



HAL
open science

Extended loop quantum gravity

L Fatibene, M Ferraris, M Francaviglia

► **To cite this version:**

L Fatibene, M Ferraris, M Francaviglia. Extended loop quantum gravity. *Classical and Quantum Gravity*, 2010, 27 (18), pp.185016. 10.1088/0264-9381/27/18/185016 . hal-00625165

HAL Id: hal-00625165

<https://hal.science/hal-00625165>

Submitted on 21 Sep 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Extended Loop Quantum Gravity*

by *L.Fatibene*^{a,b}, *M.Ferraris*^a, *M.Francaviglia*^{a,b,c}

^a Department of Mathematics, University of Torino (Italy)

^b INFN - Iniziativa Specifica Na12

^c LCS, University of Calabria (Italy)

Abstract: We discuss constraint structure of extended theories of gravitation (also known as $f(R)$ theories) in the vacuum selfdual formulation introduced in [1].

1. Introduction

We have recently investigated a formulation of $f(R)$ theories (in a metric-affine framework) based on non-linear actions similar to the Holst Lagrangian; see [1]. These actions are in fact written in terms of the scalar curvature ${}^\beta R$ of the Barbero-Immirzi connection with parameter β (see [2], [3]) and are dynamically equivalent to the corresponding “classical” $f(R)$ theory. For the linear case $f({}^\beta R) = {}^\beta R$ one obtains the standard Holst action. Hence these new actions are to be understood as Barbero-Immirzi formulations of the corresponding classical $f(R)$ theory.

This could be interesting for at least two reasons: from the point of view of LQG this new formulation provides a family of models which are classically well-understood and investigated in detail (see [4], [5]). There are many classical effects known in $f(R)$ theories that should be traced in their quantum genesis. The minisuperspace of these models is quite well-understood and should be studied in loop quantum cosmology (LQC) formulation (see [6]), to contribute to a better understanding of the classical limit of LQG models. Moreover, as in all metric-affine models, matter has a non-trivial feedback on the gravitational field which would be also interesting to trace in its quantum origin. It is often said that matter in LQG *simply* adds new labels to spin networks, while in these models one could expect a more complicated mechanism that would be certainly interesting to be discussed in detail. Finally, there are a number of equivalences, e.g. with scalar tensor models (see [7]), that again would be interesting to be discussed in detail at quantum level. Let us stress that these equivalences are known to hold at the classical level and, as usual, one should investigate whether they still hold at the full quantum level or just emerge classically.

From the classical viewpoint we shall here provide a route to define a quantization *à la* loop of $f(R)$ theories. Of course classical effects of these extended theories of gravitation have been extensively investigated. It is therefore interesting to investigate also their quantum effects. For example it would be interesting to see whether the removal of singularities that has been shown to hold in standard loop quantization of GR is preserved generically in these extended gravitational models.

For the sake of simplicity we shall here restrict our attention to the Euclidean signature and to the selfdual formulation (which in the Euclidean sector is in fact a special case of the

* This paper is published despite the effects of the Italian law 133/08 (<http://groups.google.it/group/scienceaction>). This law drastically reduces public funds to public Italian universities, which is particularly dangerous for free scientific research, and it will prevent young researchers from getting a position, either temporary or tenured, in Italy. The authors are protesting against this law to obtain its cancellation.

Barbero-Immirzi formulation) and show that one can apply LQG methods (see [8]) also to the quantization of these theories. In vacuum we shall obtain something similar to Einstein gravity with a cosmological constant. This is very well expected on the basis of a classical equivalence (see [9]); however, let us stress that our result seems to establish a stronger equivalence at the *quantum* level and not only at the classical level.

Moreover, let us stress that the classical equivalence holds only in vacuum, while the equivalence is broken when generic matter is considered and the extended models are equivalent to scalar tensor theories; see [7]. Tracing the mechanism which leads to this shift of equivalence at the quantum level would be therefore rather interesting and will be investigated in forthcoming papers. We shall follow the notation introduced in [1] and [8].

The aim of this paper is to go towards a quantum description of $f(R)$ theories; we have however to mention the reverse problem of giving a (semi)-classical account of the quantum effect of ordinary standard LQG models; see [10]. The two approaches are somehow complementary and based on similar techniques.

2. Selfdual Formulation for Extended Theories

In [1] we introduced

$${}^\beta R := R^{ab}{}_{\mu\nu} e_a^\mu e_b^\nu + \beta R^{ab}{}_{\mu\nu} e^{c\mu} e^{d\nu} \epsilon_{cdab} \quad (2.1)$$

where e_a^μ is a spin frame (see [11]), $R^{ab}{}_{\mu\nu}$ is the curvature of a spin connection ω_μ^{ab} on a 4 dimensional (spin) manifold M and $\beta \neq 0$ is a real parameter. Indices $a, c, \dots = 0..3$ and $\mu, \nu, \dots = 0..3$ while $i, j, \dots = 1..3$.

In the Euclidean sector one obtains for $\beta = \frac{1}{2}$ the standard selfdual curvature

$${}^+R := R^{ab}{}_{\mu\nu} e_a^\mu e_b^\nu + \frac{1}{2} R^{ab}{}_{\mu\nu} e^{c\mu} e^{d\nu} \epsilon_{cdab} \quad (2.2)$$

which can be written in terms of the curvature $F_{\mu\nu}^i := p_{ab}^i R^{ab}{}_{\mu\nu}$ of the usual selfdual connection $A_\mu^i := p_{ab}^i \omega_\mu^{ab} = \omega_\mu^{0i} + \frac{1}{2} \epsilon^i{}_{jk} \omega_\mu^{jk}$ as follows

$$\frac{1}{2} {}^+R = \frac{1}{2} R^{cd}{}_{\mu\nu} \left(\delta_{[c}^a \delta_{d]}^b + \frac{1}{2} \epsilon_{cd}{}^{ab} \right) e_a^\mu e_b^\nu = R^{cd}{}_{\mu\nu} p_{cd}^i p_i^{ab} e_a^\mu e_b^\nu = p_i^{ab} F_{\mu\nu}^i e_a^\mu e_b^\nu =: F \quad (2.3)$$

□

Here p_i^{ab} denotes the algebra projector $p: \mathfrak{spin}(4) \rightarrow \mathfrak{su}(2)$ on selfdual forms. It is given by

$$p_i^{0j} = \frac{1}{2} \delta_i^j \quad p_i^{j0} = -p_i^{0j} \quad p_i^{jk} = \frac{1}{2} \epsilon_i{}^{jk} \quad (2.4)$$

and the inverse projector p_{ab}^i is defined by

$$p_{0j}^i = \frac{1}{2} \delta_j^i \quad p_{j0}^i = -p_{0j}^i \quad p_{jk}^i = \frac{1}{2} \epsilon^i{}_{jk} \quad (2.5)$$

One can easily prove that

$$p_{ab}^i p_j^{ab} = \delta_j^i \quad p_{ab}^i p_i^{cd} = \frac{1}{2} \left(\delta_{[c}^a \delta_{d]}^b + \frac{1}{2} \epsilon^{ab}{}_{cd} \right) \quad (2.6)$$

□

One is then led to consider the following family of Lagrangians

$$L^+ = \frac{1}{2\kappa} e f(F) + L_m \quad (2.7)$$

where $\kappa = 8\pi G$, e is the determinant of the frame matrix, f is a generic analytic function and L_m encodes the matter contribution. Usually matter is assumed to couple only with g (and possibly to its derivatives up to some finite order; usually, in view of minimal coupling principle, at most 1) and not to the connection ω_μ^{ab} . Hereafter we shall just consider the vacuum sector, i.e. we set $L_m = 0$.

In the special case $f(F) = F$ one obtains an equivalent formulation of the usual selfdual action

$$\begin{aligned} \mathbb{L}^+ &= \frac{1}{8\kappa} R^{ab} \wedge e^c \wedge e^d \epsilon_{abcd} = \frac{1}{16\kappa} (R^{ab}{}_{\mu\nu} + \frac{1}{2} \epsilon^{ab}{}_{ef} R^{ef}{}_{\mu\nu}) e_\rho^c e_\sigma^d \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} ds = \\ &= \frac{1}{14\kappa} R^{ef}{}_{\mu\nu} \left(\delta_{[e}^a \delta_{f]}^b + \frac{1}{2} \epsilon^{ab}{}_{ef} \right) e_\rho^c e_\sigma^d \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} ds = \frac{1}{8\kappa} R^{ef}{}_{\mu\nu} p_i^{ab} p_{ef}^i e_\rho^c e_\sigma^d \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} ds = \\ &= \frac{1}{8\kappa} p_i^{ab} F^i{}_{\mu\nu} e_\rho^c e_\sigma^d \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} ds = \frac{\epsilon}{2\kappa} p_i^{ab} F^i{}_{\mu\nu} e_a^\mu e_b^\nu ds = \frac{\epsilon}{2\kappa} F ds \end{aligned} \quad (2.8)$$

where ds is the standard local basis of 4-forms on M induced by coordinates.

Field equations of the Lagrangian \mathbb{L}^+ are

$$\begin{cases} p_i^{ab} F_{\mu\nu}^i e_\rho^c e_\sigma^d \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} = 0 \\ p_{ab}^i \nabla_\mu (e_\nu^a e_\rho^b) \epsilon^{\mu\nu\rho\sigma} = 0 \end{cases} \quad (2.9)$$

Let us now consider a Cauchy (boundary) surface $i : S \rightarrow M : k^A \mapsto x^\mu(k)$, $A, B, \dots = 1..3$; in coordinates $x^\mu = (t, k^A)$ adapted to the submanifold S one has $i : k^A \mapsto k^A$ and $\partial_A x^\mu = \delta_A^\mu$.

□

The unit covector normal to S is given by $n = dx^0$. One can use antiselfdual transformations to define a canonical adapted frame $e_a = \underline{e}_a^\mu \partial_\mu$ and coframe $e^a = \bar{e}_\mu^a dx^\mu$ (see [12]) given by

$$\begin{cases} \underline{e}_0^0 = N^{-1} & \underline{e}_i^0 = 0 & \bar{e}_0^0 = N & \bar{e}_i^0 = 0 \\ \underline{e}_0^j = N^{-1} N^j & \underline{e}_i^j = \underline{\alpha}_i^j & \bar{e}_0^j = -N^l \bar{\alpha}_l^j & \bar{e}_i^j = \bar{\alpha}_i^j \end{cases} \quad (2.10)$$

Tetrads (or better spin frames; [11]) adapted to S define triads $\epsilon_i = e_i = \underline{\alpha}_i^A \partial_A$ on S . Also the selfdual connection can be projected onto S to define a connection $A_A^i = A_\mu^i \partial_A x^\mu$ on S . Let us denote by $F_{AB}^i = F_{\mu\nu}^i \partial_A x^\mu \partial_B x^\nu$ the projected curvature (which is the same as the curvature of the projected connection); for later convenience let us also define the tangent-normal projection of the curvature $F_A^i = F_{\mu\nu}^i \partial_A x^\mu n^\nu$ (of course the normal-normal projection vanishes due to the skew symmetry of F).

Let us also set $E_i^A = \epsilon \epsilon_i^A$ for the momentum conjugated to the connection A_A^i written in terms of the triad ϵ_i^A tangent to S , with ϵ the determinant of the (co)triad ϵ_i^A .

⊥

Field equations (2.9) can be projected onto S to obtain a number of evolution equations and the following constraints on S :

$$\begin{cases} \bar{\nabla}_A E_i^A = 0 \\ F_{AB}^i E_i^A = 0 \\ \epsilon_i^{jk} F_{AB}^i E_j^A E_k^B = 0 \end{cases} \quad (2.11)$$

These constitute the starting point of LQG quantization scheme; the first equation is related to gauge covariance, the second to $\text{Diff}(S)$ -covariance; while the third equation is called the

Hamiltonian constraint, when quantized it becomes the so-called *Wheeler-deWitt equation* and it encodes the (quantum) dynamics. In order to solve the first and second equation one defines an Hilbert space spanned by spin knots (see [8]) so that the Wheeler-deWitt equation is implemented as an operator on that space and it defines physical states.

On this basis one expects to be able to perform the same steps with extended models $f(F)$; since the extended models are still gauge and generally covariant, the first and second equations are expected to remain unchanged. This would mean that the definition of Area and Volume operators are unchanged and “spacetime” gets discretized in extended models *exactly* as in standard LQG. Since extended models are known to provide a modified dynamics with respect to standard GR one also expect that the Wheeler-deWitt equation has to be modified.

We shall hereafter compute the analogous of equations (2.11) for the action (2.7) in order to fully confirm our expectations.

3. Constraint Structure

Let us then consider the Lagrangian

$$L^+ = \frac{e}{2\kappa} f(F) \quad (3.1)$$

i.e. the purely gravitational part of (2.7).

Field equations are

$$\begin{cases} f' p_i^{ab} F_{\mu\nu}^i e_a^\mu - \frac{1}{2} f e_\nu^b = 0 \\ p_i^{ab} \nabla_\mu (e f' e_a^\mu e_b^\nu) = 0 \end{cases} \quad (3.2)$$

The *master equation* $f'F - 2f = 0$ is obtained by tracing the first one by means of e_b^ν ; see [1] and [9]. This can be replaced back into the first equation to obtain

$$f' (p_i^{ab} F_{\mu\nu}^i e_a^\mu - \frac{1}{4} F e_\nu^b) = 0 \quad \Rightarrow \quad p_i^{ab} F_{\mu\nu}^i e_a^\mu - \frac{1}{4} F e_\nu^b = 0 \quad (3.3)$$

where we used the fact that generically $f' \neq 0$ on the zeroes of the master equation. For simplicity let us assume that the master equation has only one (simple) zero $F = \rho$; when there are many (simple) zeroes each of them defines a sector of the quantum theory and one is supposed to sum over all sectors, which are in correspondence with the discrete zero structure of the analytic function f .

Let us also define a conformal tetrad $\tilde{e}_\mu^a = \sqrt{|f'|} e_\mu^a$, set $\sigma = \text{sgn}(f'(\rho))$ and use tilde to denote quantities depending on the conformal tetrad, e.g. $\tilde{E}_i^A = \tilde{e}_i^A = |f'| E_i^A$ and

$$\tilde{F} = p_i^{ab} F_{\mu\nu}^i \tilde{e}_a^\mu \tilde{e}_b^\nu = \frac{\sigma}{f'} F \quad (3.4)$$

Field equations are hence equivalent to

$$\begin{cases} p_i^{ab} F_{\mu\nu}^i \tilde{e}_a^\mu - \frac{1}{4} \tilde{F} \tilde{e}_\nu^b = 0 \\ f' F - 2f = 0 \quad \Rightarrow \quad F = \rho \\ p_i^{ab} \nabla_\mu (\tilde{e} \tilde{e}_a^\mu \tilde{e}_b^\nu) = 0 \end{cases} \quad (3.5)$$

The third equation implies the constraint

$$\overset{A}{\nabla}_A \tilde{E}_i^A = 0 \quad (3.6)$$

as in the standard case, though for the conformal frame \tilde{e}_μ^a .

The second equation can be now expanded as

$$\tilde{F} = p_i^{ab} F_{\mu\nu}^i \tilde{e}_a^\mu \tilde{e}_b^\nu = 2p_i^{0l} F_{\mu\nu}^i \tilde{e}_0^\mu \tilde{e}_l^\nu + p_i^{lk} F_{\mu\nu}^i \tilde{e}_l^\mu \tilde{e}_k^\nu = -\tilde{F}_A^i \tilde{\epsilon}_i^A + \frac{1}{2} \epsilon_i^{lk} F_{AB}^i \tilde{\epsilon}_l^A \tilde{\epsilon}_k^B = \frac{\sigma}{f} \rho \quad (3.7)$$

which allows us to express $\tilde{F}_A^i \tilde{\epsilon}_i^A$ as a function of constrained fields, i.e.

$$\tilde{F}_A^i \tilde{\epsilon}_i^A = \frac{1}{2} \epsilon_i^{lk} F_{AB}^i \tilde{\epsilon}_l^A \tilde{\epsilon}_k^B - \frac{\sigma}{f} \rho \quad (3.8)$$

Notice that the first equation is really different from the standard case (i.e. LQG without cosmological constant) due to the different coefficient $\frac{1}{4}$ (which in the standard case is $\frac{1}{2}$ and allows a complete cancellation of $\tilde{F}_A^i \tilde{\epsilon}_i^A$). The standard case in LQG can be recovered by setting $f(F) = F$; in this case the master equation simply implies $F = 0$ and the standard case without cosmological constant is obtained in particular. The first equation can be projected in the normal direction to the constraint to obtain

$$\left(p_i^{ab} F_{\mu\nu}^i \tilde{e}_a^\mu - \frac{1}{4} \tilde{F} \tilde{e}_\nu^b \right) \tilde{e}_b^\alpha \tilde{e}_d^\nu n_\alpha = 0 \quad \Rightarrow \quad (3.9)$$

$$p_i^{j0} F_{\mu\nu}^i \tilde{e}_j^\mu \tilde{e}_d^\nu - \frac{1}{4} \tilde{F} \delta_d^0 = 0 \quad \Rightarrow \quad (3.10)$$

$$F_{A\nu}^i \tilde{e}_i^A \tilde{e}_d^\nu + \frac{1}{2} \tilde{F} \delta_d^0 = 0 \quad (3.11)$$

For $d = k = 1..3$ one has

$$F_{AB}^i \tilde{\epsilon}_i^A \tilde{\epsilon}_k^B = 0 \quad \Rightarrow \quad F_{AB}^i \tilde{E}_i^A = 0 \quad (3.12)$$

For $d = 0$ one has instead

$$\tilde{F}_A^i \tilde{\epsilon}_i^A + \frac{1}{2} \tilde{F} = 0 \quad (3.13)$$

and, using (3.7) and (3.8), one obtains

$$\begin{aligned} \tilde{F}_A^i \tilde{\epsilon}_i^A - \frac{1}{2} \tilde{F}_A^i \tilde{\epsilon}_i^A + \frac{1}{4} \epsilon_i^{lk} F_{AB}^i \tilde{\epsilon}_l^A \tilde{\epsilon}_k^B &= \frac{1}{2} \tilde{F}_A^i \tilde{\epsilon}_i^A + \frac{1}{4} \epsilon_i^{lk} F_{AB}^i \tilde{\epsilon}_l^A \tilde{\epsilon}_k^B = \\ &= \frac{1}{4} \epsilon_i^{lk} F_{AB}^i \tilde{\epsilon}_l^A \tilde{\epsilon}_k^B - \frac{\sigma}{2f} \rho + \frac{1}{4} \epsilon_i^{lk} F_{AB}^i \tilde{\epsilon}_l^A \tilde{\epsilon}_k^B = \frac{1}{2} \epsilon_i^{lk} F_{AB}^i \tilde{\epsilon}_l^A \tilde{\epsilon}_k^B - \frac{\sigma}{2f} \rho = 0 \end{aligned} \quad (3.14)$$

$$\epsilon_i^{lk} F_{AB}^i \tilde{\epsilon}_l^A \tilde{\epsilon}_k^B = \frac{\sigma}{f} \rho \quad (3.15)$$

$$\epsilon_i^{lk} F_{AB}^i \tilde{E}_l^A \tilde{E}_k^B = \frac{\sigma}{f} \rho \tilde{\epsilon}^2 = \frac{\sigma}{f} \rho \tilde{E} \quad (3.16)$$

where $\tilde{E} := \det(\tilde{\epsilon}_i^A) = \tilde{\epsilon}^3 \tilde{\epsilon}^{-1} = \tilde{\epsilon}^2$ denotes the determinant of the conformal momentum \tilde{E}_i^A .

Let us stress that all this can be done also in the standard LQG framework, though in that case F_A^i does not enter other constraints and hence can be ignored.

Accordingly, the constraints can be written in terms of the conformal triad as follows

$$\begin{cases} \overset{A}{\nabla}_A \tilde{E}_i^A = 0 \\ F_{AB}^i \tilde{E}_i^A = 0 \\ \epsilon_i^{jk} F_{AB}^i \tilde{E}_j^A \tilde{E}_k^B = \frac{\sigma}{f} \rho \tilde{E} \end{cases} \quad (3.17)$$

As expected, the first and second constraints are unchanged with respect to (2.11), while the Wheeler-deWitt equation is modified by the “source term” $\frac{\sigma}{f}\rho\tilde{E}$, which explicitly depends on the non-linearity of $f(F)$. This is the quantum counterpart of what happens classically for $f(R)$ theories and reflects also what happens in standard LQG with the cosmological constant $\Lambda = -\frac{1}{4|f|}\rho$; see Appendix A. Let us also notice that the third constraint is a density, which is fundamental in the approach to quantization proposed by Thiemann; see [13].

4. Conclusions and Perspectives

We have shown that, in the generic extended models introduced in [1], constraints allows a loop theory approach to quantization formally similar to what one usually does in vacuum models with cosmological constant. This shows that the equivalence between $f(R)$ models and Einstein with cosmological constant (shown in [9] to hold in the classical theory) holds also at the quantum level.

Of course more attention should be paid when matter couplings are considered, when this equivalence is known to break and is replaced at least by a conformal equivalence.

Also the whole Hamiltonian structure of the theory should be verified in detail to exclude second class constraints which might add further equations to the set (3.17). These constraints (3.17) are in any case necessary conditions on the boundary S . Since from them discretization of “spacetime” follows one can claim in any event that extended spacetimes are discretized as in standard LQG.

Appendix A. LQG with Cosmological Constant

Let us here briefly review the standard result for LQG quantization in vacuum with cosmological constant in order to compare it with what we found for extended models.

Let us consider the Lagrangian

$$\begin{aligned} L_\Lambda &= ({}^+R^{ab} + \frac{\Lambda}{6}e^a \wedge e^b) \wedge e^c \wedge e^d \epsilon_{abcd} = (\frac{1}{2}{}^+R^{ab}{}_{\mu\nu} + \frac{\Lambda}{6}e_\rho^a e_\sigma^b) e_\rho^c e_\sigma^d \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} ds = \\ &= e (\frac{1}{2}{}^+R^{ab}{}_{\mu\nu} e_\rho^\mu e_\sigma^\nu \epsilon^{abcd} \epsilon_{abcd} + \frac{\Lambda}{6} \epsilon^{abcd} \epsilon_{abcd}) ds = 2e ({}^+R^{ab}{}_{\mu\nu} e_a^\mu e_b^\nu + 2\Lambda) ds \end{aligned} \quad (A.1)$$

which can also be written in terms of the selfdual curvature as

$$L_\Lambda = (2p_i^{ab} F^i + \frac{\Lambda}{6} e^a \wedge e^b) \wedge e^c \wedge e^d \epsilon_{abcd} \quad (A.2)$$

By varying this Lagrangian one gets the following field equations

$$\begin{cases} (p_i^{ab} F_{\mu\nu}^i + \frac{4\Lambda}{6} e_\mu^a e_\nu^b) e_\rho^c \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} = 0 \\ p_i^{ab} \nabla_\mu (e_\rho^c e_\sigma^d) \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} = 0 \end{cases} \quad (A.3)$$

By projecting on the boundary S one gets the following constraints

$$\begin{cases} \nabla_A E_i^A = 0 \\ F_{AB}^i E_i^A = 0 \\ \epsilon_i^{jk} F_{AB}^i E_i^A E_j^B = -4\Lambda E \end{cases} \quad (A.4)$$

which account for the value of the cosmological constant as claimed after (3.17) in which, however, the conformal frame was used.

Appendix B. Matter Fields

We want here to briefly comment about some cases of models with matter. The general treatment of matter is quite difficult and it deserves further investigation. Matter fields in this context are complicated by two different reasons: first, in classical $f(R)$ theories generic matter modifies the master equations to become $f'R - f = T$ where T is the g -trace of energy-momentum tensor of matter $T_{\mu\nu}$ (i.e. Hilbert stress tensor). This generically still allows to determine R (and thence $f'(R)$) as a function of matter fields. Accordingly, these quantities generically are not constant any longer, and they depend on the spacetime point x through the matter fields. Of course, there are special cases in which the energy-momentum tensor happens to be traceless which still can be treated easily, essentially as in the vacuum case.

Second, in LQG matter can be easily considered though the method is based on regarding it in terms of groups so that matter contribution can be suitably encoded in terms of holonomies. This is trivial for Yang-Mills fields and easy in a number of relevant examples; see [8].

The connection A_μ^i in LQG is a $SU(2)$ -connection, as described in detail in [14], [15]. Hence it is a principal connection on a suitable $SU(2)$ -bundle ${}^+P$ over the spacetime M . If one couples with a Yang-Mills matter field \mathbb{A}_μ^I a new gauge group G is introduced and the gauge field is a principal connection over a bundle \mathbb{P} with structure group G . Here the Lie algebra \mathfrak{g} of the (semisimple) Lie group G is of dimension n and T_I denotes an orthonormal basis with respect to the Cartan-Killing metric on G .

The gauge field strength is assumed to be denoted as usual by $\mathbb{F}_{\mu\nu}^I$ and the Yang-Mills Lagrangian is

$$L_m = -\frac{1}{4}\sqrt{g}\delta_{IJ}\mathbb{F}_{\mu\nu}^I\mathbb{F}_{\rho\sigma}^Jg^{\mu\rho}g^{\nu\sigma}ds \quad (B.1)$$

and as usual we shall set $\mathbb{F}_I^{\mu\nu} := \delta_{IJ}\mathbb{F}_{\rho\sigma}^Jg^{\mu\rho}g^{\nu\sigma}$, Greek world indices are lowered and raised by the associated metric g while Latin algebra indices are lowered and raised by the Cartan-Killing metric δ on G .

The energy-momentum tensor is

$$T_{\mu\nu} = \mathbb{F}_{I\mu}{}^\alpha\mathbb{F}_{\nu\alpha}^I - \frac{1}{4}\mathbb{F}_{\alpha\beta}^I\mathbb{F}_I^{\alpha\beta}g_{\mu\nu} \quad (B.2)$$

which when the spacetime is in dimension $\dim(M) = 4$ is in fact traceless.

The master equation is then exactly the same as in the vacuum case, and the conformal tetrad \tilde{e}_μ^a is defined exactly as in the vacuum case.

Field equations are then in the form

$$\begin{cases} p_i^{ab}F_{\mu\nu}^i\tilde{e}_a^\mu - \frac{1}{4}\tilde{F}\tilde{e}_\nu^b = \kappa T_{\mu\nu}\tilde{e}^{b\mu} \\ f'F - 2f = 0 & \Rightarrow F = \rho \\ p_i^{ab}\nabla_\mu(\tilde{e}\tilde{e}_a^\mu\tilde{e}_b^\nu) = 0 & \Rightarrow \overset{A}{\nabla}_A E_i^A = 0 \\ \nabla_\mu(\sqrt{g}\mathbb{F}_I^{\mu\nu}) = 0 \end{cases} \quad (B.3)$$

where, of course, now ∇_μ takes care, when necessary, also of the G -gauge transformations, besides spacetime diffeomorphisms.

For the sake of simplicity let us consider hereafter the case of electromagnetism, i.e. taking $G = U(1)$ which being of dimension 1 leads to a systematic understanding of the algebra indices. These field equations can be shown to project on the Cauchy surface S to get the following equations

$$\begin{cases} \nabla_A \tilde{E}_i^A = 0 \\ F_{AB}^i \tilde{E}_i^A = \kappa \frac{\sigma}{f} \mathbb{B} \times \mathbb{E} \\ \epsilon_i^{jk} F_{AB}^i \tilde{E}_j^A \tilde{E}_k^B = -4\Lambda \tilde{E} + \kappa \frac{\sigma}{f} (|\mathbb{E}|^2 + |\mathbb{B}|^2) \\ \nabla_A (\mathbb{E}^A) = 0 \end{cases} \quad (B.4)$$

where we defined $\mathbb{E}_A := \mathbb{F}_{\mu\nu} \partial_A x^\mu n^\nu$ and $\mathbb{B}_A := \frac{1}{2} \epsilon_A^{BC} \mathbb{F}_{\mu\nu} \partial_B x^\mu \partial_C x^\nu$ for the electric and magnetic field. These are the Hamiltonian constraints (together with $\nabla_A (\mathbb{B}^A) = 0$) and they are the starting point of LQG quantization with matter coupling. The system is described in terms of a connection of the group $SU(2) \times U(1)$ and thence in terms of its holonomies. This should lead to define spin networks with extra label of irreducible representation of $U(1)$. See [8] and references quoted therein.

Once again the conformal frame plays a preferred role. Except for that the model is equivalent to standard GR, with cosmological constant, coupled with electromagnetic field though with a modified coupling constant.

Acknowledgments

We wish to thank C. Rovelli for discussions about Barbero-Immirzi formulation. We acknowledge the contribution of INFN (Iniziativa Specifica NA12) and the local research project *Leggi di conservazione in teorie della gravitazione classiche e quantistiche* (2010) of Dipartimento di Matematica of University of Torino (Italy).

References

- [1] L.Fatibene, M.Ferraris, M.Francaviglia, *New Cases of Universality Theorem for Gravitational Theories*, (in preparation)
- [2] F. Barbero, *Real Ashtekar variables for Lorentzian signature space-time*, Phys. Rev. D51, 5507, 1996
- [3] G. Immirzi, *Quantum Gravity and Regge Calculus*, Nucl. Phys. Proc. Suppl. **57**, 65-72
- [4] S. Capozziello, V.F. Cardone, V. Salzano, *Cosmography of $f(R)$ gravity*, Phys.Rev.D**78**, 063504, 2008
- [5] S. Nojiri, S.D. Odintsov, *Modified gravity as realistic candidate for dark energy, inflation and dark matter* AIP Conf.Proc. 1115, 2009, 212-217; arXiv:0810.1557
- [6] M. Bojowald, *Consistent Loop Quantum Cosmology* Class.Quant.Grav.**26** 075020, 2009
- [7] G. Magnano, M. Ferraris, M. Francaviglia, *Nonlinear gravitational Lagrangians*, Gen.Rel.Grav. **19**(5), 1987, 465-479
- [8] C. Rovelli, *Quantum Gravity*, Cambridge University Press, Cambridge, 2004
- [9] A. Borowiec, M. Ferraris, M. Francaviglia, I. Volovich, *Universality of Einstein Equations for the Ricci Squared Lagrangians*, Class. Quantum Grav. **15**, 43-55, 1998
- [10] G.J. Olmo, P. Singh, *Covariant Effective Action for Loop Quantum Cosmology a la Palatini*, Journal of Cosmology and Astroparticle Physics 0901:030, 2009; arXiv:0806.2783
- [11] L. Fatibene, M. Francaviglia, *Natural and gauge natural formalism for classical field theories. A geometric perspective including spinors and gauge theories*, Kluwer Academic Publishers, Dordrecht, 2003
- [12] L. Fatibene, M. Francaviglia, *Spin Structures on Manifolds and Ashtekar Variables*, Int. J. Geom. Methods Mod. Phys. **2**(2), 147-157, (2005)

- [13] T. Thiemann, *LoopQuantumGravity: An InsideView*, hep-th/0608210
- [14] L. Fatibene, M.Francaviglia, C.Rovelli, *On a Covariant Formulation of the Barberi-Immirzi Connection* CQG 24 (2007) 3055-3066; gr-qc/0702134
- [15] L. Fatibene, M.Francaviglia, C.Rovelli, *Spacetime Lagrangian Formulation of Barbero-Immirzi Gravity* CQG 24 (2007) 4207-4217; gr-qc/0706.1899