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Nicolas Raymond

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Sharp asymptotics for the twisted Neumann Laplacian with magnetic field

Nicolas Raymond

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Abstract

We consider the twisted magnetic Laplacian with Neumann condition on a smooth and bounded domain of \mathbb{R}^2 in the semiclassical limit $h \rightarrow 0$. Under generic assumptions, we prove the existence of a complete asymptotics in powers of h of the lowest eigenvalues.

1 Introduction and main results

Let Ω be an open bounded and simply connected subset of \mathbb{R}^2 with smooth boundary. Let us consider a smooth vector potential \mathbf{A} such that $\beta = \nabla \times \mathbf{A} > 0$ on $\bar{\Omega}$ and a a smooth and positive function on $\bar{\Omega}$. We are interested in estimating the lowest eigenvalues $\lambda_n(h)$ of the operator $P_{h,\mathbf{A}} = (ih\nabla + \mathbf{A})a(ih\nabla + \mathbf{A})$ whose domain is given by:

$$\text{Dom}(P_{h,\mathbf{A}}) = \left\{ \psi \in L^2(\Omega) : (-ih\nabla + \mathbf{A})a(-ih\nabla + \mathbf{A})\psi \in L^2(\Omega) \right. \\ \left. \text{and } (-ih\nabla + \mathbf{A})\psi \cdot \nu = 0 \text{ on } \partial\Omega \right\}.$$

The corresponding quadratic form denoted by $Q_{h,\mathbf{A}}$ is defined on $H^1(\Omega)$ by:

$$Q_{h,\mathbf{A}}(\psi) = \int_{\Omega} a(x) |(-ih\nabla + \mathbf{A})\psi|^2 dx.$$

By gauge invariance, this is standard that the spectrum of $P_{h,\mathbf{A}}$ only depends on the magnetic field $\beta = \nabla \times \mathbf{A}$.

1.1 Motivations

Before stating our main result, we should briefly describe the context and the motivations of this paper. As much in 2D as in 3D, the magnetic Laplacian, corresponding to the case when $a = 1$, appears in the theory of superconductivity when studying the third critical field H_{C_3} after the linearization of the Ginzburg-Landau functional (see

for instance [16, 17] and also the book of Fournais and Helffer [11]). It turns out that H_{C_3} can be related to the lowest eigenvalue of the magnetic Laplacian in the regime $h \rightarrow 0$. In fact, the case which is mainly investigated is the case when the magnetic field is constant. In 2D, the two term asymptotics is done in the case of the disk in [4] (see also [5] and [9]) and is generalized in [13] to smooth and bounded domains. The asymptotics at any order of all the lowest eigenvalues is proved in [10]. In 3D, one can mention the celebrated paper [14] giving the two term asymptotics of the first eigenvalue. When the magnetic field is variable (and $a = 1$), less results are known. In 2D, the paper [16] provides a one term asymptotics of the lowest eigenvalue and [19] gives the two term asymptotics under generic assumptions. In 3D, for the one term asymptotics, one can mention [17] and for a three term asymptotics upper bound [20].

Here we consider a twist factor $a > 0$. As we will see, the presence of a will not complicate so much our analysis. The motivation to add this term comes from [3] where the twisted magnetic Laplacian appears after a Gamma limit of a 3D Ginzburg-Landau functional (in a thin film limit) so that we can think to a as the memory of a 3D geometry. Moreover, we will see that the quantity to minimize to get the lowest energy is the function $a\beta$ so that this situation reminds what happens in 3D in [17, 20] and where the three term asymptotics is still not established.

We shall now recall basic properties of the de Gennes operator in order to state our results.

The de Gennes operator For $\xi \in \mathbb{R}$, we consider the Neumann realization H_ξ in $L^2(\mathbb{R}_+)$ associated with the operator

$$-\frac{d^2}{dt^2} + (t - \xi)^2, \quad \text{Dom}(H_\xi) = \{u \in B^2(\mathbb{R}_+) : u'(0) = 0\}. \quad (1.1)$$

One knows (see [8]) that it has compact resolvent and its lowest eigenvalue is denoted $\mu(\xi)$; the associated L^2 -normalized and positive eigenstate is denoted by $u_\xi = u(\cdot, \xi)$ and is in the Schwartz class. The function $\xi \mapsto \mu(\xi)$ admits a unique minimum in $\xi = \xi_0$ and we let:

$$\Theta_0 = \mu(\xi_0), \quad (1.2)$$

$$C_1 = \frac{u_{\xi_0}^2(0)}{3}. \quad (1.3)$$

Let us also recall identities established by [5, p. 1283-1284]. For $k \in \mathbb{N}^*$, we let:

$$M_k = \int_{t>0} (t - \xi_0)^k |u_{\xi_0}(t)|^2 dt.$$

$$M_0 = 1, \quad M_1 = 0, \quad M_2 = \frac{\Theta_0}{2}, \quad M_3 = \frac{C_1}{2} \quad \text{and} \quad \frac{\mu''(\xi_0)}{2} = 3C_1 \sqrt{\Theta_0}. \quad (1.4)$$

1.2 Main result

We will assume that

$$\Theta_0 \min_{\partial\Omega} a(x)\beta(x) < \min_{\Omega} a(x)\beta(x). \quad (1.5)$$

and that

$$x \in \partial\Omega \mapsto a(x)\beta(x) \text{ admits a unique and non degenerate minimum at } x_0. \quad (1.6)$$

Let us state our first rough estimate:

Proposition 1.1 *Under Assumptions (1.5) and (1.6), for all $n \geq 1$, we have:*

$$\lambda_n(h) = \Theta_0 h a(x_0)\beta(x_0) + O(h^{5/4}).$$

From this proposition, we see that the asymptotics of $\lambda_n(h)$ is related to local properties of $P_{h,\mathbf{A}}$ near x_0 . That is why we are led to introduce a system of local coordinates near x_0 .

Local coordinates (s, t) We use the local coordinates (s, t) near $x_0 = (0, 0)$, where $t(x) = d(x, \partial\Omega)$ and $s(x)$ is the tangential coordinate of x . We choose a parametrization of the boundary:

$$\gamma : \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \rightarrow \partial\Omega.$$

Let $\nu(s)$ be the unit vector normal to the boundary, pointing inward at the point $\gamma(s)$. We choose the orientation of the parametrization γ to be counter-clockwise, so that:

$$\det(\gamma'(s), \nu(s)) = 1.$$

The curvature $k(s)$ at the point $\gamma(s)$ is given in this parametrization by:

$$\gamma''(s) = k(s)\nu(s).$$

The map Φ defined by:

$$\begin{aligned} \Phi : \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times]0, t_0[&\rightarrow \Omega \\ (s, t) &\mapsto \gamma(s) + t\nu(s), \end{aligned} \quad (1.7)$$

is clearly a diffeomorphism, when t_0 is sufficiently small, with image

$$\Phi(\mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times]0, t_0]) = \{x \in \Omega \mid d(x, \partial\Omega) < t_0\} = \Omega_{t_0}.$$

We let:

$$\begin{aligned} \tilde{A}_1(s, t) &= (1 - tk(s))\mathbf{A}(\Phi(s, t)) \cdot \gamma'(s), & \tilde{A}_2(s, t) &= \mathbf{A}(\Phi(s, t)) \cdot \nu(s), \\ \tilde{\beta}(s, t) &= \beta(\Phi(s, t)), \end{aligned}$$

and we get:

$$\partial_s \tilde{A}_2 - \partial_t \tilde{A}_1 = (1 - tk(s))\tilde{\beta}(s, t).$$

The quadratic form becomes:

$$Q_{h,\mathbf{A}}(\psi) = \int \tilde{a}(1 - tk(s))|(-ih\partial_t + \tilde{A}_2)\psi|^2 + \tilde{a}(1 - tk(s))^{-1}|(-ih\partial_s + \tilde{A}_1)\psi|^2 dsdt.$$

In a (simply connected) neighborhood of $(0, 0)$, we can choose a gauge such that:

$$\tilde{A}_1(s, t) = - \int_{t_1}^t (1 - t'k(s))\tilde{\beta}(s, t')dt', \quad \tilde{A}_2 = 0. \quad (1.8)$$

Let us introduce some notation. We write the Taylor expansions:

$$\tilde{a}(s, t) = a(\Phi(s, t)) = 1 + a_1s + a_2t + a_{11}s^2 + a_{12}st + a_{22}t^2 + O(|s|^3 + |t|^3)$$

and

$$\tilde{\beta} = \beta(\Phi(s, t)) = 1 + b_1s + b_2t + b_{11}s^2 + b_{12}st + b_{22}t^2 + O(|s|^3 + |t|^3),$$

where we have assumed the normalization:

$$a(x_0) = \beta(x_0) = 1. \quad (1.9)$$

Let us translate the generic assumptions (1.5) and (1.6). The critical point condition becomes:

$$a_1 = -b_1 \quad (1.10)$$

and the non-degeneracy property reformulates:

$$b_{11} + a_1b_1 + a_{11} = a_{11} + b_{11} - a_1^2 = \alpha > 0. \quad (1.11)$$

We can now state the main result of this paper:

Theorem 1.2 *We assume Assumptions (1.5), (1.6) and the normalization condition (1.9). For all $n \geq 1$, there exist a sequence $(\gamma_{n,j})_{j \geq 0}$ and $h_0 > 0$ such that for all $h \in (0, h_0)$, we have:*

$$\lambda_n(h) \underset{h \rightarrow 0}{\sim} h \sum_{j \geq 0} \gamma_{n,j} h^{\frac{j}{4}}.$$

Moreover, we have, for all $n \geq 1$:

$$\begin{aligned} \gamma_{n,0} &= \Theta_0, \quad \gamma_{n,1} = 0, \\ \gamma_{n,2} &= C(k_0, a_2, b_2) + (2n - 1) \left(\frac{\alpha \Theta_0 \mu''(\xi_0)}{2} \right)^{1/2}, \end{aligned}$$

with:

$$C(k_0, a_2, b_2) = -C_1 k_0 + \frac{3C_1}{2} a_2 + \left(\frac{C_1}{2} + \xi_0 \Theta_0 \right) b_2.$$

Remark 1.3 In particular, this theorem generalizes [19, Theorem 1.7]. Moreover, as a consequence of the asymptotics of the eigenvalues (which are simple for h small enough), we also get the corresponding asymptotics for the eigenfunctions. These eigenfunctions are approximated (in the L^2 sense) by the power series which we will use as quasimodes (see (2.8)).

1.3 Scheme of the proof

In Section 2, we perform a construction of quasimodes and quasi-eigenvalues thanks to a formal expansion in power series of the operator. This analysis relies on generalizations of the Feynman-Hellmann formula and of the Virial theorem which were already introduced in [20] and which are an alternative to the Grushin approach used in [10]. Then, we use the spectral theorem to infer the existence of spectrum near each constructed power series. In Section 3, we prove a rough lower bound for the lowest eigenvalues and deduce Agmon estimates with respect to the variable t which provide a localization of the lowest eigenfunctions in a neighborhood of the boundary of size $h^{1/2}$. In Section 4, we improve the lower bound of Section 3 and deduce a localization of size $h^{1/4}$ with respect to the tangential coordinate s . In Section 5, we prove a lower bound for $Q_{h,\mathbf{A}}$ thanks to the definition of "magnetic coordinates" and we reduce the study to a model operator (partially similar to the one studied through a Born-Oppenheimer approximation in [6]) for which we are able to estimate the spectral gap between the lowest eigenvalues.

2 Accurate construction of quasimodes

This section is devoted to the proof of the following theorem:

Theorem 2.1 *For all $n \geq 1$, there exists a sequence $(\gamma_{n,j})_{j \geq 0}$ such that, for all $J \geq 0$, there exist $h_0 > 0$, $C > 0$ such that:*

$$d \left(h \sum_{j=0}^J \gamma_{n,j} h^{j/4}, \sigma(P_{h,\mathbf{A}}) \right) \leq C h^{\frac{J+1}{4}}.$$

Moreover, we have, for all $n \geq 1$:

$$\begin{aligned} \gamma_{n,0} &= \Theta_0, \quad \gamma_{n,1} = 0, \\ \gamma_{n,2} &= C(k_0, a_2, b_2) + (2n-1) \left(\frac{(a_{11} + b_{11} - a_1^2) \Theta_0 \mu''(\xi_0)}{2} \right)^{1/2}. \end{aligned}$$

The proof of Theorem 2.1 is based on a construction of quasimodes for $P_{h,\mathbf{A}}$ localized near x_0 . Near x_0 and using a suitable gauge (see (1.8)), we are led to construct quasimodes for the following the operator:

$$\begin{aligned} \mathcal{L}(s, -ih\partial_s; t, -ih\partial_t) &= -h^2(1 - tk(s))^{-1} \partial_t (1 - tk(s)) \tilde{a} \partial_t \\ &\quad + (1 - tk(s))^{-1} (-ih\partial_s + \tilde{A}) (1 - tk(s))^{-1} \tilde{a} (-ih\partial_s + \tilde{A}), \end{aligned}$$

where:

$$\tilde{A}(s, t) = (t - \xi_0 h^{1/2}) + b_1 s (t - \xi_0 h^{1/2}) + (b_2 - k_0) \frac{t^2}{2} + b_{11} s^2 (t - \xi_0 h^{1/2}) + O(|t|^3 + |st^2|).$$

Let us now perform the scaling:

$$s = h^{1/4} \sigma \text{ and } t = h^{1/2} \tau.$$

The operator becomes:

$$\mathcal{L}(h) = \mathcal{L}(h^{1/4}\sigma, -ih^{3/4}\partial_\sigma; h^{1/2}\tau, -ih^{1/2}\partial_\tau).$$

We can formally write $\mathcal{L}(h)$ as a power series:

$$\mathcal{L}(h) \sim h \sum_{j \geq 0} \mathcal{L}_j h^{j/4},$$

where

$$\mathcal{L}_0 = -\partial_\tau^2 + (\tau - \xi_0)^2, \quad (2.1)$$

$$\begin{aligned} \mathcal{L}_1 &= -a_1\sigma\partial_\tau^2 - 2i\partial_\sigma(\tau - \xi_0) + a_1(\tau - \xi_0)^2\sigma + 2b_1\sigma\tau(\tau - \xi_0) \\ &= a_1\sigma H_{\xi_0} - 2i\partial_\sigma(\tau - \xi_0) + 2b_1\sigma(\tau - \xi_0)^2, \end{aligned} \quad (2.2)$$

$$\begin{aligned} \mathcal{L}_2 &= -a_2\tau\partial_\tau^2 - a_2\partial_\tau + k_0\partial_\tau + 2k_0\tau(\tau - \xi_0)^2 + a_2\tau(\tau - \xi_0)^2 \\ &\quad + (b_2 - k_0)\tau^2(\tau - \xi_0) - ia_1(\tau - \xi_0) \\ &\quad + \sigma^2(a_{11}H_{\xi_0} - a_1^2(\tau - \xi_0)^2 + 2b_{11}(\tau - \xi_0)^2) \\ &\quad - \partial_\sigma^2 - 2ia_1(\tau - \xi_0)\sigma\partial_\sigma + ia_1(\tau - \xi_0)\partial_\sigma\sigma. \end{aligned} \quad (2.3)$$

The aim is now to define good quasimodes for $\mathcal{L}(h)$. Before starting the construction, we shall recall in the next subsection a few formulas coming from perturbation theory.

2.1 Feynman-Hellmann and Virial formulas

For $\rho > 0$ and $\xi \in \mathbb{R}$, let us introduce the Neumann realization on \mathbb{R}_+ of:

$$H_{\rho,\xi} = -\rho^{-1}\partial_\tau^2 + (\rho^{1/2}\tau - \xi)^2.$$

By scaling, we observe that $H_{\rho,\xi}$ is unitarily equivalent to H_ξ and that $H_{1,\xi} = H_\xi$ (the corresponding eigenfunction is $u_{1,\xi} = u_\xi$). The form domain of $H_{\rho,\xi}$ is $B^1(\mathbb{R}_+)$ and is independent from ρ and ξ so that the family $(H_{\rho,\xi})_{\rho>0, \xi \in \mathbb{R}}$ is an holomorphic family of type (B) (see [15, p. 395]). The lowest eigenvalue of $H_{\rho,\xi}$ is $\mu(\xi)$ and we will denote by $u_{\rho,\xi}$ the corresponding normalized eigenfunction:

$$u_{\rho,\xi}(\tau) = \rho^{1/4}u_\xi(\rho^{1/2}\tau).$$

Since u_ξ satisfies the Neumann condition, we observe that $\partial_\rho^m \partial_\xi^n u_{\rho,\xi}$ also satisfies it. In order to lighten the notation and when it is not ambiguous we will write H for $H_{\rho,\xi}$, u for $u_{\rho,\xi}$ and μ for $\mu(\xi)$.

The main idea is now to take derivatives of:

$$Hu = \mu u \quad (2.4)$$

with respect to ρ and ξ . Taking the derivative with respect to ρ and ξ , we get the proposition:

Proposition 2.2 *We have:*

$$(H - \mu)\partial_\xi u = 2(\rho^{1/2}\tau - \xi)u + \mu'(\xi)u \quad (2.5)$$

and

$$(H - \mu)\partial_\rho u = -\rho^{-2}\partial_\tau^2 - \xi\rho^{-1}(\rho^{1/2}\tau - \xi) - \rho^{-1}\tau(\rho^{1/2}\tau - \xi)^2. \quad (2.6)$$

Moreover, we get:

$$(H - \mu)(Su) = Xu, \quad (2.7)$$

where

$$X = -\frac{\xi}{2}\mu'(\xi) + \rho^{-1}\partial_\tau^2 + (\rho^{1/2}\tau - \xi)^2$$

and

$$S = -\frac{\xi}{2}\partial_\xi - \rho\partial_\rho.$$

Proof: Taking the derivatives with respect to ξ and ρ of (2.4), we get:

$$(H - \mu)\partial_\xi u = \mu'(\xi)u - \partial_\xi H u$$

and

$$(H - \mu)\partial_\rho u = -\partial_\rho H.$$

We have: $\partial_\xi H = -2(\rho^{1/2}\tau - \xi)$ and $\partial_\rho H = \rho^{-2}\partial_\rho^2 + \rho^{-1/2}\tau(\rho^{1/2}\tau - \xi)$. \square

Taking $\rho = 1$ and $\xi = \xi_0$ in (2.5), we deduce, with the Fredholm alternative:

Corollary 2.3 *We have:*

$$(H_{\xi_0} - \mu(\xi_0))v_{\xi_0} = 2(t - \xi_0)u_{\xi_0},$$

with:

$$v_{\xi_0} = (\partial_\xi u_\xi)|_{\xi=\xi_0}.$$

Moreover, we have:

$$\int_{\tau>0} (\tau - \xi_0)u_{\xi_0}^2 d\sigma d\tau = 0.$$

Corollary 2.4 *We have, for all $\rho > 0$:*

$$\int_{\tau>0} (\rho^{1/2}\tau - \xi_0)u_{\rho,\xi_0}^2 d\sigma d\tau = 0$$

and:

$$\int_{\tau>0} (\tau - \xi_0) (\partial_\rho u)_{\rho=1,\xi=\xi_0} u d\sigma d\tau = -\frac{\xi_0}{4}.$$

Corollary 2.5 *We have:*

$$(H_{\xi_0} - \mu(\xi_0))S_0u = (\partial_\tau^2 + (\tau - \xi_0)^2) u_{\xi_0},$$

where:

$$S_0u = -(\partial_\rho u_{\rho,\xi})_{|\rho=1,\xi=\xi_0} - \frac{\xi_0}{2} v_{\xi_0}.$$

Moreover, we have:

$$\|\partial_\tau u_{\xi_0}\|^2 = \|(\tau - \xi_0)u_{\xi_0}\|^2 = \frac{\Theta_0}{2}.$$

The next three propositions deal with the second derivatives of (2.4) with respect to ξ and ρ .

Proposition 2.6 *We have:*

$$(H_\xi - \mu(\xi))w_{\xi_0} = 4(\tau - \xi_0)v_{\xi_0} + (\mu''(\xi_0) - 2)u_{\xi_0},$$

with

$$w_{\xi_0} = (\partial_\xi^2 u_\xi)_{|\xi=\xi_0}.$$

Moreover, we have:

$$\int_{\tau>0} (\tau - \xi_0)v_{\xi_0}u_{\xi_0} d\sigma d\tau = \frac{2 - \mu''(\xi_0)}{4}.$$

Proof: Taking the derivative of (2.5) with respect to ξ (with $\rho = 1$), we get:

$$(H_\xi - \mu(\xi))\partial_\xi^2 u_\xi = 2\mu'(\xi)\partial_\xi u_\xi + 4(\tau - \xi)\partial_\xi u_\xi + (\mu''(\xi) - 2)u_\xi.$$

It remains to take $\xi = \xi_0$ and to write the Fredholm alternative. \square

Proposition 2.7 *We have:*

$$\begin{aligned} (H - \mu)(\partial_\rho^2 u)_{\rho=1,\xi=\xi_0} &= -2(\partial_\tau^2 + (\tau - \xi_0)^2)(\partial_\rho u)_{\rho=1,\xi=\xi_0} \\ &\quad - 2\xi_0(\tau - \xi_0)(\partial_\rho u)_{\rho=1,\xi=\xi_0} + (2\partial_\tau^2 - \frac{\xi_0\tau}{2})u_{\xi_0} \end{aligned}$$

and:

$$\langle (\partial_\tau^2 + (\tau - \xi_0)^2)(\partial_\rho u)_{\rho=1,\xi=\xi_0}, u_{\xi_0} \rangle = -\frac{\Theta_0}{2}.$$

Proof: We just have to take the derivative of (2.6) with respect to ρ and $\rho = 1$, $\xi = \xi_0$. To get the second identity, we use the Fredholm alternative, Corollary 2.4 and Corollary 2.5. \square

Taking the derivative of (2.7) with respect to ρ , we find:

Lemma 2.8 *We have:*

$$\begin{aligned} (H - \mu)(\partial_\rho S_0u)_{\rho=1,\xi=\xi_0} &= (-\partial_\tau^2 + \tau(\tau - \xi_0))u_{\xi_0} - (\partial_\rho H)_{\rho=1,\xi=\xi_0}(S_0u) \\ &\quad + (\partial_\tau^2 + (\tau - \xi_0)^2)(\partial_\rho u)_{\rho=1,\xi=\xi_0} \end{aligned}$$

and

$$\langle (\partial_\rho H)_{\rho=1,\xi=\xi_0}(S_0u), u \rangle = \frac{\Theta_0}{2}.$$

Lemma 2.9 *We have:*

$$\langle (\tau - \xi_0)S_0u, u_{\xi_0} \rangle = \frac{\xi_0}{8}\mu''(\xi_0).$$

Proof: We have:

$$\mu'(\xi) = -2 \int_{\tau>0} (\rho^{1/2}\tau - \xi)u_{\rho,\xi}^2 d\sigma d\tau$$

and:

$$S_0\mu' = -2 \int_{\tau>0} S_0(\rho^{1/2}\tau - \xi)u_{\xi_0}^2 d\sigma d\tau - 4 \int_{\tau>0} (\tau - \xi_0)S_0u u_{\xi_0} d\sigma d\tau.$$

□

Combining Lemmas 2.8 and 2.9, we deduce:

Proposition 2.10 *We have:*

$$\langle (-\partial_\tau^2 - (\tau - \xi_0)^2)S_0u, u_{\xi_0} \rangle = -\frac{\Theta_0}{2} + \frac{\Theta_0}{8}\mu''(\xi_0).$$

Proposition 2.11 *We have:*

$$\langle (\partial_\tau^2 + (\tau - \xi_0)^2)v_{\xi_0}, u_{\xi_0} \rangle = \frac{\xi_0\mu''(\xi_0)}{4}.$$

Proof: We take the derivative of (2.5) with respect to ρ (after having fixed $\xi = \xi_0$):

$$(H - \mu)(\partial_\xi u)_{\xi=\xi_0} = 2(\rho^{1/2}\tau - \xi_0)u_{\rho,\xi_0}.$$

We deduce:

$$(H - \mu)(\partial_\rho \partial_\xi u)_{\rho=1,\xi=\xi_0} = -(\partial_\rho H)_{\rho=1,\xi=\xi_0}v_{\xi_0} + \tau u_{\xi_0} + 2(\tau - \xi_0)(\partial_\rho u)_{\rho=1,\xi=\xi_0}.$$

The Fredholm alternative provides:

$$\langle (\partial_\tau^2 + \tau(\tau - \xi_0))v_{\xi_0}, u_{\xi_0} \rangle = \xi_0 + 2\langle (\tau - \xi_0)(\partial_\rho u)_{\rho=1,\xi=\xi_0}, u_{\xi_0} \rangle = \frac{\xi_0}{2},$$

where we have used Corollary 2.4. □

We have now the elements to perform an accurate construction of quasimodes.

2.2 Construction

We look for quasimodes expressed as power series:

$$\psi \sim \sum_{j \geq 0} \psi_j h^{j/4}$$

and eigenvalues:

$$\lambda \sim h \sum_{j \geq 0} \lambda_j h^{j/4}$$

so that, in the sense of formal series:

$$\mathcal{L}(h)\psi \sim \lambda\psi.$$

Term in h We consider the equation:

$$(\mathcal{L}_0 - \lambda_0)\psi_0 = 0.$$

We are led to take $\lambda_0 = \Theta_0$ and $\psi_0(\sigma, \tau) = f_0(\sigma)u_{\xi_0}(\tau)$.

Term in $h^{5/4}$ We want to solve the equation:

$$(\mathcal{L}_0 - \Theta_0)\psi_1 = \lambda_1\psi_0 - \mathcal{L}_1\psi_0.$$

We have, by using that $b_1 = -a_1$ and Proposition 2.2:

$$(\mathcal{L}_0 - \Theta_0)(\psi_1 - if_0'(\sigma)v_{\xi_0} - a_1\sigma f_0(\sigma)S_0u) = \lambda_1u_{\xi_0}.$$

This implies that $\lambda_1 = 0$ and we take:

$$\psi_1(\sigma, \tau) = if_0'(\sigma)v_{\xi_0} + a_1\sigma f_0(\sigma)S_0u + f_1(\sigma)u_{\xi_0}(\tau),$$

f_0 and f_1 being to determine.

Term in $h^{3/2}$ We consider the equation:

$$(\mathcal{L}_0 - \Theta_0)\psi_2 = \lambda_2\psi_0 - \mathcal{L}_1\psi_1 - \mathcal{L}_2\psi_0.$$

Let us rewrite this equation by using the expression of ψ_1 :

$$(\mathcal{L}_0 - \Theta_0)\psi_2 = \lambda_2\psi_0 - \mathcal{L}_1(if_0'(\sigma)v_{\xi_0} + a_1\sigma f_0(\sigma)S_0u) - \mathcal{L}_1(f_1(\sigma)u_{\xi_0}) - \mathcal{L}_2\psi_0.$$

With Proposition 2.2, we deduce:

$$\begin{aligned} & (\mathcal{L}_0 - \Theta_0)(\psi_2 - if_1'(\sigma)v_{\xi_0} - a_1\sigma f_1(\sigma)S_0u) \\ &= \lambda_2\psi_0 - \mathcal{L}_1(if_0'(\sigma)v_{\xi_0} + a_1\sigma f_0(\sigma)S_0u) - \mathcal{L}_2\psi_0. \end{aligned}$$

We take the partial scalar product (with respect to τ) of the r.h.s. with u_{ξ_0} and we get the equation:

$$\langle \mathcal{L}_1(if_0'(\sigma)v_{\xi_0} + a_1\sigma f_0(\sigma)S_0u) + \mathcal{L}_2\psi_0, u_{\xi_0} \rangle_\tau = \lambda_2f_0.$$

This equation can be written in the form:

$$(AD_\sigma^2 + B_1\sigma D_\sigma + B_2D_\sigma\sigma + C\sigma^2 + D)f_0 = \lambda_2f_0.$$

Terms in D_σ^2 Let us first analyze $\langle \mathcal{L}_2u_{\xi_0}, u_{\xi_0} \rangle$. This is easy to see that this terms is 1. Let us then analyze $\langle \mathcal{L}_1\psi_1, u_{\xi_0} \rangle$. With Proposition 2.6, we deduce that this term is $-2\langle (\tau - \xi_0)v_{\xi_0}, u_{\xi_0} \rangle = \frac{\mu''(\xi_0)}{2} - 1$. We get: $A = \frac{\mu''(\xi_0)}{2} > 0$.

Terms in σ^2 Let us collect the terms of $\langle \mathcal{L}_2 u_{\xi_0}, u_{\xi_0} \rangle$. We get:

$$\Theta_0 a_{11} + 2b_{11} \langle (\tau - \xi_0)^2 u_{\xi_0}, u_{\xi_0} \rangle - a_1^2 \langle (\tau - \xi_0)^2 u_{\xi_0}, u_{\xi_0} \rangle.$$

With Corollary 2.5, this term is equal to:

$$\Theta_0 a_{11} + \Theta_0 b_{11} - \frac{\Theta_0}{2} a_1^2.$$

Let us analyze the terms coming from $\langle \mathcal{L}_1 \psi_1, u_{\xi_0} \rangle$. We obtain the term:

$$a_1^2 \langle (-\partial_\tau^2 - (\tau - \xi_0)^2) S_0 u, u_{\xi_0} \rangle = -\frac{\Theta_0}{2} a_1^2 + \Theta_0 \frac{\mu''(\xi_0)}{8} a_1^2,$$

where we have used Proposition 2.10. Thus, we have:

$$C = \Theta_0 a_{11} + \Theta_0 b_{11} - \Theta_0 a_1^2 + \frac{\Theta_0}{8} \mu''(\xi_0) a_1^2 > 0.$$

Terms in σD_σ This term only comes from $\langle \mathcal{L}_1 \psi_1, u_{\xi_0} \rangle$. It is equal to:

$$a_1 \langle (\partial_\tau^2 + (\tau - \xi_0)^2) v_{\xi_0}, u_{\xi_0} \rangle = a_1 \frac{\xi_0 \mu''(\xi_0)}{4},$$

where we have used Proposition 2.11.

Terms in $D_\sigma \sigma$ This term is:

$$2a_1 \langle (\tau - \xi_0) S_0 u, u_{\xi_0} \rangle = a_1 \frac{\xi_0 \mu''(\xi_0)}{4},$$

where we have applied Lemma 2.9.

Value of D We have:

$$D = \langle (-a_2 \tau \partial_\tau^2 - a_2 \partial_\tau + k_0 \partial_\tau + 2k_0 \tau (\tau - \xi_0)^2 + a_2 \tau (\tau - \xi_0)^2) u_{\xi_0}, u_{\xi_0} \rangle \\ + \langle ((b_2 - k_0) \tau^2 (\tau - \xi_0) - i a_1 (\tau - \xi_0)) u_{\xi_0}, u_{\xi_0} \rangle.$$

Using the relations (1.4) and the definition of C_1 given in (1.3), we get:

$$D = C(k_0, a_2, b_2).$$

Let us introduce the quadratic form which is fundamental in the analysis. We let:

$$\mathcal{Q}(\sigma, \eta) = \frac{\mu''(\xi_0)}{2} \eta^2 + a_1 \frac{\xi_0 \mu''(\xi_0)}{4} \eta \sigma + a_1 \frac{\xi_0 \mu''(\xi_0)}{4} \sigma \eta \\ + \Theta_0 \left(a_{11} + b_{11} - a_1^2 + a_1^2 \frac{\mu''(\xi_0)}{8} \right) \sigma^2.$$

Lemma 2.12 \mathcal{Q} is definite and positive.

Proof: We notice that $\mu''(\xi_0) > 0$ and $a_{11} + b_{11} - a_1^2 + a_1^2 \frac{\mu''(\xi_0)}{8} > 0$. The determinant is given by:

$$\begin{aligned} & \Theta_0 \frac{\mu''(\xi_0)}{2} \left(a_{11} + b_{11} - a_1^2 + a_1^2 \frac{\mu''(\xi_0)}{8} \right) - a_1^2 \frac{\Theta_0 \mu''(\xi_0)^2}{16} \\ &= \frac{\Theta_0 \mu''(\xi_0)}{2} (a_{11} + b_{11} - a_1^2) > 0. \end{aligned}$$

□

We immediately deduce that $\mathcal{Q}(\sigma, -i\partial_\sigma)$ is unitarily equivalent to an harmonic oscillator and that the increasing sequence of its eigenvalues is given by

$$\left\{ (2n+1) \left(\frac{\Theta_0 \mu''(\xi_0)}{2} (a_{11} + b_{11} - a_1^2) \right)^{1/2} \right\}_{n \in \mathbb{N}}.$$

The compatibility equation becomes:

$$\mathcal{Q}(\sigma, D_\sigma) f_0 = (\lambda_2 - D) f_0.$$

Thus, we choose λ_2 such that $\lambda_2 - D$ is in the spectrum of $\mathcal{Q}(\sigma, D_\sigma)$ and we take for f_0 the corresponding normalized eigenfunction (which is in the Schwartz class). For that choice of f_0 , we can consider the unique solution ψ_2^\perp (which is in the Schwartz class) of:

$$(\mathcal{L}_0 - \Theta_0) \psi_2^\perp = \lambda_2 \psi_0 - \mathcal{L}_1 (i f_0'(\sigma) v_{\xi_0} + a_1 \sigma f_0(\sigma) S_0 u) - \mathcal{L}_2 \psi_0$$

satisfying $\langle \psi_2^\perp, u_{\xi_0} \rangle = 0$. It follows that ψ_2 is in the form:

$$\psi_2 = \psi_2^\perp(\sigma, \tau) + i f_1'(\sigma) v_{\xi_0} + a_1 \sigma f_1(\sigma) S_0 u + f_2(\sigma) u_{\xi_0},$$

where f_1 and f_2 are still to be determined.

Higher order terms Let $N \geq 2$. Let us assume that, for $0 \leq j \leq N-2$, the functions ψ_j are determined and belong to the Schwartz class. Moreover, let us also assume that, for $j = N-1, N$, we can write:

$$\psi_j(\sigma, \tau) = \psi_j^\perp(\sigma, \tau) + i f_{j-1}'(\sigma) v_{\xi_0} + a_1 \sigma f_{j-1}(\sigma) S_0 u + f_j(\sigma) u_{\xi_0},$$

where the $(\psi_j^\perp)_{j=N-1, N}$ and f_{N-2} are determined functions in the Schwartz class and where the $(f_j)_{j=N-1, N}$ are not determined. Finally, we also assume that the $(\lambda_j)_{0 \leq j \leq N}$ are determined. We notice that this recursion assumption is satisfied for $N = 2$. Let us write the equation of order $N+1$:

$$\begin{aligned} (\mathcal{L}_0 - \Theta_0) \psi_{N+1} &= \lambda_{N+1} \psi_0 - \mathcal{L}_1 \psi_N + (\lambda_2 - \mathcal{L}_2) \psi_{N-1} \\ &\quad - \mathcal{L}_{N+1} \psi_0 + \sum_{j=1}^{N-2} (\lambda_{N+1-j} - \mathcal{L}_{N+1-j}) \psi_j. \end{aligned}$$

This equation takes the form:

$$(\mathcal{L}_0 - \Theta_0)\psi_{N+1} = \lambda_{N+1}\psi_0 - \mathcal{L}_1\psi_N + (\lambda_2 - \mathcal{L}_2)\psi_{N-1} + F_N(\sigma, \tau),$$

where F_N is a determined function in the Schwartz class by recursion assumption. Using Proposition 2.2, we can rewrite:

$$\begin{aligned} & (\mathcal{L}_0 - \Theta_0)(\psi_{N+1} - if'_N(\sigma)v_{\xi_0} - a_1\sigma f_N(\sigma)S_0u) \\ &= \lambda_{N+1}\psi_0 - \mathcal{L}_1\left(\psi_N^\perp(\sigma, \tau) + if'_{N-1}(\sigma)v_{\xi_0} + a_1\sigma f_{N-1}(\sigma)S_0u\right) + (\lambda_2 - \mathcal{L}_2)\psi_{N-1} \\ & \quad + F_N(\sigma, \tau) \\ &= \lambda_{N+1}\psi_0 - \mathcal{L}_1(if'_{N-1}(\sigma)v_{\xi_0} + a_1\sigma f_{N-1}(\sigma)S_0u) + (\lambda_2 - \mathcal{L}_2)(f_{N-1}u_{\xi_0}) \\ & \quad + G_N(\sigma, \tau), \end{aligned}$$

where G_N is a determined function of the Schwartz class. We now write the Fredholm condition. The same computation as previously leads to an equation in the form:

$$\mathcal{Q}(\sigma, -i\partial_\sigma)f_{N-1} = (\lambda_2 - C(a_2, b_2, k_0))f_{N-1} + \lambda_{N+1}f_0 + g_N(\sigma),$$

with $g_N = \langle G_N, u_{\xi_0} \rangle_\tau$. This can be rewritten as:

$$(\mathcal{Q}(\sigma, -i\partial_\sigma) - (\lambda_2 - C(a_2, b_2, k_0)))f_{N-1} = g_N(\sigma) + \lambda_{N+1}f_0.$$

The Fredholm condition applied to this equation provides: $\lambda_{N+1} = -\langle g_N, f_0 \rangle_\sigma$ and a unique solution f_{N-1} in the Schwartz class such that $\langle f_{N-1}, f_0 \rangle_\sigma = 0$. For this choice of f_{N-1} and λ_{N+1} , we can consider the unique solution ψ_{N+1}^\perp (in the Schwartz class) such that:

$$\begin{aligned} & (\mathcal{L}_0 - \Theta_0)\psi_{N+1}^\perp \\ &= \lambda_{N+1}\psi_0 - \mathcal{L}_1\left(\psi_N^\perp(\sigma, \tau) + if'_{N-1}(\sigma)v_{\xi_0} + a_1\sigma f_{N-1}(\sigma)S_0u\right) + (\lambda_2 - \mathcal{L}_2)\psi_{N-1} \\ & \quad + F_N(\sigma, \tau). \end{aligned}$$

This leads to take:

$$\psi_{N+1} = \psi_{N+1}^\perp + if'_N(\sigma)v_{\xi_0} + a_1\sigma f_N(\sigma)S_0u + f_{N+1}u_{\xi_0}.$$

This ends the proof of the recursion. Thus, we have constructed two sequences $(\lambda_j)_j$ and $(\psi_j)_j$ which depend on n (through the choice of f_0). Let us write $\lambda_{n,j}$ for λ_j and $\psi_{n,j}$ for ψ_j to emphasize this dependence.

Conclusion: proof of Theorem 2.1 Let us consider a smooth cutoff function χ_0 near x_0 . For $n \geq 1$ and $J \geq 0$, we let:

$$\psi_h^{[n,J]}(x) = \chi_0(x) \sum_{j=0}^J \psi_{n,j}(h^{-1/4}s(x), h^{-1/2}t(x))h^{j/4} \quad (2.8)$$

and:

$$\lambda_h^{[n,J]} = \sum_{j=0}^J \lambda_{n,j} h^{j/4}.$$

Using the fact that the ψ_j are in the Schwartz class, we get:

$$\| (P_{h,\mathbf{A}} - \lambda_h^{[n,J]}) \psi_h^{[n,J]} \| \leq C(n, J) h^{\frac{J+1}{4}} \| \psi_h^{[n,J]} \|.$$

Thanks to the spectral theorem, we deduce Theorem 2.1.

3 Rough lower bound and consequence

This section is devoted to establish a rough lower bound for $\lambda_n(h)$. In particular, we give the first term of the asymptotics and deduce the so-called normal Agmon estimates.

3.1 A first lower bound

We now aim at proving a lower bound:

Proposition 3.1 *We have:*

$$\lambda_n(h) \geq \Theta_0 h a(x_0) \beta(x_0) - Ch^{5/4}.$$

Proof: We use a partition of unity with balls D_j of size h^ρ and satisfying:

$$\sum_j \chi_j^2 = 1 \text{ and } \sum_j \|\nabla \chi_j\|^2 \leq Ch^{-2\rho}.$$

The so-called IMS formula (cf. [7]) provides:

$$Q_{h,\mathbf{A}}(\psi) = \sum_j Q_{h,\mathbf{A}}(\chi_j \psi) - h^2 \sum_j \int_{\Omega} a \|\nabla \chi_j\|^2 |\psi|^2 dx$$

and thus:

$$Q_{h,\mathbf{A}}(\psi) \geq \sum_j Q_{h,\mathbf{A}}(\chi_j \psi) - Ch^{2-2\rho} \|\psi\|^2.$$

In each ball, we approximate a by a constant:

$$Q_{h,\mathbf{A}}(\chi_j \psi) \geq (a(x_j) - Ch^\rho) \|(-ih\nabla + \mathbf{A})(\chi_j \psi)\|^2.$$

If D_j does not intersect the boundary, then:

$$\|(-ih\nabla + \mathbf{A})(\chi_j \psi)\|^2 \geq h \int_{\Omega} \beta(x) |\chi_j \psi|^2 dx.$$

We deduce:

$$Q_{h,\mathbf{A}}(\chi_j \psi) \geq (a(x_j) \beta(x_j) h - Ch^{1+\rho}) \|\chi_j \psi\|^2.$$

If D_j intersects the boundary, we can assume that its center is on the boundary and we write in the local coordinates (up to a change of gauge):

$$Q_{h,\mathbf{A}}(\chi_j\psi) \geq (1 - Ch^\rho) \int \tilde{a} \left(h^2 |\partial_t(\chi_j\psi)|^2 + |(-ih\partial_s + \tilde{A}_1)(\chi_j\psi)|^2 \right) dsdt.$$

We deduce:

$$Q_{h,\mathbf{A}}(\chi_j\psi) \geq (1 - Ch^\rho)(a(x_j) - Ch^\rho) \int h^2 |\partial_t(\chi_j\psi)|^2 + |(-ih\partial_s + \tilde{A}_1)(\chi_j\psi)|^2 dsdt.$$

We approximate A_1 by its linear approximation A_1^{lin} and we have:

$$\begin{aligned} & \int h^2 |\partial_t(\chi_j\psi)|^2 + |(-ih\partial_s + \tilde{A}_1)(\chi_j\psi)|^2 dsdt \\ & \geq (1 - \varepsilon) \int h^2 |\partial_t(\chi_j\psi)|^2 + |(-ih\partial_s + \tilde{A}_1^{\text{lin}})(\chi_j\psi)|^2 dsdt - C\varepsilon^{-1} \int |x - x_j|^4 |\chi_j\psi|^2 dx \\ & \geq ((1 - \varepsilon)\Theta_0\beta(x_j)h - C\varepsilon^{-1}h^{4\rho}) \|\chi_j\psi\|^2. \end{aligned}$$

To optimize the remainder, we choose: $\varepsilon = h^{2\rho-1/2}$. Then, we take $\rho = \frac{3}{8}$ and the conclusion follows. \square

3.2 Normal Agmon estimates

We now prove the following (weighted) localization estimates:

Proposition 3.2 *Let us consider a smooth cutoff function χ supported in a fixed neighborhood of the boundary. Let $(\lambda_n(h), \psi_h)$ be an eigenpair of $P_{h,\mathbf{A}}$. For all $\delta \geq 0$, there exist $\varepsilon_0, C \geq 0$ and h_0 such that, for $h \in (0, h_0)$:*

$$\begin{aligned} & \|e^{\chi(x)\varepsilon_0 t(x)h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4}} \psi_h\|^2 \leq C \|e^{\delta\chi(x)|s(x)|h^{-1/4}} \psi_h\|^2, \\ & Q_{h,\mathbf{A}} \left(e^{\chi(x)\varepsilon_0 t(x)h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4}} \psi_h \right) \leq Ch \|e^{\delta\chi(x)|s(x)|h^{-1/4}} \psi_h\|^2. \end{aligned}$$

Proof: The proof is based on a technique of Agmon (see for instance [1, 2, 12]). Let us recall the IMS formula ; we have, for an eigenpair $(\lambda_n(h), \psi_h)$:

$$Q_{h,\mathbf{A}}(e^\Phi \psi_h) = \lambda_n(h) \|e^\Phi \psi_h\|^2 + h^2 \|a^{1/2} \nabla \Phi e^\Phi \psi_h\|^2.$$

We take:

$$\Phi = \chi(x)\varepsilon_0 t(x)h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4},$$

where χ is a smooth cutoff function supported near the boundary. We use a partition of unity χ_j with balls of size $Rh^{1/2}$ with R large enough and we get:

$$\sum_j \left(Q_{h,\mathbf{A}}(\chi_j e^\Phi \psi_h) - \lambda_n(h) \|\chi_j e^\Phi \psi_h\|^2 - CR^{-2}h - h^2 \|\chi_j a^{1/2} \nabla \Phi e^\Phi \psi_h\|^2 \right) \leq 0.$$

We now distinguish the balls intersecting the boundary and the others. For the interior balls, we have the lower bound, for $\eta > 0$ and h small enough:

$$Q_{h,\mathbf{A}}(\chi_j e^\Phi \psi_h) \geq \left(a(x_j)\beta(x_j)h - Ch^{3/2} \right) \|\chi_j e^\Phi \psi_h\|^2.$$

For the boundary balls, we have:

$$Q_{h,\mathbf{A}}(\chi_j e^\Phi \psi_h) \geq \left(\Theta_0 a(x_j)\beta(x_j)h - Ch^{3/2} \right) \|\chi_j e^\Phi \psi_h\|^2.$$

Let us now split the sum:

$$\begin{aligned} & \sum_{j \text{int}} \int \left(a(x_j)\beta(x_j)h - \Theta_0 a(x_0)\beta(x_0)h - Ch^{3/2} - CR^{-2}h - Ch^2 \|\nabla \Phi\|^2 \right) |\chi_j e^\Phi \psi_h|^2 dx \\ & \leq Ch \sum_{j \text{bnd}} \|\chi_j e^\Phi \psi_h\|^2. \end{aligned}$$

We can notice that:

$$\|\nabla \Phi\|^2 \leq C(\varepsilon_0^2 h^{-1} + \delta^2 h^{-1/2}).$$

Taking R large enough, ε_0 and h small enough and using (1.6), we get the existence of $c > 0$ such that:

$$a(x_j)\beta(x_j)h - \Theta_0 a(x_0)\beta(x_0)h - Ch^{3/2} - CR^{-2}h - Ch^2 \|\nabla \Phi\|^2 \geq ch.$$

We deduce:

$$c \sum_{j \text{int}} \|\chi_j e^\Phi \psi_h\|^2 \leq C \sum_{j \text{bnd}} \|\chi_j e^\Phi \psi_h\|^2.$$

Due to support considerations, we can write:

$$C \sum_{j \text{bnd}} \|\chi_j e^\Phi \psi_h\|^2 \leq \tilde{C} \sum_{j \text{bnd}} \|\chi_j e^{\delta \chi(x)|s(x)|h^{-1/4}} \psi_h\|^2.$$

Thus, we infer:

$$\|e^\Phi \psi_h\|^2 \leq \tilde{C} \|e^{\delta \chi(x)|s(x)|h^{-1/4}} \psi_h\|^2.$$

We deduce that:

$$\sum_j Q_{h,\mathbf{A}}(\chi_j e^\Phi \psi_h) \leq Ch \|e^{\delta \chi(x)|s(x)|h^{-1/4}} \psi_h\|^2.$$

and thus:

$$Q_{h,\mathbf{A}}(e^\Phi \psi_h) \leq Ch \|e^{\delta \chi(x)|s(x)|h^{-1/4}} \psi_h\|^2.$$

□

Corollary 3.3 *Let $\eta \in (0, \frac{1}{2}]$. Let $(\lambda_n(h), \psi_h)$ be an eigenpair of $P_{h,\mathbf{A}}$. For all $\delta \geq 0$, there exist $\varepsilon_0, C \geq 0$ and h_0 such that, for $h \in (0, h_0)$:*

$$\|\chi_{h,\eta} e^{\chi(x)\varepsilon_0 t(x)h^{-1/2} + \delta \chi(x)|s(x)|h^{-1/4}} \psi_h\|^2 \leq C \|\chi_{h,\eta} e^{\delta \chi(x)|s(x)|h^{-1/4}} \psi_h\|^2,$$

$$Q_{h,\mathbf{A}} \left(\chi_{h,\eta} e^{\chi(x)\varepsilon_0 t(x)h^{-1/2} + \delta \chi(x)|s(x)|h^{-1/4}} \psi_h \right) \leq Ch \|\chi_{h,\eta} e^{\delta \chi(x)|s(x)|h^{-1/4}} \psi_h\|^2,$$

where $\chi_{h,\eta}(x) = \hat{\chi}(t(x)h^{-1/2+\eta})$ and with $\hat{\chi}$ a smooth cutoff function being 1 near 0.

Proof: With Proposition 3.2, we have:

$$\|\chi_{h,\eta} e^{\chi(x)\varepsilon_0 t(x)h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4}} \psi_h\|^2 \leq C \|e^{\delta\chi(x)|s(x)|h^{-1/4}} \psi_h\|^2.$$

We can write:

$$\|e^{\delta\chi(x)|s(x)|h^{-1/4}} \psi_h\|^2 = \|\chi_{h,\eta} e^{\delta\chi(x)|s(x)|h^{-1/4}} \psi_h\|^2 + \|\sqrt{1 - \chi_{h,\eta}} e^{\delta\chi(x)|s(x)|h^{-1/4}} \psi_h\|^2.$$

Using Proposition 3.2, we have the estimate:

$$\begin{aligned} & \|\sqrt{1 - \chi_{h,\eta}} e^{\delta\chi(x)|s(x)|h^{-1/4}} \psi_h\|^2 \\ &= \|\sqrt{1 - \chi_{h,\eta}} e^{-\chi(x)\varepsilon_0 t(x)h^{-1/2}} e^{\chi(x)\varepsilon_0 t(x)h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4}} \psi_h\|^2 \\ &= O(h^\infty) \|e^{\delta\chi(x)|s(x)|h^{-1/4}} \psi_h\|^2. \end{aligned}$$

The IMS formula provides:

$$Q_{h,\mathbf{A}}(e^\Phi \psi_h) = Q_{h,\mathbf{A}}(\chi_{h,\eta} e^\Phi \psi_h) + Q_{h,\mathbf{A}}(\sqrt{1 - \chi_{h,\eta}} e^\Phi \psi_h) + O(h^{1+2\eta}) \|e^\Phi \psi_h\|^2.$$

□

Corollary 3.4 *Let $\eta \in (0, \frac{1}{2}]$. Let $(\lambda_n(h), \psi_h)$ be an eigenpair of $P_{h,\mathbf{A}}$. For all $\delta \geq 0$, there exist $\varepsilon_0, C \geq 0$ and h_0 such that, for $h \in (0, h_0)$:*

$$\begin{aligned} & \|\chi_{h,\eta} e^{\chi(x)\varepsilon_0 t(x)h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4}} (-ih\partial_s + \tilde{A}_1) \psi_h\|^2 \leq Ch \|\chi_{h,\eta} e^{\delta\chi(x)|s(x)|h^{-1/4}} \psi_h\|^2, \\ & \|\chi_{h,\eta} e^{\chi(x)\varepsilon_0 t(x)h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4}} (-ih\partial_t + \tilde{A}_2) \psi_h\|^2 \leq Ch \|\chi_{h,\eta} e^{\delta\chi(x)|s(x)|h^{-1/4}} \psi_h\|^2. \end{aligned}$$

4 Order of the second term

It is well-known that the order of the second term in the asymptotics of $\lambda_n(h)$ is closely related to localization properties of the corresponding eigenfunctions. The aim of this section is to establish such properties.

Proposition 4.1 *Under the generic assumptions, there exist $C > 0$ and $h_0 > 0$ such that for $h \in (0, h_0)$:*

$$\lambda_n(h) \geq \Theta_0 a(x_0) \beta(x_0) h - Ch^{3/2}.$$

Moreover, for all $\delta \geq 0$, there exist $C > 0$ and $h_0 > 0$ such that for $h \in (0, h_0)$:

$$\int e^{2\delta\chi(x)|s| h^{-1/4}} |\psi|^2 ds dt \leq C \|\psi\|^2.$$

Proof: Let us recall the so-called IMS formula (see for instance [7]); we have, for an eigenpair $(\lambda_n(h), \psi)$:

$$Q_{h,\mathbf{A}}(e^\Phi \psi) - \lambda_n(h) \|e^\Phi \psi\|^2 - h^2 \|a^{1/2} \nabla \Phi e^\Phi \psi\|^2 = 0.$$

We take:

$$\Phi = \delta\chi(x)|s(x)|h^{-1/4}, \quad \text{with } \delta \geq 0.$$

The idea is now to prove a suitable lower bound for $Q_{h,\mathbf{A}}$. We use a partition of unity with balls of size $h^{1/4}$. We get the lower bound:

$$Q_{h,\mathbf{A}}(e^\Phi\psi) \geq \sum_j Q_{h,\mathbf{A}}(\psi_j) - Ch^{3/2}\|e^\Phi\psi\|^2,$$

where

$$\psi_j = \chi_{j,h}e^\Phi\psi$$

and we deduce:

$$\sum_j Q_{h,\mathbf{A}}(\psi_j) - Ch^{3/2}\|\psi_j\|^2 - \lambda_n(h)\|\psi_j\|^2 \leq 0, \quad (4.1)$$

since $\|\nabla\Phi\| \leq Ch^{-1/2}$.

Interior balls Considering the balls intersecting the boundary, we get:

$$\sum_{j\text{int}} Q_{h,\mathbf{A}}(\psi_j) \geq \sum_{j\text{int}} (a(x_j)\beta(x_j)h - Ch^{5/4})\|\psi_j\|^2. \quad (4.2)$$

Boundary balls Let us consider the j such that D_j intersects the boundary. Using first the normal Agmon estimates, we have the lower bound:

$$\sum_{j\text{bnd}} Q_{h,\mathbf{A}}(\psi_j) \geq \sum_{j\text{bnd}} \int \tilde{a}(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2) dsdt - Ch^{3/2}\|\psi\|^2,$$

where we have used the IMS formula to get:

$$\begin{aligned} & \sum_{j\text{bnd}} \int t\tilde{a}(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2) dsdt \\ & \leq C \int_{0 < t < t_0} t\tilde{a}(|(-ih\partial_t + \tilde{A}_2)e^\Phi\psi|^2 + |(ih\partial_s + \tilde{A}_1)e^\Phi\psi|^2) dsdt + Ch^{3/2}\|e^\Phi\psi\|^2. \end{aligned}$$

Using again the normal estimates and also the size of the balls, we get:

$$\sum_{j\text{bnd}} Q_{h,\mathbf{A}}(\psi_j) \geq \sum_{j\text{bnd}} \int \tilde{a}_j^{\text{lin}}(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2) dsdt - Ch^{3/2}\|\psi\|^2, \quad (4.3)$$

where

$$\tilde{a}_j^{\text{lin}} = a_j + (s - s_j)\partial_s\tilde{a}(x_j).$$

Case $|s_j| \geq s_0$ Let us consider the boundary balls such that $|s_j| \geq s_0$. Using the size of the balls, we get the lower bound:

$$\begin{aligned} & \int \tilde{a}_j^{\text{lin}} (|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2) dsdt \\ & \geq (\Theta_0 a(x_j)\beta(x_j)h - Ch^{5/4}) \|\psi_j\|^2 \geq \Theta_0(1 + \varepsilon)a(x_0)\beta(x_0)h \|\psi_j\|^2. \end{aligned} \quad (4.4)$$

Case $|s_j| \leq s_0$ Let us consider the boundary balls such that $|s_j| \leq s_0$. In each ball, we can use a new gauge so that:

$$\begin{aligned} & \sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \int \tilde{a}_j^{\text{lin}} (|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2) dsdt \\ & = \sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \int \tilde{a}_j^{\text{lin}} (|h\partial_t\psi_j|^2 + |(ih\partial_s + \tilde{A}_1^{\text{new}})\psi_j|^2) dsdt, \end{aligned}$$

where \tilde{A}_1^{new} satisfies:

$$|\tilde{A}_1^{\text{new}} - t\tilde{\beta}_j^{\text{lin}}| \leq C(t|s - s_j|^2 + t^2),$$

with:

$$\tilde{\beta}_j^{\text{lin}} = \tilde{\beta}_j + \partial_s \tilde{\beta}(x_j)(s - s_j).$$

We obtain, thanks to the estimates of Agmon:

$$\begin{aligned} & \sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \int \tilde{a}_j^{\text{lin}} (|h\partial_t\psi_j|^2 + |(ih\partial_s + \tilde{A}_1^{\text{new}})\psi_j|^2) dsdt \\ & \geq (1 - h^{1/2}) \sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \int \tilde{a}_j^{\text{lin}} (h^2|\partial_t\psi_j|^2 + |(ih\partial_s + t\tilde{\beta}_j^{\text{lin}})\psi_j|^2) dsdt - Ch^{3/2}\|\psi\|^2. \end{aligned} \quad (4.5)$$

In each ball, we use the change of variables:

$$\sigma = s \quad \text{and} \quad \tau = \left(\tilde{\beta}_j^{\text{lin}}\right)^{1/2} t.$$

We can write:

$$\partial_t = \left(\tilde{\beta}_j^{\text{lin}}\right)^{1/2} \partial_\tau \quad \text{and} \quad \partial_s = \partial_\sigma + \partial_s \left(\tilde{\beta}_j^{\text{lin}}\right)^{1/2} \partial_\tau$$

and

$$dsdt = \left(\tilde{\beta}_j^{\text{lin}}\right)^{-1/2} d\sigma d\tau.$$

We obtain:

$$\begin{aligned} & \int h^2 |\partial_t \hat{\psi}_j|^2 + |(ih\partial_s + t\tilde{\beta}_j^{\text{lin}})\hat{\psi}_j|^2 dsdt \\ & \geq (1 - h^{1/2}) \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \left(h^2 |\partial_t \hat{\psi}_j|^2 + |(ih \left(\tilde{\beta}_j^{\text{lin}}\right)^{-1/2} \partial_\sigma + \tau)\hat{\psi}_j|^2 \right) \left(\tilde{\beta}_j^{\text{lin}}\right)^{-1/2} d\sigma d\tau \\ & - Ch^{3/2} \int |\tau \partial_\tau \hat{\psi}_j|^2 d\sigma d\tau. \end{aligned} \quad (4.6)$$

With the normal Agmon estimates, we have:

$$\sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \int |\tau \partial_\tau \hat{\psi}_j|^2 d\sigma d\tau \leq C \|\psi\|^2.$$

We can notice that the Dirichlet realization on $(-\tilde{s}_0, \tilde{s}_0)$ of $D_\sigma \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2}$ is self-adjoint on $L^2 \left(\left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma \right)$. Thus, we shall commute D_σ and $\left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2}$ and control the remainder due to the commutator. We can write:

$$\begin{aligned} & \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \left(h^2 |\partial_t \hat{\psi}_j|^2 + |(ih \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \partial_\sigma + \tau) \hat{\psi}_j|^2 \right) \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\ &= \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} h^2 |\partial_t \hat{\psi}_j|^2 \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\ & \quad + \int a_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} |(ih \partial_\sigma \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} - ih \partial_\sigma \left\{ \tilde{\beta}_j^{\text{lin}} \right\}^{-1/2} + \tau) \hat{\psi}_j|^2 \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau. \end{aligned}$$

We can estimate the double product:

$$\begin{aligned} & 2h \Re \left(\int a_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \left(ih \partial_\sigma \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} + \tau \right) \hat{\psi}_j i \partial_\sigma \left\{ \tilde{\beta}_j^{\text{lin}} \right\}^{-1/2} \overline{\hat{\psi}_j} \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \right) \\ &= -2h^2 \Re \left(\int a_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \partial_\sigma \left\{ \tilde{\beta}_j^{\text{lin}} \right\}^{-1/2} \partial_\sigma \left(\left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \hat{\psi}_j \right) \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \overline{\hat{\psi}_j} d\sigma d\tau \right) \\ &= -h^2 \int a_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \partial_\sigma \left\{ \tilde{\beta}_j^{\text{lin}} \right\}^{-1/2} \partial_\sigma \left(\left| \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \hat{\psi}_j \right|^2 \right) d\sigma d\tau = O(h^2) \|\psi_j\|^2, \end{aligned}$$

where we have used an integration by parts for the last estimate. We deduce:

$$\begin{aligned} & \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \left(h^2 |\partial_t \hat{\psi}_j|^2 + |(ih \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \partial_\sigma + \tau) \hat{\psi}_j|^2 \right) \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \quad (4.7) \\ & \geq \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \left(h^2 |\partial_t \hat{\psi}_j|^2 + |(ih \partial_\sigma \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} + \tau) \hat{\psi}_j|^2 \right) \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\ & \quad - Ch^2 \|\psi_j\|^2. \end{aligned}$$

For s_0 small enough, we have, using the non-degeneracy, for s such that $|s| \leq \tilde{s}_0$ (with \tilde{s}_0 slightly bigger than s_0):

$$\tilde{a}_j^{\text{lin}}(s) \tilde{\beta}_j^{\text{lin}}(s) \geq a(x_0) \beta(x_0) + \frac{\alpha}{4} |s|^2.$$

Let us analyze the integral:

$$\begin{aligned} & \int |\sigma (ih \partial_\sigma \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} + \tau) \hat{\psi}_j|^2 \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\ &= \int |(ih \partial_\sigma \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} + \tau) \sigma \hat{\psi}_j - ih \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \hat{\psi}_j|^2 \left(\tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau. \end{aligned}$$

We must estimate the double product:

$$\begin{aligned}
& 2\Re \int \left((ih\partial_\sigma (\tilde{\beta}_j^{\text{lin}})^{-1/2} + \tau)\sigma\hat{\psi}_j ih (\tilde{\beta}_j^{\text{lin}})^{-1/2} \overline{\hat{\psi}_j} \right) (\tilde{\beta}_j^{\text{lin}})^{-1/2} d\sigma d\tau \\
&= -2h^2 \Re \int \left(\partial_\sigma \left((\tilde{\beta}_j^{\text{lin}})^{-1/2} \sigma\hat{\psi}_j \right) (\tilde{\beta}_j^{\text{lin}})^{-1/2} \overline{\hat{\psi}_j} \right) (\tilde{\beta}_j^{\text{lin}})^{-1/2} d\sigma d\tau \\
&= -h^2 \int \partial_\sigma \left| (\tilde{\beta}_j^{\text{lin}})^{-1/2} \hat{\psi}_j \right|^2 (\tilde{\beta}_j^{\text{lin}})^{-1/2} d\sigma d\tau + O(h^2) \|\hat{\psi}_j\|^2 \\
&= O(h^2) \|\hat{\psi}_j\|^2.
\end{aligned}$$

We infer:

$$\begin{aligned}
& \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \left(h^2 |\partial_t \hat{\psi}_j|^2 + |(ih\partial_\sigma (\tilde{\beta}_j^{\text{lin}})^{-1/2} + \tau)\hat{\psi}_j|^2 \right) (\tilde{\beta}_j^{\text{lin}})^{-1/2} d\sigma d\tau \\
&\geq a(x_0)\beta(x_0) \int \left(h^2 |\partial_t \hat{\psi}_j|^2 + |(ih\partial_\sigma (\tilde{\beta}_j^{\text{lin}})^{-1/2} + \tau)\hat{\psi}_j|^2 \right) (\tilde{\beta}_j^{\text{lin}})^{-1/2} d\sigma d\tau \\
&+ \frac{\alpha}{4} \int \left(h^2 |\partial_t (\sigma\hat{\psi}_j)|^2 + |(ih\partial_\sigma (\tilde{\beta}_j^{\text{lin}})^{-1/2} + \tau)\sigma\hat{\psi}_j|^2 \right) (\tilde{\beta}_j^{\text{lin}})^{-1/2} d\sigma d\tau \\
&- Ch^2 \|\hat{\psi}_j\|^2.
\end{aligned}$$

We recall that, for all $\xi \in \mathbb{R}$:

$$\int \left(h^2 |\partial_t \phi|^2 + |(\tau - h\xi - \xi_0 h^{1/2})\phi|^2 \right) d\tau \geq h\mu(\xi_0 + h^{1/2}\xi) \|\phi\|^2 \geq \Theta_0 \|\phi\|^2.$$

We infer with the functional calculus:

$$\begin{aligned}
& \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \left(h^2 |\partial_t \hat{\psi}_j|^2 + |(ih\partial_\sigma (\tilde{\beta}_j^{\text{lin}})^{-1/2} + \tau - \xi_0 h^{1/2})\hat{\psi}_j|^2 \right) (\tilde{\beta}_j^{\text{lin}})^{-1/2} d\sigma d\tau \\
&\geq h\Theta_0 \int \left(a(x_0)\beta(x_0) + \frac{\alpha}{4}\sigma^2 \right) |\hat{\psi}_j|^2 (\tilde{\beta}_j^{\text{lin}})^{-1/2} d\sigma d\tau - Ch^2 \|\hat{\psi}_j\|^2.
\end{aligned} \tag{4.8}$$

Lower bound for $\lambda_n(h)$ If we take $\delta = 0$, we deduce, with (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.7) and (4.8):

$$\lambda_n(h) \|\psi\|^2 \geq \sum_j \Theta_0 h a(x_0) \beta(x_0) \int |\psi_j|^2 dx - Ch^{3/2} \|\psi\|^2.$$

Tangential Agmon estimate Gathering all the estimates, we deduce the existence of $c > 0$ such that:

$$\begin{aligned}
& \sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \left(\Theta_0 h \int \left(a(x_0)\beta(x_0) + \frac{\alpha}{4}s^2 \right) |\psi_j|^2 ds dt - \Theta_0 h \|\psi_j\|^2 - Ch^{3/2} \|\psi_j\|^2 \right) \\
&+ \sum_{\substack{j \text{ bnd} \\ |s_j| \geq s_0}} ch \|\psi_j\|^2 + \sum_{j \text{ int}} ch \|\psi_j\|^2 \leq 0
\end{aligned}$$

and:

$$\sum_{\substack{j \text{ bnd} \\ 2C_0 h^{1/4} \leq |s_j| \leq s_0}} \left(\Theta_0 h \int \frac{\alpha}{4} s^2 |\psi_j|^2 ds dt - Ch^{3/2} \|\psi_j\|^2 \right) \leq Ch^{3/2} \|\psi_0\|^2 \leq Ch^{3/2} \|\psi\|^2.$$

Taking C_0 large enough, we infer:

$$\sum_{\substack{j \text{ bnd} \\ 2C_0 h^{1/4} \leq |s_j| \leq s_0}} \|\psi_j\|^2 \leq C \|\psi\|^2$$

so that:

$$\sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \|\psi_j\|^2 \leq C \|\psi\|^2$$

and:

$$\sum_j \|\psi_j\|^2 = \|e^\Phi \psi\|^2 \leq C \|\psi\|^2.$$

□

Let us write an immediate corollary (see Corollaries 3.3 and 3.4).

Corollary 4.2 *Let $(\eta_1, \eta_2) \in (0, \frac{1}{2}] \times (0, \frac{1}{4}]$. Let $(\lambda_n(h), \psi_h)$ be an eigenpair of $P_{h,A}$. For all $(k, l) \in \mathbb{N}$, there exist $C \geq 0$ and $h_0 > 0$ such that, for $h \in (0, h_0)$:*

$$\begin{aligned} \|\chi_{h,\eta_1,\eta_2} s^k t^l \psi_h\|^2 &\leq Ch^{k/2} h^l \|\psi_h\|^2, \\ \|\chi_{h,\eta_1,\eta_2} s^k t^l (-ih\partial_s + \tilde{A}_1) \psi_h\|^2 &\leq Chh^{k/2} h^l \|\psi_h\|^2, \\ \|\chi_{h,\eta_1,\eta_2} s^k t^l (-ih\partial_t + \tilde{A}_2) \psi_h\|^2 &\leq Chh^{k/2} h^l \|\psi_h\|^2. \end{aligned}$$

where $\chi_{h,\eta_1,\eta_2}(x) = \hat{\chi}(t(x)h^{-1/2+\eta_1})\hat{\chi}(s(x)h^{-1/4+\eta_2})$. Moreover, we have:

$$\begin{aligned} \|(1 - \chi_{h,\eta_1,\eta_2}) s^k t^l \psi_h\|^2 &= O(h^\infty) \|\psi_h\|^2, \\ \|(1 - \chi_{h,\eta_1,\eta_2}) s^k t^l (-ih\partial_s + \tilde{A}_1) \psi_h\|^2 &= O(h^\infty) \|\psi_h\|^2, \\ \|(1 - \chi_{h,\eta_1,\eta_2}) s^k t^l (-ih\partial_t + \tilde{A}_2) \psi_h\|^2 &= O(h^\infty) \|\psi_h\|^2. \end{aligned}$$

Remark 4.3 In the following, each reference to the "estimates of Agmon" will be a reference to this last corollary.

5 Refined lower bound

We use a cutoff function χ_h near x_0 with support of order $h^{1/4-\tilde{\eta}}$ with $\tilde{\eta} > 0$. For all $N \geq 1$, let us consider L^2 -normalized eigenpairs $(\lambda_n(h), \psi_{n,h})_{1 \leq n \leq N}$ such that $\langle \psi_{n,h}, \psi_{m,h} \rangle = 0$ when $n \neq m$. We consider the N dimensional space defined by:

$$\mathfrak{E}_N(h) = \text{span}_{1 \leq n \leq N} \tilde{\psi}_{n,h}, \quad \text{where} \quad \tilde{\psi}_{n,h} = \chi_h \psi_{n,h}.$$

Remark 5.1 The estimates of Agmon of Corollary 4.2 are satisfied by all the elements of $\mathfrak{E}_N(h)$.

We can notice that, with the estimates of Agmon, for all $\tilde{\psi} \in \mathfrak{E}_N(h)$:

$$Q_{h,\mathbf{A}}(\tilde{\psi}) \leq \lambda_N(h)\|\tilde{\psi}\|^2 + O(h^\infty)\|\tilde{\psi}\|^2. \quad (5.1)$$

In this subsection, we provide a lower bound for $Q_{h,\mathbf{A}}$ on $\mathfrak{E}_N(h)$.

Choice of gauge and new coordinates On the support of χ_h , we use a gauge such that $\tilde{A}_2 = 0$ and

$$|\tilde{A}_1 - \tilde{A}_1^{\text{app}}| \leq C(t^3 + |s|t^2 + |s|^2t),$$

where:

$$\tilde{A}_1^{\text{app}} = t(1 + b_1s + b_{11}s^2) + \frac{\hat{b}_2}{2}t^2 = t\hat{b}(s) - \xi_0\hat{b}(s)^{1/2}h^{1/2} + \frac{\hat{b}_2}{2}t^2,$$

where $\hat{b}_2 = b_2 - k_0$. We also let:

$$\tilde{a}^{\text{app}}(s, t) = 1 + a_1s + a_{11}s^2 + a_2t = \hat{a}(s) + a_2t.$$

Moreover, in this neighborhood of $(0, 0)$, we introduce new coordinates:

$$\tau = t(\hat{b}(s))^{1/2}, \quad \sigma = s. \quad (5.2)$$

In particular, we get:

$$\partial_t = (\hat{b}(\sigma))^{1/2}\partial_\tau, \quad \partial_s = \partial_\sigma + \frac{1}{2}\hat{b}^{-1}\partial_s\hat{b}\tau\partial_\tau$$

and:

$$dsdt = \hat{b}^{-1/2}d\sigma d\tau.$$

To simplify the notation, we let: $p = \hat{b}^{-1/2}$. We will also use the change of variable:

$$\check{\sigma} = \int_0^\sigma \frac{1}{p(u)} du = f(\sigma)$$

so that $L^2(p d\sigma)$ becomes $L^2(\check{p}^2 d\check{\sigma})$.

5.1 A first lower bound

This subsection is devoted to the proof of the following lower bound of $Q_{h,\mathbf{A}}$ on $\mathfrak{E}_N(h)$.

Proposition 5.2 *There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and all $\tilde{\psi} \in \mathfrak{E}_N(h)$:*

$$Q_{h,\mathbf{A}}(\tilde{\psi}) \geq \check{Q}_{h,\text{app}}(\tilde{\psi}) - Ch^{3/2+1/4}\|\tilde{\psi}\|^2, \quad (5.3)$$

where:

$$\begin{aligned} \tilde{Q}_{h,\text{app}}(\tilde{\psi}) &= \int (1 + a_2\tau)(1 - \tau k_0) |h\partial_\tau \tilde{\psi}|^2 \tilde{p}^2 d\tilde{\sigma} d\tau \\ &+ \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(ih\tilde{p}^{-1}\partial_{\tilde{\sigma}}\tilde{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau) \tilde{\psi}|^2 \tilde{p}^2 d\tilde{\sigma} d\tau \\ &+ h\alpha\Theta_0 \int \tilde{\sigma}^2 |\tilde{\psi}|^2 \tilde{p}^2 d\tilde{\sigma} d\tau, \end{aligned}$$

where $\tilde{\psi}$ denotes $\tilde{\psi}$ in the coordinates $(\tilde{\sigma}, \tau)$.

In order to prove Proposition 5.2, we will need a first lemma:

Lemma 5.3 *There exist $h_0 > 0$ and $C > 0$ s. t. for $h \in (0, h_0)$ and all $\tilde{\psi} \in \mathfrak{E}_N(h)$:*

$$Q_{h,\mathbf{A}}(\tilde{\psi}) \geq \hat{Q}_{h,\text{app}}(\hat{\psi}) - Ch^{3/2+1/4} \|\tilde{\psi}\|^2.$$

where:

$$\begin{aligned} \hat{Q}_{h,\text{app}}(\hat{\psi}) &= \int m_2(\sigma, \tau) |h\partial_\tau \hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &+ \int m_1(\sigma, \tau) |(h\Xi + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau) \hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau, \end{aligned}$$

with

$$\begin{aligned} \Xi &= i\partial_\sigma \hat{b}^{-1/2}, \quad m_1(\sigma, \tau) = (1 + \alpha\sigma^2)(1 + a_2\tau)(1 - \tau k_0)^{-1}, \\ m_2(\sigma, \tau) &= (1 + \alpha\sigma^2)(1 + a_2\tau)(1 - \tau k_0) \end{aligned}$$

and where $\hat{\psi}$ denotes $\tilde{\psi}$ in the coordinates (σ, τ) .

Proof: We have:

$$Q_{h,\mathbf{A}}(\tilde{\psi}) = \int \tilde{a}(1 - tk(s)) |(-ih\partial_t + \tilde{A}_2) \tilde{\psi}|^2 + \tilde{a}(1 - tk(s))^{-1} |(ih\partial_s + \tilde{A}_1) \tilde{\psi}|^2 ds dt.$$

Thanks to the normal and tangential Agmon estimates, we get:

$$Q_{h,\mathbf{A}}(\tilde{\psi}) \geq \int \tilde{a}(1 - tk_0) h^2 |\partial_t \tilde{\psi}|^2 + \tilde{a}(1 - tk_0)^{-1} |(ih\partial_s + \tilde{A}_1) \tilde{\psi}|^2 ds dt - Ch^{3/2+1/4} \|\tilde{\psi}\|^2.$$

The Agmon estimates imply:

$$\begin{aligned} Q_{h,\mathbf{A}}(\tilde{\psi}) &\geq \int \tilde{a}^{\text{app}}(1 - tk_0) h^2 |\partial_t \tilde{\psi}|^2 + \tilde{a}^{\text{app}}(1 - tk_0)^{-1} |(ih\partial_s + \tilde{A}_1^{\text{app}}) \tilde{\psi}|^2 ds dt \\ &\quad - Ch^{3/2+1/4} \|\tilde{\psi}\|^2. \end{aligned}$$

We get:

$$\begin{aligned} Q_{h,\mathbf{A}}(\tilde{\psi}) &\geq \\ &\int \hat{a}(1 + a_2t) \left((1 - tk_0) h^2 |\partial_t \tilde{\psi}|^2 + (1 - tk_0)^{-1} \left| (ih\partial_s + \tilde{A}_1^{\text{app}}) \tilde{\psi} \right|^2 \right) ds dt \\ &\quad - Ch^{3/2+1/4} \|\tilde{\psi}\|^2. \end{aligned}$$

With the coordinates (σ, τ) , we obtain:

$$\begin{aligned} & \int \hat{a}(1 + a_2 t)(1 - tk_0)h^2 |\partial_t \tilde{\psi}|^2 + (1 + a_2 t)(1 - tk_0)^{-1} |(ih\partial_s + \tilde{A}_1^{\text{app}})\tilde{\psi}|^2 ds dt \\ & \geq \hat{Q}_h(\hat{\psi}) - Ch^{3/2+1/4} \|\tilde{\psi}\|^2, \end{aligned}$$

where:

$$\begin{aligned} \hat{Q}_h(\hat{\psi}) &= \int \tilde{m}_2(\sigma, \tau) |h\partial_\tau \hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &+ \int \tilde{m}_1(\sigma, \tau) |(h\hat{b}^{-1/2}i\partial_\sigma + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 \hat{b}^{-1/2} - h\frac{\partial_\sigma \hat{b}}{2\hat{b}^{3/2}}\tau D_\tau)\hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau, \end{aligned}$$

where:

$$\tilde{m}_1(\sigma, \tau) = \hat{a}\hat{b}(1 + a_2\tau)(1 - \tau k_0)^{-1}, \quad \tilde{m}_2(\sigma, \tau) = \hat{a}\hat{b}(1 + a_2\tau)(1 - \tau k_0).$$

With the estimates of Agmon, we can simplify the quadratic form modulo lower order terms:

$$\begin{aligned} \hat{Q}_h(\hat{\psi}) &\geq \int \tilde{m}_2(\sigma, \tau) |h\partial_\tau \hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &+ \int \tilde{m}_1(\sigma, \tau) |(h\hat{b}^{-1/2}i\partial_\sigma + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &- Ch^{3/2+1/4} \|\tilde{\psi}\|^2. \end{aligned}$$

We recall that $\hat{a}\hat{b} = 1 + \alpha\sigma^2 + O(|\sigma|^3)$ so that, we the estimates of Agmon, we infer:

$$\begin{aligned} \hat{Q}_h(\hat{\psi}) &\geq \int m_2(\sigma, \tau) |h\partial_\tau \hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &+ \int m_1(\sigma, \tau) |(h\hat{b}^{-1/2}i\partial_\sigma + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &- Ch^{3/2+1/4} \|\tilde{\psi}\|^2. \end{aligned}$$

We now want to replace $\hat{b}^{-1/2}i\partial_\sigma$ by $i\partial_\sigma \hat{b}^{-1/2}$ which is self-adjoint on $L^2(\hat{b}^{-1/2} d\sigma d\tau)$.

Writing a commutator, we get:

$$\begin{aligned} & \int m_1(\sigma, \tau) |(h\hat{b}^{-1/2}i\partial_\sigma + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &= \int m_1(\sigma, \tau) |(hi\partial_\sigma \hat{b}^{-1/2} - ih(\partial_\sigma \hat{b}^{-1/2}) + \tau - \xi_0 h^{1/2} + \frac{b_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau. \end{aligned}$$

Let us consider the double product:

$$\begin{aligned} & 2h\Re \left(\int m_1(\sigma, \tau) \left(hi\partial_\sigma \hat{b}^{-1/2} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau \right) \hat{\psi} i(\partial_\sigma \hat{b}^{-1/2})\bar{\hat{\psi}} \hat{b}^{-1/2} d\sigma d\tau \right) \\ &= 2h\Re \left(\int m_1(\sigma, \tau) \left(hi\partial_\sigma \hat{b}^{-1/2} - h\frac{b_1}{2}\tau D_\tau \right) \hat{\psi} i(\partial_\sigma \hat{b}^{-1/2})\bar{\hat{\psi}} \hat{b}^{-1/2} d\sigma d\tau \right) \\ &= -2h^2\Re \int m_1(\sigma, \tau) \left(\partial_\sigma (\hat{b}^{-1/2}\hat{\psi}) \right) (\partial_\sigma \hat{b}^{-1/2})\bar{\hat{\psi}} \hat{b}^{-1/2} d\sigma d\tau + O(h^2) \|\hat{\psi}\|^2, \end{aligned}$$

where we have used the normal Agmon estimates. We deduce that:

$$\begin{aligned}
& 2\Re \left(\int m_1(\sigma, \tau) \left(h i \partial_\sigma \hat{b}^{-1/2} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2} \tau^2 - h \frac{b_1}{2} \tau D_\tau \right) \hat{\psi} i (\partial_\sigma \hat{b}^{-1/2}) \overline{\hat{\psi}} \hat{b}^{-1/2} d\sigma d\tau \right) \\
&= -h^2 \int m_1(\sigma, \tau) (\partial_\sigma \hat{b}^{-1/2}) \partial_\sigma \left| \hat{b}^{-1/2} \hat{\psi} \right|^2 d\sigma d\tau + O(h^2) \|\hat{\psi}\|^2 \\
&= O(h^2) \|\hat{\psi}\|^2.
\end{aligned}$$

This implies:

$$\begin{aligned}
& \int m_1(\sigma, \tau) \left| (h \hat{b}^{-1/2} i \partial_\sigma + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2} \tau^2 - h \frac{b_1}{2} \tau D_\tau) \hat{\psi} \right|^2 \hat{b}^{-1/2} d\sigma d\tau \\
&\geq \int m_1(\sigma, \tau) \left| (h i \partial_\sigma \hat{b}^{-1/2} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2} \tau^2 - h \frac{b_1}{2} \tau D_\tau) \hat{\psi} \right|^2 \hat{b}^{-1/2} d\sigma d\tau \\
&\quad - Ch^2 \|\hat{\psi}\|^2.
\end{aligned}$$

□

Proof of Proposition 5.2 We use Lemma 5.3. In the coordinates $(\check{\sigma}, \tau)$, we have:

$$\begin{aligned}
\hat{Q}_{h,\text{app}}(\hat{\psi}) &= \int m_2(f^{-1}(\check{\sigma}), \tau) |h \partial_\tau \check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau \\
&\quad + \int m_1(f^{-1}(\check{\sigma}), \tau) \left| (i h \check{p}^{-1} \partial_{\check{\sigma}} \check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2} \tau^2 - h \frac{b_1}{2} \tau D_\tau) \check{\psi} \right|^2 \check{p}^2 d\check{\sigma} d\tau,
\end{aligned}$$

where

$$\begin{aligned}
m_1(f^{-1}(\check{\sigma}), \tau) &= (1 + \alpha f^{-1}(\check{\sigma})^2) (1 + a_2 \tau) (1 - \tau k_0)^{-1}, \\
m_2(f^{-1}(\check{\sigma}), \tau) &= (1 + \alpha f^{-1}(\check{\sigma})^2) (1 + a_2 \tau) (1 - \tau k_0).
\end{aligned}$$

We notice that: $f^{-1}(\check{\sigma}) = \check{\sigma} + O(|\check{\sigma}|^2)$ so that, we can use the estimates of Agmon to get:

$$\begin{aligned}
\hat{Q}_{h,\text{app}}(\hat{\psi}) &\geq \int m_2(\check{\sigma}, \tau) |h \partial_\tau \check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau \\
&\quad + \int m_1(\check{\sigma}, \tau) \left| (i h \check{p}^{-1} \partial_{\check{\sigma}} \check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2} \tau^2 - h \frac{b_1}{2} \tau D_\tau) \check{\psi} \right|^2 \check{p}^2 d\check{\sigma} d\tau \\
&\quad - Ch^{3/2+1/4} \|\check{\psi}\|^2.
\end{aligned}$$

This inequality can be rewritten as:

$$\hat{Q}_{h,\text{app}}(\hat{\psi}) \geq \check{Q}_{h,\text{app},1}(\check{\psi}) + \check{Q}_{h,\text{app},2}(\check{\psi}) - Ch^{3/2+1/4} \|\check{\psi}\|^2,$$

where:

$$\begin{aligned}\check{Q}_{h,\text{app},1}(\check{\psi}) &= \int (1 + a_2\tau)(1 - \tau k_0) |h\partial_\tau \check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau \\ &+ \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau\end{aligned}$$

and:

$$\begin{aligned}\check{Q}_{h,\text{app},2}(\check{\psi}) &= \int (1 + a_2\tau)(1 - \tau k_0) |h\partial_\tau(\check{\sigma}\check{\psi})|^2 \check{p}^2 d\check{\sigma} d\tau \\ &+ \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |\check{\sigma}(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau\end{aligned}$$

Reduction of $\check{Q}_{h,\text{app},2}(\check{\psi})$ By the estimates of Agmon, we have:

$$\begin{aligned}\check{Q}_{h,\text{app},2}(\check{\psi}) &\geq \int |h\partial_\tau(\check{\sigma}\check{\psi})|^2 \check{p}^2 d\check{\sigma} d\tau \\ &+ \int |\check{\sigma}(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau - Ch^{3/2+1/4}\|\check{\psi}\|^2.\end{aligned}$$

Moreover, we get:

$$\begin{aligned}&\int |\check{\sigma}(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau \\ &\geq \int |\check{\sigma}(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2})\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau - Ch^{3/2+1/4}\|\check{\psi}\|^2.\end{aligned}$$

Let us analyze $\int |\check{\sigma}(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2})\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau$. We have:

$$\begin{aligned}&\int |\check{\sigma}(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2})\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau \\ &= \int |(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2})\check{\sigma}\check{\psi} - ih\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau.\end{aligned}$$

The double product is:

$$\begin{aligned}&2\Re \left(\int (ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2})\check{\sigma}\check{\psi} ih\check{\psi} \check{p}^2 d\check{\sigma} d\tau \right) \\ &= -2h^2 \Re \left(\int (\check{p}^{-1}\partial_{\check{\sigma}}\check{p})\check{\sigma}\check{\psi} \check{\psi} \check{p}^2 d\check{\sigma} d\tau \right).\end{aligned}$$

But, we have:

$$2\Re \left(\int \partial_{\check{\sigma}}(\check{\sigma}\check{p}\check{\psi}) \overline{\check{p}\check{\psi}} d\check{\sigma} d\tau \right) = 2\Re \left(\int \check{p}\check{\psi} \overline{\check{p}\check{\psi}} d\check{\sigma} d\tau \right) + \int \check{\sigma} \partial_{\check{\sigma}} |\check{p}\check{\psi}|^2 d\check{\sigma} d\tau$$

and:

$$\int \check{\sigma} \partial_{\check{\sigma}} |\check{p}\check{\psi}|^2 d\check{\sigma} d\tau = - \int |\check{p}\check{\psi}|^2 d\check{\sigma} d\tau.$$

Gathering the estimates, we obtain the lower bound:

$$\hat{Q}_{h,\text{app}}(\hat{\psi}) \geq \check{Q}_{h,\text{app}}(\check{\psi}) - Ch^{3/2+1/4}\|\check{\psi}\|^2.$$

5.2 Toward a model operator

We now define the unitary transform which diagonalizes the self-adjoint operator $\check{p}^{-1}D_{\check{\sigma}}\check{p}$ (for completeness, one should extend \check{p} by 1 away from a neighborhood of 0). As we will see, with the coordinate $\check{\sigma}$, this transform admits a nice expression.

Weighted Fourier transform Let us now introduce the weighted Fourier transform $\mathcal{F}_{\check{p}}$:

$$(\mathcal{F}_{\check{p}}\psi)(\lambda) = \int_{\mathbb{R}} e^{-i\lambda\check{\sigma}}\psi(\check{\sigma})\check{p}(\check{\sigma})d\check{\sigma} = \mathcal{F}(\check{p}\psi).$$

We observe that $\mathcal{F}_{\check{p}} : L^2(\mathbb{R}, \check{p}^2 d\check{\sigma}) \rightarrow L^2(\mathbb{R}, d\lambda)$ is unitary. Standard computations provide:

$$\mathcal{F}_{\check{p}}((\check{p}^{-1}D_{\check{\sigma}}\check{p})\psi) = \lambda\mathcal{F}_{\check{p}}(\psi)$$

and:

$$\mathcal{F}_{\check{p}}(\check{\sigma}\psi) = -D_{\lambda}\mathcal{F}_{\check{p}}(\psi).$$

Proposition 5.4 *There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and all $\check{\psi} \in \mathfrak{E}_N(h)$:*

$$\begin{aligned} \check{Q}_{h,\text{app}}(\check{\psi}) &\geq \int (1 + a_2\tau)(1 - \tau k_0) |hD_{\tau}\check{\phi}|^2 d\lambda d\tau \\ &+ \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2)\check{\phi}|^2 d\lambda d\tau \\ &+ h\alpha\Theta_0 \int |D_{\lambda}\check{\phi}|^2 d\lambda d\tau - Ch^{3/2+1/4}\|\check{\psi}\|^2, \end{aligned}$$

where:

$$\check{\phi} = e^{-i\frac{b_1}{2h}(-h\lambda\frac{\tau^2}{2} + \frac{\tau^3}{3} - \xi_0 h^{1/2}\frac{\tau^2}{2} + \frac{b_2}{8}\tau^4)} \mathcal{F}_{\check{p}}(\check{\psi}).$$

Proof: We have:

$$\begin{aligned} \check{Q}_{h,\text{app}}(\check{\psi}) &= \int (1 + a_2\tau)(1 - \tau k_0) |h\partial_{\tau}\check{\varphi}|^2 d\lambda d\tau \\ &+ \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_{\tau})\check{\varphi}|^2 d\lambda d\tau \\ &+ h\alpha\Theta_0 \int |D_{\lambda}\check{\varphi}|^2 d\lambda d\tau, \end{aligned}$$

where

$$\check{\varphi} = \mathcal{F}_{\check{p}}(\check{\psi}).$$

With the normal estimates, we can write:

$$\begin{aligned}
& \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\check{\varphi}|^2 d\lambda d\tau \\
& \geq \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2)\check{\varphi}|^2 d\lambda d\tau \\
& \quad - b_1 \Re \left(\int (1 + a_2\tau)(1 - \tau k_0)^{-1} (-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2)\check{\varphi} \overline{\tau h D_\tau \check{\varphi}} d\lambda d\tau \right) \\
& \geq \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\check{\varphi}|^2 d\lambda d\tau \\
& \quad - b_1 \Re \left(\int (-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2)\check{\varphi} \overline{\tau h D_\tau \check{\varphi}} d\lambda d\tau \right) - Ch^{3/2+1/4}\|\check{\psi}\|^2.
\end{aligned}$$

Completing a square and using the normal Agmon estimates to control the additional terms, we get:

$$\begin{aligned}
& \check{Q}_{h,\text{app}}(\check{\psi}) \\
& \geq \int (1 + a_2\tau)(1 - \tau k_0) \left| \left(hD_\tau - \frac{b_1}{2}\tau \left(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 \right) \right) \check{\varphi} \right|^2 d\lambda d\tau \\
& \quad + \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2)\check{\varphi}|^2 d\lambda d\tau \\
& \quad + h\alpha\Theta_0 \int |D_\lambda \check{\varphi}|^2 d\lambda d\tau - Ch^{3/2+1/4}\|\check{\psi}\|^2.
\end{aligned}$$

We now change the gauge by letting:

$$\check{\varphi} = e^{i\frac{b_1}{2h}(-h\lambda\frac{\tau^2}{2} + \frac{\tau^3}{3} - \xi_0 h^{1/2}\frac{\tau^2}{2} + \frac{b_2}{8}\tau^4)} \check{\phi}.$$

We deduce:

$$\begin{aligned}
\check{Q}_{h,\text{app}}(\check{\psi}) & \geq \int (1 + a_2\tau)(1 - \tau k_0) |hD_\tau \check{\phi}|^2 d\lambda d\tau \\
& \quad + \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2)\check{\phi}|^2 d\lambda d\tau \\
& \quad + h\alpha\Theta_0 \int \left| D_\lambda \left(e^{-i\frac{\lambda b_1 \tau^2}{4}} \check{\phi} \right) \right|^2 d\lambda d\tau - Ch^{3/2+1/4}\|\check{\psi}\|^2.
\end{aligned}$$

Finally, we write:

$$\begin{aligned}
& \int \left| D_\lambda \left(e^{-i\frac{\lambda b_1 \tau^2}{4}} \check{\phi} \right) \right|^2 d\lambda d\tau = \int \left| D_\lambda \check{\phi} - \frac{b_1}{4}\tau^2 \check{\phi} \right|^2 d\lambda d\tau \\
& \geq \int |D_\lambda \check{\phi}|^2 d\lambda d\tau - C\|\tau^2 \check{\phi}\| \|D_\lambda \check{\phi}\| \geq \int |D_\lambda \check{\phi}|^2 d\lambda d\tau - C\|\tau^2 \check{\psi}\| \|D_\lambda \check{\phi}\|.
\end{aligned}$$

In addition, we notice that:

$$\|D_\lambda \check{\phi}\| \leq C (\|\check{\sigma}\psi\| + \|\tau^2 \check{\psi}\|) \leq Ch^{1/4} \|\check{\psi}\|.$$

□

In order to get a good model operator, we shall add a cutoff function with respect to τ . Let $\eta \in (0, \frac{1}{100})$. Let χ a cutoff function such that:

$$\chi(t) = 1, \text{ for } |t| \leq 1, \quad 0 \leq \chi \leq 1 \text{ and } \text{supp} \chi \subset [-2, 2].$$

We define

$$l(x) = x\chi(h^\eta x).$$

Applying the normal Agmon estimates, we have:

Proposition 5.5 *There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and all $\check{\psi} \in \mathfrak{E}_N(h)$:*

$$\begin{aligned} \check{Q}_{h,\text{app}}(\check{\psi}) &\geq \int (1 + a_2 h^{1/2} l(h^{-1/2} \tau)) (1 - h^{1/2} l(h^{-1/2} \tau) k_0) |h D_\tau \check{\phi}|^2 d\lambda d\tau \\ &+ \int (1 + a_2 h^{1/2} l(h^{-1/2} \tau)) (1 - h^{1/2} l(h^{-1/2} \tau) k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2} h l(h^{-1/2} \tau)^2) \check{\phi}|^2 d\lambda d\tau \\ &+ h\alpha \Theta_0 \int |D_\lambda \check{\phi}|^2 d\lambda d\tau - Ch^{3/2+1/4} \|\check{\psi}\|^2, \end{aligned}$$

where:

$$\check{\phi} = e^{-i\frac{b_1}{2h}(-h\lambda\frac{\tau^2}{2} + \frac{\tau^3}{3} - \xi_0 h^{1/2} \frac{\tau^2}{2} + \frac{b_2}{8} \tau^4)} \mathcal{F}_{\check{p}}(\check{\psi}).$$

5.3 A simpler model

We introduce the rescaled quadratic form:

$$\begin{aligned} Q_{\eta,h}(\varphi) &= \int (1 + a_2 h^{1/2} l(x)) (1 - l(x) k_0 h^{1/2}) |\partial_x \varphi|^2 d\lambda dx \\ &+ \int (1 + a_2 l(x) h^{1/2}) (1 - l(x) k_0 h^{1/2})^{-1} |(x - \xi_0 + h^{1/2} \lambda + \frac{\hat{b}_2}{2} l(x)^2 h^{1/2}) \varphi|^2 d\lambda dx \\ &+ \alpha \Theta_0 \int |D_\lambda \varphi|^2 d\lambda dx, \end{aligned}$$

We recall that $\hat{b}_2 = b_2 - k_0$. We will denote by $H_{\eta,h}$ its corresponding Friedrichs extension. We will denote by $\nu_n(Q_{\eta,h})$ the sequence of its Rayleigh quotients. For each λ , we will need to consider the following quadratic form:

$$\begin{aligned} q_{\lambda,\eta,h}(\varphi) &= \int (1 + a_2 h^{1/2} l(x)) (1 - l(x) k_0 h^{1/2}) |\partial_x \varphi|^2 dx \\ &+ \int (1 + a_2 l(x) h^{1/2}) (1 - l(x) k_0 h^{1/2})^{-1} |(x - \xi_0 + h^{1/2} \lambda + \frac{\hat{b}_2}{2} l(x)^2 h^{1/2}) \varphi|^2 dx, \end{aligned}$$

whose domain is $B^1(\mathbb{R}^+)$. We denote by $\nu_j(q_{\lambda,\eta,h})$ the increasing sequence of the eigenvalues of the associated operator. The main proposition of this subsection is the following:

Proposition 5.6 *For all $n \geq 1$, there exist $h_0 > 0$ and $C > 0$ s. t., for $h \in (0, h_0)$:*

$$\nu_n(Q_{\eta,h}) \geq \Theta_0 + \left(C(k_0, a_2, b_2) + (2n-1) \sqrt{\frac{\alpha \mu''(\xi_0) \Theta_0}{2}} \right) h^{1/2} - C h^{1/2+1/8}.$$

Jointly with Propositions 5.5, 5.2, Inequality (5.1), the min-max principle, we first deduce the size of the spectral gap between the lowest eigenvalues of $P_{h,\mathbf{A}}$. Then, with Theorem 2.1, we deduce Theorem 1.2.

5.3.1 Elementary properties of the spectrum

This subsection is devoted to basic properties of the spectrum of $Q_{\eta,h}$. The following proposition provides a lower bound for $\nu_1(q_{\lambda,\eta,h})$.

Proposition 5.7 *There exist positive constants C, c_0, M and h_0 s.t. if $h \in (0, h_0)$, then:*

1. *If $|\lambda| \geq M h^{-1/4-\eta}$, then:*

$$\nu_1(q_{\lambda,\eta,h}) \geq \Theta_0 + c_0 \min(1, \lambda^2 h).$$

2. *If $|\lambda| \leq M h^{-1/4-\eta}$, then:*

$$\nu_1(q_{\lambda,\eta,h}) \geq \Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h - C h^{3/4-3\eta},$$

where $C(k_0, a_2, b_2)$ is given in Theorem 1.2.

Proof: The proof is left to the reader as an adaptation of [11, Proposition 5.2.1]. \square

Let us now prove a lower bound for the essential spectrum of $H_{\eta,h}$.

Proposition 5.8 *There exist $h_0 > 0$ and $\tilde{c}_0 > 0$ such that, if $h \in (0, h_0)$, then:*

$$\inf \sigma_{\text{ess}}(Q_{\eta,h}) \geq \Theta_0 + \tilde{c}_0$$

Proof: Let $\phi \in \text{Dom}(Q_{\eta,h})$ such that $\text{supp}(\phi) \subset \mathbb{R}_+^2 \setminus [-\tilde{R}, \tilde{R}]^2$. Let us use a partition of unity: $\chi_{1,R}^2 + \chi_{2,R}^2 = 1$ such that $\chi_{1,R}(x) = \chi_1(R^{-1}x)$ and where χ_1 is a smooth cutoff function being 1 near 0. We have:

$$Q_{\eta,h}(\phi) \geq Q_{\eta,h}(\chi_{1,R}\phi) + Q_{\eta,h}(\chi_{2,R}\phi) - CR^{-2}\|\phi\|^2.$$

For $R \geq 2h^{-\eta}$, we have (the metrics becomes flat and we can compare with a problem in \mathbb{R}^2):

$$Q_{\eta,h}(\chi_{2,R}\phi) \geq \|\chi_{2,R}\phi\|^2.$$

We have:

$$Q_{\eta,h}(\chi_{1,R}\phi) \geq \int_{\mathbb{R}_+^2} \nu_1(q_{\lambda,\eta,h}) |\chi_{1,R}\phi|^2 + \alpha\Theta_0 |D_\lambda(\chi_{1,R}\phi)|^2 dx d\lambda$$

Taking $h \in (0, h_0)$ (where h_0 is given by Proposition 5.7) and $\tilde{R} \geq h^{-1/2}$, we infer:

$$Q_{\eta,h}(\chi_{1,R}\phi) \geq \int_{\mathbb{R}_+^2} (\Theta_0 + c_0) |\chi_{1,R}\phi|^2 dx d\lambda.$$

This implies that:

$$Q_{\eta,h}(\phi) \geq (\min(1, \Theta_0 + c_0) - Ch^{2\eta}) \|\phi\|^2.$$

The conclusion follows from a Persson's lemma-like argument (see [18] and also [11, Appendix B.3]). \square

The following proposition provides an upper bound for the lowest eigenvalues of $H_{\eta,h}$.

Proposition 5.9 *For all $M \geq 1$, there exist $h_0 > 0$, $C > 0$ s. t. for all $1 \leq n \leq M$:*

$$\nu_n(Q_{\eta,h}) \leq h^{-1} \lambda_n(h) + O(h^\infty).$$

Proof: This is a consequence of (5.1) joint with the lower bounds of Propositions 5.2 and 5.5 and the min-max principle (see for instance [21]). \square

Remark 5.10 For h small enough, we deduce that there is at least M eigenvalues below $\Theta_0 + \tilde{c}_0$. Let us consider the M first eigenvalues $\nu_n(Q_{\eta,h})$ below $\Theta_0 + \tilde{c}_0$. With Theorem 2.1, we deduce that, for all $M \geq 1$, there exist $h_0 > 0$ and $C(M) > 0$ such that, for $1 \leq n \leq M$:

$$0 \leq \nu_n(Q_{\eta,h}) - \Theta_0 \leq C(M)h^{1/2}.$$

For $1 \leq n \leq M$, let us consider a normalized eigenfunction $f_{n,\eta,h}$ associated to $\nu_n(Q_{\eta,h})$ so that $f_{n,\eta,h}$ and $f_{m,\eta,h}$ are orthogonal if $n \neq m$. Let us introduce:

$$\mathfrak{F}_M(h) = \text{span}_{1 \leq j \leq M}(f_{j,\eta,h}).$$

5.3.2 Agmon estimates

First, let us state Agmon estimates with respect to x .

Proposition 5.11 *There exist $h_0 > 0$, $\varepsilon_0 > 0$, $C > 0$ such that, for all $f \in \mathfrak{F}_M(h)$:*

$$\int_{\mathbb{R}_+^2} e^{\varepsilon_0 x} |f|^2 dx d\lambda \leq C \|f\|^2.$$

Proof: Let us use a partition of unity: $\chi_{1,R}^2 + \chi_{2,R}^2 = 1$, with $R \geq h^{-\eta}$. We take $\Phi = \varepsilon_0 \chi\left(\frac{x}{r}\right) |x|$. This IMS formula implies (with $f = f_{n,\eta,h}$):

$$Q_{\eta,h}(\chi_{1,R} e^\Phi f) + Q_{\eta,h}(\chi_{2,R} e^\Phi f) - C\varepsilon_0^2 \|e^\Phi f\|^2 - \nu_n(Q_{\eta,h}) \|e^\Phi f\|^2 \leq 0.$$

We recall that:

$$Q_{\eta,h}(\chi_{2,R} e^\Phi f) \geq \|\chi_{2,R} e^\Phi f\|^2$$

and that:

$$Q_{\eta,h}(\chi_{1,R} e^\Phi f) \geq \int \nu_1(q_{\lambda,\eta,h}) |\chi_{1,R} e^\Phi f|^2 dx d\lambda.$$

On the one hand, we have:

$$\begin{aligned} & Q_{\eta,h}(\chi_{2,R} e^\Phi f) - C\varepsilon_0^2 \|\chi_{2,R} e^\Phi f\|^2 - (\Theta_0 + Ch^{1/2}) \|\chi_{2,R} e^\Phi f\|^2 \\ & \geq (1 - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2}) \|\chi_{2,R} e^\Phi f\|^2. \end{aligned}$$

On the other hand, we get:

$$\begin{aligned} & Q_{\eta,h}(\chi_{1,R} e^\Phi f) - C\varepsilon_0^2 \|\chi_{1,R} e^\Phi f\|^2 - (\Theta_0 + Ch^{1/2}) \|\chi_{1,R} e^\Phi f\|^2 \\ & \geq \int (\nu_1(q_{\eta,\lambda,h}) - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2}) |\chi_{1,R} e^\Phi f|^2 dx d\lambda. \end{aligned}$$

When $|\lambda| \geq Mh^{-1/4-\eta}$, we have:

$$\nu_1(q_{\eta,\lambda,h}) - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2} \geq -C\varepsilon_0^2 - Ch^{1/2}.$$

When $|\lambda| \leq Mh^{-1/4}$, we have:

$$\nu_1(q_{\eta,\lambda,h}) - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2} \geq -C\varepsilon_0^2 - \tilde{C}h^{1/2}.$$

If h and ε_0 are small enough, we deduce that:

$$(1 - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2}) \|\chi_{2,R} e^\Phi f\|^2 \leq C \|\chi_{1,R} e^\Phi f\|^2$$

so that:

$$\|\chi_{2,R} e^\Phi f\|^2 \leq \tilde{C} \|f\|^2 \text{ and } \|e^\Phi f\|^2 \leq \hat{C} \|f\|^2,$$

where \tilde{C} and \hat{C} are independent from r . It remains to make $r \rightarrow +\infty$ and apply the Fatou lemma. Finally, this is easy to extend the inequality to $f \in \mathfrak{F}_M(h)$. \square

Then, we will need Agmon estimates with respect to λ :

Proposition 5.12 *There exist $h_0 > 0$, $C > 0$ such that, for all $f \in \mathfrak{F}_M(h)$:*

$$\int_{\mathbb{R}_+^2} e^{2h^{1/4}|\lambda|} |f|^2 dx d\lambda \leq C \|f\|^2 \quad (5.4)$$

and:

$$\int_{\mathbb{R}_+^2} e^{2h^{1/4}|\lambda|} |D_\lambda f|^2 dx d\lambda \leq Ch^{1/2} \|f\|^2 \quad (5.5)$$

Remark 5.13 Heuristically, those estimates with respect to λ correspond to the phase space localization of [10, Section 5].

Proof: We take $f = f_{j,\eta,h}$ and we use the IMS formula (with $\Phi = h^{1/4}\chi(r^{-1}|\lambda|)|\lambda|$) to get:

$$Q_{\eta,h}(e^\Phi f) \leq \nu_j(Q_{\eta,h})\|e^\Phi f\|^2 + C\|\nabla\Phi e^\Phi f\|^2 \leq (\Theta_0 + C(M)h^{1/2} + Ch^{1/2})\|e^\Phi f\|^2.$$

We recall that:

$$\begin{aligned} Q_{\eta,h}(e^\Phi f) &\geq \int_{\mathbb{R}_+^2} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2 + \alpha\Theta_0|D_\lambda(e^\Phi f)|^2 dx d\lambda \\ &\geq \int_{\mathbb{R}_+^2} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2 dx d\lambda. \end{aligned}$$

We have, for all $D > 0$:

$$\begin{aligned} &\int_{\mathbb{R}_+^2} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2 dx d\lambda \\ &= \int_{|\lambda| \leq Dh^{-1/4}} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2 dx d\lambda + \int_{|\lambda| \geq Dh^{-1/4}} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2 dx d\lambda. \end{aligned}$$

Moreover, we get:

$$\begin{aligned} &\int_{|\lambda| \geq Mh^{-1/4-\eta}} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2 dx d\lambda \\ &\geq \int_{|\lambda| \geq Mh^{-1/4-\eta}} (\Theta_0 + c_0 \min(1, h\lambda^2)) |e^\Phi f|^2 dx d\lambda \end{aligned}$$

and:

$$\begin{aligned} &\int_{Dh^{-1/4} \leq |\lambda| \leq Mh^{-1/4-\eta}} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2 dx d\lambda \\ &\geq \int_{Dh^{-1/4} \leq |\lambda| \leq Mh^{-1/4-\eta}} \left(\Theta_0 + C(k_0, a_2, b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h - Ch^{3/4-3\eta} \right) |e^\Phi f|^2 dx d\lambda. \end{aligned}$$

This leads to:

$$\begin{aligned} &\int_{|\lambda| \geq Dh^{-1/4}} (c_1 \min(1, h\lambda^2) - \tilde{C}h^{1/2} - C\alpha^2 h^{1/2}) |e^\Phi f|^2 dx d\lambda \\ &\leq \tilde{C}h^{1/2} \int_{|\lambda| \leq Dh^{-1/4}} |f|^2 d\lambda dx. \end{aligned}$$

It remains to take D large enough and we get (5.4). Then, we have:

$$\int_{\mathbb{R}_+^2} (\nu_1(q_{\lambda,\eta,h}) - \Theta_0)|e^\Phi f|^2 + \alpha\Theta_0|D_\lambda(e^\Phi f)|^2 dx d\lambda \leq Ch^{1/2}\|f\|^2.$$

But, we notice that:

$$\begin{aligned}
& \int_{\mathbb{R}_+^2} (\nu_1(q_{\lambda,\eta,h}) - \Theta_0) |e^\Phi f|^2 dx d\lambda \\
& \geq \int_{Dh^{-1/4} \leq |\lambda| \leq Mh^{-1/4-\eta}} \left(C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h - Ch^{3/4-3\eta} \right) |e^\Phi f|^2 dx d\lambda \\
& + \int_{|\lambda| \leq Dh^{-1/4}} \left(C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h - Ch^{3/4-3\eta} \right) |e^\Phi f|^2 d\lambda dx.
\end{aligned}$$

Taking D larger, we get:

$$\int_{Dh^{-1/4} \leq |\lambda| \leq Mh^{-1/4-\eta}} \left(\frac{\mu''(\xi_0)}{2} \lambda^2 h - Ch^{1/2} - Ch^{3/4-3\eta} \right) |e^\Phi f|^2 dx d\lambda \geq 0.$$

Moreover, we have:

$$\left| \int_{|\lambda| \leq Dh^{-1/4}} \left(C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h - Ch^{3/4-3\eta} \right) |e^\Phi f|^2 d\lambda dx \right| \leq Ch^{1/2} \|f\|^2.$$

□

5.3.3 Approximations of eigenvectors

Let us define the quadratic form q_0 with domain $B^1(\mathbb{R}_+) \otimes L^2(\mathbb{R})$:

$$q_0(\varphi) = Q_0(\varphi) - \Theta_0 \|\varphi\|^2 = \int_{\mathbb{R}_+^2} |\partial_x \varphi|^2 + |(x - \xi_0)\varphi|^2 - \Theta_0 |\varphi|^2 dx d\lambda.$$

The Friedrichs extension of q_0 is the operator $H_{\xi_0} \otimes \text{Id}_{L^2(\mathbb{R})}$. We also define the projection on the kernel of $H_{\xi_0} \otimes \text{Id}_{L^2(\mathbb{R})}$:

$$\Pi_0 \varphi = \langle \varphi, u_{\xi_0} \rangle_x u_{\xi_0}(x).$$

The next proposition states an approximation result for the elements of $\mathfrak{F}_M(h)$ (which behave as tensor products):

Proposition 5.14 *For all $M \geq 1$, there exist $h_0 > 0$ and $C > 0$ such that, we have, for all $f \in \mathfrak{F}_M(h)$:*

$$\|f - \Pi_0 f\|_{L^2} + \|\partial_x(f - \Pi_0 f)\|_{L^2} + \|x(f - \Pi_0 f)\|_{L^2} \leq Ch^{1/8} \|f\|, \quad (5.6)$$

$$\|(\lambda f - \Pi_0 \lambda f)\|_{L^2} + \|\partial_x(\lambda f - \Pi_0 \lambda f)\|_{L^2} + \|x(\lambda f - \Pi_0 \lambda f)\|_{L^2} \leq Ch^{-1/8} \|f\|, \quad (5.7)$$

$$\|(\partial_\lambda f - \Pi_0 \partial_\lambda f)\|_{L^2} + \|\partial_x(\partial_\lambda f - \Pi_0 \partial_\lambda f)\|_{L^2} + \|x(\partial_\lambda f - \Pi_0 \partial_\lambda f)\|_{L^2} \leq Ch^{3/8} \|f\|. \quad (5.8)$$

In particular, Π_0 is an isomorphism from $\mathfrak{F}_M(h)$ onto its range.

Proof: We take $f = f_{j,\eta,h}$. By definition, we have:

$$H_{\eta,h} f = \nu_j(Q_{\eta,h}) f. \quad (5.9)$$

Approximation of f We deduce:

$$Q_{\eta,h}(f) = \nu_j(Q_{\eta,h})\|f\|^2 \leq (\Theta_0 + Ch^{1/2})\|f\|^2.$$

We have:

$$Q_{\eta,h}(f) \geq (1 - Ch^{1/2-\eta}) \int_{\mathbb{R}_+^2} |\partial_x f|^2 + |(x - \xi_0 + h^{1/2}\lambda + h^{1/2}\frac{\hat{b}_2}{2}l(x)^2)f|^2 dx d\lambda.$$

Moreover, we get (using the estimates of Agmon), for all $\varepsilon \in (0, 1)$:

$$\begin{aligned} & \int_{\mathbb{R}_+^2} |\partial_x f|^2 + |(x - \xi_0 + h^{1/2}\lambda + h^{1/2}\frac{\hat{b}_2}{2}l(x)^2)f|^2 dx d\lambda \\ & \geq (1 - \varepsilon)Q_0(f) - C\varepsilon^{-1}h^{1/2}\|f\|^2. \end{aligned}$$

Taking $\varepsilon = h^{1/4}$, we deduce:

$$q_0(f) \leq Ch^{1/4}\|f\|^2.$$

We deduce (5.6).

Approximation of λf We multiply (5.9) by λ and take the scalar product with λf :

$$Q_{\eta,h}(\lambda f) \leq (\Theta_0 + Ch^{1/2})\|\lambda f\|^2 + |\langle [H_{\eta,h}, \lambda]f, \lambda f \rangle|.$$

Thus, it follows:

$$Q_{\eta,h}(\lambda f) \leq (\Theta_0 + Ch^{1/2})\|\lambda f\|^2 + \alpha\Theta_0|\langle D_\lambda f, \lambda f \rangle| \leq \Theta_0\|\lambda f\|^2 + C\|f\|^2.$$

We get:

$$Q_{\eta,h}(\lambda f) \geq (1 - Ch^{1/2-\eta}) ((1 - \varepsilon)Q_0(\lambda f) - C\varepsilon^{-1}\|f\|^2).$$

We take $\varepsilon = h^{1/4}$ to deduce:

$$q_0(\lambda f) \leq Ch^{-1/4}\|f\|^2.$$

We infer (5.7).

Approximation of $D_\lambda f$ We take the derivative of (5.9) with respect to λ and take the scalar product with $\partial_\lambda f$:

$$Q_{\eta,h}(\partial_\lambda f) \leq (\Theta_0 + Ch^{1/2})\|\partial_\lambda f\|^2 + |\langle [H_{\eta,h}, \partial_\lambda]f, \partial_\lambda f \rangle|.$$

The estimates of Agmon give:

$$|\langle [H_{\eta,h}, \partial_\lambda]f, \partial_\lambda f \rangle| \leq Ch^{3/4}\|f\|^2.$$

We have:

$$Q_{\eta,h}(\partial_\lambda f) \geq (1 - Ch^{1/2-\eta}) ((1 - \varepsilon)Q_0(\partial_\lambda f) - C\varepsilon^{-1}h\|f\|^2).$$

We take $\varepsilon = h^{1/4}$ and we deduce:

$$q_0(\partial_\lambda f) \leq Ch^{3/4}\|f\|^2.$$

We infer (5.8). □

5.3.4 Conclusion: proof of Proposition 5.6

For all $f \in \mathfrak{F}_M(h)$, we have the lower bound:

$$\begin{aligned} Q_{\eta,h}(f) &\geq \int_{\mathbb{R}_+^2} \nu_1(q_{\lambda,\eta,h}) |f|^2 + \alpha \Theta_0 |D_\lambda f|^2 dx d\lambda \\ &\geq \int_{\mathbb{R}_+^2} \left(\nu_1(q_{\lambda,\eta,h}) - \left(\Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) \right) |f|^2 dx d\lambda \\ &\quad + \int_{\mathbb{R}_+^2} \left(\Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) |f|^2 dx d\lambda + \alpha \Theta_0 |D_\lambda f|^2 dx d\lambda \end{aligned}$$

We now estimate:

$$\begin{aligned} &\int_{\mathbb{R}_+^2} \left(\nu_1(q_{\lambda,\eta,h}) - \left(\Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) \right) |f|^2 dx d\lambda \\ &= \int_{|\lambda| \geq Mh^{-1/4-\eta}} \left(\nu_1(q_{\lambda,\eta,h}) - \left(\Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) \right) |f|^2 dx d\lambda \\ &\quad + \int_{|\lambda| \leq Mh^{-1/4-\eta}} \left(\nu_1(q_{\lambda,\eta,h}) - \left(\Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) \right) |f|^2 dx d\lambda \end{aligned}$$

Moreover, we get:

$$\begin{aligned} &\int_{|\lambda| \geq Mh^{-1/4-\eta}} \left(\nu_1(q_{\lambda,\eta,h}) - \left(\Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) \right) |f|^2 dx d\lambda \\ &\geq \int_{|\lambda| \geq Mh^{-1/4-\eta}} - \left(\Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) |f|^2 dx d\lambda = O(h^\infty) \|f\|^2, \end{aligned}$$

where the last estimate is a consequence of the estimates of Agmon. Then, we get:

$$\begin{aligned} &\int_{|\lambda| \leq Mh^{-1/4-\eta}} \left(\nu_1(q_{\lambda,\eta,h}) - \left(\Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) \right) |f|^2 dx d\lambda \\ &\geq -Ch^{3/4-3\eta} \|f\|^2. \end{aligned}$$

We deduce:

$$\begin{aligned} Q_{\eta,h}(f) &\geq \int_{\mathbb{R}_+^2} \left(C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) |f|^2 dx d\lambda + \alpha \Theta_0 |D_\lambda f|^2 dx d\lambda \\ &\quad + \Theta_0 \|f\|^2 - Ch^{3/4-3\eta} \|f\|^2. \end{aligned}$$

We now use Proposition 5.14 to get:

$$\begin{aligned} Q_{\eta,h}(f) &\geq \int_{\mathbb{R}_+^2} \left(C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) |\Pi_0 f|^2 dx d\lambda + \alpha \Theta_0 |D_\lambda \Pi_0 f|^2 dx d\lambda \\ &\quad + \Theta_0 \|f\|^2 - Ch^{1/2+1/8} \|\Pi_0 f\|^2. \end{aligned}$$

But, we notice that, for all $f \in \mathfrak{F}_M(h)$:

$$Q_{\eta,h}(f) \leq \nu_M(Q_{\eta,h}) \|f\|^2$$

and thus:

$$\begin{aligned}
& \int_{\mathbb{R}_+^2} \left(C(k_0, a_2, b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h \right) |\Pi_0 f|^2 dx d\lambda + \alpha \Theta_0 |D_\lambda \Pi_0 f|^2 dx d\lambda \\
& \leq (\nu_M(Q_{\eta, h}) - \Theta_0) \|f\|^2 + Ch^{1/2+1/8} \|\Pi_0 f\|^2 \\
& \leq (\nu_M(Q_{\eta, h}) - \Theta_0) \|\Pi_0 f\|^2 + \tilde{C}h^{1/2+1/8} \|\Pi_0 f\|^2.
\end{aligned}$$

The conclusion follows from the min-max principle.

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