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WHY NOT PROPORTIONAL?

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Abstract

This paper reviews the arguments that justify the principles of proportional and degressively proportional representation.

1   Introduction

In current practice, apportionnements are obtained through negociation, with references to principles of justice, fairness, or efficiency. This essay is devoted to the second aspect: What has been, and what should be, the content of the notions of "justice", "fairness" and "efficientcy" with respect to apportionnement problems? Usual justifications in this literature rest on intuitive appeal for equality, but the structure of the apportionnment problem is complex enough that it is not clear what exactly we should aim at equalizing, or that equality should be the ultimate goal. In this regard I first review the early literature, which considers as obvious the idea that strictly proportional representation is ideal (section 2). In the more recent literature I distinguish three major arguments.

The Poor Representativity argument in favor of degressive proportionality (section 3) echoes the ancient view that a single delegate cannot represent well a very large number of citizens. This view has been given a clean mathematical expression known as the Square Root Law. The welfarist approach (section 4) has, more recently, clearly enunciated the argument in favor of proportionality, a point which I call here the Pure Majoritarian argument in favor of proportionality. The welfarist framework also provides a simple argument in favor of degressive proportionality, which I call the Utilitarian-Egalitarian argument. This last point appears to be the mathematical expression of the commonly mentionned idea that large countries would have too much power in a strictly proportional system.

2 Proportional representation

Equality is a very basic form of justice requirement, which appears in almost all definitions of justice or just institutions. Many philosophical as well as practical controversies can be seen as debates about *what* should be equalized, but once the equalizandum is found, the determination of justice is "a matter of weight", an operation essentially quantitative. Consider, for instance Aristotle's classification of the social philosophies of his time:

> "... all men agree that what is just in distribution must be according to merit in some sense, though they do not all specify the same sort of merit, but democrats identify it with the status of freeman, supporters of oligarchy with wealth (or with noble birth), and supporters of aristocracy with excellence.
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> The just, then, is a species of the proportionate."
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> [*Nichomachean Ethics* Book V, chapter 3]

The practical question of allocating seats of an assembly to represent voters presents itself already in mathematical, quantitative, terms: "How many" seats per country, per district, per party, per voter, etc. In such a case, Aristotle's phrase "The just is a species of the proportionate" can easily be taken word for word. Then the general requirement of equality may seem, at first sight, to have a non-questionable consequence: the ultimate democratic unit is the citizen, therefore one should seek to equalize the number of seats per citizen. Many would consider this definition of a "just" apportionment as obvious, and would not ellaborate further on the notion of equality.

For instance, concerning the European Parliament, Balinski and Young (1982a) wrote:

> "Trans-national parties have formed. Members of Parliament no longer vote in national blocs; instead of representing purely national interests they represent people in one larger European Community. This new situation means, however, that representatives ought properly to represent equal numbers of constituents no matter in what nation they happen to reside."

Starting from the a priori idea that the representational ideal was proportionality, the early mathematical literature on the subject has thus neglected to provide further justification of why proportionality is desirable. Instead of this normative enquiry, it has focused on a peripheral, albeit very practical question: the question of the rounding methods. If each representative is to represent the same number of citizens, strictly proportional representation usually would recommend fractional numbers of representatives. In practice, one has thus to move away from strict proportionality in order to have a definite, integer, number of representatives. Practical apportionnement can only be proportional "up to one seat", and the different rounding methods can be compared to that respect (Balinski and Young 1982b).

The Cambridge Compromise (Grimmet et al, 2011) has tackled carefully the questions of roundings to integers, but has also put forward the idea that the principle of degressive proportionality should be conceived independently of the rounding questions.

3   Degressive proportionality

The voting power literature started with the observation that with a weighted voting rule, the possibility for the vote of a country to make a difference is not in direct relation with its weight. A well-known example is the voting rule used at the Council of the European Union in 1958-1973 after the Treaty of Rome, which is the first version of the European Union.

The European Economic Community had six member states. France, Germany and Italy had 4 votes each, Belgium and the Netherlands had 2 votes and Luxembourg had 1 vote. Acts proposed by the Commission required 12 votes to be adopted.

In this case, the single Luxembourg vote can never make a difference for reaching or not the threshold of 12 votes. Formally, Luxembourg has no power at all.

An even simpler example is a two-voter society using weighted majority rule with unequal weights for the two voters. Then the smaller voter has no power at all. Such examples make it clear that it would be a mistake to trust that proportional representation automatically ensures proportional power.

From this observation, and with the objective of designing a "fair" apportionment system and of representing people equally, one might therefore seek to equalize across citizens the voting power, defined as the probability that a citizen, through the election, makes a difference.

It is important at this point to note that the voting power approach relies on hypotheses that must be made precise to describe how individual opinions are formed and translated into votes of their representatives. Also, one should not confuse the question of how many delegates should one country send to an assembly and the question of how many votes should be given to each country in a council deliberation. In the former case, the delegates of the country may well belong to different parties and split their votes, in the latter case, the country will have one voice. In both cases, proportionality can be a theoretical answer, but the two cases are clearly not to be confused.

The most usual hypothesis is "independent voting". This means that the individual opinions are independent random variables with the same law (usually a probability .5 to favor or to object the proposal) and that, most importantly, any two individuals have the same probability to be in agreement, whether they are citizens of the same country or not.

In the case of "Bloc voting", that is when each country is represented by a single spokesman (its government) or if all the representatives vote the same way, a much-celebrated result, known as the "square-root law" is obtained. Under this set of hypotheses, the power of an individual citizen to change her

country's vote will be equalized across individuals if the weight of each country is proportional to $\sqrt{n}$, where n is the number of citizens in the country (Penrose, 1946; Lindner and Machover 2004, Słomczyński and Życzkowski, 2004, 2010). The logic behind this result is well explained by Penrose himself:

> "...small electorates are less susceptible to control by resolute blocs and are likely to obtain more representative governments than large electorates. If factors other than numerical size are constant, the spokesman for any electorate represents a section of that electorate proportional to the square root of the total number of electors."

The political argument is that the quality of national representations decreases with the size of their populations. We could call this point the **Poor Representativity argument in favor of degressive proportionality**. Under specific assumptions (independence), this decrease of quality can be expressed mathematically, leading to the concluclusion that the probability for a citizen of a country of size n to be pivotal is proportional to $1/\sqrt{n}$. Giving weight $\sqrt{n}$ to the country compensates exactly and is thus "fair" because it equalizes the probability of being decisive among citizens of the different countries.

Up to my knowledge this precise argument was first raised by Penrose and was not discussed in political science. A somehow related idea is that overly large bodies of constituents cannot form a nation. To quote again Aristotle: "You cannot make a city of ten men, and if there are a hundred thousand it is a city no longer." [*Nichomachean Ethics* Book IX, chapter 10]. But Penrose's idea is to explain the loss of representativity by the statistical flutuations of independent variables, and this particular point can hardly be found, even informally stated, in the political science literature.

Felsenthal and Machover (1998) propose another equalisandum besides voting power: the expected value of the majority deficit, where the majority deficit after a decision is 0 for all countries where a majority of voters support the decision, and is equal for the other countries to the difference between the number of individuals who object to the decision and the number of of individuals who aggree with the decision. A square-root law can then be derived, again under the assumption of independent voting.

In testing the theoretical ideas that justify the square root law, Gelman, Katz and Bafumi (2004) noticed that the empirical estimate one can produce of the probability of being pivotal is closer to $1/n$ than to $1/\sqrt{n}$. They deduce that the proportional representation is "basically fair" compared to a representation that would follow the square-root law.

Different formulas and indices have previously been proposed to quantify various kinds of voting power: see Shapley and Shubik (1954), Banzaf (1965), Owen (1972), or Laruelle and Valenciano (2005). These formulas are more or less explicitly related to models of how countries' votes derive from the opinions of the citizens and to how coalitions of countries emerge. They can be used to describe actual parliaments (see Barr and Passarelli (2009) for the EU). Surveys of the theory of voting power have been published by Felsenthal and Machover (1998) and more recently by Laruelle and Valenciano (2008).

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The square-root law is an instance of degressive proportionality (up to rounding). Degressively proportional allocations are often found in practice and advocated (Lamassoure and Severin, 2007). This may explain why the square-root law has gained some popularity among those who discuss which mathematical formula could be adopted as a rule to define the weights of representatives in commissions and/or the composition of parliaments.[1] But the argument itself upon which the square-root rule is based, the Poor Representativity argument, has not gained much endorsement. In a sense the intellectual construction of the square-root rule is appreciated because of its consequences more than as a justification of these consequences.

The Cambridge Compromise took as a constraint that the apportionnement should be degressively proportional. It recommends a Base+Prop formula: a number of seats is divided equally among the countries independently of their population figures and a number of seats is allocated in proportion of the population. The consequence of using a Base+Prop formula is a degressively proportional distribution of seats (up to roundings). The proposal is not grounded on a single principle to be applied or on a criterion to be optimized, but the idea can nevertheless be faithfully explained as resulting from two (contradictory!) principles. A principle of equality among states is served by the Base, that is the fixed number of seats allocated to each of them, and a principle of equality among citizens is served by the proportional part of the allocation.

4 Welfarism

The ideal of equalizing voting power may be criticized, as any form of egalitarianism. For instance, a trivial way to equalize voting power would be to make decisions randomly, with no regard to individuals' opinions. This obviously bad solution is egalitarian, which proves that equality cannot be the only criterion. A voting rule has consequences for each citizen. If we are able to measure how good the rule is for each citizen, be it by some "voting power" measure or any other index, we obtain, associated to each rule, a distribution of the index accross the whole population. It is this distribution which has to be evaluated.

As is common in social sciences, in evaluating such distributions, one is faced with a trade-off between efficiency and equality, two requirements which are partially, but not completely, opposed. The apportionment problem is no exception and, as an institutional design problem, requires for its solution that the social objectives be clearly stated.

For a single yes/no decision, Barbera and Jackson (2006) show that the rule that would maximize the sum, across individuals, of expected utility is a weighted voting rule, where the weight of a country is proportional to the social importance of the issue at stake for this country's citizens. Barbera and Jakson use a hypothesis of independence across countries and Beisbart and Bovens (2007) and Beisbart and Hartmann (2010) relax this hypothesis to show how

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[1] See the open letter "How to vote in the Council of the European Union", signed by several scientists in 2008. http://chaos.if.uj.edu.pl/~karol/pdf/voting08.pdf

the results then depend on how different countries' interests are correlated. (See also Laruelle and Valenciano, 2008.)

This approach ellaborates on a very natural argument in favor of proportionality which can be stated, in its simplest form, as follows:

Suppose that the citizens of a country always share the same opinion; that is the country's vote. Suppose moreover that the stake is the same for all citizens in all countries, say +1 or 0. We can think that a citizen will earn 1 Euro if the collective decision matches her will and 0 if not. Then the efficient collective decision is clear: it should follow the will of the majority, for that is the way for the whole society to benefit more. In that case, the majoritarian outcome is easily obtained by applying to each country's vote a weight proportional to its population. This provides justification for proportionality by the efficiency of majority rule in these one-shot yes/no decisions, which have the same importance for everyone. We can call this point the **Pure Majoritarian argument in favor of proportionalit**y.

It turns out that this simple point is quite robust to a number of variations. Suppose for instance that, within each country, not all citizens but a fixed fraction (larger than .5) of them agree with the country's vote. Then simple computation shows that the result is the same. Suppose now that the opinions are randomly generated. Since proportionality is optimal *in every instance*, it is also optimal ex ante, for instance in expectation.

The obvious drawback of the Pure Majoritarian argument is that, when we do not take one but several decisions, members of small countries are less often satisfied than members of large countries. The argument can then be turned into arguments in favor of decreasing proportionality in two ways. Still consider, for simplicity, the extreme case where all the citizens of a country share the same opinion.

First, consider that the objective is not efficiency but equality, among citizens of different countries, of the probability p_i that the final decision matches citizen i's preference. Give the same weight to all countries, large or small. Then if opinions are independent from one country to the other, p_i is same within each country and also the same from one country to the other. Equalty is thus achieved. One can obtain in this way a justification for the idea that countries should be represented per se, independently of their size. Because degressive proportionality is "in between" constant and proportional weights, it is justified by a compromise between efficiency and equality (Beisbart and Hartmann, 2010).

Second, if one has in mind that a constitutional decision such as the weights given to the countries will govern not only one but a long sequence of future decisions, one should consider that the utility for a citizen of the voting scheme is not simply +1 or 0 but stems from the fact that her will is going to be more or less often fulfilled. This frequency, say p_i is the object of preference of the individual i, and the social judgment should bear on the distribution of these frequencies p_i in the population. It turns out that if the utility attached to frequency p_i by the individual i is a concave function, say $\psi(p_i)$, common to all individuals,

then maximizing the sum of individual utilities ususally leads to recommend weighted voting rules whose weights are degressively proportional with respect to the countries' populations (Koriyama and Laslier 2011; Macé and Treibich 2011). The concavity of the individual evaluation of p_i can be interpreted, as usual, in terms of decreasing marginal utility. The marginal satisfaction obtained by an individual when her p_i is raised from, say, .60 to .61 is larger than the one obtained when her p_i is raised from, say, .71 to .72.

The two previous arguments, equality and concave utilitarianism, are in fact equivalent, a pattern familiar in normative economics, which links utilitarianism to the theory of inequality (see for instance Blackorby, Bossert and Donaldson, 2002). Indeed, when the social criterion is the sum of individual utilities, the concavity of utility with respect to p_i also means that the society would gain by having one p_i decresased from .71 to .70 and another p_j increased from .60 to .61. The two versions of this idea can thus be given a single name: the **Utilitarian-Egalitarian argument in favor of decreasing proportionality**.

This argument is intuitive in the case of a committee, where each country has one and only one delegate who is to be given some weight, and in the case of a parliament dealing with issues about which all the citizens share the opinion voiced by their representatives. It can be proven that the argument is still valid as soon as the citizens' opinions are not completely independent from their country of residence (Koriyama and Laslier 2011).

The idea that small countries should be given relatively more weight because, otherwise, they would systematically lose is familiar in politics. The Utilitarian-Egalitarian approach articulates this sensible concern. When a populous country has a weight proportional to its population, its citizens are better treated than the citizens of smaller countries because that country is relatively more often in a position to dictate its choice. It may therefore be fair to redistribute part of this large weight to smaller countries.

The Cambridge Compromise solution was not designed to be optimal with respect to some utilitarian-egalitarian criterion. It nevertheless shares an important qualitative property with these allocation schemes, namely degressive proportionality.

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