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# Homogenization of high-contrast two-phase conductivities perturbed by a magnetic field. Comparison between dimension two and dimension three.

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## Abstract

Homogenized laws for sequences of high-contrast two-phase non-symmetric conductivities perturbed by a magnetic field  $h$  are derived in two and three dimensions under the strong field assumption. In dimension two an extension of the Dykhne transformation to non-periodic high conductivities permits to prove that the homogenized conductivity depends on a polynomial of  $h$  through some homogenized matrix-valued function obtained in the absence of magnetic field. This result is improved in the periodic framework thanks to an alternative approach, and illustrated by a cross-like thin structure. In dimension three a fiber-reinforced medium shows in contrast a rational dependence on  $h$  of the homogenized conductivity due to the high diffusion along the fibers direction.

**Keywords:** homogenization, high-contrast conductivity, magneto-transport, strong field, two-phase composites.

**AMS classification:** 35B27, 74Q20

## 1 Introduction

The mathematical theory of homogenization for second-order elliptic partial differential equations has been widely studied since the pioneer works of Spagnolo on  $G$ -convergence [41], of Murat, Tartar on  $H$ -convergence [38, 39], and of Bensoussan, Lions, Papanicolaou on periodic structures [2], in the framework of uniformly bounded (both from below and above) sequences of conductivity matrix-valued functions. It is also known since the end of the seventies [25, 32] (see also the extensions [1, 23, 11, 33]) that the homogenization of the sequence of conductivity problems, in a bounded open set  $\Omega$  of  $\mathbb{R}^3$ ,

$$\begin{cases} \operatorname{div}(\sigma_n \nabla u_n) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

with a uniform boundedness from below but not from above for  $\sigma_n$ , may induce nonlocal effects. However, the situation is radically different in dimension two since the nature of problem (1.1) is shown [10, 13] to be preserved in the homogenization process provided that the sequence  $\sigma_n$  is uniformly bounded from below.

H-convergence theory includes the case of non-symmetric conductivities in connection with the Hall effect [29] in electrodynamics (see, e.g., [34, 40]). Indeed, in the presence of a constant magnetic field  $h$  the conductivity matrix is modified and becomes non-symmetric. Since the seminal work of Bergman [3] the influence of a low magnetic field in composites has been studied for two-dimensional

composites [35, 4, 17], and for columnar composites [7, 5, 8, 27, 28]. The case of a strong field, namely when the symmetric part and the antisymmetric part of the conductivity are of the same order, has been also investigated [6, 9]. Moreover, dimension three may induce anomalous homogenized Hall effects [21, 18, 19] which do not appear in dimension two [17].

In the context of high-contrast problems the situation is more delicate when the conductivities are not symmetric. An extension in dimension two of H-convergence for non-symmetric and non-uniformly bounded conductivities was obtained in [14] thanks to an appropriate div-curl lemma. More recently, the Keller, Dykhne [31, 24] two-dimensional duality principle which claims that the mapping

$$A \mapsto \frac{A^T}{\det A} \quad (1.2)$$

is stable under homogenization, was extended to high-contrast conductivities in [16]. However, the homogenization of both high-contrast and non-symmetric conductivities has not been precisely studied in the context of the strong field magneto-transport especially in dimension three. In this paper we establish an effective perturbation law for a mixture of two high-contrast isotropic phases in the presence of a magnetic field. The two-dimensional case is performed in a general way for non-periodic and periodic microstructures. It is then compared to the case of a three-dimensional fiber-reinforced microstructure.

In dimension two consider a sequence  $\sigma_n(h)$  of isotropic two-phase matrix-valued conductivities perturbed by a fixed  $h \in \mathbb{R}$ , constant, and defined by

$$\sigma_n(h) := (1 - \chi_n)(\alpha_1 I_2 + \beta_1 h J) + \chi_n(\alpha_{2,n} I_2 + \beta_{2,n} h J), \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (1.3)$$

where  $\chi_n$  is the characteristic function of phase 2, with volume fraction  $\theta_n \rightarrow 0$ ,  $\alpha_1 > 0$ ,  $\beta_1$  are the constants of the low phase 1, and  $\alpha_{2,n} \rightarrow \infty$ ,  $\beta_{2,n}$  are real sequences of the hard phase 2 where  $\beta_{2,n}$  is possibly unbounded. The coefficients  $\alpha_1$  and  $\beta_1$ , respectively  $\alpha_{2,n}$  and  $\beta_{2,n}$  also have the same order of magnitude according to the strong field assumption. Assuming that the sequence  $\theta_n^{-1} \chi_n$  converges weakly-\* in the sense of the Radon measures to a bounded function, and that  $\theta_n \alpha_{2,n}$ ,  $\theta_n \beta_{2,n}$  converge respectively to constants  $\alpha_2 > 0$ ,  $\beta_2$ , we prove (see Theorem 2.1) that the perturbed conductivity  $\sigma_n(h)$  converges in an appropriate sense of H-convergence (see Definition 1.1) to the homogenized matrix-valued function

$$\sigma_*(h) = \sigma_*^0(\alpha_1, \alpha_2 + \alpha_2^{-1} \beta_2^2 h^2) + \beta_1 h J, \quad (1.4)$$

for some function  $\sigma_*^0$  which depends uniquely on the microstructure  $\chi_n$  in the absence of a magnetic field, and is defined for a subsequence of  $n$ . The proof of the result is based on a Dykhne transformation of the type

$$A_n \mapsto ((p_n A_n + q_n J)^{-1} + r_n J)^{-1}, \quad (1.5)$$

which permits to change the non-symmetric conductivity  $\sigma_n(h)$  into a symmetric one. Then, extending the duality principle (1.2) established in [16], we prove that transformation (1.5) is also stable under high-contrast conductivity homogenization.

In the periodic case, i.e. when  $\sigma_n(h)(\cdot) = \Sigma_n(\cdot/\varepsilon_n)$  with  $\Sigma_n$   $Y$ -periodic and  $\varepsilon_n \rightarrow 0$ , we use an alternative approach based on an extension of Theorem 4.1 of [13] to  $\varepsilon_n Y$ -periodic but non-symmetric conductivities (see Theorem 3.1). So, it turns out that the homogenized conductivity  $\sigma_*(h)$  is the limit as  $n \rightarrow \infty$  of the constant H-limit  $(\sigma_n)_*$  associated with the periodic homogenization (see, e.g., [2]) of the oscillating sequence  $\Sigma_n(\cdot/\varepsilon)$  as  $\varepsilon \rightarrow 0$  and for a fixed  $n$ . Finally, the Dykhne transformation performed by Milton [35] (see also [36], Chapter 4) applied to the local periodic conductivity  $\Sigma_n$  and its effective conductivity  $(\sigma_n)_*$ , allows us to recover the perturbed homogenized formula (1.4). An example of a periodic cross-like thin structure provides an explicit computation of  $\sigma_*(h)$  (see Proposition 3.2).

To make a comparison with dimension three we restrict ourselves to the  $\varepsilon_n Y$ -periodic fiber-reinforced structure introduced by Fenchenco, Khruslov [25] to derive a nonlocal effect in homogenization. However, in the present context the fibers radius  $r_n$  is chosen to be super-critical, i.e.

$r_n \rightarrow 0$  and  $\varepsilon_n^2 |\ln r_n| \rightarrow 0$ , in order to avoid such an effect. Similarly to (1.3) the perturbed conductivity is defined for  $h \in \mathbb{R}^3$ , by

$$\sigma_n(h) := (1 - \chi_n)(\alpha_1 I_3 + \beta_1 \mathcal{E}(h)) + \chi_n(\alpha_{2,n} I_3 + \beta_{2,n} \mathcal{E}(h)), \quad \mathcal{E}(h) := \begin{pmatrix} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ -h_2 & h_1 & 0 \end{pmatrix}, \quad (1.6)$$

where  $\chi_n$  is the characteristic function of the fibers which are parallel to direction  $e_3$ . The isotropy is partially ensured for orthogonal transformations letting  $h$  be invariant. Under the same assumptions on the conductivity coefficients as in the two-dimensional case, with  $\theta_n = \pi r_n^2$ , the homogenized conductivity is, in contrasts with (1.4), given by (see Theorem 4.1)

$$\sigma_*(h) = \sigma_*^0 \left( \alpha_1, \frac{\alpha_2^3 + \alpha_2 \beta_2^2 |h|^2}{\alpha_2^2 + \beta_2^2 h_3^2} \right) + \beta_1 \mathcal{E}(h), \quad \text{where} \quad \sigma_*^0(\gamma_1, \gamma_2) := \gamma_1 I_3 + \gamma_2 (e_3 \otimes e_3). \quad (1.7)$$

Therefore, the dependence in  $h$  is rational while it is polynomial in the two-dimensional case (1.4), due to the fiber-reinforcement in the direction  $e_3$ .

The paper is organized as follows. Section 2 and 3 deal with dimension two. In Section 2 we study the two-dimensional general (non-periodic) case thanks to an appropriate div-curl lemma. In Section 3 an alternative approach is performed in the periodic framework. Finally, Section 4 is devoted to the three-dimensional case with the fiber-reinforced structure.

## Notations

- $\Omega$  denotes a bounded open subset of  $\mathbb{R}^d$ ;
- $I_d$  denotes the unit matrix in  $\mathbb{R}^{d \times d}$ , and  $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ;
- for any matrix  $A$  in  $\mathbb{R}^{d \times d}$ ,  $A^T$  denotes the transposed of the matrix  $A$ ,  $A^s$  denotes its symmetric part;
- for  $h \in \mathbb{R}^3$ ,  $\mathcal{E}(h)$  denotes the antisymmetric matrix in  $\mathbb{R}^{3 \times 3}$  defined by  $\mathcal{E}(h)x := h \times x$ , for  $x \in \mathbb{R}^3$ ;
- for any  $A, B \in \mathbb{R}^{d \times d}$ ,  $A \leq B$  means that for any  $\xi \in \mathbb{R}^d$ ,  $A\xi \cdot \xi \leq B\xi \cdot \xi$ ; we will use the fact that for any invertible matrix  $A \in \mathbb{R}^{d \times d}$ ,  $A \geq \alpha I_d \Rightarrow A^{-1} \leq \alpha^{-1} I_d$ ;
- $|\cdot|$  denotes both the euclidean norm in  $\mathbb{R}^d$  and the subordinate norm in  $\mathbb{R}^{d \times d}$ ;
- for any locally compact subset  $X$  of  $\mathbb{R}^d$ ,  $\mathcal{M}(X)$  denotes the space of the Radon measures defined on  $X$ ;
- for any  $\alpha, \beta > 0$ ,  $\mathcal{M}(\alpha, \beta; \Omega)$  is the set of the invertible matrix-valued functions  $A : \Omega \rightarrow \mathbb{R}^{d \times d}$  such that

$$\forall \xi \in \mathbb{R}^d, \quad A(x)\xi \cdot \xi \geq \alpha |\xi|^2 \quad \text{and} \quad A^{-1}(x)\xi \cdot \xi \geq \beta^{-1} |\xi|^2 \quad \text{a.e. in } \Omega; \quad (1.8)$$

- $C$  denotes a constant which may vary from a line to another one.

In the sequel, we will use the following extension of  $H$ -convergence introduced in [16]:

**Definition 1.1** *Let  $\alpha_n$  and  $\beta_n$  be two sequences of positive numbers such that  $\alpha_n \leq \beta_n$ , and let  $A_n$  be a sequence of matrix-valued functions in  $\mathcal{M}(\alpha_n, \beta_n; \Omega)$  (see (1.8)).*

*The sequence  $A_n$  is said to  $H(\mathcal{M}(\Omega)^2)$ -converge to the matrix-valued function  $A_*$  if for any distribution  $f$  in  $H^{-1}(\Omega)$ , the solution  $u_n$  of the problem*

$$\begin{cases} \operatorname{div}(A_n \nabla u_n) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies the convergences

$$\begin{cases} u_n \rightharpoonup u & \text{in } H_0^1(\Omega) \\ A_n \nabla u_n \rightharpoonup A_* \nabla u & \text{weakly-* in } \mathcal{M}(\Omega)^2, \end{cases}$$

where  $u$  is the solution of the problem

$$\begin{cases} \operatorname{div}(A_* \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We now give a notation for  $H(\mathcal{M}(\Omega)^2)$ -limits of high-contrast two-phase composites. We consider the characteristic function  $\chi_n$  of the hard phase, and denote  $\omega_n := \{\chi_n = 1\}$ .

**Notation 1.1** *A sequence of isotropic two-phase conductivities in the absence of a magnetic field is denoted by*

$$\sigma_n^0(\alpha_{1,n}, \alpha_{2,n}) := (1 - \chi_n)\alpha_{1,n}I_2 + \chi_n\alpha_{2,n}I_2, \quad (1.9)$$

with

$$\lim_{n \rightarrow \infty} \alpha_{1,n} = \alpha_1 > 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} |\omega_n| \alpha_{2,n} = \alpha_2 > 0, \quad (1.10)$$

and its  $H(\mathcal{M}(\Omega)^2)$ -limit is denoted by  $\sigma_*^0(\alpha_1, \alpha_2)$ .

## 2 A two-dimensional non-periodic medium

### 2.1 A div-curl approach

We extend the classical div-curl lemma.

**Lemma 2.1** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^2$ . Let  $\alpha > 0$ , let  $\bar{a} \in L^\infty(\Omega)$  and let  $A_n$  be a sequence of matrix-valued functions in  $L^\infty(\Omega)^{2 \times 2}$  (not necessarily symmetric) satisfying*

$$A_n \geq \alpha I_2 \quad \text{a.e. in } \Omega \quad \text{and} \quad \frac{\det A_n}{\det A_n^s} |A_n^s| \rightharpoonup \bar{a} \in L^\infty(\Omega) \quad \text{weakly-* in } \mathcal{M}(\Omega). \quad (2.1)$$

Let  $\xi_n$  be a sequence in  $L^2(\Omega)^2$  and  $v_n$  a sequence in  $H^1(\Omega)$  satisfying the following assumptions:

(i)  $\xi_n$  and  $v_n$  satisfy the estimate

$$\int_{\Omega} A_n^{-1} \xi_n \cdot \xi_n \, dx + \|v_n\|_{H^1(\Omega)} \leq C; \quad (2.2)$$

(ii)  $\xi_n$  satisfies the classical condition

$$\operatorname{div} \xi_n \text{ is compact in } H^{-1}(\Omega). \quad (2.3)$$

Then, there exist  $\xi$  in  $L^2(\Omega)^2$  and  $v$  in  $H^1(\Omega)$  such that the following convergences hold true up to a subsequence

$$\xi_n \rightharpoonup \xi \quad \text{weakly-* in } \mathcal{M}(\Omega)^2 \quad \text{and} \quad \nabla v_n \rightharpoonup \nabla v \quad \text{weakly in } L^2(\Omega)^2. \quad (2.4)$$

Moreover, we have the following convergence in the distribution sense

$$\xi_n \cdot \nabla v_n \rightharpoonup \xi \cdot \nabla v \quad \text{weakly in } \mathcal{D}'(\Omega).$$

**Proof of Lemma 2.1.** The proof consists in considering the "good-divergence" sequence  $\xi_n$  as a sum of a compact sequence of gradients  $\nabla u_n$  and a sequence of divergence-free functions  $J\nabla z_n$ . We then use Lemma 3.1 of [16] to obtain the strong convergence of  $z_n$  in  $L^2_{loc}(\Omega)$ . Finally, replacing  $\xi_n$  by  $\nabla u_n + J\nabla z_n$ , we conclude owing to integration by parts.

*First step:* Proof of convergences (2.4).

An easy computation gives

$$\left( (A_n^{-1})^s \right)^{-1} = \frac{\det A_n}{\det A_n^s} A_n^s. \quad (2.5)$$

The sequence  $\xi_n$  is bounded in  $L^1(\Omega)^2$  since the Cauchy-Schwarz inequality combined with the weak-\* convergence of (2.1), (2.2) and (2.5) yields

$$\left( \int_{\Omega} |\xi_n| \, dx \right)^2 \leq \int_{\Omega} \left| \left( (A_n^{-1})^s \right)^{-1} \right| \, dx \int_{\Omega} (A_n^{-1})^s \xi_n \cdot \xi_n \, dx = \int_{\Omega} \frac{\det A_n}{\det A_n^s} |A_n^s| \, dx \int_{\Omega} A_n^{-1} \xi_n \cdot \xi_n \, dx \leq C.$$

Therefore,  $\xi_n$  converges up to a subsequence to some  $\xi \in \mathcal{M}(\Omega)^2$  in the weak-\* sense of the measures. Let us prove that the vector-valued measure  $\xi$  is actually in  $L^2(\Omega)^2$ . Again by the Cauchy-Schwarz inequality combined with (2.1), (2.2) and (2.5) we have, for any  $\Phi \in \mathcal{C}_0(\Omega)^2$ ,

$$\begin{aligned} \left| \int_{\Omega} \xi(dx) \cdot \Phi \right| &= \lim_{n \rightarrow \infty} \left| \int_{\Omega} \xi_n \cdot \Phi \, dx \right| \\ &\leq \limsup_{n \rightarrow \infty} \left( \int_{\Omega} \frac{\det A_n}{\det A_n^s} |A_n^s| |\Phi|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} A_n^{-1} \xi_n \cdot \xi_n \, dx \right)^{\frac{1}{2}} \leq C \left( \int_{\Omega} \bar{a} |\Phi|^2 \, dx \right)^{\frac{1}{2}}, \end{aligned}$$

which implies that  $\xi$  is absolutely continuous with respect to the Lebesgue measure. Since  $\bar{a} \in L^\infty(\Omega)$ , we also get that

$$\left| \int_{\Omega} \xi \cdot \Phi \, dx \right| \leq \|\Phi\|_{L^2(\Omega)^2}$$

hence  $\xi \in L^2(\Omega)^2$ . Therefore, the first convergence of (2.4) holds true with its limit in  $L^2(\Omega)^2$ . The second one is immediate.

*Second step:* Introduction of a stream function.

By (2.3), the sequence  $u_n$  in  $H_0^1(\Omega)$  defined by  $u_n := \Delta^{-1}(\operatorname{div} \xi_n)$  strongly converges in  $H_0^1(\Omega)$ :

$$u_n \longrightarrow u \quad \text{in } H_0^1(\Omega). \quad (2.6)$$

Let  $\omega$  be a regular simply connected open set such that  $\omega \subset\subset \Omega$ . Since by definition  $\xi_n - \nabla u_n$  is a divergence-free function in  $L^2(\Omega)^2$ , there exists (see, e.g., [26]) a unique stream function  $z_n \in H^1(\omega)$  with zero  $\omega$ -average such that

$$\xi_n = \nabla u_n + J\nabla z_n \quad \text{a.e. in } \omega. \quad (2.7)$$

*Third step:* Convergence of the stream function  $z_n$ .

Since  $\nabla u_n$  is bounded in  $L^2(\Omega)^2$  by the second step,  $\xi_n$  is bounded in  $L^1(\Omega)^2$  by the first step and  $z_n$  has a zero  $\omega$ -average, the Sobolev embedding of  $W^{1,1}(\omega)$  into  $L^2(\omega)$  combined with the Poincaré-Wirtinger inequality in  $\omega$  implies that  $z_n$  is bounded in  $L^2(\omega)$  and thus converges, up to a subsequence still denoted by  $n$ , to a function  $z$  in  $L^2(\omega)$ .

Moreover, let us define

$$S_n := (J^{-1}(A_n^{-1})^s J)^{-1}.$$

The Cauchy-Schwarz inequality gives

$$\begin{aligned}
\int_{\omega} S_n^{-1} \nabla z_n \cdot \nabla z_n \, dx &= \int_{\omega} J^{-1} (A_n^{-1})^s J \nabla z_n \cdot \nabla z_n \, dx \\
&= \int_{\omega} (A_n^{-1})^s J \nabla z_n \cdot J \nabla z_n \, dx \\
&= \int_{\omega} (A_n^{-1})^s [\xi_n - \nabla u_n] \cdot [\xi_n - \nabla u_n] \, dx \\
&\leq 2 \int_{\omega} (A_n^{-1})^s \xi_n \cdot \xi_n \, dx + 2 \int_{\omega} (A_n^{-1})^s \nabla u_n \cdot \nabla u_n \, dx \\
&= 2 \int_{\omega} A_n^{-1} \xi_n \cdot \xi_n \, dx + 2 \int_{\omega} A_n^{-1} \nabla u_n \cdot \nabla u_n \, dx.
\end{aligned}$$

The first term is bounded by (2.2) and the last term by the inequality  $A_n^{-1} \leq \alpha^{-1} I_2$  and the convergence (2.6). Therefore, the sequences  $v_n := z_n$  and, by (2.12),  $S_n$  satisfy all the assumptions of Lemma 3.1 of [16] since, by (2.5),

$$S_n = \frac{\det A_n}{\det A_n^s} J^{-1} A_n^s J.$$

Then, we obtain the convergence

$$z_n \longrightarrow z \quad \text{strongly in } L_{\text{loc}}^2(\omega). \quad (2.8)$$

Moreover, the convergence (2.6) gives

$$\xi = \nabla u + J \nabla z \quad \text{in } \mathcal{D}'(\omega). \quad (2.9)$$

*Fourth step:* Integration by parts and conclusion.

We have, as  $J \nabla v_n$  is a divergence-free function,

$$\xi_n \cdot \nabla v_n = (\nabla u_n + J \nabla z_n) \cdot \nabla v_n = \nabla u_n \cdot \nabla v_n - \operatorname{div}(z_n J \nabla v_n). \quad (2.10)$$

The strong convergence of  $\nabla u_n$  in (2.6), the second weak convergence of (2.4) justified in the first step and (2.8) give

$$\nabla u_n \cdot \nabla v_n - \operatorname{div}(z_n J \nabla v_n) \longrightarrow \nabla u \cdot \nabla v - \operatorname{div}(z J \nabla v) \quad \text{in } \mathcal{D}'(\omega). \quad (2.11)$$

We conclude, by combining this convergence with (2.10), (2.9) and integrating by parts, to the convergence

$$\xi_n \cdot \nabla v_n \longrightarrow \nabla u \cdot \nabla v - \operatorname{div}(z J \nabla v) = (\nabla u + J \nabla z) \cdot \nabla v = \xi \cdot \nabla v \quad \text{weakly in } \mathcal{D}'(\omega).$$

for an arbitrary open subset  $\omega$  of  $\Omega$ . □

We also need the following result in order to study the  $H(\mathcal{M}(\Omega)^2)$ -convergence of non-symmetric conductivities thanks to a suitable Dykhne transformation.

**Proposition 2.1** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^2$  such that  $|\partial\Omega| = 0$ . Let  $p_n, q_n$  and  $r_n, n \in \mathbb{N}$  be sequences of real numbers converging respectively to  $p > 0, q$  and 0. Let  $\alpha > 0$ , let  $\beta_n, n \in \mathbb{N}$  be a sequence of real numbers such that  $\beta_n \geq \alpha$ , and let  $A_n$  be a sequence of matrix-valued functions in  $\mathcal{M}(\alpha, \beta_n; \Omega)$  (not necessarily symmetric) satisfying*

$$r_n A_n \text{ is bounded in } L^\infty(\Omega)^{2 \times 2} \quad \text{and} \quad \frac{\det A_n}{\det A_n^s} |A_n^s| \rightharpoonup \bar{a} \in L^\infty(\Omega) \quad \text{weakly-* in } \mathcal{M}(\Omega), \quad (2.12)$$

and that

$$B_n = ((p_n A_n + q_n J)^{-1} + r_n J)^{-1} \text{ is a sequence of symmetric matrices.}$$

Then, there exist a subsequence of  $n$ , still denoted by  $n$ , and a matrix-valued function  $A_*$  in  $\mathcal{M}(\alpha, \beta; \Omega)$ , with  $\beta = 2\|\bar{a}\|_{L^\infty(\Omega)}$ , such that

$$A_n \xrightarrow{H(\mathcal{M}(\Omega)^2)} A_* \quad \text{and} \quad ((p_n A_n + q_n J)^{-1} + r_n J)^{-1} \xrightarrow{H(\mathcal{M}(\Omega)^2)} pA_* + qJ. \quad (2.13)$$

**Remark 2.1** Proposition 2.1 is a Keller-Dykhne duality result in a high conductivity framework. This completes Theorem 2.2 of [16] performed with  $B_n = J^{-1}A_n^{-1}J$ .

**Remark 2.2** The hypothesis of symmetry on  $B_n$  is only sufficient but is adapted to the situation because we want to transform a non-symmetric problem to a symmetric one. It is only useful to obtain the condition on  $\frac{\det B_n}{\det B_n^s}|B_n|$  to apply the results of [16].

**Remark 2.3** The convergence of  $r_n$  to  $r = 0$  is not necessary but sufficient for our purpose. If  $r \neq 0$ , the different convergences are conserved but leads us to the expression  $pA_* + qJ = B_*((1 - qr)I + p r J A_*)$ .

**Proof of Proposition 2.1.** The proof is divided into two steps. In the first step, we use Lemma 2.1 to show the  $H(\mathcal{M}(\Omega)^2)$ -convergence of  $\tilde{A}_n := p_n A_n + q_n J$  to  $pA_* + qJ$ . In the second step, we build a matrix  $Q_n$  which will be used as a corrector for  $B_n$  and then use again Lemma 2.1.

First step:  $\tilde{A}_* = pA_* + qJ$ .

First of all, thanks to Theorem 2.2 [16], we already know that, up to a subsequence still denoted by  $n$ ,  $A_n$   $H(\mathcal{M}(\Omega)^2)$ -converges to  $A_*$ . We consider a corrector  $P_n$  associated with  $A_n$  in the sense of Murat-Tartar (see, e.g., [39]), such that, for  $\lambda \in \mathbb{R}^2$ ,  $P_n \lambda = \nabla w_n^\lambda$  is defined by

$$\begin{cases} \operatorname{div}(A_n \nabla w_n^\lambda) = \operatorname{div}(A_* \nabla(\lambda \cdot x)) & \text{in } \Omega \\ w_n^\lambda = \lambda \cdot x & \text{on } \partial\Omega \end{cases} \quad (2.14)$$

Again with Theorem 2.2 of [16] and Definition 1.1, we know that  $P_n \lambda$  converges weakly in  $L^2(\Omega)^2$  to  $\lambda$  and  $A_n P_n \lambda$  converges weakly-\* in  $\mathcal{M}(\Omega)$  to  $A_* \lambda$ .

Since, for any  $\lambda, \mu \in \mathbb{R}^2$ ,

$$\alpha \|\nabla w_n^\mu\|_{L^2(\Omega)^2}^2 \leq \int_{\Omega} A_n \nabla w_n^\mu \cdot \nabla w_n^\mu \, dx = \int_{\Omega} A_* \mu \cdot \nabla w_n^\mu \, dx \leq 2\|\bar{a}\|_{L^\infty(\Omega)} |\mu| |\Omega|^{1/2} \|\nabla w_n^\mu\|_{L^2(\Omega)^2}$$

and

$$\int_{\Omega} A_n^{-1} A_n \nabla w_n^\lambda \cdot A_n \nabla w_n^\lambda \, dx = \int_{\Omega} A_n \nabla w_n^\lambda \cdot \nabla w_n^\lambda \, dx,$$

the sequences  $\xi_n := A_n \nabla w_n^\lambda$  and  $v_n := w_n^\mu$  satisfy (2.2) and (2.3). This combined with (2.12) implies that we can apply Lemma 2.1 to obtain

$$\forall \lambda, \mu \in \mathbb{R}, \quad A_n P_n \lambda \cdot P_n \mu \rightharpoonup A_* \lambda \cdot \mu \text{ in } \mathcal{D}'(\Omega). \quad (2.15)$$

We denote  $\tilde{A}_n := p_n A_n + q_n J$  and consider  $\delta_n$  such that  $\delta_n J := A_n - A_n^s$ . Then, the matrix  $\tilde{A}_n$  satisfies

$$\tilde{A}_n \xi \cdot \xi = p_n A_n \xi \cdot \xi \geq p_n \alpha |\xi|^2. \quad (2.16)$$

Moreover,

$$\det \tilde{A}_n = p_n^2 \det A_n^s + (p_n \delta_n + q_n)^2 \leq p_n^2 (\det A_n^s + 2\delta_n^2) + 2q_n^2 \leq 2p_n^2 \det A_n + 2q_n^2 \leq C \det A_n,$$

the last inequality being a consequence of  $A_n \geq \alpha I_2$ . This inequality gives, by (2.12),

$$\frac{\det \tilde{A}_n}{\det \tilde{A}_n^s} |\tilde{A}_n^s| = \frac{\det \tilde{A}_n}{p_n^2 \det A_n^s} p_n |A_n^s| \leq C \frac{\det A_n}{\det A_n^s} |A_n^s| \leq C. \quad (2.17)$$

Then by (2.16), (2.17) and [16] again, up to a subsequence still denoted by  $n$ ,  $\tilde{A}_n$   $H(\mathcal{M}(\Omega)^2)$ -converges to  $\tilde{A}_*$  and we have, by the classical div-curl lemma of [39] for  $JP_n \lambda \cdot P_n \mu$  and (2.15),

$$\forall \lambda, \mu \in \mathbb{R}, \quad (p_n A_n + q_n J) P_n \lambda \cdot P_n \mu = p_n A_n P_n \lambda \cdot P_n \mu + q_n J P_n \lambda \cdot P_n \mu \xrightarrow{\mathcal{D}'(\Omega)} p A_* \lambda \cdot \mu + q J \lambda \cdot \mu,$$

that can be rewritten

$$\tilde{A}_* = p A_* + q J.$$

*Second step:  $B_* = \tilde{A}_*$ .*

Let  $\theta \in \mathcal{C}^1(\Omega)$  and  $\tilde{P}_n$  a corrector associated with  $\tilde{A}_n$ , such that, for  $\lambda \in \mathbb{R}^2$ ,  $\tilde{P}_n \lambda = \nabla \tilde{w}_n^\lambda$  is defined by

$$\begin{cases} \operatorname{div}(\tilde{A}_n \nabla \tilde{w}_n^\lambda) = \operatorname{div}(\tilde{A}_* \nabla(\theta \lambda \cdot x)) & \text{in } \Omega \\ \tilde{w}_n^\lambda = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.18)$$

By Definition 1.1, we have

$$\begin{cases} \tilde{w}_n^\lambda \rightharpoonup \theta \lambda \cdot x & \text{weakly in } H_0^1(\Omega), \\ \tilde{A}_n \nabla \tilde{w}_n^\lambda \rightharpoonup \tilde{A}_* \nabla(\theta \lambda \cdot x) & \text{weakly-* in } \mathcal{M}(\Omega)^2. \end{cases} \quad (2.19)$$

Let now consider  $B_n = (\tilde{A}_n^{-1} + r_n J)^{-1}$ .  $B_n$  is symmetric and so is its inverse :

$$B_n^{-1} = \tilde{A}_n^{-1} + r_n J = (\tilde{A}_n^{-1} + r_n J)^s = (\tilde{A}_n^{-1})^s.$$

We then have, by a little computation (like in (2.5)) and (2.17),

$$\frac{\det B_n}{\det B_n^s} |B_n^s| = |B_n| = \left| \left( (\tilde{A}_n^{-1})^s \right)^{-1} \right| = \frac{\det \tilde{A}_n}{\det \tilde{A}_n^s} |\tilde{A}_n^s| \leq C. \quad (2.20)$$

For any  $\xi \in \mathbb{R}^2$ , the sequence  $\nu_n := (I + r_n J \tilde{A}_n)^{-1} \xi$  satisfies, by (2.12),

$$|\xi| \leq \left( 1 + \|r_n \tilde{A}_n\|_{L^\infty(\Omega)^{2 \times 2}} \right) |\nu_n| \leq (1 + p_n \|r_n A_n\|_{L^\infty(\Omega)^{2 \times 2}} + q_n r_n) |\nu_n| \leq (1 + C) |\nu_n|,$$

hence

$$B_n \xi \cdot \xi = \tilde{A}_n \nu_n \cdot (I + r_n J \tilde{A}_n) \nu_n = \tilde{A}_n \nu_n \cdot \nu_n = p_n A_n \nu_n \cdot \nu_n \geq p_n \alpha |\nu_n|^2 \geq \alpha \frac{p_n}{(1 + C)^2} |\xi|^2 \geq C |\xi|^2 \quad (2.21)$$

with  $C > 0$ . Therefore, with (2.21) and (2.20), again by Theorem 2.2 of [16], up to a subsequence still denoted by  $n$ ,  $B_n$   $H(\mathcal{M}(\Omega)^2)$ -converges to  $B_*$ .

Let  $\psi \in \mathcal{C}^1(\Omega)$  and  $R_n$  be a corrector associated to  $B_n$ , such that, for  $\mu \in \mathbb{R}^2$ ,  $R_n \mu = \nabla v_n^\mu$  is defined by

$$\begin{cases} \operatorname{div}(B_n \nabla v_n^\mu) = \operatorname{div}(B_* \nabla(\psi \mu \cdot x)) & \text{in } \Omega \\ v_n^\mu = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.22)$$

By Definition 1.1, we have the convergences

$$\begin{cases} v_n^\mu \rightharpoonup \psi \mu \cdot x & \text{weakly in } H_0^1(\Omega), \\ B_n \nabla v_n^\mu \rightharpoonup B_* \nabla(\psi \mu \cdot x) & \text{weakly-* in } \mathcal{M}(\Omega)^2. \end{cases} \quad (2.23)$$

Let us define the matrix  $Q_n := (I + r_n J \tilde{A}_n) \tilde{P}_n$ . We have

$$B_n Q_n = (\tilde{A}_n^{-1} + r_n J)^{-1} (I + r_n J \tilde{A}_n) \tilde{P}_n = (\tilde{A}_n^{-1} + r_n J)^{-1} (\tilde{A}_n^{-1} + r_n J) \tilde{A}_n \tilde{P}_n = \tilde{A}_n \tilde{P}_n. \quad (2.24)$$

We are going to pass to the limit in  $\mathcal{D}'(\Omega)$  the equality given by (2.24) and the symmetry of  $B_n$ :

$$\tilde{A}_n \tilde{P}_n \lambda \cdot R_n \mu = B_n Q_n \lambda \cdot R_n \mu = Q_n \lambda \cdot B_n R_n \mu. \quad (2.25)$$

On the one hand,  $\tilde{A}_n$  satisfies (2.1) by (2.16) and (2.17). The sequences  $\xi_n := \tilde{A}_n \tilde{P}_n \lambda$  and  $v_n := v_n^\mu$  satisfy the hypothesis (2.3) by (2.18) and (2.2) because

$$\int_{\Omega} (\tilde{A}_n)^{-1} \xi_n \cdot \xi_n \, dx + \|v_n\|_{H_0^1(\Omega)} = \int_{\Omega} \tilde{A}_n \tilde{P}_n \lambda \cdot \tilde{P}_n \lambda \, dx + \|v_n^\mu\|_{H_0^1(\Omega)} \, dx \leq C$$

by (2.18) and the convergences (2.23) and (2.19). The application of Lemma 2.1, (2.19) and (2.23) give the convergence

$$\tilde{A}_n \tilde{P}_n \lambda \cdot R_n \mu \longrightarrow A^* \nabla(\theta \lambda \cdot x) \cdot \nabla(\psi \mu \cdot x) \quad \text{in } \mathcal{D}'(\Omega). \quad (2.26)$$

On the other hand, we have the equality

$$Q_n \lambda \cdot B_n R_n \mu = B_n R_n \mu \cdot \tilde{P}_n \lambda + B_n R_n \mu \cdot r_n J \tilde{A}_n \tilde{P}_n. \quad (2.27)$$

The matrix  $B_n$  satisfies (2.1) by (2.21) and (2.20). The sequences  $\xi_n := B_n R_n \mu$  and  $v_n := \tilde{w}_n^\lambda$  satisfy the hypothesis (2.3) by (2.22) and (2.2) of Lemma 2.1 because

$$\int_{\Omega} (B_n)^{-1} \xi_n \cdot \xi_n \, dx + \|v_n\|_{H_0^1(\Omega)} = \int_{\Omega} B_n R_n \mu \cdot R_n \mu \, dx + \|\tilde{w}_n^\lambda\|_{H_0^1(\Omega)} \, dx \leq C$$

by (2.22) and the convergences (2.19) and (2.23). The application of Lemma 2.1, (2.19) and (2.23) give the convergence

$$B_n R_n \mu \cdot \tilde{P}_n \lambda \longrightarrow B_* \nabla(\psi \mu \cdot x) \cdot \nabla(\theta \lambda \cdot x) \quad \text{in } \mathcal{D}'(\Omega). \quad (2.28)$$

The convergence of the right part of (2.27) is more delicate. The demonstration is the same as for Lemma 2.1. Let  $\omega$  be a simply connected open subset of  $\Omega$  such as  $\omega \subset\subset \Omega$ . The function  $\tilde{A}_n \tilde{P}_n \lambda - \tilde{A}_* \nabla(\theta \lambda \cdot x)$  is divergence-free and we can introduce a function  $z_n^\lambda$  such as

$$\tilde{A}_n \tilde{P}_n \lambda = \tilde{A}_* \nabla(\theta \lambda \cdot x) + J \nabla z_n^\lambda, \quad (2.29)$$

$$z_n^\lambda \longrightarrow 0 \quad \text{strongly in } L_{\text{loc}}^2(\omega). \quad (2.30)$$

The equality

$$\begin{aligned} B_n R_n \mu \cdot r_n J \tilde{A}_n \tilde{P}_n \lambda &= r_n B_n R_n \mu \cdot J \tilde{A}_* \nabla(\theta \lambda \cdot x) - r_n B_n R_n \mu \cdot \nabla z_n^\lambda \\ &= r_n B_n R_n \mu \cdot J \tilde{A}_* \nabla(\theta \lambda \cdot x) - r_n \operatorname{div}(z_n^\lambda B_n R_n \mu) + r_n z_n^\lambda \operatorname{div}(B_* \nabla(\theta \lambda \cdot x)) \end{aligned}$$

leads us, by (2.23), (2.30) and the convergence to 0 of  $r_n$ , like in the demonstration of Lemma 2.1, to

$$B_n R_n \mu \cdot r_n J \tilde{A}_n \tilde{P}_n \lambda \longrightarrow 0 \quad \text{in } \mathcal{D}'(\omega). \quad (2.31)$$

Finally, by combining (2.25), (2.26), (2.28) and (2.31), we obtain, for any simply connected open subset  $\omega$  of  $\Omega$  such as  $\omega \subset\subset \Omega$ ,

$$\tilde{A}_* \nabla(\theta \lambda \cdot x) \cdot \nabla(\psi \mu \cdot x) = B_* \nabla(\psi \mu \cdot x) \cdot \nabla(\theta \lambda \cdot x) \quad \text{in } \mathcal{D}'(\omega).$$

We conclude, by taking  $\theta = 1$  and  $\psi = 1$  on  $\omega$  and taking into account that  $B_*$  is symmetric and  $\omega$ ,  $\lambda$ ,  $\mu$  are arbitrary, that:

$$B_* = \tilde{A}_* = p A_* + q J.$$

□

## 2.2 An application to isotropic two-phase media

In this section, we study the homogenization of a two-phase isotropic medium with high contrast and non-necessarily symmetric conductivities. The study of the symmetric case in Proposition 2.2 permits to obtain Theorem 2.1 by applying the transformation of Proposition 2.1. We use Notation 1.1.

**Proposition 2.2** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^2$  such that  $|\partial\Omega| = 0$ . Let  $\omega_n$ ,  $n$  in  $\mathbb{N}$ , be a sequence of open subsets of  $\Omega$  with characteristic function  $\chi_n$ , satisfying  $\theta_n := |\omega_n| < 1$ ,  $\theta_n$  converges to 0, and*

$$\frac{\chi_n}{\theta_n} \rightharpoonup a \in L^\infty(\Omega) \quad \text{weakly-* in } \mathcal{M}(\Omega). \quad (2.32)$$

We assume that there exists  $\alpha_1, \alpha_2 > 0$  and two positive sequences  $\alpha_{1,n}, \alpha_{2,n} \geq a_0 > 0$  verifying

$$\lim_{n \rightarrow \infty} \alpha_{1,n} = \alpha_1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \theta_n \alpha_{2,n} = \alpha_2, \quad (2.33)$$

and that the conductivity takes the form

$$\sigma_n^0(\alpha_{1,n}, \alpha_{2,n}) = \alpha_{1,n}(1 - \chi_n)I_2 + \chi_n \alpha_{2,n}I_2.$$

Then, there exists a subsequence of  $n$ , still denoted by  $n$ , and a locally Lipschitz function

$$\sigma_*^0 : (0, \infty)^2 \longrightarrow \mathcal{M}(a_0, 2\|a\|_\infty; \Omega)$$

such that

$$\forall (\alpha_1, \alpha_2) \in (0, \infty)^2, \quad \sigma_n^0(\alpha_{1,n}, \alpha_{2,n}) \xrightarrow{H(\mathcal{M}(\Omega)^2)} \sigma_*^0(\alpha_1, \alpha_2). \quad (2.34)$$

**Proof of Proposition 2.2.** The proof is divided into two parts. We first prove the theorem for  $\alpha_{1,n} = \alpha_1$ ,  $\alpha_{2,n} = \theta_n^{-1}\alpha_2$ , and then treat the general case.

*First step:* The case  $\alpha_{1,n} = \alpha_1$ ,  $\alpha_{2,n} = \theta_n^{-1}\alpha_2$ .

In this step we denote  $\sigma_n^0(\alpha) := \sigma_n^0(\alpha_1, \theta_n^{-1}\alpha_2)$ , for  $\alpha = (\alpha_1, \alpha_2) \in (0, \infty)^2$ . Theorem 2.2 of [16] implies that for any  $\alpha \in (0, \infty)^2$ , there exists a subsequence of  $n$  such that  $\sigma_n^0(\alpha)$   $H(\mathcal{M}(\Omega)^2)$ -converges in the sense of Definition 1.1 to some matrix-valued function in  $\mathcal{M}(a_0, 2\|a\|_\infty; \Omega)$ .

By a diagonal extraction, there exists a subsequence of  $n$ , still denoted by  $n$ , such that

$$\forall \alpha \in \mathbb{Q}^2 \cap (0, \infty)^2, \quad \sigma_n^0(\alpha) \xrightarrow{H(\mathcal{M}(\Omega)^2)} \sigma_*^0(\alpha). \quad (2.35)$$

We are going to show that this convergence is true any pair  $\alpha \in (0, \infty)^2$ .

We have, by (2.32), for any  $\alpha \in \mathbb{Q}^2 \cap (0, \infty)^2$ ,

$$|\sigma_n^0(\alpha)| = \alpha_1(1 - \chi_n) + \chi_n \frac{\alpha_2}{\theta_n} \rightharpoonup \alpha_1 + \alpha_2 \quad a \in L^\infty(\Omega) \quad \text{weakly-* in } \mathcal{M}(\Omega) \quad (2.36)$$

and, since  $\theta_n \in (0, 1)$ ,

$$\forall \xi \in \mathbb{R}^2, \quad \sigma_n^0(\alpha)\xi \cdot \xi = \alpha_1(1 - \chi_n)|\xi|^2 + \chi_n \frac{\alpha_2}{\theta_n} |\xi|^2 \geq \min(\alpha_1, \alpha_2)|\xi|^2 \quad \text{a.e. in } \Omega. \quad (2.37)$$

By applying Theorem 2.2 of [16] with (2.36), we have the inequality

$$|\sigma_*^0(\alpha)\lambda| \leq 2|\lambda|(\alpha_1 + \alpha_2\|a\|_\infty). \quad (2.38)$$

For any  $\alpha \in \mathbb{Q}^2 \cap (0, \infty)^2$  and  $\lambda \in \mathbb{R}^2$ , consider the corrector  $w_n^{\alpha, \lambda}$  associated with  $\sigma_n^0(\alpha)$  defined by

$$\begin{cases} \operatorname{div} \left( \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \right) = \operatorname{div} \left( \sigma_*^0(\alpha) \lambda \right) & \text{in } \Omega, \\ w_n^{\alpha, \lambda} = \lambda \cdot x & \text{on } \partial\Omega, \end{cases} \quad (2.39)$$

which depends linearly on  $\lambda$ .

Let  $\alpha \in \mathbb{Q}^2 \cap (0, \infty)^2$ . Let us show that the energies

$$\int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha, \lambda} \, dx \quad (2.40)$$

are bounded. We have, by (2.39), (2.38) and the Cauchy-Schwarz inequality

$$\begin{aligned} & \int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha, \lambda} \, dx \\ &= \int_{\Omega} \sigma_*^0(\alpha) \lambda \cdot (\nabla w_n^{\alpha, \lambda} - \lambda) \, dx + \int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \lambda \, dx \\ &= \int_{\Omega} \sigma_*^0(\alpha) \lambda \cdot \nabla w_n^{\alpha, \lambda} \, dx - \underbrace{\int_{\Omega} \sigma_*^0(\alpha) \lambda \cdot \lambda \, dx}_{\geq 0} + \int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \lambda \, dx \end{aligned}$$

which leads us to

$$\int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha, \lambda} \, dx \leq \int_{\Omega} |\sigma_*^0(\alpha) \lambda \cdot \nabla w_n^{\alpha, \lambda}| \, dx + \int_{\Omega} |\sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \lambda| \, dx. \quad (2.41)$$

On the one hand, the Cauchy-Schwarz inequality gives

$$\left( \int_{\Omega} |\sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \lambda| \, dx \right)^2 \leq |\lambda|^2 \int_{\Omega} |\sigma_n^0(\alpha)| \, dx \int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha, \lambda} \, dx$$

that is

$$\left( \int_{\Omega} |\sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \lambda| \, dx \right)^2 \leq |\lambda|^2 |\alpha| \int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha, \lambda} \, dx. \quad (2.42)$$

On the other hand, by (2.37) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_{\Omega} |\sigma_*^0(\alpha) \lambda \cdot \nabla w_n^{\alpha, \lambda}| \, dx &\leq 2|\lambda|(\alpha_1 + \alpha_2 \|a\|_{\infty}) \sqrt{\int_{\Omega} |\nabla w_n^{\alpha, \lambda}|^2 \, dx} \\ &\leq 2|\lambda|(\alpha_1 + \alpha_2 \|a\|_{\infty}) \sqrt{\frac{1}{\alpha_1} + \frac{1}{\alpha_2}} \sqrt{\int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha, \lambda} \, dx} \end{aligned}$$

that is

$$\int_{\Omega} |\sigma_*^0(\alpha) \lambda \cdot \nabla w_n^{\alpha, \lambda}| \, dx \leq C |\lambda|^2 |\alpha| \sqrt{\frac{1}{\alpha_1} + \frac{1}{\alpha_2}} \sqrt{\int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha, \lambda} \, dx} \quad (2.43)$$

where  $C$  does not depend on  $n$  nor  $\alpha$ .

By combining (2.41), (2.42) and (2.43), we have

$$\int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha, \lambda} \, dx \leq C |\lambda|^2 \underbrace{(|\alpha| + |\alpha|^2(\alpha_1^{-1} + \alpha_2^{-1}))}_{=: M(\alpha)} \quad (2.44)$$

where  $C$  does not depend on  $n$  nor  $\alpha$ .

Let  $\alpha' \in \mathbb{Q}^2 \cap (0, \infty)^2$ . The sequences  $\xi_n := \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda}$  and  $v_n := w_n^{\alpha', \lambda}$  satisfy the assumptions (2.2) and (2.3) of Lemma 2.1. By symmetry, we have the convergences

$$\begin{cases} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha', \lambda} \rightharpoonup \sigma_*^0(\alpha) \lambda \cdot \lambda & \text{weakly in } \mathcal{D}'(\Omega), \\ \sigma_n^0(\alpha') \nabla w_n^{\alpha', \lambda} \cdot \nabla w_n^{\alpha, \lambda} \rightharpoonup \sigma_*^0(\alpha') \lambda \cdot \lambda & \text{weakly in } \mathcal{D}'(\Omega). \end{cases} \quad (2.45)$$

As the matrices are symmetric, we have

$$(\sigma_n^0(\alpha) - \sigma_n^0(\alpha')) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha', \lambda} = \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha', \lambda} - \sigma_n^0(\alpha') \nabla w_n^{\alpha', \lambda} \cdot \nabla w_n^{\alpha, \lambda},$$

hence

$$(\sigma_n^0(\alpha) - \sigma_n^0(\alpha')) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha', \lambda} \rightharpoonup (\sigma_*^0(\alpha) - \sigma_*^0(\alpha')) \lambda \cdot \lambda \quad \text{weakly in } \mathcal{D}'(\Omega). \quad (2.46)$$

Let  $\lambda \in \mathbb{R}^2$ . We have, by the Cauchy-Schwarz inequality, with the Einstein convention

$$\begin{aligned} & \int_{\Omega} |(\sigma_n^0(\alpha) - \sigma_n^0(\alpha')) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha', \lambda}| \, dx \\ &= \int_{\Omega \setminus \omega_n} |\alpha_1 - \alpha'_1| |\nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha', \lambda}| \, dx + \int_{\omega_n} |\alpha_2 - \alpha'_2| |\nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha', \lambda}| \, dx \\ &\leq |\alpha_1 - \alpha'_1| \sqrt{\int_{\Omega \setminus \omega_n} |\nabla w_n^{\alpha, \lambda}|^2 \, dx} \sqrt{\int_{\Omega \setminus \omega_n} |\nabla w_n^{\alpha', \lambda}|^2 \, dx} \\ &\quad + |\alpha_2 - \alpha'_2| \sqrt{\int_{\omega_n} |\nabla w_n^{\alpha, \lambda}|^2 \, dx} \sqrt{\int_{\omega_n} |\nabla w_n^{\alpha', \lambda}|^2 \, dx} \\ &\leq |\alpha_i - \alpha'_i| \sqrt{\frac{1}{\alpha_i} \int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha, \lambda} \, dx} \sqrt{\frac{1}{\alpha'_i} \int_{\Omega} \sigma_n^0(\alpha) \nabla w_n^{\alpha', \lambda} \cdot \nabla w_n^{\alpha', \lambda} \, dx}. \end{aligned}$$

This combined with (2.44) yields

$$\int_{\Omega} |(\sigma_n^0(\alpha) - \sigma_n^0(\alpha')) \nabla w_n^{\alpha, \lambda} \cdot \nabla w_n^{\alpha', \lambda}| \leq C |\lambda|^2 \frac{|\alpha_i - \alpha'_i|}{\sqrt{|\alpha_i| |\alpha'_i|}} M(\alpha) M(\alpha')$$

The sequence of (2.46) is thus bounded in  $L^1(\Omega)^2$  which implies that (2.46) holds weakly-\* in  $\mathcal{M}(\Omega)$ . Hence, we get, for any  $\varphi \in \mathcal{C}_c(\Omega)$ , that

$$\int_{\Omega} |(\sigma_*^0(\alpha) - \sigma_*^0(\alpha')) \lambda \cdot \lambda| \varphi \, dx \leq C |\lambda|^2 \frac{|\alpha_i - \alpha'_i|}{\sqrt{|\alpha_i| |\alpha'_i|}} M(\alpha) M(\alpha') \|\varphi\|_{\infty}. \quad (2.47)$$

Then, the Riesz representation theorem implies that

$$\| \sigma_*^0(\alpha) - \sigma_*^0(\alpha') \|_{L^1(\Omega)^{2 \times 2}} \leq C \frac{|\alpha_i - \alpha'_i|}{\sqrt{|\alpha_i| |\alpha'_i|}} M(\alpha) M(\alpha').$$

Therefore, by the definition of  $M$  in (2.44), for any compact subset  $K \subset (0, \infty)^2$ ,

$$\exists C > 0, \quad \forall \alpha, \alpha' \in \mathbb{Q}^2 \cap K, \quad \| \sigma_*^0(\alpha) - \sigma_*^0(\alpha') \|_{L^1(\Omega)^{2 \times 2}} \leq C |\alpha - \alpha'|. \quad (2.48)$$

This estimate permits to extend the definition (2.35) of  $\sigma_*^0$  on  $(0, \infty)^2$  by

$$\forall \alpha \in (0, \infty)^2, \quad \sigma_*^0(\alpha) = \lim_{\substack{\alpha' \rightarrow \alpha \\ \alpha' \in \mathbb{Q}^2 \cap (0, \infty)^2}} \sigma_*^0(\alpha') \quad \text{strongly in } L^1(\Omega)^{2 \times 2}. \quad (2.49)$$

Let  $\alpha \in (0, \infty)^2$ . Theorem 2.2 of [16] implies that there exists a subsequence of  $n$ , denoted by  $n'$ , and a matrix-valued function  $\tilde{\sigma}_* \in \mathcal{M}(a_0, 2\|a\|_\infty; \Omega)$  such that

$$\sigma_{n'}(\alpha) \xrightarrow{H(\mathcal{M}(\Omega)^2)} \tilde{\sigma}_*. \quad (2.50)$$

Repeating the arguments leading to (2.48), for any positive sequence of rational pair  $(\alpha^q)_{q \in \mathbb{N}}$  converging to  $\alpha$ , we have

$$\exists C > 0, \quad \|\tilde{\sigma}_* - \sigma_*^0(\alpha^q)\|_{L^1(\Omega)^{2 \times 2}} \leq C |\alpha - \alpha^q|, \quad (2.51)$$

hence, by (2.49),  $\tilde{\sigma}_* = \sigma_*^0(\alpha)$ . Therefore by the uniqueness of the limit in (2.50), we obtain for the whole sequence satisfying (2.35)

$$\forall \alpha \in (0, \infty)^2, \quad \sigma_n(\alpha) \xrightarrow{H(\mathcal{M}(\Omega)^2)} \sigma_*^0(\alpha). \quad (2.52)$$

In particular, the function  $\sigma_*^0$  satisfies (2.48) and (2.49), i.e.  $\sigma_*^0$  is a locally Lipschitz function on  $(0, \infty)^2$ .

*Second step:* The general case.

We denote  $\alpha^n = (\alpha_{1,n}, \alpha_{2,n})$  and  $\sigma_n^0(\alpha^n) = \sigma_n^0(\alpha_{1,n}, \alpha_{2,n})$ . Theorem 2.2 of [16] implies that there exists a subsequence of  $n$ , denoted by  $n'$ , such that  $\sigma_{n'}^0(\alpha^{n'})$   $H(\mathcal{M}(\Omega)^2)$ -converges to some  $\tilde{\sigma}_* \in \mathcal{M}(a_0, 2\|a\|_\infty; \Omega)$  in the sense of Definition 1.1.

As in the first step, for any  $\alpha^{n'} \in (0, \infty)^2$  and  $\lambda \in \mathbb{R}^2$ , we can consider the corrector  $w_{n'}^{\alpha^{n'}, \lambda}$  associated with  $\sigma_{n'}^0(\alpha^{n'})$  defined by

$$\begin{cases} \operatorname{div} \left( \sigma_{n'}^0(\alpha^{n'}) \nabla w_{n'}^{\alpha^{n'}, \lambda} \right) = \operatorname{div}(\tilde{\sigma}_* \lambda) & \text{in } \Omega, \\ w_{n'}^{\alpha^{n'}, \lambda} = \lambda \cdot x & \text{on } \partial\Omega, \end{cases} \quad (2.53)$$

which depends linearly on  $\lambda$ . Proceeding as in the first step, we obtain like in (2.46), with  $\alpha = (\alpha_1, \alpha_2)$  the limit of  $\alpha^n$  according to (2.33),

$$\left( \sigma_{n'}^0(\alpha) - \sigma_{n'}^0(\alpha^{n'}) \right) \nabla w_{n'}^{\alpha^{n'}, \lambda} \cdot \nabla w_{n'}^{\alpha, \lambda} \rightharpoonup (\sigma_*^0(\alpha) - \tilde{\sigma}_*) \lambda \cdot \lambda \quad \text{weakly in } \mathcal{D}'(\Omega). \quad (2.54)$$

Moreover, by the energy bound (2.44), which also holds for  $\alpha^{n'}$ , we have, for any  $\varphi \in \mathcal{D}(\Omega)$ ,

$$\int_{\Omega} \left( \sigma_{n'}^0(\alpha) - \sigma_{n'}^0(\alpha^{n'}) \right) \nabla w_{n'}^{\alpha^{n'}, \lambda} \cdot \nabla w_{n'}^{\alpha, \lambda} \varphi \, dx \xrightarrow{n' \rightarrow \infty} 0.$$

This combined with (2.54), yields

$$\int_{\Omega} (\sigma_*^0(\alpha) - \tilde{\sigma}_*) \lambda \cdot \lambda \varphi \, dx = 0,$$

which implies that  $\sigma_*^0(\alpha) = \tilde{\sigma}_*$ . We conclude by a uniqueness argument.  $\square$

We can now obtain a result for (perturbed) non-symmetric conductivities. Then, we will use a Dykhne transformation to recover the symmetric case following the Milton approach [36] (pp. 61–65). This will allow us to apply Proposition 2.2.

**Theorem 2.1** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^2$  such that  $|\partial\Omega| = 0$ . Let  $\omega_n$ ,  $n \in \mathbb{N}$ , be a sequence of open subsets of  $\Omega$  and denote by  $\chi_n$  their characteristic function. We assume that  $\theta_n = |\omega_n| < 1$  converges to 0 and*

$$\frac{\chi_n}{\theta_n} \rightharpoonup a \in L^\infty(\Omega) \quad \text{weakly-* in } \mathcal{M}(\Omega). \quad (2.55)$$

Consider the conductivity defined by

$$\sigma_n(h) = (1 - \chi_n)\sigma_1(h) + \frac{\chi_n}{\theta_n}\sigma_2(h) \quad (2.56)$$

where for  $j = 1, 2$ ,  $\sigma_j(h) = \alpha_j + h\beta_j J \in \mathbb{C}$  with  $\alpha_1, \alpha_2 > 0$  and  $(\beta_1, \beta_2) \neq (0, 0)$ .

Then, there exists a subsequence of  $n$ , still denoted by  $n$ , and a locally Lipschitz function

$$\sigma_*^0 : (0, \infty)^2 \longrightarrow \mathcal{M}\left(\min(\alpha_1, \alpha_2), 2(|\sigma_1| + |\sigma_2| \|a\|_\infty); \Omega\right)$$

such that

$$\sigma_n(h) \xrightarrow{H(\mathcal{M}(\Omega)^2)} \sigma_*^0(\alpha_1, \alpha_2 + \alpha_2^{-1}\beta_2^2 h^2) + h\beta_1 J.$$

**Proof of Theorem 2.1.** We have

$$\forall \xi \in \mathbb{R}^2, \quad \sigma_n(h)\xi \cdot \xi = (1 - \chi_n)\alpha_1|\xi|^2 + \frac{\chi_n}{\theta_n}\alpha_2|\xi|^2 \geq \min(\alpha_1, \alpha_2)|\xi|^2 \quad \text{a.e. in } \Omega$$

and, by (2.55),

$$|\sigma_n(h)| = (1 - \chi_n)|\sigma_1(h)| + \frac{\chi_n}{\theta_n}|\sigma_2(h)| \rightharpoonup |\sigma_1(h)| + a|\sigma_2(h)| \in L^\infty(\Omega) \quad \text{weakly-* in } \mathcal{M}(\Omega).$$

In order to make a Dykhne transformation like in p.62 of [36], we consider two real coefficients  $a_n$  and  $b_n$  in such a way that

$$B_n := (a_n\sigma_n(h) + b_n J)(a_n I_2 + J\sigma_n(h))^{-1} = ((p_n\sigma_n(h) + q_n J)^{-1} + r_n J)^{-1}$$

is symmetric. An easy computation shows that the previous equality holds when

$$p_n := \frac{a_n^2}{a_n^2 + b_n}, \quad q_n := \frac{a_n b_n}{a_n^2 + b_n} \quad \text{and} \quad r_n := \frac{1}{a_n}.$$

On the one hand, the estimates (3.39) and (3.40) with  $\alpha_{2,n} = \theta_n^{-1}\alpha_2$ ,  $\beta_{2,n} = \theta_n^{-1}\beta_2$ , yield (note that they are independent of  $\chi_n$ )

$$p_n \underset{n \rightarrow \infty}{\sim} 1, \quad q_n \underset{n \rightarrow \infty}{\longrightarrow} -h\beta_1, \quad r_n \underset{n \rightarrow \infty}{\longrightarrow} 0 \quad \text{and} \quad \|r_n\sigma_n(h)\|_\infty \leq C(|\sigma_1(h)| + |\sigma_2(h)|). \quad (2.57)$$

On the other hand, as in Section 3.2, with Notation 1.1 and (3.34), we have

$$B_n = \sigma_n^0(\alpha'_{1,n}(h), \alpha'_{2,n}(h)), \quad (2.58)$$

where

$$\alpha'_{1,n}(h) = \frac{a_n(\alpha_1 + ih\beta_1) + ib_n}{a_n + i(\alpha_1 + ih\beta_1)} \quad \text{and} \quad \alpha'_{2,n}(h) = \frac{a_n(\alpha_2/\theta_n + ih\beta_2/\theta_n) + ib_n}{a_n + i(\alpha_2/\theta_n + ih\beta_2/\theta_n)}. \quad (2.59)$$

Hence, like in (3.41), we have

$$\lim_{n \rightarrow \infty} \alpha'_{1,n}(h) = \alpha_1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \theta_n \alpha'_{2,n}(h) = \alpha_2 + \alpha_2^{-1}\beta_2^2 h^2. \quad (2.60)$$

We can first apply Proposition 2.2 with the conditions (2.58) and (2.60) to have the  $H(\mathcal{M}(\Omega)^2)$ -convergence of  $B_n$ . Then, by virtue of Proposition 2.1, with (2.57) we get that

$$\sigma_n(h) \xrightarrow{H(\mathcal{M}(\Omega)^2)} \sigma_*^0(\alpha_1, \alpha_2 + \alpha_2^{-1}\beta_2^2 h^2) + h\beta_1 J.$$

□

### 3 A two-dimensional periodic medium

In this section we consider a sequence  $\Sigma_n$  of matrix valued functions (not necessarily symmetric) in  $L^\infty(\mathbb{R}^2)^{2 \times 2}$ , which satisfies the following assumptions:

1 .  $\Sigma_n$  is  $Y$ -periodic, where  $Y := (0, 1)^2$ , i.e.,

$$\forall n \in \mathbb{N}, \forall \kappa \in \mathbb{Z}^2, \quad \Sigma_n(\cdot + \kappa) = \Sigma_n(\cdot) \quad \text{a.e. in } \mathbb{R}^2, \quad (3.1)$$

2 .  $\Sigma_n$  is equi-coercive in  $\mathbb{R}^2$ , i.e.,

$$\exists \alpha > 0 \quad \text{such that} \quad \forall n \in \mathbb{N}, \forall \xi \in \mathbb{R}^2, \quad \Sigma_n \xi \cdot \xi \geq \alpha |\xi|^2 \quad \text{a.e. in } \mathbb{R}^2. \quad (3.2)$$

Let  $\varepsilon_n$  be a sequence of positive numbers which tends to 0. From the sequences  $\Sigma_n$  and  $\varepsilon_n$  we define the highly oscillating sequence of matrix-valued functions  $\sigma_n$  by

$$\sigma_n(x) = \Sigma_n \left( \frac{x}{\varepsilon_n} \right), \quad \text{a.e. } x \in \mathbb{R}^2. \quad (3.3)$$

By virtue of (3.1) and (3.2),  $\sigma_n$  is an equi-coercive sequence of  $\varepsilon_n$ -periodic matrix-valued functions in  $L^\infty(\mathbb{R}^2)^{2 \times 2}$ . For a fixed  $n \in \mathbb{N}$ , let  $(\sigma_n)_*$  be the constant matrix defined by

$$\forall \lambda, \mu \in \mathbb{R}^2, \quad (\sigma_n)_* \lambda \cdot \mu = \int_Y \Sigma_n \nabla W_n^\lambda \cdot \nabla W_n^\mu \, dy, \quad (3.4)$$

where, for any  $\lambda \in \mathbb{R}^2$ ,  $W_n^\lambda \in H_\#^1(Y)$ , the set of  $Y$ -periodic functions belonging to  $H_{loc}^1(\mathbb{R}^2)$ , is the solution of the auxiliary problem

$$\int_Y (W_n^\lambda - \lambda \cdot y) \, dy = 0 \quad \text{and} \quad \operatorname{div}(\Sigma_n \nabla W_n^\lambda) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2) \quad (3.5)$$

or equivalently

$$\begin{cases} \int_Y \Sigma_n \nabla W_n^\lambda \cdot \nabla \varphi \, dy = 0, & \forall \varphi \in H_\#^1(Y) \\ \int_Y (W_n^\lambda(y) - \lambda \cdot y) \, dy = 0. \end{cases} \quad (3.6)$$

Set

$$w_n^\lambda(x) := \varepsilon_n W_n^\lambda \left( \frac{x}{\varepsilon_n} \right), \quad \text{for } x \in \Omega, \quad (3.7)$$

and

$$w_n := (w_n^{e_1}, w_n^{e_2}) = (w_n^1, w_n^2). \quad (3.8)$$

#### 3.1 A uniform convergence result

**Theorem 3.1** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^2$  with a Lipschitz boundary. Consider a highly oscillating sequence of matrix-valued functions  $\sigma_n$  satisfying (3.1), (3.2), (3.3) and the constant matrix  $(\sigma_n)_*$  defined by (3.4). We assume that*

$$(\sigma_n)_* \longrightarrow \sigma_* \quad \text{in } \mathbb{R}^{2 \times 2}. \quad (3.9)$$

*Consider, for  $f \in H^{-1}(\Omega) \cap W^{-1,q}(\Omega)$  with  $q > 2$ , the solution  $u_n$  of the problem*

$$\mathcal{P}_n \begin{cases} -\operatorname{div}(\sigma_n \nabla u_n) & = f & \text{in } \Omega \\ u_n & = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.10)$$

Then,  $u_n$  converges uniformly to the solution  $u \in H_0^1(\Omega)$  of

$$\mathcal{P} \begin{cases} -\operatorname{div}(\sigma_* \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.11)$$

Moreover we have the corrector result, with the  $\varepsilon_n Y$ -periodic sequence  $w_n$  defined in (3.8):

$$\nabla u_n - \sum_{i=1}^2 \partial_i u \nabla w_n^i \longrightarrow 0 \quad \text{in } L^1(\Omega)^2. \quad (3.12)$$

**Remark 3.1** The first point of Theorem 3.1 is an extension to the non-symmetric case of the results of [13] and [15]. The uniform convergence of  $u_n$  is a straightforward consequence of Theorem 2.7 of [15] taking into account that in the present case  $\sigma_n \in L^\infty(\Omega)^{2 \times 2}$  for a fixed  $n$ . The fact that  $f \in W^{-1,q}(\Omega)$  with  $q > 2$  ensures the uniform convergence.

### Proof of Theorem 3.1.

Derivation of the limit problem  $\mathcal{P}$ .

We only have to show that  $u$  is the solution of  $\mathcal{P}$  in (3.11). We consider a corrector  $D\tilde{w}_n : \mathbb{R}^2 \longrightarrow \mathbb{R}^{2 \times 2}$  associated with  $\sigma_n^T$  defined by

$$\tilde{w}_n(x) := \varepsilon_n \widetilde{W}_n \left( \frac{x}{\varepsilon_n} \right) = \left( \varepsilon_n \widetilde{W}_n^1 \left( \frac{x}{\varepsilon_n} \right), \varepsilon_n \widetilde{W}_n^2 \left( \frac{x}{\varepsilon_n} \right) \right)$$

where for  $i = 1, 2$ ,  $\widetilde{W}_n^i \in H_{\#}^1(Y)$  is the solution of the auxiliary problem

$$\int_Y (\widetilde{W}_n^i - e_i \cdot x) \, dx = 0 \quad \text{and} \quad \operatorname{div} \left( \sigma_n^T \nabla \widetilde{W}_n^i \right) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \quad (3.13)$$

Again, thanks to Theorem 2.7 of [15],  $\tilde{w}_n$  converges uniformly to the identity in  $\Omega$  by the integral condition (3.13). Let  $\varphi \in \mathcal{D}(\Omega)$ . We have, using the Einstein convention, by integrating by parts and by the Schwarz theorem ( $\partial_{i,j}^2 \varphi = \partial_{j,i}^2 \varphi$ )

$$\begin{aligned} & \int_{\Omega} \sigma_n \nabla u_n \cdot \nabla (\varphi(\tilde{w}_n)) \, dx \\ &= \int_{\Omega} \nabla u_n \cdot \sigma_n^T \nabla \tilde{w}_n^i (\partial_i \varphi)(\tilde{w}_n) \, dx \\ &= \underbrace{\int_{\Omega} \sigma_n^T \nabla \tilde{w}_n^i \cdot \nabla (u_n \partial_i \varphi(\tilde{w}_n)) \, dx}_{=0} - \int_{\Omega} \sigma_n^T \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j \partial_{i,j}^2 \varphi(\tilde{w}_n) u_n \, dx \\ &= - \int_{\Omega} \sigma_n \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^i \partial_{i,i}^2 \varphi(\tilde{w}_n) u_n \, dx - \int_{\Omega} \sigma_n^T \nabla \tilde{w}_n^2 \cdot \nabla \tilde{w}_n^1 \partial_{2,1}^2 \varphi(\tilde{w}_n) u_n \, dx \\ & \quad - \int_{\Omega} \sigma_n^T \nabla \tilde{w}_n^1 \cdot \nabla \tilde{w}_n^2 \partial_{1,2}^2 \varphi(\tilde{w}_n) u_n \, dx \\ &= - \int_{\Omega} \sigma_n \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^i \partial_{i,i}^2 \varphi(\tilde{w}_n) u_n \, dx - \int_{\Omega} \sigma_n \nabla \tilde{w}_n^1 \cdot \nabla \tilde{w}_n^2 \partial_{1,2}^2 \varphi(\tilde{w}_n) u_n \, dx \\ & \quad - \int_{\Omega} \sigma_n^T \nabla \tilde{w}_n^1 \cdot \nabla \tilde{w}_n^2 \partial_{1,2}^2 \varphi(\tilde{w}_n) u_n \, dx \\ &= - \int_{\Omega} \sigma_n^s \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^i \partial_{i,i}^2 \varphi(\tilde{w}_n) u_n \, dx - 2 \int_{\Omega} \sigma_n^s \nabla \tilde{w}_n^1 \cdot \nabla \tilde{w}_n^2 \partial_{1,2}^2 \varphi(\tilde{w}_n) u_n \, dx. \end{aligned}$$

This leads us to the equality

$$\langle f, \varphi(\tilde{w}_n) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_{\Omega} \sigma_n \nabla u_n \cdot \nabla (\varphi(\tilde{w}_n)) \, dx = - \int_{\Omega} \sigma_n^s \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j \partial_{i,j}^2 \varphi(\tilde{w}_n) u_n \, dx. \quad (3.14)$$

To study the convergence of the last term of (3.14), we first show that  $\sigma_n^s \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j$  is bounded in  $L^1(\Omega)$ . We have, by periodicity and the Cauchy-Schwarz inequality

$$\begin{aligned} \int_{\Omega} |\sigma_n^s \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j| \, dx &= \int_{\Omega} |\Sigma_n^s \nabla \tilde{W}_n^i \cdot \nabla \tilde{W}_n^j| \left( \frac{x}{\varepsilon_n} \right) \, dx \\ &\leq C \int_Y |\Sigma_n^s \nabla \tilde{W}_n^i \cdot \nabla \tilde{W}_n^j| \, dx \\ &\leq C \sqrt{\int_Y |\Sigma_n^s \nabla \tilde{W}_n^i \cdot \nabla \tilde{W}_n^i| \, dx} \sqrt{\int_Y |\Sigma_n^s \nabla \tilde{W}_n^j \cdot \nabla \tilde{W}_n^j| \, dx} \\ &\leq C \sqrt{(\sigma_n)_* e_i \cdot e_i} \sqrt{(\sigma_n)_* e_j \cdot e_j} \end{aligned}$$

which is bounded by the hypothesis (3.9). Therefore,

$$\sigma_n^s \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j \text{ is bounded in } L^1(\Omega). \quad (3.15)$$

Due to the periodicity, we know that for  $i, j = 1, 2$ ,

$$2\sigma_n^s \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j = \sigma_n^T \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j + \sigma_n^T \nabla \tilde{w}_n^j \cdot \nabla \tilde{w}_n^i \rightharpoonup (\sigma_*)^T e_i \cdot e_j + (\sigma_*)^T e_j \cdot e_i = 2(\sigma_*)^s e_i \cdot e_j$$

weakly-\* in  $\mathcal{M}(\Omega)$ . Hence, we get that

$$\sigma_n^s \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^j \rightharpoonup (\sigma_*)^s e_i \cdot e_j \text{ weakly-* in } \mathcal{M}(\Omega). \quad (3.16)$$

Moreover,  $\partial_{i,j}^2 \varphi(\tilde{w}_n) u_n$  converges uniformly to  $\partial_{i,j}^2 \varphi u$ . Thus, by passing to the limit in (3.14), we have, again with the Einstein convention

$$\langle f, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = - \int_{\Omega} (\sigma_*)^s e_i \cdot e_j \partial_{i,j}^2 \varphi u \, dx = - \int_{\Omega} \sigma_* : \nabla^2 \varphi u \, dx.$$

Therefore, by integrating by parts and using  $\varphi = 0$  on  $\partial\Omega$ ,

$$\int_{\Omega} \sigma_* \nabla u \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \quad (3.17)$$

*Proof of the corrector result*

First of all, we show that the corrector function  $w_n$  is bounded in  $H^1(\Omega)^2$ . By the definition (3.8) of  $w_n$ , the  $Y$ -periodicity of  $W_n^{e_i}$  and the equi-coercivity of  $\Sigma_n$ , we have, for  $i = 1, 2$ ,

$$\alpha \|\nabla w_n^i\|_{L^2(\Omega)^2}^2 \leq C\alpha \|\nabla W_n^{e_i}\|_{L^2(Y)^2}^2 \leq C \int_Y \Sigma_n \nabla W_n^i \cdot \nabla W_n^i \, dx = C (\sigma_n)_* e_i \cdot e_i \quad (3.18)$$

which is bounded. This inequality combined with the uniform convergence of  $w_n$  yields to the boundedness of  $w_n$  in  $H^1(\Omega)^2$ .

Let us consider an approximation  $u^\delta \in \mathcal{D}(\Omega)$  of  $u$  such that

$$\|u - u^\delta\|_{H_0^1(\Omega)} \leq \delta. \quad (3.19)$$

On the one hand, we have

$$\int_{\Omega} \sigma_n \nabla u_n \cdot \nabla (u_n - u^\delta(w_n)) \, dx = \langle f, (u_n - u^\delta(w_n)) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Since  $w_n$  converges uniformly to identity on  $\Omega$  and is bounded in  $H^1(\Omega)$  (see (3.18)), with  $u^\delta \in \mathcal{D}(\Omega)$ ,  $u^\delta(w_n)$  converges weakly to  $u^\delta$  in  $H_0^1(\Omega)$ . Hence, by the weak convergence of  $u_n$  to  $u$  in  $H_0^1(\Omega)$  and (3.19), we can pass to the limit the previous inequality and obtain, for any  $\delta > 0$ ,

$$\limsup_{n \rightarrow \infty} \left| \int_{\Omega} \sigma_n \nabla u_n \cdot \nabla (u_n - u^\delta(w_n)) \, dx \right| = \left| \langle f, u - u^\delta \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \right| \leq C\delta. \quad (3.20)$$

On the other hand, similarly to the proof of the first point (3.14), we are led to the equality

$$\int_{\Omega} \sigma_n \nabla(u^\delta(w_n)) \cdot \nabla(u_n - u^\delta(w_n)) \, dx = - \int_{\Omega} \sigma_n^s \nabla w_n^i \cdot \nabla w_n^j \, \partial_{i,j}^2 u^\delta(w_n) (u_n - u^\delta(w_n)) \, dx. \quad (3.21)$$

As in the first point,  $\sigma_n^s \nabla w_n^i \cdot \nabla w_n^j$  is bounded in  $L^1(\Omega)$  (see (3.15)),  $u_n$  converges uniformly to  $u$  and  $\partial_{i,j} u^\delta(w_n)$  converges uniformly to  $\partial_{i,j} u^\delta$  because  $u^\delta$  is a  $\mathcal{D}(\Omega)$  function. By passing to the limit in (3.21)

$$\int_{\Omega} \sigma_n \nabla(u^\delta(w_n)) \cdot \nabla(u_n - u^\delta(w_n)) \, dx \xrightarrow{n \rightarrow \infty} - \int_{\Omega} (\sigma_*)^s e_i \cdot e_j \, \partial_{i,j}^2 u^\delta (u - u^\delta) \, dx. \quad (3.22)$$

Moreover, like in (3.17) we have

$$\int_{\Omega} (\sigma_*)^s e_i \cdot e_j \, \partial_{i,j}^2 u^\delta (u - u^\delta) \, dx = \int_{\Omega} \sigma_* \nabla u^\delta \cdot \nabla(u - u^\delta) \, dx. \quad (3.23)$$

By combining this equality with the convergence (3.22), we obtain the inequality

$$\lim_{n \rightarrow \infty} \left| \int_{\Omega} \sigma_n \nabla(u^\delta(w_n)) \cdot \nabla(u_n - u^\delta(w_n)) \, dx \right| \leq \left| \int_{\Omega} \sigma_* \nabla u^\delta \cdot \nabla(u - u^\delta) \right| \quad (3.24)$$

$$\leq C |\sigma_*| \|\nabla u^\delta\|_{L^2(\Omega)^2} \|\nabla(u - u^\delta)\|_{L^2(\Omega)^2} \leq C\delta. \quad (3.25)$$

Thus, by adding (3.20) and (3.25), we have

$$\limsup_{n \rightarrow \infty} \int_{\Omega} \sigma_n \nabla(u_n - u^\delta(w_n)) \cdot \nabla(u_n - u^\delta(w_n)) \, dx \leq C\delta$$

which leads us, by equi-coercivity, to

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \alpha \|\nabla(u_n - u^\delta(w_n))\|_{L^2(\Omega)^2}^2 \\ & \leq \limsup_{n \rightarrow \infty} \left| \int_{\Omega} \sigma_n \nabla(u_n - u^\delta(w_n)) \cdot \nabla(u_n - u^\delta(w_n)) \, dx \right| \leq C\delta. \end{aligned} \quad (3.26)$$

Thus, the Cauchy-Schwarz inequality, the boundedness of  $\nabla w_n^i$  in  $L^2(\Omega)^2$  (3.18) and the Einstein convention give, for any  $\delta > 0$ ,

$$\begin{aligned} & \|\nabla u_n - \nabla w_n^i \partial_i u\|_{L^1(\Omega)^2} \\ & \leq \|\nabla u_n - \nabla w_n^i \partial_i u^\delta\|_{L^1(\Omega)^2} + \|\nabla w_n^i \partial_i (u^\delta - u)\|_{L^1(\Omega)^2} \\ & \leq \|\nabla u_n - \nabla w_n^i \partial_i u^\delta\|_{L^1(\Omega)^2} + \|\nabla w_n^i\|_{L^2(\Omega)^2} \|\partial_i (u^\delta - u)\|_{L^2(\Omega)} \\ & \leq \|\nabla u_n - \nabla w_n^i \partial_i u^\delta\|_{L^1(\Omega)^2} + C\delta \\ & \leq \|\nabla u_n - \nabla w_n^i \partial_i u^\delta(w_n)\|_{L^1(\Omega)^2} + \|\nabla w_n^i (\partial_i u^\delta - \partial_i u^\delta(w_n))\|_{L^1(\Omega)^2} + C\delta \\ & \leq \|\nabla u_n - \nabla w_n^i \partial_i u^\delta(w_n)\|_{L^1(\Omega)^2} + \|\nabla w_n^i\|_{L^2(\Omega)^2} \|\partial_i u^\delta - \partial_i u^\delta(w_n)\|_{L^2(\Omega)} + C\delta \\ & \leq \|\nabla u_n - \nabla w_n^i \partial_i u^\delta(w_n)\|_{L^1(\Omega)^2} + C \|\partial_i u^\delta - \partial_i u^\delta(w_n)\|_{L^2(\Omega)} + C\delta. \end{aligned}$$

Since  $u^\delta \in \mathcal{D}(\Omega)$  and  $w_n$  converges uniformly to the identity on  $\Omega$ , the second term of the last inequality converges to 0. Hence, we get that

$$\limsup_{n \rightarrow \infty} \|\nabla u_n - \nabla w_n^i \partial_i u\|_{L^1(\Omega)^2} \leq \limsup_{n \rightarrow \infty} \|\nabla u_n - \nabla w_n^i \partial_i u^\delta(w_n)\|_{L^1(\Omega)^2} + C\delta. \quad (3.27)$$

Finally, this inequality combined with (3.26) gives, for any  $\delta > 0$ ,

$$0 \leq \limsup_{n \rightarrow \infty} \|\nabla u_n - \nabla w_n^i \partial_i u\|_{L^1(\Omega)^2} \leq C\sqrt{\delta} + C\delta,$$

which implies the corrector result (3.12).  $\square$

**Remark 3.2** If the solution  $u$  is a  $\mathcal{C}^2$  function, then the convergence (3.12) holds true in  $L_{loc}^2(\Omega)$  since we may take  $u = u^\delta$ .

### 3.2 A two-phase result

Here, we recall a two-phase result due to G.W. Milton (see [36] pp. 61–65) using the Dykhne transformation.

In order to apply the previous theorem, we reformulate Milton's calculus in such a way that every coefficient depends on  $n$ . We then consider, for a fixed  $n$ , the periodic homogenization of a conductivity  $\sigma_n(h)$  to obtain  $(\sigma_n)_*(h)$  through the link between the homogenization of the transformed conductivity and  $(\sigma_n)_*(h)$  given by formula (4.16) in [36]. Finally, we study the limit of  $(\sigma_n)_*(h)$  through the asymptotic behavior of the coefficients of the transformation, and apply Theorem 3.1 in the example Section 3.3.

In this section we consider a two-phase periodic isotropic medium. Let  $\chi_n$  be a sequence of characteristic functions of subsets of  $Y$ . We define for any  $\alpha_1 > 0$ ,  $\beta_1 \in \mathbb{R}$ , any sequences  $\alpha_{2,n} > 0$ ,  $\beta_{2,n} \in \mathbb{R}$  and any  $h \in \mathbb{R}$ , a parametrized conductivity  $\Sigma_n(h)$ :

$$\Sigma_n(h) = (1 - \chi_n)(\alpha_1 I_2 + h\beta_1 J) + \chi_n(\alpha_{2,n} I_2 + h\beta_{2,n} J) \quad \text{in } Y. \quad (3.28)$$

We still denote by  $\Sigma_n(h)$  the periodic extension to  $\mathbb{R}^2$  of  $\Sigma_n(h)$  (which satisfies (3.1)). We assume that  $\Sigma_n(h)$  satisfies (3.2), and define  $\sigma_n(h)$  by (3.3) and  $(\sigma_n)_*(h)$  by (3.4).

We have the following result based on an analysis of [36] (pp. 61–65).

**Proposition 3.1** *Let  $\chi_n$  be a sequence of characteristic functions of subsets of  $Y$ ,  $\alpha_1, \alpha_2 > 0$ , a positive sequence  $\alpha_{2,n}$ ,  $\beta_1, \beta_2 \in \mathbb{R}$ , and a sequence  $\beta_{2,n}$  such that*

$$\lim_{n \rightarrow \infty} \alpha_{2,n} = \infty, \quad \liminf_{n \rightarrow \infty} |\beta_{2,n} - \beta_1| > 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\beta_{2,n}}{\alpha_{2,n}} = \frac{\beta_2}{\alpha_2}. \quad (3.29)$$

*Assume that the effective conductivity in the absence of a magnetic field*

$$(\sigma_n^0)_*(\gamma_{1,n}, \gamma_{2,n}) \quad \text{is bounded when} \quad \lim_{n \rightarrow \infty} \gamma_{1,n} = \alpha_1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\gamma_{2,n}}{\alpha_{2,n}} = \gamma_2 > 0. \quad (3.30)$$

*Then, there exist two parametrized positive sequences  $\alpha'_{1,n}(h), \alpha'_{2,n}(h)$  such that*

$$\lim_{n \rightarrow \infty} \alpha'_{1,n}(h) = \alpha_1 \quad \text{and} \quad \alpha'_{2,n}(h) \underset{n \rightarrow \infty}{\sim} \frac{\alpha_2^2 + h^2 \beta_2^2}{\alpha_2^2} \alpha_{2,n}, \quad (3.31)$$

*and*

$$(\sigma_n)_*(h) = (\sigma_n^0)_*(\alpha'_{1,n}(h), \alpha'_{2,n}(h)) + h\beta_1 J + \underset{n \rightarrow \infty}{o}(1) \quad (3.32)$$

*where  $(\sigma_n^0)_*(\alpha'_{1,n}(h), \alpha'_{2,n}(h))$  is bounded.*

**Remark 3.3** *In view of condition (3.29), the case where  $\beta_{2,n}$  tends to  $\beta_1$  corresponds to perturb the symmetric conductivity*

$$\sigma_n^s = (1 - \chi_n)\alpha_1 I_2 + \chi_n \alpha_{2,n} I_2$$

*by*

$$\sigma_n^s + \beta_1 J + \underset{n \rightarrow \infty}{o}(1).$$

*Then it is clear that*

$$(\sigma_n)_*(h) = (\sigma_n^s)_* + \beta_1 J + \underset{n \rightarrow \infty}{o}(1).$$

**Proof of Proposition 3.1.** The proof is divided into two parts. After applying Milton's computation (pp. 61–64 of [36]), we study the asymptotic behavior of the different coefficients.

*First step:* Applying Dykhne's transformation through Milton's computations.

In order to make the Dykhne's transformation following Milton [36] (pp. 62–64), we consider two real coefficients  $a_n$  and  $b_n$  such that

$$\sigma'_n := (a_n \sigma_n(h) + b_n J)(a_n I_2 + J \sigma_n(h))^{-1} = a_n (\sigma_n(h) + (a_n)^{-1} b_n J)(a_n I_2 + J \sigma_n(h))^{-1} \quad (3.33)$$

is symmetric and, more precisely, according to Notation 1.1, reads as

$$\sigma'_n = (1 - \chi_n) \alpha'_{1,n}(h) I_2 + \chi_n \alpha'_{2,n}(h) I_2 = \sigma_n^0(\alpha'_{1,n}(h), \alpha'_{2,n}(h)). \quad (3.34)$$

Then, using the complex representation

$$\alpha I_2 + \beta J \longleftrightarrow \alpha + \beta i \quad (3.35)$$

suggested by Tartar in [20], the constants  $a_n, b_n$  must satisfy

$$\alpha'_{1,n}(h) = \frac{a_n(\alpha_1 + ih\beta_1) + ib_n}{a_n + i(\alpha_1 + ih\beta_1)} \in \mathbb{R} \quad \text{and} \quad \alpha'_{2,n}(h) = \frac{a_n(\alpha_{2,n} + ih\beta_{2,n}) + ib_n}{a_n + i(\alpha_{2,n} + ih\beta_{2,n})} \in \mathbb{R}, \quad (3.36)$$

which implies that

$$b_n = \frac{-a_n^2 h \beta_1 + a_n \Delta_1}{a_n - h \beta_1} = \frac{-a_n^2 h \beta_{2,n} + a_n \Delta_{2,n}}{a_n - h \beta_{2,n}}. \quad (3.37)$$

Denoting  $\Delta_1 := \alpha_1^2 + h^2 \beta_1^2$  and  $\Delta_{2,n} := \alpha_{2,n}^2 + h^2 \beta_{2,n}^2$  (thanks to (3.29),  $n$  is considered to be larger enough such that  $\beta_{2,n} - \beta_1 \neq 0$  and  $a_n$  is real), the equality (3.37) provides two non-zero solutions for  $a_n$ :

$$a_n = \frac{\Delta_{2,n} - \Delta_1 + \sqrt{(\Delta_{2,n} - \Delta_1)^2 + 4h^2(\beta_{2,n} - \beta_1)(\beta_{2,n}\Delta_1 - \beta_1\Delta_{2,n})}}{2h(\beta_{2,n} - \beta_1)}, \quad (3.38)$$

and

$$a_n^- = \frac{\Delta_{2,n} - \Delta_1 - \sqrt{(\Delta_{2,n} - \Delta_1)^2 + 4h^2(\beta_{2,n} - \beta_1)(\beta_{2,n}\Delta_1 - \beta_1\Delta_{2,n})}}{2h(\beta_{2,n} - \beta_1)}.$$

The value (3.38) is associated with a positive matrix  $\sigma'_n$ , while  $a_n^-$  leads us to the negative matrix  $\sigma_n^- = -J(\sigma'_n)^{-1}J^{-1}$  to exclude (see [35] for more details).

*Second step:* asymptotic behavior of the coefficients and the homogenized matrix.

One the one hand, by the equality (3.38) combined with (3.29), we have

$$\lim_{n \rightarrow \infty} a_n \frac{h(\beta_{2,n} - \beta_1)}{\alpha_{2,n}^2} = \frac{\alpha_2^2 + h^2 \beta_2^2}{\alpha_2^2}$$

which clearly implies that

$$a_n \underset{n \rightarrow \infty}{\sim} \frac{\alpha_2^2 + h^2 \beta_2^2}{\alpha_2^2} \frac{\alpha_{2,n}^2}{h(\beta_{2,n} - \beta_1)} \quad \text{and} \quad a_n - h\beta_{2,n} \underset{n \rightarrow \infty}{\sim} \frac{\alpha_{2,n}^2}{h(\beta_{2,n} - \beta_1)}. \quad (3.39)$$

On the other hand, (3.29), (3.39) and the first equality of (3.37) give

$$b_n = -a_n h \beta_1 + \Delta_1 + \underset{n \rightarrow \infty}{o}(1). \quad (3.40)$$

From (3.29), (3.38), (3.39) and (3.40) we deduce the following asymptotic behavior for the modified phases:

$$\lim_{n \rightarrow \infty} \alpha'_{1,n}(h) = \alpha_1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\alpha'_{2,n}(h)}{\alpha_{2,n}} = \frac{\alpha_2^2 + h^2 \beta_2^2}{\alpha_2^2}. \quad (3.41)$$

To consider  $(\sigma'_n)_*$ , we need to verify that  $\sigma'_n$  is equi-coercive. We have, by denoting for any  $\xi \in \mathbb{R}^2$ ,  $\nu_n = (a_n I_2 + J \sigma_n(h))^{-1} \xi$ ,

$$\forall \xi \in \mathbb{R}^2, \quad \sigma'_n \xi \cdot \xi = (a_n \sigma_n(h) + b_n J) \nu_n \cdot (a_n I_2 + J \sigma_n(h)) \nu_n = (a_n^2 + b_n) \sigma_n(h) \nu_n \cdot \nu_n$$

and, because  $a_n^{-1}\sigma_n(h)$  is bounded in  $L^\infty(\Omega)^{2 \times 2}$  by (3.39),

$$\forall \xi \in \mathbb{R}^2, \quad |\xi| = |a_n \nu_n + J \sigma_n(h) \nu_n| \leq a_n(1+C)|\nu_n|.$$

The equi-coercivity of  $\sigma_n(h)$  gives

$$\exists C > 0, \quad \forall \xi \in \mathbb{R}^2, \quad \sigma'_n \xi \cdot \xi \geq \frac{C}{(1+C)^2} \frac{a_n^2 + b_n}{a_n^2} |\xi|^2 \quad (3.42)$$

that is, for  $n$  larger enough, by (3.39) and (3.40),  $\sigma'_n$  is equi-coercive.

We can now apply the Keller-Dykhne duality theorem (see, e.g., [31, 24]) to equality (3.33) to obtain

$$(\sigma'_n)_* = (a_n(\sigma_n)_* + b_n J)(a_n I_2 + J(\sigma_n)_*)^{-1}. \quad (3.43)$$

Moreover, by inverting this transformation, we have

$$(\sigma_n)_*(h) = (a_n I_2 - (\sigma'_n)_* J)^{-1} (a_n (\sigma'_n)_* - b_n J).$$

Considering (3.29), (3.39), (3.40), and the boundedness of  $(\sigma'_n)_*$  (as a consequence of the bound (3.30)) we get that

$$(\sigma_n)_*(h) = (\sigma'_n)_* - \frac{b_n}{a_n} J + \underset{n \rightarrow \infty}{o}(1) = (\sigma'_n)_* + h \beta_1 J + \underset{n \rightarrow \infty}{o}(1), \quad (3.44)$$

which concludes the proof taking into account (3.34).  $\square$

To derive the limit of  $(\sigma_n^0)_*(\alpha'_{1,n}(h), \alpha'_{2,n}(h))$ , we need more information on the geometry of the high conductive phase. To this end, we study the following example.

### 3.3 A cross-like thin structure

We consider a bounded open subset  $\Omega$  of  $\mathbb{R}^2$  with a Lipschitz boundary, a real sequence  $\varepsilon_n$  converging to 0, and  $f \in H^{-1}(\Omega) \cap W^{-1,q}(\Omega)$  with  $q > 2$ . We define, for any  $h \in \mathbb{R}$ ,  $\alpha_1, \beta_1 > 0$  and positive sequences  $t_n \in (0, 1/2]$ ,  $\alpha_{2,n}, \beta_{2,n}$ , a parametrized matrix-valued function  $\Sigma_n(h)$  from the unit rectangular cell period  $Y := (-\frac{\ell}{2}, \frac{\ell}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ , with  $\ell \geq 1$ , to  $\mathbb{R}^{2 \times 2}$ , by (cf. figure 3.1)

$$\Sigma_n(h) := \begin{cases} \alpha_{2,n} I_2 + \beta_{2,n} h J & \text{in } \omega_n := \{(x_1, x_2) \in Y \mid |x_1|, |x_2| \leq t_n\} \\ \alpha_1 I_2 + \beta_1 h J & \text{in } Y \setminus \omega_n \end{cases} \quad (3.45)$$

Denoting again by  $\Sigma_n(h)$  its periodic extension to  $\mathbb{R}^2$ , we finally consider the conductivity

$$\sigma_n(h)(x) = \Sigma_n(h) \begin{pmatrix} x \\ \varepsilon_n \end{pmatrix}, \quad x \in \Omega, \quad (3.46)$$

and the associated homogenization problem:

$$\mathcal{P}_n \begin{cases} -\operatorname{div}(\sigma_n(h) \nabla u_n) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.47)$$

By virtue of Theorem 3.1 and Proposition 3.1, we focus on the study of the limit of  $(\sigma_n^0)_*(\alpha'_{1,n}(h), \alpha'_{2,n}(h))$ .

**Proposition 3.2** *Let  $\sigma_n(h)$  be the conductivity defined by (3.45) and (3.46) and its homogenization problem (3.47). We assume that:*

$$2t_n(\ell+1)\alpha_{2,n} \xrightarrow{n \rightarrow \infty} \alpha_2 > 0 \quad \text{and} \quad 2t_n(\ell+1)\beta_{2,n} \xrightarrow{n \rightarrow \infty} \beta_2 > 0. \quad (3.48)$$

*Then, the homogenized conductivity is given by*

$$\sigma_*(h) = \begin{pmatrix} \alpha_1 + \frac{\alpha_2^2 + \beta_2^2 h^2}{(\ell+1)\alpha_2} & -h\beta_1 \\ h\beta_1 & \alpha_1 + \frac{\alpha_2^2 + \beta_2^2 h^2}{\ell(\ell+1)\alpha_2} \end{pmatrix}.$$

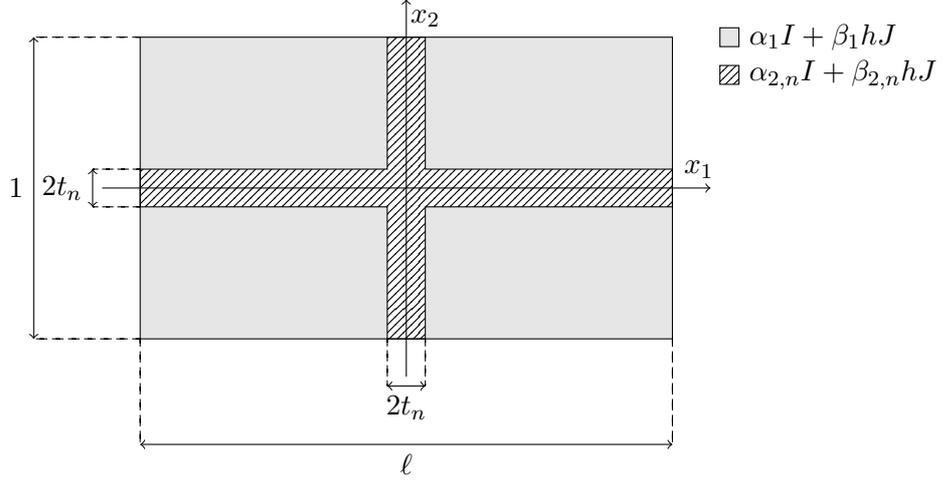


Figure 3.1: The period of the cross-like thin structure

**Remark 3.4** *The previous proposition does not respect exactly the framework defined at the beginning of this section because the period cell is not the unit square  $Y = (0, 1)^2$ : we can nevertheless extend all this section to any type of period cells.*

**Remark 3.5** *The condition (3.48) is a condition of boundedness in  $L^1(\Omega)^{2 \times 2}$  of  $\sigma_n$  because*

$$|\omega_n| = 2t_n(\ell + 1) - 4t_n^2 \sim 2t_n(\ell + 1),$$

*which will ensure the convergence of  $(\sigma_n^0)_*$ .*

**Proof of Proposition 3.2.** In order to apply Proposition 3.1, we consider two positive sequences  $\alpha'_{1,n}(h), \alpha'_{2,n}(h)$  satisfying

$$\lim_{n \rightarrow \infty} \alpha'_{1,n}(h) = \alpha_1 \quad \text{and} \quad \alpha'_{2,n}(h) \underset{n \rightarrow \infty}{\sim} \frac{\alpha_2^2 + h^2 \beta_2^2}{\alpha_2^2} \alpha_{2,n}. \quad (3.49)$$

We will study the homogenization of  $\sigma'_n := \sigma_n^0(\alpha'_{1,n}(h), \alpha'_{2,n}(h))$ .

To this end, consider a corrector  $W_n^\lambda = \lambda \cdot x - X_n^\lambda$  in the Murat-Tartar sense (see, e.g., [39]) associated with

$$\Sigma'_n := \begin{cases} \alpha'_{2,n}(h) I_2 & \text{in } \omega_n = \{(x_1, x_2) \in Y \mid |x_1|, |x_2| \leq t_n\} \\ \alpha'_{1,n}(h) I_2 & \text{in } Y \setminus \omega_n \end{cases} \quad (3.50)$$

and defined by

$$\begin{cases} \operatorname{div}(\Sigma'_n \nabla X_n^\lambda) = \operatorname{div}(\Sigma'_n \lambda) & \text{in } \mathcal{D}'(\mathbb{R}^2) \\ X_n^\lambda & \text{is } Y\text{-periodic} \\ \int_Y X_n^\lambda \, dy = 0. \end{cases} \quad (3.51)$$

On one hand, the extra diagonal coefficients of  $(\sigma'_n)_*$  are equal to 0 because, as  $\Sigma'_n$  is an even function on  $Y$ , we have, for  $i = 1, 2$ ,

$$\begin{cases} y_i \mapsto W_n^{e_i}(y) & \text{is an odd function,} \\ y_i \mapsto W_n^{e_j}(y) & \text{is an even function for } i \neq j, \end{cases}$$

which implies that  $y_1 \mapsto \Sigma'_n \nabla W_n^{e_1} \cdot \nabla W_n^{e_2}$  is an odd function. Then, by symmetry of  $Y$  with respect to 0,

$$(\sigma'_n)_* e_i \cdot e_j = \int_Y \Sigma'_n \nabla W_n^{e_i} \cdot \nabla W_n^{e_j} \, dy = 0.$$

On the other hand, as  $\Sigma'_n$  is isotropic, for the diagonal coefficients, we use the Voigt-Reuss inequalities (see, e.g., [30] p.44 or [37]): for any  $i = 1, 2$  and  $j \neq i$ ,

$$\langle \langle (\Sigma'_n e_i \cdot e_i)^{-1} \rangle_i^{-1} \rangle_j \leq (\sigma'_n)_* e_i \cdot e_i \leq \langle \langle (\Sigma'_n e_i \cdot e_i)^{-1} \rangle_j^{-1} \rangle_i \quad (3.52)$$

where  $\langle \cdot \rangle_i$  denotes the average with respect to  $y_i$  at a fixed  $y_j$  for  $j \neq i$ .

An easy computation gives, for the direction  $e_1$ ,

$$(1 - 2t_n) \left( \frac{\ell - 2t_n}{\ell \alpha'_{1,n}(h)} + \frac{2t_n}{\ell \alpha'_{2,n}(h)} \right)^{-1} + 2t_n \left( \frac{\ell}{\ell \alpha'_{2,n}(h)} \right)^{-1} \leq (\sigma'_n)_* e_1 \cdot e_1$$

and

$$(\sigma'_n)_* e_1 \cdot e_1 \leq \ell \left( \frac{\ell - 2t_n}{(1 - 2t_n) \alpha'_{1,n}(h) + 2t_n \alpha'_{2,n}(h)} + \frac{2t_n}{\alpha'_{2,n}(h)} \right)^{-1}.$$

By (3.48) and (3.49), we have the convergence

$$\lim_{n \rightarrow \infty} (\sigma'_n)_* e_1 \cdot e_1 = \alpha_1 + \frac{\alpha_2^2 + \beta_2^2 h^2}{(\ell + 1) \alpha_2}.$$

A similar computation on the direction  $e_2$  gives the asymptotic behavior:

$$\lim_{n \rightarrow \infty} (\sigma'_n)_* = \lim_{n \rightarrow \infty} (\sigma_n^0)_* (\alpha'_{1,n}(h), \alpha'_{2,n}(h)) = \begin{pmatrix} \alpha_1 + \frac{\alpha_2^2 + \beta_2^2 h^2}{(\ell + 1) \alpha_2} & 0 \\ 0 & \alpha_1 + \frac{\alpha_2^2 + \beta_2^2 h^2}{\ell(\ell + 1) \alpha_2} \end{pmatrix}. \quad (3.53)$$

Moreover, the matrix  $\sigma_n(h)$  clearly satisfies all the hypothesis of Theorem 3.1. By Theorem 3.1 and (3.53), we have

$$\lim_{n \rightarrow \infty} (\sigma_n)_*(h) = \lim_{n \rightarrow \infty} (\sigma_n^0)_* (\alpha'_{1,n}(h), \alpha'_{2,n}(h)) + \beta_1 h J = \begin{pmatrix} \alpha_1 + \frac{\alpha_2^2 + \beta_2^2 h^2}{(\ell + 1) \alpha_2} & -h \beta_1 \\ h \beta_1 & \alpha_1 + \frac{\alpha_2^2 + \beta_2^2 h^2}{\ell(\ell + 1) \alpha_2} \end{pmatrix}.$$

We finally apply Theorem 3.1 to get that  $\sigma_*(h) = \lim_{n \rightarrow \infty} (\sigma_n)_*(h)$ .  $\square$

## 4 A three-dimensional fibered microstructure

In this section we study a particular two-phase composite in dimension three. One of the phases is composed by a periodic set of high conductivity fibers embedded in an isotropic medium (figure 4.1a). The conductivity  $\sigma_n(h)$  is not symmetric due to the perturbation of a magnetic field.

First, describe the geometry of the microstructure. Let  $Y := (-\frac{1}{2}, \frac{1}{2})^3$  be the unit cube centered at the origin of  $\mathbb{R}^3$ . For  $r_n \in (0, \frac{1}{2})$ , consider the closed cylinder  $\omega_n$  parallel to the  $x_3$ -axis, of radius  $r_n$  and centered in  $Y$ :

$$\omega_n := \{y \in Y \mid y_1^2 + y_2^2 \leq r_n^2\}. \quad (4.1)$$

Let  $\Omega = \tilde{\Omega} \times (0, 1)$  be an open cylinder of  $\mathbb{R}^3$ , where  $\tilde{\Omega}$  is a bounded domain of  $\mathbb{R}^2$  with a Lipschitz boundary. For  $\varepsilon_n \in (0, 1)$ , consider the closed subset  $\Omega_n$  of  $\Omega$  defined by the intersection with  $\Omega$  of the  $\varepsilon_n Y$ -periodic network in  $\mathbb{R}^3$  composed by the closed cylinders parallel to the  $x_3$ -axis, centered on the points  $\varepsilon_n k$ ,  $k \in \mathbb{Z}^2$ , in the  $x_1$ - $x_2$  plane, and of radius  $\varepsilon_n r_n$ , namely:

$$\Omega_n := \Omega \cap \bigcup_{\nu \in \mathbb{Z}^3} \varepsilon_n (\omega_n + \nu). \quad (4.2)$$

The period cell of the microstructure is represented in figure 4.1b.

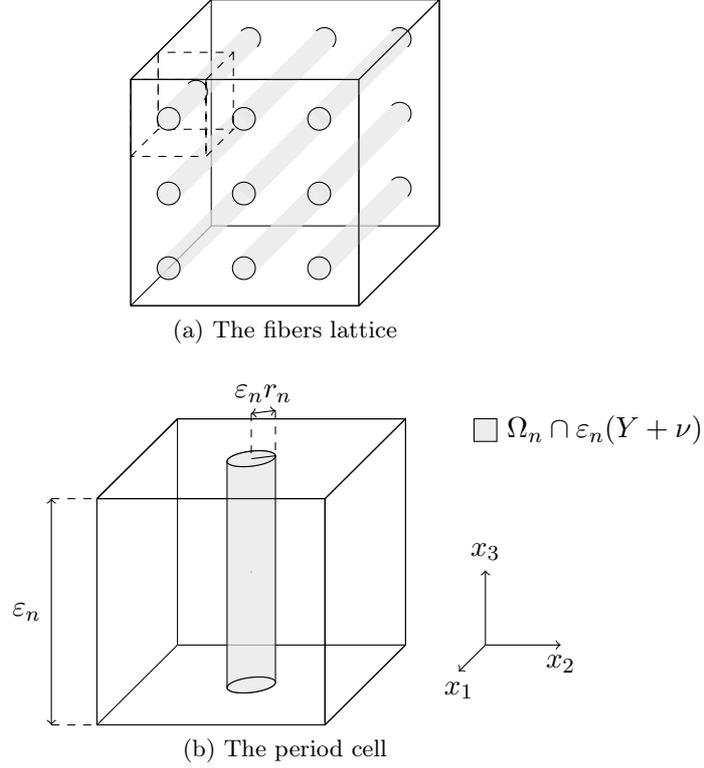


Figure 4.1: The fibered structure in dimension 3

We then define the two-phase conductivity by

$$\sigma_n(h) = \begin{cases} \alpha_1 I_3 + \beta_1 \mathcal{E}(h) & \text{in } \Omega \setminus \Omega_n \\ \alpha_{2,n} I_3 + \beta_{2,n} \mathcal{E}(h) & \text{in } \Omega_n, \end{cases} \quad (4.3)$$

where  $\alpha_1 > 0$ ,  $\beta_1 \in \mathbb{R}$ ,  $\alpha_{2,n} > 0$  and  $\beta_{2,n}$  are real sequences, and

$$\mathcal{E}(h) := \begin{pmatrix} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ -h_2 & h_1 & 0 \end{pmatrix}, \quad \text{for } h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \in \mathbb{R}^3.$$

Our aim is to study the homogenization problem

$$\mathcal{P}_{\Omega,n} \begin{cases} -\operatorname{div}(\sigma_n(h)\nabla u_n) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4)$$

**Theorem 4.1** *Let  $\alpha_1 > 0$ ,  $\beta_1 \in \mathbb{R}$ , and let  $\varepsilon_n, r_n, \alpha_{2,n}, \beta_{2,n}$ ,  $n \in \mathbb{N}$ , be real sequences such that  $\varepsilon_n, r_n > 0$  converge to 0,  $\alpha_{2,n} > 0$ , and*

$$\lim_{n \rightarrow \infty} \varepsilon_n^2 |\ln r_n| = 0, \quad \lim_{n \rightarrow \infty} |\omega_n| \alpha_{2,n} = \alpha_2 > 0, \quad \lim_{n \rightarrow \infty} |\omega_n| \beta_{2,n} = \beta_2 \in \mathbb{R}. \quad (4.5)$$

*Consider, for  $h \in \mathbb{R}^3$ , the conductivity  $\sigma_n(h)$  defined by (4.3).*

*Then, there exists a subsequence of  $n$ , still denoted by  $n$ , such that, for any  $f \in H^{-1}(\Omega)$  and any  $h \in \mathbb{R}^3$ , the solution  $u_n$  of  $\mathcal{P}_{\Omega,n}$  converges weakly in  $H_0^1(\Omega)$  to the solution  $u$  of*

$$\mathcal{P}_{\Omega,*} \begin{cases} -\operatorname{div}(\sigma_*(h)\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.6)$$

where  $\sigma_*(h)$  is given by

$$\sigma_*(h) = \alpha_1 I_3 + \left( \frac{\alpha_2^3 + \alpha_2 \beta_2^2 |h|^2}{\alpha_2^2 + \beta_2^2 h_3^2} \right) e_3 \otimes e_3 + \beta_1 \mathcal{E}(h). \quad (4.7)$$

**Remark 4.1** Theorem 4.1 can be actually extended to fibers with a more general cross-section. More precisely, we can replace the disk  $r_n D$  of radius  $r_n$  by the homothetic  $r_n Q$  of any connected open set  $Q$  included in the unit disk  $D$ , such that the present fiber  $\omega_n$  is replaced by the new fiber  $r_n Q \times (-\frac{1}{2}, \frac{1}{2})$  in the period cell of the microstructure.

On the one hand, this change allows us to use the same test function  $v_n$  (4.8) defined in the proof of Theorem 4.1, since  $v_n$  remains equal to 1 in the new fibers due to the inclusion  $Q \subset D$ . On the other hand, Lemma 4.1 allows us to replace the disk  $D$  by the open set  $Q \subset D$ .

**Remark 4.2** We can also extend the result of Theorem 4.1 to an isotropic fibered microstructure composed by three similar periodic fibers lattices arranged in the three orthogonal directions  $e_1, e_2, e_3$ , namely

$$\omega_n := \bigcup_{j=1}^3 \left\{ y \in Y \mid \sum_{i \neq j} y_i^2 \leq r_n^2 \right\} \quad \text{and} \quad \Omega_n := \Omega \cap \bigcup_{\nu \in \mathbb{Z}^3} \varepsilon_n(\omega_n + \nu),$$

as represented in figure 4.2. Then, we derive the following homogenization conductivity:

$$\sigma_*(h) = \alpha_1 I_3 + \sum_{i=1}^3 \left( \frac{\alpha_2^3 + \alpha_2 \beta_2^2 |h|^2}{\alpha_2^2 + \beta_2^2 h_i^2} \right) e_i \otimes e_i + \beta_1 \mathcal{E}(h).$$

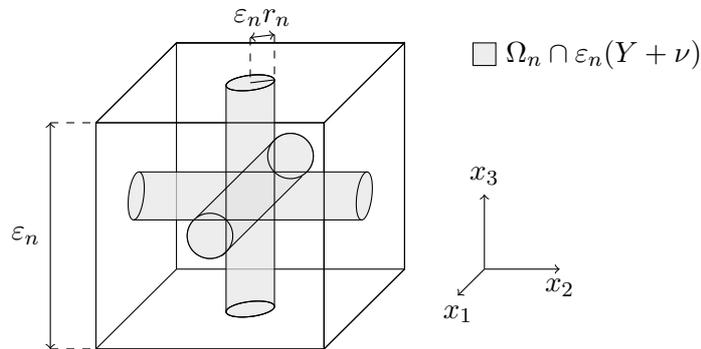


Figure 4.2: The period cell of the isotropic fibered structure in dimension 3

**Remark 4.3** We can check that when the volume fraction  $\theta_n = \theta$  and the hard phase of the conductivity  $\alpha_{2,n} = \alpha_\theta$  and  $\beta_{2,n} = \beta_\theta$  are independent of  $n$ , the explicit formula of [28] denoted by  $\sigma_*(\theta, h)$ , for the classical (since the period cell is now independent of  $n$ ) periodically homogenized conductivity (see (3.4)) has a limit as  $\theta \rightarrow 0$  when  $\theta \alpha_\theta$  and  $\theta \beta_\theta$  converge. Indeed, we may replace in the computations of [28] the optimal Vigdergauz shape by the circular cross-section in the previous asymptotic regime. Therefore, Theorem 4.1 validates the double process characterized by the homogenization at a fixed volume fraction  $\theta$  combined with the limit as  $\theta \rightarrow 0$ , by one homogenization process in which both the period and the volume fraction  $\theta_n = \pi r_n^2$  of the high conductivity phase tend to 0 as  $n \rightarrow \infty$ .

**Remark 4.4** The hypothesis on the convergence of  $\varepsilon_n^2 |\ln r_n|$  (4.5) allows us to avoid nonlocal effects in dimension three (see [25, 1]). These effects do not appear in dimension two as shown in [12]. Therefore, we can make a comparison between dimension two and dimension three based on the strong field perturbation in the absence of nonlocal effects.

**Proof of Theorem 4.1** The proof will be divided into four parts. We first prove the weak-\* convergence in  $\mathcal{M}(\Omega)$  of  $\sigma_n(h)\nabla u_n$  in  $\Omega_n$ . Then we establish a linear system satisfied by the limits defined by

$$\frac{\mathbf{1}_{\Omega_n} \partial u_n}{|\omega_n| \partial x_i} \rightharpoonup \xi_i \quad \text{weakly-* in } \mathcal{M}(\Omega).$$

Moreover, we deduce from Lemma 4.1 that

$$\frac{\mathbf{1}_{\Omega_n} \partial u_n}{|\omega_n| \partial x_3} \rightharpoonup \frac{\partial u}{\partial x_3} \quad \text{weakly-* in } \mathcal{M}(\Omega).$$

We finally calculate the homogenized matrix.

We first remark that, classically, the sequence of solutions  $u_n$  of  $\mathcal{P}_{\Omega,n}$  (see (4.4)) is bounded in  $H_0^1(\Omega)$  because, since  $\alpha_{2,n}$  diverges to  $\infty$  :

$$\|\nabla u_n\|_{L^2(\Omega)^3}^2 \leq C \int_{\Omega} (\alpha_1 \mathbf{1}_{\Omega \setminus \Omega_n} I_3 + \alpha_{2,n} \mathbf{1}_{\Omega_n} I_3) \nabla u_n \cdot \nabla u_n \, dx = \int_{\Omega} \sigma_n(h) \nabla u_n \cdot \nabla u_n \, dx.$$

By the Poincaré inequality, the previous inequality and (4.4) lead us to

$$\|u_n\|_{H_0^1(\Omega)}^2 \leq C \|\nabla u_n\|_{L^2(\Omega)^3}^2 \leq C |\langle f, u_n \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}| \leq C \|f\|_{H^{-1}(\Omega)} \|u_n\|_{H_0^1(\Omega)}$$

and then to

$$\|u_n\|_{H_0^1(\Omega)} \leq C \|f\|_{H^{-1}(\Omega)}.$$

Thus, up to a subsequence still denoted by  $n$ ,  $u_n$  converges weakly to some function  $u$  in  $H_0^1(\Omega)$ .

*First step:* Weak-\* convergence in  $\mathcal{M}(\Omega)$  of the conductivity in the fibers  $\mathbf{1}_{\Omega_n} (\alpha_{2,n} I_3 + \beta_{2,n} \mathcal{E}(h)) \nabla u_n$ . We proceed as in [23] with a suitable oscillating test function. For  $R \in (0, 1/2)$ , define the  $Y$ -periodic (independent of  $y_3$ ) function  $V_n$  by

$$V_n(y_1, y_2, y_3) = \begin{cases} 1 & \text{if } \sqrt{y_1^2 + y_2^2} \leq r_n \\ \frac{\ln R - \ln \sqrt{y_1^2 + y_2^2}}{\ln R - \ln r_n} & \text{if } r_n \leq \sqrt{y_1^2 + y_2^2} \leq R \\ 0 & \text{if } \sqrt{y_1^2 + y_2^2} \geq R, \end{cases} \quad \text{for } y \in Y,$$

and the rescaled function

$$v_n(x) = V_n\left(\frac{x}{\varepsilon_n}\right), \quad \text{for } x \in \mathbb{R}^3. \quad (4.8)$$

In particular, by using the cylindrical coordinates and the fact that  $r_n$  converges to 0, this function satisfies the inequalities

$$\begin{aligned} \|v_n\|_{L^2(\Omega)}^2 &\leq C \|V_n\|_{L^2(Y)}^2 = C \left| \ln \frac{R}{r_n} \right|^{-2} \left( \pi r_n^2 + \int_0^{2\pi} \int_{r_n}^R r \ln^2 \frac{R}{r} \, dr d\theta \right) \\ &= C \left| \ln \frac{R}{r_n} \right|^{-2} \left( \pi \frac{R^2 - r_n^2}{2} - \pi r_n^2 \ln^2 \frac{R}{r_n} - \pi \ln \frac{R}{r_n} \right) \leq C \left| \ln \frac{R}{r_n} \right|^{-2}, \end{aligned}$$

$$\|\nabla v_n\|_{L^2(\Omega)^3}^2 \leq \frac{C}{\varepsilon_n^2} \|\nabla V_n\|_{L^2(Y)^3}^2 = \frac{C}{\varepsilon_n^2} \left| \ln \frac{R}{r_n} \right|^{-2} \int_0^{2\pi} \int_{r_n}^R \frac{1}{r} \, dr d\theta \leq \frac{C}{\varepsilon_n^2} \left| \ln \frac{R}{r_n} \right|^{-1}$$

and, consequently

$$\|v_n\|_{L^2(\Omega)} + \varepsilon_n \|\nabla v_n\|_{L^2(\Omega)^3} \leq C \sqrt{\left| \ln \frac{R}{r_n} \right|^{-1}} \xrightarrow{n \rightarrow \infty} 0. \quad (4.9)$$

Let  $\lambda$  be a vector in  $\mathbb{R}^3$  perpendicular to the  $x_3$ -axis. Define the  $Y$ -periodic function  $\tilde{X}_n$  by  $\nabla \tilde{X}_n = \lambda$  in  $\omega_n$ , such that  $\tilde{X}_n \in \mathcal{D}(Y)$  and is  $Y$ -periodic, and the rescaled function  $X_n$  by

$$X_n(x) = \varepsilon_n \tilde{X}_n \left( \frac{x}{\varepsilon_n} \right). \quad (4.10)$$

In particular,  $X_n$  satisfies

$$\|X_n\|_\infty = \varepsilon_n \|\tilde{X}_n\|_\infty \leq C\varepsilon_n \quad , \quad \|\nabla X_n\|_\infty = \|\nabla \tilde{X}_n\|_\infty \leq C \quad \text{and} \quad \nabla X_n = \lambda \quad \text{in} \quad \Omega_n. \quad (4.11)$$

We have, by (4.11) and (4.9),

$$\begin{aligned} \|v_n X_n\|_{H^1(\Omega)} &\leq \|X_n\|_\infty \|v_n\|_{L^2(\Omega)} + \|X_n\|_\infty \|\nabla v_n\|_{L^2(\Omega)^3} + \|\nabla X_n\|_\infty \|v_n\|_{L^2(\Omega)} \\ &\leq C(\|v_n\|_{L^2(\Omega)} + \varepsilon_n \|\nabla v_n\|_{L^2(\Omega)^3}) \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

which gives

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \varphi v_n X_n \xrightarrow{n \rightarrow \infty} 0 \quad \text{strongly in} \quad H_0^1(\Omega). \quad (4.12)$$

Let  $\varphi \in \mathcal{D}(\Omega)$ . By the strong convergence (4.12), we have

$$\int_{\Omega} \sigma_n(h) \nabla u_n \cdot \nabla (\varphi v_n X_n) \, dx = \langle f, \varphi v_n X_n \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \xrightarrow{n \rightarrow \infty} 0. \quad (4.13)$$

Let us decompose this integral which converges to 0, into the integral on the fibers set  $\Omega_n$  and the integral on its complementary:

$$\int_{\Omega} \sigma_n(h) \nabla u_n \cdot \nabla (\varphi v_n X_n) \, dx = \int_{\Omega \setminus \Omega_n} (\alpha_1 I_3 + \beta_1 \mathcal{E}(h)) \nabla u_n \cdot \nabla (\varphi v_n X_n) \, dx \quad (4.14a)$$

$$+ \int_{\Omega_n} (\alpha_{2,n} I_3 + \beta_{2,n} \mathcal{E}(h)) \nabla u_n \cdot \nabla (\varphi v_n X_n) \, dx. \quad (4.14b)$$

The expression (4.14a) converges to 0 since, by the Cauchy-Schwarz inequality, the boundedness of  $u_n$  in  $H_0^1(\Omega)$  and (4.12), we have

$$\left| \int_{\Omega \setminus \Omega_n} (\alpha_1 I_3 + \beta_1 \mathcal{E}(h)) \nabla u_n \cdot \nabla (\varphi v_n X_n) \, dx \right| \leq |\alpha_1 I_3 + \beta_1 \mathcal{E}(h)| \|\nabla u_n\|_{L^2(\Omega)^3} \|\varphi v_n X_n\|_{H_0^1(\Omega)} \xrightarrow{n \rightarrow \infty} 0. \quad (4.15)$$

Consequently, as  $v_n = 1$  and  $\nabla X_n = \lambda$  on  $\Omega_n$ , by (4.13), (4.14a), (4.14b) and (4.15), we have

$$\int_{\Omega_n} \sigma_n(h) \nabla u_n \cdot \lambda \varphi \, dx + \int_{\Omega_n} \sigma_n(h) \nabla u_n \cdot \nabla \varphi X_n \, dx \xrightarrow{n \rightarrow \infty} 0. \quad (4.16)$$

To prove the convergence to 0 of the right term, we now show that  $\mathbf{1}_{\Omega_n} (\alpha_{2,n} I_3 + \beta_{2,n} \mathcal{E}(h)) \nabla u_n$  is bounded in  $L^1(\Omega)^3$ . We have, by the Cauchy-Schwarz inequality, (4.5) and the classical equivalent  $|\Omega_n| \underset{n \rightarrow \infty}{\sim} |\Omega| |\omega_n|$ ,

$$\begin{aligned} \left( \int_{\Omega_n} |(\alpha_{2,n} I_3 + \beta_{2,n} \mathcal{E}(h)) \nabla u_n| \, dx \right)^2 &\leq |I_3 + \alpha_{2,n}^{-1} \beta_{2,n} \mathcal{E}(h)|^2 |\Omega_n| \alpha_{2,n} \int_{\Omega_n} \alpha_{2,n} |\nabla u_n|^2 \, dx \\ &\leq C \int_{\Omega} \sigma_n(h) \nabla u_n \cdot \nabla u_n \, dx \\ &\leq C \|f\|_{H^{-1}(\Omega)} \|u_n\|_{H_0^1(\Omega)}. \end{aligned}$$

This combined with the boundedness of  $u_n$  in  $H_0^1(\Omega)$  implies that  $\mathbf{1}_{\Omega_n} (\alpha_{2,n} I_3 + \beta_{2,n} \mathcal{E}(h)) \nabla u_n$  is bounded in  $L^1(\Omega)^3$ . This bound and the uniform convergence to 0 of  $X_n$  (see (4.11)) imply the convergence to 0 of the right term of (4.16), hence

$$\int_{\Omega_n} (\alpha_{2,n} I_3 + \beta_{2,n} \mathcal{E}(h)) \nabla u_n \cdot \lambda \varphi \, dx \xrightarrow{n \rightarrow \infty} 0.$$

We rewrite this condition as

$$\forall \lambda \perp e_3, \quad \mathbb{1}_{\Omega_n} (\alpha_{2,n} I_3 + \beta_{2,n} \mathcal{E}(h)) \nabla u_n \cdot \lambda \rightharpoonup 0 \quad \text{weakly-* in } \mathcal{M}(\Omega). \quad (4.17)$$

*Second step:* Linear relations between weak-\* limits of  $\frac{\mathbb{1}_{\Omega_n} \partial u_n}{|\omega_n| \partial x_i}$ .

Thanks to the Cauchy-Schwarz inequality, we have

$$\left\| \frac{\mathbb{1}_{\Omega_n} \partial u_n}{|\omega_n| \partial x_i} \right\|_{L^1(\Omega)} \leq \frac{1}{|\omega_n|} \int_{\Omega_n} |\nabla u_n| \, dx \leq \frac{1}{\sqrt{\alpha_{2,n} |\omega_n|}} \sqrt{\frac{|\Omega_n|}{|\omega_n|}} \sqrt{\int_{\Omega_n} \alpha_{2,n} |\nabla u_n|^2 \, dx}$$

which leads us, by (4.5) and the asymptotic behavior  $|\Omega_n| \underset{n \rightarrow \infty}{\sim} |\Omega| |\omega_n|$ , to

$$\left\| \frac{\mathbb{1}_{\Omega_n} \partial u_n}{|\omega_n| \partial x_i} \right\|_{L^1(\Omega)} \leq \frac{C}{\sqrt{\alpha_{2,n} |\omega_n|}} \int_{\Omega} \sigma_n(h) \nabla u_n \cdot \nabla u_n \, dx \leq C \left| \langle f, u_n \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \right|$$

which is bounded by the boundedness of  $u_n$  in  $H_0^1(\Omega)$ . This allows us to define, up to a subsequence, the following limits

$$\frac{\mathbb{1}_{\Omega_n} \partial u_n}{|\omega_n| \partial x_i} \rightharpoonup \xi_i \quad \text{weakly-* in } \mathcal{M}(\Omega), \quad \text{for } i = 1, 2, 3. \quad (4.18)$$

Then, by (4.17) we have

$$(\alpha_{2,n} I_3 + \beta_{2,n} \mathcal{E}(h)) \mathbb{1}_{\Omega_n} \nabla u_n \cdot \lambda = (\alpha_{2,n} |\omega_n| I_3 + \beta_{2,n} |\omega_n| \mathcal{E}(h)) \frac{\mathbb{1}_{\Omega_n} \nabla u_n \cdot \lambda}{|\omega_n|} \rightharpoonup 0 \quad \text{weakly-* in } \mathcal{M}(\Omega).$$

Therefore, putting  $\lambda = e_1, e_2$  in this limit and using condition (4.5), we obtain the linear system

$$\begin{cases} \alpha_2 \xi_1 + \beta_2 h_2 \xi_3 - \beta_2 h_3 \xi_2 = 0 \\ \alpha_2 \xi_2 + \beta_2 h_3 \xi_1 - \beta_2 h_1 \xi_3 = 0 \end{cases} \quad \text{in } \mathcal{M}(\Omega),$$

which is equivalent to

$$\begin{cases} \xi_1 = \frac{\beta_2^2 h_1 h_3 - \alpha_2 \beta_2 h_2}{\alpha_2^2 + \beta_2^2 h_3^2} \xi_3 \\ \xi_2 = \frac{\beta_2^2 h_2 h_3 + \alpha_2 \beta_2 h_1}{\alpha_2^2 + \beta_2^2 h_3^2} \xi_3 \end{cases} \quad \text{in } \mathcal{M}(\Omega). \quad (4.19)$$

*Third step:* Proof of  $\xi_3 = \frac{\partial u}{\partial x_3}$ .

We need the following result which is an extension of the estimate (3.13) of [22]. The statement of this lemma is more general than necessary for our purpose but is linked to Remark 4.1.

**Lemma 4.1** *Let  $Q$  be a non-empty connected open subset of the unit disk  $D$ . Then, there exists a constant  $C > 0$  such that any function  $U \in H^1(Y)$  satisfies the estimate*

$$\left| \frac{1}{|r_n Q|} \int_{r_n Q \times (-\frac{1}{2}, \frac{1}{2})} U \, dy - \int_Y U \, dy \right| \leq C \sqrt{|\ln r_n|} \|\nabla U\|_{L^2(Y)^3}. \quad (4.20)$$

**Proof of Lemma 4.1.** Let  $U \in H^1(Y)$ . To prove Lemma 4.1, we compare the average value of  $U$  on  $r_n Q$  and  $r_n D$ . Denoting  $\tilde{y} = (y_1, y_2)$ , we have, for any  $y_3 \in (-\frac{1}{2}, \frac{1}{2})$ ,

$$\begin{aligned} \left| \int_{r_n Q} U(\tilde{y}, y_3) \, d\tilde{y} - \int_{r_n D} U(\tilde{y}, y_3) \, d\tilde{y} \right| &= \left| \int_Q U(r_n \tilde{y}, y_3) \, d\tilde{y} - \int_D U(r_n \tilde{y}, y_3) \, d\tilde{y} \right| \\ &\leq \int_Q \left| U(r_n \tilde{y}, y_3) - \int_D U(r_n \tilde{y}, y_3) \, d\tilde{y} \right| \, d\tilde{y}, \end{aligned}$$

and, since  $Q \subset D$ ,

$$\begin{aligned} \left| \int_{r_n Q} U(\tilde{y}, y_3) \, d\tilde{y} - \int_{r_n D} U(\tilde{y}, y_3) \, d\tilde{y} \right| &\leq \frac{|D|}{|Q|} \int_D \left| U(r_n \tilde{y}, y_3) - \int_D U(r_n \tilde{y}, y_3) \, d\tilde{y} \right| \, d\tilde{y} \\ &\leq C \int_D r_n \left( \left| \frac{\partial U}{\partial x_1} \right| + \left| \frac{\partial U}{\partial x_2} \right| \right) (r_n \tilde{y}, y_3) \, d\tilde{y} \\ &= \frac{C}{\pi r_n} \int_{r_n D} \left( \left| \frac{\partial U}{\partial x_1} \right| + \left| \frac{\partial U}{\partial x_2} \right| \right) (\tilde{y}, y_3) \, d\tilde{y}, \end{aligned}$$

the last inequality being a consequence of the Poincaré-Wirtinger inequality. Hence, integrating the previous inequality with respect to  $y_3 \in (-\frac{1}{2}, \frac{1}{2})$  and applying the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} \left| \int_{r_n Q \times (-\frac{1}{2}, \frac{1}{2})} U(y) \, dy - \int_{r_n D \times (-\frac{1}{2}, \frac{1}{2})} U(y) \, dy \right| &\leq \frac{C}{\pi r_n} \int_{r_n D \times (-\frac{1}{2}, \frac{1}{2})} |\nabla U|(y) \, dy \\ &\leq C \sqrt{\int_{r_n D \times (-\frac{1}{2}, \frac{1}{2})} |\nabla U|^2(y) \, dy} \\ &\leq C \|\nabla U\|_{L^2(Y)^3}. \end{aligned}$$

This combined with the estimate (3.13) of [22], i.e. (4.20) for  $Q = D$ , and the fact that  $\sqrt{|\ln r_n|}$  diverges to  $\infty$  give the thesis.  $\square$

Let  $\varphi \in \mathcal{D}(\Omega)$ . A rescaling of (4.20) with  $Q = D$  implies the inequality

$$\left| \frac{1}{|\omega_n|} \int_{\Omega_n} u_n \varphi \, dx - \int_{\Omega} u_n \varphi \, dx \right| \leq C \varepsilon_n \sqrt{|\ln r_n|} \|\nabla(u_n \varphi)\|_{L^2(\Omega)^3}.$$

Combining this estimate and the first condition of (4.5) with

$$\|\nabla(u_n \varphi)\|_{L^2(\Omega)^3} \leq \|\nabla u_n\|_{L^2(\Omega)^3} \|\varphi\|_{\infty} + \|u_n\|_{L^2(\Omega)} \|\nabla \varphi\|_{\infty} \leq C,$$

it follows that

$$\frac{\mathbf{1}_{\Omega_n}}{|\omega_n|} u_n - u_n \rightharpoonup 0 \text{ in } \mathcal{D}'(\Omega).$$

This convergence does not hold true when  $\varepsilon_n^2 |\ln r_n|$  converges to some positive constant. Under this critical regime, non-local effects appear (see Remark 4.4).

Finally, as  $\mathbf{1}_{\Omega_n}$  does not depend on the  $x_3$  variable, we have

$$\frac{\mathbf{1}_{\Omega_n}}{|\omega_n|} \frac{\partial u_n}{\partial x_3} = \frac{\partial}{\partial x_3} \frac{\mathbf{1}_{\Omega_n}}{|\omega_n|} u_n = \frac{\partial}{\partial x_3} \left( \frac{\mathbf{1}_{\Omega_n}}{|\omega_n|} u_n - u_n \right) + \frac{\partial u_n}{\partial x_3} \rightharpoonup \frac{\partial u}{\partial x_3} = \xi_3 \text{ in } \mathcal{D}'(\Omega).$$

*Fourth step:* Derivation of the homogenized matrix.

We now study the limit of  $\sigma_n(h)\nabla u_n$  in order to obtain  $\sigma_*(h)$ . We have

$$\begin{aligned}\sigma_n(h)\nabla u_n \cdot e_1 &= \mathbb{1}_{\Omega \setminus \Omega_n} \left( \alpha_1 \frac{\partial u_n}{\partial x_1} - \beta_1 h_3 \frac{\partial u_n}{\partial x_2} + \beta_1 h_2 \frac{\partial u_n}{\partial x_3} \right) \\ &\quad + \alpha_{2,n} |\omega_n| \frac{\mathbb{1}_{\Omega_n}}{|\omega_n|} \frac{\partial u_n}{\partial x_1} - \beta_{2,n} h_3 |\omega_n| \frac{\mathbb{1}_{\Omega_n}}{|\omega_n|} \frac{\partial u_n}{\partial x_2} + \beta_{2,n} h_2 |\omega_n| \frac{\mathbb{1}_{\Omega_n}}{|\omega_n|} \frac{\partial u_n}{\partial x_3}.\end{aligned}\tag{4.21}$$

Hence, passing to the weak-\* limit in  $\mathcal{M}(\Omega)$  this equality and using the linear system (4.19),  $\sigma_n(h)\nabla u_n \cdot e_1$  weakly-\* converges in  $\mathcal{M}(\Omega)$  to

$$\begin{aligned}&\left( \alpha_1 \frac{\partial u}{\partial x_1} - \beta_1 h_3 \frac{\partial u}{\partial x_2} + \beta_1 h_2 \frac{\partial u}{\partial x_3} \right) + \alpha_2 \xi_1 - \beta_2 h_3 \xi_2 + \beta_2 h_2 \xi_3 \\ &= (\alpha_1 I_3 + \beta_1 \mathcal{E}(h)) \nabla u \cdot e_1 + \alpha_2 \frac{\beta_2^2 h_1 h_3 - \alpha_2 \beta_2 h_2}{\alpha_2^2 + \beta_2^2 h_3^2} \xi_3 - \beta_2 h_3 \frac{\beta_2^2 h_2 h_3 + \alpha_2 \beta_2 h_1}{\alpha_2^2 + \beta_2^2 h_3^2} \xi_3 + \beta_2 h_2 \xi_3 \\ &= (\alpha_1 I_3 + \beta_1 \mathcal{E}(h)) \nabla u \cdot e_1 \\ &\quad + \underbrace{\frac{\alpha_2 (\beta_2^2 h_1 h_3 - \alpha_2 \beta_2 h_2) - \beta_2 h_3 (\beta_2^2 h_2 h_3 + \alpha_2 \beta_2 h_1) + \beta_2 h_2 (\alpha_2^2 + \beta_2^2 h_3^2)}{\alpha_2^2 + \beta_2^2 h_3^2}}_{=0} \xi_3,\end{aligned}$$

that is

$$\sigma_n(h)\nabla u_n \cdot e_1 \rightharpoonup (\alpha_1 I_3 + \beta_1 \mathcal{E}(h)) \nabla u \cdot e_1 \quad \text{weakly-* in } \mathcal{M}(\Omega).\tag{4.22}$$

The same calculus leads us to

$$\sigma_n(h)\nabla u_n \cdot e_2 \rightharpoonup (\alpha_1 I_3 + \beta_1 \mathcal{E}(h)) \nabla u \cdot e_2 \quad \text{weakly-* in } \mathcal{M}(\Omega).\tag{4.23}$$

We have, for the last direction  $e_3$ ,

$$\sigma_n(h)\nabla u_n \cdot e_3 \rightharpoonup \left( \alpha_1 \frac{\partial u}{\partial x_3} - \beta_1 h_2 \frac{\partial u}{\partial x_1} + \beta_1 h_1 \frac{\partial u}{\partial x_2} \right) + \alpha_2 \xi_3 + \beta_2 h_2 \xi_1 - \beta_2 h_1 \xi_2 \quad \text{weakly-* in } \mathcal{M}(\Omega).$$

Hence, again with the linear system (4.19),

$$\begin{aligned}&\left( \alpha_1 \frac{\partial u}{\partial x_3} - \beta_1 h_2 \frac{\partial u}{\partial x_1} + \beta_1 h_1 \frac{\partial u}{\partial x_2} \right) + \alpha_2 \xi_3 - \beta_2 h_2 \xi_1 + \beta_2 h_1 \xi_2 \\ &= (\alpha_1 I_3 + \beta_1 \mathcal{E}(h)) \nabla u \cdot e_3 + \alpha_2 \xi_3 - \beta_2 h_2 \frac{\beta_2^2 h_1 h_3 - \alpha_2 \beta_2 h_2}{\alpha_2^2 + \beta_2^2 h_3^2} \xi_3 + \beta_2 h_1 \frac{\beta_2^2 h_2 h_3 + \alpha_2 \beta_2 h_1}{\alpha_2^2 + \beta_2^2 h_3^2} \xi_3.\end{aligned}$$

Finally, by the previous equality, (4.22) and (4.23), we get that

$$\sigma_*(h) = \alpha_1 I_3 + \left( \frac{\alpha_2^3 + \alpha_2 \beta_2^2 |h|^2}{\alpha_2^2 + \beta_2^2 h_3^2} \right) e_3 \otimes e_3 + \beta_1 \mathcal{E}(h).$$

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## References

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