} denote the restriction of y (resp. v) to μ and of c to π . Then we have

$$R\bar{y} = \bar{c} \quad \text{and} \quad \hat{v}(c) = \langle \bar{v}, \bar{y} \rangle.$$

Putting $\bar{x}^T := \bar{v}^T R^{-1}$, we therefore obtain

$$\hat{v}(c) = \bar{v}^T \bar{y} = \bar{x}^T R\bar{y} = \bar{x}^T \bar{c} = \langle \bar{c}, \bar{x} \rangle. \quad (2)$$

We extend \bar{x} to the vector $x^\pi \in \mathbb{R}^N$ by setting $x_p^\pi = \bar{x}_p$ if p occurs in π and $x^\pi = 0$ otherwise. x^π is the *marginal vector* of $\Gamma = (\mathcal{F}, v)$ associated with $c \in \mathbb{R}^N$.

Lemma 4.1 *The marginal vector x^π can be computed as follows:*

- (0) $x_{p_k}^\pi = v(M_k)$;
- (1) $x_{p_s}^\pi = v(M_s) - \sum_{G \prec M_s} v(G)$, for $s = 1, \dots, k-1$
(where $G \prec M_s$ means that G is a maximal member of the family $\mathcal{M}_s(\mu) = \{G' \in \mathcal{M}(\mu) \setminus \{M_s\} \mid G' \subset M_s, \}$).

Moreover, $x^\pi(M_t) = v(M_t)$ holds for $t = 1, \dots, k$.

Proof. (1) follows immediately from the relation

$$x_{p_s}^\pi = v(M_s) - \sum \{\bar{x}_{p_t} \mid t > s, p_t \in M_s\} \quad (s = k-1, k-2, \dots, 1).$$

◇

In the case $N \in \mathcal{F}$, we have $M_1 = N$ and observe (from Lemma 4.1) that $x^\pi(N) = v(M_1) = v(N)$ holds for any marginal vector x^π . Note furthermore that Γ admits only a finite number of marginal vectors (since Π is finite).

Example 4.1 *Let us take again the communication structure of Example 3.1. Then the corresponding (π, μ) -incidence matrix is*

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

and one obtains the solution $\bar{x} = [\bar{x}_4, \bar{x}_2, \bar{x}_3, \bar{x}_5]$ of the system

$$\begin{aligned} x_4 + x_2 + x_3 + x_5 &= v(12345) \\ x_2 &= v(12) \\ x_3 + x_5 &= v(35) \\ x_5 &= v(5) \end{aligned}$$

as $\bar{x} = [v(12345) - v(12) - v(35), v(12), v(35) - v(5), v(5)]$.

4.2 Weber Set

We associate with the cooperation structure $\Gamma = (\mathcal{F}, v)$ the convex hull $\mathcal{W}(v)$ of all marginal vectors x^π , *i.e.*,

$$\mathcal{W}(v) := \text{conv}\{x^\pi \mid \pi \in \Pi\}$$

and call the polytope $\mathcal{W}(v) \subseteq \mathbb{R}^N$ the *Weber set* of Γ .

Theorem 4.2 *Assume $N \in \mathcal{F}$. Then $\text{core}(v) \subseteq \mathcal{W}(v)$.*

Proof. Suppose that the claim of the Theorem were false and a vector $z \in \text{core}(v) \setminus \mathcal{W}(v)$ existed. Since $\mathcal{W}(v)$ is a closed convex set, we could now separate z from $\mathcal{W}(v)$ by a hyperplane, *i.e.*, there would be a parameter vector $c \in \mathbb{R}^N$ such that

$$\langle c, z \rangle < \langle c, x \rangle \quad \text{for all marginal vectors } x.$$

But then the marginal vector $x^\pi \in \mathcal{W}(v)$ associated with c would yield a contradiction:

$$\langle c, z \rangle \geq \hat{v}(c) = \langle c, x^\pi \rangle.$$

◇

REMARK. For the classical case $\mathcal{F} = 2^N \setminus \emptyset$, Theorem 4.2 is due to Weber [30].

4.2.1 Shapley Value

It appears natural to define the "Shapley value" $\Phi(v)$ of a cooperation structure as the average of its marginal vectors:

$$\Phi(v) := \frac{1}{|\Pi|} \sum_{\pi \in \Pi} x^\pi \in \mathcal{W}(v), \quad (3)$$

where Π is the collection of all possible rankings π produced by the Monge algorithm. In the classical case $\Gamma = (2^N \setminus \emptyset, v)$, $\Phi(v)$ coincides with the value introduced by Shapley [27].

5 Convexity

We say that cooperation structure $\Gamma = (\mathcal{F}, v)$ is *convex* (or simply, that v is convex) if its Monge extension $\hat{v} : \mathbb{R}^N \rightarrow \mathbb{R}$ is a concave (a.k.a. *convex down*) function, *i.e.*, satisfies for all parameter vectors $c, d \in \mathbb{R}^N$ and real scalars $0 < t < 1$,

$$t\hat{v}(c) + (1-t)\hat{v}(d) \leq \hat{v}(tc + (1-t)d).$$

Theorem 5.1 *Assume $N \in \mathcal{F}$ and v is monotone. Then $\Gamma = (\mathcal{F}, v)$ is convex if and only if for all $c \in \mathbb{R}^N$,*

$$\begin{aligned} \hat{v}(c) &= \min\{\langle c, x \rangle \mid x \in \text{core}(v)\} \\ &= \max\{\langle v, y \rangle \mid y_F \geq 0, \forall F \in \mathcal{F} \setminus \{N\}, \sum_{F \ni j} y_F \leq c_j, \forall j \in N\}. \end{aligned}$$

Proof. It is straightforward to check in the Monge algorithm that \hat{v} is *positively homogeneous* in the sense

$$\hat{v}(\lambda c) = \lambda \hat{v}(c) \quad \text{for all } c \in \mathbb{R}^N \text{ and real scalars } \lambda \geq 0.$$

A well-known result from convex analysis (see, *e.g.*, Rockafellar [26]) therefore asserts that the concavity of \hat{v} is equivalent with \hat{v} being the lower support function of its core, which is the first equality claimed.

The second equality follows from linear programming duality with respect to the core representation (1) of Theorem 4.1.

◇

For the proof of an alternative characterization in Theorem 5.2, we need a technical fact.

Lemma 5.1 *Let $\pi = p_1 \dots p_k \in \Pi$ be an arbitrary ranking sequence. Then there exists some $\tilde{c} \in \mathbb{R}^N$ such that the Monge algorithm produces the output (μ, π, \tilde{y}) with the properties:*

- (i) $\tilde{y}_{M_s} > 0$ for all $s = 1, \dots, k$.
- (ii) $\sum_{F \ni j} \tilde{y}_F < \tilde{c}_j$ for each $j \notin \pi$.

Proof. Let $c \in \mathbb{R}^N$ be a parameter vector so that the Monge algorithm produces the output (μ, π, y) . We now modify c to a weighting $\tilde{c} \in \mathbb{R}^N$ as follows.

We choose some $c_0 > \max\{|c_p| \mid p \in N\}$ and replace each c_p by $c'_p = c_p + c_0 \geq 0$. Relative to c' , the Monge algorithm then clearly produces the output (μ, π, y') with $y' \geq 0$.

Each component c'_{p_s} with $p_s \in \pi$ is now replaced by $\tilde{c}_{p_s} = c'_{p_s} + 2^s$ for $s = 1, \dots, k$. Each of the remaining components c'_j with $j \notin \pi$ is replaced by a large positive constant $K \gg 0$ (e.g., $K > 2c_0 + 2^n$).

It is straightforward to verify that \tilde{c} produces the same ranking sequence π as c' and therefore as c . Moreover, the latter modification ensures property (ii) to hold while the former modification guarantees (i).

◇

Theorem 5.2 *Assume $N \in \mathcal{F}$ and v monotone. Then $\Gamma = (\mathcal{F}, v)$ is convex if and only if $x^\pi \in \text{core}(v)$ holds for each marginal vector x^π , i.e., if and only if*

$$\text{core}(v) = \mathcal{W}(v).$$

Proof. Assume first $\mathcal{W}(v) \subseteq \text{core}(v)$ and consider an arbitrary $c \in \mathbb{R}^N$ with associated marginal vector $x^\pi \in \mathcal{W}(v)$. Then we have

$$\begin{aligned} \hat{v}(c) &\geq \min\{\langle c, x \rangle \mid x \in \text{core}(v)\} \\ &= \max\{\langle v, y \rangle \mid y_F \geq 0, \forall F \in \mathcal{F} \setminus N, \sum_{F \ni j} y_F \leq c_j, \forall j \in N\} \\ &\geq \hat{v}(c). \end{aligned}$$

So equality holds throughout and exhibits Γ as convex by Theorem 5.1.

Conversely, consider the marginal vector x^π . By Lemma 5.1, x^π arises from the MA-output $(\mu, \pi = p_1 \dots p_k, \tilde{y})$ relative to some input c such that

- (i) $\tilde{y}_{M_s} > 0$ for all $s = 1, \dots, k$.
- (ii) $\sum_{F \ni j} \tilde{y}_F < c_j$ for each $j \notin \pi$.

If Γ is convex, \tilde{y} is an optimal solution for the linear program

$$\max\{\langle v, y \rangle \mid y_F \geq 0, \forall F \in \mathcal{F} \setminus N, \sum_{F \ni j} y_F \leq c_j, \forall j \in N\}.$$

Let \tilde{x} be an optimal solution for the dual linear program

$$\min\{\langle c, x \rangle \mid x \geq 0, x(N) = v(N), x(F) \geq v(F), \forall F \in \mathcal{F}\}.$$

Being optimal, \tilde{x} and \tilde{y} must satisfy the complementary slackness conditions:

$$\begin{aligned} \tilde{y}_{M_s} > 0 &\implies \tilde{x}(M_s) = v(M_s) \quad (M_s \neq N) \\ \tilde{x}_j > 0 &\implies \sum_{F \ni j} \tilde{y}_F = c_j. \end{aligned}$$

By (ii), the latter conditions imply $\tilde{x}_j = 0$ if $j \notin \pi$. Because $\tilde{x}(N) = v(N)$ is true for the core vectors \tilde{x} , we conclude from (i) and the former conditions that \tilde{x} is identical to the marginal vector x^π , which means $x^\pi \in \text{core}(v)$ in particular. Since $\text{core}(v)$ is a convex subset of \mathbb{R}^N , we therefore find in view of Theorem 4.2:

$$\mathcal{W}(c) = \text{conv}\{x^\pi \mid \pi \in \Pi\} \subseteq \text{core}(v) \subseteq \mathcal{W}(v).$$

◇

REMARK. For the special case of classical cooperative games, Theorem 5.1 was observed by Schmeidler [29], while Theorem 5.2 is due to Lovász [23].

6 Communication Structures

We say that the cooperation structure $\Gamma = (\mathcal{F}, v)$ (with possibly $N \notin \mathcal{F}$) is a *communication structure* if \mathcal{F} is *weakly union-closed*, i.e., satisfies

(WU) $F \cup F' \in \mathcal{F}$ for all $F, F' \in \mathcal{F}$ with $F \cap F' \neq \emptyset$.

Note that the set systems \mathcal{F} with property (WU) coincide with the *union-stable* systems investigated by Algaba *et al.* [1].

Example 6.1 Let $G = (N, E)$ a graph with node set N and edge set E . Consider any non-empty node sets F and F' that induce connected subgraphs of G . Then $F \cup F'$ induces a connected subgraph if $F \cap F' \neq \emptyset$ holds. So the Myerson games on graphs (cf. Ex. 2.1) form a special class of communication structures.

It follows from (WU) that the maximal feasible coalitions of a communication structure are pairwise disjoint. Hence a communication structure naturally decomposes into pairwise disjoint communication structures, each of them exhibiting a unique maximal feasible coalition. Without loss of generality, we therefore assume $N \in \mathcal{F}$ in our subsequent analysis of communication structures.

A special case of a communication structure is given when \mathcal{F} is closed under arbitrary unions. Examples arise from cooperative games under precedence constraints (Faigle and Kern [14]), games with permission structure (Gilles *et al.* [17]), or antimatroids, which are the complements of discrete convex geometries (see, e.g., Korte *et al.* [21]). In view of $\mathcal{F}_0 = 2^N$, every classical cooperative game can be understood as a union closed communication structure.

REMARK. Algaba *et al.* [1] have proposed a "Myerson value" for union-stable structures as the (classical) Shapley value of an associated classical cooperative game. This value, however, does *not* coincide with the Shapley value (3) that arises naturally from the Monge algorithm for this class. The notion of games on *regular set systems* introduced by Lange and Grabisch (see [22], where a Shapley-like value is proposed) is also closely related to Myerson games.

Example 6.2 A communication structure (\mathcal{F}, v) is an augmenting system in the sense of Bilbao [5] if it satisfies for all $F, G \in \mathcal{F}$ with $F \subseteq G$,

$$G \setminus F \neq \emptyset \implies F \cup \{i\} \in \mathcal{F} \text{ for some } i \in G \setminus F.$$

The class of union-closed augmenting systems is exactly the class of antimatroids.

6.1 Greedy Communication Structures

We want to characterize convex communication structures (with $N \in \mathcal{F}$). To this end, we relax the definition and call an arbitrary communication

structure $\Gamma = (\mathcal{F} = \{F_1, \dots, F_m\}, v)$ *greedy* if the Monge algorithm (viewed as a greedy algorithm) is guaranteed to produce an optimal solution for the linear program

$$\max_{y \geq 0} \langle v, y \rangle \quad \text{s.t.} \quad \sum_{F \ni p} y_F \leq c_p, \quad \forall p \in N \quad (4)$$

for any (non-negative) $c \geq 0$. Hence a convex communication structure (\mathcal{F}, v) is necessarily greedy (cf. Theorem 5.1).

We call the valuation $v : \mathcal{F} \rightarrow \mathbb{R}_+$ *strongly monotone* if it satisfies for any $F \in \mathcal{F}$ and pairwise disjoint feasible sets $G_1, \dots, G_f \in \mathcal{F}(F)$ the inequality

$$\sum_{\ell=1}^f v(G_\ell) \leq v(F).$$

Note that $f = 1$ exhibits every strongly monotone v to be also monotone in the usual sense.

Example 6.3 Assume that \mathcal{F} is closed under taking arbitrary unions. Then $v : \mathcal{F} \rightarrow \mathbb{R}_+$ is strongly monotone if and only if v is monotone and superadditive.

Lemma 6.1 If the communication structure (\mathcal{F}, v) is greedy, then v is necessarily strongly monotone.

Proof. Take $F \in \mathcal{F}$ and suppose that $v(F) < \sum_{\ell=1}^f v(G_\ell)$ holds. Take $c = \mathbf{1}_F$ and y the output of the Monge algorithm, whose only non-zero component is $y_F = 1$. Then y' defined by

$$\begin{aligned} y'_F &:= 0 \\ y'_{G_1} = \dots = y'_{G_f} &:= y_F = 1 \\ y'_G &:= y_G = 0 \text{ otherwise} \end{aligned}$$

is feasible but $\langle v, y' \rangle > \langle v, y \rangle$, a contradiction to the fact that y is optimal for (4). \diamond

For the next definition, it is convenient to augment the family \mathcal{F} to $\mathcal{F}_0 = \{F_1, \dots, F_m, F_{m+1}\}$ with $F_{m+1} = \emptyset$ and to set $v(\emptyset) = 0$.

Let $F, F' \in \mathcal{F}$ be *intersecting*, i.e., $F \cap F' \neq \emptyset$. Then $F \cup F' \in \mathcal{F}$ follows from the weak union property (WU), while $F \cap F' \notin \mathcal{F}$ may be

possible. Nevertheless, (WU) implies that the maximal sets in the family $\mathcal{F}_0(F \cap F') = \mathcal{F}(F \cap F') \cup \{\emptyset\}$ are pairwise disjoint. So we arrive at the well-defined parameter

$$v(F \cap F') := \sum \{v(G) \mid G \in \mathcal{F}_0(F \cap F') \text{ maximal}\}. \quad (5)$$

Example 6.4 Assume that \mathcal{F} is closed under arbitrary unions. Then for any $F, F' \in \mathcal{F}$, there is a unique maximal feasible set

$$F \wedge F' = \cup \{G \in \mathcal{F}_0 \mid G \subseteq F \cap F'\} \in \mathcal{F}_0$$

and $v(F \cap F') = v(F \wedge F')$ follows for any intersecting $F, F' \in \mathcal{F}$.

We now say that the communication structure (\mathcal{F}, v) is *supermodular* (or simply, v is supermodular) if for any intersecting feasible sets $F, F' \in \mathcal{F}$ the following inequality holds:

$$v(F \cup F') + v(F \cap F') \geq v(F) + v(F'), \quad (6)$$

where $v(F \cap F')$ is understood as in (5) if $F \cap F' \notin \mathcal{F}$ holds.

Lemma 6.2 If the communication structure (\mathcal{F}, v) is greedy, then v is necessarily supermodular.

Proof. Let $F, F' \in \mathcal{F}$ be intersecting. Then the supermodular inequality is trivial if $F \subset F'$ or $F' \subset F$ holds. We thus assume that neither is the case and consider the nonnegative parameter vector $c = \mathbf{1}_{F \cup F'} + \mathbf{1}_{F \cap F'}$. The greedy solution \tilde{y} for (4) yields

$$\langle v, \tilde{y} \rangle = v(F \cup F') + v(F \cap F').$$

On the other hand, the vector $\overline{y} \in \mathbb{R}^{\mathcal{F}}$ with the components

$$\overline{y}_G = \begin{cases} 1/2 & \text{if } G \in \{F \cup F', F, F'\} \\ 1/2 & \text{if } G \text{ maximal in } \mathcal{F}(F \cap F') \\ 0 & \text{otherwise} \end{cases}$$

is also a feasible solution with the objective value

$$\langle v, \overline{y} \rangle = \frac{1}{2}[v(F \cup F') + v(F) + v(F') + v(F \cap F')].$$

So \tilde{y} can only be optimal if the supermodular inequality holds.

◇

We now investigate sufficient conditions and first recall that the *support* of a vector $y \in \mathbb{R}^{\mathcal{F}}$ is defined as the set

$$\text{supp}(y) = \{F \in \mathcal{F} \mid y_F \neq 0\}.$$

Lemma 6.3 *Assuming that $(\mathcal{F} = \{F_1, \dots, F_m\}, v)$ is a supermodular communication structure, let y^* be the lexicographically maximal optimal solution for the linear program (4). Then $\text{supp}(y^*)$ is a nested family, i.e., one has for any $F_i, F_j \in \text{supp}(y^*)$ with $i < j$,*

$$\text{either } F_i \cap F_j = \emptyset \text{ or } F_i \supset F_j.$$

Proof. Suppose $F_i, F_j \in \text{supp}(y^*)$ are intersecting and $F_s = F_i \cup F_j$. If $s < i$ were true, we could modify y^* to the vector \bar{y} with the components

$$\bar{y}_G = \begin{cases} y_G^* + \varepsilon & \text{if } G = F_s \\ y_G^* + \varepsilon & \text{if } G \text{ is maximal in } \mathcal{F}(F_i \cap F_j) \\ y_G^* - \varepsilon & \text{if } G = F_i \text{ or } G = F_j \\ y_G^* & \text{otherwise} \end{cases}$$

and obtain a feasible solution that is lexicographically strictly larger than y^* . Moreover,

$$\langle v, \bar{y} \rangle = \langle v, y^* \rangle + \varepsilon (v(F_i \cup F_j) + v(F_i \cap F_j) - v(F_i) - v(F_j)).$$

Supermodularity of v implies that also \bar{y} must be optimal and we arrive at a contradiction to the choice of y^* . So $s = i$ and hence $F_i \supset F_j$ must hold.

◇

Theorem 6.1 *The communication structure $\Gamma = (\mathcal{F} = \{F_1, \dots, F_m\}, v)$ is greedy if and only if the valuation $v : \mathcal{F} \rightarrow \mathbb{R}_+$ is strongly monotone and supermodular.*

Proof. The necessity of the conditions follows from Lemma 6.1 and Lemma 6.2. We prove sufficiency by induction on the number $|\mathcal{F}|$ of feasible coalitions.

Let y be the greedy solution and denote by y^* the (with respect to the index order of \mathcal{F}) lexicographically maximal optimal solution.

CLAIM: $y_{F_1} = y_{F_1}^*$.

Now $y_{F_1} \geq y_{F_1}^*$ is a direct consequence of the Monge algorithm. So it suffices to show that strict dominance $y_{F_1} > y_{F_1}^*$ is impossible. Recalling from Lemma 6.3 that $\text{supp}(y^*)$ is a nested family, let $\{G_1, G_2, \dots, G_k\}$ be the collection of (inclusion-wise) maximal proper subsets of F_1 in the support $\text{supp}(y^*)$. So the G_i s are pairwise disjoint.

Let $\varepsilon = \min\{y_{F_1} - y_{F_1}^*, y_{G_1}^*, \dots, y_{G_k}^*\} \geq 0$ and define \bar{y} by

$$\begin{aligned}\bar{y}_{F_1} &= y_{F_1}^* + \varepsilon \\ \bar{y}_{G_i} &= y_{G_i}^* - \varepsilon, \quad i = 1, \dots, k \\ \bar{y}_G &= y_G^* \text{ otherwise.}\end{aligned}$$

Then \bar{y} is a feasible solution and satisfies

$$\langle v, \bar{y} \rangle = \langle v, y^* \rangle + \varepsilon \left(v(N) - \sum_{i=1}^k v(G_i) \right) \geq \langle v, y^* \rangle$$

by the strong monotonicity of v . Hence also \bar{y} is optimal and lexicographically maximal, which implies $\bar{y} = y^*$ and hence $\varepsilon = 0$, as claimed.

To finish the proof, consider the representative $p_1 \in F_1$ chosen by the Monge algorithm. Because of $y_{F_1}^* = y_{F_1} = c_{p_1} = \min\{c_p \mid p \in F_1\}$, we find:

$$y_F^* = y_F \quad \text{holds for all } F \in \mathcal{F} \text{ with } p_1 \in F. \quad (7)$$

Let $\mathcal{F}' = \{F \in \mathcal{F} \mid p_1 \notin F\}$. Then \mathcal{F}' is weakly union closed. Moreover, the Monge algorithm produces the value

$$\sum_{F \in \mathcal{F}'} y_F v(F) \leq \sum_{F \in \mathcal{F}'} y_F^* v(F)$$

on \mathcal{F}' . On the other hand, $|\mathcal{F}'| \leq |\mathcal{F}| - 1$ holds. So we know by induction that the Monge algorithm is optimal on \mathcal{F}' . Taking (7) into account, we therefore conclude

$$\sum_{F \in \mathcal{F}} y_F^* v(F) \leq \sum_{F \in \mathcal{F}} y_F v(F) \leq \sum_{F \in \mathcal{F}} y_F^* v(F)$$

and hence optimality of y . ◇

REMARK. Theorem 6.1 generalizes Theorem 4 in [13].

6.2 Convex Communication Structures

Theorem 6.1 allows us to characterize convex communication structures as follows.

Theorem 6.2 *Assume $N \in \mathcal{F}$. Then the communication structure $\Gamma = (\mathcal{F}, v)$ is convex if and only if v is strongly monotone and supermodular.*

Proof. If Γ is convex, then Γ is greedy. Hence (by Theorem 6.1) the conditions are necessary. Conversely, we show that a greedy communication structure is convex if $N \in \mathcal{F}$ holds. It suffices to argue that the greedy algorithm is optimal for the linear program

$$\max \langle v, y \rangle \quad \text{s.t.} \quad y_F \geq 0 \quad \forall F \neq N, \quad \sum_{F \ni p} y_F \leq c_p, \quad \forall p \in N. \quad (8)$$

Indeed, let $C \gg 0$ be a large constant and modify c to the vector \bar{c} with components $\bar{c}_j = c_j + C > 0$. Then the greedy solution \tilde{y} is optimal relative to \bar{c} . On the other hand we have

$$\langle c, y \rangle = \langle \bar{c}, y \rangle - Cv(N).$$

for each feasible solution y for (8). Since $Cv(N)$ is constant, we conclude that \tilde{y} is also optimal for c .

◇

Corollary 6.1 (cf. [6]) *Let (\mathcal{F}, v) be an augmenting system with a monotone characteristic function $v : \mathcal{F}_0 \rightarrow \mathbb{R}$. Then (\mathcal{F}, v) is convex if and only if v is supermodular.*

Proof. Any monotone function v on an augmenting system \mathcal{F} is necessarily strongly monotone.

◇

REMARK. Convexity of augmenting systems is defined without reference to any Monge-type extensions and relative to a different model for Weber sets in [6]. Our Corollary 6.1 shows that the two notions of convexity coincide on this special class.

Classical Cooperative Games. Assume $\mathcal{F}_0 = 2^N$, i.e., (\mathcal{F}, v) is a classical cooperative game. Then \mathcal{F}_0 is the union- and intersection-stable Boolean lattice of all subsets of N with the operations $F \wedge F' = F \cap F'$

and $F \vee F' = F \cup F'$. In this context, the supermodular inequality (6) is the defining property for "convex games" in the sense of Shapley [28].

The equivalence of v being supermodular and $\text{core}(v)$ containing all marginal vectors was first realized in the classical context by Edmonds [10] (see also Faigle [11] and Ichiishi [20]). It is easy to see that a classical non-negative supermodular function is necessarily monotone.

REMARK. We do not know of a characterization of general convex co-operation structures (\mathcal{F}, v) in terms of an appropriately generalized notion of "supermodularity". (For some sufficient conditions, see, *e.g.*, Faigle and Peis [16].)

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