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The density of the ruin time for a renewal-reward process perturbed by a diffusion

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Abstract

Let X be a mixed process, sum of a Brownian motion and a renewal-reward process, and τ_x be the first passage time of a fixed level $x < 0$ by X . We prove that τ_x has a density and we give a formula for it. Links with ruin theory are presented. Our result may be computed in classical settings (for a Lévy or Sparre Andersen process) and also in a non markovian context with possible positive and negative jumps. Some numerical applications illustrate the interest of this density formula.

Keywords: Renewal-reward process, Brownian motion, Jump-diffusion process, Time of ruin.

JEL Classification codes: G22, C60.

1. Introduction

Computing the first passage time law for a process X is an old problem. Indeed, the first result was obtained by Levy [18] for X a Brownian motion with a drift. If X has jumps, Zolotarev [30] and Borokov [2] obtained some results for X a spectrally negative Lévy process and Doney [7] for X a spectrally positive Lévy process. Recently Coutin and Dorobantu [5] show the existence of the density for the first passage time of a level x by X , where X is a Lévy process with a compound Poisson process and a Gaussian component.

At our knowledge a such result does not exist in the literature when X is a renewal-reward process perturbed by a Brownian motion with drift. The law of the first passage time for such a process may be used in the financial and actuarial field : it allows to compute default or ruin probabilities. Many authors were interested to compute the ruin (default) time distribution in the particular case of a Lévy process. Indeed, Gerber [10] introduced the risk model (a compound Poisson process with drift) perturbed by a Brownian motion. Then, a lot of works deal with such a model where both interarrival times and claims amount are exponentially distributed (see for exemple [4, 8]). Some particular cases of this risk process were applied in finance or in insurance by many authors (see for example [15, 16, 17, 22, 29]).

A renewal-reward process perturbed by a diffusion (with only negative jumps) was studied for exemple by [9, 22]. Other results involving inequalities, renewal equations, moments and distributions are given in [25, 26, 27, 28]. At last, some results involving discount factors are given in [11, 23, 24]. When considering both Erlang(n) waiting times and perturbation by a diffusion risk process, [19] studied the expected discounted penalty function.

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Here, we investigate a much more general setting, where claims size and waiting times are two independent iid sequences, with a perturbation by a diffusion process, and when we can observe both positive or negative jumps. In this quite general setting, we study the question of the existence of a density of the ruin time. We easily deduce the same result for a Sparre Andersen process. The Sparre Andersen risk model corresponds to the situation where there is not a gaussian component and both interarrival times and claim sizes are two independent iid sequences. In this model, some results are available when claims size or waiting times are exponential, Erlang(n) or even Phase-Type distributed (see for example [12]). Our result is a generalization of Coutin and Dorobantu's article [5] when we consider no more a compound Poisson process but a renewal-reward process.

For the numerical applications, we derive some relations allowing to build the conditional density when we observe the paths of the risk process at only jump times. A particular specification of this model will allow to deal with both usual jump-diffusion risk model, and also with dual model where reserves are decreasing and suddenly increasing. The main advantage of the density computation is that we may obtain the ruin probability with less simulations than for an empirical cumulative distribution function of the ruin time.

2. Notations and Assumptions

Let X be the process defined by $X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i$, $t \geq 0$, where $m \in \mathbb{R}$, W is a standard Brownian motion, N is a renewal counting process and $(Y_i, i \in \mathbb{N}^*)$ is a sequence of i.i.d. random variables with cumulative distribution function F_Y defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We suppose that W , N and $(Y_i, i \in \mathbb{N}^*)$ are independent.

Let τ_x be the first passage time of level $x < 0$ by X : $\tau_x = \inf\{u \geq 0 : X_u \leq x\}$.

Let $(T_n, n \in \mathbb{N}^*)$ be the sequence of the jump times of the process N and $(S_i = T_i - T_{i-1}, i \in \mathbb{N}^*)$ be the i.i.d. inter-renewal random variables. By convention, $T_0 = 0$.

We suppose that $(S_i, i \in \mathbb{N}^*)$ check the following assumption :

Assumption 2.1. *For all $i \in \mathbb{N}^*$, S_i is an absolutely continuous positive random variable with bounded continuous density f_S .*

For example, Exponential distribution, Gamma distribution $\Gamma(\alpha, \beta)$ with $\alpha > 1$, Beta distribution $B(\alpha, \beta)$ with $\alpha, \beta > 1$, Log-normal distribution... satisfy Assumption 2.1.

Let us denote by F_S the cumulative distribution function of S_i ($F'_S = f_S$). Since f_S is bounded, then there exists $M > 0$, such that $|f_S(y)| \leq M \forall y \in \mathbb{R}^+$.

Let $(\mathcal{F}_t)_{t \geq 0}$ be $\mathcal{F}_t = \sigma(W_s, s \leq t) \vee \sigma(N_s, s \leq t, Y_1, \dots, Y_{N_t}) \vee \mathcal{N}$ where \mathcal{N} is the set of negligible sets of $(\mathcal{F}, \mathbb{P})$.

From now on, $\mathbb{E}(\cdot | \mathcal{F}_{T_i})$ is denoted \mathbb{E}^{T_i} , $i \in \mathbb{N}^*$ and $\Delta F_Y(z) = F_Y(z) - F_Y(z_-)$.

3. The main result

The main result of this paper is the following theorem. It gives the law of τ_x .

Theorem 3.1. *The cumulative distribution function of τ_x has a right derivative at 0 and is differentiable at every point of $]0, \infty[$. The derivative, denoted $f(\cdot, x)$, is equal to*

$$f(0, x) = f_S(0) \frac{F_Y(x) + 3F_Y(x^-)}{4} \quad \text{and for every } t > 0$$

$$f(t, x) = \mathbb{E} \left(\mathbf{1}_{\{\tau_x > T_{N_t}\}} \tilde{f} \left(t - T_{N_t}, x - X_{T_{N_t}} \right) \right)$$

$$+ \sum_{i \geq 0} \mathbb{E} \left(\mathbf{1}_{T_i < t \wedge \tau_x} f_S(t - T_i) \mathbb{E}^{T_i} \left(\mathbf{1}_{\tilde{\tau}_{x - X_{T_i}} > t - T_i} F_Y(x - X_{T_i} - W_{t - T_i} - m(t - T_i)) \right) \right),$$

where $\tilde{\tau}_z = \inf\{t \geq 0 : mt + W_t \leq z\}$, $z < 0$ and $\tilde{f}(u, z) = \frac{|z|}{\sqrt{2\pi u^3}} e^{-\frac{(z-mu)^2}{2u}} \mathbf{1}_{]0, \infty[}(u)$ its density.

The first term of the density comes from the crossing of the barrier x by the Brownian component. The second one comes from the crossing of the level x because of a jump of X .

Proposition 3.2. *Suppose that S_1 and Y_1 are integrable, then $\mathbb{P}(\tau_x = \infty) = 0$ if and only if $m\mathbb{E}(S_1) + \mathbb{E}(Y_1) \leq 0$.*

The proof of Theorem 3.1 is a direct consequence of Propositions 7.1 and 7.5 given in Appendix 7.

Remark 3.3. *This result is already known when N is a Poisson process (see [5]) or when X is a renewal-reward process without Brownian component (see for example [20]).*

Note that if Y_1 is an absolutely continuous random variable, then the expression of $f(0, x)$ is much simpler :

$$f(0, x) = f_S(0)F_Y(x).$$

In the same way, we can obtain the law of $\tau^x = \inf\{u > 0 : X_u \geq x\}$, $x > 0$. Using the same reasoning, we can obtain the law of a passage time for a process without Brownian component :

Lemma 3.4. *Let X be a Sparre Andersen risk process $X_t = mt - \sum_{i=1}^{N_t} Y_i$ where $m > 0$ and $(Y_i, i \in \mathbb{N}^*)$ is a sequence of i.i.d. absolutely continuous positive random variables. The cumulative distribution function of τ_x has a right derivative at 0 and is differentiable at every point of $]0, \infty[$. The derivative, denoted $f(\cdot, x)$, is equal to*

$$\begin{aligned} f(0, x) &= f_S(0)(1 - F_Y(-x)) \quad \text{and for every } t > 0 \\ f(t, x) &= \sum_{i \geq 0} \mathbb{E}[\mathbf{1}_{T_i < t \wedge \tau_x} f_S(t - T_i) (1 - F_Y(-x + X_{T_i} + m(t - T_i)))]. \end{aligned}$$

This result can easily be obtained from Theorem 3.1. Indeed, it suffices to remove the first term of the density (because it is due to the Brownian component) and to replace Y by $-Y$ (thus $F_Y(y) = 1 - F_{-Y}(-y)$).

4. Link with ruin theory

In the previous section, the density of the first passage time τ_x ($x < 0$) was given, with :

$$X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i, \quad t \geq 0 \text{ and } \tau_x = \inf\{u \geq 0, X_u \leq x\}.$$

In some particular settings, the passage time τ_x directly corresponds to a ruin time in both models : renewal classical risk model and renewal dual risk model (with additional Brownian motion).

The classical renewal risk process (also refereed as *positive risk sums*) corresponds to the reserves of an insurance company, which are increasing with premiums, and suddenly decreasing at each time when a claim occurs. The settings corresponding to the classical risk model are the following ones :

$$\begin{cases} R_t = X_t - x \\ Y_i < 0, \quad i \geq 1 \\ mt = ct, \quad c > 0 \end{cases}$$

So that $R_t = u + ct + W_t - \sum_{i=1}^{N_t} |Y_i|$, with $u = -x$ the initial reserves, and c the premium rate. The passage time τ_x is then the time to ruin: $\tau_x = \inf\{t > 0, R_t \leq 0\}$.

Remark that τ_x is unchanged if the money unit is changed, so that we can easily consider a Brownian motion with volatility σ_W by setting premium rate $m' = m/\sigma_W$, threshold $x' = x/\sigma_W$, and claims amount $Y' = Y/\sigma_W$.

Remark 4.1. *The dual risk process, also called negative risk sums [13], corresponds to a situation where the company's reserves are decreasing (e.g. due to annuity paiements or to research investments) and suddenly increasing when a positive event occurs (e.g. paiements are stopped or research was successful). This kind of situation can be easily imagined for oil prospection for example.*

For the dual risk model, we can consider the following situation :

$$R_t = X_t - x, \quad Y_i > 0 (i \geq 1), \quad mt = ct, \quad c < 0.$$

So that $R_t = u - |c|t + W_t + \sum_{i=1}^{N_t} |Y_i|$, with $u = -x$ the initial reserves, and c the decreasing rate. τ_x is then the time to ruin : $\tau_x = \inf\{t \geq 0, R_t \leq 0\}$.

What is noticeable is that the given formula offers the opportunity to deal with both positive and negative risk sums, which is rather new in the actuarial field, even if some links between both models have been done [20].

5. Numerical applications

5.1. Density from discrete observations

In this section we present some numerical applications of our density. The difficulty of the formula of Theorem 3.1 comes from the fact that the density is given as an expectation which involves ruin time, which would require the computation of the ruin time density.

The following proposition allows to compute the density of ruin given the path of the process X at only jump times :

Proposition 5.1. *Let $t > 0$ and $x < 0$. The density of the ruin time can be written as :*

$$f(t, x) = \mathbb{E} \left[\tilde{f}(t - T_{N_t}, X_{T_{N_t}} - x) \gamma_{N_t} \right] + \mathbb{E} \left[\sum_{i=0}^{N_t} f_S(t - T_i) g(t - T_i, X_{T_i} - x) \gamma_i \right]$$

Where $g(\cdot)$, $\tilde{f}(\cdot)$ and $\{\gamma_i\}_{i \geq 0}$ are:

$$\begin{aligned} \tilde{f}(u, z) &= \frac{|z|}{\sqrt{2\pi u^3}} e^{-\frac{(z-mu)^2}{2u}} \mathbf{1}_{u>0, z<0}, \\ \beta(a, b, d) &= \left(1 - e^{-\frac{2ab}{d}}\right) \mathbf{1}_{a>0, b>0}, \\ \gamma_i &= \prod_{j=1}^i \beta(X_{T_{j-1}} - x, X_{T_j}^- - x, T_j - T_{j-1}) \mathbf{1}_{X_{T_j} - x > 0}, \\ g(t, \alpha) &= \int_{-\alpha}^0 \int_a^{+\infty} f_{m,t}(a, u) F_Y(-\alpha - u) du da, \\ f_{m,t}(a, u) &= \frac{\sqrt{2}(u - 2a)}{\sqrt{\pi t^3}} \exp\left(-\frac{(u - 2a)^2}{2t} - \frac{m^2 t}{2} - mu\right). \end{aligned}$$

Proof Let x be a given real, $x < 0$. Since $T_i \leq t$ if and only if $i \leq N_t$, formula of Theorem 3.1 can be written :

$$f(t, x) = \mathbb{E} \left[\mathbf{1}_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, X_{T_{N_t}} - x) \right] + \mathbb{E} \left[\sum_{i=0}^{N_t} \mathbf{1}_{\tau_x > T_i} f_S(t - T_i) g(t - T_i, X_{T_i} - x) \right],$$

where $g(u, z)$ and $\tilde{f}(u, z)$ are deterministic functions of u and z .

Let $\mathcal{G}_t = \sigma(N_t, T_1, \dots, T_{N_t}, X_{T_1}, \dots, X_{T_{N_t}}, Y_1, \dots, Y_{N_t})$. We can write:

$$f(t, x) = \mathbb{E} \left[\mathbb{E}(\mathbf{1}_{\tau_x > T_{N_t}} \tilde{f}(t - T_{N_t}, X_{T_{N_t}} - x) | \mathcal{G}_t) \right] + \mathbb{E} \left[\mathbb{E} \left(\sum_{i=0}^{N_t} \mathbf{1}_{\tau_x > T_i} f_S(t - T_i) g(t - T_i, X_{T_i} - x) | \mathcal{G}_t \right) \right].$$

And we easily show that:

$$f(t, x) = \mathbb{E} \left[\tilde{f}(t - T_{N_t}, X_{T_{N_t}} - x) \mathbb{E}(\mathbf{1}_{\tau_x > T_{N_t}} | \mathcal{G}_t) \right] + \mathbb{E} \left[\sum_{i=0}^{N_t} f_S(t - T_i) g(t - T_i, X_{T_i} - x) \mathbb{E}(\mathbf{1}_{\tau_x > T_i} | \mathcal{G}_t) \right].$$

Now if $B_i = \{\inf_{u \in]T_{i-1}, T_i[} X_u - x > 0\}$, then $\mathbb{P}(B_i | \mathcal{G}_t) = \beta(X_{T_{i-1}} - x, X_{T_i^-} - x, T_i - T_{i-1})$, where $X_{T_i^-} = X_{T_i} - Y_i$. One can show that $\beta(a, b, d) = (1 - e^{-2ab/d}) \mathbf{1}_{a>0, b>0}$. Since conditionally to \mathcal{G}_t , $\{B_i\}_{i=1, \dots, N_t}$ are mutually independent, one shows :

$$\mathbb{E}(\mathbf{1}_{\tau_x > T_i} | \mathcal{G}_t) = \prod_{j=1}^i \mathbb{P}(B_j | \mathcal{G}_t) \mathbf{1}_{X_{T_j} - x > 0}$$

and the result holds. □

Remark 5.2. Let $t > 0$ and $\alpha > 0$.

- In the exponential case, when $-Y$ is exponentially distributed with parameter λ , then Proposition 5.1 holds with:

$$g(t, \alpha) = e^{\lambda t(\lambda/2+m) - \lambda \alpha} \left[\phi \left(\frac{\alpha - t(m + \lambda)}{\sqrt{t}} \right) - e^{2\alpha(m+\lambda)} \phi \left(\frac{-\alpha - t(m + \lambda)}{\sqrt{t}} \right) \right].$$

- In the double exponential case, when Y has the density $f_Y(y) = p\eta_1 e^{-\eta_1 y} \mathbf{1}_{y>0} + q\eta_2 e^{\eta_2 y} \mathbf{1}_{y<0}$ where $p + q = 1$, $p, q > 0$, $\eta_1 > 1$ and $\eta_2 > 0$ then :

$$g(t, \alpha) = qe^{\eta_2 t(\eta_2/2+m) - \eta_2 \alpha} \left[\phi \left(\frac{\alpha - t(m + \eta_2)}{\sqrt{t}} \right) - e^{2\alpha(m+\eta_2)} \phi \left(\frac{-\alpha - t(m + \eta_2)}{\sqrt{t}} \right) \right].$$

- When Y is a Bernoulli r.v : $\mathbb{P}(Y = 1) = p$ and $\mathbb{P}(Y = -1) = 1 - p$ where $p > 0$, then :

$$g(t, \alpha) = (1 - p) \left[\phi \left(\frac{-\alpha + 1 - mt}{\sqrt{t}} \right) - \phi \left(\frac{-\alpha - mt}{\sqrt{t}} \right) - e^{2\alpha m} \phi \left(\frac{\alpha + 1 - mt}{\sqrt{t}} \right) + e^{2\alpha m} \phi \left(\frac{\alpha - mt}{\sqrt{t}} \right) \right].$$

5.2. Numerical simulations

Empirical cdf of the ruin time. Without the results of this paper, one can get an empirical cumulative distribution function of the ruin time using stochastic simulations. We will see in our case that, even when ruin events are not too rare, this approach would lead to less precise results and longer calculation times. It could however constitute a benchmark to check the results, by comparing the numerically integrated proposed density with the empirical cumulative distribution function of the ruin time. Consider a path of X observed only at the jump times: given all values of the jump times up to time t and all values of X at these jump times, the conditional probability $\psi_{N_t}(t)$ that a ruin occurs before the last jump time T_{N_t} is $1 - \gamma_{N_t}$, so that the conditional probability that a ruin occurs before t is : $(1 - \gamma_{N_t}) + \gamma_{N_t} \mathbb{P} \left(X_{T_{N_t}} + \inf_{0 \leq s \leq t - T_{N_t}} ms + W_s < x \right)$, which can be written :

$$\begin{aligned} \psi_{N_t}(t) &= (1 - \gamma_{N_t}) + \gamma_{N_t} s_m(t - T_{N_t}, X_{T_{N_t}} - x), \\ \text{with } s_m(t, r) &= \Phi \left(\frac{-mt - r}{\sqrt{t}} \right) + e^{-2mr} \Phi \left(\frac{mt - r}{\sqrt{t}} \right). \end{aligned}$$

Thus, evaluating the conditional probability of ruin at time t for each path, we can estimate the probability of ruin before time t . Again, this method does not involve results of this paper, and is not suited to the case where ruin probability is small, since the empirical distribution of τ_x relies mainly only on ruined path of the process. Furthermore, this method does not permit to get a density without supplementary assumptions, since empirical cdf is not differentiable. It is here only presented in the purpose of checking the coherency of the developed methodology in the case where ruin events are not too rare.

Sparre Andersen Model. The Sparre Andersen model corresponds to the situation where there are no Brownian component into the process X (see [12]).

In this case, we can show that (see Lemma 3.4):

$$\begin{aligned} f(t, x) &= \mathbb{E} \left(\sum_{i=0}^{N_t} f_S(t - T_i) g(t - T_i, X_{T_i} - x) \gamma_i \right), \\ \text{with } g(t, \alpha) &= 1 - F_Y(\alpha + mt) \\ \gamma_i &= \prod_{j=1}^i \beta(X_{T_{j-1}} - x, X_{T_j}^- - x) \mathbf{1}_{X_{T_j} - x > 0} \\ \beta(a, b) &= \mathbf{1}_{a > 0, b > 0} \end{aligned}$$

In order to check the interest and the coherency of the proposed formula, we have first consider the most simple case where both claims amount $-Y$ and interarrival times S are set to be exponential r.v. with respective parameters $\lambda_Y = \lambda_S = 1$. Using Proposition 5.1 with these values, we obtain the results of Figure 1. One can see on this figure the shape of obtained density of τ_x , when premium rate is $m = 1.1$ and threshold $x = -1$. Despite the evaluation of the right part of the expression in 5.1 by using simulations, we get very rapidly a smooth shape of the density. Denote by n the number of ruin paths that have been simulated. Due to Proposition 5.1, we have to simulate these paths at only jump times. In Figure 2, we compare the numerical integration of the obtained density (rectangle method, continuous blue line), with the proportion of paths where ruin occurs before time t (red dotted line, which is the empirical cdf of τ_x). On the left part of this figure, we used $n = 500$

trajectories, and 100000 on the right part. One can see that with enough simulations, both curves are merged. With few simulations ($n = 500$), the proposed formula behaves very well whereas the empirical distribution of τ_x is still erratic. We also check that these curves were corresponding to known theoretical and numerical results for the finite ruin time probabilities when both S and $-Y$ are exponentials r.v. (cf. [6, 21]).

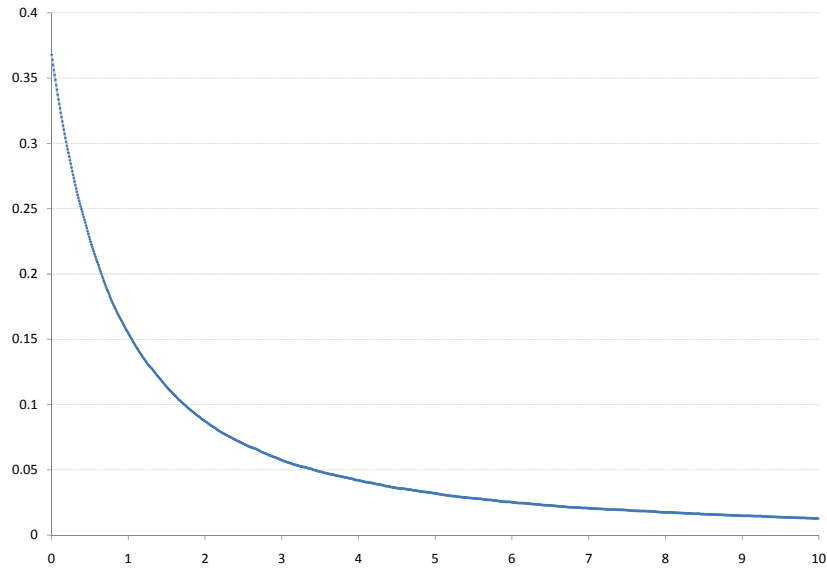


Figure 1: Density of the ruin time in the Sparre-Andersen model, $\lambda_Y = \lambda_S = 1$.

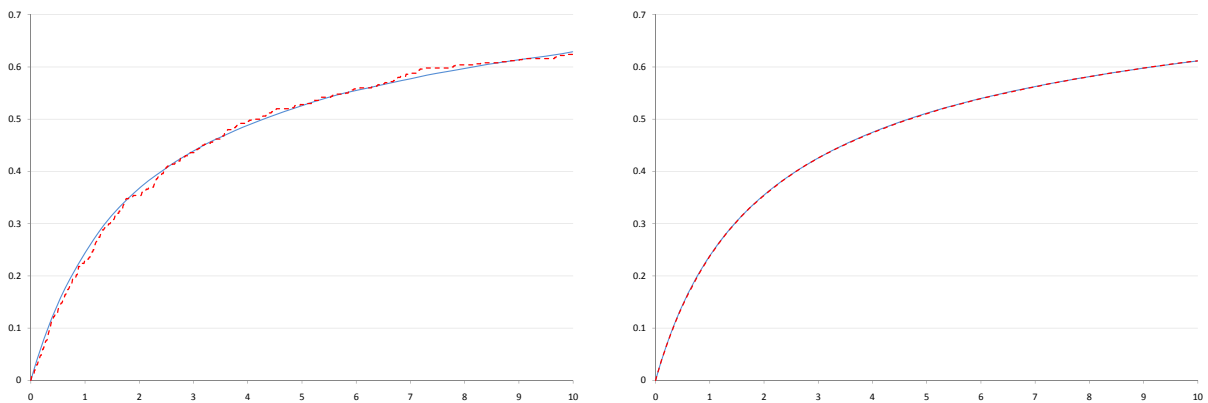


Figure 2: Numerical integrated density (continuous blue line) and empirical cdf of the ruin time (red dotted line), from 500 (left) or 100000 (right) paths of X .

Pure Brownian motion. When λ_S tends to 0, the number of claims occurred before t converges to 0, so that $f(u, t) = \tilde{f}(u, t)$. In this case indeed, X becomes a drifted Brownian motion and the density of the passage time τ_x becomes the one of a Brownian motion with drift m .

Let imagine the following assumptions : $\lambda_S > 0$ is non negligible and $-Y$ is exponentially distributed. Hence when λ_Y increases, so that claims get smaller and smaller and the residual process X gets closer to a a random motion, then the density of τ_x will also converge toward the one of $\tilde{\tau}_x$. In this case, when $g(t, \alpha)$ becomes very small, one check numerically that

$$\lim_{\lambda_Y \rightarrow +\infty} f(t, x) = \lim_{\lambda_Y \rightarrow +\infty} \mathbb{E}(\tilde{f}(t - T_{N_t}, X_{T_{N_t}} - x) \gamma_{N_t}) = \tilde{f}(t, x).$$

In both cases ($\lambda_S \rightarrow 0$ and non negligible λ_S), we checked numerically, that the formula was leading to the same distributions of τ_x and $\tilde{\tau}_x$. The obtained result is presented in Figure 3.

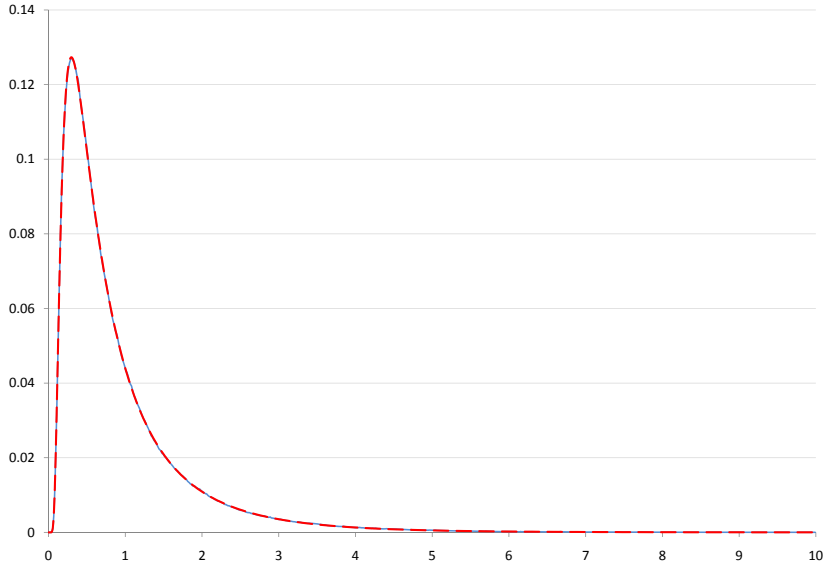


Figure 3: Density of time to ruin for the Pure-Brownian process ($\tilde{f}(t, x)$ in red dotted line, density with large value of λ_Y in blue).

Mixed process - Markovian case. We consider now a mixed process, that is the renewal model perturbed by a Brownian motion. Using Proposition 5.1, we have drawn the density of the ruin time and the corresponding integrated cumulative distribution function (see Figure 4).

With these settings, in the exponential case, one can see that a great part of the risk relies within the first two years: the knowledge of the whole density may involve some changes concerning asset-liability management. The need of liquidity may be analyzed differently when the whole distribution is known, and risk indicators based on such a density may be built. In the special case where $m \leq 0$, we also check that the ruin probability before time t goes to one when t grows.

Mixed process - Non markovian case. Until now, S was exponentially distributed, in order to check the numerical results in the most simple case. Since formula of Proposition 5.1 does not rely on this particular assumption, we can obtain the shape of the density even if the interarrival times have other distributions satisfying Assumption 2.1. In Figure 5, we consider the case where interarrival times are log-normally distributed, with $\mathbb{E}(S) = 1$ and $\text{Var}(S) = 1$ and the jumps follow a double-exponential law, i.e, the density of the claims is $f_Y(y) = p\eta_1 e^{-\eta_1 y} \mathbf{1}_{y>0} + (1-p)\eta_2 e^{\eta_2 y} \mathbf{1}_{y<0}$. We have taken here $\eta_1 = \eta_2 = 1$. Without positive jumps (when $q = 0$), we get very similar results than the

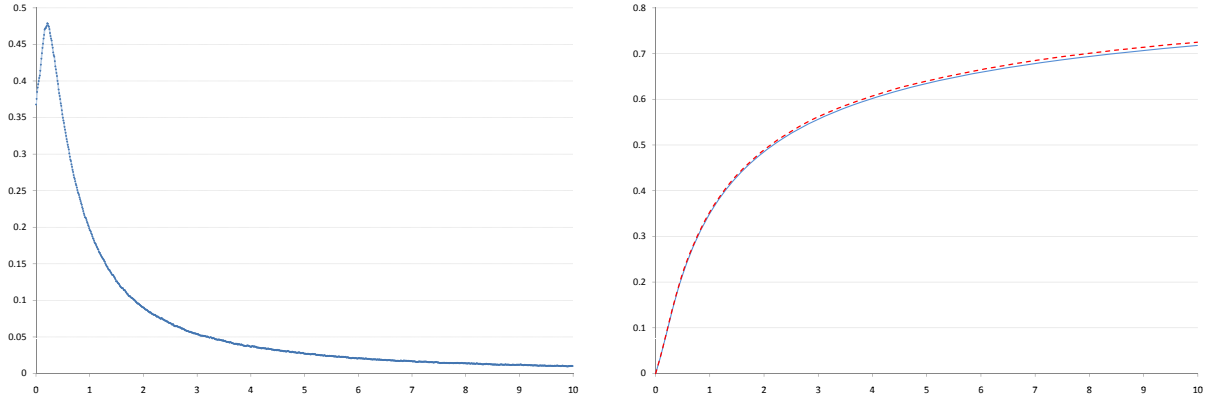


Figure 4: Left: Density of the ruin time for the mixed process (renewal model perturbed by a Brownian motion) (exponential case $\lambda_Y = \lambda_S = 1$, $m = 1.1$). Right: numerical integrated density (continuous blue line) and empirical cdf (red dotted line).

one of Figure 4. Nevertheless, the previous exponential distribution of S would not allow an increase of the variation coefficient $\sigma_S/\mathbb{E}(S)$, whereas the actual log-normal distribution allows it. With a positive probability of positive jumps (e.g. for $p = 0.1$), one can see in Figure 5 that the density has the same shape, even if cumulative distribution function is lower than in the exponential case, since the positive jumps are reducing the probability of ruin. We see here that Proposition 5.1 allows to

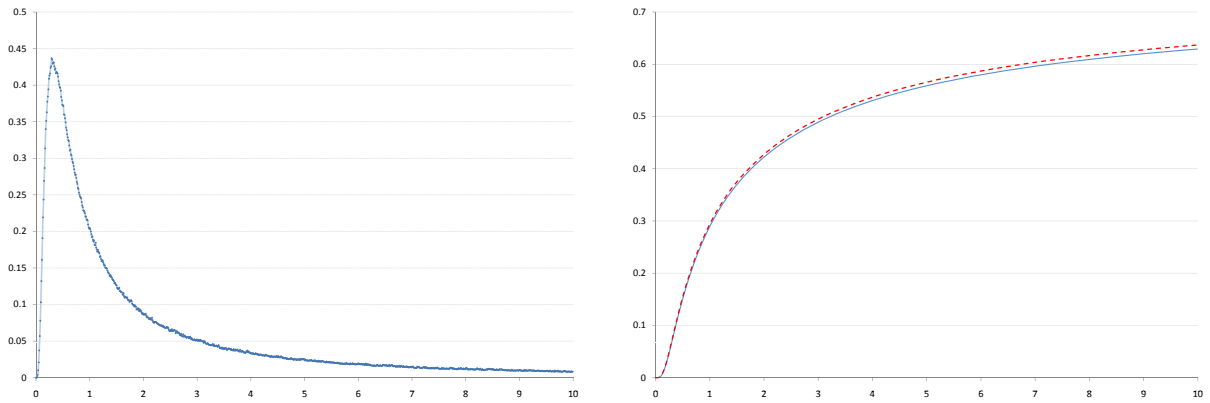


Figure 5: Left: Density of time to ruin for the mixed process in a non Markovian settings (Lognormal waiting times, double-exponential jumps, $\eta_1 = \eta_2 = 1$, $p = 0.1$). Right: numerical integrated density (continuous blue line) and empirical cdf (red dotted line).

build other densities in the non Markovian case, even when Y is no more exponentially distributed, and even when Y allows both positive and negative jumps. The double-exponential distribution for the jumps is also used by other authors, but in a markovian case (see for exemple [3, 15]).

6. Conclusion

We proposed here a formula to compute the density of the time to ruin for a renewal-reward process perturbed by a Brownian motion. Finding such a density may be important in the actuarial field, because it allows to get a more precise idea of the location of dangerous periods, and is thus of great importance for asset-liability management. One can imagine many risk indicators that can be expressed as a function of the ruin time. The knowledge of the ruin time density allows a direct derivation of these risk indicators. As an example, taking into account a discount factor is easy when the density of the ruin time is given. Furthermore, in some specific cases, the analytical expression of the density can be simplified, thus leading to theoretical results on the ruin time.

The formula is not straightforward to evaluate, so that we used simulations. However, when we need to simulate some paths of the risk process, the proposed formula offers several advantages:

- Simulating some paths over a period $[0, t_{max}]$ allows to estimate the density of the ruin time at any date t of the interval.
- When some paths of the risk process are simulated, the cumulative distribution function can be easily deduced from the empirical density of the ruin time (by numerical integration with rectangles method, for example). On the contrary, building a density from an empirical cumulative distribution function, which is non differentiable, requires some supplementary assumptions (e.g. using a smooth estimator of the cumulative distribution function, with a suitable kernel density function)
- The simulation of some paths may lead to frequent enough observations of events like $\{\tau_x < t\}$, thus sufficient for estimating the empirical cumulative distribution function. But it can lead to too rare observations of events like $\{t \leq \tau_x < t + h\}$ for a given $h > 0$, especially for small values of h . Direct estimation of the density, without using proposed formula, might require a huge number of simulations. One advantage of the proposed formula is that it only relies on events of kind $\{\tau_x > t\}$, with a frequency which does not depend on a discretization length, and which is in principle quite high with usual risk solvency settings in insurance.
- As it was stated in Proposition 5.1, one can show that the density can be estimated with simulations of the process only at the jump times. This will avoid problematic discretizations of the time, and save some computational time : in particular, the Brownian motion do not have to be simulated, only a finite set of gaussian increases of this motion are sufficient.

Concerning drawbacks and perspectives, in some limit cases, investigations are to be continued in order to cope with numerical difficulties for the evaluation of this density, for example getting approximations of $g(t, \alpha)$ when α is large. When we use simulations, the reduction of the variance of the basic estimator of this density may lead to an even faster convergence of the estimated density.

Finally, the proposed formula offers an easy way to build the density of the ruin time even in some difficult cases, allowing to take into account non-markovian settings with non-exponential waiting times, and to take into account situations where both positive and negative jumps can occur.

7. Proofs

In this section, we show that the cumulative distribution function of τ_x has a right derivative at 0 (see Proposition 7.1) and at every $t > 0$ (see Proposition 7.5). Furthermore, we compute these derivatives. Then, we give a necessary and sufficient conditions for the finitude of τ_x . The proofs presented here are inspired by Coutin-Dorobantu's proofs. The main difference is that our process

X is not markovian, but the Markov property can be applied at any jump time.

In the following, we omit the S subscript for the sake of clarity: $f(x)$ and $F(x)$ denote respectively the density function and the cumulative distribution function of the generic random variable S , so that for any x , $f(x) = f_S(x)$ and $F(x) = F_S(x)$. We will also write $\bar{F}(x) = 1 - F(x)$.

Recall that $\tilde{\tau}_z = \inf\{t \geq 0 : W_t + mt \leq z\}$, $z < 0$. By (5.12) page 197 of [14], its density is given by where

$$\tilde{f}(u, z) = \frac{|z|}{\sqrt{2\pi u^3}} \exp\left[-\frac{(z - mu)^2}{2u}\right] \mathbf{1}_{]0, \infty[}(u), \quad u \in \mathbb{R}, \quad \text{and } \mathbb{P}(\tilde{\tau}_z = \infty) < 1. \quad (1)$$

The function $\tilde{f}(\cdot, z)$ and all its derivatives admit 0 as right limit at 0 and is \mathcal{C}^∞ on \mathbb{R} .

Proposition 7.1. *The cumulative distribution function of τ_x has a right derivative at 0. The derivative, denoted $f(0, x)$, is equal to $f(0, x) = f(0) \frac{F_Y(x) + 3F_Y(x^-)}{4}$.*

Proof We split the probability $\mathbb{P}(\tau_x \leq h)$ according to the values of N_h :

$$\mathbb{P}(\tau_x \leq h) = \mathbb{P}(\tau_x \leq h, N_h = 0) + \mathbb{P}(\tau_x \leq h, N_h = 1) + \mathbb{P}(\tau_x \leq h, N_h \geq 2).$$

• **Limit of $\frac{\mathbb{P}(\tau_x \leq h, N_h \geq 2)}{h}$ when h goes to 0 :** Note that

$$\begin{aligned} \mathbb{P}(\tau_x \leq h, N_h \geq 2) &\leq \mathbb{P}(N_h \geq 2) = 1 - \mathbb{P}(N_h = 0) - \mathbb{P}(N_h = 1) \\ &= F(h) + \mathbb{P}(T_1 \leq h < T_1 + S_2) = F(h) - \int_0^h f(u) \bar{F}(h - u) du = \int_0^h f(u) F(h - u) du. \end{aligned}$$

Since $|f(u)| \leq M$, then $F(h - u) \leq M(h - u)$. Hence, $\mathbb{P}(\tau_x \leq h, N_h \geq 2) \leq \frac{M^2 h^2}{2}$ and $\lim_{h \rightarrow 0} \frac{\mathbb{P}(\tau_x \leq h, N_h \geq 2)}{h} = 0$.

• $\frac{\mathbb{P}(\tau_x \leq h, N_h = 0)}{h} \leq \frac{\mathbb{P}(\tilde{\tau}_x \leq h)}{h} \xrightarrow{h \rightarrow 0} 0$.

• **Limit of $\frac{\mathbb{P}(\tau_x \leq h, N_h = 1)}{h}$ when h goes to 0 :** Like in [5], we split $\mathbb{P}(\tau_x \leq h, N_h = 1)$ into:

$$\begin{aligned} \mathbb{P}(\tau_x \leq h, N_h = 1) &= \mathbb{P}(\tau_x < T_1, N_h = 1) + \mathbb{P}(\tau_x = T_1, N_h = 1) + \mathbb{P}(T_1 < \tau_x \leq h, N_h = 1) \\ &= A_1(h) + A_2(h) + A_3(h). \end{aligned}$$

Since $\frac{A_1(h)}{h} \leq \frac{\mathbb{P}(\tilde{\tau}_x \leq h)}{h} \xrightarrow{h \rightarrow 0} 0$, it remains to compute $\lim_{h \rightarrow 0} \frac{A_2(h)}{h}$ and $\lim_{h \rightarrow 0} \frac{A_3(h)}{h}$. Note that

$$\begin{aligned} A_2(h) &= \mathbb{P}(\tilde{\tau}_x > T_1, mT_1 + W_{T_1} + Y_1 \leq x, T_1 \leq h < T_1 + S_2) \\ &= \int_0^h f(t) \mathbb{P}(Y_1 \leq x - W_t - mt) \bar{F}(h - t) dt - \int_0^h f(t) \mathbb{P}(\tilde{\tau}_x \leq t, Y_1 \leq x - W_t - mt) \bar{F}(h - t) dt. \end{aligned}$$

On the one hand $\frac{1}{h} \int_0^h f(t) \mathbb{P}(\tilde{\tau}_x \leq t, Y_1 \leq x - W_t - mt) \bar{F}(h - t) dt \leq M \mathbb{P}(\tilde{\tau}_x \leq h) \xrightarrow{h \rightarrow 0} 0$, and on the other hand since f is bounded by M , $\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h f(t) \mathbb{P}(Y_1 \geq x - W_t - mt) F(h - t) dt = 0$.

We conclude using the same reasoning like in [5] (i.e. F_Y is a càdlàg bounded function and W is a continuous symmetric process) and the fact that f is continuous :

$$\lim_{h \rightarrow 0} \frac{A_2(h)}{h} = f(0) \left(\frac{F_Y(x) + F_Y(x^-)}{2} \right).$$

For the last term, we apply Markov property in T_1 :

$$A_3(h) = \mathbb{E} \left(\mathbf{1}_{\tau_x > T_1} \mathbf{1}_{h \geq T_1} \bar{F}(h - T_1) \mathbb{E}^{T_1} \left(\mathbf{1}_{\tilde{\tau}_x - X_{T_1} \leq h - T_1} \right) \right).$$

Since $\mathbb{E} \left(\mathbf{1}_{\tilde{\tau}_x \leq T_1 \leq h} \mathbf{1}_{X_{T_1} < x} \bar{F}(h - T_1) \mathbb{E}^{T_1} \left(\mathbf{1}_{\tilde{\tau}_x - X_{T_1} \leq h - T_1} \right) \right) \leq \mathbb{P}(\tilde{\tau}_x \leq h)$, then it remains to show that $\lim_{h \downarrow 0} \frac{G(h)}{h} = \frac{f(0)}{4} \Delta F_Y(x)$, where $G(h) = \mathbb{E}(\mathbf{1}_{h \geq T_1} \mathbf{1}_{X_{T_1} < x} \bar{F}(h - T_1) \mathbb{E}^{T_1}(\mathbf{1}_{\tilde{\tau}_x - X_{T_1} \leq h - T_1}))$.

Following the same arguments as [5] (integrating with respect to T_1 , using the fact that $\tilde{f}(\cdot, z)$ is the derivative of the cumulative distribution function of $\tilde{\tau}_z$ and Lemma 8.1 and then making the following change of variable $s = th$, $u = hv$), we get :

$$\frac{G(h)}{h} = \frac{1}{\sqrt{2\pi}} \int_0^1 \int_0^{1-t} f(th) \bar{F}(h - th) \mathbb{E} \left[e^{-\frac{(x - Y_1 - mT_1)^2}{2h(v+t)}} \left(\frac{x - Y_1 - mT_1}{\sqrt{h(v+t)}} + \frac{G\sqrt{t}}{\sqrt{v(v+t)}} \right)^+ \right] dt dv.$$

Since

$$\sup_{0 \leq h \leq 1} f(th) \bar{F}(h - th) e^{-\frac{(x - mh(t+v) - Y_1 - mT_1)^2}{2h(t+v)}} \left(\frac{x - Y_1 - mT_1}{\sqrt{h(t+v)}} + \frac{G\sqrt{v}}{\sqrt{1-v}} \right)^+ \leq M \frac{\sup_{z \geq 0} z e^{-\frac{z^2}{2}}}{\sqrt{t+v}} + M \frac{\sqrt{t}}{\sqrt{v(t+v)}} |G|,$$

then from Lebesgue's Dominated Convergence Theorem we obtain :

$$\lim_{h \rightarrow 0} \frac{G(h)}{h} = f(0) \Delta F_Y(x) \frac{\mathbb{E}(G_+)}{\sqrt{2\pi}} \int_0^1 \int_0^{1-t} \frac{\sqrt{t}}{\sqrt{v(t+v)}} dv dt = \frac{f(0)}{4} \Delta F_Y(x).$$

This identity achieves the proof. \square

Now we will show that the cumulative distribution function of τ_x is differentiable on \mathbb{R}_+^* and we will compute its derivative. The following Lemmas will be used to prove this result.

Lemma 7.2. *The following limit is null $\lim_{h \rightarrow 0} \frac{1}{h} \sum_{i \geq 0} \mathbb{P}(T_i \leq t < \tau_x < T_{i+1} \leq t + h < T_{i+2}) = 0$.*

Proof Applying Markov property at T_i , we obtain

$$\begin{aligned} \mathbb{P}(T_i \leq t < \tau_x < T_{i+1} \leq t + h < T_{i+2}) &= \mathbb{E} \left(\mathbf{1}_{T_i \leq t} \mathbf{1}_{T_i < \tau_x} \mathbb{E}^{T_i} \left(\mathbf{1}_{t - T_i < \tilde{\tau}_x - X_{T_i} < S_{i+1} \leq t + h - T_i < S_{i+1} + S_{i+2}} \right) \right) \\ &= \mathbb{E} \left(\mathbf{1}_{T_i \leq t} \mathbf{1}_{T_i < \tau_x} \int_0^h f(s + t - T_i) \bar{F}(h - s) \mathbb{E}^{T_i} \left(\mathbf{1}_{t - T_i < \tilde{\tau}_x - X_{T_i} < s + t - T_i} \right) ds \right) \leq M \int_0^h f^i(s) ds. \end{aligned}$$

where $f^i(s) = \mathbb{E} \left(\mathbf{1}_{T_i \leq t} \mathbb{E}^{T_i} \left(\mathbf{1}_{t - T_i < \tilde{\tau}_x - X_{T_i} < s + t - T_i} \right) \right) ds$.

Note that $\frac{1}{h} \int_0^h f^i(s) ds \leq \mathbb{E}(\mathbf{1}_{T_i \leq t})$ and $\sum_{i \geq 0} \mathbb{P}(T_i \leq t) = \mathbb{E}(N_t) < \infty$, so switching the limit and the sum, one has $\lim_{h \rightarrow 0} \frac{1}{h} \sum_{i \geq 0} \int_0^h f^i(s) ds = \sum_{i \geq 0} \mathbb{E} \left(\mathbf{1}_{T_i \leq t} \mathbb{E}^{T_i} \left(\mathbf{1}_{\tilde{\tau}_x - X_{T_i} = t - T_i} \right) \right) = 0$. \square

Lemma 7.3. *The following equality holds*

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h} \sum_{i \geq 0} \mathbb{P}(T_i \leq t < \tau_x = T_{i+1} \leq t + h < T_{i+2}) \\ = \sum_{i \geq 0} \mathbb{E} \left(\mathbf{1}_{T_i < t \wedge \tau_x} f(t - T_i) \mathbb{E}^{T_i} \left(\mathbf{1}_{\tilde{\tau}_x - X_{T_i} > t - T_i} (F_Y(x - X_{T_i} - W_{t - T_i} - m(t - T_i))) \right) \right) < +\infty. \end{aligned}$$

Proof Following the same method as previously, $\mathbb{P}(T_i \leq t < \tau_x = T_{i+1} \leq t + h < T_{i+2})$ equals

$$\begin{aligned} & \mathbb{E} \left(\mathbf{1}_{T_i \leq t} \mathbf{1}_{T_i < \tau_x} \mathbb{E}^{T_i} \left(\mathbf{1}_{S_{i+1} < \tilde{\tau}_x - X_{T_i}} \mathbf{1}_{W_{S_{i+1}} + mS_{i+1} + Y_{i+1} \leq x - X_{T_i}} \mathbf{1}_{t - T_i < S_{i+1} \leq t + h - T_i < S_{i+1} + S_{i+2}} \right) \right) \\ &= \mathbb{E} \left(\mathbf{1}_{T_i \leq t} \mathbf{1}_{T_i < \tau_x} \int_0^h f(y + t - T_i) \bar{F}(h - y) \mathbb{E}^{T_i} \left(\mathbf{1}_{\tilde{\tau}_x - X_{T_i} > y + t - T_i} \mathbf{1}_{W_{y+t-T_i} + m(y+t-T_i) + Y_{i+1} \leq x - X_{T_i}} \right) dy \right) \\ &= \int_0^h f_i^h(y) dy. \end{aligned}$$

As T_i admit a density $\mathbf{1}_{T_i \leq t} \mathbf{1}_{T_i < \tau_x} = \mathbf{1}_{T_i < t \wedge \tau_x}$ and $\sum_{i \geq 0} (\frac{1}{h} \int_0^h f_i^h(y) dy)$ converges uniformly in h , we only have to compute $\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h f_i^h(y) dy$.

The Lebesgue's Dominated Convergence Theorem, implies that $\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h f_i(h, y) dy$ is equal to $\mathbb{E} \left(\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h Z_1(y) dy + \lim_{h \rightarrow 0} \frac{1}{h} \int_0^h Z_2(y) dy \right)$ where

$$\begin{aligned} Z_1(u) &= \mathbf{1}_{T_i < t \wedge \tau_x} f(u + t - T_i) \bar{F}(u) \mathbb{E}^{T_i} \left(\mathbf{1}_{\tilde{\tau}_x - X_{T_i} > u + t - T_i} F_Y(x - X_{T_i} - W_{u+t-T_i} - m(u + t - T_i)) \right), \\ Z_2(u) &= \mathbf{1}_{T_i < t \wedge \tau_x} f(u + t - T_i) (F(u) - F(h - u)) \mathbb{E}^{T_i} \left(\mathbf{1}_{\tilde{\tau}_x - X_{T_i} > u + t - T_i} F_Y(x - X_{T_i} - W_{u+t-T_i} - m(u + t - T_i)) \right). \end{aligned}$$

However $\mathbb{E} \left(\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h Z_1(u) du \right)$ equals

$$\mathbb{E} \left(\mathbf{1}_{T_i < t \wedge \tau_x} f(t - T_i) \mathbb{E}^{T_i} \left(\mathbf{1}_{\tilde{\tau}_x - X_{T_i} > t - T_i} \frac{F_Y(x - X_{T_i} - W_{t-T_i} - m(t - T_i)) + F_Y(x - X_{T_i} - W_{t-T_i} - m(t - T_i))^-}{2} \right) \right)$$

and $\mathbb{E} \left(\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h Z_2(u) du \right) \leq M^2 \lim_{h \rightarrow 0} h = 0$.

Remark that $\mathbb{E}^{T_i} \left(\mathbf{1}_{\tilde{\tau}_x - X_{T_i} > t - T_i} \frac{F_Y(x - X_{T_i} - W_{t-T_i} - m(t - T_i)) + F_Y(x - X_{T_i} - W_{t-T_i} - m(t - T_i))^-}{2} \right)$ may be written as

$$\mathbb{E}^{T_i} \left(\mathbf{1}_{\tilde{\tau}_x - X_{T_i} > t - T_i} F_Y(x - X_{T_i} - W_{t-T_i} - m(t - T_i)) \right) - \frac{1}{2} \mathbb{E}^{T_i} \left(\mathbf{1}_{\tilde{\tau}_x - X_{T_i} > t - T_i} \Delta F_Y(x - X_{T_i} - W_{t-T_i} - m(t - T_i)) \right).$$

Note that the second expectation is 0 (indeed the jumps set of F_Y is countable and (M_t, W_t) where $M_t = \sup_{s \leq t} W_s$ has a density). \square

Lemma 7.4. *The following limit is null $\lim_{h \rightarrow 0} \frac{1}{h} \sum_{i \geq 0} \mathbb{P}(T_i \leq t < T_{i+1} < \tau_x \leq t + h < T_{i+2}) = 0$.*

Proof Markov Property at T_i (and then at S_{i+1}) gives

$$\begin{aligned} \mathbb{P}(T_i \leq t < T_{i+1} < \tau_x \leq t + h < T_{i+2}) &= \mathbb{E} \left(\mathbf{1}_{T_i \leq t} \mathbf{1}_{T_i < \tau_x} \mathbb{E}^{T_i} \left(\mathbf{1}_{t - T_i < \tau_x - X_{T_i} \leq t + h - T_i < S_{i+1} + S_{i+2}} \right) \right) \\ &= \mathbb{E} \left(\mathbf{1}_{T_i \leq t} \mathbf{1}_{T_i < \tau_x} \mathbb{E}^{T_i} \left(\mathbf{1}_{t - T_i < S_{i+1} \leq t + h - T_i} \mathbf{1}_{S_{i+1} < \tau_x - X_{T_i}} \mathbb{E}^{S_{i+1}} \left(\mathbf{1}_{t - T_i < \tau_x - X_{T_i} - W_{S_{i+1}} - mS_{i+1} - Y_{i+1} \leq t + h - T_i - S_{i+1} < S_{i+2}} \right) \right) \right) \end{aligned}$$

Let us note that $\mathbf{1}_{t - T_i < S_{i+1} \leq t + h - T_i} \mathbf{1}_{S_{i+1} < \tau_x - X_{T_i}} = \mathbf{1}_\Gamma - \mathbf{1}_\Gamma \mathbf{1}_{S_{i+1} > \tilde{\tau}_x - X_{T_i}}$ where

$$\Gamma = \{t - T_i < S_{i+1} \leq t + h - T_i, W_{S_{i+1}} + mS_{i+1} + Y_{i+1} > x - X_{T_i}, t - T_i < \tilde{\tau}_x - X_{T_i}\}.$$

Thus $\mathbb{P}(T_i \leq t < T_{i+1} < \tau_x \leq t + h < T_{i+2}) = G(i, h) - A(i, h)$ where

$$\begin{aligned} G(i, h) &= \mathbb{E} \left(\mathbf{1}_{T_i < t \wedge \tau_x} \mathbb{E}^{T_i} \left(\mathbf{1}_\Gamma \mathbb{E}^{S_{i+1}} \left(\mathbf{1}_{t - T_i < \tau_x - X_{T_i} - W_{S_{i+1}} - mS_{i+1} - Y_{i+1} \leq t + h - T_i - S_{i+1} < S_{i+2}} \right) \right) \right) \\ A(i, h) &= \mathbb{E} \left(\mathbf{1}_{T_i < t \wedge \tau_x} \mathbb{E}^{T_i} \left(\mathbf{1}_\Gamma \mathbf{1}_{\tilde{\tau}_x - X_{T_i} < S_{i+1}} \mathbb{E}^{S_{i+1}} \left(\mathbf{1}_{t - T_i < \tau_x - X_{T_i} - W_{S_{i+1}} - mS_{i+1} - Y_{i+1} \leq t + h - T_i - S_{i+1} < S_{i+2}} \right) \right) \right) \end{aligned}$$

$$\begin{aligned} \text{Since } \sum_{i \geq 0} \frac{A(i, h)}{h} &\leq \frac{1}{h} \sum_{i \geq 0} \mathbb{E} \left(\mathbf{1}_{T_i < t \wedge \tau_x} \mathbb{E}^{T_i} \left(\mathbf{1}_{t - T_i < S_{i+1} \leq t + h - T_i} \mathbf{1}_{t - T_i < \tilde{\tau}_x - X_{T_i} < S_{i+1}} \right) \right) \\ &\leq M \sum_{i \geq 0} \mathbb{E} \left(\mathbf{1}_{T_i < t \wedge \tau_x} \mathbb{E}^{T_i} \left(\mathbf{1}_{t - T_i < \tilde{\tau}_x - X_{T_i} < t + h - T_i} \right) \right) \leq M \mathbb{E}(N_t) < \infty, \end{aligned}$$

then $\lim_{h \rightarrow 0} \sum_{i \geq 0} \frac{A(i, h)}{h} = \sum_{i \geq 0} \lim_{h \rightarrow 0} \frac{A(i, h)}{h} = 0$ because $\tilde{\tau}_x - X_{T_i}$ has a density.

Here again $\sum_{i \geq 0} \frac{G(i, h)}{h} \leq \mathbb{E}(N_t)$, hence $\lim_{h \rightarrow 0} \sum_{i \geq 0} \frac{G(i, h)}{h} = \sum_{i \geq 0} \lim_{h \rightarrow 0} \frac{G(i, h)}{h}$.

Integrating with respect to S_{i+1} , using the fact that $\tilde{f}(\cdot, z)$ is the derivative of the cumulative distribution function of $\tilde{\tau}_z$, and then making a change of variable, we obtain that $G(i, h)$ is :
 $\int_0^h \int_0^{h-s} \mathbb{E}(\mathbf{1}_{T_i < t \wedge \tau_x} f(s + t - T_i) \bar{F}(h - s) \mathbb{E}^{T_i}(\mathbf{1}_{t - T_i < \tilde{\tau}_x - X_{T_i}} \mathbf{1}_{W_{s+t-T_i} + Y_{i+1} > x - X_{T_i}} \tilde{f}(u, x - X_{T_i} - W_{s+t-T_i} + Y_{i+1}))) du ds$.

Conditioning to $\mathcal{F}_{t-T_i}^W$ (where $\mathcal{F}_t^W = \sigma(W_s, s \leq t)$) in the second expectation, we can use Lemma 8.1. Then making the following change of variable $s = rh, u = hv$, we get :

$$\begin{aligned} \frac{G(i, h)}{h} &= \frac{1}{\sqrt{2\pi}} \int_0^1 \int_0^{1-s} \mathbb{E}(\mathbf{1}_{T_i < t \wedge \tau_x} f(rh + t - T_i) \bar{F}(h - rh) \times \\ &\mathbb{E}^{T_i}(\mathbf{1}_{t - T_i < \tilde{\tau}_x - X_{T_i}} e^{-\frac{(x - X_{T_i} - W_{t-T_i} - m(t - T_i) - Y_{i+1})^2}{2h(v+r)}} \left(\frac{x - X_{T_i} - W_{t-T_i} - m(t - T_i) - Y_{i+1}}{\sqrt{h}(v+r)^{3/2}} + \frac{G\sqrt{r}}{\sqrt{v}(v+r)} \right)^+)) dr dv. \end{aligned}$$

We conclude using the same arguments as for the density at $t=0$:

$$\lim_{h \rightarrow 0} \frac{G(i, h)}{h} = \frac{1}{4} \mathbb{E} \left(\mathbf{1}_{T_i < t \wedge \tau_x} f(t - T_i) \mathbb{E}^{T_i} \left(\mathbf{1}_{t - T_i < \tilde{\tau}_x - X_{T_i}} \Delta F_Y(x - X_{T_i} - W_{t-T_i} - m(t - T_i)) \right) \right) = 0.$$

□

Proposition 7.5. *The cumulative distribution function of τ_x is differentiable at every point of $]0, \infty[$. The derivative, denoted $f(t, x)$, $t > 0$ is equal to*

$$\begin{aligned} f(t, x) &= \mathbb{E} \left(\mathbf{1}_{\{\tau_x > T_{N_t}\}} \tilde{f} \left(t - T_{N_t}, x - X_{T_{N_t}} \right) \right) \\ &+ \sum_{i \geq 0} \mathbb{E} \left(\mathbf{1}_{T_i < t \wedge \tau_x} f(t - T_i) \mathbb{E}^{T_i} \left(\mathbf{1}_{\tilde{\tau}_x - X_{T_i} > t - T_i} F_Y(x - X_{T_i} - W_{t-T_i} - m(t - T_i)) \right) \right). \end{aligned}$$

Proof We split the probability $\mathbb{P}(t < \tau_x \leq t + h)$ according to the values of $N_{t+h} - N_t$:

$$\mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t = 0) + \mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t = 1) + \mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t \geq 2).$$

• **Limit of $\frac{B_3(h)}{h} = \frac{\mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t \geq 2)}{h}$ when h goes to 0 :** Remark that

$$\begin{aligned} B_3(h) &\leq \mathbb{P}(N_{t+h} - N_t \geq 2) = \sum_{i \geq 0} \mathbb{P}(T_i \leq t < T_i + S_{i+1} \leq T_i + S_{i+1} + S_{i+2} \leq t + h) \\ &= \sum_{i \geq 0} \mathbb{E} \left(\mathbf{1}_{T_i \leq t} \int_{t-T_i}^{t+h-T_i} f(s) F(t + h - T_i - s) ds \right). \end{aligned}$$

With following change of variable $s = u + t - T_i$, and then the fact that $|f| \leq M$:

$$B_3(h) \leq \frac{M^2 h^2}{2} \sum_{i \geq 0} \mathbb{E}(\mathbf{1}_{T_i \leq t}) = \frac{M^2 h^2}{2} \mathbb{E}(N_t), \text{ so } \lim_{h \rightarrow 0} \frac{B_3(h)}{h} = 0.$$

• **Limit of $\frac{B_1(h)}{h} = \frac{\mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 0)}{h}$ when h goes to 0** : We use the same reasoning like in [5] and we split $B_1(h)$ as

$$\begin{aligned} & \mathbb{P}(t < \tau_x \leq t+h < T_1) + \sum_{k=1}^{\infty} \mathbb{P}(t < \tau_x \leq t+h, T_k < t < t+h < T_{k+1}) \\ &= \mathbb{P}(t < \tilde{\tau}_x \leq t+h) \bar{F}(t+h) + \sum_{k=1}^{\infty} \mathbb{E} \left(\mathbf{1}_{T_k < t \wedge \tau_x} \bar{F}(t+h-T_k) \mathbb{E}^{T_k} \left(\mathbf{1}_{\{t-T_k < \tilde{\tau}_x - X_{T_k} \leq t+h-T_k\}} \right) \right) \\ &= \bar{F}(t+h) \int_t^{t+h} \tilde{f}(u, x) du + \sum_{k=1}^{\infty} \mathbb{E} \left(\mathbf{1}_{T_k < t \wedge \tau_x} \bar{F}(t+h-T_k) \int_{t-T_k}^{t+h-T_k} \tilde{f}(u, x - X_{T_k}) du \right). \end{aligned}$$

Remark that $B_1(h)$ may be written as $B_1(h) = \int_t^{t+h} \mathbb{E}(\mathbf{1}_{\{T_{N_t} < \tau_x\}} \tilde{f}(u - T_{N_t}, x - X_{T_{N_t}})) + R(h)$, where $R(h) \leq Mh \sum_{k=1}^{\infty} \int_t^{t+h} \mathbb{E} \left(\mathbf{1}_{T_k < t} \mathbf{1}_{\tau_x > T_k} \tilde{f}(u - T_k, x - X_{T_k}) du \right)$. Using the fact that $\mathbf{1}_{\tau_x > T_k} \leq \mathbf{1}_{X_{T_k} < x}$, and Corollary 8.3, then $\lim_{h \rightarrow 0} \frac{R(h)}{h} = 0$.

According to Proposition 8.2, the family of r.v. $(\frac{1}{h} \int_t^{t+h} \tilde{f}(u - T_{N_t}, x - X_{T_{N_t}}) du)_{0 < h \leq 1}$ is uniformly integrable. Since \tilde{f} is continuous with respect to u , for all $t > 0$, the result holds.

• **Limit of $\frac{B_2(h)}{h} = \frac{\mathbb{P}(t < \tau_x \leq t+h, N_{t+h} - N_t = 1)}{h}$ when h goes to 0** : We split $B_2(h)$ into:
 $\sum_{i \geq 0} \mathbb{P}(T_i \leq t < \tau_x < T_{i+1} \leq t+h < T_{i+2}) + \sum_{i \geq 0} \mathbb{P}(T_i \leq t < \tau_x = T_{i+1} \leq t+h < T_{i+2})$
 $+ \sum_{i \geq 0} \mathbb{P}(T_i \leq t < T_{i+1} < \tau_x \leq t+h < T_{i+2})$

We conclude using Lemmas 7.2, 7.3 and 7.4. □

Proof of Proposition 3.2 The result is a consequence of the following properties of X (see [1])

:

- If $m + \frac{\mathbb{E}(Y_1)}{\mathbb{E}(T_1)} < 0$, then $X_t \xrightarrow{t \rightarrow \infty} -\infty$.
- If $m + \frac{\mathbb{E}(Y_1)}{\mathbb{E}(T_1)} > 0$, then $X_t \xrightarrow{t \rightarrow \infty} \infty$.
- If $m + \frac{\mathbb{E}(Y_1)}{\mathbb{E}(T_1)} = 0$, then $\liminf_{t \rightarrow \infty} X_t = -\infty$ and $\limsup_{t \rightarrow \infty} X_t = \infty$.

□

8. Appendix

We recall here the following result on \tilde{f} given in (1) (see [5]).

Lemma 8.1. *Let G be a Gaussian random variable $\mathcal{N}(0, 1)$ and let $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, $p \geq 1$ and $x^+ = \max\{x, 0\}$. Then for every $u \in \mathbb{R}$*

$$\mathbb{E}[\tilde{f}(u, \mu + \sigma G)^p \mathbf{1}_{\{\mu + \sigma G > 0\}}] = \frac{1}{\sqrt{2^p \pi^p}} \frac{u^{\frac{1-2p}{2}} e^{-\frac{p\mu^2}{2(p\sigma^2+u)}}}{(p\sigma^2+u)^{\frac{p+1}{2}}} \mathbb{E} \left[\left(\sigma G + \sqrt{\frac{u}{p\sigma^2+u}} \mu \right)_+^p \right].$$

The following proposition is proved in [5] for the particular case where S_i , $i \geq 1$ are exponential i.i.d.r.v. The result may be generalized for a sequence S_i , $i \geq 1$ which satisfies Assumption 2.1.

Proposition 8.2. For every $t > 0$ and $1 \leq p < 3/2$

$$\sup_{0 < h \leq 1} \mathbb{E} \left[\left(\frac{1}{h} \int_t^{t+h} \mathbf{1}_{\{T_{N_t} < \tau_x\}} \tilde{f}(u - T_{N_t}, x - X_{T_{N_t}}) du \right)^p \right] < +\infty.$$

Proof Let $I(h)$ be $I(h) = \frac{1}{h} \int_t^{t+h} \mathbf{1}_{\{T_{N_t} < \tau_x\}} \tilde{f}(u - T_{N_t}, x - X_{T_{N_t}}) du$. Coutin et al. proved that

$$\mathbb{E}(I(h)^p) \leq \frac{3^{p-1}}{\sqrt{2^p \pi^p}} \mathbb{E} \left(\frac{T_{N_t}^{\frac{p}{2}}}{(t - T_{N_t})^{p-\frac{1}{2}} t^{\frac{p+1}{2}}} \mathbb{E}(|G|^p) + \frac{1}{(t - T_{N_t})^{\frac{p-1}{2}} t^{\frac{1}{2}+p}} C_p \right).$$

Observe that for every $t > 0$ and $(\alpha, \gamma) \in]-1, 0] \times [0, +\infty[$, the r.v. $(t - T_{N_t})^\alpha T_{N_t}^\gamma$ are integrable (see details below), which achieve the proof of Proposition 8.2.

$$\text{Note that } \mathbb{E}((t - T_{N_t})^\alpha T_{N_t}^\gamma) \leq t^\alpha + \sum_{i=1}^{\infty} \mathbb{E}(\mathbf{1}_{\{T_i > t\}} (t - T_i)^\alpha T_i^\gamma), \quad (2)$$

so $\mathbb{E}((t - T_{N_t})^\alpha T_{N_t}^\gamma) \leq t^\alpha + \sum_{i=1}^{\infty} \int_0^t \int_0^{t-s_1} \dots \int_0^{t-s_1 \dots s_{i-1}} f(s_1) \dots f(s_i) (t - s_1 \dots s_i)^\alpha (s_1 + \dots s_i)^\gamma ds_i \dots ds_1$.

Using the change of variable $u = tv$, $\int_{s_1 + \dots s_{i-1}}^t (t - u)^\alpha u^\gamma du \leq t^{\alpha+\gamma+1} B(\alpha + 1, \gamma + 1)$ and $\int_0^t \int_0^{t-s_1} \dots \int_0^{t-s_1 \dots s_{i-2}} f(s_1) \dots f(s_{i-1}) ds_{i-1} \dots ds_1 = \mathbb{P}(T_{i-1} < t)$, we get

$$\mathbb{E}((t - T_{N_t})^\alpha T_{N_t}^\gamma) \leq t^\alpha + M t^{\alpha+\gamma+1} B(\alpha + 1, \gamma + 1) \mathbb{E}(N_t) < \infty.$$

Consequently, the sum in the right term of inequality (2) is finite. \square

Corollary 8.3. The following limit is null :

$$\lim_{h \rightarrow 0} \sum_{k \geq 0} \int_t^{t+h} \mathbb{E}(\mathbf{1}_{x - X_{T_k} > 0} \mathbf{1}_{T_k < t} \tilde{f}(u - T_k, x - X_{T_k})) = 0.$$

Proof Using Lemma 8.1, then the inequality $(x + y)^+ \leq |x| + |y|$, $\forall x, y \in \mathbb{R}$ and the fact that $\exists C > 0$ such that $e^{-x^2} x \leq C$, $\forall x \in \mathbb{R}$, we obtain that :

$$\mathbb{E}(\mathbf{1}_{x - X_{T_k} > 0} \mathbf{1}_{T_k < t} \tilde{f}(u - T_k, x - X_{T_k})) \leq \frac{1}{\sqrt{2\pi}} \mathbb{E} \left(\frac{(u - T_k)^{-1/2} T_k^{1/2} |G| + C}{u} \mathbf{1}_{T_k < t} \right).$$

Since $u \in [t, t + h]$, then

$$\int_t^{t+h} \mathbb{E}(\mathbf{1}_{x - X_{T_k} > 0} \mathbf{1}_{T_k < t} \tilde{f}(u - T_k, x - X_{T_k})) \leq \frac{h}{\sqrt{2\pi}} \mathbb{E} \left(\frac{(t - T_k)^{-1/2} T_k^{1/2} |G| + C}{t} \mathbf{1}_{T_k < t} \right).$$

We conclude using the fact that $\sum_{k \geq 0} \mathbb{E}(\mathbf{1}_{T_k < t} (t - T_k)^{-1/2} T_k^{1/2})$ converges (see proof of Proposition 8.2). \square

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