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ON THE STABILITY IN WEAK TOPOLOGY OF THE SET OF GLOBAL SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

HAJER BAHOURI AND ISABELLE GALLAGHER

Abstract. Let X be a suitable function space and let $\mathcal{G} \subset X$ be the set of divergence free vector fields generating a global, smooth solution to the incompressible, homogeneous three dimensional Navier-Stokes equations. We prove that a sequence of divergence free vector fields converging in the sense of distributions to an element of $\mathcal{G}$ belongs to $\mathcal{G}$ if n is large enough, provided the convergence holds "anisotropically" in frequency space. Typically that excludes self-similar type convergence. Anisotropy appears as an important qualitative feature in the analysis of the Navier-Stokes equations; it is also shown that initial data which does not belong to $\mathcal{G}$ (hence which produces a solution blowing up in finite time) cannot have a strong anisotropy in its frequency support.

1. Introduction and statement of results

1.1. **Setting of the problem.** We are interested in the three dimensional, incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\ \text{div}\, u = 0 \\ u_{|t=0} = u_0\,, \end{cases}$$

where $u(t,x)$ and $p(t,x)$ are respectively the velocity and the pressure of the fluid at time $t \geq 0$ and position $x \in \mathbb{R}^3$. We recall that the pressure may be eliminated by projection onto divergence free vector fields, hence we shall consider the following version of the equations:

$$\text{(NS)} \quad \begin{cases} \partial_t u + \mathbb{P}(u \cdot \nabla u) - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\ \text{div}\, u = 0 \\ u_{|t=0} = u_0\,, \end{cases}$$

where $\mathbb{P} := \text{Id} - \nabla \Delta^{-1} \text{div}$.

Note also that the Navier-Stokes system may be written as

$$\begin{cases} \partial_t u + \text{div}(u \otimes u) - \Delta u = -\nabla p \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\ \text{div}\, u = 0 \\ u_{|t=0} = u_0\,, \end{cases}$$

where $\text{div}(u \otimes u)^j = \sum_{k=1}^{d} \partial_k (u^j u^k) = \text{div}(u^j u)$. The advantage of this weak formulation is that it makes sense for singular vector fields and allows to consider weak solutions. The question of the existence of global, smooth (and unique) solutions is a long-standing open problem, and we shall only recall here a few of the many results on this question. We refer for instance to [3] or [45] and the references therein, for a precise definition of weak solutions

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and recent surveys on the subject. An important point in the study of (NS) is its scale invariance: if u is a solution of (NS) on $\mathbb{R}^+ \times \mathbb{R}^d$ associated with the data u_0, then for any $\lambda > 0$, $u_\lambda(t,x) := \lambda u(\lambda^2 t, \lambda x)$ is a solution on $\mathbb{R}^+ \times \mathbb{R}^d$, associated with the data

$$u_{0,\lambda}(x) := \lambda u_0(\lambda x)\,. \tag{1.1}$$

In two space dimensions, $L^2(\mathbb{R}^2)$ is scale invariant, while in three space dimensions that is the case for $L^3(\mathbb{R}^3)$, the (smaller) Sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, or the Besov spaces $\dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$, with $1 \leq p \leq \infty$ and $0 < q \leq \infty$. We refer to Appendix B for all necessary definitions and properties of those spaces. Note that anisotropic spaces such as $L^2(\mathbb{R}^2; \dot{H}^{\frac{1}{2}}(\mathbb{R}))$ can also be scale invariant under (1.1), but also more generally under the anisotropic scaling

$$f_{\lambda,\mu}(x) := \lambda f(\lambda x_1, \lambda x_2, \mu x_3)\,, \quad \forall \lambda, \mu > 0\,. \tag{1.2}$$

Of course (NS) is not invariant through that transformation if $\lambda \neq \mu$.

It is well-known that (NS) is globally wellposed if the initial data is small in $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ as long as $p < \infty$ (see the successive results by [46], [24], [38], [12] and [52]). Let us emphasize that in all those results, the global solution lies in $\mathcal{C}(\mathbb{R}^+; X)$ when the Cauchy data belongs to the Banach space X. We note that the proof of uniqueness may require the use of more refined spaces. In [42], H. Koch and D. Tataru obtained a unique (in a space we shall not detail here) global in time solution for data small enough in the larger space BMO^{-1}, consisting of vector fields whose components are derivatives of BMO functions.

The smallness assumption is not necessary in order to obtain global solutions to (NS), as pointed out for instance in [13]-[15]. We also recall that in two space dimensions, (NS) is globally wellposed as soon as the initial data belongs to $L^2(\mathbb{R}^2)$, with no restriction on its size (see [47]); this is due to the fact that the $L^2(\mathbb{R}^d)$ norm is controlled a priori globally in time. This estimate also allowed J. Leray in [46] to prove the existence of global in time weak solutions in two and three dimensions. J. Leray's result extends to any dimension, as shown in [18] for instance.

In this article we are interested in the structure of the set $\mathcal{G}$ of initial data giving rise to a unique, global solution to the Navier-Stokes equations. More precisely our interest will be in the **global** nature of the solution, as the uniqueness of the solution will not be an issue. The solutions will be obtained via a fixed point procedure in an adequate function space. It is known that the set $\mathcal{G}$ contains small balls in BMO^{-1} centered at the origin. But it is known to include many more classes of functions. We recall that it was proved in [2] (see [27] for the setting of Besov spaces) that $\mathcal{G}$ is **open** for the strong topology of BMO^{-1}, provided one restricts the setting to the closure of Schwartz-class functions for the BMO^{-1} norm. In this paper we address the same question for **weak topology**. More precisely we wish to understand under what conditions a sequence of divergence free vector fields converging in the sense of distributions to an initial data in $\mathcal{G}$, will itself be in $\mathcal{G}$ (up to a finite number of terms in the sequence).

Before going into more details let us discuss some examples. If a sequence converges not only weakly but strongly in $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$, say, to an element of $\mathcal{G}$ then the result is known, see [27]. To give another example, consider a sequence of divergence free vector fields $u_{0,n}$, bounded in $L^3(\mathbb{R}^3)$, converging in the sense of distributions to some vector field u_0 in $L^3(\mathbb{R}^3) \cap \mathcal{G}$. If $(1+|\cdot|)^{1+\varepsilon} u_{0,n}$ is bounded in L^∞ for some $\varepsilon > 0$, then it is easy to see that $u_{0,n}$ generates a global unique solution to (NS) for n large enough. This can be seen using the "stability of singular points" of [37, 53], or more directly using the fact that such a sequence is actually

compact[1] in $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ for $p > 3$ and applying the strong stability result [27]. This example shows that in some cases, the weak convergence assumption implies the strong convergence in spaces where stability results are available. Here we consider a situation where such a reduction does not occur. One way to achieve this is considering sequences bounded in a scale-invariant space only, with no additional bound in a non-scale-invariant space. However in that case clearly some restrictions have to be imposed to hope to prove such a weak openness result; indeed consider for instance the sequence

$$\phi_n(x) := 2^n \phi(2^n x)\,, \quad n \in \mathbb{N}\,, \tag{1.3}$$

where ϕ is any smooth, divergence free vector field. This sequence converges to zero in the sense of distributions as n goes to infinity, and zero belongs to $\mathcal{G}$. If one could infer, by weak stability, that ϕ_n gives rise to a global unique solution for large enough n, then so would ϕ by scale invariance and that would solve the problem of global regularity for the Navier-Stokes equations. Note that the same can be said of the sequence

$$\widetilde{\phi}_n(x) := \phi(x - x_n)\,, \quad |x_n| \to \infty\,. \tag{1.4}$$

Since the global regularity problem seems out of reach, we choose here to add assumptions on the **spectral structure** of the sequences converging weakly to an element of $\mathcal{G}$, which in particular forbid sequences such as ϕ_n or $\widetilde{\phi}_n$ which in a way are "too isotropic".

Actually one has the following interesting and rather easy result, which highlights the role anisotropy can play in the study of the Navier-Stokes equations. This result shows that initial data generating a solution blowing up in finite time cannot be too anisotropic in frequency space, meaning that the set of its horizontal and vertical frequency sizes cannot be too separated; the threshold depends only on the norm of the initial data. The result is proved in Appendix B; its proof relies on elementary inequalities on the Littlewood-Paley decomposition, which are all recalled in that appendix. The notation $\Delta^h_k \Delta^v_j$ appearing in the statement stands for horizontal and vertical Littlewood-Paley truncations at scale 2^k and 2^j respectively, and is also introduced in Appendix B. The space $\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)$ is a scale invariant space, slightly smaller than $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$.

Theorem 1. *Let $\rho > 0$ be given. There is a constant $N_0 \in \mathbb{N}$ such that any divergence free vector field u_0 of norm ρ in $\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)$ satisfying $u_0 = \sum_{|j-k|\geq N_0} \Delta^h_k \Delta^v_j u_0$ gives rise to a global, unique solution to* (NS) *in $\mathcal{C}(\mathbb{R}^+; L^3(\mathbb{R}^3))$.*

Let us now define the function spaces we shall be working with. As explained above we want to work in anisotropic spaces, invariant through the scaling (1.2). For technical reasons we shall assume quite a lot of smoothness on the sequence of initial data: we choose the sequence bounded in essentially the smallest anisotropic Besov space $\dot{B}^{s,s'}_{p,q}$ invariant through (1.2). It is likely that this smoothness could be relaxed somewhat, but perhaps not with the method we follow. We shall point out as we go along where those restrictions appear, see in particular Remark 4.7 page 20.

Definition 1.1. *We define, for $0 < q \leq \infty$, the space $\mathcal{B}^1_q$ by the (quasi)-norm*

$$\|f\|_{\mathcal{B}^1_q} := \Big(\sum_{j,k\in\mathbb{Z}} 2^{(j+k)q} \|\Delta^h_k \Delta^v_j f\|^q_{L^1(\mathbb{R}^3)} \Big)^{\frac{1}{q}}\,, \tag{1.5}$$

[1]This fact can readily be seen by applying a profile decomposition technique and eliminating all profiles except for the weak limit, thanks to the additional bounds satisfied by the sequence.

where Δ_k^h and Δ_j^v are horizontal and vertical frequency localization operators (see Appendix B).

This corresponds to the space $\dot{B}^{1,1}_{1,q}$ defined in Appendix B, where the reader will also find its properties used in this text. More generally we define in Appendix B

$$\|f\|_{\dot{B}^{s,r}_{p,q}} := \Big(\sum_{j,k\in\mathbb{Z}} 2^{(rj+sk)q} \|\Delta_k^h \Delta_j^v f\|^q_{L^p(\mathbb{R}^3)} \Big)^{\frac{1}{q}} .$$

The norm (1.5) is equivalent to the norm (B.3) which is clearly invariant by the scaling (1.2), and is slightly larger (if $q \leq 1$) than the more classical $\dot{B}^{0,\frac{1}{2}}_{2,1}$ norm (for the role of $\dot{B}^{0,\frac{1}{2}}_{2,1}$ in the study of the Navier-Stokes equations see for instance [17],[51]). Moreover the space $\mathcal{B}^1_q$ is **anisotropic** by essence, which as pointed out above, will be an important feature of our analysis.

It is proved in Appendix A that any initial data small enough in $\mathcal{B}^1_1$ generates a unique, global solution to (NS) in the space $\mathcal{S}_{1,1} := \widetilde{L^\infty}(\mathbb{R}^+; \mathcal{B}^1_1) \cap L^1(\mathbb{R}^+; \dot{B}^{3,1}_{1,1} \cap \dot{B}^{1,3}_{1,1})$, and if the data is not small then there is a unique solution in the local space

$$\mathcal{S}_{1,1}(T) := \widetilde{L^\infty_{loc}}([0,T); \mathcal{B}^1_1) \cap L^1_{loc}((0,T); \dot{B}^{3,1}_{1,1} \cap \dot{B}^{1,3}_{1,1})$$

for some $T > 0$.

We provide also in Appendix A a **strong** stability result in $\mathcal{B}^1_1$, whose proof follows a classical procedure, and the main goal of this text is to prove a stability result in the **weak** topology for data in $\mathcal{B}^1_q$ for $0 < q < 1$.

Now let us define our notion of an anisotropically oscillating sequence. We shall need another more technical assumption later, which is stated in Section 2 (see Assumption 2 page 10).

Assumption 1. *Let $0 < q \leq \infty$ be given. We say that a sequence $(f_n)_{n\in\mathbb{N}}$, bounded in $\mathcal{B}^1_q$, is **anisotropically oscillating** if the following property holds. There exists $p \geq 2$ such that for all sequences (k_n, j_n) in $\mathbb{Z}^{\mathbb{N}} \times \mathbb{Z}^{\mathbb{N}}$,*

$$\liminf_{n\to\infty} 2^{k_n(-1+\frac{2}{p})+\frac{j_n}{p}} \|\Delta^h_{k_n} \Delta^v_{j_n} f_n\|_{L^p(\mathbb{R}^3)} = C > 0 \quad \Longrightarrow \lim_{n\to\infty} |j_n - k_n| = \infty . \tag{1.6}$$

Remark 1.2. It is easy to see (see Appendix B) that any function f in $\mathcal{B}^1_q$ belongs also to $\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,\infty}$ for any $p \geq 1$ hence

$$f \in \mathcal{B}^1_q \Longrightarrow \sup_{(k,j)\in\mathbb{Z}^2} 2^{k(-1+\frac{2}{p})+\frac{j}{p}} \|\Delta^h_k \Delta^v_j f\|_{L^p} < \infty .$$

The left-hand side of (1.6) indicates which ranges of frequencies are predominant in the sequence (f_n): if $\limsup_{n\to\infty} 2^{k_n(-1+\frac{2}{p})+\frac{j_n}{p}} \|\Delta^h_{k_n} \Delta^v_{j_n} f_n\|_{L^p}$ is zero for a couple of frequencies $(2^{k_n}, 2^{j_n})$, then the sequence $(f_n)_{n\in\mathbb{N}}$ is "unrelated" to those frequencies, with the vocabulary of [31] (see also Lemma 5.2 in this paper). The right–hand side of (1.6) is then an **anisotropy** property. Indeed one sees easily that a sequence such as $(\phi_n)_{n\in\mathbb{N}}$ defined in (1.3) is precisely not anisotropically oscillating: for the left-hand side of (1.6) to hold for ϕ_n one would need $j_n \sim k_n \sim n$, which is precisely not the condition required on the right-hand side of (1.6). A typical sequence satisfying Assumption 1 is rather (for $a \in \mathbb{R}^3$)

$$f_n(x) := 2^{\alpha n} f\big(2^{\alpha n}(x_1 - a_1), 2^{\alpha n}(x_2 - a_2), 2^{\beta n}(x_3 - a_3)\big), \quad (\alpha, \beta) \in \mathbb{R}^2, \quad \alpha \neq \beta$$

with f smooth. One of the results of this paper states that any sequence satisfying Assumption 1 may be written as the superposition of such sequences, up to a small remainder term (see Proposition 2.4 page 7).

1.2. **Main results.** We prove in this article that $\mathcal{G}$ is open for weak topology, provided the weakly converging sequence is of the type described in Assumption 1.

Theorem 2. *Let $q \in]0,1[$ be given and let $(u_{0,n})_{n\in\mathbb{N}}$ be a sequence of divergence free vector fields bounded in $\mathcal{B}^1_q$, converging towards $u_0 \in \mathcal{B}^1_q$ in the sense of distributions, and assume that u_0 generates a unique solution in $\mathcal{S}_{1,1}(\infty)$. If $u_0 - (u_{0,n})_{n\in\mathbb{N}}$ is anisotropically oscillating and satisfies Assumption 2 page 10, then for n large enough, $u_{0,n}$ generates a unique, global solution to* (NS) *in $\mathcal{S}_{1,1}(\infty)$.*

Remark 1.3. Theorem 2 may be generalized by adding two more sequences to $u_{0,n}$, where in each additional sequence the "privileged" direction is not x_3 but x_1 or x_2. It is clear from the proof that the same result holds, but we choose not to present the proof of that more general result due to its technical cost. Actually a more interesting generalization would consist in considering more geometrical assumptions, but that requires more work and ideas, and will not be addressed here.

Remark 1.4. Assumption 2 is stated page 10, along with some comments (see in particular Remarks 2.8, 2.9 and 2.10). Its statement requires the introduction of the profile decomposition of the sequence of initial data and it requires that some of the profiles vanish at zero.

Remark 1.5. Theorem 2 generalizes the result of [15], where it is shown that the initial data

$$u_0(x) + \sum_{j=1}^{J} (v_0^{1(j)} + \varepsilon_j w_0^{1(j)}, v_0^{2(j)} + \varepsilon_j w_0^{2(j)}, w_0^{3(j)})(x_1, x_2, \varepsilon_j x_3)$$

generates a global solution if u_0 belongs to $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap \mathcal{G}$, if the profiles $(v_0^{1(j)}, v_0^{2(j)}, 0)$ and $w_0^{(j)}$ are divergence free and in $L^2(\mathbb{R}_{x_3}; \dot{H}^{-1}(\mathbb{R}^2))$, as well as all their derivatives, if $\varepsilon_1, \dots \varepsilon_J > 0$ are small enough, and finally under the assumption that $v_0^{1(j)}(x_1, x_2, 0) \equiv v_0^{2(j)}(x_1, x_2, 0) \equiv 0$ and $w_0^{3(j)}(x_1, x_2, 0) \equiv 0$. Those last requirements are analogous to Assumption 2. Note that even in the case when $u_0 \equiv 0$, such initial data cannot be dealt with simply using Theorem 1 since it is not bounded in $\dot{B}^{\frac{1}{2}}_{2,1}$. Note also that as in [15], the special structure of (NS) is used in the proof of Theorem 2.

Remark 1.6. Notice that it is not assumed that the global solution associated with u_0 satisfies uniform, global in time integral bounds. Similarly to [2] and [27] such bounds may be derived a posteriori from the fact that the solution is global: see Appendix A, Corollary 3.

Remark 1.7. One can see from the proof of Theorem 2 that the solution $u_n(t)$ associated with $u_{0,n}$ converges for all times, in the sense of distributions to the solution associated with u_0. In this sense the Navier-Stokes equations are stable by weak convergence.

The proof of Theorem 2 enables us to infer easily the following results. The first corollary generalizes the statement of Theorem 2 to the case when $u_0 \notin \mathcal{G}$.

Corollary 1. *Let $(u_{0,n})_{n\in\mathbb{N}}$ be a sequence of divergence free vector fields bounded in the space $\mathcal{B}^1_q$ for some $0 < q < 1$, converging towards some $u_0 \in \mathcal{B}^1_q$ in the sense of distributions, with $u_0 - (u_{0,n})_{n\in\mathbb{N}}$ anisotropically oscillating and satisfying Assumption 2. Let u be the solution to the Navier-Stokes equations associated with u_0 and assume that the life span of u is $T^* < \infty$. Then for all $T < T^*$, there is a subsequence such that the life span of the solution associated with $u_{0,n}$ is at least T.*

The second corollary deals with the case when the **sequence** belongs to $\mathcal{G}$, with an a priori boundedness assumption on the solution (which could actually be generalized but we choose

not to complicate things too much at this stage; see Appendix B for definitions), and infers that the weak limit also belongs to $\mathcal{G}$.

Corollary 2. *Assume $(u_n^0)_{n\in\mathbb{N}}$ is a sequence of initial data, such that the associate solution u_n is uniformly bounded in $\widetilde{L^2}\big(\mathbb{R}^+;\dot{B}^{\frac{2}{3},\frac{1}{3}}_{3,1}\big)$. If u_n^0 converges in the sense of distributions to some u_0, with $u_0-(u_{0,n})_{n\in\mathbb{N}}$ anisotropically oscillating and satisfying Assumption 2, then u_0 gives rise to a unique, global solution in $\mathcal{S}_{3,1}(\infty)$.*

1.3. **Notation.** For all points $x=(x_1,x_2,x_3)$ in $\mathbb{R}^3$ and all vector fields $v=(v^1,v^2,v^3)$, we shall denote by

$$x_h := (x_1,x_2) \quad \text{and} \quad v^h := (v^1,v^2)$$

their horizontal parts. We shall also define horizontal differentiation operators $\nabla^h := (\partial_1,\partial_2)$ and $\mathrm{div}_h := \nabla^h\cdot$, as well as $\Delta_h := \partial_1^2+\partial_2^2$.

We shall also use the shorthand notation for function spaces X (defined on $\mathbb{R}^2$) and Y (defined on $\mathbb{R}$): $X_hY_v := X(\mathbb{R}^2;Y(\mathbb{R}))$.

Finally we shall denote by C a constant which does not depend on the various parameters appearing in this paper, and which may change from line to line. We shall also denote sometimes $x\le Cy$ by $x\lesssim y$.

1.4. **General scheme of the proof and organization of the paper.** The main arguments leading to Theorem 2 are the following: by a profile decomposition argument, the sequence of initial data is decomposed into the sum of the weak limit u_0 and a sequence of "orthogonal" profiles, up to a small remainder term. Under Assumptions 1 and 2 and using scaling arguments it is proved that each individual profile belongs to $\mathcal{G}$; this step relies crucially on the results of [14] and [15]. The mutual orthogonality of each term in the decomposition of the initial data implies finally that the sum of the solutions associated to each profile is itself an approximate solution to (NS), globally in time, which concludes the proof.

The paper is organized in the following way:

- In Section 2 we provide an "anisotropic profile decomposition" of the sequence of initial data, based on a general result, Theorem 3 stated and proved in Section 5 page 26. This enables us to replace the sequence of initial data, up to an arbitrarily small remainder term, by a finite (but large) sum of profiles.
- Section 3 is then devoted to the construction of an approximate solution by propagating globally in time each individual profile of the decomposition. The propagation is through either the Navier-Stokes flow or transport-diffusion equations.
- In Section 4 we prove that the construction of the previous step does provide an approximate solution to the Navier-Stokes equations, thus completing the proof of Theorem 2, while Corollaries 1 and 2 are proved at the end of Section 4. That section is the most technical part of the proof, as one must check that the nonlinear interactions of all the functions constructed in the previous step are negligible. It also relies on results proved in Appendix A, on the global regularity for the Navier-Stokes equation (and perturbed versions of that equation) for small data and forces in anisotropic Besov spaces.
- Finally in Appendix B we collect useful results on isotropic and anisotropic spaces which are used in this text, and we prove Theorem 1.

2. Profile decomposition of the initial data

In this section we consider a sequence of initial data as given in Theorem 2, and write down an anisotropic profile decomposition for that sequence. We shall constantly be using the following scaling operators.

Definition 2.1. *For any two sequences $\boldsymbol{\varepsilon} = (\varepsilon_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\gamma} = (\gamma_n)_{n\in\mathbb{N}}$ of positive real numbers and any sequence $\boldsymbol{x} = (x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^3$ we define the scaling operator*

$$\Lambda^n_{\boldsymbol{\varepsilon},\boldsymbol{\gamma},\boldsymbol{x}}\phi(x) := \frac{1}{\varepsilon_n}\phi\left(\frac{x_h - x_{n,h}}{\varepsilon_n}, \frac{x_3 - x_{n,3}}{\gamma_n}\right).$$

Remark 2.2. The operator $\Lambda^n_{\boldsymbol{\varepsilon},\boldsymbol{\gamma},\boldsymbol{x}}$ is an isometry in the space $\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,q}$ for any $1 \le p \le \infty$ and $0 < q \le \infty$.

Then we define the notion of orthogonal cores/scales as follows (see also Section 5).

Definition 2.3. *We say that two triplets of sequences $(\boldsymbol{\varepsilon}^{\boldsymbol{\ell}}, \boldsymbol{\gamma}^{\boldsymbol{\ell}}, \boldsymbol{x}^{\boldsymbol{\ell}})$ for $\ell \in \{1,2\}$, where $(\boldsymbol{\varepsilon}^{\boldsymbol{\ell}}, \boldsymbol{\gamma}^{\boldsymbol{\ell}})$ are two sequences of positive real numbers and $\boldsymbol{x}^{\boldsymbol{\ell}}$ are sequences in $\mathbb{R}^3$, are **orthogonal** if*

$$\text{either} \quad \frac{\varepsilon^1_n}{\varepsilon^2_n} + \frac{\varepsilon^2_n}{\varepsilon^1_n} + \frac{\gamma^1_n}{\gamma^2_n} + \frac{\gamma^2_n}{\gamma^1_n} \to \infty\,, \quad n \to \infty$$

$$\text{or} \quad (\varepsilon^1_n, \gamma^1_n) = (\varepsilon^2_n, \gamma^2_n) \quad \text{and} \quad |(x^1_n)^{\varepsilon^1,\gamma^1} - (x^2_n)^{\varepsilon^1,\gamma^1}| \to \infty\,, \quad n \to \infty\,,$$

where we have denoted $(\boldsymbol{x}^{\boldsymbol{\ell}})^{\boldsymbol{\varepsilon}^{\boldsymbol{k}},\boldsymbol{\gamma}^{\boldsymbol{k}}} := \left(\frac{\boldsymbol{x}^{\boldsymbol{\ell}}_{\boldsymbol{h}}}{\boldsymbol{\varepsilon}^{\boldsymbol{k}}}, \frac{\boldsymbol{x}^{\boldsymbol{\ell}}_{\boldsymbol{3}}}{\boldsymbol{\gamma}^{\boldsymbol{k}}}\right)$.

Note that up to extracting a subsequence, any sequence of positive real numbers can be assumed to converge either to 0, to ∞, or to a constant. In the rest of this paper, up to rescaling the profiles by a fixed constant, we shall assume that if the limit of any one of the sequences $\boldsymbol{\varepsilon}^{\boldsymbol{\ell}}, \boldsymbol{\gamma}^{\boldsymbol{\ell}}, \boldsymbol{\eta}^{\boldsymbol{\ell}}, \boldsymbol{\delta}^{\boldsymbol{\ell}}$ is a constant, then it is one.

The main result of this section is the following.

Proposition 2.4. *Under the assumptions of Theorem 2, the following holds. Let $2 \le p \le \infty$ be given as in Assumption 1. For all integers ℓ there are two sets of orthogonal sequences in the sense of Definition 2.3, $(\boldsymbol{\varepsilon}^{\boldsymbol{\ell}}, \boldsymbol{\gamma}^{\boldsymbol{\ell}}, \boldsymbol{x}^{\boldsymbol{\ell}})$ and $(\boldsymbol{\eta}^{\boldsymbol{\ell}}, \boldsymbol{\delta}^{\boldsymbol{\ell}}, \tilde{\boldsymbol{x}}^{\boldsymbol{\ell}})$ and for all $\alpha \in (0,1)$ there are arbitrarily smooth divergence free vector fields $(\tilde{\phi}^{h,\ell}_\alpha, 0)$ and $(-\nabla_h\Delta_h^{-1}\partial_3\phi^\ell_\alpha, \phi^\ell_\alpha)$ such that up to extracting a subsequence, one can write*

$$u_{0,n} = u_0 + \sum_{\ell=1}^{\tilde{L}} \Lambda^n_{\boldsymbol{\eta}^{\boldsymbol{\ell}},\boldsymbol{\delta}^{\boldsymbol{\ell}},\tilde{\boldsymbol{x}}^{\boldsymbol{\ell}}}\left(\tilde{\phi}^{h,\ell}_\alpha + \tilde{r}^{h,\ell}_\alpha, 0\right) + \sum_{\ell=1}^{L} \Lambda^n_{\boldsymbol{\varepsilon}^{\boldsymbol{\ell}},\boldsymbol{\gamma}^{\boldsymbol{\ell}},\boldsymbol{x}^{\boldsymbol{\ell}}}\left(-\frac{\varepsilon^\ell_n}{\gamma^\ell_n}\nabla_h\Delta_h^{-1}\partial_3(\phi^\ell_\alpha + r^\ell_\alpha), \phi^\ell_\alpha + r^\ell_\alpha\right)$$
$$+ (\tilde{\psi}^{h,\tilde{L}}_n - \nabla_h\Delta_h^{-1}\partial_3\psi^L_n, \psi^L_n)\,, \quad \operatorname{div}\tilde{r}^{h,\ell}_\alpha = 0\,, \quad \|\tilde{r}^{h,\ell}_\alpha\|_{\mathcal{B}^1_q} + \|r^\ell_\alpha\|_{\mathcal{B}^1_q} \le \alpha\,,$$

with $\tilde{\psi}^{h,\tilde{L}}_n$ and ψ^L_n independent of α and uniformly bounded (in n and L) in $\mathcal{B}^1_q$, and

$$\limsup_{n\to\infty}\left(\|\tilde{\psi}^{h,L}_n\|_{\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}} + \|\psi^L_n\|_{\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}\right) \to 0\,, \quad L \to \infty\,. \tag{2.1}$$

Moreover the following properties hold:

$$\forall \ell \in \mathbb{N}, \quad \lim_{n\to\infty}(\delta^\ell_n)^{-1}\eta^\ell_n \in \{0,\infty\}\,, \quad \lim_{n\to\infty}(\gamma^\ell_n)^{-1}\varepsilon^\ell_n = 0\,, \tag{2.2}$$

as well as the following stability result:

$$\sum_{\ell\in\mathbb{N}}\left(\|\tilde{\phi}^{h,\ell}_\alpha\|_{\mathcal{B}^1_q} + \|\phi^\ell_\alpha\|_{\mathcal{B}^1_q}\right) \lesssim \sup_n \|u_{0,n}\|_{\mathcal{B}^1_q} + \|u_0\|_{\mathcal{B}^1_q}\,. \tag{2.3}$$

Proof of Proposition 2.4. The proof is divided into two steps. First we decompose the third component $u^3_{0,n}$ according to Theorem 3 in Section 5, and then we decompose the horizontal component $u^h_{0,n}$ using both the first step and Theorem 3 again (for the divergence free part of $u^h_{0,n}$).

Step 1. Decomposition of $u^3_{0,n}$. Let us apply Theorem 3 of Section 5 (see page 26) to the sequence $u^3_{0,n} - u^3_0$. With the notation of Theorem 3, we define

$$\begin{aligned}
\varepsilon^\ell_n &:= 2^{-j_1(\lambda_\ell(n))}\\
\gamma^\ell_n &:= 2^{-j_2(\lambda_\ell(n))}\\
x^\ell_{n,h} &:= 2^{-j_1(\lambda_\ell(n))}k_1(\lambda_\ell(n))\\
x^\ell_{n,3} &:= 2^{-j_2(\lambda_\ell(n))}k_2(\lambda_\ell(n))\,.
\end{aligned}$$

The orthogonality of the sequences $(\boldsymbol{\varepsilon^\ell}, \boldsymbol{\gamma^\ell}, \boldsymbol{x^\ell})$, as given in Definition 2.3, is a consequence of the orthogonality property stated in Theorem 3 along with Remark 5.1. According to that theorem we can write

$$u^3_{0,n} - u^3_0 = \sum_{\ell=1}^{L} \Lambda^n_{\boldsymbol{\varepsilon^\ell},\boldsymbol{\gamma^\ell},\boldsymbol{x^\ell}}\varphi^\ell + \psi^L_n\,, \tag{2.4}$$

where due to (5.10) in Theorem 3,

$$\sum_{\ell\in\mathbb{N}} \|\varphi^\ell\|_{\mathcal{B}^1_q} \lesssim \sup_n \|u^3_{0,n} - u^3_0\|_{\mathcal{B}^1_q} < \infty\,.$$

In particular ψ^L_n is uniformly bounded (in n and L) in $\mathcal{B}^1_q \subset \dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,q}$, and Theorem 3 gives

$$\limsup_{n\to\infty} \|\psi^L_n\|_{\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,p}} \to 0\,, \quad L\to\infty\,.$$

The result (2.1) then follows by Hölder's inequality for sequences. Note that we have used here the fact that $q < 1$.

Using horizontal and vertical frequency truncations, given any $\alpha > 0$ we may further decompose φ^ℓ into

$$\varphi^\ell = \phi^\ell_\alpha + r^\ell_\alpha\,, \quad \text{with } \phi^\ell_\alpha \text{ arbitrarily smooth and } \|r^\ell_\alpha\|_{\mathcal{B}^1_q} \le \alpha\,, \tag{2.5}$$

and we have, by this choice of regularization,

$$\|\phi^\ell_\alpha\|_{\mathcal{B}^1_q} + \|r^\ell_\alpha\|_{\mathcal{B}^1_q} \le 2\|\varphi^\ell\|_{\mathcal{B}^1_q}\,.$$

This implies (2.3) for ϕ^ℓ_α.

Now let us prove that

$$\forall \ell\in\mathbb{N}, \quad \lim_{n\to\infty} (\gamma^\ell_n)^{-1}\varepsilon^\ell_n = 0\,.$$

Assumption 1 along with Lemma 5.2 page 30 imply that the limit of $(\gamma^\ell_n)^{-1}\varepsilon^\ell_n$ belongs to $\{0,\infty\}$. Moreover by the divergence free condition on $u_{0,n}$ we have $\mathrm{div}_h\, u^h_{0,n} = -\partial_3 u^3_{0,n}$ and since $u^h_{0,n}$ is bounded in $\mathcal{B}^1_q$ we infer that $\partial_3 u^3_{0,n}$ is bounded in $\dot{B}^{0,1}_{1,q}$ and $\partial_3 u^3_0$ also belongs to $\dot{B}^{0,1}_{1,q}$. This in turn, due to Lemma 5.3, implies that

$$\lim_{n\to\infty} (\gamma^\ell_n)^{-1}\varepsilon^\ell_n = 0\,.$$

Step 2. Decomposition of $u^h_{0,n}$. The divergence free assumption on the initial data enables us to recover from the previous step a profile decomposition for $u^h_{0,n}$. Indeed there is a two-dimensional, divergence free vector field $\nabla^\perp_h C_{0,n}$ such that

$$u^h_{0,n} = \nabla^\perp_h C_{0,n} - \nabla_h \Delta_h^{-1} \partial_3 u^3_{0,n}\,,$$

where $\nabla^\perp_h = (-\partial_1, \partial_2)$. Similarly there is some function φ such that

$$u^h_0 = \nabla^\perp_h \varphi - \nabla_h \Delta_h^{-1} \partial_3 u^3_0\,.$$

Furthermore as recalled in the previous step $\partial_3 u^3_{0,n}$ is bounded in $\dot B^{0,1}_{1,q}$. This implies that the sequence $\nabla^\perp_h C_{0,n}$ is bounded in $\mathcal{B}^1_q$ and arguing similarly for $\nabla^\perp_h \varphi$, the profile decomposition of Section 5 may also be applied to $\nabla^\perp_h C_{0,n}(x) - \nabla^\perp_h \varphi$: we get

$$\nabla^\perp_h C_{0,n} - \nabla^\perp_h \varphi = \sum_{\ell=1}^{L} \Lambda^n_{\boldsymbol{\eta}^\ell, \boldsymbol{\delta}^\ell, \tilde{\boldsymbol{x}}^\ell} \tilde\phi^{h,\ell} + \tilde\psi^{h,L}_n$$

with $\limsup_{n\to\infty} \|\tilde\psi^L_n\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,p}} \to 0$ as $L \to \infty$ and $\operatorname{div}_h \tilde\phi^{h,\ell} = 0$ thanks to Lemma 5.4. Finally $\eta^\ell_n/\delta^\ell_n \to 0$ or ∞ due to the anisotropy assumption as in the previous step. The rest of the construction is identical to Step 1. The proposition is proved. □

Before evolving in time the decomposition provided in Proposition 2.4 we notice that it may happen that the cores and scales of concentration $(\boldsymbol{\eta}^\ell, \boldsymbol{\delta}^\ell, \tilde{\boldsymbol{x}}^\ell)$ appearing in the decomposition of $\nabla^\perp_h C_{0,n}$ coincide with (or more generally are non orthogonal to) those appearing in the decomposition of $u^3_{0,n}$, namely $(\boldsymbol{\varepsilon}^\ell, \boldsymbol{\gamma}^\ell, \boldsymbol{x}^\ell)$. In that case the corresponding profiles should be evolved together in time. This leads naturally to the next definition.

Definition 2.5. *For each $\ell \in \mathbb{N}$, we define $\kappa(\ell)$ by the condition (with the notation of Definition 2.3)*

$$\lim_{n\to\infty} \Big(\frac{\boldsymbol{\varepsilon}^{\boldsymbol{\kappa}(\boldsymbol{\ell})}}{\boldsymbol{\eta}^{\boldsymbol{\ell}}}, \frac{\boldsymbol{\gamma}^{\boldsymbol{\kappa}(\boldsymbol{\ell})}}{\boldsymbol{\delta}^{\boldsymbol{\ell}}}, (\boldsymbol{x}^{\boldsymbol{\kappa}(\boldsymbol{\ell})} - \tilde{\boldsymbol{x}}^{\boldsymbol{\ell}})^{\boldsymbol{\varepsilon}^{\boldsymbol{\kappa}(\boldsymbol{\ell})}, \boldsymbol{\gamma}^{\boldsymbol{\kappa}(\boldsymbol{\ell})}} \Big) = (\lambda_1, \lambda_2, a)\,, \quad \lambda_1, \lambda_2 > 0\,,\ a \in \mathbb{R}^3\,. \tag{2.6}$$

We also define for each $L \in \mathbb{N}$ the set

$$\mathcal{K}(L) := \Big\{ \ell \in \mathbb{N} \,/\, \ell = \kappa(\tilde\ell)\,,\ \tilde\ell \in \{1, \dots, L\} \Big\}\,.$$

Remark 2.6. Note that for each ℓ there is at most one such $\kappa(\ell)$ by orthogonality. Moreover up to rescaling-translating the profiles we can assume that $\lambda_1 = \lambda_2 = 1$ and $a = 0$.

The decomposition of Proposition 2.4 can now be written, for any $L \in \mathbb{N}$ in the following way. The interest of the next formulation is that as we shall see, each profile is either small, or orthogonal to all the others. In the next formula we decide, to simplify notation that the profile $\phi^{\kappa(\ell)}_\alpha$ is equal to zero if (2.6) does not hold. We also have changed slightly the remainder terms r^ℓ_α and ψ^L_n, without altering their smallness properties (and keeping their notation for simplicity), due to the fact that in Definition 2.5 the ratios converge to a fixed

limit but are in fact not strictly equal to the limit. So we write

$$
\begin{aligned}
u_{0,n} \;=\;& u_0 + \sum_{\ell=1}^{L} \Lambda^n_{\boldsymbol{\eta}^\ell,\boldsymbol{\delta}^\ell,\tilde{\boldsymbol{x}}^\ell}\left(\tilde\phi^{h,\ell}_\alpha + \tilde r^{h,\ell}_\alpha - \frac{\eta^\ell_n}{\delta^\ell_n}\nabla_h\Delta_h^{-1}\partial_3(\phi^{\kappa(\ell)}_\alpha + r^{\kappa(\ell)}_\alpha), \phi^{\kappa(\ell)}_\alpha + r^{\kappa(\ell)}_\alpha\right)\\
&+ \sum_{\substack{\kappa(\ell)=1\\ \kappa(\ell)\in\mathcal{K}(\infty)\setminus\mathcal{K}(L)}}^{L} \Lambda^n_{\boldsymbol{\eta}^\ell,\boldsymbol{\delta}^\ell,\tilde{\boldsymbol{x}}^\ell}\left(\tilde\phi^{h,\ell}_\alpha + \tilde r^{h,\ell}_\alpha - \frac{\eta^\ell_n}{\delta^\ell_n}\nabla_h\Delta_h^{-1}\partial_3(\phi^{\kappa(\ell)}_\alpha + r^{\kappa(\ell)}_\alpha), \phi^{\kappa(\ell)}_\alpha + r^{\kappa(\ell)}_\alpha\right)\\
&+ \sum_{\substack{\ell=1\\ \ell\notin\mathcal{K}(\infty)}}^{L} \Lambda^n_{\boldsymbol{\varepsilon}^\ell,\boldsymbol{\gamma}^\ell,\boldsymbol{x}^\ell}\left(-\frac{\varepsilon^\ell_n}{\gamma^\ell_n}\nabla_h\Delta_h^{-1}\partial_3(\phi^\ell_\alpha + r^\ell_\alpha), \phi^\ell_\alpha + r^\ell_\alpha\right)\\
&- \sum_{\substack{\ell>L\\ \ell\in\mathcal{K}(L)}} \Lambda^n_{\boldsymbol{\varepsilon}^\ell,\boldsymbol{\gamma}^\ell,\boldsymbol{x}^\ell}\left(-\frac{\varepsilon^\ell_n}{\gamma^\ell_n}\nabla_h\Delta_h^{-1}\partial_3\phi^\ell, \phi^\ell\right) - \sum_{\substack{\ell>L\\ 1\le\kappa(\ell)\le L}} \Lambda^n_{\boldsymbol{\eta}^\ell,\boldsymbol{\delta}^\ell,\tilde{\boldsymbol{x}}^\ell}\big(\tilde\phi^{h,\ell}_\alpha + \tilde r^{h,\ell}_\alpha, 0\big)\\
&+ \big(\tilde\psi^{h,L}_n - \nabla_h\Delta_h^{-1}\partial_3\psi^L_n, \psi^L_n\big)\,.
\end{aligned}
\tag{2.7}
$$

Before moving on to the time evolution of (2.7), we are now in position to state the second assumption entering in the statement of Theorem 2.

Assumption 2. *With the notation of Proposition* 2.4*, there is* L_0 *such that for every* $L \ge L_0$*, the following holds.*

- *Suppose there are two indexes* $\ell_1 \neq \ell_2$ *in* $\{1,\dots,L\}$ *such that the following properties are satisfied:*

$$
\begin{gathered}
\eta^{\ell_1}_n = \eta^{\ell_2}_n\,, \quad \delta^{\ell_1}_n \to \infty\,, \quad \delta^{\ell_2}_n \to 1 \ \text{ or } \infty \quad \text{with} \quad \delta^{\ell_1}_n/\delta^{\ell_2}_n \to \infty\,,\\
\text{and } (\tilde x^{\ell_1}_n - \tilde x^{\ell_2}_n)^{\eta^{\ell_2}_n,\delta^{\ell_2}_n} \to a^{\ell_1,\ell_2} \in \mathbb{R}^3\,, \quad \frac{\tilde x^{\ell_2}_{n,3}}{\delta^{\ell_2}_n} \to a^k_3 \in \mathbb{R}\,.
\end{gathered}
\tag{2.8}
$$

Then one has $\tilde\phi^{h,\ell_1}(\cdot,0) := (\tilde\phi^{h,\ell_1}_\alpha + \tilde r^{h,\ell_1}_\alpha)(\cdot,0) \equiv 0$.

- *If* $u_0 \not\equiv 0$ *and if there are* $\ell_1 \neq \ell_2 \in \{1,\dots,L\}$ *such that for* $i \in \{1,2\}$, $\eta^{\ell_i}_n = 1$ *with* $\delta^{\ell_i}_n \to \infty$ *while* $\tilde x^{\ell_i}_{n,h}$ *is bounded and* $\tilde x^{\ell_i}_{n,3}/\delta^{\ell_i}_n \to \tilde a^{\ell_i}_3 \in \mathbb{R}$*, then* $\tilde\phi^{h,\ell_i}(\cdot,-\tilde a^{\ell_i}_3) \equiv 0$ *for each* $i \in \{1,2\}$.

- *A similar result holds for the profiles* $\phi^\ell := \tilde\phi^{h,\ell}_\alpha + r^{h,\ell}_\alpha$*, with the corresponding assumptions on the scales and cores.*

Proposition 2.7. *With the notation of Proposition* 2.4 *assume the following:*

- *If* $\ell_1 \neq \ell_2$ *in* $\{1,\dots,L\}$ *are two indexes satisfying* (2.8)*, then a weak limit of the sequence* $\eta^{\ell_2}_n(u^h_{0,n} - u^h_0 - \tilde\psi^{h,L}_n + \nabla_h\Delta_h^{-1}\partial_3\psi^L_n)(\eta^{\ell_2}_n y_h + \tilde x^{\ell_2}_{n,h}, \delta^{\ell_2}_n y_3 + \tilde x^{\ell_2}_{n,3})$ *is* $\tilde\phi^{h,\ell_2}(y)$.
- *A similar result holds for* $\varepsilon^{\ell_2}_n(u^3_{0,n} - u^3_0 - \tilde\psi^{3,\tilde L}_n - \psi^L_n)(\varepsilon^{\ell_2}_n y_h + x^{\ell_2}_{n,h}, \gamma^{\ell_2}_n y_3 + x^{\ell_2}_{n,3})$*, with the corresponding assumptions on the scales and cores.*

Then Assumption 2 *holds.*

Proof of Proposition 2.7. • We shall start by proving the result for a couple $\ell_1 \neq \ell_2$ chosen in $\{1,\dots,L\}$ so that $\delta^{\ell_1}_n$ is the largest vertical scale among the vertical scales associated with all couples satisfying (2.8).

We begin by noticing that the limit (after extraction) of $\frac{\tilde{x}^{\ell_1}_{n,3}}{\delta^{\ell_1}_n}$ is necessarily zero since

$$\frac{\tilde{x}^{\ell_1}_{n,3}}{\delta^{\ell_1}_n} = \left(\frac{\tilde{x}^{\ell_1}_{n,3} - \tilde{x}^{\ell_2}_{n,3}}{\delta^{\ell_2}_n} + \frac{\tilde{x}^{\ell_2}_{n,3}}{\delta^{\ell_2}_n}\right)\frac{\delta^{\ell_2}_n}{\delta^{\ell_1}_n} \to 0\,. \tag{2.9}$$

Without loss of generality we may also assume that for the index ℓ_1 we have chosen, $\delta^{\ell_2}_n$ is the largest vertical scale satisfying (2.8). By the hypothesis of Proposition 2.7 we know that the weak limit of $\eta^{\ell_2}_n\big(u^h_{0,n} - u^h_0 - \tilde{\psi}^{h,\tilde{L}}_n + \nabla_h\Delta_h^{-1}\partial_3\psi^L_n\big)(\eta^{\ell_2}_n y_h + \tilde{x}^{\ell_2}_{n,h}, \delta^{\ell_2}_n y_3 + \tilde{x}^{\ell_2}_{n,3})$ is $\tilde{\phi}^{h,\ell_2}(y)$. This weak limit may be explicitly computed: noticing that for any integer k,

$$\eta^{\ell_2}_n\big(\Lambda^n_{\boldsymbol{\eta^k},\boldsymbol{\delta^k},\boldsymbol{\tilde{x}^k}}\tilde{\phi}^{h,k}\big)(\eta^{\ell_2}_n y_h + \tilde{x}^{\ell_2}_{n,h}, \delta^{\ell_2}_n y_3 + \tilde{x}^{\ell_2}_{n,3}) = \Lambda^n_{\frac{\boldsymbol{\eta^k}}{\boldsymbol{\eta^{\ell_2}}},\frac{\boldsymbol{\delta^k}}{\boldsymbol{\delta^{\ell_2}}},\boldsymbol{\tilde{x}^{k,\ell_2}}}\tilde{\phi}^{h,k}(y),$$

with $\tilde{x}^{k,\ell_2}_n := (\tilde{x}^k_n - \tilde{x}^{\ell_2}_n)^{\eta^{\ell_2}_n,\delta^{\ell_2}_n}$, we find that the weak limit of such a term is zero except in three situations : if $k = \ell_2$, if $k = \ell_1$, or if

$$\eta^k_n = \eta^{\ell_2}_n\,, \quad \delta^k_n/\delta^{\ell_2}_n \to \infty\,, \quad (\tilde{x}^k_n - \tilde{x}^{\ell_2}_n)^{\eta^k_n,\delta^k_n} \to a^{k,\ell_2} \in \mathbb{R}^3\,. \tag{2.10}$$

If $k = \ell_2$, then the function is simply equal to $\tilde{\phi}^{h,\ell_2}(y)$, and if $k = \ell_1$ then by (2.8) the weak limit is equal to $\tilde{\phi}^{h,\ell_1}(y_h + a^{\ell_1,\ell_2}_h, 0)$. Finally if (2.10) were to hold then in particular k would satisfy the same properties as ℓ_2 in the statement of the proposition, while $\delta^k_n/\delta^{\ell_2}_n \to \infty$, and that is impossible by choice of ℓ_2 as corresponding to the largest vertical scale satisfying (2.8). So finally the weak limit of $\eta^{\ell_2}_n\big(u^h_{0,n} - u^h_0 - \tilde{\psi}^{h,\tilde{L}}_n + \nabla_h\Delta_h^{-1}\partial_3\psi^L_n\big)(\eta^{\ell_2}_n y_h + \tilde{x}^{\ell_2}_{n,h}, \delta^{\ell_2}_n y_3 + \tilde{x}^{\ell_2}_{n,3})$ is $\tilde{\phi}^{h,\ell_2}(y) + \tilde{\phi}^{\ell_1}(y_h + a^{\ell_1,\ell_2}_h, 0)$, hence necessarily by the assumptions of Proposition 2.7, we have that $\tilde{\phi}^{h,\ell_1}(y_h + a^{\ell_1,\ell_2}_h, 0) \equiv 0$ so the result is proved in the case of the largest possible vertical scale.

Now we can argue by induction for the other possible ℓ^1's: suppose that ℓ^1 corresponds to the second largest for instance, then calling $\delta^{\ell_0}_n$ the largest one, the same argument implies that the weak limit of the sequence $\eta^{\ell_2}_n\big(u^h_{0,n} - u^h_0 - \tilde{\psi}^{h,\tilde{L}}_n + \nabla_h\Delta_h^{-1}\partial_3\psi^L_n\big)(\eta^{\ell_2}_n y_h + \tilde{x}^{\ell_2}_{n,h}, \delta^{\ell_2}_n y_3 + \tilde{x}^{\ell_2}_{n,3})$ is the function $\tilde{\phi}^{\ell_2}(y) + \tilde{\phi}^{h,\ell_1}(y_h + a^{\ell_1,\ell_2}_h, 0) + \tilde{\phi}^{h,\ell_0}(y_h + a^{\ell_0,\ell_2}_h, 0) = \tilde{\phi}^{\ell_2}(y) + \tilde{\phi}^{h,\ell_1}(y_h + a^{\ell_1,\ell_2}_h, 0)$ hence $\tilde{\phi}^{\ell_1}(y_h + a^{\ell_1,\ell_2}_h, 0) \equiv 0$ and by induction, the result is proved.

- The proof of the second point is very similar: we first consider ℓ_1 corresponding to the largest vertical scale among the indexes satsfying $\eta^\ell_n = 1$, $\delta^\ell_n \to \infty$, $\tilde{x}^\ell_{n,h} \to \tilde{a}^\ell_h$ bounded and $\tilde{x}^\ell_{n,3}/\delta^\ell_n \to \tilde{a}^\ell_3 \in \mathbb{R}$. If there is no other index satisfying those requirements then we notice that the weak limit of $u^h_{0,n} - u^h_0 - \tilde{\psi}^{h,\tilde{L}}_n + \nabla_h\Delta_h^{-1}\partial_3\psi^L_n$ is $\tilde{\phi}^\ell(y_h - \tilde{a}^\ell_h, -\tilde{a}^\ell_3)$, while we also know that it is zero, so the result follows. If there is a second index satisfying those requirements, then we consider $\delta^{\ell_2}_n$ the next largest vertical scale (by orthogonality it cannot be equal to $\delta^{\ell_1}_n$) and we use the assumption of Proposition 2.7, which implies that the weak limit of the sequence $(u^h_{0,n} - u^h_0 - \tilde{\psi}^{h,\tilde{L}}_n + \nabla_h\Delta_h^{-1}\partial_3\psi^L_n(y_h + \tilde{x}^{\ell_2}_{n,h}, \delta^{\ell_2}_n y_3 + \tilde{x}^{\ell_2}_{n,3})$ is the function $\tilde{\phi}^{h,\ell_2}(y)$ while a direct computation gives the limit $\tilde{\phi}^{h,\ell_2}(y) + \tilde{\phi}^{h,\ell_1}(y_h - \tilde{a}^{\ell_1}_h + \tilde{a}^{\ell_2}_h, -\tilde{a}^\ell_3)$ and again we get the result.
The rest of the argument is as above, by induction on the size of the vertical scales.

- The proof is identical for the profiles ϕ^ℓ.

Proposition 2.7 is proved. □

Remark 2.8. Assuming the hypotheses of Proposition 2.7 is actually quite natural. Indeed for any choice of sequences of cores $(x^\ell_{n,h})_{n\in\mathbb{N}}$ and of scales $(\eta^\ell_n)_{n\in\mathbb{N}}$, one has that the sequence $\eta^\ell_n(u^h_{0,n} - u^h_0 - \tilde\psi^{h,\tilde L}_n + \nabla_h\Delta_h^{-1}\partial_3\psi^L_n)(\eta^\ell_n y_h + x^\ell_{n,h}, \delta^\ell_n y_3 + x_{n,3})$ converges in $\mathcal{S}'$, and it is assumed here that the weak limit is precisely the profile $\tilde\phi^{h,\ell}$. Note that for a profile decomposition in the space $\dot B^{s,s'}_{p,q}$ that is obvious as soon as $s < 2/p$ and $s' < 1/p$. Here we have $s' = 1/p$ so this is a true assumption (in the same way as the sequence $f(x_h, \varepsilon x_3)$ does not necessarily converge weakly to zero with ε).

For example the sequence provided in Remark 1.2 satisfies Assumption 2 since there is only one profile involved.

More generally consider the sequence (assuming that $0 \neq \alpha \neq \gamma$, and $\beta_1, \beta_2 \neq \alpha, \beta_1 \neq \gamma$)

$$2^{\alpha n}\Big(f_1\big(2^{\alpha n}x_1, 2^{\alpha n}x_2, 2^{\beta_1 n}x_3 - a_3\big) + f_2\big(2^{\alpha n}x_1, 2^{\alpha n}x_2, 2^{\beta_2 n}x_3\big)\Big) + 2^{\gamma n} f_3\big(2^{\gamma n}x_1, 2^{\gamma n}x_2, 2^{\beta_1 n}x_3\big)\,.$$

It clearly satisfies Assumption 1. If $\beta_2 = \beta_1$ then Assumption 2 is also satisfied. If $\beta_1 < \beta_2 < 0$ then one must have $f_1(\cdot, -a_3) \equiv 0$ to ensure Assumption 2: if there are two profiles with the same horizontal scale (here $2^{-\alpha n}$) and different vertical scales going both to infinity (since $\beta_2 \neq \beta_1$ and both are negative), then the profile with the largest vertical scale (here $f_1\big(2^{\alpha n}x_1, 2^{\alpha n}x_2, 2^{\beta_1 n}x_3 - a_3\big)$ since $\beta_1 < \beta_2$), must vanish at $x_3 = 0$.

Remark 2.9. If it is assumed that the initial data is bounded also in $L^2(\mathbb{R}^3)$, then the same arguments as those leading to Lemma 5.3 allow to infer that the vertical scales γ^ℓ_n and δ^ℓ_n must all go to zero. In particular Assumption 2 is unnecessary in that case since the hypotheses are never met.

Remark 2.10. Assumption 2 is used in the following to show that profiles do not interact one with another (see Paragraph 4.3).

3. Time evolution of each profile, construction of an approximate solution

In this section we shall construct an approximate solution to the Navier-Stokes equations by evolving in time each individual profile provided in Proposition 2.4 – or rather the version written in (2.7) – either by the Navier-Stokes flow or by a linear transport-diffusion equation, depending on the profiles. First we shall be needing a time-dependent version of the scaling operator $\Lambda^n_{\boldsymbol{\varepsilon},\boldsymbol{\gamma},\boldsymbol{x}}$ given in Definition 2.1.

Definition 3.1. *For any two sequences $\boldsymbol{\varepsilon} = (\varepsilon_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\gamma} = (\gamma_n)_{n\in\mathbb{N}}$ of positive real numbers and any sequence $\boldsymbol{x} = (x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^3$ we define the scaling operator*

$$\widetilde\Lambda^n_{\boldsymbol{\varepsilon},\boldsymbol{\gamma},\boldsymbol{x}}\phi(t,x) := \frac{1}{\varepsilon_n}\phi\left(\frac{t}{\varepsilon_n^2}, \frac{x_h - x_{n,h}}{\varepsilon_n}, \frac{x_3 - x_{n,3}}{\gamma_n}\right)\,.$$

Next let us introduce some notation for function spaces naturally associated with the resolution of the Navier-Stokes equations. We refer to Appendix B for definitions.

Definition 3.2. *We define the following function spaces, for $1 \le p \le \infty$ and $0 < q \le \infty$:*

$$\mathcal{I}_{p,q} := \bigcap_{r=1}^{\infty} \widetilde{L^r}\big(\mathbb{R}^+; \dot B^{-1+\frac{3}{p}+\frac{2}{r}}_{p,q}(\mathbb{R}^3)\big)\,,$$

$$\mathcal{A}_{p,q} := \bigcap_{r=1}^{\infty} \widetilde{L^r}(\mathbb{R}^+; \dot B^{-1+\frac{2}{p}+\frac{2}{r},\frac{1}{p}}_{p,q})\,,$$

$$\mathcal{S}_{p,q} := \widetilde{L^\infty}(\mathbb{R}^+; \dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,q}) \cap \widetilde{L^1}(\mathbb{R}^+; \dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,q} \cap \dot B^{-1+\frac{2}{p},2+\frac{1}{p}}_{p,q})\,.$$

Remark 3.3. The spaces defined above are natural spaces for the resolution of the Navier-Stokes equations: for instance $\mathcal{I}_{p,\infty}$ is associated with small data in $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ (see [12],[52], as well as [3]) and $\mathcal{S}_{p,1}$ with small data in $\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}$ (see Appendix A). Note that $\mathcal{A}_{p,q}$ contains strictly $\mathcal{S}_{p,q}$ and $\mathcal{A}_{p_1,q}$ is embedded in $\mathcal{A}_{p_2,q}$ as soon as $p_1 \leq p_2$, and similarly for $\mathcal{S}_{p_1,q}$ and $\mathcal{S}_{p_2,q}$.

Remark 3.4. The operator $\widetilde{\Lambda}^n_{\boldsymbol{\varepsilon},\boldsymbol{\gamma},\boldsymbol{x}}$ is an isometry in $\mathcal{A}_{p,q}$ for all $1 \leq p \leq \infty$ and $0 < q \leq \infty$. That is however not the case for the space $\mathcal{S}_{p,q}$.

Now let us consider the decomposition (2.7), and evolve each term in time so as to construct by superposition an approximate solution to the Navier-Stokes equations with data $u_{0,n}$. We leave to Section 4 the proof that the superposition is indeed an approximate solution to (NS).

- The first term of the decomposition (2.7) is the weak limit $u_0 \in \mathcal{B}^1_q$, which gives rise to a unique, global solution by assumption: we define $u \in \mathcal{S}_{1,1}(\infty)$ the associate global solution. Due to Corollary 3 stated page 35, we know that actually u belongs to $\mathcal{S}_{1,1}$.

- Let us turn to the profiles in the decomposition (2.7), namely first the terms

$$\tilde{\varphi}^\ell_{0,n} := \Lambda^n_{\boldsymbol{\eta}^\ell,\boldsymbol{\delta}^\ell,\tilde{\boldsymbol{x}}^\ell}\left(\tilde{\phi}^{h,\ell}_\alpha - \frac{\eta^\ell_n}{\delta^\ell_n}(\nabla_h\Delta_h^{-1}\partial_3\phi^{\kappa(\ell)}_\alpha), \phi^{\kappa(\ell)}_\alpha\right)$$

for any $\ell \in \mathbb{N}$. We use the notation of Appendix A, and in particular that of Theorem 4.

Lemma 3.5. *Let $\ell \in \mathbb{N}$. There is $\tilde{L}_0$, independent of n and α, such that the following properties hold.*

- *If $\ell \geq \tilde{L}_0$ and $\kappa(\ell) \geq \tilde{L}_0$, then for all $\alpha \in (0,1)$ and n large enough, $\tilde{\varphi}^\ell_{0,n}$ belongs to $\mathcal{G}$ and the associate solution $\tilde{u}^\ell_n$ to* (NS) *satisfies*

$$\forall \ell \geq \tilde{L}_0 \text{ s.t. } \kappa(\ell) \geq \tilde{L}_0\,, \quad \|\tilde{u}^\ell_n\|_{\mathcal{S}_{3,1}} \leq 2\big(\|\tilde{\phi}^{h,\ell}_\alpha\|_{\dot{B}^{-\frac{1}{3},\frac{1}{3}}_{3,1}} + \|\phi^{\kappa(\ell)}_\alpha\|_{\dot{B}^{-\frac{1}{3},\frac{1}{3}}_{3,1}}\big) \leq 2c_0\,. \tag{3.1}$$

- *For every $\ell \in \mathbb{N}$, if $\eta^\ell_n/\delta^\ell_n$ converges to ∞ when n goes to infinity, then for all $\alpha \in (0,1)$ and for n large enough $\tilde{\varphi}^\ell_{0,n}$ belongs to $\mathcal{G}$: the associate solution $\tilde{u}^\ell_n$ to* (NS) *is bounded in $\mathcal{S}_{3,1}$ and satisfies for all $1 \leq r \leq \infty$ and all $\frac{1}{3} \leq \sigma \leq \frac{1}{3} + \frac{2}{r}$*

$$\tilde{u}^\ell_n \to 0 \quad \text{in} \quad \widetilde{L^r}(\mathbb{R}^+; \dot{B}^{-\frac{1}{3}+\sigma,\frac{2}{r}-\sigma+\frac{1}{3}}_{3,1})\,, \quad n \to \infty\,. \tag{3.2}$$

- *For every $\ell \in \mathbb{N}$, if $\eta^\ell_n/\delta^\ell_n$ converges to 0 when n goes to infinity, then for all $\alpha \in (0,1)$ and for n large enough $\tilde{\varphi}^\ell_{0,n}$ belongs to $\mathcal{G}$: the associate solution $\tilde{u}^\ell_n$ to* (NS) *is uniformly bounded in the space $\mathcal{S}_{1,1}$ and satisfies for all $\alpha \in (0,1)$*

$$\begin{aligned} &\tilde{u}^\ell_n = \widetilde{\Lambda}^n_{\boldsymbol{\eta}^\ell,\boldsymbol{\delta}^\ell,\tilde{\boldsymbol{x}}^\ell}\left(\tilde{U}^{h,\ell} + \frac{\eta^\ell_n}{\delta^\ell_n}U^{\kappa(\ell),h}_n, U^{\kappa(\ell),3}_n\right) + \tilde{R}^\ell_n\,, \quad \tilde{R}^\ell_n \text{ bounded in } \mathcal{S}_{3,1} \\ &\quad \text{with} \quad \tilde{R}^\ell_n \to 0 \text{ in } \widetilde{L^2}(\mathbb{R}^+; \dot{B}^{\frac{2}{3},\frac{1}{3}}_{3,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{5}{3},\frac{1}{3}}_{3,1} \cap \dot{B}^{\frac{2}{3},\frac{4}{3}}_{3,1})\,, \quad n \to \infty\,, \end{aligned} \tag{3.3}$$

while $\tilde{U}^{h,\ell}$, $U^{\kappa(\ell),3}_n$ and $\frac{\eta^\ell_n}{\delta^\ell_n}U^{\kappa(\ell),h}_n$ are smooth and bounded in $\mathcal{S}_{1,1}$.

Finally if $\tilde{\phi}^{h,\ell}(\cdot,z_3) \equiv \phi^{\kappa(\ell)}(\cdot,z_3) \equiv 0$ for some $z_3 \in \mathbb{R}$, then for all $s \geq 0$,

$$\limsup_{n\to\infty} \left\|\tilde{U}^{h,\ell}(\cdot,z_3) + U^{\kappa(\ell),3}_n(\cdot,z_3)\right\|_{L^\infty(\mathbb{R}^+;H^s(\mathbb{R}^2))\cap L^2(\mathbb{R}^+;\dot{H}^{s+1}(\mathbb{R}^2))} \lesssim \alpha\,. \tag{3.4}$$

Proof of Lemma 3.5. • By the stability property (2.3), for all $\beta > 0$ there is $\tilde{L}(\beta)$ such that if $\ell \geq \tilde{L}(\beta)$ and $\kappa(\ell) \geq \tilde{L}(\beta)$, then

$$\|\tilde{\phi}_\alpha^{h,\ell}\|_{\mathcal{B}_q^1} + \|\phi_\alpha^{\kappa(\ell)}\|_{\mathcal{B}_q^1} \leq \beta\,.$$

Then if β is small enough, in particular $\tilde{\phi}_\alpha^{h,\ell}$ is smaller than, say $c_0/2$ in $\dot{B}_{3,1}^{-\frac{1}{3},\frac{1}{3}}$ (by Sobolev embeddings).

Now let $\alpha > 0$ be given and let us consider the initial data $(-\frac{\eta_n^\ell}{\delta_n^\ell}\nabla_h\Delta_h^{-1}\partial_3\phi_\alpha^{\kappa(\ell)}, \phi_\alpha^{\kappa(\ell)})$. Notice that the only possible limit for the ratio of scales associated with $\phi_\alpha^{\kappa(\ell)}$ is zero by Proposition 2.4, so we can restrict our attention here to the case when $\eta_n^\ell/\delta_n^\ell \to 0$. By construction of $\phi_\alpha^{\kappa(\ell)}$ in (2.5), the vector field $\nabla_h\Delta_h^{-1}\partial_3\phi_\alpha^{\kappa(\ell)}$ belongs to $\mathcal{B}_q^1$ for each given α, hence since $\eta_n^\ell/\delta_n^\ell$ converges to 0 when n goes to infinity, then for n large enough and for $\kappa(\ell) \geq \tilde{L}(\beta)$

$$\Big\| -\frac{\eta_n^\ell}{\delta_n^\ell}(\nabla_h\Delta_h^{-1}\partial_3\phi_\alpha^{\kappa(\ell)}), \phi_\alpha^{\kappa(\ell)}\Big\|_{\mathcal{B}_q^1} \leq 2\beta\,.$$

Finally choosing $\beta \leq c_0/4$, for $\ell \geq \tilde{L}(\beta)$, $\kappa(\ell) \geq \tilde{L}(\beta)$ and n large enough (depending on ℓ and α) Theorem 4 applies (using also Remark 2.2) to yield that $\tilde{\varphi}_{0,n}^\ell$ belongs to $\mathcal{G}$ and (3.1) holds.

• If $\eta_n^\ell/\delta_n^\ell$ converges to ∞, then we observe that $\phi_\alpha^{\kappa(\ell)} \equiv 0$ (since as recalled above the only possible limit for the ratio of scales associated with $\phi_\alpha^{\kappa(\ell)}$ is zero) and we have by a direct computation

$$\Big\|\Lambda^n_{\boldsymbol{\eta^\ell},\boldsymbol{\delta^\ell},\tilde{\boldsymbol{x}}^\ell}\left(\tilde{\phi}_\alpha^{h,\ell}, 0\right)\Big\|_{\dot{B}^0_{3,1}} \lesssim \left(\frac{\delta_n^\ell}{\eta_n^\ell}\right)^{\frac{1}{3}}\,.$$

In particular for n large enough the data is small in $\dot{B}^0_{3,1}$ so small data theory of [38] and [52] (see also [3]) gives the result: there is a global solution to (NS) associated with that initial data, which goes to zero (like $(\delta_n^\ell/\eta_n^\ell)^{\frac{1}{3}}$) in $\widetilde{L^\infty}(\mathbb{R}^+;\dot{B}^0_{3,1}) \cap \widetilde{L^1}(\mathbb{R}^+;\dot{B}^2_{3,1})$. By Proposition B.3 and interpolation, it therefore goes to zero in $\widetilde{L^r}(\mathbb{R}^+;\dot{B}_{3,1}^{-\frac{1}{3}+\sigma,\frac{2}{r}-\sigma+\frac{1}{3}})$ for all $1 \leq r \leq \infty$ and all $\sigma \in [\frac{1}{3}, \frac{1}{3} + \frac{2}{r}]$, as expected.

In particular $\widetilde{u}_n^\ell$ is bounded in $\widetilde{L^2}(\mathbb{R}^+;\dot{B}_{3,1}^{\frac{2}{3},\frac{1}{3}})$ which controls the Navier-Stokes equation for data in $\dot{B}_{3,1}^{-\frac{1}{3},\frac{1}{3}}$ (see Theorem 4), so we get in particular that $\widetilde{u}_n^\ell$ is bounded in $\mathcal{S}_{3,1}$.

• Conversely let us suppose that $\eta_n^\ell/\delta_n^\ell$ converges to 0. Then by (isotropic) scale and translation invariance of (NS) we can first rescale by η_n^ℓ and translate by $\widetilde{x}_n^\ell$, hence consider the initial data

$$\begin{aligned}\widetilde{\phi}_{0,n}^\ell(x) &:= \Lambda^n_{\mathbf{1},\frac{\boldsymbol{\delta^\ell}}{\boldsymbol{\eta^\ell}},\mathbf{0}}\left(\tilde{\phi}_\alpha^{h,\ell} - \frac{\eta_n^\ell}{\delta_n^\ell}(\nabla_h\Delta_h^{-1}\partial_3\phi_\alpha^{\kappa(\ell)}), \phi_\alpha^{\kappa(\ell)}\right)(x)\\ &= \left(\tilde{\phi}_\alpha^{h,\ell} - \frac{\eta_n^\ell}{\delta_n^\ell}(\nabla_h\Delta_h^{-1}\partial_3\phi_\alpha^{\kappa(\ell)}), \phi_\alpha^{\kappa(\ell)}\right)(x_h, \frac{\eta_n^\ell}{\delta_n^\ell}x_3)\,.\end{aligned}$$

Since $\eta_n^\ell/\delta_n^\ell \to 0$ as n goes to infinity, we can rely on Theorem 3 in [14] which states that as soon as $\eta_n^\ell/\delta_n^\ell$ is small enough (depending on norms of the profiles $\tilde{\phi}_\alpha^{h,\ell}, \phi_\alpha^{\kappa(\ell)}$), then $\widetilde{\phi}_{0,n}^\ell$ belongs to $\mathcal{G}$ and according to [14] the solution to (NS) associated with $\widetilde{\phi}_{0,n}^\ell$ is of the form

$$\big(\tilde{U}^{h,\ell} + \frac{\eta_n^\ell}{\delta_n^\ell}U_n^{\kappa(\ell),h}, U_n^{\kappa(\ell),3}\big)(t, x_h, \frac{\eta_n^\ell}{\delta_n^\ell}x_3) + \tilde{r}_n^\ell(t,x)$$

where for each $z_3 \in \mathbb{R}$, $\tilde{U}^{h,\ell}(\cdot, z_3)$ is the global solution to the two-dimensional Navier-Stokes equations with data $\tilde{\phi}_\alpha^{h,\ell}(\cdot, z_3)$, while $U_n^{\kappa(\ell)}$ is a divergence-free vector field solving the linear transport-diffusion equation $(T_{\underline{v}}^\varepsilon)$ of [14] with $\underline{v} = \tilde{U}^{h,\ell}$ and $\varepsilon = \eta_n^\ell/\delta_n^\ell$, with data $\big(-\nabla_h \Delta_h^{-1}\partial_3 \phi_\alpha^{\kappa(\ell)}, \phi_\alpha^{\kappa(\ell)}\big)$: we have, for some pressure $p_n^{\kappa(\ell)}$

$$\partial_t U_n^{\kappa(\ell)} + \tilde{U}^{h,\ell}\cdot\nabla^h U_n^{\kappa(\ell)} - \Delta_h U_n^{\kappa(\ell)} - \left(\frac{\eta_n^\ell}{\delta_n^\ell}\right)^2 \partial_3^2 U_n^{\kappa(\ell)} = -\left(\nabla^h, \left(\frac{\eta_n^\ell}{\delta_n^\ell}\right)^2 \partial_3\right) p_n^{\kappa(\ell)}\,.$$

Both $\tilde{U}^{h,\ell}$ and $U_n^{\kappa(\ell)}$ are as smooth as needed.

In particular relying on [14] Proposition 3.2, and [32] (where estimates in the – more difficult – inhomogeneous situation are obtained), we have that $\tilde{U}^{h,\ell}$, $U_n^{\kappa(\ell),3}$ and $\frac{\eta_n^\ell}{\delta_n^\ell} U_n^{\kappa(\ell),h}$ are bounded in $\mathcal{S}_{2,1}$. It is not difficult to prove also (for instance using the estimates of Appendix A) that they are bounded in $\mathcal{S}_{1,1}$.

Furthermore $\widetilde{r}_n^\ell$ goes to zero in $\mathcal{I}_{2,1}$ by [14] (actually the result of [14] only states the convergence to zero in $L^\infty(\mathbb{R}^+; \dot{H}^{\frac12}) \cap L^2(\mathbb{R}^+; \dot{H}^{\frac32})$ but it is clear from the proof that it can be extended all the way to $\mathcal{I}_{2,1}$). It then suffices to unscale to the original data to find the form (3.3), with $\widetilde{R}_n^\ell$ going to zero in $\mathcal{I}_{2,1}$. We infer in particular by Proposition B.3 and Sobolev embeddings that $\widetilde{R}_n^\ell$ goes to zero in $\widetilde{L^2}(\mathbb{R}^+; \dot{B}_{3,1}^{\frac23,\frac13}) \cap L^1(\mathbb{R}^+; \dot{B}_{3,1}^{\frac53,\frac13} \cap \dot{B}_{3,1}^{\frac23,\frac43})$ as required.

Finally let us prove that $\widetilde{R}_n^\ell$ is bounded in $\mathcal{S}_{3,1}$. We notice that due to the above bounds, the function $\widetilde{u}_n^\ell$ solves (NS) and is bounded in $\widetilde{L^2}(\mathbb{R}^+; \dot{B}_{3,1}^{\frac23,\frac13})$ since that holds for the right-hand side of (3.3) by direct inspection. By Theorem 4 this implies that $\widetilde{u}_n^\ell$ is bounded in particular in $\widetilde{L^\infty}(\mathbb{R}^+; \dot{B}_{3,1}^{-\frac13,\frac13})$, which proves the result for $\widetilde{R}_n^\ell$ again inspecting the formula (3.3) giving $\widetilde{u}_n^\ell - \widetilde{R}_n^\ell$ and recalling that $\eta_n^\ell/\delta_n^\ell \to 0$ as n goes to infinity.

To conclude suppose that $\tilde{\phi}^{h,\ell}(\cdot, z_3) \equiv \phi^{\kappa(\ell)}(\cdot, z_3) \equiv 0$ for some $z_3 \in \mathbb{R}$. Then by construction of ϕ_α^ℓ in (2.5) and that of $\tilde{U}^{h,\ell}$ recalled above, the result follows for $\tilde{U}^{h,\ell}(t, \cdot, z_3)$. For $U_n^{\kappa(\ell)}(t, \cdot, z_3)$ we get the result from Proposition 3.2 of [15].

Lemma 3.5 is proved. □

- Now let us consider $\Lambda^n_{\boldsymbol{\varepsilon}^\ell, \boldsymbol{\gamma}^\ell, \boldsymbol{x}^\ell}\left(-\frac{\varepsilon_n^\ell}{\gamma_n^\ell}(\nabla_h \Delta_h^{-1}\partial_3 \phi_\alpha^\ell)(x), \phi_\alpha^\ell(x)\right)$, when $\ell \notin \mathcal{K}(\infty)$.

Lemma 3.6. *Assume $\ell \notin \mathcal{K}(\infty)$. Then there is L_0, independent of n such that the following result holds. For any ℓ and for n large enough, $\Lambda^n_{\boldsymbol{\varepsilon}^\ell, \boldsymbol{\gamma}^\ell, \boldsymbol{x}^\ell}\left(-\frac{\varepsilon_n^\ell}{\gamma_n^\ell}(\nabla_h \Delta_h^{-1}\partial_3 \phi_\alpha^\ell)(x), \phi_\alpha^\ell(x)\right)$ belongs to $\mathcal{G}$ and the associate solution u_n^ℓ to* (NS) *enjoys the following properties.*

- *For every $\ell \geq L_0$, $\alpha \in (0,1)$ and $n \in \mathbb{N}$ large enough,*

$$\|u_n^\ell\|_{\mathcal{S}_{3,1}} \leq 2\|\phi_\alpha^\ell\|_{\dot{B}_{3,1}^{-\frac13,\frac13}} \leq 2c_0\,. \tag{3.5}$$

- *For every $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and n large enough, the sequence u_n^ℓ is uniformly bounded in the space $\widetilde{L^\infty}(\mathbb{R}^+; \dot{B}_{3,1}^{-\frac13,\frac13}) \cap L^1(\mathbb{R}^+; \dot{B}_{3,1}^{\frac53,\frac13} \cap \dot{B}_{3,1}^{\frac23,\frac43})$ and satisfies*

$$\begin{gathered} u_n^\ell = \widetilde{\Lambda}^n_{\boldsymbol{\varepsilon}^\ell, \boldsymbol{\gamma}^\ell, \boldsymbol{x}^\ell}\left(\frac{\varepsilon_n^\ell}{\gamma_n^\ell} U_n^{\ell,h}, U_n^{\ell,3}\right) + R_n^\ell \quad \textit{where} \\ R_n^\ell \to 0 \ \textit{in}\ \widetilde{L^2}(\mathbb{R}^+; \dot{B}_{3,1}^{\frac23,\frac13}) \cap L^1(\mathbb{R}^+; \dot{B}_{3,1}^{\frac53,\frac13} \cap \dot{B}_{3,1}^{\frac23,\frac43})\,, \quad n \to \infty\,, \end{gathered} \tag{3.6}$$

and all the properties stated in Lemma 3.5 hold.

Proof of Lemma 3.6. The proof follows the lines of the proof of Lemma 3.5, and is in fact easier. One first uses the stability property (2.3) to obtain the existence of L_0 such that for all $\ell \geq L_0$, for each $\alpha \in (0,1)$ and for n large enough,

$$\|(\nabla_h \Delta_h^{-1}\partial_3 \phi_\alpha^\ell, \phi_\alpha^\ell)\|_{\dot{B}^{0,\frac{1}{2}}_{2,1}} \leq c_0$$

and Theorem 4 applies. Then we notice again that by rescaling and translation it is enough to consider the vector field $\Lambda^n_{\mathbf{1},\frac{\boldsymbol{\gamma}^\ell}{\boldsymbol{\varepsilon}^\ell},\mathbf{0}}\Big(-\frac{\varepsilon_n^\ell}{\gamma_n^\ell}(\nabla_h\Delta_h^{-1}\partial_3\phi_\alpha^\ell)(x), \phi_\alpha^\ell(x)\Big)$, and again [14] gives the result (recalling that $\varepsilon_n^\ell/\gamma_n^\ell$ goes to zero by Proposition 2.4). Compared with the proof of Lemma 3.5, in this case the profile U_n^ℓ is simply a solution to the heat equation in $\mathbb{R}^3$ with viscosity $(\varepsilon_n^\ell/\gamma_n^\ell)^2$ in the third direction (see [14] system $(T^\varepsilon_{\underline{v}})$, with $\underline{v} \equiv 0$ and $\varepsilon = \varepsilon_n^\ell/\gamma_n^\ell$). The lemma is proved. □

In the following we define, with the notation of Lemmas 3.5 and 3.6,

$$\begin{aligned} \mathcal{U}_n^L &:= \sum_{1\leq \ell\leq L} \tilde{u}_n^\ell + \sum_{\substack{1\leq\kappa(\ell)\leq L\\ \ell>L}} \tilde{u}_n^\ell + \sum_{\ell=1}^L u_n^\ell\,, \quad \text{and} \\ \mathcal{R}_n^L &:= \sum_{1\leq \ell\leq L} \tilde{R}_n^\ell + \sum_{\substack{1\leq\kappa(\ell)\leq L\\ \ell>L}} \tilde{R}_n^\ell + \sum_{\ell=1}^L R_n^\ell\,, \end{aligned} \tag{3.7}$$

and we recall that

$$\forall L\,, \quad \lim_{n\to\infty} \|\mathcal{R}_n^L\|_{\widetilde{L^2}(\mathbb{R}^+;\dot{B}^{\frac{2}{3},\frac{1}{3}}_{3,1})\cap L^1(\mathbb{R}^+;\dot{B}^{\frac{5}{3},\frac{1}{3}}_{3,1}\cap \dot{B}^{\frac{2}{3},\frac{4}{3}}_{3,1})} = 0\,. \tag{3.8}$$

- Finally we propagate all the remaining terms in (2.7) by the heat equation: we define

$$\mathcal{V}_n^L := \rho_n^L + \Psi_n^L \tag{3.9}$$

with

$$\begin{aligned} \Psi_n^L(t) := e^{t\Delta}\Big((\tilde{\psi}_n^{h,L} - \nabla_h\Delta_h^{-1}\partial_3\psi_n^L, \psi_n^L) - \sum_{\substack{\ell>L\\ \ell\in\mathcal{K}(L)}} \Lambda^n_{\boldsymbol{\varepsilon}^\ell,\boldsymbol{\gamma}^\ell,\boldsymbol{x}^\ell}\big(-\frac{\varepsilon_n^\ell}{\gamma_n^\ell}\nabla_h\Delta_h^{-1}\partial_3\phi^\ell, \phi^\ell\big) \\ - \sum_{\substack{\ell>L\\ 1\leq\kappa(\ell)\leq L}} \Lambda^n_{\boldsymbol{\eta}^\ell,\boldsymbol{\delta}^\ell,\tilde{\boldsymbol{x}}^\ell}\big(\tilde{\phi}_\alpha^{h,\ell}, 0\big)\Big) \end{aligned}$$

and

$$\begin{aligned} \rho_n^L(t) := e^{t\Delta}\Big(\sum_{\ell=1}^L \Lambda^n_{\boldsymbol{\eta}^\ell,\boldsymbol{\delta}^\ell,\tilde{\boldsymbol{x}}^\ell}(\tilde{r}_\alpha^{h,\ell} - \frac{\eta_n^\ell}{\delta_n^\ell}\nabla_h\Delta_h^{-1}\partial_3 r_\alpha^{\kappa(\ell)}, r_\alpha^{\kappa(\ell)}) \\ + \sum_{\substack{\kappa(\ell)=1\\ \kappa(\ell)\in\mathcal{K}(\infty)\setminus\mathcal{K}(L)}}^L \Lambda^n_{\boldsymbol{\eta}^\ell,\boldsymbol{\delta}^\ell,\tilde{\boldsymbol{x}}^\ell}(\tilde{r}_\alpha^{h,\ell} - \frac{\eta_n^\ell}{\delta_n^\ell}\nabla_h\Delta_h^{-1}\partial_3 r_\alpha^{\kappa(\ell)}), r_\alpha^{\kappa(\ell)}) \\ + \sum_{\substack{\ell=1\\ \ell\notin\mathcal{K}(\infty)}}^L \Lambda^n_{\boldsymbol{\varepsilon}^\ell,\boldsymbol{\gamma}^\ell,\boldsymbol{x}^\ell}(-\frac{\varepsilon_n^\ell}{\gamma_n^\ell}\nabla_h\Delta_h^{-1}\partial_3 r_\alpha^\ell, r_\alpha^\ell) - \sum_{\substack{\ell>L\\ 1\leq\kappa(\ell)\leq L}} \Lambda^n_{\boldsymbol{\eta}^\ell,\boldsymbol{\delta}^\ell,\tilde{\boldsymbol{x}}^\ell}(\tilde{r}_\alpha^{h,\ell}, 0)\Big) \end{aligned}$$

We notice that by (2.5)

$$\begin{aligned}&\forall L\in\mathbb{N}\,,\quad \limsup_{n\to\infty}\|\rho_n^L\|_{\mathcal{S}_{3,1}}\le C(L)\,\alpha\,,\\ &\text{and}\quad \limsup_{n\to\infty}\big(\|\Psi_n^{L,h}\|_{\mathcal{S}_{3,1}+\widetilde{\mathcal{S}}_{3,1}}+\|\Psi_n^{L,3}\|_{\mathcal{S}_{3,1}}\big)\to 0\,,\quad L\to\infty \text{ uniformly in }\alpha\,,\end{aligned}\tag{3.10}$$

where $\widetilde{\mathcal{S}}_{3,1}:=\bigcap_{r=1}^{\infty}\bigcap_{\sigma=0}^{\frac{2}{r}}\widetilde{L^r}(\mathbb{R}^+;\dot B_{3,1}^{\frac{2}{3}+\sigma,\frac{2}{r}-\sigma-\frac{2}{3}})$. The presence of that space is due to terms of the type $\nabla_h\Delta_h^{-1}\partial_3\phi^\ell$ and those bounds are due to (2.1) as well as the stability property (2.3) and the fact that $\varepsilon_n^\ell/\gamma_n^\ell\to 0$. In particular

$$\limsup_{n\to\infty}\big(\|\Psi_n^L\|_{L^1(\mathbb{R}^+;\dot B_{3,1}^{\frac{5}{3},\frac{1}{3}}\cap\dot B_{3,1}^{\frac{2}{3},\frac{4}{3}})\cap\widetilde{L^2}(\mathbb{R}^+;\dot B_{3,1}^{\frac{2}{3},\frac{1}{3}})}\big)\to 0\,,\quad L\to\infty \text{ uniformly in }\alpha\,.\tag{3.11}$$

4. Global regularity for the profiles superposition

Now we need to superpose each of the solutions constructed in the previous section, and check that the superposition is indeed a good approximate solution. This will prove Theorem 2, and at the end of this section we shall show how the methods developed here give easily Corollaries 1 and 2.

4.1. Statement of the superposition result and main steps of its proof.

The main result is the following, where we use the notation of the previous section.

Proposition 4.1. *For n and L large enough, α small enough and up to an extraction, we have*

$$u_n=u+\mathcal{U}_n^L+\mathcal{V}_n^L+w_n^L\,,\tag{4.1}$$

where w_n^L belongs to $\mathcal{S}_{3,1}$ with $\lim_{\alpha\to 0}(\limsup_{n\to\infty}\|w_n^L\|_{\mathcal{S}_{3,1}})\to 0$ as $L\to\infty$.

Remark 4.2. The choice of the function space $\mathcal{S}_{3,1}$ in the statement of Proposition 4.1 is for convenience, we have not tried to optimize on the integrability index here and other spaces would certainly do as well.

Remark 4.3. This proposition proves Theorem 2. Indeed the sequence (u_n) belongs in particular to the space $\widetilde{L^2}(\mathbb{R}^+;\dot B_{3,1}^{\frac{2}{3},\frac{1}{3}})$, since the results of the previous section show that this is the case for all the terms in the right-hand side of (4.1). But we know from Theorem 4 that this norm controls the equation so the result follows.

Proof of Proposition 4.1. Let u_n be the solution of (NS) associated with the data $u_{0,n}$, which a priori has a finite life span T_n^*, and define

$$w_n^L:=u_n-G_n^L\quad\text{with}\quad G_n^L:=u+F_n^L\quad\text{and}\quad F_n^L:=\mathcal{U}_n^L+\mathcal{V}_n^L\,.$$

The vector field w_n^L satisfies

$$\partial_t w_n^L+\mathbb{P}(w_n^L\cdot\nabla w_n^L+G_n^L\cdot\nabla w_n^L+w_n^L\cdot\nabla G_n^L)-\Delta w_n^L=-\mathbb{P}Z_n^L\,,\quad \operatorname{div} w_n^L=0$$

with initial data $w_{n|t=0}^L=0$, and where, recalling the definitions of $\mathcal{U}_n^L$ and $\mathcal{V}_n^L$ in (3.7) and (3.9) respectively,

$$\begin{aligned}Z_n^L:=&\sum_{\ell\neq k}u_n^\ell\cdot\nabla u_n^k+\sum_{\ell\neq k}\tilde u_n^\ell(\mathbf{1}_{1\le\ell\le L}+\mathbf{1}_{\substack{1\le\kappa(\ell)\le L\\ \ell>L}})\cdot\nabla\tilde u_n^k(\mathbf{1}_{1\le k\le L}+\mathbf{1}_{\substack{1\le\kappa(k)\le L\\ \ell>L}})\\ &+\sum_{\ell\neq k}\big(\tilde u_n^\ell(\mathbf{1}_{1\le\ell\le L}+\mathbf{1}_{\substack{1\le\kappa(\ell)\le L\\ \ell>L}})\cdot\nabla u_n^k+u_n^\ell\cdot\nabla\tilde u_n^k(\mathbf{1}_{1\le k\le L}+\mathbf{1}_{\substack{1\le\kappa(k)\le L\\ \ell>L}})\big)\\ &+u\cdot\nabla F_n^L+F_n^L\cdot\nabla u+\mathcal{U}_n^L\cdot\nabla\mathcal{V}_n^L+\mathcal{V}_n^L\cdot\nabla\mathcal{U}_n^L+\mathcal{V}_n^L\cdot\nabla\mathcal{V}_n^L\,.\end{aligned}$$

The proposition follows from the two following lemmas.

Lemma 4.4. *Define* $\mathcal{Y} := L^2(\mathbb{R}^+; \dot{B}^{\frac{2}{3},\frac{1}{3}}_{3,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{5}{3},\frac{1}{3}}_{3,1} \cap \dot{B}^{\frac{2}{3},\frac{4}{3}}_{3,1})$. *With the notation of Lemmas 3.5 and 3.6, there is a constant* K *(depending on* $L_0, \widetilde{L}_0$ *and bounds on* u_0, (u_n) *and* u*) such that one can decompose* $G^L_n = \mathcal{G}^{L,1}_n + \mathcal{G}^{L,2}_n$, *with the following properties: for each* $L \in \mathbb{N}$ *and each* $\alpha \in (0,1)$ *there is* $N(L,\alpha)$ *such that*

$$\|\mathcal{G}^{L,1}_n\|_{\mathcal{Y}} \leq K \quad \text{for } n \geq N(L,\alpha)\,,$$

while for all $L \in \mathbb{N}$ *there is* $\alpha_0 > 0$ *such that*

$$\forall\, 0 < \alpha \leq \alpha_0\,, \quad \|\mathcal{G}^{L,2}_n\|_{\mathcal{Y}} \leq K \quad \text{uniformly in } n\,.$$

Lemma 4.5. *Define*

$$\mathcal{X} := L^1(\mathbb{R}^+; \dot{B}^{-\frac{1}{3},\frac{1}{3}}_{3,1}) + \widetilde{L^2}(\mathbb{R}^+; \dot{B}^{-\frac{1}{3},-\frac{2}{3}}_{3,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{2}{3},-\frac{2}{3}}_{3,1})\,.$$

We can write $Z^L_n = \mathcal{Z}^{L,1}_n + \mathcal{Z}^{L,2}_n + \mathcal{Z}^{L,3}_n$ *with*

$$\limsup_{L\to\infty} \|\mathcal{Z}^{L,1}_n\|_{\mathcal{X}} = 0 \text{ uniformly in } n, \alpha\,, \tag{4.2}$$

$$\forall L, \quad \limsup_{\alpha\to 0} \|\mathcal{Z}^{L,2}_n\|_{\mathcal{X}} = 0 \text{ uniformly in } n\,, \tag{4.3}$$

$$\text{and} \quad \forall L\,, \forall \alpha\,, \quad \limsup_{n\to\infty} \|\mathcal{Z}^{L,3}_n\|_{\mathcal{X}} = 0\,. \tag{4.4}$$

Assume indeed for the time being that those two lemmas are true. Then we start by choosing L large enough so that uniformly in α and N one has

$$\|\mathcal{Z}^{L,1}_n\|_{\mathcal{X}} \leq \frac{c_0}{12} \exp\big(-2Kc_0^{-1}\big) \quad \text{uniformly in } n, \alpha \tag{4.5}$$

with the notation of Theorem 5 stated and proved in Appendix A, and Lemma 4.5. Then now that L is fixed we choose $\alpha \in (0, \alpha_0)$ small enough so that

$$\|\mathcal{Z}^{L,2}_n\|_{\mathcal{X}} \leq \frac{c_0}{12} \exp\big(-2Kc_0^{-1}\big) \quad \text{uniformly in } n \tag{4.6}$$

and

$$\|\mathcal{G}^{L,2}_n\|_{\mathcal{Y}} \leq K \quad \text{uniformly in } n\,,$$

with the notation of Lemma 4.4. Finally now that L and α are fixed we take $N_0 \geq N(L,\alpha)$ so that for all $n \geq N_0$,

$$\|\mathcal{Z}^{L,3}_n\|_{\mathcal{X}} \leq \frac{c_0}{12} \exp\big(-2Kc_0^{-1}\big) \tag{4.7}$$

and

$$\|\mathcal{G}^{L,1}_n\|_{\mathcal{Y}} \leq K\,.$$

It then suffices to apply Theorem 5 in Appendix A with $U = G^L_n$, $F = Z^L_n$ and data $u_0 \equiv 0$, noticing that $\mathcal{X} = \mathcal{X}_{3,1}$ and $\mathcal{Y} \subset \mathcal{Y}_{3,1}$. The result follows immediately: we get that w^L_n belongs to $\mathcal{S}_{3,1}$, and the fact that $\lim_{\alpha\to 0}(\limsup_{n\to\infty} \|w^L_n\|_{\mathcal{S}_{3,1}}) \to 0$ as $L \to \infty$ is due to the fact that one can choose the bounds in (4.5)-(4.7) as small as one want, provided L and n are large enough, and α is small enough. □

The two coming paragraphs are devoted to the proofs of Lemmas 4.4 and 4.5, thus achieving the proof of Theorem 2. The final paragraph of this section contains the proofs of Corollaries 1 and 2.

4.2. **Study of the drift term G_n^L.**

Proof of Lemma 4.4. Recall that $G_n^L = u + F_n^L = u + \mathcal{U}_n^L + \mathcal{V}_n^L$ with the notation of Section 3, so since we know that u belongs to $\mathcal{S}_{2,1}$, which embeds continuously in $\mathcal{Y}$, and u depends neither on L, on α nor on n, we need to study F_n^L. According to Lemmas 3.5 and 3.6 and recalling the notation (3.7), we can split $F_n^L = \mathcal{U}_n^L + \mathcal{V}_n^L$ into $F_n^L := F_n^{L,1} + F_n^{L,2} + \mathcal{V}_n^L$, with

$$F_n^{L,1} := \sum_{\substack{1\le\ell\le L\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell + \sum_{\substack{1\le\kappa(\ell)\le L\\ \ell>L\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell + \sum_{\ell=1}^{L} u_n^\ell \quad \text{and} \quad F_n^{L,2} := \sum_{\substack{\ell=1\\ \eta_n^\ell/\delta_n^\ell\to\infty}}^{L} \tilde{u}_n^\ell\,. \tag{4.8}$$

The result (3.2) deals with $F_n^{L,2}$, since according to (3.2), $\tilde{u}_n^\ell$ goes to zero in $\mathcal{Y}$ for each ℓ as n goes to infinity. So that term is incorporated in the term $\mathcal{G}_n^{L,1}$.
Now let us consider $F_n^{L,1}$. We can decompose the sum again into several pieces, writing with the notation of Lemmas 3.5 and 3.6, for all $L > \max(L_0, \tilde{L}_0)$,

$$\sum_{\ell=1}^{L} u_n^\ell = \sum_{\ell=1}^{L_0} u_n^\ell + \sum_{\ell=L_0+1}^{L} u_n^\ell\,,$$

$$\sum_{\substack{1\le\ell\le L\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell = \sum_{\substack{1\le\ell\le \tilde{L}_0\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell + \sum_{\substack{\tilde{L}_0<\ell\le L\\ 1\le\kappa(\ell)\le\tilde{L}_0\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell + \sum_{\substack{\tilde{L}_0<\ell\le L\\ \tilde{L}_0<\kappa(\ell)\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell\,,$$

$$\text{and} \quad \sum_{\substack{1\le\kappa(\ell)\le L\\ \ell>L\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell = \sum_{\substack{1\le\kappa(\ell)\le \tilde{L}_0\\ \ell>L\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell + \sum_{\substack{\tilde{L}_0<\kappa(\ell)\le L\\ \ell>L\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell\,.$$

In all three right-hand-sides, the easiest term to deal with is the last one: indeed we can write

$$\Big\| \sum_{\ell=L_0+1}^{L} u_n^\ell + \sum_{\substack{\tilde{L}_0<\ell\le L\\ \tilde{L}_0<\kappa(\ell)\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell + \sum_{\substack{\tilde{L}_0<\kappa(\ell)\le L\\ \ell>L\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell \Big\|_{\mathcal{Y}} \lesssim \sum_{\ell=L_0+1}^{L} \|u_n^\ell\|_{\mathcal{Y}} + \sum_{\substack{\tilde{L}_0<\ell\\ \tilde{L}_0<\kappa(\ell)}} \|\tilde{u}_n^\ell\|_{\mathcal{Y}}\,.$$

Then by (3.1) and (3.5) we infer that as soon as n is large enough (depending on the choice of L and α)

$$\Big\| \sum_{\ell=L_0+1}^{L} u_n^\ell + \sum_{\substack{\tilde{L}_0<\ell\lesssim L\\ \tilde{L}_0<\kappa(\ell)\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell + \sum_{\substack{\tilde{L}_0<\kappa(\ell)\lesssim L\\ \ell>L\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell \Big\|_{\mathcal{Y}} \le \sum_{L_0<\ell} \|\phi_\alpha^\ell\|_{\dot{B}^{-\frac13,\frac13}_{3,1}} + \sum_{\tilde{L}_0<\ell} \|\tilde{\phi}_\alpha^{h,\ell}\|_{\dot{B}^{-\frac13,\frac13}_{3,1}}$$

and the conclusion comes from the embedding of $\mathcal{B}_q^1$ into $\dot{B}^{-\frac13,\frac13}_{3,1}$ along with the stability property (2.3): for $n \ge N(L,\alpha)$

$$\Big\| \sum_{\ell=L_0+1}^{L} u_n^\ell + \sum_{\substack{\tilde{L}_0<\ell\le L\\ \tilde{L}_0<\kappa(\ell)\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell + \sum_{\substack{\tilde{L}_0<\kappa(\ell)\le L\\ \ell>L\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell \Big\|_{\mathcal{Y}} \lesssim \sum_{L_0<\ell} \|\phi_\alpha^\ell\|_{\mathcal{B}_q^1} + \sum_{\tilde{L}_0<\ell} \|\tilde{\phi}_\alpha^{h,\ell}\|_{\mathcal{B}_q^1} \le C\,.$$

So $\displaystyle\sum_{\ell=L_0+1}^{L} u_n^\ell + \sum_{\substack{\tilde{L}_0<\ell\le L\\ \tilde{L}_0<\kappa(\ell)\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell + \sum_{\substack{\tilde{L}_0<\kappa(\ell)\le L\\ \ell>L\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell$ is of the type $\mathcal{G}_n^{L,1}$.

Now let us estimate $\sum_{\ell=1}^{L_0} u_n^\ell$ and $\sum_{\substack{1\le\ell\le\tilde L_0\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde u_n^\ell$. There is of course no uniformity problem in L and we simply use the uniform bound in $\mathcal{Y}$ provided in Lemmas 3.5 and 3.6. The terms $\sum_{\substack{\tilde L_0+1\le\ell\le L\\ 1\le\kappa(\ell)\le L_0\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde u_n^\ell$ and $\sum_{\substack{1\le\kappa(\ell)\le L_0\\ \ell>L\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde u_n^\ell$ are dealt with similarly and all those three terms are also of the type $\mathcal{G}_n^{L,1}$. Choosing $\mathcal{G}_n^{L,2} := \mathcal{V}_n^L$ and using (3.10) and (3.11) concludes the proof of Lemma 4.4. □

Remark 4.6. This argument shows that $\mathcal{U}_n^L$ is uniformly bounded in the space $\mathcal{S}_{3,1}$.

Remark 4.7. It is important to have chosen the initial data bounded in a space of the type $\dot B^{s,s'}_{p,q}$ with $p=1>q$ (hence in particular with $p=1=q$ by embedding), as it enables us to prove easily the uniform bound on $F_n^{L,1}$. As seen for instance in [28], it is indeed possible to prove such a bound when $p=q$ and it is not clear how to prove it in the general case, when $p\neq q$. Then it is very natural to pick $q\le 1$ as explained in the introduction in order to have a good Cauchy theory for the Navier-Stokes equations in anisotropic spaces, and finally the choice $q<1$ implies by interpolation that the remainders are small precisely in a space where the Cauchy theory for (NS) is satisfactory (namely $q=1$).

4.3. **Study of the forcing term.**

Proof of Lemma 4.5. We recall that

$$\begin{aligned}
Z_n^L := &\sum_{\ell\neq k} u_n^\ell\cdot\nabla u_n^k + \sum_{\ell\neq k}\tilde u_n^\ell(\mathbf{1}_{1\le\ell\le L}+\mathbf{1}_{\substack{1\le\kappa(\ell)\le L\\ \ell>L}})\cdot\nabla\tilde u_n^k(\mathbf{1}_{1\le k\le L}+\mathbf{1}_{\substack{1\le\kappa(k)\le L\\ \ell>L}})\\
&+\sum_{\ell\neq k}\big(\tilde u_n^\ell(\mathbf{1}_{1\le\ell\le L}+\mathbf{1}_{\substack{1\le\kappa(\ell)\le L\\ \ell>L}})\cdot\nabla u_n^k + u_n^\ell\cdot\nabla\tilde u_n^k(\mathbf{1}_{1\le k\le L}+\mathbf{1}_{\substack{1\le\kappa(k)\le L\\ \ell>L}})\big)\\
&\qquad + u\cdot\nabla F_n^L + F_n^L\cdot\nabla u + \mathcal{U}_n^L\cdot\nabla\mathcal{V}_n^L + \mathcal{V}_n^L\cdot\nabla\mathcal{U}_n^L + \mathcal{V}_n^L\cdot\nabla\mathcal{V}_n^L\,.
\end{aligned}$$

We define

$$\begin{aligned}
H_n^{L,1} := &\sum_{\ell\neq k} u_n^\ell\cdot\nabla u_n^k + \sum_{\ell\neq k}\tilde u_n^\ell(\mathbf{1}_{1\le\ell\le L}+\mathbf{1}_{\substack{1\le\kappa(\ell)\le L\\ \ell>L}})\cdot\nabla\tilde u_n^k(\mathbf{1}_{1\le k\le L}+\mathbf{1}_{\substack{1\le\kappa(k)\le L\\ \ell>L}})\\
&+\sum_{\ell\neq k}\big(\tilde u_n^\ell(\mathbf{1}_{1\le\ell\le L}+\mathbf{1}_{\substack{1\le\kappa(\ell)\le L\\ \ell>L}})\cdot\nabla u_n^k + u_n^\ell\cdot\nabla\tilde u_n^k(\mathbf{1}_{1\le k\le L}+\mathbf{1}_{\substack{1\le\kappa(k)\le L\\ \ell>L}})\big)\,,
\end{aligned}$$

$$H_n^{L,2} := \mathcal{U}_n^L\cdot\nabla\mathcal{V}_n^L + \mathcal{V}_n^L\cdot\nabla\mathcal{U}_n^L + \mathcal{V}_n^L\cdot\nabla\mathcal{V}_n^L \quad\text{and}\quad H_n^{L,3} := u\cdot\nabla F_n^L + F_n^L\cdot\nabla u\,.$$

Let start by discussing $H_n^{L,1}$. We shall actually only deal with

$$\sum_{1\le\ell\neq k\le L}\tilde u_n^\ell\cdot\nabla\tilde u_n^k = \sum_{1\le\ell\neq k\le L}\mathrm{div}\,\big(\tilde u_n^\ell\otimes\tilde u_n^k\big)\,,$$

as all the other terms in $H_n^{L,1}$ can be dealt with similarly. Referring to Lemma 3.5, we know that this term can in turn be split into two parts, defining

$$H_n^{L,1,1} := \sum_{\substack{1\le \ell\neq k\le L\\ \eta_n^\ell/\delta_n^\ell\to\infty}} \operatorname{div}\left(\tilde u_n^\ell\otimes\tilde u_n^k+\tilde u_n^k\otimes\tilde u_n^\ell\right)+\sum_{\substack{1\le \ell\neq k\le L\\ \eta_n^\ell/\delta_n^\ell+\eta_n^k/\delta_n^k\to 0}} \operatorname{div}\left(\tilde R_n^\ell\otimes\tilde u_n^k+\tilde u_n^k\otimes\tilde R_n^\ell\right),$$

$$H_n^{L,1,2} := \sum_{\substack{1\le \ell\neq j\le L\\ \eta_n^\ell/\delta_n^\ell+\eta_n^j/\delta_n^j\to 0}} \operatorname{div}\Big(\widetilde\Lambda^n_{\boldsymbol{\eta^\ell},\boldsymbol{\delta^\ell},\boldsymbol{\tilde x^\ell}}\big(\tilde U^{h,\ell}+\frac{\eta_n^\ell}{\delta_n^\ell}U_n^{\kappa(\ell),h},U_n^{\kappa(\ell),3}\big)$$
$$\otimes\widetilde\Lambda^n_{\boldsymbol{\eta^j},\boldsymbol{\delta^j},\boldsymbol{\tilde x^j}}\big(\tilde U^{h,j}+\frac{\eta_n^j}{\delta_n^j}U_n^{\kappa(j),h},U_n^{\kappa(j),3}\big)\Big).$$

The first term $H_n^{L,1,1}$ is dealt with using product laws in anisotropic Besov spaces (see Appendix B). On the one hand we have for any $j\in\{1,2\}$, by (B.4),

$$\begin{aligned}\|\partial_j(fg)\|_{\widetilde{L^1}(\mathbb{R}^+;\dot B^{-\frac13,\frac13}_{3,1})} &\lesssim \|fg\|_{\widetilde{L^1}(\mathbb{R}^+;\dot B^{\frac23,\frac13}_{3,1})}\\ &\lesssim \|f\|_{\widetilde{L^2}(\mathbb{R}^+;\dot B^{\frac23,\frac13}_{3,1})}\|g\|_{\widetilde{L^2}(\mathbb{R}^+;\dot B^{\frac23,\frac13}_{3,1})},\end{aligned}\tag{4.9}$$

and on the other hand estimate (B.5) gives

$$\begin{aligned}\|\partial_3(fg)\|_{\widetilde{L^2}(\mathbb{R}^+;\dot B^{-\frac13,-\frac23}_{3,1})} &\lesssim \|fg\|_{\widetilde{L^2}(\mathbb{R}^+;\dot B^{-\frac13,\frac13}_{3,1})}\\ &\lesssim \|f\|_{\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{-\frac13,\frac13}_{3,1})}\|g\|_{\widetilde{L^2}(\mathbb{R}^+;\dot B^{\frac23,\frac13}_{3,1})}.\end{aligned}\tag{4.10}$$

and by (B.4) again

$$\begin{aligned}\|\partial_3(fg)\|_{\widetilde{L^1}(\mathbb{R}^+;\dot B^{\frac23,-\frac23}_{3,1})} &\lesssim \|fg\|_{\widetilde{L^1}(\mathbb{R}^+;\dot B^{\frac23,\frac13}_{3,1})}\\ &\lesssim \|f\|_{\widetilde{L^2}(\mathbb{R}^+;\dot B^{\frac23,\frac13}_{3,1})}\|g\|_{\widetilde{L^2}(\mathbb{R}^+;\dot B^{\frac23,\frac13}_{3,1})}.\end{aligned}\tag{4.11}$$

So using (3.2) along with the uniform bounds provided by Lemma 3.5 gives

$$\forall L,\quad \lim_{n\to\infty}\Big\|\sum_{\substack{1\le \ell\neq k\le L\\ \eta_n^\ell/\delta_n^\ell\to\infty}}\operatorname{div}\left(\tilde u_n^\ell\otimes\tilde u_n^k\right)\Big\|_{\mathcal{X}}=0.\tag{4.12}$$

The terms $\tilde R_n^\ell\otimes\tilde u_n^k$ are dealt with in the same way using Lemma 3.5: we find that $\widetilde H_n^{L,1,1}$ satisfies the bound (4.2).

The same product laws (using the structure of the nonlinear term) enable us to deal with $H_n^{L,2}$, recalling that

$$H_n^{L,2} := \mathcal{U}_n^L\cdot\nabla\mathcal{V}_n^L+\mathcal{V}_n^L\cdot\nabla\mathcal{U}_n^L+\mathcal{V}_n^L\cdot\nabla\mathcal{V}_n^L$$

using (3.10)-(3.11) to estimate $\mathcal{V}_n^L$, and Remark 4.6 for $\mathcal{U}_n^L$. To control $\mathcal{V}_n^L\cdot\nabla\mathcal{V}_n^L$ for instance, we notice that the horizontal component does not belong a priori to $\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{-\frac13,\frac13}_{3,1})$ (see (3.10)) but that is not a problem as in (4.10), due to the structure of the nonlinear term, one of the two functions is necessarily a third component, which does belong to $\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{-\frac13,\frac13}_{3,1})$. We argue similarly for all the other terms.

Next let us consider the term $H_n^{L,1,2}$ and prove it satisfies the bounds (4.3)-(4.4). Let us define a typical term

$$U_n^{j,\ell} := \widetilde\Lambda^n_{\boldsymbol{\eta^\ell},\boldsymbol{\delta^\ell},\boldsymbol{\tilde x^\ell}}\tilde U^{h,\ell}\otimes\widetilde\Lambda^n_{\boldsymbol{\eta^j},\boldsymbol{\delta^j},\boldsymbol{\tilde x^j}}\tilde U^{h,j},$$

and first show that

$$\text{div } U_n^{j,\ell} \text{ is bounded in } L^1(\mathbb{R}^+;\dot B^{1,1}_{1,1})\cap\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{-1,1}_{1,1})+L^1(\mathbb{R}^+;\dot B^{2,0}_{1,1})\cap\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{0,0}_{1,1})\,. \tag{4.13}$$

This follows from the fact that $\tilde U^{h,\ell}$ belongs to $L^2(\mathbb{R}^+;\dot B^{2,1}_{1,1})\cap\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{1,1}_{1,1})$ (see Lemma 3.5 for that result): we know indeed that $\dot B^{2,1}_{1,1}$ is an algebra and that the product of two functions in $\dot B^{1,1}_{1,1}$ belongs to $\dot B^{0,1}_{1,1}$ (see Appendix B). Since $\widetilde{L^2}(\mathbb{R}^+;\dot B^{2,1}_{1,1})\cap\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{1,1}_{1,1})$ is invariant through the action of $\widetilde\Lambda^n_{\boldsymbol{\eta^\ell},\boldsymbol{\delta^\ell},\boldsymbol{\tilde x^\ell}}$ (see Remark 3.4) the result (4.13) follows.

Now let us prove that $U_n^{j,\ell}$ goes to zero in $L^1(\mathbb{R}^+;\dot B^{2,1}_{1,1})\cap\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{0,1}_{1,1})$, as in (4.3)-(4.4): $\operatorname{div} U_n^{j,\ell}$ will then go to zero in $L^1(\mathbb{R}^+;\dot B^{1,1}_{1,1})\cap\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{-1,1}_{1,1})+L^1(\mathbb{R}^+;\dot B^{2,0}_{1,1})\cap\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{0,0}_{1,1})$ which is contained in $\mathcal{X}$.

Let us start by the $L^1(\mathbb{R}^+;\dot B^{2,1}_{1,1})$ norm. By the equivalent formulation in terms of the heat flow (B.3), we know that $\tau^{-2}\tau'^{-\frac32}K_h(\tau)K_v(\tau')U_n^{j,\ell}(t,x)$ is uniformly bounded in L^1 in all variables. To prove the result, by Lebesgue's dominated convergence theorem we shall therefore prove the pointwise convergence of $\tau^{-2}\tau'^{-\frac32}K_h(\tau)K_v(\tau')U_n^{j,\ell}(t,x)$ to zero for almost every (τ,τ',t,x), as n goes to infinity.

We shall use the well-known bounds

$$\begin{aligned}\|K_h(\tau)K_v(\tau')f(t,x)\|_{L^\infty_{t,x}} &\le \tau^{-1}\tau'^{-\frac12}\|f(t,x)\|_{L^\infty_tL^1_x}\quad\text{and}\\ \|K_h(\tau)K_v(\tau')f(t,x)\|_{L^\infty_{t,x}} &\le \|f(t,x)\|_{L^\infty_{t,x}}\,,\end{aligned}\tag{4.14}$$

as well as their interpolates, in the horizontal and vertical space variables: for instance denoting $L^p_hL^r_v:=L^p(\mathbb{R}^2;L^r(\mathbb{R}))$ we have also

$$\|K_h(\tau)K_v(\tau')f(t,x)\|_{L^\infty_{t,x}}\le\tau^{-1}\|f(t,x)\|_{L^\infty_tL^1_hL^\infty_v}\,.$$

We first notice that

$$\begin{aligned}\|U_n^{j,\ell}\|_{L^\infty_tL^1_x}&\le\big\|\widetilde\Lambda^n_{\boldsymbol{\eta^\ell},\boldsymbol{\delta^\ell},\boldsymbol{\tilde x^\ell}}\tilde U^{h,\ell}\big\|_{L^\infty_tL^2_hL^1_v}\big\|\widetilde\Lambda^n_{\boldsymbol{\eta^j},\boldsymbol{\delta^j},\boldsymbol{\tilde x^j}}\tilde U^{h,j}\big\|_{L^\infty_tL^2_hL^\infty_v}\\&\le C\delta_n^\ell\end{aligned}$$

so the a.e. pointwise convergence of $\tau^{-2}\tau'^{-\frac32}K_h(\tau)K_v(\tau')U_n^{j,\ell}(t,x)$ to zero follows, using (4.14), if (by symmetry in ℓ and j) either δ_n^ℓ or δ_n^j go to zero. So from now on we assume that δ_n^ℓ and δ_n^j go to infinity or 1. Next we write

$$\begin{aligned}\|U_n^{j,\ell}\|_{L^\infty_tL^1_hL^\infty_v}&\le\|\widetilde\Lambda^n_{\boldsymbol{\eta^\ell},\boldsymbol{\delta^\ell},\boldsymbol{\tilde x^\ell}}\tilde U^{h,\ell}\|_{L^\infty_tL^1_hL^\infty_v}\|\widetilde\Lambda^n_{\boldsymbol{\eta^j},\boldsymbol{\delta^j},\boldsymbol{\tilde x^j}}\tilde U^{h,j}\|_{L^\infty_tL^\infty_hL^\infty_v}\\&\le C\frac{\eta_n^\ell}{\eta_n^j}\end{aligned}$$

so again from now on we may assume that $\eta_n^\ell=\eta_n^j$, if not the result is proved (if one or the other ratio goes to zero). But in that case

$$\|U_n^{j,\ell}\|_{L^\infty_tL^\infty_x}\le C\frac{1}{(\eta_n^\ell)^2}$$

hence from now on we restrict our attention to the case when $\eta_n^\ell=\eta_n^j\to0$ or 1. We notice that by the change of variables

$$y_h:=\frac{x_h-\tilde x^\ell_{n,h}}{\eta_n^\ell}\,,\quad y_3:=\frac{x_3-\tilde x^\ell_{n,3}}{\delta_n^\ell}\,,\quad\sigma:=(\eta_n^\ell)^{-2}\tau\,,\quad\sigma':=(\delta_n^\ell)^{-2}\tau'\,,\quad s:=(\eta_n^\ell)^{-2}t\,,$$

we have after an easy computation

$$\int \tau^{-2}\tau'^{-\frac{3}{2}}\Big|K_h(\tau)K_v(\tau')U_n^{j,\ell}(t,x)\Big|d\tau d\tau' dxdt = \int \sigma^{-2}\sigma'^{-\frac{3}{2}}\Big|K_h(\sigma)K_v(\sigma')\tilde{U}_n^{j,\ell}(s,y)\Big|d\sigma d\sigma' dsdy$$

where

$$\tilde{U}_n^{j,\ell}(s,y) := \tilde{U}^{h,\ell}(s,y)\otimes \tilde{U}^{h,j}\Big(s, y_h + \frac{\tilde{x}_{n,h}^\ell - \tilde{x}_{n,h}^j}{\eta_n^j}, \frac{\delta_n^\ell}{\delta_n^j}y_3 + \frac{\tilde{x}_{n,3}^\ell - \tilde{x}_{n,3}^j}{\delta_n^j}\Big),$$

so if $\delta_n^\ell = \delta_n^j$ then the orthogonality assumption on the cores of concentration implies the result, so we may assume for instance that δ_n^ℓ/δ_n^j goes to infinity, and since neither goes to zero, that in particular δ_n^ℓ goes to infinity. The same argument lets us assume that $(\tilde{x}_{n,h}^\ell - \tilde{x}_{n,h}^j)/\eta_n^j$ is bounded.

Next we notice that the change of variables

$$y_h := \frac{x_h - \tilde{x}_{n,h}^\ell}{\eta_n^\ell}, \quad y_3 := \frac{x_3 - \tilde{x}_{n,3}^\ell}{\delta_n^j}, \quad \sigma := (\eta_n^\ell)^{-2}\tau, \quad \sigma' := (\delta_n^j)^{-2}\tau', \quad s := (\eta_n^\ell)^{-2}t\,,$$

gives

$$\int \tau^{-2}\tau'^{-\frac{3}{2}}\Big|K_h(\tau)K_v(\tau')U_n^{j,\ell}(t,x)\Big|d\tau d\tau' dxdt = \int \sigma^{-2}\sigma'^{-\frac{3}{2}}\Big|K_h(\sigma)K_v(\sigma')\widetilde{V}_n^{j,\ell}(s,y)\Big|d\sigma d\sigma' dsdy$$

where

$$\widetilde{V}_n^{j,\ell}(s,y) := \tilde{U}^{h,\ell}(s,y_h,\frac{\delta_n^j}{\delta_n^\ell}y_3)\otimes \tilde{U}^{h,j}\Big(s, y_h + \frac{\tilde{x}_{n,h}^\ell - \tilde{x}_{n,h}^j}{\eta_n^j}, y_3 + \frac{\tilde{x}_{n,3}^\ell - \tilde{x}_{n,3}^j}{\delta_n^j}\Big).$$

So if $(\tilde{x}_{n,3}^\ell - \tilde{x}_{n,3}^j)/\delta_n^j$ is not bounded, then for each fixed y_3 the limit of $\widetilde{V}_n^{j,\ell}(s,y)$ is zero hence we may from now on assume that $(\tilde{x}_{n,3}^\ell - \tilde{x}_{n,3}^j)/\delta_n^j$ is bounded, and similarly for $\tilde{x}_{n,3}^j/\delta_n^j$ and $\tilde{x}_{n,3}^\ell/\delta_n^j$ by translation invariance. Notice that repeating the argument (2.9) we get that $\tilde{x}_{n,3}^\ell/\delta_n^\ell$ must go to zero. According to Assumption 2, we may therefore now assume that

$$\tilde{\varphi}^{h,\ell}(\cdot,0)\equiv 0\,,$$

which implies by Lemma 3.5, (3.4), that

$$\Big|\tilde{U}^{h,\ell}\big(t,y_h,\frac{\delta_n^j}{\delta_n^\ell}y_3\big)\Big| \le \Big(\frac{\delta_n^j}{\delta_n^\ell}|y_3| + \alpha\Big)f(t,y_h) \tag{4.15}$$

where $f(t,y_h)$ is a smooth function in $L^\infty(\mathbb{R}^+;L^2\cap L^\infty(\mathbb{R}^2))$. We obtain finally that

$$\|\widetilde{V}_n^{j,\ell}(\cdot,\cdot,y_3)\|_{L_t^\infty L_h^1} \lesssim \Big|\alpha + \frac{\delta_n^j}{\delta_n^\ell}\Big|.$$

The result in $L^1(\mathbb{R}^+;\dot{B}_{1,1}^{2,1})$ follows.

The same argument gives actually also the result in $\widetilde{L^\infty}(\mathbb{R}^+;\dot{B}_{1,1}^{0,1})$ since all convergences to zero above are uniform in t.

All other terms of $H_n^{L,1,2}$ are dealt with in a similar fashion hence $H_n^{L,1}$ satisfies the bounds (4.3) and (4.4).

Recalling that $H_n^{L,2}$ was already dealt with, let us finally consider $H_n^{L,3}$ with

$$H_n^{L,3} := u\cdot\nabla F_n^L + F_n^L\cdot\nabla u\,.$$

Using the decomposition (4.8) of F_n^L and the same arguments as above give

$$\forall L,\quad \limsup_{n\to\infty}\big(u\cdot\nabla\widetilde{F}_n^{L,1} + \widetilde{F}_n^{L,1}\cdot\nabla u\big) = 0 \ \text{ in } \ L^1(\mathbb{R}^+;\dot{B}_{1,1}^{1,1}+\dot{B}_{1,1}^{2,0})\cap\widetilde{L^\infty}(\mathbb{R}^+;\dot{B}_{1,1}^{-1,1}+\dot{B}_{1,1}^{0,0})$$

where

$$\widetilde{F}_n^{L,1} := \sum_{\substack{1\le \ell\le L\\ \eta_n^\ell/\delta_n^\ell\to 0}} \tilde{u}_n^\ell + \sum_{\ell=1}^{L} u_n^\ell \quad \text{and} \quad F_n^{L,2} := \sum_{\substack{\ell=1\\ \eta_n^\ell/\delta_n^\ell\to \infty}}^{L} \tilde{u}_n^\ell\,.$$

while the terms $F_n^{L,1} - \widetilde{F}_n^{L,1}$ and $F_n^{L,2}$ are dealt with using the product laws (4.9)-(4.11). We leave the details to the reader. Lemma 4.5 is proved. □

4.4. **Proof of Corollaries 1 and 2.**

4.4.1. *Proof of Corollary* 1. If the solution u associated with u_0 only has a finite life span T^*, then we can retrace the following steps, replacing everywhere $\mathbb{R}^+$ by $[0,T]$ for $T<T^*$ and it is obvious that the result of Corollary 1 holds as soon as n is large enough (depending on T).

4.4.2. *Proof of Corollary* 2. The proof of that corollary is very close to the proof of a similar result in the isotropic context (see [25], Theorem 2(ii)). Under the assumptions of Corollary 2, we can apply the previous results (in particular Corollary 1) to write that as long as the solution u associated with u_0 exists, it may be decomposed into

$$u = u_n - \mathcal{U}_n^L - \mathcal{V}_n^L - w_n^L\,,$$

and we know that for all $T<T^*$, denoting by $\mathcal{L}_2(T) := \widetilde{L^2}([0,T];\dot{B}^{\frac{2}{3},\frac{1}{3}}_{3,1})$,

$$\lim_{\alpha\to 0}\big(\limsup_{n\to\infty}\|w_n^L\|_{\mathcal{L}_2(T)}\big)\to 0\,,\quad L\to\infty\,. \tag{4.16}$$

Moreover we also have, for n large enough, α small enough and all L (due to the assumption on u_n and to Lemma 4.4),

$$\|u+w_n^L\|_{\mathcal{L}_2(T)}\le C\,,$$

uniformly in L, α and n. Next recalling that if a solution blows up at time T^*, then its norm in $\mathcal{L}_2(T)$ blows up when T goes to T^* (see Appendix A), we can therefore choose $T<T^*$ such that

$$\|u\|_{\mathcal{L}_2(T)}\ge 2C\,.$$

We conclude by noticing that

$$\|u\|_{\mathcal{L}_2(T)}\le C+\|w_n^L\|_{\mathcal{L}_2(T)}$$

so choosing n and L large enough and α small enough gives a contradiction due to (4.16), whence the result.

5. Profile decompositions in $\mathcal{B}^1_q$

5.1. **Introduction and statement of the theorem.** After the pioneering works of P. -L. Lions [48] and [49], the lack of compactness in critical Sobolev embeddings was investigated for different types of examples through several angles. For instance, in [30] the lack of compactness in the critical Sobolev embedding $\dot{H}^s(\mathbb{R}^d)\hookrightarrow L^p(\mathbb{R}^d)$ in the case where $d\ge 3$ with $0\le s<d/2$ and $p=2d/(d-2s)$ is described in terms of microlocal defect measures and in [31], it is characterized by means of profiles. More generally for Sobolev spaces in the L^q framework, this question is treated in [36] (see also the more recent work [41]) by the use of nonlinear wavelet approximation theory. In [6], the authors look into the lack of compactness of the critical embedding $H^1_{rad}(\mathbb{R}^2)\hookrightarrow\mathcal{L}$, where $\mathcal{L}$ denotes the Orlicz space associated to the function $\phi(s)=\mathrm{e}^{s^2}-1$. Other studies were conducted in various works (see among others [7, 11, 23, 54, 55, 56]) supplying us with a large amount of information on solutions of nonlinear partial differential equations, both in the elliptic or the evolution framework; among other applications, one can mention [5, 25, 26, 28, 39, 40, 57]. Recently in [4], the wavelet-based profile decomposition introduced by S. Jaffard in [36] was revisited in order to

treat a larger range of examples of critical embedding of function spaces $X \hookrightarrow Y$ including Sobolev, Besov, Triebel-Lizorkin, Lorentz, Hölder and BMO spaces. For that purpose, two generic properties on the spaces X and Y were identified to build the profile decomposition in a unified way. These properties concern wavelet decompositions in the spaces X and Y supposed to have the same scaling, and endowed with an unconditional wavelet basis $(\psi_\lambda)_{\lambda\in\Lambda}$.

The first property is related to the existence of a *nonlinear projector* Q_M satisfying

$$\lim_{M\to+\infty} \max_{\|f\|_X\leq 1} \|f - Q_M f\|_Y = 0\,.$$

More precisely, if f may be decomposed in the following way (the notation will be made precise below): $f = \sum_{\lambda\in\nabla} d_\lambda \psi_\lambda$, then $Q_M f$, sometimes called the *best M-term approximation*, takes the general form

$$Q_M f := \sum_{\lambda\in E_M} d_\lambda \psi_\lambda\,, \tag{5.1}$$

where the sets $E_M = E_M(f)$ of cardinality M depend on f and satisfy $E_M(f) \subset E_{M+1}(f)$. The existence of such a nonlinear projector was extensively studied in nonlinear approximation theory and for many cases, like Sobolev spaces, it turns out that the set $E_M = E_M(f)$ can be chosen as the subset of ∇ that corresponds to the M largest values of $|d_\lambda|$. It is in fact known (see [50] for instance) that in homogeneous Besov spaces $\dot{B}^\sigma_{r,r}$, we have the following norm equivalence :

$$\|f\|_{\dot{B}^\sigma_{r,r}} \sim \|(d_\lambda)_{\lambda\in\nabla}\|_{\ell^r}\,, \tag{5.2}$$

for $f = \sum_{\lambda\in\nabla} d_\lambda \psi_\lambda$ with wavelets normalized in $\dot{B}^\sigma_{r,r}$. Therefore, in the particular case where $X = \dot{B}^s_{p,p}$ and $Y = \dot{B}^t_{q,q}$, with $\frac{1}{p} - \frac{1}{q} = \frac{s-t}{d}$, the nonlinear projector Q_M defined by (5.1), where $E_M = E_M(f)$ is the subset of ∇ of cardinality M that corresponds to the M largest values of $|d_\lambda|$, is appropriate and satisfies (see [4] for instance):

$$\sup_{\|f\|_{\dot{B}^s_{p,p}}\leq 1} \|f - Q_M f\|_{\dot{B}^t_{q,q}} \leq C M^{-\frac{s-t}{d}}\,. \tag{5.3}$$

The second property concerns the *stability* of wavelet expansions in the function space X with respect to certain operations such as "shifting" the indices of wavelet coefficients, as well as disturbing the value of these coefficients. In practice and for most cases of interest, this property derives from the fact that the X norm of a function is equivalent to the norm of its wavelet coefficients in a certain sequence space, by invoking Fatou's lemma.

Under these assumptions, it is proved in [4] that, as in the previous works [30] and [36], translation and scaling invariance are the sole responsible for the defect of compactness of the embedding of $X \hookrightarrow Y$.

In what follows, we shall apply the same lines of reasoning, taking advantage of an anisotropic wavelet setting to describe the lack of compactness of the Sobolev embedding $\mathcal{B}^1_q \hookrightarrow \dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,p}$ with $p > \max(1,q)$ in terms of an asymptotic anisotropic profile decomposition. We recall that as defined in the introduction of this paper, $\mathcal{B}^1_q := \dot{B}^{1,1}_{1,q}$. Our presentation is essentially based on ideas and methods developed for the isotropic setting in [4]. Because of the anisotropy, we use a two-parameter wavelet basis. More precisely, wavelet decompositions of a function have the form

$$f = \sum_{\lambda=(\lambda_1,\lambda_2)\in\nabla} d_\lambda \psi_\lambda\,, \tag{5.4}$$

where the wavelets ψ_λ are assumed to be normalized in the space $X = \mathcal{B}^1_q$, and where the notation $\lambda_1 = (j_1, k_1) \in \mathbb{Z} \times \mathbb{Z}^2$ (resp. $\lambda_2 = (j_2, k_2) \in \mathbb{Z} \times \mathbb{Z}$) concatenates the scale index $j_1 = j_1(\lambda_1)$ (resp. $j_2 = j_2(\lambda_2)$) and the space index $k_1 = k_1(\lambda_1)$ (resp. $k_2 = k_2(\lambda_2)$) for the horizontal variable (resp. the vertical variable). Thus the index set ∇ in (5.4) is defined as $\nabla := (\mathbb{Z} \times \mathbb{Z}^2) \times (\mathbb{Z} \times \mathbb{Z})$ and the wavelets ψ_λ write under the form

$$\psi_\lambda = \psi_{(\lambda_1,\lambda_2)} = 2^{j_1}\psi(2^{j_1} \cdot -k_1, 2^{j_2} \cdot -k_2)$$

where ψ the so-called "mother wavelet" is generated by a finite dimensional inner product of one variable functions ψ^e, for $e \in E$ a finite set. It is known (see for instance [8]) that wavelet bases are unconditional bases, i.e. there exists a constant D such that for any finite subset $E \subset \nabla$ and coefficients vectors $(c_\lambda)_{\lambda\in E}$ and $(d_\lambda)_{\lambda\in E}$ such that $|c_\lambda| \le |d_\lambda|$ for all λ, one has

$$\Big\| \sum_{\lambda\in E} c_\lambda \psi_\lambda \Big\|_{\mathcal{B}^1_q} \le D \Big\| \sum_{\lambda\in E} d_\lambda \psi_\lambda \Big\|_{\mathcal{B}^1_q} \tag{5.5}$$

and similarly for $\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,p}$. In addition $\mathcal{B}^1_q$ and $\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,p}$ may be characterized by simple properties on wavelet coefficients: for $f = \sum_{\lambda\in\nabla} d_\lambda\psi_\lambda = \sum_{(\lambda_1,\lambda_2)\in\nabla} d_{(\lambda_1,\lambda_2)}\psi_{(\lambda_1,\lambda_2)}$ with normalized wavelets, we have the following norm equivalences:

$$\|f\|^q_{\mathcal{B}^1_q} \sim \sum_{j_1\in\mathbb{Z}} \Big(\sum_{|\lambda_1|=j_1} \Big(\sum_{j_2\in\mathbb{Z}} \big(\sum_{|\lambda_2|=j_2} |d_{(\lambda_1,\lambda_2)}| \big)^q \Big)^{1/q} \Big)^q \tag{5.6}$$

and

$$\|f\|_{\dot{B}^{-1+\frac{2}{p}}_{p,p}(\mathbb{R}^2;\dot{B}^{\frac{1}{p}}_{p,p}(\mathbb{R}))} \sim \|(d_\lambda)_{\lambda\in\nabla}\|_{\ell^p}\,. \tag{5.7}$$

Moreover as proved in [4, 43], there exists a nonlinear projector Q_M of the form (5.1) such that

$$\lim_{M\to+\infty} \max_{\|f\|_{\mathcal{B}^1_q}\le 1} \|f - Q_M f\|_{\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,p}} = 0\,. \tag{5.8}$$

We refer to [1, 8, 10, 19, 20, 21, 22, 29, 34, 44, 58] and the references therein for more details on the construction of wavelet bases and on the characterization of function spaces by expansions in such bases.

In the sequel, for any function ϕ, not necessarily a wavelet, and any scale-space index λ defined by $\lambda = (\lambda_1, \lambda_2) = ((j_1, k_1), (j_2, k_2)) \in \nabla$, we shall use the notation

$$\phi_\lambda(x) := 2^{j_1}\phi(2^{j_1}x_h - k_1, 2^{j_2}x_3 - k_2),$$

and to avoid heaviness, we shall define for $i \in \{1, 2\}$ and $\lambda = (\lambda_1, \lambda_2) = ((j_1, k_1), (j_2, k_2))$, by $j_i = j_i(\lambda)$ and $k_i = k_i(\lambda)$.

We shall prove the following theorem, characterizing the lack of compactness in the critical embedding $\mathcal{B}^1_q \hookrightarrow \dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,p}$, $p > \max(q, 1)$. The result actually holds for many such embeddings, but for the sake of readability we choose to only state and prove it in this particular case.

Theorem 3. *Let $(u_n)_{n\ge0}$ be a bounded sequence in $\mathcal{B}^1_q$. Then, up to a subsequence extraction, there exists a family of functions $(\phi^\ell)_{\ell\ge0}$ in $\mathcal{B}^1_q$ and sequences of scale-space indices $(\lambda_\ell(n))_{n\ge0}$ for each $\ell > 0$ such that for all $p > \max(q, 1)$,*

$$u_n = \sum_{\ell=1}^{L} \phi^\ell_{\lambda_\ell(n)} + \psi^L_n\,, \quad \textit{where} \quad \limsup_{n\to\infty} \|\psi^L_n\|_{\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,p}} \to 0 \quad \textit{as} \quad L \to \infty\,.$$

The decomposition is asymptotically orthogonal in the sense that for any $k \neq \ell$, as $n \to +\infty$, either

$$|j_1(\lambda_k(n)) - j_1(\lambda_\ell(n))| + |j_2(\lambda_k(n)) - j_2(\lambda_\ell(n))| \to +\infty \tag{5.9}$$

or

$$|k_1(\lambda_k(n)) - 2^{j_1(\lambda_k(n))-j_1(\lambda_\ell(n))}k_1(\lambda_\ell(n))| + |k_2(\lambda_k(n)) - 2^{j_2(\lambda_k(n))-j_2(\lambda_\ell(n))}k_2(\lambda_\ell(n))| \to +\infty\,.$$

Moreover, we have the following stability estimates

$$\sum_{\ell=1}^{\infty} \|\phi^\ell\|_{\mathcal{B}^1_q} \leq C \sup_{n\geq 0} \|u_n\|_{\mathcal{B}^1_q}\,, \tag{5.10}$$

where C is a constant which only depends on the choice of the wavelet basis.

Remark 5.1. Up to rescaling the profiles, if (5.9) does not hold then one may assume that $j_i(\lambda_\ell(n)) = j_i(\lambda_k(n))$ for $i \in \{1, 2\}$.

5.2. **Proof of Theorem 3.** Along the same lines as in [4], the anisotropic profile decomposition construction proceeds in several steps.

5.2.1. *Step 1: rearrangements.* According to the notation (5.4), we first introduce the wavelet decompositions of the sequence u_n, namely $u_n = \sum_{\lambda\in\nabla} d_{\lambda,n}\psi_\lambda$. Then we use the nonlinear projector Q_M to write for each $M > 0$

$$u_n = Q_M u_n + R_M u_n\,, \quad \text{with} \quad \lim_{M\to+\infty} \sup_{n>0} \|R_M u_n\|_{\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,p}} = 0\,,$$

in view of (5.8) and the boundedness of the sequence u_n in $\mathcal{B}^1_q$. Noting

$$Q_M u_n = \sum_{m=1}^{M} d_{m,n}\psi_{\lambda(m,n)}\,,$$

it is obvious that the coefficients $d_{m,n}$ are uniformly bounded in n and m, so up to a diagonal subsequence extraction procedure in n, we can reduce to the case where for all m, the sequence $(d_{m,n})_{n>0}$ converges towards a finite limit that depends on m,

$$d_m := \lim_{n\to+\infty} d_{m,n}\,.$$

We may thus write

$$u_n = \sum_{m=1}^{M} d_m\psi_{\lambda(m,n)} + t_{n,M}\,, \quad \text{where} \quad t_{n,M} := \sum_{m=1}^{M} (d_{m,n} - d_m)\psi_{\lambda(m,n)} + R_M u_n\,.$$

5.2.2. *Step 2: construction of approximate profiles.* The profiles ϕ^ℓ will be built as limits of sequences $\phi^{\ell,i}$ resulting by the following algorithm. At the first iteration $i = 1$, we define

$$\phi^{1,1} = d_1\psi\,, \;\; \lambda_1(n) := \lambda(1, n)\,, \;\; \varphi_1(n) := n\,.$$

Now, supposing that after iteration step $i - 1$, we have constructed $L - 1$ functions denoted by $(\phi^{1,i-1}, \ldots, \phi^{L-1,i-1})$ and scale-space index sequences $(\lambda_1(n), \ldots, \lambda_{L-1}(n))$ with $L \leq i$, as well as an increasing sequence of positive integers $\varphi_{i-1}(n)$ such that

$$\sum_{m=1}^{i-1} d_m\psi_{\lambda(m,\varphi_{i-1}(n))} = \sum_{\ell=1}^{L-1} \phi^{\ell,i-1}_{\lambda_\ell(\varphi_{i-1}(n))}\,,$$

we shall use the i-th component $d_i\psi_{\lambda(i,\varphi_{i-1}(n))}$ to either modify one of these functions or construct a new one at iteration i according to the following dichotomy.

(i) First case: assume that we can extract $\varphi_i(n)$ from $\varphi_{i-1}(n)$ such that for $\ell = 1, \dots, L-1$ at least one of the following holds:

$$\lim_{n\to+\infty} |j_1(\lambda(i,\varphi_i(n))) - j_1(\lambda_\ell(\varphi_i(n)))| + |j_2(\lambda(i,\varphi_i(n))) - j_2(\lambda_\ell(\varphi_i(n)))| = +\infty\,, \tag{5.11}$$

or

$$\begin{aligned}\lim_{n\to+\infty} &\Big|k_1\big(\lambda(i,\varphi_i(n))\big) - 2^{j_1(\lambda(i,\varphi_i(n)))-j_1(\lambda_\ell(\varphi_i(n)))}k_1\big(\lambda_\ell(\varphi_i(n))\big)\Big| \\ &+\Big|k_1\big(\lambda(i,\varphi_i(n))\big) - 2^{j_1(\lambda(i,\varphi_i(n)))-j_1(\lambda_\ell(\varphi_i(n)))}k_1\big(\lambda_\ell(\varphi_i(n))\big)\Big| = +\infty\,.\end{aligned} \tag{5.12}$$

In such a case, we create a new profile and scale-space index sequence by defining

$$\phi^{L,i} := d_i\psi\,, \quad \lambda_L(n) := \lambda(i,n)\,, \quad \phi^{\ell,i} := \phi^{\ell,i-1}\,\forall \ell \in \{1,\dots,L-1\}\,.$$

(ii) Second case: assume that for some subsequence $\varphi_i(n)$ of $\varphi_{i-1}(n)$ and for some ℓ belonging to $\{1,\dots,L-1\}$ neither (5.11) nor (5.12) holds. Then it follows that for i in $\{1,2\}$, the quantities $j_i(\lambda_\ell(\varphi_i(n))) - j_i(\lambda(i,\varphi_i(n)))$ and $k_i\big(\lambda(i,\varphi_i(n))\big) - 2^{j_i(\lambda(i,\varphi_i(n)))-j_i(\lambda_\ell(\varphi_i(n)))}k_i\big(\lambda_\ell(\varphi_i(n))\big)$ only take a finite number of values as n varies. Therefore, up to an additional subsequence extraction, we may assume that there exists numbers a_1, a_2, b_1 and b_2 such that for all $n > 0$ and for $i \in \{1,2\}$,

$$j_i(\lambda(i,\varphi_i(n))) - j_i(\lambda_\ell(\varphi_i(n))) = a_i\,,$$

and

$$k_i(\lambda(i,\varphi_i(n))) - 2^{j_i(\lambda(i,\varphi_i(n)))-j_i(\lambda_\ell(\varphi_i(n)))}k_i(\lambda_\ell(\varphi_i(n))) = b_i\,.$$

We then update the function $\phi^{\ell,i-1}$ according to

$$\phi^{\ell,i} := \phi^{\ell,i-1} + d_i 2^{a_1}\psi(2^{a_1}\cdot - b_1, 2^{a_2}\cdot - b_2)\,, \quad \phi^{\ell',i} := \phi^{\ell',i-1}\,\forall \ell' \in \{1,\dots,L-1\}\,,\ \ell' \neq \ell.$$

Up to a diagonal subsequence extraction procedure in n, it derives from this construction that for each value of M there exists $L = L(M) \le M$ such that

$$\sum_{m=1}^{M} d_m\psi_{\lambda(m,n)} = \sum_{\ell=1}^{L} \phi^{\ell,M}_{\lambda_\ell(n)}$$

with for each $\ell = 1,\dots,L$

$$\phi^{\ell,M}_{\lambda_\ell(n)} = \sum_{m\in E(\ell,M)} d_m\psi_{\lambda(m,n)}\,,$$

and where the sets $E(\ell,M)$ for $\ell = 1,\dots,L$ form a partition of $\{1,\dots,M\}$. Moreover, for $i \in \{1,2\}$ and for any $m, m' \in E(\ell,M)$ we have

$$j_i(\lambda(m,n)) - j_i(\lambda(m',n)) = a_i(m,m')\,, \tag{5.13}$$

and

$$k_i(\lambda(m,n)) - 2^{j_i(\lambda(m,n))-j_i(\lambda(m',n))}k_i(\lambda(m',n)) = b_i(m,m')\,, \tag{5.14}$$

where $a_i(m,m')$ and $b_i(m,m')$ do not depend on n.

5.2.3. *Step 3: construction of the exact profiles.* The profiles ϕ^ℓ will be obtained as the limits in $\mathcal{B}^1_q$ of $\phi^{\ell,M}$ as $M \to +\infty$. To this end, we shall use (5.6) and the fact that the wavelet basis $(\psi_\lambda)_{\lambda\in\nabla}$ is an unconditional basis of $\mathcal{B}^1_q$. So let us define for fixed ℓ and M such that $\ell \le L(M)$ the functions $g^{\ell,M} := \sum_{m\in E(\ell,M)} d_m\psi_{\lambda(m)}$ and $f^{\ell,M,n} := \sum_{m\in E(\ell,M)} d_{m,n}\psi_{\lambda(m)}$, with $\lambda(m) := \lambda(m,1)$. In view of (5.13), (5.14) and the scaling invariance of the space $\mathcal{B}^1_q$, we have

$$\|f^{\ell,M,n}\|_{\mathcal{B}^1_q} = \Big\|\sum_{m\in E(\ell,M)} d_{m,n}\psi_{\lambda(m,n)}\Big\|_{\mathcal{B}^1_q}\,.$$

Since $\sum_{m\in E(\ell,M)} d_{m,n}\psi_{\lambda(m,n)}$ is a part of the expansion of u_n, we deduce the existence of a constant C which depends neither on n nor on ℓ and M such that

$$\|f^{\ell,M,n}\|_{\mathcal{B}^1_q} \leq C\,.$$

Now, according to the first step of the proof of the theorem, the coefficients d_m are the limits of $d_{m,n}$ when n tends to infinity. Therefore, (5.6) and Fatou's lemma imply that

$$\|g^{\ell,M}\|_{\mathcal{B}^1_q} \leq \liminf_{n\to+\infty} \|f^{\ell,M,n}\|_{\mathcal{B}^1_q}\,,$$

which ensures the convergence in $\mathcal{B}^1_q$ of the sequence $g^{\ell,M}$ towards a limit g^ℓ as $M\to+\infty$. Finally, since by construction the $g^{\ell,M}$ are rescaled versions of the $\phi^{\ell,M}$, there exists numbers $A_1>0$, $A_2>0$, $B_1\in\mathbb{R}^2$ and $B_2\in\mathbb{R}$ such that

$$\phi^{\ell,M} = 2^{A_1} g^{\ell,M}(2^{A_1}\cdot -B_1, 2^{A_2}\cdot -B_2)\,.$$

Therefore $\phi^{\ell,M}$ converges in $\mathcal{B}^1_q$ towards $\phi^\ell := 2^{A_1} g^\ell(2^{A_1}\cdot -B_1, 2^{A_2}\cdot -B_2)$ as $M\to+\infty$.

To conclude the construction, we argue exactly as in the proof of Theorem 1.1 in [4].

Finally, let us prove that the decomposition derived in Theorem 3 is stable. The argument is again similar to the one followed in [4], we reproduce it here for the convenience of the reader. We shall use the following property: if $E_1,\dots,E_L$ are disjoint finite sets in ∇, then for any coefficient sequence (d_λ), one has

$$\sum_{\ell=1}^L \|\sum_{\lambda\in E_\ell} d_\lambda\psi_\lambda\|_{\mathcal{B}^1_q} \leq C\|\sum_{\ell=1}^L\sum_{\lambda\in E_\ell} d_\lambda\psi_\lambda\|_{\mathcal{B}^1_q}\,. \tag{5.15}$$

Such an estimate was proved in [4] for Besov spaces $\dot{B}^s_{p,a}(\mathbb{R}^d)$ and generalizes easily to our framework. Let us then consider for $\ell=1,\dots,L$ the functions

$$\phi^{\ell,M,n} := \sum_{m\in E(\ell,M)} d_{m,n}\psi_{\lambda(m,n)}\,,$$

where $E(\ell,M)$ are the sets introduced in the second step of the proof of the decomposition. These functions are linear combinations of wavelets with indices in disjoint finite sets $E_1,\dots,E_L$ (that vary with n), which implies by (5.15) that

$$\sum_{\ell=1}^L \|\phi^{\ell,M,n}\|_{\mathcal{B}^1_q} \leq C\Big\|\sum_{\ell=1}^L \phi^{\ell,M,n}\Big\|_{\mathcal{B}^1_q}\,.$$

Since the functions $\phi^{\ell,M,n}$ are part of the wavelet expansion of u_n, we deduce that

$$\sum_{\ell=1}^L \|\phi^{\ell,M,n}\|_{\mathcal{B}^1_q} \leq C\sup_{n\geq 0}\|u_n\|_{\mathcal{B}^1_q}\,.$$

Now, by construction the sequence $(\phi^{\ell,M,n})_{n>0}$ converges in $\mathcal{B}^1_q$ towards the approximate profiles $\phi^{\ell,M}_{\lambda_\ell(n)} = \sum_{m\in E(\ell,M)} d_m\psi_{\lambda(m,n)}$ as $n\to\infty$. It follows that for any $\varepsilon>0$ we have

$$\sum_{\ell=1}^L \|\phi^{\ell,M}_{\lambda_\ell(n)}\|_{\mathcal{B}^1_q} \leq C\sup_{n\geq 0}\|u_n\|_{\mathcal{B}^1_q} + \varepsilon\,,$$

for n large enough. Thanks to the scaling invariance, we thus find that

$$\sum_{\ell=1}^{L} \|\phi^{\ell,M}\|_{\mathcal{B}^1_q} \leq C \sup_{n\geq 0} \|u_n\|_{\mathcal{B}^1_q}.$$

Letting M go to $+\infty$, we obtain the same inequality for the exact profiles and we conclude by letting $L \to +\infty$. The theorem is proved. □

5.3. **Some additional properties.** The following result is very useful.

Lemma 5.2. *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{B}^1_q$, which does not converge strongly to zero in $\mathcal{B}^1_q$ and which may be decomposed with the notation of Theorem 3 into*

$$u_n = \sum_{\ell=1}^{L} \phi^{\ell}_{\lambda_\ell(n)} + \psi^L_n\,. \tag{5.16}$$

Let $p \geq 2$ be given. For any $\ell \in \{1, ..., L\}$, there are three constants $C > 0$ and $(a^1_\ell, a^2_\ell) \in \mathbb{Z}^2$ such that

$$\limsup_{n\to\infty} 2^{j_1(\lambda_\ell(n))(-1+\frac{2}{p})+\frac{j_2(\lambda_\ell(n))}{p}} \Big\|\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}\, u_n\Big\|_{L^p(\mathbb{R}^3)} = C\,. \tag{5.17}$$

Proof of Lemma 5.2. We start by noticing that the existence of $C < \infty$ satisfying (5.17) is obvious, the only difficulty is to prove that $C > 0$.

• Let us first estimate one individual contribution, meaning let us show that there is $C^{\ell,p} > 0$ and $(a^1_\ell, a^2_\ell) \in \mathbb{Z}^2$ such that

$$\limsup_{n\to\infty} 2^{j_1(\lambda_\ell(n))(-1+\frac{2}{p})+\frac{j_2(\lambda_\ell(n))}{p}} \Big\|\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}\, \phi^{\ell}_{\lambda_\ell(n)}\Big\|_{L^p(\mathbb{R}^3)} = C^{\ell,p}\,. \tag{5.18}$$

By definition $\Delta^h_{j_1+a_1} u = 2^{2(j_1+a_1)}\Psi(2^{j_1+a_1}\cdot)*_h u$ and $\Delta^v_{j_2+a_2} u = 2^{j_2+a_2}\Psi(2^{j_2+a_2}\cdot)*_v u$, where Ψ is the frequency localization function introduced in Appendix B and $*_h$ (resp. $*_v$) denotes the convolution operator in the horizontal (resp. vertical) variable. Writing

$$\phi^{\ell}_{\lambda_\ell(n)} = 2^{j_1(\lambda_\ell(n))}\phi^{\ell}\Big(2^{j_1(\lambda_\ell(n))}(\cdot_h - x^{\ell}_{n,h}), 2^{j_2(\lambda_\ell(n))}(\cdot_3 - x^{\ell}_{n,3})\Big),$$

we easily prove that

$$\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}\phi^{\ell}_{\lambda_\ell(n)} = 2^{j_1(\lambda_\ell(n))}\big(\widetilde{\Psi}^{\ell} * \phi^{\ell}\big)\big(2^{j_1(\lambda_\ell(n))}(\cdot_h - x^{\ell}_{n,h}), 2^{j_2(\lambda_\ell(n))}(\cdot_3 - x^{\ell}_{n,3})\big)$$

where $\widetilde{\Psi}^{\ell}(x) := 2^{2a^1_\ell+a^2_\ell}\Psi(2^{a^1_\ell}x_h)\Psi(2^{a^2_\ell}x_3)$, which ensures that

$$\limsup_{n\to\infty} 2^{j_1(\lambda_\ell(n))(-1+\frac{2}{p})+\frac{j_2(\lambda_\ell(n))}{p}} \Big\|\Delta^h_{j_1(\lambda_\ell(n))}\Delta^v_{j_2(\lambda_\ell(n))}u_n\Big\|_{L^p(\mathbb{R}^3)} = \|\widetilde{\Psi}^{\ell} * \phi^{\ell}\|_{L^p(\mathbb{R}^3)} \neq 0\,, \tag{5.19}$$

as soon as (a^1_ℓ, a^2_ℓ) are conveniently chosen so that the supports of $\widehat{\widetilde{\Psi}^{\ell}}$ and $\widehat{\phi}^{\ell}$ are not disjoint.

• Next let us prove that for $\ell' \neq \ell$

$$2^{j_1(\lambda_\ell(n))(-1+\frac{2}{p})+\frac{j_2(\lambda_\ell(n))}{p}} \Big\|\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}\phi^{\ell'}_{\lambda_{\ell'}(n)}\Big\|_{L^p(\mathbb{R}^3)} \to 0 \quad \text{as} \quad n\to\infty\,,$$

when the scales $j(\lambda_\ell(n))$ and $j(\lambda_{\ell'}(n))$ are orthogonal, meaning $2^{j_i(\lambda_\ell(n))-j_i(\lambda_{\ell'}(n))} \to 0$ or ∞ as $n \to \infty$, for i equal either to 1 or 2. Noticing that

$$\Delta^h_k\Delta^v_j\big(\phi(2^{k'}x_h, 2^{j'}x_3)\big) = (\Delta^h_{k-k'}\Delta^v_{j-j'}\phi)(2^{k'}x_h, 2^{j'}x_3)$$

we deduce that

$$2^{j_1(\lambda_\ell(n))(-1+\frac{2}{p})+\frac{j_2(\lambda_\ell(n))}{p}}\Big\|\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}\,\phi^{\ell'}_{\lambda_{\ell'}(n)}\Big\|_{L^p(\mathbb{R}^3)}$$
$$= 2^{j_1^{\ell,\ell'}(n)(-1+\frac{2}{p})+\frac{j_2^{\ell,\ell'}(n)}{p}}\Big\|\Delta^h_{j_1^{\ell,\ell'}(n)+a^1_\ell}\Delta^v_{j_2^{\ell,\ell'}(n)+a^2_\ell}\,\phi^{\ell'}\Big\|_{L^p(\mathbb{R}^3)}$$

where

$$j_1^{\ell,\ell'}(n) := j_1(\lambda_\ell(n)) - j_1(\lambda_{\ell'}(n)) \quad \text{and} \quad j_2^{\ell,\ell'}(n) := j_2(\lambda_\ell(n)) - j_2(\lambda_{\ell'}(n))\,.$$

Since $\phi^{\ell'} \in \dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,q}$, we deduce that

$$2^{j_1(\lambda_\ell(n))(-1+\frac{2}{p})+\frac{j_2(\lambda_\ell(n))}{p}}\Big\|\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}\phi^{\ell'}_{\lambda_{\ell'}(n)}\Big\|_{L^p(\mathbb{R}^3)} \to 0, \quad \text{as} \quad n\to\infty\,. \tag{5.20}$$

• Finally, let us regroup in (5.16) all the profiles corresponding to the same scales: namely let us write, for a given $\ell \in \mathbb{N}$

$$u_n - \psi^L_n = u^\ell_{n,1} + u^\ell_{n,2}\,,$$

where (up to conveniently re-ordering the profiles $\phi^{\ell_1}_{\lambda_{\ell_1}(n)},\dots,\phi^{\ell_L}_{\lambda_{\ell_L}(n)}$),

$$u^\ell_{n,1} := \sum_{k=1}^{L_\ell}\phi^{\ell_k}_{\lambda_{\ell_k}(n)} \quad \text{with} \quad j_i(\lambda_{\ell_k}(n)) = j_i(\lambda_\ell(n))\,, \quad \forall i\in\{1,2\}\,,$$

and on the other hand, writing to simplify $j_i(\lambda_\ell(n)) =: j_i(n)$,

$$u^\ell_{n,2} := \sum_{k=L_\ell+1}^{L}\phi^{\ell_k}_{\lambda_{\ell_k}(n)}\,,$$

with scales $j_i(\lambda_{\ell_k}(n))$ orthogonal to the scale $j_i(n)$ for every $k\in\{L_\ell+1,\dots,L\}$. The result (5.20) enables us to take care of the term $u^\ell_{n,2}$ which satisfies

$$2^{j_1(\lambda_\ell(n))(-1+\frac{2}{p})+\frac{j_2(\lambda_\ell(n))}{p}}\Big\|\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}u^\ell_{n,2}\Big\|_{L^p(\mathbb{R}^3)} \to 0, \quad \text{as} \quad n\to\infty\,,$$

so let us prove that

$$\limsup_{n\to\infty} 2^{j_1(n)(-1+\frac{2}{p})+\frac{j_2(n)}{p}}\Big\|\Delta^h_{j_1(n)+a^1_\ell}\Delta^v_{j_2(n)+a^2_\ell}u^\ell_{n,1}\Big\|_{L^p(\mathbb{R}^3)} = C>0\,.$$

By Hölder's inequality if $2\le p\le\infty$, we have

$$\begin{aligned} 2^{\frac{j_2(n)}{2}}\Big\|\Delta^h_{j_1(n)}\Delta^v_{j_2(n)}\,u^\ell_{n,1}\Big\|_{L^2(\mathbb{R}^3)} &\le \Big(2^{j_1(n)+j_2(n)}\Big\|\Delta^h_{j_1(n)}\Delta^v_{j_2(n)}\,u^\ell_{n,1}\Big\|_{L^1(\mathbb{R}^3)}\Big)^{\frac{p-2}{2(p-1)}} \\ &\quad\times\Big(2^{j_1(n)(-1+\frac{2}{p})+\frac{j_2(n)}{p}}\Big\|\Delta^h_{j_1(n)}\Delta^v_{j_2(n)}\,u^\ell_{n,1}\Big\|_{L^p(\mathbb{R}^3)}\Big)^{\frac{p}{2(p-1)}} \end{aligned} \tag{5.21}$$

and since both terms on the right-hand side are bounded, the result will follow if we prove that

$$\limsup_{n\to\infty} 2^{\frac{j_2(n)}{2}}\Big\|\Delta^h_{j_1(n)+a^1_\ell}\Delta^v_{j_2(n)+a^2_\ell}\,u^\ell_{n,1}\Big\|_{L^2(\mathbb{R}^3)} = C>0.$$

But this is a simple orthogonality argument, noticing that

$$\begin{aligned} \Big\|\Delta^h_{j_1(n)+a^1_\ell}\Delta^v_{j_2(n)+a^2_\ell}\,u_{n,1}\Big\|^2_{L^2(\mathbb{R}^3)} &= \sum_{k=1}^{L_\ell}\|\Delta^h_{j_1(n)+a^1_\ell}\Delta^v_{j_2(n)+a^2_\ell}\,\phi^{\ell_k}_{\lambda_{\ell_k}(n)}\|^2_{L^2(\mathbb{R}^3)} \\ &\quad+\sum_{k\neq k'}(\Delta^h_{j_1(n)+a^1_\ell}\Delta^v_{j_2(n)+a^2_\ell}\,\phi^{\ell_k}_{\lambda_{\ell_k}(n)}|\Delta^h_{j_1(n)+a^1_\ell}\Delta^v_{j_2(n)+a^2_\ell}\,\phi^{\ell_{k'}}_{\lambda_{\ell_{k'}}(n)})_{L^2(\mathbb{R}^3)}\,. \end{aligned} \tag{5.22}$$

Indeed we know from (5.18) that

$$\begin{aligned} 2^{\frac{j_2(n)}{2}}\left(\sum_{k=1}^{L_\ell}\|\Delta^h_{j_1(n)+a^1_\ell}\Delta^v_{j_2(n)+a^2_\ell}\,\phi^{\ell_k}_{\lambda_{\ell_k}(n)}\|^2_{L^2(\mathbb{R}^3)}\right)^{\frac12} &\geq 2^{\frac{j_2(n)}{2}}\|\Delta^h_{j_1(n)+a^1_\ell}\Delta^v_{j_2(n)+a^2_\ell}\,\phi^{\ell}_{\lambda_{\ell}(n)}\|_{L^2(\mathbb{R}^3)} \\ &\geq C^{\ell,2}>0 \end{aligned} \tag{5.23}$$

so it is enough to prove that

$$2^{j_2(n)}\sum_{k\neq k'}(\Delta^h_{j_1(n)+a^1_\ell}\Delta^v_{j_2(n)+a^2_\ell}\,\phi^{\ell_k}_{\lambda_{\ell_k}(n)}|\Delta^h_{j_1(n)+a^1_\ell}\Delta^v_{j_2(n)+a^2_\ell}\,\phi^{\ell_{k'}}_{\lambda_{\ell_{k'}}(n)})_{L^2(\mathbb{R}^3)}\to 0\,. \tag{5.24}$$

This is a finite sum so it suffices to prove the result for each individual term, which writes after a change of variables

$$\int_{\mathbb{R}^3}(\Delta^h_{a^1_\ell}\Delta^v_{a^2_\ell}\phi^{\ell_k})(x)\times(\Delta^h_{a^1_\ell}\Delta^v_{a^2_\ell}\phi^{\ell_{k'}})(x+2^{j_2(n)}(x^{\ell_k}_{n,h}-x^{\ell_{k'}}_{n,h}))\,dx$$

which goes to zero when n goes to infinity, due to the orthogonality of the cores of concentration (see Theorem 3), so (5.24) holds.

• Finally we need to take the remainder into account. But a reverse triangle inequality gives trivially the result, since the remainder ψ^L_n may be made arbitrarily small in $\dot B^{-1+\frac2p,\frac1p}_{p,\infty}$ as soon as L is large enough, uniformly in n, whereas (5.22)-(5.23) guarantee that making L larger does not decrease the norm of the sum of the profiles.

The lemma is proved. □

Lemma 5.3. *Let us consider a sequence $(v_n)_{n\in\mathbb{N}}$, bounded in $\mathcal{B}^1_q$, which may be decomposed with the notation of Theorem 3 into*

$$v_n=\sum_{\ell=1}^{L}\phi^\ell_{\lambda_\ell(n)}+\psi^L_n\,.$$

Assume moreover that $\lim\limits_{n\to\infty}2^{-j_1(\lambda_\ell(n))+j_2(\lambda_\ell(n))}\in\{0,\infty\}$. If $(\partial_3 v_n)_{n\in\mathbb{N}}$ is bounded in $\dot B^{0,1}_{1,q}$, then

$$\lim_{n\to\infty}2^{-j_1(\lambda_\ell(n))+j_2(\lambda_\ell(n))}=0\,.$$

Proof of Lemma 5.3. By definition of $\dot B^{0,1}_{1,q}$, we have

$$\|\partial_3 v_n\|_{\dot B^{0,1}_{1,q}}=\Big(\sum_{j,k\in\mathbb{Z}}2^{jq}\|\Delta^h_k\Delta^v_j\partial_3 v_n\|^q_{L^1(\mathbb{R}^3)}\Big)^{1/q}<\infty\quad\text{uniformly in } n\,.$$

In particular, for any $\ell\in\{1,...,L\}$, we have

$$2^{j_2(\lambda_\ell(n))}\Big\|\Delta^h_{j_1(\lambda_\ell(n))}\Delta^v_{j_2(\lambda_\ell(n))}\partial_3 v_n\Big\|_{L^1(\mathbb{R}^3)}<\infty\quad\text{uniformly in } n\,. \tag{5.25}$$

Now reasoning as in the proof of Lemma 5.2 and taking into account that $\partial_3 v_n$ is also bounded in $\dot B^{1,0}_{1,q}$, we find that there are two integers a^1_ℓ and a^2_ℓ such that

$$\limsup_{n\to\infty}2^{j_1(\lambda_\ell(n))}\Big\|\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}\partial_3\phi^\ell_{\lambda_\ell(n)}\Big\|_{L^1(\mathbb{R}^3)}=C>0\,,$$

and for any $\ell'\neq\ell$

$$2^{j_1(\lambda_\ell(n))}\Big\|\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}\partial_3\phi^{\ell'}_{\lambda_{\ell'}(n)}\Big\|_{L^1(\mathbb{R}^3)}\to 0\quad\text{as}\quad n\to\infty\,.$$

Finally, we argue as in the proof of Lemma 5.2 and write

$$v_n=v_{n,1}+v_{n,2}+\psi^L_n\,,$$

where $v_{n,1}$ contains all the profiles with scale $j_i(\lambda_\ell(n))$, meaning (up to re-ordering the profiles)

$$v_{n,1} := \sum_{k=1}^{L_\ell} \phi^{\ell_k}_{\lambda_{\ell_k}(n)}\,,$$

with $\phi^{\ell_k}_{\lambda_{\ell_k}(n)} = 2^{j_1(\lambda_\ell(n))}\phi^{\ell_k}\Big(2^{j_1(\lambda_\ell(n))}(x_h - x^{\ell_k}_{n,h}), 2^{j_2(\lambda_\ell(n))}(x_3 - x^{\ell_k}_{n,3})\Big)$ and where, denoting $j_i(n) := j_i(\lambda_\ell(n))$,

$$v_{n,2} := \sum_{k=L_\ell+1}^{L} \phi^{\ell_k}_{\lambda_{\ell_k}(n)}\,,$$

with scales $j(\lambda_{\ell_k}(n))$ orthogonal to the scale $j_i(n)$ for any $k \in \{L_1+1,\dots,L\}$. Using the same argument as in the proof of Lemma 5.2, we easily prove that for any $\ell \in \{1,\dots,L\}$

$$2^{j_2(\lambda_\ell(n))}\Big\|\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}\partial_3 v_n\Big\|_{L^1(\mathbb{R}^3)} \sim 2^{-j_1(\lambda_\ell(n))+j_2(\lambda_\ell(n))}\,C, \quad \text{as} \quad n\to\infty\,,$$

with $C>0$, which concludes the proof of the lemma due to (5.25). □

Lemma 5.4. *Let us consider $(v^h_n = (v^1_n, v^2_n))_{n\in\mathbb{N}}$ a bounded sequence of vector fields in $\mathcal{B}^1_q$ and let us suppose, with the notation of Theorem 3, that*

$$v^h_n = \sum_{\ell=1}^{L} \tilde{\phi}^{\ell,h}_{\lambda_\ell(n)} + \psi^{L,h}_n\,.$$

If $\operatorname{div}_h v^h_n = 0$, then for any $\ell \in \{1,...,L\}$ we have $\operatorname{div}_h \tilde{\phi}^{\ell,h}_{\lambda_\ell(n)} = 0$.

Proof of Lemma 5.4. We use the notation of the proof of Lemma 5.2. Taking advantage of the fact that the operator div_h is continuous from $\mathcal{B}^1_q$ into $\dot{B}^{0,1}_{1,q}$, we get, along the same lines as (5.19) in the proof of Lemma 5.2 and recalling that $\dot{B}^{0,1}_{1,q}$ embeds in $\dot{B}^{\frac{2}{p}-2,\frac{1}{p}}_{p,q}$,

$$\limsup_{n\to\infty} 2^{j_1(\lambda_\ell(n))(\frac{2}{p}-2)}2^{\frac{j_2(\lambda_\ell(n))}{p}}\Big\|\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}\operatorname{div}_h \tilde{\phi}^{\ell,h}_{\lambda_\ell(n)}\Big\|_{L^p} = \|\widetilde{\Psi}^\ell * \operatorname{div}_h \tilde{\phi}^{\ell,h}_{\lambda_\ell(n)}\|_{L^p}$$

and for any $\ell' \neq \ell$, as in (5.20),

$$2^{j_2(\lambda_\ell(n))}\Big\|\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}\operatorname{div}_h \tilde{\phi}^{\ell'}_{\lambda_{\ell'}(n)}\Big\|_{L^1(\mathbb{R}^3)} \to 0 \quad \text{as} \quad n\to\infty\,.$$

Moreover as in (5.24),

$$2^{-2j_1(\lambda_\ell(n))}2^{j_2(\lambda_\ell(n))}\sum_{k\neq k'}(\Delta^h_{j_1(n)+a^1_\ell}\Delta^v_{j_2(n)+a^2_\ell}\,\phi^{\ell_k}_{\lambda_{\ell_k}(n)}|\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}\,\phi^{\ell_{k'}}_{\lambda_{\ell_{k'}}(n)})_{L^2} \to 0\,.$$

Then we follow the method giving Lemma 5.2 which yields

$$\begin{aligned}
0 &= 2^{-2j_1(\lambda_\ell(n))}2^{j_2(\lambda_\ell(n))}\Big\|\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}\operatorname{div}_h v^h_n\Big\|^2_{L^2(\mathbb{R}^3)}\\
&\geq 2^{-2j_1(\lambda_\ell(n))}2^{j_2(\lambda_\ell(n))}\sum_{k=1}^{L_\ell}\|\Delta^h_{j_1(\lambda_\ell(n))+a^1_\ell}\Delta^v_{j_2(\lambda_\ell(n))+a^2_\ell}\operatorname{div}_h \tilde{\phi}^{\ell,h}_{\lambda_\ell(n)}\|^2_{L^2(\mathbb{R}^3)} + o(1), \quad n\to\infty\\
&\geq \|\widetilde{\Psi}^\ell * \operatorname{div}_h \tilde{\phi}^{\ell,h}_{\lambda_\ell(n)}\|^2_{L^2} + o(1), \quad n\to\infty
\end{aligned}$$

so finally $\widetilde{\Psi}^\ell * \operatorname{div}_h \tilde{\phi}^{\ell,h}_{\lambda_\ell(n)} \equiv 0$ for all couples (a^1_ℓ, a^2_ℓ), hence $\operatorname{div}_h \tilde{\phi}^{\ell,h}_{\lambda_\ell(n)} \equiv 0$. □

Appendix A. The (perturbed) Navier-Stokes equation in $\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}$

A.1. **Statement of the results.** In this appendix it proved that (NS) is globally wellposed for small data in $\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}$, using anisotropic techniques (note that in [35] such a study was undertaken in the framework of Sobolev spaces). We also study a perturbed Navier-Stokes equation in such spaces.

We use the following notation:

$$\begin{aligned}
\mathcal{S}_{p,q} &:= \widetilde{L^\infty}(\mathbb{R}^+;\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,q}) \cap \widetilde{L^1}(\mathbb{R}^+;\dot{B}^{1+\frac{2}{p},\frac{1}{p}}_{p,q} \cap \dot{B}^{-1+\frac{2}{p},2+\frac{1}{p}}_{p,q}),\\
\mathcal{S}_{p,q}(T) &:= \widetilde{L^\infty_{loc}}([0,T[;\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,q}) \cap \widetilde{L^1_{loc}}([0,T[;\dot{B}^{1+\frac{2}{p},\frac{1}{p}}_{p,q} \cap \dot{B}^{-1+\frac{2}{p},2+\frac{1}{p}}_{p,q}),\\
\mathcal{X}_{p,q} &:= \widetilde{L^1}(\mathbb{R}^+;\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,q}) + \widetilde{L^2}(\mathbb{R}^+;\dot{B}^{-1+\frac{2}{p},-1+\frac{1}{p}}_{p,q}) \cap \widetilde{L^1}(\mathbb{R}^+;\dot{B}^{\frac{2}{p},-1+\frac{1}{p}}_{p,q}),\\
\mathcal{Y}_{p,q} &:= L^2(\mathbb{R}^+;\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,q}) \cap L^1(\mathbb{R}^+;\dot{B}^{\frac{2}{p},1+\frac{1}{p}}_{p,q} \cap \dot{B}^{1+\frac{2}{p},\frac{1}{p}}_{p,q}).
\end{aligned}$$

Theorem 4. *Let $1 \le p < \infty$ be given. There is a constant c_0 such that the following result holds. Let $u_0 \in \dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}$ verifying the smallness condition $\|u_0\|_{\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}} \le c_0$. Then, there exists a unique, global solution u to* (NS) *in $\mathcal{Y}_{p,1}$, and it satisfies*

$$\|u\|_{\mathcal{Y}_{p,1}} \le 2\|u_0\|_{\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}.$$

If the initial data belongs to $\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}$ with no smallness condition, then there is a maximal time of existence $T^* > 0$ such that there is a unique solution in $\mathcal{Y}_{p,1}(T^*)$ and if $T^* < \infty$ then

$$\lim_{T\to T^*} \|u\|_{\widetilde{L^2}([0,T];\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1})} = \infty. \tag{A.1}$$

If the initial data belongs moreover to $\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,q}$ with $q < 1$ then the solution belongs to the space $\mathcal{Y}_{p,q}(T^*)$, on the same life span.

Moreover if $p < 4$ then the spaces $\mathcal{Y}_{p,q}$ can be replaced by $\mathcal{S}_{p,q}$ everywhere.

The next result deals with a perturbed Navier-Stokes system:

$$\text{(NSP)}\quad \begin{cases} \partial_t u + \mathbb{P}(u\cdot\nabla u + U\cdot\nabla u + u\cdot\nabla U) - \Delta u = F \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^3\\ u_{|t=0} = u_0\,,\\ \operatorname{div} u_0 = \operatorname{div} F = 0\,. \end{cases}$$

Theorem 5. *Let $1 \le p < 4$ be given. There is a constant c_0 such that the following result holds. Consider three divergence free vector fields $u_0 \in \dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}$, $F \in \mathcal{X}_{p,1}$ and $U \in \mathcal{Y}_{p,1}$. If*

$$\|u_0\|_{\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}} + \|F\|_{\mathcal{X}_{p,1}} \le c_0 \exp\big(-c_0^{-1}\|U\|_{\mathcal{Y}_{p,1}}\big),$$

then there is a unique, global solution to (NSP)*, in the space*

$$\widetilde{L^2}(\mathbb{R}^+;\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1} \cap \dot{B}^{-1+\frac{2}{p},1+\frac{1}{p}}_{p,1}) \cap \widetilde{L^1}(\mathbb{R}^+;\dot{B}^{1+\frac{2}{p},\frac{1}{p}}_{p,1} \cap \dot{B}^{\frac{2}{p},1+\frac{1}{p}}_{p,1}).$$

The proofs of those two theorems allow to obtain the following strong stability result, which to simplify we only state in the case $p = 1$ since it is the setting of the stability result by weak convergence proved in this paper. We recall that $\mathcal{B}^1_1 = \dot{B}^{1,1}_{1,1}$.

Corollary 3 (Strong stability in $\mathcal{B}^1_1$)**.** *Let $u_0 \in \mathcal{B}^1_1$ be a divergence free vector field generating a unique solution u in $\widetilde{L^\infty_{loc}}(\mathbb{R}^+;\mathcal{B}^1_1) \cap \widetilde{L^1_{loc}}(\mathbb{R}^+;\dot{B}^{3,1}_{1,1} \cap \dot{B}^{1,3}_{1,1})$. Then u belongs to $\mathcal{S}_{1,1}$ and $\|u(t)\|_{\mathcal{B}^1_1} \to 0$ as $t \to \infty$.*
Moreover there is ε_0 such that any $v_0 \in \mathcal{B}^1_1$ satisfying $\|u_0 - v_0\|_{\mathcal{B}^1_1} \leq \varepsilon_0$ generates a unique global solution in $\mathcal{S}_{1,1}$.

A.2. **Proof of Theorem 4.** We shall proceed in several steps:

(1) If u_0 belongs to $\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}$, we prove that a fixed point may be performed in the Banach space $\widetilde{L^2}(\mathbb{R}^+;\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1})$, which implies the existence and uniqueness of a solution in that space for small data.
(2) We then prove that the solution constructed in the previous step actually belongs to $\mathcal{Y}_{p,1}$, and to $\mathcal{S}_{p,1}$ if $p < 4$, and we prove that any "almost global solution" belongs to $\mathcal{S}_{p,1}$ and decays to zero at infinity.
(3) We deduce from the estimates leading to the above steps the result for large data.
(4) We prove the propagation of regularity in $\mathcal{S}_{p,q}$ for $q < 1$.

(1) Let us start by applying a fixed point theorem in the Banach space $\widetilde{L^2}(\mathbb{R}^+;\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1})$, to (NS) written in integral form:

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-t')\Delta}\,\mathbb{P}\mathrm{div}\,(u\otimes u)(t')\,dt'\,,$$

recalling that $\mathbb{P} := \mathrm{I} - \nabla\Delta^{-1}\mathrm{div}$ is the Leray projector onto divergence free vector fields. We first notice that (see Proposition B.2)

$$\|e^{t\Delta}\Delta^h_k\Delta^v_j u_0\|_{L^p} \lesssim e^{-ct(2^{2k}+2^{2j})}\|\Delta^h_k\Delta^v_j u_0\|_{L^p}\,,$$

so one sees immediately that for any $1 \leq r \leq \infty$ and for any $0 \leq \sigma \leq 2/r$,

$$\|e^{t\Delta}u_0\|_{\widetilde{L^r}(\mathbb{R}^+;\dot{B}^{-1+\frac{2}{p}+\sigma,\frac{2}{r}-\sigma+\frac{1}{p}}_{p,1})} \lesssim \|u_0\|_{\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}\,. \tag{A.2}$$

Now let us turn to the non linear term. Defining

$$B(u,u)(t) := -\int_0^t e^{(t-t')\Delta}\,\mathbb{P}\mathrm{div}\,(u\otimes u)(t')\,dt'\,,$$

we have

$$2^{\frac{2k}{p}+\frac{j}{p}}\|\Delta^h_k\Delta^v_j B(u,u)(t)\|_{L^p} \lesssim \int_0^t e^{-c(t-t')(2^{2k}+2^{2j})}(2^k+2^j)2^{\frac{2k}{p}+\frac{j}{p}}\|\Delta^h_k\Delta^v_j(u\otimes u)(t')\|_{L^p}\,dt'\,.$$

The space $\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1}$ is an algebra according to (B.4) so we have

$$\|u\otimes u\|_{L^1(\mathbb{R}^+;\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1})} \lesssim \|u\|^2_{\widetilde{L^2}(\mathbb{R}^+;\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1})}\,. \tag{A.3}$$

It follows that

$$2^{\frac{2k}{p}+\frac{j}{p}}\|\Delta^h_k\Delta^v_j B(u,u)(t)\|_{L^p} \lesssim \|u\|^2_{\widetilde{L^2}(\mathbb{R}^+;\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1})}\int_0^t e^{-c(t-t')(2^{2k}+2^{2j})}(2^k+2^j)c_{jk}(t')\,dt'\,, \tag{A.4}$$

where $c_{jk}(t')$ belongs to $\ell^1_{jk}(L^1_{t'})$ and Young's inequality in time gives

$$\|B(u,u)\|_{\widetilde{L^2}(\mathbb{R}^+;\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1})} \lesssim \|u\|^2_{\widetilde{L^2}(\mathbb{R}^+;\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1})}\,. \tag{A.5}$$

The small data result follows classically from (A.2) and (A.5) by a fixed point in $\widetilde{L^2}(\mathbb{R}^+;\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1})$.

(2) Now let us prove that the solution actually belongs to $\mathcal{Y}_{p,1}$. We first notice that the above computations actually imply that the solution u belongs to $L^1(\mathbb{R}^+;\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1}\cap\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1})$. Indeed that holds for the term $e^{t\Delta}u_0$ due to (A.2) so we just need to concentrate on the bilinear term. We return to estimate (A.4) and consider any real number $r\in[1,\infty]$. Using (A.3), we can write for any $\sigma\in\mathbb{R}$

$$\begin{aligned}I_{jk}(t)&:=2^{k(-1+\frac{2}{p}+\sigma)}2^{j(\frac{2}{r}-\sigma+\frac{1}{p})}\|\Delta^h_k\Delta^v_jB(u,u)\|_{L^p}\\&\lesssim\|u\|^2_{\widetilde{L^2}(\mathbb{R}^+;\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}\int_0^t e^{-c(t-t')(2^{2k}+2^{2j})}(2^k+2^j)2^{k(-1+\frac{2}{p}+\sigma-\frac{2}{p})}2^{j(\frac{2}{r}-\sigma+\frac{1}{p}-\frac{1}{p})}c_{jk}(t')\,dt'\,,\end{aligned}$$

where again $c_{jk}(t')$ belongs to $\ell^1_{jk}(L^1_{t'})$. We want to prove that $I_{jk}(t)$ belongs to $\ell^1_{jk}(L^r_{t'})$. We apply a Young inequality in the time variable, which produces

$$\|I_{jk}\|_{L^r}\lesssim\|u\|^2_{\widetilde{L^2}(\mathbb{R}^+;\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}(2^{2k}+2^{2j})^{-\frac{1}{r}}(2^k+2^j)2^{k(-1+\frac{2}{p}+\sigma-\frac{2}{p})}2^{j(\frac{2}{r}-\sigma+\frac{1}{p}-\frac{1}{p})}d_{jk}\,,\tag{A.6}$$

with $d_{jk}\in\ell^1_{jk}$. An easy computation shows that the sequence bounding $\|I_{jk}\|_{L^r}$ is bounded in ℓ^1_{jk} as soon as one has $1\le\sigma\le 2/r$. This implies in particular that u belongs to the space $L^1(\mathbb{R}^+;\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1}\cap\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1})$ as claimed.

Remark A.1. Note in passing that if 2^k+2^j was replaced by 2^k on the right-hand side of (A.6), then one would recover directly the whole range $0\le\sigma\le 2/r$. Here we need an extra step because of the presence of 2^j.

From now on we assume that $p<4$, and we want to extend this result to any degree of integrability in time, as well as to the space $L^1(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},2+\frac{1}{p}}_{p,1})$. Let us start with the case $r=\infty$. Due to the smallness of u_0 and to the result we just found, it is enough to prove that

$$\|B(u,u)\|_{\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}\lesssim\|u\|_{\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}\|u\|_{L^1(\mathbb{R}^+;\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1}\cap\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1})}\tag{A.7}$$

since (A.2) takes care of $e^{t\Delta}u_0$. But we have, if $p<4$,

$$\begin{aligned}\|u\cdot\nabla u\|_{L^1(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}&\le\|u^h\cdot\nabla^h u\|_{L^1(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}+\|u^3\partial_3u\|_{L^1(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}\\&\lesssim\|u\|_{\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}\Big(\|u\|_{L^1(\mathbb{R}^+;\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1})}+\|u\|_{L^1(\mathbb{R}^+;\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1})}\Big)\end{aligned}\tag{A.8}$$

by the product laws (B.5) recalled in Appendix B, and the result follows exactly as above: on the one hand (A.8) gives

$$\begin{aligned}J_{jk}(t)&:=2^{k(-1+\frac{2}{p})}2^{\frac{j}{p}}\|\Delta^h_k\Delta^v_jB(u,u)\|_{L^p}\\&\lesssim\int_0^t e^{-c(t-t')(2^{2k}+2^{2j})}2^{k(-1+\frac{2}{p})}2^{\frac{j}{p}}2^{-k(-1+\frac{2}{p})}2^{-\frac{j}{p}}c_{jk}(t')\,dt'\\&\qquad\times\|u\|_{\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}\Big(\|u\|_{\widetilde{L^1}(\mathbb{R}^+;\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1})}+\|u\|_{L^1(\mathbb{R}^+;\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1})}\Big)\,,\end{aligned}$$

with $c_{jk}(t)\in\ell^1_{jk}(L^1_t)$, hence

$$\begin{aligned}\|B(u,u)\|_{\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}&\le\|J_{jk}\|_{\ell^1_{jk}((L^\infty_t)}\\&\lesssim\|u\|_{\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}\Big(\|u\|_{L^1(\mathbb{R}^+;\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1})}+\|u\|_{L^1(\mathbb{R}^+;\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1})}\Big)\,,\end{aligned}$$

which proves (A.7). On the other hand

$$\begin{aligned}K_{jk}(t) &:= 2^{k(-1+\frac{2}{p})}2^{j(2+\frac{1}{p})}\|\Delta_k^h\Delta_j^v B(u,u)\|_{L^p}\\ &\lesssim \int_0^t e^{-c(t-t')(2^{2k}+2^{2j})}2^{k(-1+\frac{2}{p})}2^{j(2+\frac{1}{p})}2^{-k(-1+\frac{2}{p})}2^{-\frac{j}{p}}c_{jk}(t')\,dt'\\ &\qquad\times\|u\|_{\widetilde{L^\infty}(\mathbb{R}^+;\dot B_{p,1}^{-1+\frac{2}{p},\frac{1}{p}})}\Big(\|u\|_{L^1(\mathbb{R}^+;\dot B_{p,1}^{1+\frac{2}{p},\frac{1}{p}})}+\|u\|_{L^1(\mathbb{R}^+;\dot B_{p,1}^{\frac{2}{p},1+\frac{1}{p}})}\Big),\end{aligned}$$

with $c_{jk}(t)\in\ell^1_{jk}(L^1_t)$, hence

$$\begin{aligned}\|B(u,u)\|_{L^1(\mathbb{R}^+;\dot B_{p,1}^{-1+\frac{2}{p},2+\frac{1}{p}})} &\le \|K_{jk}\|_{\ell^1_{jk}(L^1_t)}\\ &\lesssim \|u\|_{\widetilde{L^\infty}(\mathbb{R}^+;\dot B_{p,1}^{-1+\frac{2}{p},\frac{1}{p}})}\Big(\|u\|_{L^1(\mathbb{R}^+;\dot B_{p,1}^{1+\frac{2}{p},\frac{1}{p}})}+\|u\|_{L^1(\mathbb{R}^+;\dot B_{p,1}^{\frac{2}{p},1+\frac{1}{p}})}\Big).\end{aligned}$$

We conclude that if the initial data is small enough, then the solution belongs to $\mathcal{S}_{p,1}$.

Remark A.2. It is easy to see, using Remark A.1 for instance, that one could add an exterior force, small enough in $\widetilde{L^1}(\mathbb{R}^+;\dot B_{p,q}^{-1+\frac{2}{p},\frac{1}{p}})$, and the small data result would be identical.

Remark A.3. Note that all the estimates can be restricted to a time interval $[a,b]$ of $\mathbb{R}^+$.

Remark A.4. The $\widetilde{L^\infty}(\mathbb{R}^+;\dot B_{p,1}^{-1+\frac{2}{p},\frac{1}{p}})$ norm on the right-hand side of (A.8) can be replaced by the (smaller) $L^\infty(\mathbb{R}^+;\dot B_{p,1}^{-1+\frac{2}{p},\frac{1}{p}})$ norm. The same goes for the $\widetilde{L^2}(\mathbb{R}^+;\dot B_{p,1}^{\frac{2}{p},\frac{1}{p}})$ norm in (A.3), which can be replaced by the $L^2(\mathbb{R}^+;\dot B_{p,1}^{\frac{2}{p},\frac{1}{p}})$ norm. This will be useful in the proof of Theorem 5.

(3) It is classical that the previous estimates can be adapted to the case of large initial data (for instance by solving first the heat equation and then a perturbed Navier-Stokes equation, of the same type as in the proof of Theorem 5 below) and we leave this to the reader.

(4) Now we are left with the proof of the propagation of regularity result. Again this is an easy exercise based on the fact that Young's inequality for sequences are true in ℓ^q with $q>0$ so we can simply copy the above arguments.

Theorem 4 is proved. □

A.3. **Proof of Theorem 5.** We shall follow the proof of Theorem 4 above, writing (NSP) under the integral form

$$u(t)=e^{t\Delta}u_0-\int_0^t e^{(t-t')\Delta}\,\mathbb{P}\Big(\operatorname{div}(u\otimes u+U\otimes u+u\otimes U)+F\Big)(t')\,dt'.$$

The linear term $e^{t\Delta}u_0$ and the term involving $\operatorname{div}(u\otimes u)$ (called $B(u,u)$ in the previous proof) have already been dealt with and we know that in particular for any $a<b$ and any $1\le r\le\infty$,

$$\forall\, 0\le\sigma\le\frac{2}{r},\quad \|e^{t\Delta}u_0\|_{\widetilde{L^r}([a,b];\dot B_{p,1}^{-1+\frac{2}{p}+\sigma,\frac{2}{r}-\sigma+\frac{1}{p}})}\lesssim\|u_0\|_{\dot B_{p,1}^{-1+\frac{2}{p},\frac{1}{p}}}. \tag{A.9}$$

We have as well

$$\|B(u,u)\|_{\widetilde{L^2}([a,b];\dot B_{p,1}^{\frac{2}{p},\frac{1}{p}})}+\|B(u,u)\|_{L^1([a,b];\dot B_{p,1}^{\frac{2}{p},1+\frac{1}{p}}\cap\dot B_{p,1}^{1+\frac{2}{p},\frac{1}{p}})}\lesssim\|u\|^2_{\widetilde{L^2}([a,b];\dot B_{p,1}^{\frac{2}{p},\frac{1}{p}})}, \tag{A.10}$$

and if $1 \le p < 4$,

$$\begin{aligned}\|B(u,u)\|_{\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})} + \|B(u,u)\|_{\widetilde{L^2}(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},1+\frac{1}{p}}_{p,1})} + \|B(u,u)\|_{L^1(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},2+\frac{1}{p}}_{p,1})} \\ \lesssim \|u\|_{L^\infty(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}\|u\|_{L^1(\mathbb{R}^+;\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1}\cap \dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1})}\,.\end{aligned} \tag{A.11}$$

Note that the estimate in $\widetilde{L^2}(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},1+\frac{1}{p}}_{p,1})$ appearing in (A.11) is a consequence of an interpolation between the spaces $\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})$ and $L^1(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},2+\frac{1}{p}}_{p,1})$.
Now let us study the term containing the force F. We define

$$\mathcal{F}(t) := \int_0^t e^{(t-t')\Delta}\,\mathbb{P}F(t')\,dt'\,, \quad \text{with} \quad F_1 \in L^1(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}) \quad \text{and}$$
$$F_2 \in \widetilde{L^2}(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},-1+\frac{1}{p}}_{p,1}) \cap L^1(\mathbb{R}^+;\dot B^{\frac{2}{p},-1+\frac{1}{p}}_{p,1})\,.$$

On the one hand the above arguments (see the estimates of I_{jk} and K_{jk}, or simply Remark A.1) enable us to write directly that for all $\sigma \in [0,2]$,

$$\|\mathcal{F}\|_{\widetilde{L^\infty}([a,b];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})} + \|\mathcal{F}\|_{L^1([a,b];\dot B^{-1+\sigma+\frac{2}{p},2-\sigma+\frac{1}{p}}_{p,1})} \lesssim \|F_1\|_{L^1([a,b];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})} \tag{A.12}$$

while for all $1 \le \sigma \le 2$,

$$\|\mathcal{F}\|_{\widetilde{L^2}([a,b];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})} + \|\mathcal{F}\|_{L^1([a,b];\dot B^{-1+\sigma+\frac{2}{p},2-\sigma+\frac{1}{p}}_{p,1})} \lesssim \|F_2\|_{L^1([a,b];\dot B^{\frac{2}{p},-1+\frac{1}{p}}_{p,1})}\,. \tag{A.13}$$

On the other hand the same computations as in the proof of Theorem 4 give easily

$$\|\mathcal{F}\|_{\widetilde{L^\infty}([a,b];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})\cap\widetilde{L^2}([a,b];\dot B^{-1+\frac{2}{p},1+\frac{1}{p}}_{p,1})} \lesssim \|F_2\|_{\widetilde{L^2}(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},-1+\frac{1}{p}}_{p,1})}\,. \tag{A.14}$$

Finally let us turn to the contribution of U. We define

$$\mathcal{U}(t) := -\int_0^t e^{t\Delta}\,\mathbb{P}\mathrm{div}\,(u\otimes U + U\otimes u)(t')\,dt'\,.$$

We can write using (B.5) (and Remark A.4)

$$\begin{aligned}\|u^h\cdot\nabla^h U + u^3\partial_3 U\|_{L^1([a,b];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})} &\lesssim \|u\|_{L^\infty([a,b];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}\|U\|_{L^1([a,b];\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1}\cap\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1})} \\ &\lesssim \|u\|_{\widetilde{L^\infty}([a,b];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}\|U\|_{L^1([a,b];\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1}\cap\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1})}\end{aligned}$$

and using (B.5) again,

$$\|U^h\cdot\nabla^h u + U^3\partial_3 u\|_{L^1([a,b];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})} \lesssim \|u\|_{\widetilde{L^2}([a,b];\dot B^{-1+\frac{2}{p},1+\frac{1}{p}}_{p,1}\cap\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}\|U\|_{L^2([a,b];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}\,.$$

This enables us to write

$$\begin{aligned}\|\mathcal{U}\|_{\widetilde{L^\infty}([a,b];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})\cap\widetilde{L^2}([a,b];\dot B^{-1+\frac{2}{p},1+\frac{1}{p}}_{p,1})} \lesssim \Big(&\|u\|_{\widetilde{L^\infty}([a,b];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}\|U\|_{L^1([a,b];\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1}\cap\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1})} \\ &+\|u\|_{\widetilde{L^2}([a,b];\dot B^{-1+\frac{2}{p},1+\frac{1}{p}}_{p,1}\cap\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}\|U\|_{L^2([a,b];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}\Big)\,.\end{aligned} \tag{A.15}$$

Putting estimates (A.9), (A.10), (A.12), (A.13), (A.15) together we infer that

$$\begin{aligned}\|u\|_{\widetilde{L^2}([a,b];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})\cap L^1([a,b];\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1}\cap\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1})} \le C\Big(&\|u\|^2_{\widetilde{L^2}([a,b];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})} \\ +\|u\|_{\widetilde{L^2}([a,b];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}\|U\|_{L^2([a,b];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})} &+ \|u(a)\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}} + \|F\|_{\mathcal{X}_{p,1}}\Big)\,,\end{aligned} \tag{A.16}$$

while estimates (A.9), (A.11), (A.14), (A.15) give

$$\begin{aligned}\|u\|_{\widetilde{L^\infty}([a,b];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})\cap\widetilde{L^2}([a,b];\dot B^{-1+\frac{2}{p},1+\frac{1}{p}}_{p,1})} \le C\Big(&\|u\|_{\widetilde{L^\infty}([a,b];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}\|u\|_{L^1([a,b];\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1}\cap\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1})}\\ &+\|u\|_{\widetilde{L^\infty}([a,b];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}\|U\|_{L^1([a,b];\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1}\cap\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1})}\\ &+\|u\|_{\widetilde{L^2}([a,b];\dot B^{-1+\frac{2}{p},1+\frac{1}{p}}_{p,1}\cap\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}\|U\|_{L^2([a,b];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}+\|u(a)\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}+\|F\|_{\mathcal{X}_{p,1}}\Big)\,.\end{aligned}\tag{A.17}$$

To conclude we resort to a Gronwall-type argument (see for instance [27] for a similar argument): there exist N real numbers $(T_i)_{1\le i\le N}$ such that $T_1=0$ and $T_N=+\infty$, such that $\mathbb{R}_+=\bigcup_{i=1}^{N-1}[T_i,T_{i+1}]$ and satisfying

$$\|U\|_{L^2([T_i,T_{i+1}];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}+\|U\|_{L^1([T_i,T_{i+1}];\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1}\cap\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1})}\le\frac{1}{8C}\quad\forall i\in\{1,\dots,N-1\}\,.\tag{A.18}$$

Then suppose that

$$\|u_0\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}+\|F\|_{\mathcal{X}_{p,1}}\le\frac{1}{8CN(2C)^N}\,.\tag{A.19}$$

By time continuity we can define a maximal time $T\in\mathbb{R}^+\cup\{\infty\}$ such that

$$\|u\|_{\widetilde{L^2}([0,T];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}+\|u\|_{L^1([0,T];\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1}\cap\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1})}\le\frac{1}{4C}\,.\tag{A.20}$$

If $T=\infty$ then the theorem is proved. Suppose now that $T<+\infty$. Then we can define an integer $k\in\{1,\dots,N-1\}$ such that

$$T_k\le T<T_{k+1}\,,$$

and plugging (A.18) and (A.20) into (A.16) we get for any $i\le k-1$

$$\begin{aligned}\|u\|_{\widetilde{L^2}([T_i,T_{i+1}];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}+\|u\|_{L^1([T_i,T_{i+1}];\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1}\cap\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1})}&\le C\|u(T_i)\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}\\ &+C\|F\|_{\mathcal{X}_{p,1}}+\frac14\|u\|_{\widetilde{L^2}([T_i,T_{i+1}];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}+\frac14\|u\|_{\widetilde{L^2}([T_i,T_{i+1}];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}\,,\end{aligned}$$

so finally

$$\begin{aligned}\|u\|_{\widetilde{L^2}([T_i,T_{i+1}];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}+\|u\|_{L^1([T_i,T_{i+1}];\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1}\cap\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1})}&\\ \le 2C\Big(\|u(T_i)\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}+\|F\|_{\mathcal{X}_{p,1}}\Big)\,.&\end{aligned}\tag{A.21}$$

From relations (A.16) and (A.17) we also get

$$\|u\|_{\widetilde{L^\infty}([T_i,T_{i+1}];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}\le 2C\Big(\|u(T_i)\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}+\|F\|_{\mathcal{X}_{p,1}}\Big).\tag{A.22}$$

Since $\widetilde{L^\infty}(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})\subset L^\infty(\mathbb{R}^+;\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})$, we further infer that

$$\|u(T_{i+1})\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}\le 2C\Big(\|u(T_i)\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}+\|F\|_{\mathcal{X}_{p,1}}\Big)\,.$$

A trivial induction now shows that for all $i\in\{1,\dots,k-1\}$,

$$\|u(T_i)\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}\le(2C)^{i-1}\Big(\|u_0\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}+\|F\|_{\mathcal{X}_{p,1}}\Big)\,.$$

We conclude from (A.21) and (A.22) that

$$\|u\|_{\widetilde{L^2}([T_i,T_{i+1}];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}+\|u\|_{L^1([T_i,T_{i+1}];\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1}\cap\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1})}\leq(2C)^i\Big(\|u_0\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}+\|F\|_{\mathcal{X}_{p,1}}\Big)$$

and

$$\|u\|_{\widetilde{L^\infty}([T_i,T_{i+1}];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}\leq(2C)^i\Big(\|u_0\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}+\|F\|_{\mathcal{X}_{p,1}}\Big)$$

for all $i\leq k-1$. The same arguments as above also apply on the interval $[T_k,T]$ and yield

$$\|u\|_{\widetilde{L^2}([T_k,T];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}\leq(2C)^N\Big(\|u_0\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}+\|F\|_{\mathcal{X}_{p,1}}\Big)$$

and

$$\|u\|_{\widetilde{L^\infty}([T_k,T];\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})}\leq(2C)^N\Big(\|u_0\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}+\|F\|_{\mathcal{X}_{p,1}}\Big)\,.$$

Then it is easy to see that (see for instance [27])

$$\begin{aligned}\|u\|_{\widetilde{L^2}([0,T];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}&\leq\|u\|_{\widetilde{L^2}([T_1,T_2];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}+\cdots+\|u\|_{\widetilde{L^2}([T_k,T];\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}\\&\leq N(2C)^N\|u_0\|_{\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}}+N(2C)^N\|F\|_{\mathcal{X}_{p,1}}\,.\end{aligned}$$

Under assumption (A.19) this contradicts the maximality of T as defined in (A.20). Since the integer N can be chosen of size equivalent to $\|U\|_{L^2(\mathbb{R}^+;\dot B^{\frac{2}{p},\frac{1}{p}}_{p,1})}+\|U\|_{L^1(\mathbb{R}^+;\dot B^{1+\frac{2}{p},\frac{1}{p}}_{p,1}\cap\dot B^{\frac{2}{p},1+\frac{1}{p}}_{p,1})}$, the theorem is proved. □

Remark A.5. Note that we have obtained also that u belongs to $\widetilde{L^\infty}(\mathbb{R}^+,\dot B^{-1+\frac{2}{p},\frac{1}{p}}_{p,1})$.

A.4. **Proof of Corollary 3.** Let $u\in\widetilde{L^\infty_{loc}}(\mathbb{R}^+;\mathcal{B}^1_1)\cap L^1_{loc}(\mathbb{R}^+;\dot B^{3,1}_{1,1}\cap\dot B^{1,3}_{1,1})$ solve (NS) with initial data $u_0\in\mathcal{B}^1_1$. Let us start by proving that $u\in\mathcal{S}_{1,1}$ and that $\|u(t)\|_{\mathcal{B}^1_1}\to0$ as t goes to ∞. Actually it is enough to prove the convergence to zero result in large times, since the fact that $u\in\mathcal{S}_{1,1}$ is then a consequence of Theorem 4 since for T large enough we have $\|u(T)\|_{\mathcal{B}^1_1}\leq c_0$.

We shall only sketch the proof as it is very similar to the same result in the isotropic case, proved in [27]. The idea is to use a frequency truncation to decompose $u_0=v_0+w_0$ with $\|w_0\|_{\mathcal{B}^1_1}\leq\varepsilon_0$ for some arbitrarily small ε_0 and with $v_0\in\mathcal{B}^1_1\cap L^2$. We then solve globally (NS) in $\mathcal{S}_{1,1}$ with data w_0, and we know from Theorem 4 that

$$\|w\|_{\mathcal{S}_{1,1}}\leq2\varepsilon_0\,.$$

It is easy to see (using the same arguments as in Proposition B.3) that $\|w_0\|_{\dot B^{-1}_{\infty,\infty}}\lesssim\varepsilon_0$ so the arguments of Proposition A.2 of [27] imply that

$$\sup_{t>0}\sqrt{t}\|w(t)\|_{L^\infty}\lesssim\varepsilon_0\,. \tag{A.23}$$

Now let us consider v: it satisfies the perturbed (NSP) equation with data v_0, with $F=0$ and with $U=w$, and it belongs to $\widetilde{L^\infty_{loc}}(\mathbb{R}^+;\mathcal{B}^1_1)\cap L^1_{loc}(\mathbb{R}^+;\dot B^{3,1}_{1,1}\cap\dot B^{1,3}_{1,1})$ since that holds for u and w. We claim that there is $T>0$ such that

$$v\in\widetilde{L^\infty}([0,T];L^2)\cap L^2([0,T];\dot H^1)\,.$$

Indeed we have by product laws the following analogue of (A.8):

$$\begin{aligned}\|u\cdot\nabla u\|_{L^1([0,T];\dot B^{1,\frac{1}{2}}_{1,1})}&\leq\|u^h\cdot\nabla^h u\|_{L^1([0,T];\dot B^{1,\frac{1}{2}}_{1,1})}+\|u^3\partial_3u\|_{L^1([0,T];\dot B^{1,\frac{1}{2}}_{1,1})}\\&\lesssim\|u\|_{\widetilde{L^\infty}([0,T];\dot B^{1,\frac{1}{2}}_{1,1})}\Big(\|u\|_{L^1([0,T];\dot B^{3,1}_{1,1})}+\|u\|_{L^1([0,T];\dot B^{2,2}_{1,1})}\Big)\,,\end{aligned}$$

which implies as in (A.7) that

$$\|B(u,u)\|_{\widetilde{L^\infty}([0,T];\dot B^{1,\frac12}_{1,1})} \lesssim \|u\|_{\widetilde{L^\infty}([0,T];\dot B^{1,\frac12}_{1,1})}\|u\|_{L^1([0,T];\dot B^{3,1}_{1,1}\cap \dot B^{2,1+1}_{1,1})}$$

so as in Lemma A.2 of [27] we get $v \in \widetilde{L^\infty}([0,T];\dot B^{1,\frac12}_{1,1}) \subset \widetilde{L^\infty}([0,T];\dot B^{0,0}_{2,1}) \subset \widetilde{L^\infty}([0,T];L^2)$. The bound in $L^2([0,T];\dot H^1)$ is obtained in a similar way, noticing that if f is in $L^1([0,T];\dot B^{1,\frac12}_{1,1})$, then $F := \int_0^t e^{(t-t')\Delta}\mathbb{P}(f)(t')\,dt'$ satisfies, by similar computations to the proof of Theorem 4,

$$\|F\|_{\widetilde{L^2}([0,T];\dot B^1_{2,1}(\mathbb{R}^3))} \lesssim \|f\|_{L^1([0,T];\dot B^{0,0}_{2,1})} \lesssim \|f\|_{L^1([0,T];\dot B^{1,\frac12}_{1,1})}.$$

Then we conclude exactly as in the proof of Theorem 2.1 in [27]: we find, writing an energy estimate in L^2 and using (A.23) that v can be made arbitrarily small in $\dot H^{\frac12}$ as time goes to infinity, hence by Proposition B.3 the same holds in $\dot B^{0,\frac12}_{2,1}$. It follows that $u(t) = v(t) + w(t)$ is arbitrarily small in $\dot B^{0,\frac12}_{2,1}$ (say smaller than c_0, if ε_0 is small enough) for t large enough, hence there is a global solution in $\mathcal{S}_{2,1}$ associated with u_0, which can be shown to also belong to $\mathcal{S}_{1,1}$ by a propagation of regularity argument. We know indeed by Theorem 4 that u belongs to $\mathcal{S}_{1,1}(T)$ for some time T so we just need to check that the $\widetilde{L^2}([0,T];\dot B^{2,1}_{1,1})$ norm of u remains bounded uniformly in T. But product laws give

$$\|u\otimes u\|_{L^1([0,T];\dot B^{2,1}_{1,1})} \lesssim \|u\|_{\widetilde{L^2}([0,T];\dot B^{2,1}_{1,1})}\|u\|_{\widetilde{L^2}([0,T];\dot B^{1,\frac12}_{2,1})}$$

so as in (A.5) we get

$$\|B(u,u)\|_{\widetilde{L^2}([0,T];\dot B^{2,1}_{1,1})} \lesssim \|u\|_{\widetilde{L^2}([0,T];\dot B^{2,1}_{1,1})}\|u\|_{\widetilde{L^2}([0,T];\dot B^{1,\frac12}_{2,1})},$$

which allows to prove the result.

Then the strong stability result is obtained using Theorem 5. Indeed we can solve (NS) with initial data v_0 for a short time and the solution v can be written as $u-w$. The vector field w then satisfies (PNS) with initial data w_0, with forcing term zero, and with $U = u$. We know that $u \in \mathcal{S}_{1,1} \subset \mathcal{Y}_{1,1}$ so the result is a direct consequence of Theorem 5.

Corollary 3 is proved. □

Appendix B. Anisotropic Littlewood-Paley decomposition

In this section we recall the definition of the isotropic and anisotropic Littlewood-Paley decompositions and associated function spaces, and give their main properties that are used in this paper. We refer for instance to [3], [17], [33], [32], [35], [51] and [59] for all necessary details.

B.1. **Isotropic decomposition and function spaces.** Let $\widehat{\chi}$ (the Fourier transform of χ) be a radial function in $\mathcal{D}(\mathbb{R})$ such that $\widehat{\chi}(t) = 1$ for $|t| \le 1$ and $\widehat{\chi}(t) = 0$ for $|t| > 2$, and we define (in d space dimensions) $\chi_\ell := 2^{d\ell}\chi(2^\ell|\cdot|)$. Then the frequency localization operators used in this paper are defined by

$$S_\ell := \chi_\ell * \cdot \quad \text{and} \quad \Delta_\ell := S_{\ell+1} - S_\ell =: \Psi_\ell * \cdot.$$

Now let us define Besov spaces on $\mathbb{R}^d$ using this decomposition. We start by defining, as in [3],

$$\mathcal{S}'_h := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \,/\, \|\Delta_j f\|_{L^\infty} \to 0,\ j \to -\infty \right\}. \tag{B.1}$$

Let f be in $\mathcal{S}'(\mathbb{R}^d)$, let p belong to $[1,\infty]$ and q to $]0,\infty]$, and let $s\in\mathbb{R}, s<d/p$. We say that f belongs to $\dot B^s_{p,q}(\mathbb{R}^d)$ if the sequence $\varepsilon_\ell := 2^{\ell s}\|\Delta_\ell f\|_{L^p}$ belongs to $\ell^q(\mathbb{Z})$, and we have

$$\|f\|_{\dot B^s_{p,q}(\mathbb{R}^d)} := \|\varepsilon_\ell\|_{\ell^q(\mathbb{Z})}\,.$$

If $s=d/p$ and $q=1$, then the same definition holds as soon as one assumes moreover that $f\in\mathcal{S}'_h$ — or equivalently after taking the quotient with polynomials. Finally in all other cases then $\dot B^s_{p,q}(\mathbb{R}^d)$ is defined by the above norm, after taking the quotient with polynomials (see [9] and the references therein for a discussion).

It is well-known that an equivalent norm is given by

$$\forall s\in\mathbb{R},\ \forall (p,q)\in[1,\infty],\quad \|f\|_{\dot B^s_{p,q}(\mathbb{R}^d)} = \Big\| t^{-\frac{s}{2}}\|K(t)f\|_{L^p(\mathbb{R}^d)}\Big\|_{L^q(\mathbb{R}^+;\frac{dt}{t})} \tag{B.2}$$

with $K(t):=t\partial_t e^{t\Delta}$. We recall also that Sobolev spaces are defined by the norm $\|\cdot\|_{\dot B^s_{2,2}(\mathbb{R}^d)}$ and

$$\forall s<\frac{d}{2},\quad \|f\|_{\dot H^s(\mathbb{R}^d)} := \Big(\int |\xi|^{2s}|\hat f(\xi)|^2\,d\xi\Big)^{\frac12}$$

where $\hat f$ is the Fourier transform of f.

Finally it is useful, in the context of the Navier-Stokes equations, to introduce the following space-time norms (see [16]):

$$\|f\|_{\widetilde{L^r}([0,T];\dot B^s_{p,q})} := \big\|2^{js}\|\Delta_j f\|_{L^r([0,T];L^p(\mathbb{R}^d))}\big\|_{\ell^q}$$

or equivalently

$$\|f\|_{\widetilde{L^r}([0,T];\dot B^s_{p,q})} = \Big\| t^{-\frac{s}{2}}\|K(t)f\|_{L^r([0,T];L^p(\mathbb{R}^d))}\Big\|_{L^q(\mathbb{R}^+;\frac{dt}{t})}\,.$$

The following proposition lists a few useful inequalities related to those spaces.

Proposition B.1. *If $1\le p\le q\le\infty$, then*

$$\|\partial^\alpha\Delta_j f\|_{L^q(\mathbb{R}^d)} \lesssim 2^{j(|\alpha|+d(1/p-1/q))}\|\Delta_j f\|_{L^p(\mathbb{R}^d)}\,,$$

$$\textit{and}\quad \|e^{t\Delta}\Delta_j f\|_{L^q(\mathbb{R}^d)} \lesssim e^{-ct2^{2j}}\|\Delta_j f\|_{L^q(\mathbb{R}^d)}\,.$$

Finally let us recall product laws in Besov spaces:

$$\|fg\|_{\dot B^{s_1+s_2-\frac{d}{p}}_{p,q}(\mathbb{R}^d)} \lesssim \|f\|_{\dot B^{s_1}_{p,q}(\mathbb{R}^d)}\|g\|_{\dot B^{s_2}_{p,q}(\mathbb{R}^d)}\,,$$

as soon as

$$s_1+s_2>0\quad\text{and}\quad s_j<\frac{d}{p}\,,\ j\in\{1,2\}\,.$$

B.2. **Anisotropic decomposition and function spaces.** Similarly we define a three dimensional, anisotropic decomposition as follows. For $(j,k)\in\mathbb{Z}^2$, we define the horizontal decomposition as

$S^h_k f := \mathcal{F}^{-1}\big(\widehat\chi(2^{-k}|\xi_h|)\hat f(\xi)\big)$ and $\Delta^h_k := S^h_{k+1}-S^h_k$, which writes $\mathcal{F}(\Delta^h_k f) := \widehat\Psi(2^{-k}|\xi_h|)\hat f(\xi)$

and the vertical decomposition as

$S^v_j f := \mathcal{F}^{-1}\big(\widehat\chi(2^{-j}|\xi_3|)\hat f(\xi)\big)$ and $\Delta^v_j := S^v_{j+1}-S^v_j$, which writes $\mathcal{F}(\Delta^v_j f) := \widehat\Psi(2^{-j}|\xi_3|)\hat f(\xi)\,.$

Now let us define anisotropic Besov spaces. We define, for all $(s,s')\in\mathbb{R}^2, s<2/p, s'<1/p$ and all $p\in[1,\infty]$ and $q\in]0,\infty]$,

$$\dot B^{s,s'}_{p,q} := \Big\{ f\in\mathcal{S}'\ /\ \|f\|_{\dot B^{s,s'}_{p,q}} := \Big\|2^{ks+js'}\|\Delta^h_k\Delta^v_j f\|_{L^p}\Big\|_{\ell^q}<\infty\Big\}\,.$$

In all other cases one defines the same norm, and one needs to take the quotient with polynomials.

As in (B.2) an equivalent definition using the heat flow is

$$\|f\|_{\dot{B}^{s,s'}_{p,q}} = \Big\| t^{-\frac{s}{2}} t'^{-\frac{s'}{2}} \|K_h(t)K_v(t')f\|_{L^p} \Big\|_{L^q(\mathbb{R}^+\times\mathbb{R}^+;\frac{dt}{t}\frac{dt'}{t'})} \tag{B.3}$$

where $K_h(t) := t\partial_t e^{t\Delta_h^2}$ and $K_v(t) := t\partial_t e^{t\partial_3^2}$.

As in the isotropic case we introduce the following space-time norms:

$$\|f\|_{\widetilde{L^r}([0,T];\dot{B}^{s,s'}_{p,q})} := \big\| 2^{ks+js'} \|\Delta_k^h \Delta_j^v f\|_{L^r([0,T];L^p)} \big\|_{\ell^q}$$

or equivalently

$$\|f\|_{\widetilde{L^r}([0,T];\dot{B}^{s,s'}_{p,q})} = \Big\| t^{-\frac{s}{2}} t'^{-\frac{s'}{2}} \|K_h(t)K_v(t')f\|_{L^r([0,T];L^p)} \Big\|_{L^q(\mathbb{R}^+\times\mathbb{R}^+;\frac{dt}{t}\frac{dt'}{t'})} .$$

Notice that of course $\widetilde{L^r}([0,T];\dot{B}^{s,s'}_{p,r}) = L^r([0,T];\dot{B}^{s,s'}_{p,r})$, and by Minkowski's inequality, we have the embedding $\widetilde{L^r}([0,T];\dot{B}^{s,s'}_{p,q}) \subset L^r([0,T];\dot{B}^{s,s'}_{p,q})$ if $r \geq q$.

The anisotropic counterpart of Proposition B.1 is the following.

Proposition B.2. *If $1 \leq p_1 \leq p_2 \leq \infty$, then*

$$\begin{aligned}
\|\partial_{x_h}^\alpha \Delta_k^h f\|_{L^{p_2}(\mathbb{R}^2;L^r(\mathbb{R}))} &\lesssim 2^{k(|\alpha|+2(1/p_1-1/p_2))} \|\Delta_k^h f\|_{L^{p_1}(\mathbb{R}^2;L^r(\mathbb{R}))} ,\\
\|\partial_{x_3}^\alpha \Delta_j^v f\|_{L^r(\mathbb{R}^2;L^{p_2}(\mathbb{R}))} &\lesssim 2^{j(|\alpha|+1/p_1-1/p_2)} \|\Delta_j^v f\|_{L^r(\mathbb{R}^2;L^{p_1}(\mathbb{R}))} ,\\
\|e^{t\Delta}\Delta_k^h\Delta_j^v f\|_{L^q} &\lesssim e^{-ct(2^{2k}+2^{2j})} \|\Delta_k^h\Delta_j^v f\|_{L^q} .
\end{aligned}$$

In this paper we use product laws in anisotropic Besov spaces, which read as follows:

$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{2}{p},s_2'}_{p,q}} \lesssim \|f\|_{\dot{B}^{s_1,\frac{1}{p}}_{p,1}} \|g\|_{\dot{B}^{s_2,s_2'}_{p,q}} + \|f\|_{\dot{B}^{s_1,s_2'}_{p,q}} \|g\|_{\dot{B}^{s_2,\frac{1}{p}}_{p,1}} ,$$

as soon as

$$\frac{1}{p} \leq s_2', \quad s_1+s_2>0 \quad \text{and} \quad s_j < \frac{2}{p} ,\ j\in\{1,2\} ,$$

and

$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{2}{p},s_1'+s_1'-\frac{1}{p}}_{p,q}} \lesssim \|f\|_{\dot{B}^{s_1,s_1'}_{p,q}} \|g\|_{\dot{B}^{s_2,s_2'}_{p,q}} ,$$

as soon as

$$s_1'+s_2'>0 \quad \text{and} \quad s_j' < \frac{1}{p} ,\ j\in\{1,2\}$$

and with the same conditions on s_1, s_2. Finally

$$\|fg\|_{\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1}} \lesssim \|f\|_{\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1}} \|g\|_{\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1}} , \tag{B.4}$$

and if $p<4$,

$$\|fg\|_{\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}} \lesssim \|f\|_{\dot{B}^{-1+\frac{2}{p},\frac{1}{p}}_{p,1}} \|g\|_{\dot{B}^{\frac{2}{p},\frac{1}{p}}_{p,1}} . \tag{B.5}$$

The following result compares some isotropic and anisotropic Besov spaces.

Proposition B.3. *Let s and t be two nonnegative real numbers. Then for any $(p,q)\in[1,\infty]^2$ one has*

$$\|f\|_{\dot{B}^{s,t}_{p,q}} \lesssim \|f\|_{\dot{B}^{s+t}_{p,q}} .$$

Proof of Proposition B.3. We recall that

$$\|f\|^q_{\dot B^{s,t}_{p,q}} = \sum_{j,k} 2^{ksq}2^{jtq}\|\Delta^h_k\Delta^v_j f\|^q_{L^p}\,.$$

We separate the sum into two parts, depending on whether $j<k$ or $j\geq k$ and we shall only detail the first case (the second one is identical). We notice indeed that if $j<k$, then

$$\begin{aligned}\|\Delta^h_k\Delta^v_j f\|_{L^p} &= \|\sum_\ell \Delta_\ell\Delta^h_k\Delta^v_j f\|_{L^p}\\ &\sim \|\Delta_k\Delta^h_k\Delta^v_j f\|_{L^p}\,.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{j<k} 2^{ksq}2^{jtq}\|\Delta^h_k\Delta^v_j f\|^q_{L^p} &\lesssim \sum_{j<k} 2^{ksq}2^{jtq}\|\Delta_k f\|^q_{L^p}\\ &\lesssim \sum_k 2^{k(s+t)q}\|\Delta_k f\|^q_{L^p}\end{aligned}$$

and the result follows. □

Finally let us prove the following easy lemma, which implies that $(u_{0,n})_{n\in\mathbb{N}}$ is bounded in $\mathcal{B}^1_q$ if it is bounded in a space of the type $\dot B^{1\pm\varepsilon_1,1\pm\varepsilon_2}_{1,1}$ for some $\varepsilon_1,\varepsilon_2>0$.

Lemma B.4. *Let $s_1,s_2\in\mathbb{R}$, $p\in[1,\infty]$, $0<q_1\leq q_2\leq\infty$ be given, as well as two positive real numbers ε_1 and ε_2. The space $\dot B^{s_1\pm\varepsilon_1,s_2\pm\varepsilon_2}_{1,q_2}$ is continuously embedded in $\dot B^{s_1,s_2}_{p,q_1}$.*

Proof. Let f be an element of $\dot B^{s_1\pm\varepsilon_1,s_2\pm\varepsilon_2}_{1,q_2}$ and let us prove that f belongs to $\dot B^{s_1,s_2}_{p,q_1}$. We write

$$\|f\|^{q_1}_{\dot B^{s_1,s_2}_{p,q_2}} = \sum_{j,k} 2^{ks_1q_1}2^{js_2q_1}\|\Delta^h_k\Delta^v_j f\|^{q_1}_{L^p}$$

and we decompose the sum into four terms, depending on the sign of j and k. For instance we have

$$\begin{aligned}\mathcal{F}_1 &:= \sum_{\substack{j\leq 0\\ k\geq 0}} 2^{ks_1q_1}2^{js_2q_1}\|\Delta^h_k\Delta^v_j f\|^{q_1}_{L^p}\\ &\leq \sum_{\substack{j\leq 0\\ k\geq 0}} 2^{-k\varepsilon_1q_1}2^{j\varepsilon_2q_1}2^{k(s_1+\varepsilon_1)q_1}2^{j(s_2-\varepsilon_2)q_1}\|\Delta^h_k\Delta^v_j f\|^{q_1}_{L^p}\end{aligned}$$

and we apply Hölder's inequality for sequences which gives

$$\mathcal{F}_1 \lesssim \|f\|_{\dot B^{s_1+\varepsilon_1,s_2-\varepsilon_2}_{1,q_2}}\,.$$

The other terms are dealt with similarly. □

B.3. **On the role of anisotropy in the Navier-Stokes equations.** In this final short paragraph, we shall prove Theorem 1 stated in the introduction.

Proof of Theorem 1. The proof follows from the small data theory recalled in Appendix A. Let us first consider $v_0 := \sum_{j-k<-N_0}\Delta^h_k\Delta^v_j u_0$. We have

$$\begin{aligned}\|v_0\|_{\dot B^{0,\frac12}_{2,1}} &\sim \sum_{j-k<-N_0} 2^{\frac{j}{2}}\|\Delta^h_k\Delta^v_j u_0\|_{L^2(\mathbb{R}^3)}\\ &\sim \sum_{j-k<-N_0} 2^{\frac{j-k}{2}}2^{\frac{k}{2}}\|\Delta^h_k\Delta^v_j u_0\|_{L^2(\mathbb{R}^3)} \leq C2^{-\frac{N_0}{2}}\rho\end{aligned}$$

due to Proposition B.3 which states in particular that $\dot{B}^{\frac{1}{2}}_{2,1} \subset \dot{B}^{\frac{1}{2},0}_{2,1}$. So v_0 can be made arbitrarily small in $\dot{B}^{0,\frac{1}{2}}_{2,1}$, for N_0 large enough (depending only on ρ).
Now let us consider $w_0 = \sum_{j-k>N_0} \Delta^h_k \Delta^v_j u_0$. We shall prove that in this case $\|w_0\|_{L^3}$ is small. Indeed we know (see for instance [3]) that $\dot{B}^0_{3,1} \subset L^3$, and moreover we have as soon as N_0 is large enough (depending only on the choice of the Littlewood-Paley decomposition)

$$\|\Delta_\ell w_0\|_{L^3} \sim \Big\| \sum_{k-\ell<-N_0} \Delta^h_k \Delta^v_\ell u_0 \Big\|_{L^3}.$$

It follows that

$$\begin{aligned}\|\Delta_\ell w_0\|_{L^3} &\leq \sum_{k-\ell<-N_0} \|\Delta^h_k \Delta^v_\ell u_0\|_{L^3} \\ &\leq C \sum_{k-\ell<-N_0} 2^{\frac{k}{3}} 2^{\frac{\ell}{6}} \|\Delta^h_k \Delta^v_\ell u_0\|_{L^2}\end{aligned}$$

by Bernstein's inequalities (see Proposition B.2, applying successively the inequalities for the horizontal and the vertical truncations). So using Proposition B.3 again which states in particular that $\dot{B}^{\frac{1}{2}}_{2,1} \subset \dot{B}^{0,\frac{1}{2}}_{2,1}$, we get

$$\|\Delta_\ell w_0\|_{L^3} \leq C \sum_{k-\ell<-N_0} 2^{\frac{k-\ell}{3}} 2^{\frac{\ell}{2}} \|\Delta^h_k \Delta^v_\ell u_0\|_{L^2} \leq C 2^{-\frac{N_0}{3}} \rho c_\ell\,,$$

where c_ℓ is a sequence in the unit ball of $\ell^1(\mathbb{Z})$. So again if N_0 is large enough (depending only on ρ) then we find that w_0 is small in $\dot{B}^0_{3,1}$ hence in L^3.

To conclude we can start by solving (NS) associated with the data w_0 which yields a global, unique solution w that by Proposition B.3 belongs to $\mathcal{Y}_{3,1}$, with norm smaller than $2\|w_0\|_{L^3}$ (by small data theory, as soon as N_0 is large enough). Then since $\dot{B}^{0,\frac{1}{2}}_{2,1}$ embeds in $\dot{B}^{-\frac{1}{3},\frac{1}{3}}_{3,1}$ we can apply Theorem 5 with $F = 0$ and $U = w$ which solves the perturbed equation satisfied by $u - w$ globally in time, as soon as N_0 again is large enough. The solution belongs to $\mathcal{C}(\mathbb{R}^+; L^3(\mathbb{R}^3))$ by classical propagation of regularity arguments, and that proves the theorem. □

Remark B.5. Contrary to Theorem 2, the proof of Theorem 1 does not require the special structure of the nonlinear term in (NS) as it reduces to checking that the initial data is small in an adequate scale-invariant space.

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