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Invariant and Some New Exact Solutions of Burgers Equation

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Abstract. In this work, first we use the classification of one-dimensional subalgebras of Lie point symmetries admitted by Burgers Equation and the corresponding reduced differential equations to construct a large class of new exact solutions. Second, by using the Riccati transformation method, we obtain some new solutions of the Burgers equation namely, exponential, rational and periodic solutions.

Keywords: Lie symmetries, Riccati transformation, Invariant solutions, Burgers equation.

1 Introduction

Lie's classical method of group invariants initiated by Sophus Lie [1, 2] in the second half of 19th century, can be provide new and simple solution to differential equations arising from any application (see e.g. [3, 4, 5]). The symmetry group of a system of differential equations is, roughly speaking, a group of transformations of independent and dependent variables leaving the set of all solutions invariant. Once the symmetry group of a system of equations is known, it can be used to generate new solutions from the old ones, often interesting ones from trivial ones. It can be used to classify solutions as invariant solutions and to classify and simplify differential equations. An important application is the symmetry reduction: the reduction of an ordinary differential equation (ODE) to a lower order one, the reduction of a partial differential equation (PDE) to one with fewer independent variables. Consequently, for a first order equation one parameter symmetry group allows us to integrate it by a single quadrature. The symmetry method is often used for reduction of partial differential equations to the equations with fewer number of independent variables and thus for construction of exact solutions for different mathematical physics phenomena.

On the other hand the Burger's equation is a partial differential equation introduced by Johannes Martinus Burgers in 1948 [6] as a simplification of the Navier-Stokes equation where the velocity is a function of only one spacial dimension it represents the simplest wave equation combining both dissipative and nonlinear effects, and it appears in a wide variety of physical applications [7, 8, 9]. It can be considered to be integrable in the sense that it is linearizable under the Cole-Hopf transformation. Here we consider the Burger's equation with the form :

$$u_t = u_{xx} + uu_x. \quad (1)$$

which is an important case of the general non linear diffusion equation with convection term

$$u_t = (A(u)u_x)_x + B(u)u_x + C(u),$$

where $u = u(x, t)$ is the unknown function and $A(u), B(u), C(u)$ are arbitrary smooth functions on u . The indices t and x denotes differentiating with respect to the variables t and x respectively. Eq.(1) is obtained where $A(u) = 1, B(u) = u$ and $C(u) = 0$.

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In our previous paper [10] the Lie point symmetries of the Burger's equation and the corresponding optimal system of one dimensional subalgebras are performed. A few particular exact solutions are also given for some reduced equations. In this present paper we complete the resolution of all reduced differential equations corresponding to infinitesimal generators listed in the optimal system below. The Riccati transformation method is also applied to construct exponential, rational and periodic solutions.

The paper is organized as follows. In section 2, we recall the optimal system of the Burgers equation. In section 3, reduced equations of Burgers equation are solved then a large families of exact solutions are obtained in terms of special functions namely Airy, AiryBi functions, Laguerre, Hermite polynomials and hypergeometric function . In section 4, a Riccati approach has been applied to construct exponential, rational and periodic solutions. Finally, we conclude with some discussion.

2 The infinitesimal symmetries and the optimal system of Burgers equation

If we note by \mathcal{G} the Lie algebra of symmetries admitted by the Burgers equation Eq.(1) then a general element of this algebra is written as

$$V = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5,$$

where

$$\begin{aligned} V_1 &= tx \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} - (tu + x) \frac{\partial}{\partial u}, & V_2 &= x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u}, \\ V_3 &= t \frac{\partial}{\partial x} - \frac{\partial}{\partial u}, & V_4 &= \frac{\partial}{\partial x}, & V_5 &= \frac{\partial}{\partial t}. \end{aligned}$$

In [10] by use of commutation and adjoint tables the classification of one dimensional subalgebra of \mathcal{G} called also the optimal system is obtained to constitute a list of inequivalent generators which are:

$$\begin{aligned} V_3 + V_5 &= t \frac{\partial}{\partial x} - \frac{\partial}{\partial u} + \frac{\partial}{\partial t}; \\ -V_3 + V_5 &= -t \frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \frac{\partial}{\partial t}; \\ V_5 &= \frac{\partial}{\partial t}; \\ V_1 + V_5 &= tx \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} - (tu + x) \frac{\partial}{\partial u} + \frac{\partial}{\partial t}; \\ -V_1 + V_5 &= -tx \frac{\partial}{\partial x} - t^2 \frac{\partial}{\partial t} + (tu + x) \frac{\partial}{\partial u} + \frac{\partial}{\partial t}; \\ \beta V_1 + V_2 + V_4 &= tx \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} - (tu + x) \frac{\partial}{\partial u} + x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} + \frac{\partial}{\partial x}; \\ \beta V_1 - V_2 + V_4 &= \beta tx \frac{\partial}{\partial x} + \beta t^2 \frac{\partial}{\partial t} - \beta (tu + x) \frac{\partial}{\partial u} - x \frac{\partial}{\partial x} - 2t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} + \frac{\partial}{\partial x}; \\ \beta V_1 + V_4 &= \beta tx \frac{\partial}{\partial x} + \beta t^2 \frac{\partial}{\partial t} - \beta (tu + x) \frac{\partial}{\partial u} + \frac{\partial}{\partial x}; \\ V_3 &= t \frac{\partial}{\partial x} - \frac{\partial}{\partial u}; \\ V_1 &= tx \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} - (tu + x) \frac{\partial}{\partial u}; \\ V_2 &= x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u}; \end{aligned}$$

where β is an arbitrary constant.

3 Invariants and reduced equations and the corresponding solutions

In this section we analysis all reduced differential equations corresponding to infinitesimal generators listed in the above optimal system and we give the corresponding solutions in each cases.

3.1 Reduced differential equations corresponding to infinitesimal generator: $V = V_3 + V_5$ and the corresponding solution

In this case the invariants are:

$$\xi = 2x - t^2, \quad \text{and} \quad w = t + u,$$

and the Burger's equation (1) is reduced to the ordinary differential equation

$$4w'' + 2ww' + 1 = 0. \quad (2)$$

where the primes denotes the differentiation with respect ξ . The general solution of equation (3) is written in terms of Airy and Airy Bi functions as :

$$w(x) = \frac{2\left(\frac{-1}{2}\right)^{1/3} \left(Bi \left(1, (-1)^{1/3} 2^{2/3} \left(\frac{\xi}{4} + c_1 \right) \right) + Ai \left(1, (-1)^{1/3} 2^{2/3} \left(\frac{\xi}{4} + c_1 \right) \right) c_2 \right)}{Bi \left((-1)^{1/3} 2^{2/3} \left(\frac{\xi}{4} + c_1 \right) \right) + Ai \left((-1)^{1/3} 2^{2/3} \left(\frac{\xi}{4} + c_1 \right) \right) c_2},$$

where c_1 and c_2 are arbitrary constants. From $c_1 = c_2 = 0$, we obtain an exact solution of (1) written in terms of AiryBi functions as :

$$u(x, t) = \frac{2\left(\frac{-1}{2}\right)^{1/3} Bi \left(1, (-1)^{1/3} 2^{2/3} \left(\frac{(2x-t^2)}{4} \right) \right)}{Bi \left((-1)^{1/3} 2^{2/3} \left(\frac{(2x-t^2)}{4} \right) \right)} - t. \tag{3}$$

3.2 Reduced differential equation corresponding to infinitesimal generator: $V = -V_3 + V_5$ and the corresponding solution

The invariants are:

$$\xi = 2x + t^2, \quad \text{and} \quad w = t - u.$$

The reduced equation obtained here is:

$$4w'' - 2ww' + 1 = 0. \tag{4}$$

In this case the corresponding invariant solution of (1) is also written in terms of Airy functions in the following form :

$$u(x, t) = t - \frac{2^{2/3} \left(Bi \left(1, (2)^{2/3} \left(\frac{(2x+t^2)}{4} + c_1 \right) \right) + Ai \left(1, (2)^{2/3} \left(\frac{(2x+t^2)}{4} + c_1 \right) \right) c_2 \right)}{Bi \left((2)^{2/3} \left(\frac{(2x+t^2)}{4} + c_1 \right) \right) + Ai \left((2)^{2/3} \left(\frac{(2x+t^2)}{4} + c_1 \right) \right) c_2},$$

where c_1 and c_1 are arbitrary constants. Then for $c_1 = c_1 = 0$, the solution $u(x, t)$, becomes

$$u(x, t) = t - \frac{2^{2/3} \left(Bi \left(1, (2)^{2/3} \left(\frac{(2x+t^2)}{4} \right) \right) \right)}{Bi \left((2)^{2/3} \left(\frac{(2x+t^2)}{4} \right) \right)}. \tag{5}$$

3.3 Reduced differential equation corresponding to infinitesimal generator: $V = V_5$ and the corresponding solution

In this case the invariants are :

$$\xi = x, \quad \text{and} \quad w = u.$$

In this case the equation (1) is taken into the differential equation

$$w'' + ww' = 0. \tag{6}$$

Solving the equation (6) leads to an exact solution of Eq.(1) which is given by

$$u(x, t) = \begin{cases} \frac{2}{x+k_1} & b_3 = 0; \\ (k_2 \exp(\sqrt{b_3}x) - \frac{1}{2\sqrt{b_3}})^{-1} + \sqrt{b_3} & b_3 > 0; \\ \sqrt{-2b_3} \tan\left(-\frac{\sqrt{-2b_3}}{2}x + k_3\right) & b_3 < 0; \end{cases} \tag{7}$$

where k_1, k_2 and k_3 are arbitrary constants.

3.4 Reduced differential equation corresponding to infinitesimal generator $V = V_1 + V_5$ and the corresponding solution

Here the invariants are:

$$\xi = x(1+t^2)^{-\frac{1}{2}}, \quad \text{and} \quad w = (1+t^2)^{\frac{1}{2}}u + \xi t.$$

The reduced corresponding equation is :

$$w'' + ww' + \xi = 0. \quad (8)$$

The general solution of equation (9) is written in terms of Hermite polynomial $H(n, z)$

$$\begin{aligned} w(\xi) = & \frac{i 2^{ic_1/2} \xi c_2 H\left(-\frac{1}{2} - \frac{ic_1}{2}, \left(\frac{1}{2} + \frac{i}{2}\right) \xi\right)}{c_2 2^{ic_1/2} H\left(-\frac{1}{2} - \frac{ic_1}{2}, \left(\frac{1}{2} + \frac{i}{2}\right) \xi\right) + \exp\left(\frac{i\xi^2}{2}\right) H\left(-\frac{1}{2} + \frac{ic_1}{2}, \left(-\frac{1}{2} + \frac{i}{2}\right) \xi\right)} \\ & + \frac{(-i-1) 2^{ic_1/2} c_2 H\left(\frac{1}{2} - \frac{ic_1}{2}, \left(\frac{1}{2} + \frac{i}{2}\right) \xi\right)}{c_2 2^{ic_1/2} H\left(-\frac{1}{2} - \frac{ic_1}{2}, \left(\frac{1}{2} + \frac{i}{2}\right) \xi\right) + \exp\left(\frac{i\xi^2}{2}\right) H\left(-\frac{1}{2} + \frac{ic_1}{2}, \left(-\frac{1}{2} + \frac{i}{2}\right) \xi\right)} \\ & - \frac{i \exp\left(\frac{i\xi^2}{2}\right) (1+i) H\left(\frac{1}{2} + \frac{ic_1}{2}, \left(-\frac{1}{2} + \frac{i}{2}\right) \xi\right)}{c_2 2^{ic_1/2} H\left(-\frac{1}{2} - \frac{ic_1}{2}, \left(\frac{1}{2} + \frac{i}{2}\right) \xi\right) + \exp\left(\frac{i\xi^2}{2}\right) H\left(-\frac{1}{2} + \frac{ic_1}{2}, \left(-\frac{1}{2} + \frac{i}{2}\right) \xi\right)} \\ & - \frac{i \exp\left(\frac{i\xi^2}{2}\right) \xi H\left(-\frac{1}{2} + \frac{ic_1}{2}, \left(-\frac{1}{2} + \frac{i}{2}\right) \xi\right)}{c_2 2^{ic_1/2} H\left(-\frac{1}{2} - \frac{ic_1}{2}, \left(\frac{1}{2} + \frac{i}{2}\right) \xi\right) + \exp\left(\frac{i\xi^2}{2}\right) H\left(-\frac{1}{2} + \frac{ic_1}{2}, \left(-\frac{1}{2} + \frac{i}{2}\right) \xi\right)}, \end{aligned}$$

where c_1 and c_2 are arbitrary constants. Hence, specific values of constants $c_1 = -i$ and $c_2 = 0$ yield to an exact solution of Eq.(1) given in complex forme

$$u(x, t) = -\frac{t}{1+t^2} + i \frac{x}{1+t^2}. \quad (9)$$

3.5 Reduced differential equation corresponding to infinitesimal generator : $V = -V_1 + V_5$ and the corresponding solution

The invariants are :

$$\xi = x(t^2 - 1)^{-\frac{1}{2}}, \quad \text{and} \quad w = (t^2 - 1)^{\frac{1}{2}}u + \xi t.$$

One finds that the reduced ordinary differential equation is equivalent to the equation:

$$w' = -\frac{1}{2}w^2 + \frac{1}{2}\xi^2 + b_5; \quad (10)$$

where b_5 is an arbitrary constant. Equation Eq.(10) can be integrated also in terms of Hermite polynomials

$$\begin{aligned} w(\xi) = & \frac{2^{(1+b_5)/2} \xi c_1 H\left(\frac{-1}{2} - \frac{b_5}{2}, \frac{\xi}{\sqrt{2}}\right)}{\sqrt{2} \left(2^{b_5/2} c_1 H\left(\frac{-1}{2} - \frac{b_5}{2}, \frac{\xi}{\sqrt{2}}\right) + \exp\left(\frac{\xi^2}{2}\right) H\left(\frac{-1}{2} + \frac{b_5}{2}, \frac{i\xi}{\sqrt{2}}\right) \right)} \\ & - \frac{2^{1+\frac{b_5}{2}} c_1 H\left(\frac{1}{2} - \frac{b_5}{2}, \frac{\xi}{\sqrt{2}}\right)}{\sqrt{2} \left(2^{b_5/2} c_1 H\left(\frac{-1}{2} - \frac{b_5}{2}, \frac{\xi}{\sqrt{2}}\right) + \exp\left(\frac{\xi^2}{2}\right) H\left(\frac{-1}{2} + \frac{b_5}{2}, \frac{i\xi}{\sqrt{2}}\right) \right)} \\ & - \frac{\exp\left(\frac{\xi^2}{2}\right) \sqrt{2} \xi H\left(-\frac{1}{2} + \frac{b_5}{2}, \frac{i\xi}{\sqrt{2}}\right)}{\sqrt{2} \left(2^{b_5/2} c_1 H\left(\frac{-1}{2} - \frac{b_5}{2}, \frac{\xi}{\sqrt{2}}\right) + \exp\left(\frac{\xi^2}{2}\right) H\left(\frac{-1}{2} + \frac{b_5}{2}, \frac{i\xi}{\sqrt{2}}\right) \right)} \\ & - \frac{\exp\left(\frac{\xi^2}{2}\right) 2i H\left(\frac{1}{2} + \frac{b_5}{2}, \frac{i\xi}{\sqrt{2}}\right)}{\sqrt{2} \left(2^{b_5/2} c_1 H\left(\frac{-1}{2} - \frac{b_5}{2}, \frac{\xi}{\sqrt{2}}\right) + \exp\left(\frac{\xi^2}{2}\right) H\left(\frac{-1}{2} + \frac{b_5}{2}, \frac{i\xi}{\sqrt{2}}\right) \right)}, \end{aligned}$$

where c_1 is an arbitrary constant. In this case if we put $b_5 = 1$ and $c_1 = 0$ we get an exact solution of equation (1) given by

$$u(x, t) = -\frac{x}{1+t}. \quad (11)$$

3.6 Reduced differential equation corresponding to infinitesimal generator : $V = \beta V_1 + V_2 + V_4$ and the corresponding solution

The invariants are:

$$\xi = \frac{x + \beta t + 1}{\sqrt{\beta t^2 + 2t}}, \quad \text{and} \quad w = \sqrt{\beta t^2 + 2t}u + \frac{\beta t(x - 1)}{\sqrt{\beta t^2 + 2t}}.$$

Then the reduced differential equation is equivalent to the Riccati type equation

$$w' = -\frac{1}{2}w^2 - \xi w + b_6; \tag{12}$$

where b_6 is an arbitrary constant. The general solution of equation Eq.(12) is given by

$$\begin{aligned} w(\xi) = & \frac{-\sqrt{2}(2 + b_6)c_1 H\left(-2 - \frac{b_6}{2}, \frac{\xi}{\sqrt{2}}\right) - 2\xi c_1 H\left(-1 - \frac{b_6}{2}, \frac{\xi}{\sqrt{2}}\right)}{c_1 H\left(-1 - \frac{b_6}{2}, \frac{\xi}{\sqrt{2}}\right) + \exp\left(\frac{\xi^2}{2}\right) F1\left(\frac{-d}{4}, \frac{1}{2}, \frac{-\xi^2}{2}\right)} \\ & + \frac{-2\xi \exp\left(\frac{\xi^2}{2}\right) F1\left(\frac{-b_6}{4}, \frac{1}{2}, \frac{-\xi^2}{2}\right)}{c_1 H\left(-1 - \frac{b_6}{2}, \frac{\xi}{\sqrt{2}}\right) + \exp\left(\frac{\xi^2}{2}\right) F1\left(\frac{-b_6}{4}, \frac{1}{2}, \frac{-\xi^2}{2}\right)} \\ & + \frac{\xi \exp\left(\frac{\xi^2}{2}\right) (2 + b_6) F1\left(\frac{-b_6}{4}, \frac{3}{2}, \frac{-\xi^2}{2}\right)}{c_1 H\left(-1 - \frac{b_6}{2}, \frac{\xi}{\sqrt{2}}\right) + \exp\left(\frac{\xi^2}{2}\right) F1\left(\frac{-b_6}{4}, \frac{1}{2}, \frac{-\xi^2}{2}\right)}, \end{aligned}$$

where H is the Hermite polynomials and $F1$ is the confluent hypergeometric function. For $b_6 = -2$ and $c_1 = 0$ we obtain an exact solution of (1) that is

$$u(x, t) = \frac{\beta(1 - x)}{\beta t + 2} - \frac{2(x + \beta t + 1)}{\beta t^2 + 2t}. \tag{13}$$

3.7 Reduced differential equation corresponding to infinitesimal generator : $V = \beta V_1 - V_2 + V_4$ and the corresponding solution

The invariants are :

$$\xi = \frac{x + \beta t - 1}{\sqrt{\beta t^2 - 2t}}, \quad \text{and} \quad w = \sqrt{\beta t^2 - 2t}u + \frac{\beta t(x + 1)}{\sqrt{\beta t^2 - 2t}}.$$

We obtain that the reduced differential equation is also equivalent to the Riccati equation

$$w' = -\frac{1}{2}w^2 + \xi w + b_7; \tag{14}$$

where b_7 is an arbitrary constant. Equation (14) is integrated by

$$w(\xi) = \frac{-\sqrt{2}b_7c_1 H\left(-1 - \frac{b_7}{2}, \frac{\xi}{\sqrt{2}}\right) + b_7\xi F1\left(1 + \frac{b_7}{4}, \frac{3}{2}, \frac{\xi^2}{2}\right)}{c_1 H\left(-\frac{b_7}{2}, \frac{\xi}{\sqrt{2}}\right) + F1\left(\frac{b_7}{4}, \frac{1}{2}, \frac{\xi^2}{2}\right)},$$

where H is the Hermite polynomial and $F1$ is the confluent hypergeometric function. for $b_7 = -4$ and $c_1 = 0$, an explicit solution of (1) is given by:

$$u(x, t) = \frac{4(x + \beta t - 1)}{(x + \beta t - 1)^2 - (\beta t^2 - 2t)} - \frac{\beta t(x + 1)}{\beta t^2 - 2t}.$$

3.8 Reduced differential equation corresponding to infinitesimal generator: $V = \beta V_1 + V_4$ and the corresponding solution

The invariants are:

$$\xi = \frac{\sqrt{2\beta t x + 1}}{t}, \quad \text{and} \quad w = \frac{\xi^2}{2\beta}t + \frac{1}{2\beta t} + tu.$$

The equation (1) is reduced to the ordinary differential equation:

$$\beta^3 \xi w'' - \beta^3 w' + \beta^2 \xi^2 w w' - \xi^3 = 0. \quad (15)$$

The solution of equation (20) is of the forme

$$w(\xi) = \frac{2^{2/3} \left(\frac{1}{\beta^4}\right)^{1/3} \beta \left(Bi \left(1, \frac{\xi^2 + 2\beta^2 c_1}{(2\beta)^{4/3}} \right) + Ai \left(1, \frac{\xi^2 + 2\beta^2 c_1}{(2\beta)^{4/3}} \right) c_2 \right)}{Bi \left(\frac{\xi^2 + 2\beta^2 c_1}{(2\beta)^{4/3}} \right) + Ai \left(\frac{\xi^2 + 2\beta^2 c_1}{(2\beta)^{4/3}} \right) c_2},$$

where c_1 and c_2 are arbitrary constants. Then it gives rise of an exact solution of equation (1)

$$u(x, t) = t^{-1} w(\xi) - \frac{\xi^2}{2\beta} - \frac{1}{2\beta t^2}, \quad \text{with } \xi = \frac{\sqrt{2\beta t x + 1}}{t}. \quad (16)$$

3.9 Reduced differential equation corresponding to infinitesimal generator: $V = V_3$ and the corresponding solution

The invariants are:

$$\xi = t, \quad \text{and} \quad w = u + \frac{x}{\xi}.$$

So the reduced equation is:

$$\xi w' + w = 0. \quad (17)$$

The general solution of equation (17) is solved by :

$$w = \frac{\alpha}{\xi},$$

with α an arbitrary constant, then the equation (1) has as solution

$$u(x, t) = \frac{\alpha - x}{t}. \quad (18)$$

3.10 Reduced differential equation corresponding to infinitesimal generator: $V = V_1$ and the corresponding solution

The invariants are:

$$\xi = \frac{x}{t}, \quad \text{and} \quad w = tu + t\xi.$$

We find the reduced equation to be equivalent to the Riccati type equation

$$w' = -\frac{1}{2}w^2 + b_8; \quad (19)$$

where b_8 is an arbitrary constant.

In this case the general solution of (1) is

$$u(x, t) = \begin{cases} \frac{2}{x+c_1 t} - \frac{x}{t} & b_8 = 0; \\ \left\{ (c_2 t \exp(\sqrt{b_8} \frac{x}{t}) - \frac{t}{2\sqrt{b_8}}) \right\}^{-1} + \frac{\sqrt{b_8-x}}{t} & b_8 > 0; \\ \frac{\sqrt{-2b_8} \tan(-\frac{\sqrt{-2b_8}}{2t} x + c_3) - x}{t} & b_8 < 0; \end{cases} \quad (20)$$

where c_1, c_2 and c_3 are arbitrary constants.

3.11 Reduced differential equation corresponding to infinitesimal generator: $V = V_2$ and the corresponding solution

In this case the invariants are :

$$\xi = \frac{x^2}{t}, \quad \text{and} \quad w = xu.$$

Hence, the corresponding reduced equation is

$$-4\xi^2 w'' + (2\xi - \xi^2)w' - 2\xi w w' - 2w + w^2 = 0. \tag{21}$$

The integration of the last reduced equation (21) leads to the general solution:

$$w(\xi) = \frac{4\xi(C_2(\frac{U[1-C_1, \frac{3}{2}, \frac{\xi}{4}]}{2\sqrt{\xi}} - \frac{1}{4}\sqrt{\xi}U[1-C_1, \frac{3}{2}, \frac{\xi}{4}])}{C_2\sqrt{\xi}U[1-C_1, \frac{3}{2}, \frac{\xi}{4}] + \sqrt{\xi}L[C_1-1, \frac{1}{2}, \frac{\xi}{4}]} - \frac{\frac{1}{4}\sqrt{\xi}(1-C_1)U[2-C_1, \frac{5}{2}, \frac{\xi}{4}]}{C_2\sqrt{x}U[1-C_1, \frac{3}{2}, \frac{\xi}{4}] + \sqrt{\xi}L[C_1-1, \frac{1}{2}, \frac{\xi}{4}]} - \frac{\frac{1}{4}\sqrt{\xi}L[C_1-2, \frac{3}{2}, \frac{\xi}{4}]}{C_2\sqrt{x}U[1-C_1, \frac{3}{2}, \frac{\xi}{4}] + \sqrt{\xi}L[C_1-1, \frac{1}{2}, \frac{\xi}{4}]} + \frac{\frac{L[C_1-1, \frac{1}{2}, \frac{\xi}{4}]}{2\sqrt{\xi}} - \frac{1}{4}\sqrt{\xi}L[C_1-1, \frac{1}{2}, \frac{\xi}{4}])}{C_2\sqrt{\xi}U[1-C_1, \frac{3}{2}, \frac{\xi}{4}] + \sqrt{\xi}L[C_1-1, \frac{1}{2}, \frac{\xi}{4}]},$$

written in terms of hypergeometric function U and Laguerre polynomial L . Hence for $C_2 = 0$ and $C_1 = 2$, we obtain $w(\xi) = 2 - \xi - \frac{4\xi}{6-\xi}$. Consequently, an exact solution of equation (1) is given by:

$$u(x, t) = -\frac{4t}{6t - x^2} + \frac{2t}{x^2} - 1. \tag{22}$$

4 Exponential, rational and periodic solutions of Burgers equation

In this section we adopt the Riccati transformation method to Burgers equation Eq.(1) then we obtain new solutions. First recall that the Riccati method consist to search the solution of the Burgers equation with the form:

$$u(x, t) = U(\xi), \quad \xi = x - \lambda t. \tag{23}$$

If we substitute the above expression into equation (1) and after integration it will be reduced into the Riccati equation type:

$$U'(\xi) = -\frac{1}{2}U^2(\xi) - \lambda U(\xi) + b, \tag{24}$$

where b is an arbitrary constant. It is known that Eq.(24) posses the following solutions:

- Case 1, $\Delta = \lambda^2 + 2b > 0$, the solutions are;

$$U(\xi) = -\lambda - \sqrt{\Delta} \frac{1 + e^{\sqrt{\Delta}\xi}}{1 - e^{\sqrt{\Delta}\xi}},$$

- Case 2, $\Delta = \lambda^2 + 2b = 0$,

$$U(\xi) = -\lambda + \frac{2}{\xi},$$

- Case 3, $\Delta = \lambda^2 + 2b < 0$,

$$U(\xi) = -\lambda - \sqrt{-\Delta} \tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right), \quad \text{if} \quad \left|\frac{\sqrt{-\Delta}}{2}\xi\right| < \frac{\pi}{2},$$

or

$$U(\xi) = -\lambda + \sqrt{-\Delta} \cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right), \quad \text{if} \quad 0 < \frac{\sqrt{-\Delta}}{2}\xi < \pi.$$

Then from the above solutions and Eq.(23), we get a families of exact solutions of Burgers equation which are :

1. Exponential solutions

$$u(x, t) = -\lambda - \sqrt{\Delta} \frac{1 + e^{\sqrt{\Delta}(x-\lambda t)}}{1 - e^{\sqrt{\Delta}(x-\lambda t)}},$$

2. Rational solutions

$$u(x, t) = -\lambda + \frac{2}{(x - \lambda t)},$$

3. Periodic solutions

$$u(x, t) = -\lambda - \sqrt{-\Delta} \tan\left(\frac{\sqrt{-\Delta}}{2}(x - \lambda t)\right), \quad \text{if} \quad \left|\frac{\sqrt{-\Delta}}{2}(x - \lambda t)\right| < \frac{\pi}{2},$$

or

$$u(x, t) = -\lambda + \sqrt{-\Delta} \cot\left(\frac{\sqrt{-\Delta}}{2}(x - \lambda t)\right), \quad \text{if} \quad 0 < \frac{\sqrt{-\Delta}}{2}(x - \lambda t) < \pi.$$

Example for $\lambda = \pm 1$ and $b = \frac{-1}{2}$, then $\Delta = 0$ and the two rational solutions are :

$$u(x, t) = 1 + \frac{2}{(x + t)}, \quad \text{and} \quad u(x, t) = -1 + \frac{2}{(x - t)}$$

5 Conclusion

The Lie symmetry analysis and Riccati methods are a powerful methods to search exact solutions for nonlinear partial differential equations. In this work, first, by adopting Lie symmetry approach we have used the classification of one-dimensional subalgebras of Lie point symmetries admitted by Burgers Equation and the corresponding reduced differential equations to construct a large class of new exact solutions in terms of specials functions. Second, by using the Riccati method, we obtain some new solutions of the Burgers equation namely, exponential, rational and periodic solutions.

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