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Quantum Hypothesis Testing and Non-Equilibrium Statistical Mechanics

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Abstract. We extend the mathematical theory of quantum hypothesis testing to the general W^*-algebraic setting and explore its relation with recent developments in non-equilibrium quantum statistical mechanics. In particular, we relate the large deviation principle for the full counting statistics of entropy flow to quantum hypothesis testing of the arrow of time.

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1 Introduction

Starting with the works [JP1, Ru1, Pi], the mathematical theory of non-equilibrium quantum statistical mechanics has developed rapidly in recent years [Ab, AF, AP, AJPP1, AJPP2, FMS, FMU, JKP, JOP1, JOP2, JOP3, JOP4, JP2, JP3, JP4, MMS1, MMS2, MO, Na, Og1, Og2, Ro, Ru2, Tas, TM1, TM2]. The current research efforts are centered around the theory of entropic fluctuations (see [JOPP1, JOPP2]) and it is these developments that will concern us here.

Since Shannon's rediscovery of Gibbs-Boltzmann entropy there has been a close interplay between information theory and statistical mechanics. One of the deepest links is provided by the theory of large deviations [DZ, El]. We refer the reader to [Me] for a beautiful and easily accessible account of this interplay. In this context, it was natural to try to interpret recent results in non-equilibrium statistical mechanics in terms of quantum information theory.

The link can be roughly summarized as follows[1]. Consider the large deviation principle for the full counting statistics for the repeated quantum measurement of the energy/entropy flow over the time interval $[0, t]$ [LL, Ro, JOPP1]. Let $I(\theta)$ be the rate function and $e(s)$ its Legendre transform. Let $\hat{e}(s)$ be the Chernoff error exponent in the quantum hypothesis testing of the arrow of time, *i.e.*, of the family of states $\{(\omega_{t/2}, \omega_{-t/2})\}_{t>0}$, where $\omega_{\pm t/2}$ is the state of our quantum system at the time $\pm t/2$. Then $e(s) = \hat{e}(s)$. In this paper we prove this result and elaborate on the relation between quantum hypothesis testing and non-equilibrium statistical mechanics.

Hypothesis testing has a long tradition in theoretical and applied statistics [Pe, Ch, LR]. During the last decade many results of classical hypothesis testing have been extended to the quantum domain [ACM, ANS, BDK1, BDK2, BDK3, Ha1, Ha2, HMO1, HMO2, HMO3, Ka, Mo, MHO, NS1, NS2, OH, ON]. The culmination of these efforts was the proof of a long standing conjecture regarding the quantum Chernoff bound [ACM, ANS]. The following trace inequality of [ACM, ANS] played a key role in the resolution of this conjecture.

Proposition 1.1 *Let $A > 0$ and $B > 0$ be matrices on $\mathbb{C}^n$. Then*

$$\frac{1}{2}\left(\operatorname{Tr} A + \operatorname{Tr} B - \operatorname{Tr}|A - B|\right) \leq \operatorname{Tr} A^{1-s} B^s \tag{1.1}$$

holds for any $s \in [0, 1]$.

Proof. (Communicated by N. Ozawa, unpublished). For X self-adjoint, $X_\pm$ denotes its positive/negative part. Decomposing $A - B = (A - B)_+ - (A - B)_-$ one gets

$$\frac{1}{2}\left(\operatorname{Tr} A + \operatorname{Tr} B - \operatorname{Tr}|A - B|\right) = \operatorname{Tr} A - \operatorname{Tr}(A - B)_+,$$

and (1.1) is equivalent to

$$\operatorname{Tr} A - \operatorname{Tr} B^s A^{1-s} \leq \operatorname{Tr}(A - B)_+. \tag{1.2}$$

Note that $B + (A - B)_+ \geq B$ and $B + (A - B)_+ = A + (A - B)_- \geq A$. Since, for $s \in [0, 1]$, the function $x \mapsto x^s$ is operator monotone (*i.e.*, $X \leq Y \Rightarrow X^s \leq Y^s$ for any positive matrices X, Y), we can write

$$\begin{aligned}
\operatorname{Tr} A - \operatorname{Tr} B^s A^{1-s} = \operatorname{Tr}(A^s - B^s)A^{1-s} &\leq \operatorname{Tr}\left((B + (A - B)_+)^s - B^s\right)A^{1-s} \\
&\leq \operatorname{Tr}\left((B + (A - B)_+)^s - B^s\right)(B + (A - B)_+)^{1-s} \\
&= \operatorname{Tr} B + \operatorname{Tr}(A - B)_+ - \operatorname{Tr} B^s (B + (A - B)_+)^{1-s} \\
&\leq \operatorname{Tr}(A - B)_+.
\end{aligned}$$

□

[1] Needless to say, all the notions discussed in this paragraph will be defined later in the paper.

We have singled out this result for the following reason. W^*-algebras and modular theory provide a natural general mathematical framework for quantum hypothesis testing. For example, Inequality (1.2) can be formulated in W^*-algebraic language as

$$\frac{1}{2}(\omega(1)+\nu(1)-\|\omega-\nu\|) \leq (\Omega_\omega|\Delta^s_{\nu|\omega}\Omega_\omega) \tag{1.3}$$

for $0 \leq s \leq 1$, where ω and ν are faithful, normal, positive linear functionals on a W^* algebra $\mathfrak{M}$ in standard form, Ω_ω is the vector representative of ω in the natural cone and $\Delta_{\nu|\omega}$ is the relative modular operator. The extension of quantum hypothesis testing to W^*-algebras was hindered by the fact that the original proof [ACM, ANS] of the inequality (1.2) could not be extended/generalized to a proof of (1.3). Ozawa's proof, however, can and the inequality (1.3) was recently proven in [Og3]. An alternative proof, which links (1.3) to Araki-Masuda theory of non-commutative L^p-spaces, is given in Section 6.1.

In Section 6 we extend the mathematical theory of quantum hypothesis testing to the W^*-algebraic setting and prove the Chernoff bound, the Hoeffding bound, and Stein's Lemma. We develop a model independent axiomatic approach to quantum hypothesis testing which clarifies its mathematical structure and reduces the study of concrete models to the verification of the proposed axioms. We emphasize that apart from Inequality (1.3), whose proof is technically involved and has no classical counterpart, the proofs follow essentially line by line the classical arguments. The verification of the large deviation axioms that underline the proofs leads to a novel class of analytic problems in quantum statistical mechanics.

To make the paper and our main points accessible to a reader without prior knowledge of modular theory, we describe in Sections 3 and 4 quantum hypothesis testing, non-equilibrium quantum statistical mechanics, and their relation in the context of finite quantum systems. Typical examples the reader should keep in mind are a quantum spin system or a Fermi gas confined to a finite part Λ of some infinite lattice L. Needless to say, the thermodynamic limit $\Lambda \to L$ has to be taken before the large time limit $t \to \infty$. The reader not familiar with (or not interested in) the algebraic theory may directly proceed to Section 8 after reading Sections 3 and 4.

For reasons of space we have not attempted to prove quantum hypothesis testing results under the most general conditions possible. In particular, we shall only discuss hypothesis testing of faithful states (this restriction is inconsequential as far as statistical mechanics is concerned). The extensions to non-faithful states follow typically by straightforward limiting arguments (see [Og3] for an example).

This work is of a review nature. Our goal is to point to a surprising link between two directions of research which were largely unaware of each other, in a hope that they both may benefit from this connection. We shall discuss here only the single parameter full counting statistics for entropy flow and its relation to binary hypothesis testing. The multi-parameter full counting statistics describing the energy flow between different parts of the system is a well understood object but its relation with quantum hypothesis testing is unclear and would presumably involve multiple quantum state discrimination which is poorly understood at the moment (the proposal of [NS1] appears unsuitable in this context).

We also point out that for obvious space reasons this paper is not a review of either quantum statistical mechanics or quantum hypothesis testing, but of the link between the two of them. The reader who wishes to learn more about these topics individually may consult [AJPP1, JOPP1, ANS].

The paper is organized as follows. In Section 2 we review the results of Large Deviation Theory that we will need. Since these results are not stated/proven in this form in the classical references [DZ, El], we provide proofs for the reader's convenience. In Section 3 we review the existing results in quantum hypothesis testing of finite quantum systems. The non-equilibrium statistical mechanics of finite quantum systems is described in Section 4. Its relation with quantum hypothesis testing is discussed in Section 4.5. In Section 5 we review the results of modular theory that we need and give a new proof of the key preliminary inequality needed to prove (1.3) (Proposition 5.5). Sections 6 and 7 are devoted to quantum hypothesis testing and non-equilibrium statistical mechanics of infinitely extended quantum systems described by W^*-algebras and W^*-dynamical systems. Finally, in Section 8

we describe several physical models for which the existence of the large deviation functionals have been proven and to which the results described in this paper apply.

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2 Prologue

2.1 Fenchel-Legendre transform

Let $[a,b]$ be a finite closed interval in $\mathbb{R}$ and $e:[a,b]\to\mathbb{R}$ a convex continuous function. Convexity implies that for every $s\in]a,b[$ the limits

$$D^{\pm}e(s)=\lim_{h\downarrow 0}\frac{e(s\pm h)-e(s)}{\pm h}$$

exist and are finite. Moreover, $D^-e(s)\le D^+e(s)$, $D^+e(s)\le D^-e(t)$ for $s<t$, and $D^-e(s)=D^+e(s)$ outside a countable set in $]a,b[$. $D^+e(a)$ and $D^-e(b)$ also exist, although they may not be finite. If $e'(s)$ exists for all $s\in]a,b[$, then the mean value theorem implies that $e'(s)$ is continuous on $]a,b[$ and that

$$\lim_{s\downarrow a}e'(s)=D^+e(a),\qquad \lim_{s\uparrow b}e'(s)=D^-e(b).$$

We set $e(s)=\infty$ for $s\notin[a,b]$. Then $e(s)$ is a lower semi-continuous convex function on $\mathbb{R}$. The subdifferential of $e(s)$, $\partial e(s)$, is defined by

$$\partial e(s)=\begin{cases}]-\infty,D^+e(a)] & \text{if } s=a;\\ [D^-e(s),D^+e(s)] & \text{if } s\in]a,b[;\\ [D^-e(b),\infty[& \text{if } s=b;\\ \emptyset & \text{if } s\notin[a,b].\end{cases}$$

The function

$$\varphi(\theta)=\sup_{s\in[a,b]}(\theta s-e(s))=\sup_{s\in\mathbb{R}}(\theta s-e(s)) \tag{2.1}$$

is called the Fenchel-Legendre transform of $e(s)$. $\varphi(\theta)$ is finite and convex (hence continuous) on $\mathbb{R}$. Obviously, $a\ge 0\Rightarrow\varphi(\theta)$ is increasing, and $b\le 0\Rightarrow\varphi(\theta)$ is decreasing. The subdifferential of $\varphi(\theta)$ is $\partial\varphi(\theta)=[D^-\varphi(\theta),D^+\varphi(\theta)]$. The basic properties of the pair (e,φ) are summarized in:

Theorem 2.1 (1) $s\theta\le e(s)+\varphi(\theta)$.

(2) $s\theta=e(s)+\varphi(\theta)\Leftrightarrow\theta\in\partial e(s)$.

(3) $\theta\in\partial e(s)\Leftrightarrow s\in\partial\varphi(\theta)$.

(4) $e(s)=\sup_{\theta\in\mathbb{R}}(s\theta-\varphi(\theta))$.

(5) *If* $0 \in]a,b[$, *then* $\varphi(\theta)$ *is decreasing on* $]-\infty, D^-e(0)]$, *increasing on* $[D^+e(0),\infty[$, $\varphi(\theta) = -e(0)$ *for* $\theta \in \partial e(0)$, *and* $\varphi(\theta) > -e(0)$ *for* $\theta \notin \partial e(0)$.

(6)
$$\varphi(\theta) = \begin{cases} a\theta - e(a) & \text{if } \theta \le D^+e(a); \\ b\theta - e(b) & \text{if } \theta \ge D^-e(b). \end{cases}$$

The proofs of these results are simple and can be found in [El, JOPP1].

The function
$$\hat{\varphi}(\theta) = \sup_{s\in[a,b]} (\theta(s-b) - e(s)) = \varphi(\theta) - b\theta \tag{2.2}$$
will also play an important role in the sequel. Its properties are easily deduced from the properties of $\varphi(\theta)$. In particular, $\hat{\varphi}$ is convex, continuous and decreasing. It is a one-to-one map from $]-\infty, D^-e(b)]$ to $[-e(b),\infty[$. We denote by $\hat{\varphi}^{-1}$ the inverse map.

Suppose now that $a = 0$ and $b = 1$. For $r \in \mathbb{R}$ let
$$\psi(r) = \begin{cases} -\varphi(\hat{\varphi}^{-1}(r)) & \text{if } r \ge -e(1); \\ -\infty & \text{otherwise.} \end{cases} \tag{2.3}$$

Proposition 2.2 (1)
$$\psi(r) = -\sup_{s\in[0,1[} \frac{-sr - e(s)}{1-s}. \tag{2.4}$$

(2) $\psi(r)$ *is concave, increasing, finite for* $r > -e(1)$, *and* $\psi(-e(1)) = e(1) - D^-e(1)$.

(3) $\psi(r)$ *is continuous on* $]-e(1),\infty[$ *and upper-semicontinuous on* $\mathbb{R}$.

The proof of this proposition is elementary and we will omit it.

2.2 Large deviation bounds

In this paper we shall make use of some results of Large Deviation Theory, and in particular of a suitable variant of the Gärtner-Ellis theorem [DZ, El]. In this subsection we describe and prove the results we will need.

Let $\mathcal{I} \subset \mathbb{R}_+$ be an unbounded index set, $(M_t, \mathcal{F}_t, P_t)_{t\in\mathcal{I}}$ a family of measure spaces, and $X_t : M_t \to \mathbb{R}$ a family of measurable functions. We assume that the measures P_t are finite for all t. The functions
$$\mathbb{R} \ni s \mapsto e_t(s) = \log \int_{M_t} \mathrm{e}^{sX_t} \mathrm{d}P_t,$$
are convex (by Hölder's inequality) and take values in $]-\infty,\infty]$. We need

Assumption (LD). For some weight function $\mathcal{I} \ni t \mapsto w_t > 0$ such that $\lim_{t\to\infty} w_t = \infty$ the limit
$$e(s) = \lim_{t\to\infty} \frac{1}{w_t} e_t(s),$$
exists and is finite for $s \in [a,b]$. Moreover, the function $[a,b] \ni s \mapsto e(s)$ is continuous.

Until the end of this section we shall assume that (LD) holds and set $e(s) = \infty$ for $s \notin [a, b]$. The Legendre-Fenchel transform $\varphi(\theta)$ of $e(s)$ is defined by (2.1).

Proposition 2.3 (1) *Suppose that* $0 \in [a, b[$. *Then*

$$\limsup_{t\to\infty} \frac{1}{w_t} \log P_t(\{x \in M_t \,|\, X_t(x) > \theta w_t\}) \le \begin{cases} -\varphi(\theta) & \text{if } \theta \ge D^+e(0); \\ e(0) & \text{if } \theta < D^+e(0). \end{cases} \tag{2.5}$$

(2) *Suppose that* $0 \in]a, b]$. *Then*

$$\limsup_{t\to\infty} \frac{1}{w_t} \log P_t(\{x \in M_t \,|\, X_t(x) < \theta w_t\}) \le \begin{cases} -\varphi(\theta) & \text{if } \theta \le D^-e(0); \\ e(0) & \text{if } \theta > D^-e(0). \end{cases} \tag{2.6}$$

Proof. We shall prove (1); (2) follows from (1) applied to $-X_t$. For any $s \in]0, b]$, the Chebyshev inequality yields

$$P_t(\{x \in M_t \,|\, X_t(x) > \theta w_t\}) = P_t(\{x \in M_t \,|\, \mathrm{e}^{sX_t(x)} > \mathrm{e}^{s\theta w_t}\}) \le \mathrm{e}^{-s\theta w_t} \int_{M_t} \mathrm{e}^{sX_t} \mathrm{d}P_t,$$

and hence

$$\limsup_{t\to\infty} \frac{1}{w_t} \log P_t(\{x \in M_t \,|\, X_t(x) > \theta w_t\}) \le -\sup_{s\in[0,b]} (s\theta - e(s)).$$

Since

$$\sup_{s\in[0,b]} (s\theta - e(s)) = \begin{cases} \varphi(\theta) & \text{if } \theta \ge D^+e(0); \\ -e(0) & \text{if } \theta < D^+e(0), \end{cases}$$

the statement follows. □

Proposition 2.4 *Suppose that* $0 \in]a, b[$, $e(0) \le 0$, *and that* $e(s)$ *is differentiable at* $s = 0$. *Then for any* $\delta > 0$ *there is* $\gamma > 0$ *such that for* t *large enough*

$$P_t(\{x \in M_t \,|\, |w_t^{-1} X_t(x) - e'(0)| \ge \delta\}) \le \mathrm{e}^{-\gamma w_t}.$$

Proof. Since $\varphi(e'(0)) = -e(0)$, Theorem 2.1 (5) implies that $\varphi(\theta) > \varphi(e'(0)) \ge 0$ for $\theta \ne e'(0)$. Proposition 2.3 implies

$$\limsup_{t\to\infty} \frac{1}{w_t} \log P_t(\{x \in M_t \,|\, |w_t^{-1} X_t(x) - e'(0)| \ge \delta\}) \le -\min\{\varphi(e'(0) + \delta), \varphi(e'(0) - \delta)\},$$

and the statement follows. □

Proposition 2.5 *Suppose that* $e(s)$ *is differentiable on* $]a, b[$. *Then for* $\theta \in]D^+e(a), D^-e(b)[$,

$$\liminf_{t\to\infty} \frac{1}{w_t} \log P_t(\{x \in M_t \,|\, X_t(x) > \theta w_t)\} \ge -\varphi(\theta). \tag{2.7}$$

Proof. Let $\theta \in]D^+e(a), D^-e(b)[$ be given and let $\alpha > \theta$ and $\epsilon > 0$ be such that $\theta < \alpha - \epsilon < \alpha < \alpha + \epsilon < D^-e(b)$. Let $s_\alpha \in]a, b[$ be such that $e'(s_\alpha) = \alpha$ (so $\varphi(\alpha) = \alpha s_\alpha - e(s_\alpha)$). Let

$$\mathrm{d}\hat{P}_t = \mathrm{e}^{-e_t(s_\alpha)} \mathrm{e}^{s_\alpha X_t} \mathrm{d}P_t.$$

Then $\hat{P}_t$ is a probability measure on $(M_t, \mathcal{F}_t)$ and

$$\begin{aligned} P_t(\{x \in M_t \,|\, X_t(x) > \theta w_t\}) &\geq P_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) \in [\alpha - \epsilon, \alpha + \epsilon]\}) \\ &= \mathrm{e}^{e_t(s_\alpha)} \int_{\{w_t^{-1} X_t \in [\alpha-\epsilon, \alpha+\epsilon]\}} \mathrm{e}^{-s_\alpha X_t} \mathrm{d}\hat{P}_t \\ &\geq \mathrm{e}^{e_t(s_\alpha) - s_\alpha \alpha w_t - |s_\alpha| w_t \epsilon} \hat{P}_t(\{x \in M_t \,|\, w_t^{-1} X_t \in [\alpha - \epsilon, \alpha + \epsilon]\}). \end{aligned} \tag{2.8}$$

Now, if $\hat{e}_t(s) = \log \int_{M_t} \mathrm{e}^{sX_t} \mathrm{d}\hat{P}_t$, then $\hat{e}_t(s) = e_t(s + s_\alpha) - e_t(s_\alpha)$ and so for $s \in [a - s_\alpha, b - s_\alpha]$,

$$\lim_{t \to \infty} \frac{1}{w_t} \hat{e}_t(s) = e(s + s_\alpha) - e(s_\alpha).$$

Since $\hat{e}'(0) = e'(s_\alpha) = \alpha$, it follows from Proposition 2.4 that

$$\lim_{t \to \infty} \frac{1}{w_t} \log \hat{P}_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) \in [\alpha - \epsilon, \alpha + \epsilon]\}) = 0,$$

and (2.8) yields

$$\liminf_{t \to \infty} \frac{1}{w_t} \log P_t(\{x \in M_t \,|\, X_t(x) > \theta w_t\}) \geq -s_\alpha \alpha + e(s_\alpha) - |s_\alpha| \epsilon = -\varphi(\alpha) - |s_\alpha| \epsilon.$$

The statement follows by taking first $\epsilon \downarrow 0$ and then $\alpha \downarrow \theta$. □

The following local version of the Gärtner-Ellis theorem is a consequence of Propositions 2.3 and 2.5.

Theorem 2.6 *If $e(s)$ is differentiable on $]a, b[$ and $0 \in]a, b[$ then, for any open set $\mathbb{J} \subset]D^+e(a), D^-e(b)[$,*

$$\lim_{t \to \infty} \frac{1}{w_t} \log P_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) \in \mathbb{J}\}) = -\inf_{\theta \in \mathbb{J}} \varphi(\theta).$$

Proof. *Lower bound.* For any $\theta \in \mathbb{J}$ and $\delta > 0$ such that $]\theta - \delta, \theta + \delta[\subset \mathbb{J}$ one has

$$P_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) \in \mathbb{J}\}) \geq P_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) \in]\theta - \delta, \theta + \delta[\}),$$

and it follows from Proposition 2.5 that

$$\liminf_{t \to \infty} \frac{1}{w_t} \log P_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) \in \mathbb{J}\}) \geq -\varphi(\theta - \delta).$$

Letting $\delta \downarrow 0$ and optimizing over $\theta \in \mathbb{J}$, we obtain

$$\liminf_{t \to \infty} \frac{1}{w_t} \log P_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) \in \mathbb{J}\}) \geq -\inf_{\theta \in \mathbb{J}} \varphi(\theta). \tag{2.9}$$

Upper bound. By Part (5) of Theorem 2.1, we have $\varphi(\theta) = -e(0)$ for $\theta = e'(0)$ and $\varphi(\theta) > -e(0)$ otherwise. Hence, if $e'(0) \in \overline{\mathbb{J}}$ (the closure of $\mathbb{J}$), then

$$\limsup_{t \to \infty} \frac{1}{w_t} \log P_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) \in \mathbb{J}\}) \leq e(0) = -\inf_{\theta \in \mathbb{J}} \varphi(\theta).$$

In the case $e'(0) \notin \overline{\mathbb{J}}$, there exist $\alpha, \beta \in \overline{\mathbb{J}}$ such that $e'(0) \in]\alpha, \beta[\subset \mathbb{R} \setminus \overline{\mathbb{J}}$. It follows that

$$\begin{aligned} &P_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) \in \mathbb{J}\}) \\ &\quad \leq P_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) < \alpha\}) + P_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) > \beta\}) \\ &\quad \leq 2 \max\left(P_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) < \alpha\}), P_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) > \beta\})\right), \end{aligned}$$

and Proposition 2.3 yields

$$\limsup_{t \to \infty} \frac{1}{w_t} \log P_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) \in \mathbb{J}\}) \leq -\min(\varphi(\alpha), \varphi(\beta)).$$

Finally, by Part (5) of Proposition 2.1, one has

$$\inf_{\theta \in \mathbb{J}} \varphi(\theta) = \min(\varphi(\alpha), \varphi(\beta)),$$

and therefore

$$\limsup_{t \to \infty} \frac{1}{w_t} \log P_t(\{x \in M_t \,|\, w_t^{-1} X_t(x) \in \mathbb{J}\}) \leq -\inf_{\theta \in \mathbb{J}} \varphi(\theta) \tag{2.10}$$

holds for any $\mathbb{J} \subset]D^+e(a), D^-e(b)[$. The result follows from the bounds (2.9) and (2.10). □

3 Approximately finite quantum hypothesis testing

3.1 Setup

Consider a quantum system $\mathcal{Q}$ described by a finite dimensional Hilbert space $\mathcal{H}$. We denote by $\mathcal{O}_{\mathcal{H}}$ (or $\mathcal{O}$ whenever the meaning is clear within the context) the $*$-algebra of all linear operators on $\mathcal{H}$, equipped with the usual operator norm. The symbol $1_{\mathcal{H}}$ (or simply 1) stands for the unit of $\mathcal{O}_{\mathcal{H}}$. The spectrum of $A \in \mathcal{O}_{\mathcal{H}}$ is denoted by $\mathrm{sp}(A)$. $\mathcal{O}_{\mathcal{H},\mathrm{self}}$ (or simply $\mathcal{O}_{\mathrm{self}}$) is the set of all self-adjoint elements of $\mathcal{O}_{\mathcal{H}}$. $A \in \mathcal{O}_{\mathcal{H},\mathrm{self}}$ is positive, written $A \geq 0$, if $\mathrm{sp}(A) \subset [0, \infty[$. For $A \in \mathcal{O}_{\mathcal{H},\mathrm{self}}$ and $\lambda \in \mathrm{sp}(A)$, $P_\lambda(A)$ denotes the spectral projection of A. We adopt the shorthand notation

$$\mathrm{s}_A = \sum_{0 \neq \lambda \in \mathrm{sp}(A)} P_\lambda(A), \qquad A_\pm = \pm \sum_{\lambda \in \mathrm{sp}(A), \pm\lambda > 0} \lambda P_\lambda(A), \qquad |A| = A_+ - A_-.$$

s_A is the orthogonal projection on the range of A and $A_\pm$ are the positive/negative parts of A.

A linear functional $\nu : \mathcal{O} \to \mathbb{C}$ is called:

(1) hermitian if $\nu(A) \in \mathbb{R}$ for all $A \in \mathcal{O}_{\mathrm{self}}$;

(2) positive if $\nu(A^*A) \geq 0$ for all $A \in \mathcal{O}$;

(3) a state if it is positive and normalized by $\nu(1) = 1$;

(4) faithful if it is positive and such that $\nu(A^*A) = 0$ implies $A = 0$ for any $A \in \mathcal{O}$.

For any $\nu \in \mathcal{O}$, $A \mapsto \mathrm{Tr}\, \nu A$ defines a linear functional on $\mathcal{O}$, *i.e.*, an element of the dual space $\mathcal{O}^*$. Any linear functional on $\mathcal{O}$ arises in this way. In the following, we shall identify $\nu \in \mathcal{O}$ with the corresponding linear functional and write $\nu(A) = \mathrm{Tr}\, \nu A$. With this identification, the norm of the dual space $\mathcal{O}^*$ is just the trace norm on $\mathcal{O}$, *i.e.*, $\|\nu\| = \mathrm{Tr}\, |\nu| = |\nu|(1)$ with $|\nu| = (\nu^*\nu)^{1/2}$. A functional ν is hermitian iff $\nu \in \mathcal{O}_{\mathrm{self}}$ and positive iff

$\nu \geq 0$ as an operator. If $\nu \geq 0$ then it is faithful iff $\mathrm{sp}(\nu) \subset]0, \infty[$, which we denote by $\nu > 0$. The functional ν is a state iff the operator ν is a density matrix, *i.e.*, $\nu \geq 0$ and $\nu(1) = \mathrm{Tr}\,\nu = 1$. If ν is a positive linear functional then s_ν is its support projection, the smallest orthogonal projection P such that $\nu(1-P) = 0$. In particular $\|\nu\| = \nu(1) = \nu(\mathrm{s}_\nu)$ and ν is faithful iff $\mathrm{s}_\nu = 1$. If ν is a Hermitian linear functional then $\nu_\pm$ are positive linear functionals such that $\mathrm{s}_{\nu_+}\mathrm{s}_{\nu_-} = 0$, $\nu = \nu_+ - \nu_-$ (the Jordan decomposition of ν) and $\|\nu\| = \nu_+(1) + \nu_-(1)$.

Let ν and ω be positive linear functionals. The relative entropy of ν w.r.t. ω is given by

$$\mathrm{Ent}(\nu|\omega) = \begin{cases} \mathrm{Tr}\,\nu(\log\omega - \log\nu), & \text{if } \mathrm{s}_\nu \leq \mathrm{s}_\omega, \\ -\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

Rényi's relative entropy of ν w.r.t. ω is the convex function

$$[0,1] \ni s \mapsto \mathrm{Ent}_s(\nu|\omega) = \begin{cases} \log \mathrm{Tr}\,\nu^s\omega^{1-s}, & \text{if } \mathrm{s}_\nu\mathrm{s}_\omega \neq 0, \\ -\infty & \text{otherwise,} \end{cases}$$

which clearly satisfies $\mathrm{Ent}_s(\nu|\omega) = \mathrm{Ent}_{1-s}(\omega|\nu)$. If $\mathrm{s}_\nu \leq \mathrm{s}_\omega$ then this function has a real analytic extension to $s \in]0, \infty[$ and

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Ent}_s(\nu|\omega)\bigg|_{s=1} = -\mathrm{Ent}(\nu|\omega). \tag{3.2}$$

The observables of the quantum system $\mathcal{Q}$ are described by elements of $\mathcal{O}_{\mathrm{self}}$ and its physical states are states on $\mathcal{O}$, *i.e.*, density matrices. The possible outcomes of a measurement of A are the eigenvalues $a \in \mathrm{sp}(A)$. If the system is in the state ν, then the probability to observe a is $\nu(P_a(A))$. In particular, the expectation value of A is $\nu(A)$.

The setup of quantum hypothesis testing is a direct generalization of the corresponding setup in classical statistics [LR]. Let $\nu \neq \omega$ be two faithful[2] states such that one of the following two competing hypotheses holds:

Hypothesis I : $\mathcal{Q}$ is in the state ω;

Hypothesis II : $\mathcal{Q}$ is in the state ν.

A *test* is a projection $T \in \mathcal{O}_{\mathrm{self}}$ and the result of a measurement of the corresponding observable is either 0 or 1. The purpose of a test is to discriminate between the two hypotheses. Given the outcome of the measurement of T one chooses Hypothesis I or II. More precisely, if the outcome is 1 one accepts I and rejects II. Otherwise, if the outcome is 0, one accepts II and rejects I. To a given test T one can associate two kinds of errors. A type-I error occurs when the system is in the state ω but the outcome of the test is 0. The conditional probability of such an error, given that the state of the system is ω, is $\omega(1-T)$. If the system is in the state ν and the outcome of the test is 1, we get a type-II error, with conditional probability $\nu(T)$.

Assuming that Bayesian probabilities can be assigned to the states ω and ν, *i.e.*, that the state of the system is ω with probability $p \in]0,1[$ and ν with probability $1-p$, the total error probability is equal to

$$p\,\omega(1-T) + (1-p)\,\nu(T),$$

which we wish to minimize over T. It is convenient to absorb the scalar factors into ν and ω and consider the quantities

$$\begin{aligned} D(\nu,\omega,T) &= \nu(T) + \omega(1-T), \\ D(\nu,\omega) &= \inf_T D(\nu,\omega,T), \end{aligned} \tag{3.3}$$

[2] In this paper, for simplicity of the exposition, we shall only consider faithful states. We note, however, that most results extend to the general case by a straightforward limiting argument.

for given faithful linear functionals ν, ω on $\mathcal{O}$. Note that $D(\nu, \omega)$ is unitary invariant, *i.e.*,

$$D(U\nu U^*, U\omega U^*) = D(\nu, \omega),$$

for any unitary $U \in \mathcal{O}$. Since $D(\nu, \omega, T) = D(\omega, \nu, 1 - T)$, one also has $D(\nu, \omega) = D(\omega, \nu)$.

The following result is known as the Quantum Neyman-Pearson Lemma [ANS].

Proposition 3.1

$$D(\nu, \omega) = D(\nu, \omega, \mathrm{s}_{(\omega-\nu)_+}) = \frac{1}{2}(\omega(1) + \nu(1) - \mathrm{Tr}\,|\omega - \nu|).$$

Proof. For any test T,

$$\nu(T) + \omega(1 - T) = \omega(1) - (\omega - \nu)(T) \geq \omega(1) - (\omega - \nu)_+(T)$$

$$\geq \omega(1) - (\omega - \nu)_+(1) = \frac{1}{2}(\omega(1) + \nu(1) - \mathrm{Tr}\,|\omega - \nu|).$$

On the other hand,

$$D(\nu, \omega, \mathrm{s}_{(\omega-\nu)_+}) = \omega(1) - (\omega - \nu)_+(1) = \frac{1}{2}(\omega(1) + \nu(1) - \mathrm{Tr}\,|\omega - \nu|).$$

□

In the literature one often considers generalized tests defined as $T \in \mathcal{O}_{\mathrm{self}}$ satisfying $0 \leq T \leq 1$. Proposition 3.1 holds with the same proof if $D(\nu, \omega)$ is defined by taking the infimum in (3.3) over all generalized tests. The same remark applies to all other results discussed in this paper.

3.2 Bounds

In this section we discuss lower and upper bounds on the minimal error probability $D(\nu, \omega)$ that will play a key role in the sequel. These bounds are most easily described in terms of modular operators which act on the complex vector space $\mathcal{O}$ equipped with the Hilbert-Schmidt inner product

$$(A|B) = \mathrm{Tr}\, A^* B.$$

Operators acting on this Hilbert space are sometimes called *superoperators* in the physics literature.

The relative modular operator $\Delta_{\nu|\omega}$ associated with two faithful linear functionals ν, ω on $\mathcal{O}$ is defined by

$$\mathcal{O} \ni A \mapsto \Delta_{\nu|\omega} A = \nu A \omega^{-1}.$$

As a linear operator on $\mathcal{O}$, $\Delta_{\nu|\omega}$ is positive. Its spectrum consists of the eigenvalues λ/μ, $\lambda \in \mathrm{sp}(\nu)$, $\mu \in \mathrm{sp}(\omega)$. The corresponding spectral projections are the maps $A \mapsto P_\lambda(\nu) A P_\mu(\omega)$.

Set $\Omega_\omega = \omega^{1/2}$ and let $\mu_{\nu|\omega}$ be the spectral measure for $-\log \Delta_{\nu|\omega}$ and Ω_ω. Then one has

$$(\Omega_\omega | \Delta^s_{\nu|\omega} \Omega_\omega) = \int \mathrm{e}^{-sx} \mathrm{d}\mu_{\nu|\omega}(x) = \mathrm{Tr}\, \nu^s \omega^{1-s},$$

and

$$(\Omega_\omega | \Delta_{\nu|\omega}(1 + \Delta_{\nu|\omega})^{-1} \Omega_\omega) = \int \frac{\mathrm{d}\mu_{\nu|\omega}(x)}{1 + \mathrm{e}^x} = \sum_{(\lambda,\mu) \in \mathrm{sp}(\nu) \times \mathrm{sp}(\omega)} \frac{\mathrm{Tr} P_\lambda(\nu) P_\mu(\omega)}{\lambda^{-1} + \mu^{-1}}.$$

The main advantage of writing these quantities in terms of modular operators is that the following Proposition carries over without change to the infinite dimensional case (see Theorem 6.1).

Proposition 3.2 (1) *Upper bound: for any* $s \in [0,1]$,

$$D(\nu,\omega) \leq (\Omega_\omega | \Delta_{\nu|\omega}^s \Omega_\omega).$$

(2) *Lower bound:*

$$D(\nu,\omega) \geq (\Omega_\omega | \Delta_{\nu|\omega}(1+\Delta_{\nu|\omega})^{-1}\Omega_\omega).$$

Proof. Part (1) follows from Proposition 1.1 and we only need to prove Part (2). For any test T, one has

$$\nu(T) = \sum_{\lambda \in \mathrm{sp}(\nu)} \lambda \mathrm{Tr}\, T P_\lambda(\nu) T = \sum_{(\lambda,\mu)\in \mathrm{sp}(\nu)\times \mathrm{sp}(\omega)} \lambda \mu^{-1} \omega(T P_\lambda(\nu) T P_\mu(\omega)),$$

$$\omega(1-T) = \sum_{(\lambda,\mu)\in \mathrm{sp}(\nu)\times \mathrm{sp}(\omega)} \omega((1-T)P_\lambda(\nu)(1-T)P_\mu(\omega)).$$

Since, for $\kappa \geq 0$,

$$\kappa T P_\lambda(\nu) T + (1-T)P_\lambda(\nu)(1-T) = \frac{\kappa}{1+\kappa} P_\lambda(\nu) + \frac{1}{1+\kappa}(1-(1+\kappa)T)P_\lambda(\nu)(1-(1+\kappa)T) \geq \frac{\kappa}{1+\kappa} P_\lambda(\nu),$$

we derive

$$\nu(T) + \omega(1-T) \geq \sum_{(\lambda,\mu)\in \mathrm{sp}(\nu)\times \mathrm{sp}(\omega)} \frac{\lambda/\mu}{1+\lambda/\mu} \omega(P_\lambda(\nu)P_\mu(\omega)) = (\Omega_\omega | \Delta_{\nu|\omega}(1+\Delta_{\nu|\omega})^{-1}\Omega_\omega).$$

□

3.3 Asymptotic hypothesis testing

After these preliminaries, we turn to asymptotic hypothesis testing. For each $n \in \mathbb{N}$, let $\mathcal{Q}_n$ be a quantum system described by the finite dimensional Hilbert spaces $\mathcal{H}_n$ and let (ν_n, ω_n) be a pair of faithful linear functionals on $\mathcal{O}_{\mathcal{H}_n}$. Let $w_n > 0$ be given *weights* such that $\lim_{n\to\infty} w_n = \infty$. Error exponents of the Chernoff type associated to (ν_n, ω_n, w_n) are defined by

$$\overline{D} = \limsup_{n\to\infty} \frac{1}{w_n} \log D(\nu_n, \omega_n),$$

$$\underline{D} = \liminf_{n\to\infty} \frac{1}{w_n} \log D(\nu_n, \omega_n).$$

An immediate consequence of Proposition 3.2 (1) is:

Proposition 3.3

$$\overline{D} \leq \inf_{s\in[0,1]} \limsup_{n\to\infty} \frac{1}{w_n} \mathrm{Ent}_s(\nu_n|\omega_n).$$

Lower bounds in asymptotic hypothesis testing are intimately linked with the theory of large deviations and to discuss them we need

Assumption (AF1). The limit

$$e(s) = \lim_{n\to\infty} \frac{1}{w_n} \mathrm{Ent}_s(\nu_n|\omega_n)$$

exists and is finite for $s \in [0,1]$. The function $s \mapsto e(s)$ is continuous on $[0,1]$, differentiable on $]0,1[$, and $D^+e(0) < D^-e(1)$.

Note that the limiting function $s \mapsto e(s)$ is convex on $[0,1]$.

The following result is known as the Chernoff bound.

Theorem 3.4 *Suppose that Assumption (AF1) holds. Then*

$$\underline{D} = \overline{D} = \inf_{s\in[0,1]} e(s).$$

Proof. By Proposition 3.3, we only need to prove that

$$\underline{D} \geq \inf_{s\in[0,1]} e(s).$$

Applying Chebyshev's inequality to Part (2) of Proposition 3.2 one easily shows that

$$\underline{D} \geq \liminf_{n\to\infty} \frac{1}{w_n} \log \mu_{\nu_n|\omega_n}(]-\infty, -\theta w_n[), \tag{3.4}$$

for any $\theta \geq 0$. Suppose first that $e(0) \leq e(1)$. Then $D^- e(1) > 0$ and since

$$e(s) = \lim_{n\to\infty} \frac{1}{w_n} \log \int \mathrm{e}^{-sx} \mathrm{d}\mu_{\nu_n|\omega_n}(x),$$

Proposition 2.5 implies that for $\theta \in]D^+e(0), D^-e(1)[$,

$$\liminf_{n\to\infty} \frac{1}{w_n} \log \mu_{\nu_n|\omega_n}(]-\infty, -\theta w_n[) \geq -\varphi(\theta), \tag{3.5}$$

with

$$\varphi(\theta) = \sup_{s\in[0,1]} (\theta s - e(s)).$$

If $0 > D^+e(0)$ then $0 \in]D^+e(0), D^-e(1)[$ and it follows from (3.4) and (3.5) that

$$\underline{D} \geq -\varphi(0) = \inf_{s\in[0,1]} e(s).$$

If $0 \leq D^+e(0)$, then (3.4) and (3.5) imply that $\underline{D} \geq -\varphi(\theta)$ for any $\theta \in]D^+e(0), D^-e(1)[$ and since φ is continuous, one has

$$\underline{D} \geq -\varphi(D^+e(0)) = e(0) \geq \inf_{s\in[0,1]} e(s).$$

If $e(0) > e(1)$, one derives the result by exchanging the roles of ν_n and ω_n, using the fact that $D(\nu_n, \omega_n) = D(\omega_n, \nu_n)$. □

We now turn to asymmetric hypothesis testing. Until the end of this section we assume that ν_n and ω_n are states. The asymmetric hypothesis testing concerns individual error probabilities $\omega_n(1 - T_n)$ (type-I error) and $\nu(T_n)$ (type-II error). For $r \in \mathbb{R}$, error exponents of the Hoeffding type are defined by

$$\overline{B}(r) = \inf_{\{T_n\}} \left\{ \limsup_{n\to\infty} \frac{1}{w_n} \log \omega_n(1 - T_n) \;\middle|\; \limsup_{n\to\infty} \frac{1}{w_n} \log \nu_n(T_n) < -r \right\},$$

$$\underline{B}(r) = \inf_{\{T_n\}} \left\{ \liminf_{n\to\infty} \frac{1}{w_n} \log \omega_n(1 - T_n) \;\middle|\; \limsup_{n\to\infty} \frac{1}{w_n} \log \nu_n(T_n) < -r \right\},$$

$$B(r) = \inf_{\{T_n\}} \left\{ \lim_{n\to\infty} \frac{1}{w_n} \log \omega_n(1 - T_n) \;\middle|\; \limsup_{n\to\infty} \frac{1}{w_n} \log \nu_n(T_n) < -r \right\},$$

where in the last case the infimum is taken over all families of tests $\{T_n\}$ for which $\lim_n \frac{1}{w_n} \log \omega_n(1-T_n)$ exists. These exponents give the best exponential convergence rate of type-I error under the exponential convergence constraint on the type-II error. An alternative interpretation is in terms of state concentration: as $n \to \infty$ the states ω_n and ν_n are concentrating along orthogonal subspaces and the Hoeffding exponents quantify the degree of this separation on the exponential scale. In classical statistics the Hoeffding exponents can be traced back to [Ho, CL, Bl].

The following result is known as the Hoeffding bound [Ha2, Na]:

Theorem 3.5 *Suppose that Assumption (AF1) holds. Then for all $r \in \mathbb{R}$,*

$$\underline{B}(r) = \overline{B}(r) = B(r) = -\sup_{s\in[0,1[} \frac{-sr - e(s)}{1-s}.$$

For $\epsilon \in]0,1[$, error exponents of the Stein type are defined by

$$\overline{B}_\epsilon = \inf_{\{T_n\}} \left\{ \limsup_{n\to\infty} \frac{1}{w_n} \log \omega_n(1-T_n) \;\middle|\; \nu_n(T_n) \le \epsilon \right\},$$

$$\underline{B}_\epsilon = \inf_{\{T_n\}} \left\{ \liminf_{n\to\infty} \frac{1}{w_n} \log \omega_n(1-T_n) \;\middle|\; \nu_n(T_n) \le \epsilon \right\},$$

$$B_\epsilon = \inf_{\{T_n\}} \left\{ \lim_{n\to\infty} \frac{1}{w_n} \log \omega_n(1-T_n) \;\middle|\; \nu_n(T_n) \le \epsilon \right\},$$

where in the last case the infimum is taken over all families of tests $\{T_n\}$ for which $\lim_n \frac{1}{w_n} \log \omega_n(1-T_n)$ exists. Note that if

$$\beta_n(\epsilon) = \inf_{T:\nu_n(T)\le\epsilon} \omega_n(1-T),$$

then

$$\liminf_{n\to\infty} \frac{1}{w_n} \log \beta_n(\epsilon) = \underline{B}_\epsilon, \qquad \limsup_{n\to\infty} \frac{1}{w_n} \log \beta_n(\epsilon) = \overline{B}_\epsilon.$$

The study of Stein's exponents in the quantum setting goes back to [HP]. To discuss these exponents we need

Assumption (AF2). For some $\delta > 0$, the limit

$$e(s) = \lim_{n\to\infty} \frac{1}{w_n} \mathrm{Ent}_s(\nu_n|\omega_n),$$

exists and is finite for all $s \in [0, 1+\delta[$. The function $s \mapsto e(s)$ is differentiable at $s = 1$.

Relation (3.2), Assumption (AF2) and convexity imply

$$e'(1) = -\lim_{n\to\infty} \frac{1}{w_n} \mathrm{Ent}(\nu_n|\omega_n).$$

In accordance with the terminology used in non-equilibrium statistical mechanics, we shall call

$$\Sigma^+ = e'(1)$$

the *entropy production* of the hypothesis testing. The following result is known as Stein's Lemma [HP, ON].

Theorem 3.6 *Suppose that Assumptions (AF1)–(AF2) hold. Then for all* $\epsilon \in]0,1[$,

$$\underline{B}_\epsilon = \overline{B}_\epsilon = B_\epsilon = -\Sigma^+.$$

Remark. This result holds under more general conditions then (AF1) and (AF2), see Section 6.5.

Just like the Chernoff bound, the Hoeffding bound and Stein's Lemma (Theorems 3.5 and 3.6) are easy consequences of Proposition 3.2 and the Gärtner-Ellis theorem. To avoid repetitions, we refer the reader to Sections 6.4 and 6.5 for their proofs in the general W^*-algebraic setting. As we have already mentioned, given Proposition 3.2 (and its W^*-algebraic generalization), the proofs follow line by line the classical arguments. The non-trivial aspects of non-commutativity emerge only in the verification of Assumptions (AF1)–(AF2) in the context of concrete quantum statistical models.

3.4 Examples

3.4.1 Quantum i.i.d. states

Quantum i.i.d. states are the simplest (and most widely studied) examples of quantum hypothesis testing. Asymptotic hypothesis testing for such states can be interpreted in terms of multiple measurements on independent, statistically equivalent systems. Let $\mathcal{K}$ be a finite dimensional Hilbert space, ν and ω two faithful linear functionals on $\mathcal{O}_\mathcal{K}$, and

$$\mathcal{H}_n = \otimes_{j=1}^n \mathcal{K}, \qquad \nu_n = \otimes_{j=1}^n \nu, \qquad \omega_n = \otimes_{j=1}^n \omega.$$

For $s \in \mathbb{R}$, one has

$$\mathrm{Tr}\, \nu_n^s \omega_n^{1-s} = (\mathrm{Tr}\, \nu^s \omega^{1-s})^n$$

and, taking the weights $w_n = n$, we see that

$$e(s) = \lim_{n\to\infty} \frac{1}{n} \mathrm{Ent}_s(\nu_n|\omega_n) = \mathrm{Ent}_s(\nu|\omega) = \sum_{(\lambda,\mu)\in \mathrm{sp}(\nu)\times\mathrm{sp}(\omega)} \lambda^s \mu^{1-s} \mathrm{Tr}\, P_\lambda(\nu) P_\mu(\omega).$$

The function $e(s)$ is real-analytic on $\mathbb{R}$ and is strictly convex iff $\nu \neq \omega$. In particular, Theorems 3.4, 3.5 and 3.6 hold for quantum i.i.d. states.

3.4.2 Quantum spin systems

Let $\mathcal{P}$ be the collection of all finite subsets of $\mathbb{Z}^d$. For $X \in \mathcal{P}$, $|X|$ denotes the cardinality of X (the number of elements of X), $\mathrm{diam}\, X = \max\{|x-y| \,|\, x, y \in X\}$ is the diameter of X, and $X + a = \{x + a \,|\, x \in X\}$ is the translate of X by $a \in \mathbb{Z}^d$.

Suppose that a single spin is described by the finite dimensional Hilbert space $\mathcal{K}$. We attach a copy $\mathcal{K}_x$ of $\mathcal{K}$ to each site $x \in \mathbb{Z}^d$. For $X \in \mathcal{P}$ we define $\mathcal{H}_X = \otimes_{x\in X} \mathcal{K}_x$ and $\mathcal{O}_X = \mathcal{O}_{\mathcal{H}_X}$. $\|A\|$ is the usual operator norm of $A \in \mathcal{O}_X$. For $X \subset Y$, the identity $\mathcal{H}_Y = \mathcal{H}_X \otimes \mathcal{H}_{Y\setminus X}$ yields a natural identification of $\mathcal{O}_X$ with a $*$-subalgebra of $\mathcal{O}_Y$. For $a \in \mathbb{Z}^d$, we denote by $T^a : \mathcal{K}_x \to \mathcal{K}_{x+a}$ the identity map. T^a extends to a unitary map $\mathcal{H}_X \to \mathcal{H}_{X+a}$.

An interaction is a collection $\Phi = \{\Phi_X\}_{X\in\mathcal{P}}$ such that $\Phi_X \in \mathcal{O}_{X,\mathrm{self}}$ and $T^a \Phi_X T^{-a} = \Phi_{X+a}$ for any $a \in \mathbb{Z}^d$. We set

$$|||\Phi||| = \sum_{X\ni 0} |X|^{-1} \|\Phi_X\|, \qquad \|\Phi\| = \sum_{X\ni 0} \|\Phi_X\|.$$

The interaction Φ is finite range if for some $R > 0$ and any $X \in \mathcal{P}$ such that $\mathrm{diam}\, X > R$, $\Phi_X = 0$.

To a given box $\Lambda_n = [-n, n]^d$ in $\mathbb{Z}^d$ one associates the Hamiltonian

$$H_{\Lambda_n}(\Phi) = \sum_{X \subset \Lambda_n} \Phi_X.$$

If $|||\Phi||| < \infty$, then the limit

$$P(\Phi) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \log \operatorname{Tr} \mathrm{e}^{-H_{\Lambda_n}(\Phi)}$$

exists and is called the pressure of Φ. The bound $|P(\Phi) - P(\Psi)| \leq |||\Phi - \Psi|||$ holds and for $s \in [0, 1]$,

$$P(s\Phi + (1-s)\Psi) \leq sP(\Phi) + (1-s)P(\Psi) \tag{3.6}$$

(see [Si]). Two interactions Φ and Ψ are called physically equivalent (denoted $\Phi \sim \Psi$) if equality holds in (3.6) for all $s \in [0, 1]$. For further information about the notion of physical equivalence we refer the reader to [Is].

Let Φ and Ψ be interactions and ν_n, ω_n the states defined by

$$\nu_n = \frac{\mathrm{e}^{-H_{\Lambda_n}(\Phi)}}{\operatorname{Tr} \mathrm{e}^{-H_{\Lambda_n}(\Phi)}}, \qquad \omega_n = \frac{\mathrm{e}^{-H_{\Lambda_n}(\Psi)}}{\operatorname{Tr} \mathrm{e}^{-H_{\Lambda_n}(\Psi)}}.$$

Let $w_n = |\Lambda_n|$ be the weights.

Theorem 3.7 *Suppose that $d = 1$ and that Φ and Ψ are finite range. Then*

(1) *The limit*

$$e(s) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \mathrm{Ent}_s(\nu_n, \omega_n)$$

exists for all s.

(2) *$e(s)$ is a real analytic function on $\mathbb{R}$.*

(3) *If $\Phi \sim \Psi$, then $e(s) = 0$ for all s.*

(4) *If $\Phi \not\sim \Psi$, then $e(s)$ is strictly convex on $\mathbb{R}$.*

Proof. Part (1) is proven in [LRB] (see also [NR]) and Part (2) in [Og2]. To prove (3), note that the Golden-Thompson inequality implies that for all $s \in \mathbb{R}$,

$$e(s) \geq p(s) = P(s\Phi + (1-s)\Psi) - sP(\Phi) - (1-s)P(\Psi).$$

Since $e(0) = e(1) = 0$ and $e(s) \leq 0$ for $s \in [0, 1]$, if $\Phi \sim \Psi$ then $e(s) = 0$ on $[0, 1]$, and so, by analyticity, $e(s) = 0$ for all s. To prove (4), note that $p(s)$ is also a real analytic convex function satisfying $p(0) = p(1) = 0$. If $\Phi \not\sim \Psi$, then $p(s)$ is not identically equal to zero, and so $e(s)$ is also not identically equal to zero. The analyticity then implies that $e(s)$ is strictly convex. □

For $d > 1$ the following is known.

Theorem 3.8 *Given $r > 0$, there is $\delta > 0$ such that, for any two finite-range interactions Φ, Ψ satisfying $\|\Phi\|, \|\Psi\| < \delta$, one has:*

(1)

$$e(s) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \mathrm{Ent}_s(\nu_n, \omega_n),$$

exists for $|s| < r$.

(2) *$e(s)$ is real analytic on* $]-r,r[$.

(3) *If $\Phi \sim \Psi$, then $e(s) = 0$ for $s \in]-r,r[$.*

(4) *If $\Phi \not\sim \Psi$, then $e(s)$ is strictly convex* $]-r,r[$.

Remark 1. The existence of $e(s)$ is established in [NR] (see also [LRB]). The real analyticity follows from Proposition 7.10 in [NR]. Parts (3) and (4) are proven in the same way as in Theorem 3.7.

Remark 2. One can get an explicit estimate on δ in terms of r by combining Theorem 6.2.4 in [BR2] and Proposition 7.10 in [NR].

Remark 3. Theorem 3.8 is a high-temperature result. Although, in general, for $d > 1$ and low temperatures one does not expect analyticity (or even differentiability) of $e(s)$, one certainly expects that $e(s)$ exists. Remarkably, it is *not known* in general whether $e(s)$ exists outside the high temperature regime.

3.4.3 Quasi-free CAR states

Quantum hypothesis testing for translation invariant quasi-free states on the algebra of fermionic creation/annihilation operators on $\mathbb{Z}^d$ has been studied in [MHO]. For later applications we start with a more general setup.

Let $\mathfrak{h}$ be the single particle Hilbert space and let $\mathcal{H} = \Gamma_{\mathrm{f}}(\mathfrak{h})$ be the antisymmetric (fermionic) Fock space over $\mathfrak{h}$. The CAR algebra $\mathrm{CAR}(\mathfrak{h})$ is the C^*-algebra generated by $\{a^{\#}(f) \,|\, f \in \mathfrak{h}\}$, where $a^{\#}(f)$ stands for $a(f)$ (the annihilation operator) or $a^*(f)$ (the creation operator). $\varphi(f) = \frac{1}{\sqrt{2}}(a(f) + a^*(f))$ denotes the field operator. Every self-adjoint operator T on $\mathfrak{h}$ satisfying $0 \leq T \leq 1$ generates a state ω_T on $\mathrm{CAR}(\mathfrak{h})$ by

$$\omega_T(a^*(f_n) \cdots a^*(f_1) a(g_1) \cdots a(g_m)) = \delta_{nm} \det\{(g_i|Tf_j)\}.$$

T is called the density operator and ω_T is the gauge invariant quasi-free state with density T. This state is completely determined by its two point function

$$\omega_T(a^*(f)a(g)) = (g|Tf).$$

If h is the one-particle Hamiltonian, then the thermal equilibrium state at inverse temperature $\beta > 0$ and chemical potential $\mu \in \mathbb{R}$ of the corresponding ideal Fermi gas is described by the gauge invariant quasi-free state with the Fermi-Dirac density

$$T_{\beta,\mu} = \left(1 + \mathrm{e}^{\beta(h-\mu)}\right)^{-1}.$$

Let now $A \neq B$ be two generators such that $\delta < A, B < 1 - \delta$ for some $\delta > 0$. Suppose that $\dim \mathfrak{h} = \infty$. Quantum hypothesis testing is set with respect to a sequence of finite dimensional orthogonal projections p_n on $\mathfrak{h}$ such that $\mathrm{s} - \lim_n p_n = 1$. Let $\mathfrak{h}_n = \mathrm{Ran}\, p_n$, $A_n = p_n A p_n$, $B_n = p_n B p_n$, and let ν_n and ω_n be quasi-free states on $\mathrm{CAR}(\mathfrak{h}_n)$ generated by A_n and B_n. A straightforward computation yields

$$\mathrm{Tr}\, \nu_n^s \omega_n^{1-s} = \det\left[B_n^{(1-s)/2} A_n^s B_n^{(1-s)/2} + (1 - A_n)^{s/2}(1 - B_n)^{1-s}(1 - A_n)^{s/2}\right],$$

and

$$\mathrm{Ent}_s(\nu_n|\omega_n) = \mathrm{Tr}\, \log\left[B_n^{(1-s)/2} A_n^s B_n^{(1-s)/2} + (1 - A_n)^{s/2}(1 - B_n)^{1-s}(1 - A_n)^{s/2}\right].$$

The natural choice for weights is $w_n = \dim \mathfrak{h}_n$. At this point one needs to specify the model further. The case considered in [MHO] is $\mathfrak{h} = \ell^2(\mathbb{Z}^d)$, $\mathfrak{h}_n = \ell^2(\Lambda_n)$, with A and B translation invariant. If $\mathfrak{F} : \mathfrak{h} \to L^2([0, 2\pi]^d, \mathrm{d}k)$ is the usual Fourier transform, then $\mathfrak{F}A\mathfrak{F}^{-1}$ and $\mathfrak{F}B\mathfrak{F}^{-1}$ are operators of multiplication by bounded measurable functions $A(k)$ and $B(k)$ whose essential ranges are contained in $[\delta, 1 - \delta]$. An application of Szëgo's theorem (see [MHO] for details) yields

Theorem 3.9 *For all $s \in \mathbb{R}$,*

$$e(s) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \mathrm{Ent}_s(\nu_n|\omega_n) = \int_{[0,2\pi]^d} \log\left[A(k)^s B(k)^{1-s} + (1-A(k))^s (1-B(k))^{1-s}\right] \frac{\mathrm{d}k}{(2\pi)^d},$$

and the function $e(s)$ is real analytic and strictly convex on $\mathbb{R}$.

Hypothesis testing for translation invariant quasi-free CCR states on $\mathbb{Z}^d$ has been studied in [Mo].

4 Entropy production and full counting statistics

4.1 Setup

We follow [JOPP1]. Let $\mathcal{Q}$ be a quantum system described by a finite dimensional Hilbert space $\mathcal{H}$, Hamiltonian H, and initial state $\omega > 0$. The state evolves in time as $\omega_t = \mathrm{e}^{-\mathrm{i}tH}\omega\mathrm{e}^{\mathrm{i}tH}$ while the observables evolve as $A_t = \mathrm{e}^{\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH}$. Obviously, $\omega_t(A) = \omega(A_t)$.

We define the *entropy observable* of $\mathcal{Q}$ by $S = -\log\omega$. Note that

$$S_t = -\log\omega_{-t}.$$

The *entropy production observable*, defined by

$$\sigma = \frac{\mathrm{d}}{\mathrm{d}t} S_t \bigg|_{t=0} = -\mathrm{i}[H, \log\omega],$$

is the quantum analog of the phase space contraction rate in classical non-equilibrium statistical mechanics [JPR]. In terms of the *relative Hamiltonian*

$$\ell_{\omega_t|\omega} = \log\omega_t - \log\omega = \int_0^t \sigma_{-u}\mathrm{d}u,$$

which satisfies the cocycle relation

$$\ell_{\omega_{t+s}|\omega} = \ell_{\omega_t|\omega} + (\ell_{\omega_s|\omega})_{-t},$$

the relative entropy of ω_t w.r.t. ω (recall Equ. (3.1)) is given by

$$\mathrm{Ent}(\omega_t|\omega) = -\omega_t(\ell_{\omega_t|\omega}).$$

Defining the *mean entropy production rate observable* over the time interval $[0,t]$ by

$$\Sigma^t = \frac{1}{t}(S_t - S) = \frac{1}{t}\int_0^t \sigma_u \mathrm{d}u,$$

we can write the entropy balance equation

$$\omega(\Sigma^t) = \frac{1}{t}\int_0^t \omega(\sigma_u)\mathrm{d}u, = -\frac{1}{t}\mathrm{Ent}(\omega_t|\omega) \geq 0, \tag{4.1}$$

which is a finite time expression of the second law of thermodynamics.

We shall call

$$(t,s) \mapsto e_t(s) = \mathrm{Ent}_s(\omega_t|\omega)$$

the *Rényi entropic functional* of the quantum system $\mathcal{Q}$. Since $\mathrm{Ent}_s(\omega_{-t}|\omega) = \mathrm{Ent}_s(\omega|\omega_t) = \mathrm{Ent}_{1-s}(\omega_t|\omega)$, one has

$$e_{-t}(s) = e_t(1-s), \tag{4.2}$$

for any $t, s \in \mathbb{R}$. With the notations of Section 3.2

$$e_t(s) = \log(\Omega_\omega|\Delta^s_{\omega_t|\omega}\Omega_\omega) = \log \int \mathrm{e}^{-sx}\mathrm{d}\mu_{\omega_t|\omega}(x), \tag{4.3}$$

and Relation (3.2) yields

$$\frac{\mathrm{d}}{\mathrm{d}s}e_t(s)\bigg|_{s=0} = \mathrm{Ent}(\omega|\omega_t), \qquad \frac{\mathrm{d}}{\mathrm{d}s}e_t(s)\bigg|_{s=1} = -\mathrm{Ent}(\omega_t|\omega). \tag{4.4}$$

Moreover, Relation (4.2) translates into

$$\mathrm{d}\mu_{\omega_{-t}|\omega}(x) = \mathrm{e}^x \mathrm{d}\mu_{\omega_t|\omega}(-x). \tag{4.5}$$

The physical interpretation of the functional $e_t(s)$ and the spectral measure $\mu_{\omega_t|\omega}$ is in terms of the full counting statistics. At time $t = 0$, with the system in the state ω, one performs a measurement of the entropy observable S. The possible outcomes of this measurement are eigenvalues of S and $\alpha \in \mathrm{sp}(S)$ is observed with probability $\omega(P_\alpha)$, where P_α is the spectral projection of S for its eigenvalue α. After the measurement, the state of $\mathcal{Q}$ is

$$\frac{\omega P_\alpha}{\omega(P_\alpha)},$$

and this state evolves over the time interval $[0, t]$ to

$$\frac{\mathrm{e}^{-\mathrm{i}tH}\omega P_\alpha \mathrm{e}^{\mathrm{i}tH}}{\omega(P_\alpha)}.$$

A second measurement of S at time t yields the result $\alpha' \in \mathrm{sp}(S)$ with probability

$$\frac{\mathrm{Tr}\left(\mathrm{e}^{-\mathrm{i}tH}\omega P_\alpha \mathrm{e}^{\mathrm{i}tH} P_{\alpha'}\right)}{\omega(P_\alpha)}.$$

The joint probability distribution of the two measurements is given by

$$\mathrm{Tr}\left(\mathrm{e}^{-\mathrm{i}tH}\omega P_\alpha \mathrm{e}^{\mathrm{i}tH} P_{\alpha'}\right),$$

and the probability distribution of the mean rate of change of the entropy, $\phi = (\alpha' - \alpha)/t$, is given by

$$\mathrm{d}\mathbb{P}_t(\phi) = \sum_{\alpha,\alpha'} \mathrm{Tr}\left(\mathrm{e}^{-\mathrm{i}tH}\omega P_\alpha \mathrm{e}^{\mathrm{i}tH} P_{\alpha'}\right) \delta(\phi - (\alpha' - \alpha)/t)\mathrm{d}\phi.$$

We shall say that the probability measure $\mathbb{P}_t$ is the *Full Counting Statistics* (FCS) of the system $\mathcal{Q}$. Since

$$\mathrm{Tr}\,\omega^s_{-t}\omega^{1-s} = \mathrm{Tr}\,\omega_t^{1-s}\omega^s = \sum_{\alpha,\alpha'\in\mathrm{sp}(S)} \mathrm{e}^{-s(\alpha'-\alpha)}\mathrm{Tr}\left(\mathrm{e}^{-\mathrm{i}tH}\omega P_\alpha \mathrm{e}^{\mathrm{i}tH} P_{\alpha'}\right) = \int \mathrm{e}^{-st\phi}\mathrm{d}\mathbb{P}_t(\phi),$$

we conclude that

$$e_{-t}(s) = e_t(1-s) = \log \int \mathrm{e}^{-st\phi}\mathrm{d}\mathbb{P}_t(\phi).$$

Comparing this relation with Equ. (4.3) and using Equ. (4.5) we show that

$$\mathrm{d}\mathbb{P}_t(\phi) = \mathrm{e}^{t\phi}\mathrm{d}\mu_{\omega_t|\omega}(-t\phi) = \mathrm{d}\mu_{\omega_{-t}|\omega}(t\phi), \tag{4.6}$$

i.e., that $\mathbb{P}_t$ is the spectral measure of $-\frac{1}{t}\log\Delta_{\omega_{-t}|\omega}$.

The expectation and variance of ϕ w.r.t. $\mathbb{P}_t$ are given by

$$\begin{aligned} \mathbb{E}_t(\phi) &= -\frac{1}{t}\partial_s e_t(1-s)\Big|_{s=0} = \frac{1}{t}\partial_s e_t(s)\Big|_{s=1} = \omega(\Sigma^t), \\ \mathbb{E}_t(\phi^2) - \mathbb{E}_t(\phi)^2 &= \frac{1}{t^2}\partial_s^2 e_t(1-s)\Big|_{s=0} = \frac{1}{t^2}\partial_s^2 e_t(s)\Big|_{s=1} = \omega({\Sigma^t}^2) - \omega(\Sigma^t)^2. \end{aligned} \tag{4.7}$$

They coincide with the expectation and variance of Σ^t w.r.t. ω. However, in general such a relation does not hold for higher order cumulants.

The quantum system $\mathcal{Q}$ is *time reversal invariant* (TRI) if there exists an orthogonal basis of $\mathcal{H}$ for which the matrix representatives of H and ω are both real. Denoting by θ the complex conjugation in such a basis, $\Theta(A) = \theta A\theta$ defines an anti-linear $*$-automorphism of $\mathcal{O}$ for which[3]

$$\Theta(A_t) = (\Theta(A))_{-t}, \qquad \omega(\Theta(A)) = \overline{\omega}(A),$$

and hence $\omega_t(\Theta(A)) = \overline{\omega_{-t}}(A)$ holds for any $A \in \mathcal{O}$. Since, for any states ρ, ν, one has $\mathrm{Ent}_s(\overline{\rho\circ\Theta}|\overline{\nu\circ\Theta}) = \mathrm{Ent}_s(\rho|\nu)$, we can write

$$e_t(s) = \mathrm{Ent}_s(\omega_t|\omega) = \mathrm{Ent}_s(\overline{\omega_t\circ\Theta}|\overline{\omega\circ\Theta}) = \mathrm{Ent}_s(\omega_{-t}|\omega) = e_{-t}(s).$$

Thus, Equ. (4.2) and the time-reversal invariance of $\mathcal{Q}$ implies the transient *Evans-Searles symmetry* of its Rényi entropic functional

$$e_t(s) = e_t(1-s). \tag{4.8}$$

This relation has the following equivalent reformulations:

(1) $\log\int \mathrm{e}^{-st\phi}\mathrm{d}\mathbb{P}_t(\phi) = e_t(s)$;

(2) $\mathrm{d}\mathbb{P}_t(\phi) = \mathrm{d}\mu_{\omega_t|\omega}(t\phi)$;

(3) $\mathrm{d}\mathbb{P}_t(-\phi) = \mathrm{e}^{-t\phi}\mathrm{d}\mathbb{P}_t(\phi)$;

(4) $\mathrm{d}\mu_{\omega_t|\omega}(-x) = \mathrm{e}^{-x}\mathrm{d}\mu_{\omega_t|\omega}(x)$.

4.2 Open systems

To shed some light on the definitions introduced in the previous section, let us describe them in the more concrete setup of open quantum systems.

Let $\mathcal{R}_j$, $j = 1, \ldots, n$, be quantum systems described by finite dimensional Hilbert spaces $\mathcal{H}_j$ and Hamiltonians H_j. We denote by $\mathcal{O}_j$ the corresponding $*$-algebras. Let $N_j \in \mathcal{O}_{j,\mathrm{self}}$, $[H_j, N_j] = 0$, be given "conserved charges". We assume that each system $\mathcal{R}_j$ is in thermal equilibrium at inverse temperature $\beta_j > 0$ and chemical potential $\mu_j \in \mathbb{R}$, namely that its initial state is

$$\omega_j = \frac{\mathrm{e}^{-\beta_j(H_j-\mu_j N_j)}}{\mathrm{Tr}\,\mathrm{e}^{-\beta_j(H_j-\mu_j N_j)}}.$$

[3] $\overline{\omega}$ denotes the anti-linear functional $\overline{\omega}(A) = \overline{\omega(A)} = \omega(A^*)$

In our context the systems $\mathcal{R}_j$ are individual thermal reservoirs. The Hilbert space, $*$-algebra, Hamiltonian, and initial state of the full reservoir system $\mathcal{R} = \mathcal{R}_1 + \cdots + \mathcal{R}_n$ are

$$\mathcal{H}_{\mathcal{R}} = \otimes_{j=1}^n \mathcal{H}_j, \qquad \mathcal{O}_{\mathcal{R}} = \otimes_{j=1}^n \mathcal{O}_j, \qquad H_{\mathcal{R}} = \sum_{j=1}^n H_j, \qquad \omega_{\mathcal{R}} = \otimes_{j=1}^n \omega_j$$

(whenever the meaning is clear within the context we identify $A \in \mathcal{O}_j$ with the element $(\otimes_{k\neq j} 1_{\mathcal{H}_k}) \otimes A$ of $\mathcal{O}_{\mathcal{R}}$).

Let $\mathcal{S}$ be another quantum system described by the finite dimensional Hilbert space $\mathcal{H}_{\mathcal{S}}$, $*$-algebra $\mathcal{O}_{\mathcal{S}}$ and Hamiltonian $H_{\mathcal{S}}$. In our context $\mathcal{S}$ will be a small quantum system coupled to the full reservoir $\mathcal{R}$. The thermodynamic limit discussed in next section concerns only reservoirs while $\mathcal{S}$ will remain unchanged. A convenient reference state of $\mathcal{S}$ is the chaotic state

$$\omega_{\mathcal{S}} = \frac{1_{\mathcal{H}_{\mathcal{S}}}}{\dim \mathcal{H}_{\mathcal{S}}},$$

but under normal circumstances none of the final results (after the thermodynamic and the large time limit are taken) depends on this choice.

The interaction of $\mathcal{S}$ with $\mathcal{R}$ is described by the self-adjoint operator

$$V = \sum_{j=1}^n V_j,$$

where $V_j = V_j^* \in \mathcal{O}_{\mathcal{S}} \otimes \mathcal{O}_j$. The Hilbert space, $*$-algebra, Hamiltonian, and initial state of the coupled joint system $\mathcal{Q} = \mathcal{S} + \mathcal{R}$ are

$$\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}, \qquad \mathcal{O} = \mathcal{O}_{\mathcal{S}} \otimes \mathcal{O}_{\mathcal{R}}, \qquad H = H_{\mathcal{S}} + H_{\mathcal{R}} + V, \qquad \omega = \omega_{\mathcal{S}} \otimes \omega_{\mathcal{R}}.$$

The entropy observable of $\mathcal{Q}$ is

$$S = \sum_{j=1}^n \beta_j (H_j - \mu_j N_j) - \sum_{j=1}^n \log\left(\mathrm{Tr}\, \mathrm{e}^{-\beta_j (H_j - \mu_j N_j)}\right) - \log(\dim\, \mathcal{H}_{\mathcal{S}}), \tag{4.9}$$

and one easily derives that its entropy production observable is

$$\sigma = -\sum_{j=1}^n \beta_j (\Phi_j - \mu_j \mathcal{J}_j),$$

where

$$\Phi_j = -\mathrm{i}[H, H_j] = -\mathrm{i}[V, H_j], \qquad \mathcal{J}_j = -\mathrm{i}[H, N_j] = -\mathrm{i}[V, N_j]. \tag{4.10}$$

Since

$$\omega(H_{jt}) - \omega(H_j) = -\int_0^t \omega(\Phi_{ju})\mathrm{d}u, \qquad \omega(N_{jt}) - \omega(N_j) = -\int_0^t \omega(\mathcal{J}_{ju})\mathrm{d}u,$$

the observables $\Phi_j / \mathcal{J}_j$ describe energy/charge currents out of the reservoir $\mathcal{R}_j$.

In the framework of open quantum systems the FCS can be naturally generalized. Consider the commuting set of observables

$$\mathbf{S} = (\beta_1 H_1, \cdots, \beta_n H_n, -\beta_1 \mu_1 N_1, \cdots, -\beta_n \mu_n N_n).$$

Let $P_{\boldsymbol{\alpha}}$ be the joint spectral projection of $\mathbf{S}$ corresponding to the eigenvalue $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{2n}) \in \mathrm{sp}(\mathbf{S})$. Then

$$\mathrm{Tr}\, \mathrm{e}^{-\mathrm{i}tH} \omega P_{\boldsymbol{\alpha}} \mathrm{e}^{\mathrm{i}tH} P_{\boldsymbol{\alpha}'},$$

is the joint probability distribution of the measurement of $\mathbf{S}$ at time $t = 0$ followed by a later measurement at time t. Let $\mathbb{P}_t(\boldsymbol{\varepsilon}, \boldsymbol{\nu})$ be the induced probability distribution of the vector

$$(\boldsymbol{\varepsilon}, \boldsymbol{\nu}) = (\varepsilon_1, \cdots, \varepsilon_n, \nu_1, \cdots, \nu_n) = (\boldsymbol{\alpha}' - \boldsymbol{\alpha})/t,$$

which describes the mean rate of change of energy and charge of each reservoir.

Expectation and covariance of $(\boldsymbol{\varepsilon}, \boldsymbol{\nu})$ w.r.t. $\mathbb{P}_t$ are given by

$$\begin{aligned} \mathbb{E}_t(\varepsilon_j) &= -\frac{\beta_j}{t} \int_0^t \omega(\Phi_{js}) \mathrm{d}s, \\ \mathbb{E}_t(\nu_j) &= \frac{\beta_j \mu_j}{t} \int_0^t \omega(\mathcal{J}_{js}) \mathrm{d}s, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \mathbb{E}_t(\varepsilon_j \varepsilon_k) - \mathbb{E}_t(\varepsilon_j)\mathbb{E}_t(\varepsilon_k) &= \frac{\beta_j \beta_k}{t^2} \int_0^t \int_0^t \omega\left((\Phi_{js} - \omega(\Phi_{js}))(\Phi_{ku} - \omega(\Phi_{ku}))\right) \mathrm{d}s\mathrm{d}u, \\ \mathbb{E}_t(\nu_j \nu_k) - \mathbb{E}_t(\nu_j)\mathbb{E}_t(\nu_k) &= \frac{\beta_j \mu_j \beta_k \mu_k}{t^2} \int_0^t \int_0^t \omega\left((\mathcal{J}_{js} - \omega(\mathcal{J}_{js}))(\mathcal{J}_{ku} - \omega(\mathcal{J}_{ku}))\right) \mathrm{d}s\mathrm{d}u, \\ \mathbb{E}_t(\varepsilon_j \nu_k) - \mathbb{E}_t(\varepsilon_j)\mathbb{E}_t(\nu_k) &= -\frac{\beta_j \beta_k \mu_k}{t^2} \int_0^t \int_0^t \omega\left((\Phi_{js} - \omega(\Phi_{js}))(\mathcal{J}_{ku} - \omega(\mathcal{J}_{ku}))\right) \mathrm{d}s\mathrm{d}u. \end{aligned} \tag{4.12}$$

The cumulant generating function is

$$\mathbb{R}^{2n} \ni \boldsymbol{s} \mapsto e_t(\boldsymbol{s}) = \log \sum_{(\boldsymbol{\varepsilon}, \boldsymbol{\nu})} \mathrm{e}^{-t\boldsymbol{s}\cdot(\boldsymbol{\varepsilon}, \boldsymbol{\nu})} \mathbb{P}_t(\boldsymbol{\varepsilon}, \boldsymbol{\nu}).$$

If the open quantum system $\mathcal{Q}$ is TRI, then the fluctuation relation

$$e_t(\boldsymbol{s}) = e_t(\mathbf{1} - \boldsymbol{s})$$

holds ($\mathbf{1} = (1, \cdots, 1)$). $e_t(\boldsymbol{s})$ and $\mathbb{P}_t$ can be related to the modular structure (see [JOPP1] for details). However, their relation with quantum hypothesis testing is unclear at the moment and we will restrict ourselves to the FCS of the entropy observable defined by (4.9).

The above discussion of open quantum systems needs to be adjusted if the particle statistics (fermionic/bosonic) is taken into the account. These adjustments are minor and we shall discuss them only in the concrete example of the electronic black box model (see Section 8.2).

Remark. To justify the name "entropy" for the observable S, note that $\omega(S) = -\mathrm{Tr}\, \omega \log \omega$ is the Gibbs-von Neumann entropy of ω. It is well known that if ω is a thermal equilibrium state then this quantity coincides with the Boltzmann entropy of the system.

4.3 Thermodynamic limit

The dynamics of a finite (or more generally confined) quantum system being quasi-periodic, the large time asymptotics of its FCS is trivial. In order to get interesting information from this asymptotics and to relate it to quantum hypothesis testing it is necessary to further idealize the system by making it infinitely extended. In this section we briefly discuss the thermodynamic (TD) limit of the quantum system $\mathcal{Q}$.

Let $\mathcal{Q}_m$, $m = 1, 2, \ldots$, be a sequence of finite quantum systems described by Hilbert spaces $\mathcal{H}_m$, Hamiltonians H_m, and states $\omega_m > 0$, with the understanding that $m \to \infty$ corresponds to the TD limit (for example, in the case of quantum spin systems $\mathcal{Q}_m$ will be the finite spin system in the box $\Lambda_m = [-m, m]^d$ as discussed in Section 3.4.2).

We shall indicate the dependence of various objects on m by the subscript $_m$, *e.g.*, $\mathbb{P}_{mt}$ denotes the FCS of $\mathcal{Q}_m$ and $e_{mt}(s)$ its Rényi entropic functional.

$$e_{mt}(1-s) = \log \int \mathrm{e}^{-ts\phi} \mathrm{d}\mathbb{P}_{mt}(\phi)$$

is the cumulant generating function of $\mathbb{P}_{mt}$ and $\mathrm{e}^{-t\phi}\mathrm{d}\mathbb{P}_{mt}(\phi) = \mathrm{d}\mu_{\omega_{mt}|\omega_m}(-t\phi)$.

Assumption (TD). There is an open interval $\mathbb{I} \supset [0, 1]$ such that, for all $t > 0$, the limit

$$e_t(s) = \lim_{m\to\infty} e_{mt}(s)$$

exists and is finite for all $s \in \mathbb{I}$. W.l.o.g. we may assume that $\mathbb{I}$ is symmetric around the point $s = 1/2$.

Proposition 4.1 *Suppose that Assumption (TD) holds. Then there exist Borel probability measures $\mathbb{P}_t$ on $\mathbb{R}$ such that, for all s in the complex strip $\mathbb{S} = \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s \in \mathbb{I}\}$,*

$$\lim_{m\to\infty} \int \mathrm{e}^{-ts\phi} \mathrm{d}\mathbb{P}_{mt}(\phi) = \int \mathrm{e}^{-ts\phi} \mathrm{d}\mathbb{P}_t(\phi).$$

Moreover, $\mathbb{P}_{mt} \to \mathbb{P}_t$ and $\mathrm{e}^{-t\phi}\mathrm{d}\mathbb{P}_{mt} \to \mathrm{e}^{-t\phi}\mathrm{d}\mathbb{P}_t$ weakly as $m \to \infty$.

Proof. Set $k_{mt}(s) = \int \mathrm{e}^{-st\phi}\mathrm{d}\mathbb{P}_{mt}(\phi)$. The functions $s \mapsto k_{mt}(s)$ are entire analytic and for any compact set $K \subset \mathbb{S}$ and any $s \in K$,

$$\begin{aligned} |k_{mt}(s)| &\leq \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{Re}\,(s)t\phi}\mathrm{d}\mathbb{P}_{mt}(\phi) + \int_{0}^{\infty} \mathrm{e}^{-\mathrm{Re}\,(s)t\phi}\mathrm{d}\mathbb{P}_{mt}(\phi) \\ &\leq \int_{-\infty}^{0} \mathrm{e}^{-bt\phi}\mathrm{d}\mathbb{P}_{mt}(\phi) + \int_{0}^{\infty} \mathrm{e}^{-at\phi}\mathrm{d}\mathbb{P}_{mt}(\phi) \leq \mathrm{e}^{e_{mt}(b)} + \mathrm{e}^{e_{mt}(a)}, \end{aligned}$$

where $a = \min \mathrm{Re}\,(K)$ and $b = \max \mathrm{Re}\,(K)$ belong to $\mathbb{I}$. It follows that

$$\sup_{m>0, s\in K} |k_{mt}(s)| < \infty.$$

Vitali's convergence theorem implies that the sequence k_{mt} converges uniformly on compacts subsets of $\mathbb{S}$ to an analytic function k_t. Since the functions $\mathbb{R} \ni u \mapsto k_{mt}(\mathrm{i}u)$ are positive definite, so is $\mathbb{R} \ni u \mapsto k_t(\mathrm{i}u)$. Hence, by Bochner's theorem, there exists a Borel probability measure $\mathbb{P}_t$ on $\mathbb{R}$ such that

$$k_t(\mathrm{i}u) = \int \mathrm{e}^{-\mathrm{i}tu\phi}\mathrm{d}\mathbb{P}_t(\phi).$$

By construction, $\mathbb{P}_{mt} \to \mathbb{P}_t$ weakly. Analytic continuation yields that for $s \in \mathbb{S}$,

$$\lim_{m\to\infty} k_{mt}(s) = \int \mathrm{e}^{-st\phi}\mathrm{d}\mathbb{P}_t(\phi).$$

Replacing s with $1 - s$ and repeating the above argument we deduce that $\mathrm{e}^{-t\phi}\mathrm{d}\mathbb{P}_{mt}(\phi) \to \mathrm{e}^{-t\phi}\mathrm{d}\mathbb{P}_t(\phi)$ weakly. □

Note that

$$\mathbb{I} \ni s \mapsto e_t(s) = \log \int \mathrm{e}^{-t(1-s)\phi} \mathrm{d}\mathbb{P}_t(\phi)$$

is convex, $e_t(0) = e_t(1) = 0$, $e_t(s) \leq 0$ for s $\in]0,1[$, $e_t(s) \geq 0$ for $s \notin [0,1]$. Moreover, it follows from Vitali's convergence theorem in the proof of Proposition 4.1 that the derivatives of $e_{mt}(s)$ converge to the corresponding derivatives of $e_t(s)$. In particular, Equ. (4.4) implies

$$e_t'(1) = -\lim_{m\to\infty} \mathrm{Ent}(\omega_{mt}|\omega_m) = \lim_{m\to\infty} \omega_m(\Sigma_m^t). \tag{4.13}$$

Suppose now that each quantum system $\mathcal{Q}_m$ is TRI. The symmetries $e_{mt}(s) = e_{mt}(1-s)$ imply the finite-time Evans-Searles symmetry

$$e_t(s) = e_t(1-s), \tag{4.14}$$

which holds for $s \in \mathbb{I}$. This fluctuation relation has the following equivalent reformulations:

(1) For $s \in \mathbb{I}$,

$$e_t(s) = \log \int \mathrm{e}^{-ts\phi} \mathrm{d}\mathbb{P}_t(\phi).$$

(2)

$$\mathrm{d}\mathbb{P}_t(-\phi) = \mathrm{e}^{-t\phi} \mathrm{d}\mathbb{P}_t(\phi).$$

We remark that in all examples we will consider the limiting quantum dynamical system will actually exist and that $\mathbb{P}_t(s)$ and $e_t(s)$ can be expressed, via modular structure, in terms of states and dynamics of this limiting infinitely extended system. However, a passage through the TD limit is necessary for the physical interpretation of the FCS of infinitely extended systems.

4.4 Large time limit

We shall now make the

Assumption (LT). In addition to Assumption (TD), the limit

$$e(s) = \lim_{t\to\infty} \frac{1}{t} e_t(s)$$

exists and is finite for any $s \in \mathbb{I}$. Moreover, the function $\mathbb{I} \ni s \mapsto e(s)$ is differentiable.

As an immediate consequence, we note that the limiting function $e(s)$ inherits the following basic properties from $e_t(s)$:

(1) $e(s)$ is convex on $\mathbb{I}$;

(2) $e(0) = e(1) = 0$;

(3) $e(s) \leq 0$ for $s \in [0,1]$, and $e(s) \geq 0$ for $s \notin [0,1]$.

(4) If all the $\mathcal{Q}_m$'s are TRI, then the Evans-Searles symmetry

$$e(s) = e(1-s) \tag{4.15}$$

holds for $s \in \mathbb{I}$.

Together with convexity, the Evans-Searles symmetry implies

$$e(1/2) = \inf_{s\in[0,1]} e(s).$$

The asymptotic mean entropy production rate is defined as the double limit (recall the entropy balance equation (4.1))

$$\Sigma^+ = \lim_{t\to\infty}\lim_{m\to\infty} \omega_m(\Sigma_m^t) = -\lim_{t\to\infty}\lim_{m\to\infty} \frac{1}{t}\mathrm{Ent}(\omega_{mt}|\omega_m).$$

Equ. (4.13) and convexity (see, *e.g.*, Theorem 25.7 in [R]) imply that

$$\Sigma^+ = e'(1) = \lim_{t\to\infty} \mathbb{E}_t(\phi).$$

Clearly, $\Sigma^+ \geq 0$ and under normal conditions $\Sigma^+ > 0$ (that is, systems out of equilibrium under normal conditions are entropy producing). The strict positivity of entropy production is a detailed (and often difficult) dynamical question that can be answered only in the context of concrete models.

Assumption (LT) implies that the TD limit FCS $\mathbb{P}_t$ converges weakly to the Dirac measure δ_{Σ^+} as $t \to \infty$. The Gärtner-Ellis theorem (more precisely, Theorem 2.6) allows us to control the fluctuations of $\mathbb{P}_t$ in this limit. Namely, the large deviation principle

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_t(\mathbb{J}) = -\inf_{\theta\in\mathbb{J}} \varphi(\theta)$$

holds for any open set $\mathbb{J} \subset]\underline{\theta}, \overline{\theta}[$ with

$$\varphi(\theta) = \sup_{s\in\mathbb{I}}(-\theta s - e(s)),$$
$$\underline{\theta} = \inf_{s\in\mathbb{I}} e'(s), \qquad \overline{\theta} = \sup_{s\in\mathbb{I}} e'(s).$$

Under our current assumptions the functions $s \mapsto e_t(s)$ are analytic in the strip $\mathbb{S}$. Suppose that for some $\epsilon > 0$ one has

$$\sup_{t>1, s\in\mathbb{C}, |s-1|<\epsilon} \frac{1}{t}|e_t(s)| < \infty.$$

Vitali's theorem then implies that $e(s)$ is analytic for $|s-1| < \epsilon$ and

$$\lim_{t\to\infty} t^{k-1} \lim_{m\to\infty} C_{mt}^{(k)} = \partial_s^k e(s)|_{s=1},$$

where $C_{mt}^{(k)}$ is the k-th cumulant of $\mathbb{P}_{mt}$. Moreover, Brick's theorem [Br] implies the Central Limit Theorem: for any Borel set $\mathbb{J} \subset \mathbb{R}$,

$$\lim_{t\to\infty}\lim_{m\to\infty} \mathbb{P}_{mt}(\phi - \mathbb{E}_{mt}(\phi) \in t^{-1/2}\mathbb{J}) = \mu_D(\mathbb{J}),$$

where μ_D denotes the centered Gaussian of variance $D = e''(1)$.

Note that if (4.15) holds, then

$$\varphi(-\theta) = \varphi(\theta) - \theta, \qquad \overline{\theta} = -\underline{\theta}, \qquad \Sigma^+ = -e'(0), \qquad D = e''(0).$$

The material described in this and the previous section belongs to a body of structural results known as *Quantum Evans-Searles Fluctuation Theorem* [JOPP2].

4.5 Testing the arrow of time

In this section, we establish a connection between the Evans-Searles Fluctuation Theorem and hypothesis testing. To this end, we consider the problem of distinguishing the past from the future. More precisely, we shall apply the results of Section 3.3 to the family of pairs $\{(\omega_{mt}, \omega_{m(-t)})\}_{t>0}$ and investigate the various error exponents associated with them. As in the previous section, the thermodynamic limit $m \to \infty$ has to be taken prior to the limit $t \to \infty$ in order to achieve significant results.

Define the exponents of Chernoff type by

$$\underline{D} = \liminf_{t\to\infty} \frac{1}{2t} \liminf_{m\to\infty} \log D(\omega_{mt}, \omega_{m(-t)}), \qquad \overline{D} = \limsup_{t\to\infty} \frac{1}{2t} \limsup_{m\to\infty} \log D(\omega_{mt}, \omega_{m(-t)}),$$

and, for $r \in \mathbb{R}$, the exponents of Hoeffding type by

$$\overline{B}(r) = \inf_{\{T_{mt}\}} \left\{ \limsup_{t\to\infty} \frac{1}{2t} \limsup_{m\to\infty} \log \omega_{mt}(1 - T_{mt}) \;\middle|\; \limsup_{t\to\infty} \frac{1}{2t} \limsup_{m\to\infty} \log \omega_{m(-t)}(T_{mt}) < -r \right\},$$

$$\underline{B}(r) = \inf_{\{T_{mt}\}} \left\{ \liminf_{t\to\infty} \frac{1}{2t} \liminf_{m\to\infty} \log \omega_{mt}(1 - T_{mt}) \;\middle|\; \limsup_{t\to\infty} \frac{1}{2t} \limsup_{m\to\infty} \log \omega_{m(-t)}(T_{mt}) < -r \right\},$$

$$B(r) = \inf_{\{T_{mt}\}} \left\{ \lim_{t\to\infty} \frac{1}{2t} \lim_{m\to\infty} \log \omega_{mt}(1 - T_{mt}) \;\middle|\; \limsup_{t\to\infty} \frac{1}{2t} \limsup_{m\to\infty} \log \omega_{m(-t)}(T_{mt}) < -r \right\},$$

where in the last case the infimum is taken over all families of tests $\{T_{mt}\}$ for which

$$\lim_{t\to\infty} \frac{1}{2t} \lim_{m\to\infty} \log \omega_{mt}(1 - T_{mt}) \tag{4.16}$$

exists. Finally, for $\epsilon \in]0, 1[$, define the exponents of Stein type as

$$\overline{B}_\epsilon = \inf_{\{T_{mt}\}} \left\{ \limsup_{t\to\infty} \frac{1}{2t} \limsup_{m\to\infty} \log \omega_{mt}(1 - T_{mt}) \;\middle|\; \omega_{m(-t)}(T_{mt}) \le \epsilon \right\},$$

$$\underline{B}_\epsilon = \inf_{\{T_{mt}\}} \left\{ \liminf_{t\to\infty} \frac{1}{2t} \liminf_{m\to\infty} \log \omega_{mt}(1 - T_{mt}) \;\middle|\; \omega_{m(-t)}(T_{mt}) \le \epsilon \right\},$$

$$B_\epsilon = \inf_{\{T_{mt}\}} \left\{ \lim_{t\to\infty} \frac{1}{2t} \log \omega_{mt}(1 - T_{mt}) \;\middle|\; \omega_{m(-t)}(T_{mt}) \le \epsilon \right\},$$

where in the last case the infimum is taken over all families of tests $\{T_{mt}\}$ for which the limit (4.16) exists.

Theorem 4.2 *Suppose that Assumption (LT) holds and that* $0 \in]e'(0), e'(1)[$. *Then,*

(1)
$$\underline{D} = \overline{D} = \inf_{s\in[0,1]} e(s).$$

(2) *For all* r,
$$\underline{B}(r) = \overline{B}(r) = B(r) = -\sup_{0\le s<1} \frac{-sr - e(s)}{1 - s}.$$

(3) *For all* $\epsilon \in]0, 1[$,
$$\underline{B}_\epsilon = \overline{B}_\epsilon = B_\epsilon = -\Sigma^+.$$

Remark 1. These results hold under more general conditions then (LT). We have stated them in the present form for a transparent comparison with the results described in Section 4.4.

Remark 2. If the Evans-Searles symmetry $e(s) = e(1-s)$ holds, then $]e'(0), e'(1)[=]-\Sigma^+, \Sigma^+]$ so that the condition $0 \in]e'(0), e'(1)[$ is equivalent to $\Sigma^+ > 0$. In addition, $\inf_{s\in[0,1]} e(s) = e(1/2)$ in this case.

Proof. We will again prove only (1) to indicate the strategy of the argument. The proofs of (2) and (3) follow by a straightforward adaption of the general W^*-algebraic proofs described in Section 6.

Proposition 3.2 (1) implies that for $s \in [0,1]$,

$$\log D(\omega_{mt}, \omega_{m(-t)}) = \log D(\omega_{m2t}, \omega_m) \leq e_{m2t}(s),$$

and Assumption (LT) yields the upper bound

$$\overline{D} \leq \inf_{s\in[0,1]} e(s).$$

Let $\mathrm{d}\hat{\mathbb{P}}_{mt}(\phi) = e^{-t\phi}\mathrm{d}\mathbb{P}_{mt}(\phi)$ and $\mathrm{d}\hat{\mathbb{P}}_t(\phi) = \mathrm{e}^{-t\phi}\mathrm{d}\mathbb{P}_t(\phi)$. By Equ. (4.6) and Proposition 3.2 (2) one has

$$D(\omega_{mt}, \omega_{m(-t)}) = D(\omega_{m2t}, \omega_m) \geq \int \frac{\mathrm{d}\hat{\mathbb{P}}_{m2t}(\phi)}{1+\mathrm{e}^{-2t\phi}}.$$

By Proposition 4.1, $\hat{\mathbb{P}}_{m2t} \to \hat{\mathbb{P}}_{2t}$ weakly and hence

$$\liminf_{m\to\infty} \log D(\omega_{mt}, \omega_{m(-t)}) \geq \lim_{m\to\infty} \log \int \frac{\mathrm{d}\hat{\mathbb{P}}_{m2t}(\phi)}{1+\mathrm{e}^{-2t\phi}} = \log \int \frac{\mathrm{d}\hat{\mathbb{P}}_{2t}(\phi)}{1+\mathrm{e}^{-2t\phi}} \geq \log\left(\frac{1}{2}\,\hat{\mathbb{P}}_{2t}(]0,\infty[)\right),$$

where we used the Chebyshev inequality in the last step. Since

$$e(s) = \lim_{t\to\infty} \frac{1}{t} \log \int \mathrm{e}^{st\phi}\mathrm{d}\hat{\mathbb{P}}_t(\phi),$$

Assumption (LT) and Proposition 2.5 imply

$$\liminf_{t\to\infty} \frac{1}{2t} \log \hat{\mathbb{P}}_{2t}(]0,\infty[) \geq -\varphi(0) = \inf_{s\in[0,1]} e(s),$$

and hence

$$\underline{D} \geq \inf_{s\in[0,1]} e(s).$$

□

The results described in this section shed some light on the relation between hypothesis testing and non-equilibrium statistical mechanics. If the systems $\mathcal{Q}_m$ converge, as $m \to \infty$, to a limiting infinitely extended open quantum system, then this relation can be elaborated further, see Section 7.4 for details.

5 Algebraic preliminaries

5.1 Notation

Let $\mathfrak{M}$ be a W^*-algebra with unit 1, dual $\mathfrak{M}^*$, and predual $\mathfrak{M}_* \subset \mathfrak{M}^*$. The elements of $\mathfrak{M}_*$ are called normal functionals on $\mathfrak{M}$. Hermitian, positive and faithful functionals as well as states on $\mathfrak{M}$ are defined as in the finite

dimensional case (Section 3.1). $\mathfrak{M}_*^+$ denotes the set of positive elements of $\mathfrak{M}_*$. For $\nu, \omega \in \mathfrak{M}_*$, we write $\nu \geq \omega$ whenever $\nu - \omega \in \mathfrak{M}_*^+$.

An element $P \in \mathfrak{M}$ is called a projection if $P^2 = P = P^*$. If P_1, P_2 are projections, then $P_1 \leq P_2$ stands for $P_1 P_2 = P_1$ and $P_1 \perp P_2$ for $P_1 P_2 = 0$. The support of $\omega \in \mathfrak{M}_*^+$ is the projection s_ω defined by

$$\mathrm{s}_\omega = \inf\{P \in \mathfrak{M} \,|\, P \text{ is a projection and } \omega(1-P) = 0\}.$$

For any $A \in \mathfrak{M}$ and $P \geq \mathrm{s}_\omega$, $\omega(A) = \omega(AP) = \omega(PA)$ and

$$\|\omega\| = \sup_{A \in \mathfrak{M}, \|A\|=1} |\omega(A)| = \omega(1) = \omega(\mathrm{s}_\omega).$$

ω is faithful iff $\mathrm{s}_\omega = 1$.

Any hermitian $\omega \in \mathfrak{M}_*$ has a unique Jordan decomposition $\omega = \omega_+ - \omega_-$ where $\omega_\pm \in \mathfrak{M}_*^+$ and $\mathrm{s}_{\omega_+} \perp \mathrm{s}_{\omega_-}$. In particular, $\|\omega\| = \|\omega_+\| + \|\omega_-\| = \omega_+(1) + \omega_-(1)$ (see [Tak]).

If $\nu, \omega \in \mathfrak{M}_*^+$ and $\mathrm{s}_\nu \leq \mathrm{s}_\omega$ we shall say that ν is normal w.r.t. ω, denoted $\nu \ll \omega$. If $\mathrm{s}_\nu = \mathrm{s}_\omega$ then ν and ω are called equivalent, denoted $\nu \sim \omega$.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$. If $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, then $\mathcal{A}^* = \{A^* \,|\, A \in \mathcal{A}\}$ is the adjoint of $\mathcal{A}$ and $\mathcal{A}' = \{B \in \mathcal{B}(\mathcal{H}) \,|\, BA = AB \text{ for all } A \in \mathcal{A}\}$ denotes its commutant. $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ is a W^*-algebra iff $\mathfrak{M} = \mathcal{A}''$ for some $\mathcal{A} = \mathcal{A}^* \subset \mathcal{B}(\mathcal{H})$. Such a W^*-algebra is called von Neumann algebra.

5.2 Modular structure

A W^*-algebra in standard form is a quadruple $(\mathfrak{M}, \mathcal{H}, J, \mathcal{H}^+)$ where $\mathcal{H}$ is a Hilbert space, $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ is a W^*-algebra, J is an anti-unitary involution on $\mathcal{H}$ and $\mathcal{H}^+$ is a cone in $\mathcal{H}$ such that:

(1) $\mathcal{H}^+$ is self-dual, *i.e.*, $\mathcal{H}^+ = \{\Psi \in \mathcal{H} \,|\, (\Phi|\Psi) \geq 0 \text{ for all } \Phi \in \mathcal{H}^+\}$;

(2) $J\mathfrak{M}J = \mathfrak{M}'$;

(3) $JAJ = A^*$ for $A \in \mathfrak{M} \cap \mathfrak{M}'$;

(4) $J\Psi = \Psi$ for $\Psi \in \mathcal{H}^+$;

(5) $AJA\mathcal{H}^+ \subset \mathcal{H}^+$ for $A \in \mathfrak{M}$.

The quadruple $(\pi, \mathcal{H}, J, \mathcal{H}^+)$ is a standard representation of the W^*-algebra $\mathfrak{M}$ if $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$ is a faithful representation and $(\pi(\mathfrak{M}), \mathcal{H}, J, \mathcal{H}^+)$ is in standard form. A standard representation always exists. Moreover, if $(\pi_1, \mathcal{H}_1, J_1, \mathcal{H}_1^+)$ and $(\pi_2, \mathcal{H}_2, J_2, \mathcal{H}_2^+)$ are two standard representations of $\mathfrak{M}$ then there exists a unique unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(A)U^* = \pi_2(A)$ for all $A \in \mathfrak{M}$, $UJ_1U^* = J_2$, and $U\mathcal{H}_1^+ = \mathcal{H}_2^+$.

In what follows, without loss of generality, we will assume that all W^*-algebras are in standard form. For later reference we recall the following classical result (see, *e.g.*, [St]).

Theorem 5.1 *Let $\mathfrak{M}$ be a W^*-algebra in standard form. For any $\omega \in \mathfrak{M}_*^+$ there exists a unique $\Omega_\omega \in \mathcal{H}^+$ such that*

$$\omega(A) = (\Omega_\omega | A\Omega_\omega),$$

for all $A \in \mathfrak{M}$. The map $\mathfrak{M}_*^+ \ni \omega \mapsto \Omega_\omega \in \mathcal{H}^+$ is a bijection and

$$\|\Omega_\omega - \Omega_\nu\|^2 \leq \|\omega - \nu\| \leq \|\Omega_\omega - \Omega_\nu\| \, \|\Omega_\omega + \Omega_\nu\|. \tag{5.1}$$

Remark 1. The upper bound in (5.1) is trivial to prove and the interesting part is the lower bound. At the end of Section 6.1 we will give a new simple proof of this lower bound.

Remark 2. The support s_ω projects on the closure of $\mathfrak{M}'\Omega_\omega$. It follows that $\mathrm{s}'_\omega = J\mathrm{s}_\omega J \in \mathfrak{M}'$ is the orthogonal projection on the closure of $\mathfrak{M}\Omega_\omega$.

We now recall the definition of Araki's relative modular operator [Ar1, Ar2]. For $\nu, \omega \in \mathfrak{M}_*^+$ define $S_{\nu|\omega}$ on the domain $\mathfrak{M}\Omega_\omega + (\mathfrak{M}\Omega_\omega)^\perp$ by

$$S_{\nu|\omega}(A\Omega_\omega + \Theta) = \mathrm{s}_\omega A^*\Omega_\nu,$$

where $\Theta \in (\mathfrak{M}\Omega_\omega)^\perp$. $S_{\nu|\omega}$ is a densely defined anti-linear operator. It is closable and we denote its closure by the same symbol. The positive operator

$$\Delta_{\nu|\omega} = S^*_{\nu|\omega} S_{\nu|\omega}$$

is called *relative modular operator.* We denote $\Delta_\omega = \Delta_{\omega|\omega}$. The basic properties of the relative modular operator are (see [Ar1, Ar2])

Proposition 5.2 (1) $\Delta_{\lambda_1\nu|\lambda_2\omega} = \frac{\lambda_1}{\lambda_2}\Delta_{\nu|\omega}$, *for any* $\lambda_1, \lambda_2 > 0$.

(2) $\operatorname{Ker}\Delta_{\nu|\omega} = \operatorname{Ker}\mathrm{s}'_\omega\mathrm{s}_\nu$.

(3) $S_{\nu|\omega} = J\Delta^{1/2}_{\nu|\omega}$ *is the polar decomposition of* $S_{\nu|\omega}$.

(4) $\Delta^{1/2}_{\nu|\omega}\Omega_\omega = \Delta^{1/2}_{\nu|\omega}\mathrm{s}_\nu\Omega_\omega = \mathrm{s}'_\omega\Omega_\nu$.

(5) $J\Delta_{\omega|\nu}J\Delta_{\nu|\omega} = \Delta_{\nu|\omega}J\Delta_{\omega|\nu}J = \mathrm{s}'_\omega\mathrm{s}_\nu$.

We note in particular that if ν and ω are faithful, then $\Delta_{\nu|\omega} > 0$, $\Delta^{-1}_{\nu|\omega} = J\Delta_{\omega|\nu}J$ and $\Delta^{1/2}_{\nu|\omega}\Omega_\omega = \Omega_\nu$.

5.3 Relative entropies

For $\nu, \omega \in \mathfrak{M}_*^+$ we shall denote by $\mu_{\omega|\nu}$ the spectral measure for $-\log\Delta_{\omega|\nu}$ and Ω_ν. The relative entropy of ν w.r.t. ω is defined by

$$\operatorname{Ent}(\nu|\omega) = \begin{cases} (\Omega_\nu|\log\Delta_{\omega|\nu}\Omega_\nu) = -\displaystyle\int x\,\mathrm{d}\mu_{\omega|\nu}(x) & \text{if } \nu \ll \omega, \\ -\infty & \text{otherwise.} \end{cases} \tag{5.2}$$

Proposition 5.3 *Suppose that* $\nu \ll \omega$.

(1) *For* $\lambda_1, \lambda_2 > 0$,

$$\operatorname{Ent}(\lambda_1\nu|\lambda_2\omega) = \lambda_1\left(\operatorname{Ent}(\nu|\omega) + \nu(1)\log\frac{\lambda_2}{\lambda_1}\right).$$

(2)

$$\operatorname{Ent}(\nu|\omega) \le \nu(1)\log\frac{\omega(\mathrm{s}_\nu)}{\nu(1)}.$$

In particular, if $\omega(\mathrm{s}_\nu) \le \nu(1)$, *then* $\operatorname{Ent}(\nu|\omega) \le 0$.

(3) *If* $\mathfrak{N} \subset \mathfrak{M}$ *is a* W^**-subalgebra then*

$$\operatorname{Ent}(\nu|\omega) \le \operatorname{Ent}(\nu_{|\mathfrak{N}}|\omega_{|\mathfrak{N}}).$$

Part (1) follows from Proposition 5.2 (1). The proof of (2) in the special case $\omega(\mathrm{s}_\nu) = \nu(1)$ follows from the inequality $\log x \leq x - 1$. The general case is obtained by applying Part (1). Part (3) is the celebrated monotonicity of the relative entropy [LiR, Uh]. Modern proofs of (3) can be found in [OP] or [DJP].

Let $\nu, \omega \in \mathfrak{M}_*^+$. Rényi's relative entropy of ν w.r.t. ω is defined by

$$\mathrm{Ent}_s(\nu|\omega) = \log(\Omega_\omega|\Delta^s_{\nu|\omega}\Omega_\omega) = \log \int \mathrm{e}^{-sx}\mathrm{d}\mu_{\nu|\omega}(x).$$

We list below some of its properties. Their proof is simple and can be found in [JOPP2].

Proposition 5.4 (1) *If $\nu \ll \omega$ then the function $\mathbb{R} \ni s \mapsto \mathrm{Ent}_s(\nu|\omega) \in]-\infty, \infty]$ is convex. It is finite and continuous on $[0,1]$ and real analytic on $]0,1[$.*

In the remaining statements we assume that $\nu \sim \omega$.

(2) $\mathrm{Ent}_s(\nu|\omega) = \mathrm{Ent}_{1-s}(\omega|\nu)$.

(3) $\mathrm{Ent}_s(\nu|\omega) \geq s\,\mathrm{Ent}(\omega|\nu)$.

(4) $\lim\limits_{s\uparrow 1} \dfrac{1}{s-1}\mathrm{Ent}_s(\nu|\omega) = -\mathrm{Ent}(\nu|\omega)$.

We finish with some estimates that will be used in the proof of the quantum Chernoff bound.

Proposition 5.5 (1) *Let $\nu_1, \nu_2, \omega \in \mathfrak{M}_*^+$ and $s \in [0,1]$. If $\nu_2 \leq \nu_1$, then $\mathrm{Dom}\,(\Delta^{s/2}_{\nu_1|\omega}) \subset \mathrm{Dom}\,(\Delta^{s/2}_{\nu_2|\omega})$ and*

$$\|\Delta^{s/2}_{\nu_2|\omega}\Psi\| \leq \|\Delta^{s/2}_{\nu_1|\omega}\Psi\|,$$

for any $\Psi \in \mathrm{Dom}(\Delta^{s/2}_{\nu_1|\omega})$.

(2) *Let $\nu, \omega_1, \omega_2 \in \mathfrak{M}_*^+$ and $s \in [0,1]$. If $\omega_2 \leq \omega_1$, then*

$$\|\Delta^{s/2}_{\nu|\omega_2}A\Omega_{\omega_2}\| \leq \|\Delta^{s/2}_{\nu|\omega_1}A\Omega_{\omega_1}\|,$$

for any $A \in \mathfrak{M}$.

(3) *Let $\nu_1, \nu_2, \omega_1, \omega_2 \in \mathfrak{M}_*^+$ and $s \in [0,1]$. If $\nu_2 \leq \nu_1$ and $\omega_2 \leq \omega_1$, then*

$$\|\Delta^{s/2}_{\nu_2|\omega_2}A\Omega_{\omega_2}\| \leq \|\Delta^{s/2}_{\nu_1|\omega_1}A\Omega_{\omega_1}\|,$$

for any $A \in \mathfrak{M}$.

(4) *Let $\nu_1, \nu_2, \omega_1, \omega_2 \in \mathfrak{M}_*^+$ be faithful and $s \in [0,1]$. If $\nu_2 \leq \nu_1$ and $\omega_2 \leq \omega_1$, then*

$$(\Omega_{\omega_2}|(\Delta^s_{\nu_1|\omega_2} - \Delta^s_{\nu_2|\omega_2})\Omega_{\omega_2}) \leq (\Omega_{\omega_1}|(\Delta^s_{\nu_1|\omega_1} - \Delta^s_{\nu_2|\omega_1})\Omega_{\omega_1}). \tag{5.3}$$

Remark 1. Part (4) was recently established in [Og3] with a proof based on Connes Radon-Nikodym cocycles. We will give below an alternative proof which emphasizes the connection of the above estimate with Araki-Masuda's theory of non-commutative L^p-spaces [AM].

Remark 2. By an additional approximation argument the assumption that ν_i, ω_i are faithful can be removed, see [Og3] for details.

Remark 3. In the special case $s = 1/2$, Part (4) follows easily from well-known properties of the cone $\mathcal{H}^+$. Indeed, for $s = 1/2$, the inequality (5.3) is equivalent to the inequality

$$(\Omega_{\omega_1} - \Omega_{\omega_2} | \Omega_{\nu_1} - \Omega_{\nu_2}) \geq 0. \tag{5.4}$$

By the ordering property of the cone $\mathcal{H}^+$, $\Omega_{\omega_1} - \Omega_{\omega_2}, \Omega_{\nu_1} - \Omega_{\nu_2} \in \mathcal{H}^+$ and, since $\mathcal{H}^+$ is a self-dual cone, (5.4) follows.

The rest of this section is devoted to the proof of Proposition 5.5. We start with some preliminaries.

Lemma 5.6 *Let $\phi, \nu, \omega \in \mathfrak{M}_*^+$ and suppose that ν and ω are faithful. Then*

$$J\Delta_{\nu|\omega}^{\frac{1}{2}-s} A \Delta_{\phi|\omega}^{s} \Omega_\omega = \Delta_{\phi|\nu}^{s} A^* \Omega_\nu$$

holds for $s \in [0, 1/2]$ and $A \in \mathfrak{M}$.

Proof. We just sketch the well-known argument (see, *e.g.*, [AM]). We recall that for $\phi_1, \phi_2 \in \mathfrak{M}_*^+$ the Araki-Connes cocycle is defined by

$$[D\phi_1 : D\phi_2]_t = \Delta_{\phi_1|\phi_2}^{\mathrm{i}t} \Delta_{\phi_2}^{-\mathrm{i}t}.$$

For a detailed discussion of this class of operators we refer the reader to Appendices B and C of [AM]. We only need that the cocycles are elements of $\mathfrak{M}$ and that the chain rule

$$[D\phi_1 : D\phi_2]_t [D\phi_2 : D\phi_3]_t = [D\phi_1 : D\phi_3]_t$$

holds whenever $\mathrm{s}_{\phi_2} \geq \mathrm{s}_{\phi_1}$ or $\mathrm{s}_{\phi_2} \geq \mathrm{s}_{\phi_3}$. For t real, one has

$$\begin{aligned}
J\Delta_{\nu|\omega}^{\frac{1}{2}-\mathrm{i}t} A \Delta_{\phi|\omega}^{\mathrm{i}t} \Omega_\omega &= \Delta_{\omega|\nu}^{-\mathrm{i}t} J \Delta_{\nu|\omega}^{\frac{1}{2}} A [D\phi : D\omega]_t \Omega_\omega \\
&= \Delta_{\omega|\nu}^{-\mathrm{i}t} [D\phi : D\omega]_t^* A^* \Omega_\nu \\
&= \Delta_{\nu}^{-\mathrm{i}t} [D\omega : D\nu]_t^* [D\phi : D\omega]_t^* A^* \Omega_\nu \\
&= \Delta_{\nu}^{-\mathrm{i}t} ([D\phi : D\omega]_t [D\omega : D\nu]_t)^* A^* \Omega_\nu \\
&= \Delta_{\nu}^{-\mathrm{i}t} [D\phi : D\nu]_t^* A^* \Omega_\nu \\
&= \Delta_{\phi|\nu}^{-\mathrm{i}t} A^* \Omega_\nu.
\end{aligned}$$

For $B \in \mathfrak{M}$, the functions

$$f(z) = (A\Delta_{\phi|\omega}^{\overline{z}} \Omega_\omega | \Delta_{\nu|\omega}^{\frac{1}{2}-z} B \Omega_\omega), \qquad g(z) = (JB\Omega_\omega | \Delta_{\phi|\nu}^{z} A^* \Omega_\nu)$$

are analytic in the open strip $0 < \operatorname{Re} z < \frac{1}{2}$ and bounded and continuous on its closure. Since, for t real,

$$f(-\mathrm{i}t) = (JB\Omega_\omega | J\Delta_{\nu|\omega}^{\frac{1}{2}-\mathrm{i}t} A \Delta_{\phi|\omega}^{\mathrm{i}t} \Omega_\omega), \qquad g(-\mathrm{i}t) = (JB\Omega_\omega | \Delta_{\phi|\nu}^{-\mathrm{i}t} A^* \Omega_\nu),$$

one concludes from the above calculation that $f(z) = g(z)$ holds in the closed strip $0 \leq \operatorname{Re} z \leq \frac{1}{2}$. In particular

$$(A\Delta_{\phi|\omega}^{s} \Omega_\omega | \Delta_{\nu|\omega}^{\frac{1}{2}-s} B \Omega_\omega) = (JB\Omega_\omega | \Delta_{\phi|\nu}^{s} A^* \Omega_\nu),$$

for $s \in [0, 1/2]$. Since $\mathfrak{M}\Omega_\omega$ is a core for $\Delta_{\nu|\omega}^{\frac{1}{2}-s}$, we conclude that $A\Delta_{\phi|\omega}^{s}\Omega_\omega$ is in the domain of $\Delta_{\nu|\omega}^{\frac{1}{2}-s}$ and that

$$(JB\Omega_\omega | J\Delta_{\nu|\omega}^{\frac{1}{2}-s} A\Delta_{\phi|\omega}^{s}\Omega_\omega) = (\Delta_{\nu|\omega}^{\frac{1}{2}-s} A\Delta_{\phi|\omega}^{s}\Omega_\omega | B\Omega_\omega) = (JB\Omega_\omega | \Delta_{\phi|\nu}^{s} A^*\Omega_\nu),$$

from which the result follows. □

Lemma 5.7 *Let $\nu, \omega \in \mathfrak{M}_*^+$ and suppose that ω is faithful. Let $U \in \mathfrak{M}$ be a partial isometry such that $U^*U = \mathrm{s}_\nu$ and let $\nu_U(\,\cdot\,) = \nu(U^* \cdot U)$. Then*

$$\Delta_{\nu_U|\omega} = U\Delta_{\nu|\omega}U^*.$$

Proof. The vector representative of ν_U in $\mathcal{H}^+$ is $UJUJ\Omega_\nu$. Hence, for $A \in \mathfrak{M}$,

$$J\Delta_{\nu_U|\omega}^{\frac{1}{2}} A\Omega_\omega = A^*UJUJ\Omega_\nu = JUJA^*U\Omega_\nu = JU\Delta_{\nu|\omega}^{\frac{1}{2}} U^*A\Omega_\omega.$$

□

Proof of Proposition 5.5. Part (1) is well known (see Lemma C.3 in [AM]). To prove Part (2), we use Proposition 5.2 (2) and (5) to obtain

$$\Delta_{\nu|\omega_i}^{s/2} A\Omega_{\omega_i} = \Delta_{\nu|\omega_i}^{s/2} \mathrm{s}_\nu A\Omega_{\omega_i} = \Delta_{\nu|\omega_i}^{s/2} J\Delta_{\omega_i|\nu}^{1/2} A^*\Omega_\nu = J\Delta_{\omega_i|\nu}^{1/2-s/2} A^*\Omega_\nu,$$

and the statement follows from Part (1). One proves Part (3) by successive applications of Parts (1) and (2).

To prove part (4), we first notice that it suffices to prove the statement for $s \in [0, 1/2]$. Indeed, if this case is established, then the case $s \in [1/2, 1]$ follows by exchanging the roles of the ν and ω and using the identities

$$\|\Delta_{\omega_i|\nu_j}^{s/2}\Omega_{\nu_j}\| = \|\Delta_{\nu_j|\omega_i}^{(1-s)/2}\Omega_{\omega_i}\|.$$

One can further assume that $s \in]0, 1/2[$ (the case $s = 0$ is trivial and the case $s = 1/2$ follows from $s < 1/2$ by continuity).

Consider the Araki-Masuda space $L^p(\mathfrak{M}, \omega_i)$ (constructed w.r.t. the reference vector Ω_{ω_i}) for $p = 1/s \in]2, \infty[$. Let

$$\mathcal{H}_{\Omega_{\omega_i}}^{\alpha} = \{\Delta_{\omega_i}^{\alpha}\mathfrak{M}_+\Omega_{\omega_i}\}^{\mathrm{cl}}$$

be the usual α-cone in $\mathcal{H}$ (note that $\mathcal{H}_{\Omega_{\omega_i}}^{1/4} = \mathcal{H}^+$). The positive part of $L^p(\mathfrak{M}, \omega_i)$ is defined by

$$L_+^p(\mathfrak{M}, \omega_i) = L^p(\mathfrak{M}, \omega_i) \cap \mathcal{H}_{\Omega_{\omega_i}}^{1/2p},$$

and by the Lemma 4.3 in [AM],

$$L_+^p(\mathfrak{M}, \omega_i) = \{\Delta_{\nu|\omega_i}^{1/p}\Omega_{\omega_i} \mid \nu \in \mathfrak{M}_*^+\}.$$

The polar decomposition in $L^p(\mathfrak{M}, \omega_i)$ (Theorem 3 in [AM]) implies there exist unique $\phi_i \in \mathfrak{M}_*^+$ and unique partial isometries $U_i \in \mathfrak{M}$ satisfying $U_i^*U_i = \mathrm{s}_{\phi_i}$ such that

$$\Delta_{\nu_1|\omega_i}^{s}\Omega_{\omega_i} - \Delta_{\nu_2|\omega_i}^{s}\Omega_{\omega_i} = U_i\Delta_{\phi_i|\omega_i}^{s}\Omega_{\omega_i}. \tag{5.5}$$

After applying $J\Delta_{\omega_i}^{1/2-s}$ to both sides of this identity we deduce from Lemmata 5.6 and 5.7 that

$$\Delta_{\nu_1|\omega_i}^{s}\Omega_{\omega_i} - \Delta_{\nu_2|\omega_i}^{s}\Omega_{\omega_i} = U_i^*\Delta_{\phi_i\,U_i|\omega_i}^{s}\Omega_{\omega_i}.$$

By the uniqueness of the polar decomposition, $U_i^* = U_i$ and $\phi_{i\,U_i} = \phi_i$. It follows that U_i is a self-adjoint partial isometry and so its spectrum is contained in the set $\{-1, 0, 1\}$. Hence, $U_i = P_i^+ - P_i^-$ and $\mathrm{s}_{\phi_i} = P_i^+ + P_i^-$, where $P_i^\pm$ are the spectral projections corresponding to the eigenvalues ± 1. It then follows from (5.5) that

$$P_i^- \Delta^s_{\nu_1|\omega_i} \Omega_{\omega_i} - P_i^- \Delta^s_{\nu_2|\omega_i} \Omega_{\omega_i} = -P_i^- \Delta^s_{\phi_i|\omega_i} \Omega_{\omega_i}.$$

Again applying $J\Delta_{\omega_i}^{1/2-s}$ to both sides we get

$$\Delta^s_{\nu_1|\omega_i} P_i^- \Omega_{\omega_i} - \Delta^s_{\nu_2|\omega_i} P_i^- \Omega_{\omega_i} = -\Delta^s_{\phi_i|\omega_i} P_i^- \Omega_{\omega_i},$$

and so

$$\|\Delta^{s/2}_{\nu_1|\omega_i} P_i^- \Omega_{\omega_i}\|^2 - \|\Delta^{s/2}_{\nu_2|\omega_i} P_i^- \Omega_{\omega_i}\|^2 = -\|\Delta^{s/2}_{\phi_i|\omega_i} P_i^- \Omega_{\omega_i}\|^2 \leq 0.$$

Since $\nu_1 \geq \nu_2$, Proposition 5.5 (1) implies that the left hand side is positive and we deduce

$$\Delta^{s/2}_{\phi_i|\omega_i} P_i^- \Omega_{\omega_i} = 0.$$

Since $J\Delta_{\omega_i}^{1/2-s} P_i^- \Delta^s_{\phi_i|\omega_i} \Omega_{\omega_i} = \Delta^s_{\phi_i|\omega_i} P_i^- \Omega_{\omega_i} = 0$, we obtain $P_i^- \Delta^s_{\phi_i|\omega_i} \Omega_{\omega_i} = 0$ and therefore

$$\Delta^s_{\phi_i|\omega_i} \Omega_{\omega_i} = \mathrm{s}_{\phi_i} \Delta^s_{\phi_i|\omega_i} \Omega_{\omega_i} = P_i^+ \Delta^s_{\phi_i|\omega_i} \Omega_{\omega_i} = U_i \Delta^s_{\phi_i|\omega_i} \Omega_{\omega_i}.$$

Hence, Equ. (5.5) becomes

$$\Delta^s_{\nu_1|\omega_i} \Omega_{\omega_i} - \Delta^s_{\nu_2|\omega_i} \Omega_{\omega_i} = \Delta^s_{\phi_i|\omega_i} \Omega_{\omega_i}. \tag{5.6}$$

Acting on both sides of the above relation with $J\Delta_{\omega_1|\omega_2}^{1/2-s}$ and applying Lemma 5.6 we get

$$\Delta^s_{\nu_1|\omega_1} \Omega_{\omega_1} - \Delta^s_{\nu_2|\omega_1} \Omega_{\omega_1} = \Delta^s_{\phi_2|\omega_1} \Omega_{\omega_1}.$$

Comparing the last relation with the Equ. (5.6) for $i = 1$ yields

$$\Delta^s_{\phi_1|\omega_1} \Omega_{\omega_1} = \Delta^s_{\phi_2|\omega_1} \Omega_{\omega_1},$$

and so, by the uniqueness of the polar decomposition, $\phi_1 = \phi_2 = \phi$. Finally, Proposition 5.5 (2) allows us to conclude

$$(\Omega_{\omega_2}|(\Delta^s_{\nu_1|\omega_2} - \Delta^s_{\nu_2|\omega_2})\Omega_{\omega_2}) = (\Omega_{\omega_2}|\Delta^s_{\phi|\omega_2}\Omega_{\omega_2}) \leq (\Omega_{\omega_1}|\Delta^s_{\phi|\omega_1}\Omega_{\omega_1}) = (\Omega_{\omega_1}|(\Delta^s_{\nu_1|\omega_1} - \Delta^s_{\nu_2|\omega_1})\Omega_{\omega_1}).$$

□

For additional information about quantum relative entropies we refer the reader to [OP].

5.4 Classical systems

Let $(M, \mathcal{F}, P)$ be a probability space, where M is a set, $\mathcal{F}$ a σ-algebra in M, and P a probability measure on $(M, \mathcal{F})$. The classical probabilistic setup fits into the algebraic framework as follows. Let $\mathcal{M}$ be the vector space of all complex measures on $(M, \mathcal{F})$ which are absolutely continuous w.r.t. P. If $\|\nu\|$ is the total variation of $\nu \in \mathcal{M}$, then $\|\cdot\|$ is a norm on $\mathcal{M}$ and $\mathcal{M}$ is a Banach space isomorphic to $L^1(M, \mathrm{d}P)$. $\mathfrak{M} = L^\infty(M, \mathrm{d}P) = \mathcal{M}^*$, and the standard representation $(\pi, \mathcal{H}, J, \mathcal{H}^+)$ is identified as $\mathcal{H} = L^2(M, \mathrm{d}P)$, $\pi(f)g = fg$, $J(f) = \overline{f}$, and $\mathcal{H}^+ = \{f \in \mathcal{H} \,|\, f \geq 0\}$. $\mathfrak{M}_* = \mathcal{M}$ ($\nu(f) = \int f \mathrm{d}\nu$), $\mathfrak{M}_*^+$ consists of positive measures in $\mathcal{M}$ and for $\omega \in \mathfrak{M}_*^+$,

$$\Omega_\omega = \left(\frac{\mathrm{d}\omega}{\mathrm{d}P}\right)^{1/2}.$$

Let $\nu, \omega \in \mathfrak{M}_*^+$ and denote by $\nu = \nu_{\omega,\mathrm{ac}} + \nu_{\omega,\mathrm{sing}}$, with $\nu_{\omega,\mathrm{ac}} \ll \omega$ and $\nu_{\omega,\mathrm{sing}} \perp \omega$, the Lebesgue decomposition of ν w.r.t. ω. Then

$$\Delta_{\nu|\omega}(f) = \frac{\mathrm{d}\nu_{\omega,\mathrm{ac}}}{\mathrm{d}\omega} f.$$

In particular, $\Delta_{\omega|\omega} = 1$ and $\log \Delta_{\omega|\omega} = 0$. If $\nu \ll \omega$ then the relative entropy and the Rényi relative entropy take the familiar forms

$$\mathrm{Ent}(\nu|\omega) = -\int_M \log \frac{\mathrm{d}\nu}{\mathrm{d}\omega} \mathrm{d}\nu = -\int_M \frac{\mathrm{d}\nu}{\mathrm{d}\omega} \log \frac{\mathrm{d}\nu}{\mathrm{d}\omega} \mathrm{d}\omega, \qquad \mathrm{Ent}_s(\nu|\omega) = \log \int_M \left(\frac{\mathrm{d}\nu}{\mathrm{d}\omega}\right)^s \mathrm{d}\omega.$$

5.5 Finite systems

Let $\mathcal{K}$ be a finite dimensional Hilbert space and $\mathfrak{M} = \mathcal{O}_\mathcal{K}$, the $*$-algebra of all linear operators on $\mathcal{K}$ equipped with the usual operator norm. $\mathfrak{M}^* = \mathfrak{M}_*$ is identified with $\mathcal{O}_\mathcal{K}$ equipped with the trace norm $\|\omega\| = \mathrm{Tr}|\omega|$, and $\mathfrak{M}_*^+$ identified with positive linear operators on $\mathcal{K}$.

The standard representation of $\mathfrak{M}$ is constructed as follows. $\mathcal{H} = \mathcal{O}_\mathcal{K}$ equipped with the inner product $(A|B) = \mathrm{Tr}\, A^*B$, $\pi(A)B = AB$, $J(A) = A^*$, and $\mathcal{H}^+ = \{A \in O_\mathcal{K} \mid A \geq 0\}$. If $\nu, \omega \in \mathfrak{M}_*^+$ then $\Omega_\omega = \omega^{1/2}$ and

$$\Delta_{\nu|\omega}(A) = \nu A \omega^{-1} \mathrm{s}_\omega.$$

If in addition $\nu \ll \omega$ we recover the formulas

$$\mathrm{Ent}_s(\nu|\omega) = \log \mathrm{Tr}\, \nu^s \omega^{1-s} \mathrm{s}_\omega, \qquad \mathrm{Ent}(\nu|\omega) = \mathrm{Tr}\, \nu(\log \omega - \log \nu).$$

6 Hypothesis testing in W^*-algebras

6.1 Preliminaries

Let $\mathcal{Q}$ be a quantum system described by a W^*-algebra $\mathfrak{M}$ in standard form on a Hilbert space $\mathcal{H}$. Let (ν, ω) be a pair of faithful normal states on $\mathfrak{M}$ and suppose that one of the following two hypotheses holds:

Hypothesis I : $\mathcal{Q}$ is in the state ω;

Hypothesis II : $\mathcal{Q}$ is in the state ν.

A priori, Hypothesis I is realized with probability $p \in]0, 1[$ and hypothesis II with probability $1 - p$. A *test* is a projection $T \in \mathfrak{M}$ and the result of a measurement of the corresponding observable is a number in $\mathrm{sp}(T) = \{0, 1\}$. If the outcome is 1, one accepts hypothesis I, otherwise one accepts hypothesis II. The total error probability is

$$(1 - p)\nu(T) + p\,\omega(1 - T).$$

We again absorb the scalar factors into the functionals and consider the quantities

$$D(\nu, \omega, T) = \nu(T) + \omega(1 - T)$$

and

$$D(\nu, \omega) = \inf_T D(\nu, \omega, T)$$

for a pair of faithful normal functionals $\nu, \omega \in \mathfrak{M}_*^+$.

Theorem 6.1 (1)

$$D(\nu,\omega) = D(\nu,\omega,\mathrm{s}_{(\omega-\nu)_+}) = \frac{1}{2}(\omega(1)+\nu(1)-\|\omega-\nu\|).$$

(2) *For* $s\in[0,1]$,

$$D(\nu,\omega) \le (\Omega_\omega|\Delta^s_{\nu|\omega}\Omega_\omega) = \int \mathrm{e}^{-sx}\mathrm{d}\mu_{\nu|\omega}(x). \tag{6.1}$$

(3)

$$D(\nu,\omega) \ge (\Omega_\omega|\Delta_{\nu|\omega}(1+\Delta_{\nu|\omega})^{-1}\Omega_\omega) = \int \frac{\mathrm{d}\mu_{\nu|\omega}(x)}{1+\mathrm{e}^x}. \tag{6.2}$$

Proof. (1) For any test T, one has

$$\begin{aligned}\nu(T)+\omega(1-T) &= \omega(1)-(\omega-\nu)(T) \ge \omega(1)-(\omega-\nu)_+(T)\\ &\ge \omega(1)-(\omega-\nu)_+(1) = \frac{1}{2}(\omega(1)+\nu(1)-\|\omega-\nu\|).\end{aligned}$$

On the other hand,

$$D(\nu,\omega,\mathrm{s}_{(\omega-\nu)_+}) = \omega(1)-(\omega-\nu)_+(1) = \frac{1}{2}(\omega(1)+\nu(1)-\|\omega-\nu\|),$$

and (1) follows.

(2) Writing the Jordan decomposition of $\phi=\omega-\nu$ as $\phi=\phi_+-\phi_-$, and using

$$2\phi_+(1) = \|\phi\|+\phi(1),$$

we rewrite Inequality (6.1) in the equivalent form

$$(\Omega_\omega|(\Delta^s_{\omega|\omega}-\Delta^s_{\nu|\omega})\Omega_\omega) \le \phi_+(1). \tag{6.3}$$

Now, since $\omega=\nu+\phi\le\nu+\phi_+$ and $\nu\le\nu+\phi_+$, applying successively Proposition 5.5 (1), (4) and (2) yields

$$\begin{aligned}(\Omega_\omega|(\Delta^s_{\omega|\omega}-\Delta^s_{\nu|\omega})\Omega_\omega) &\le (\Omega_\omega|(\Delta^s_{\nu+\phi_+|\omega}-\Delta^s_{\nu|\omega})\Omega_\omega)\\ &\le (\Omega_{\nu+\phi_+}|(\Delta^s_{\nu+\phi_+|\nu+\phi_+}-\Delta^s_{\nu|\nu+\phi_+})\Omega_{\nu+\phi_+})\\ &\le (\Omega_{\nu+\phi_+}|\Delta^s_{\nu+\phi_+|\nu+\phi_+}\Omega_{\nu+\phi_+})-(\Omega_\nu|\Delta^s_{\nu|\nu}\Omega_\nu)\\ &= (\nu+\phi_+)(1)-\nu(1) = \phi_+(1).\end{aligned}$$

(3) We start with an observation. Let S and $P\ge 0$ be bounded self-adjoint operators on $\mathcal{H}$ and $\lambda\ge 0$. Then

$$(1-S)P(1-S)+\lambda SPS = \frac{\lambda}{1+\lambda}P+\frac{1}{1+\lambda}(1-(1+\lambda)S)P(1-(1+\lambda)S) \ge \frac{\lambda}{1+\lambda}P. \tag{6.4}$$

Set $S=\mathrm{s}_{\phi_+}$. Since

$$\nu(S) = (S\Omega_\nu|S\Omega_\nu) = (\Delta^{1/2}_{\nu|\omega}S\Omega_\omega|\Delta^{1/2}_{\nu|\omega}S\Omega_\omega),$$

we have

$$D(\nu,\omega) = \nu(S)+\omega(1-S) = (\Omega_\omega|(1-S+S\Delta_{\nu|\omega}S)\Omega_\omega).$$

Let $P(\cdot)$ be the spectral resolution of $\Delta_{\nu|\omega}$. Then,

$$D(\nu,\omega) = \int_0^\infty (\Omega_\omega \,|[(1-S)\mathrm{d}P(\lambda)(1-S) + \lambda S\mathrm{d}P(\lambda)S]\,\Omega_\omega)$$

$$\geq \int_0^\infty \frac{\lambda}{1+\lambda}\,(\Omega_\omega\,|\mathrm{d}P(\lambda)\,\Omega_\omega) = (\Omega_\omega|\Delta_{\nu|\omega}(1+\Delta_{\nu|\omega})^{-1}\Omega_\omega),$$

where in the second step we used the estimate (6.4). □

We finish this section with some remarks.

Remark 1. Using the scaling law of Proposition 5.2 (1) and the easy fact that $\Omega_{\mathrm{e}^\theta\omega} = \mathrm{e}^{\theta/2}\Omega_\omega$, one deduces from Part (3) of Proposition 6.1 that

$$D(\nu,\mathrm{e}^\theta\omega) \geq \int \frac{\mathrm{e}^\theta \mathrm{d}\mu_{\nu|\omega}(x)}{1+\mathrm{e}^{x+\theta}},$$

for any $\nu,\omega \in \mathfrak{M}_*^+$ and $\theta \in \mathbb{R}$. The Chebyshev inequality yields that, for any $\vartheta \in \mathbb{R}$,

$$D(\nu,\mathrm{e}^\theta\omega) \geq \frac{\mathrm{e}^\theta}{1+\mathrm{e}^{\vartheta+\theta}}\,\mu_{\nu|\omega}(]-\infty,\vartheta]). \tag{6.5}$$

In particular, for $\theta = \vartheta = 0$, one has

$$D(\nu,\omega) \geq \frac{1}{2}\mu_{\nu|\omega}(]-\infty,0]). \tag{6.6}$$

Remark 2. As we have already mentioned, the essential difference between the classical and the non-commutative setting lies in the proof of Parts (2) and (3) of Theorem 6.1. In the classical setup these results are elementary and are proven as follows. Suppose that ν and ω are two equivalent measures in $\mathfrak{M}_*^+$. Then by Theorem 6.1 (1) (which is a simple result with the same proof in the classical and the non-commutative cases),

$$D(\nu,\omega) = \frac{1}{2}(\omega(1)+\nu(1)-\|\omega-\nu\|) = \int \min\left(1,\frac{\mathrm{d}\nu}{\mathrm{d}\omega}\right)\mathrm{d}\omega. \tag{6.7}$$

One easily deduces the classical version of the lower bound (6.2)

$$D(\nu,\omega) \geq \int \frac{\mathrm{d}\mu_{\nu|\omega}(x)}{1+\mathrm{e}^x},$$

where $\mu_{\nu|\omega}$ is the probability distribution of the random variable $-\log \mathrm{d}\nu/\mathrm{d}\omega$ w.r.t. ω. Note however that the lower bound

$$D(\nu|\omega) \geq \mu_{\nu|\omega}(]-\infty,0]),$$

which also follows directly from (6.7), yields a stronger version of the estimate (6.6). In fact, this stronger bound does not hold in the quantum case, it is easy to find a 2×2 matrix example violating this stronger inequality.

Since for $s \in [0,1]$ and $a,b \geq 0$, $\min(a,b) \leq a^{1-s}b^s$, (6.7) also implies (2) in the classical case. (This argument easily generalizes to the case where ν and ω are not necessarily equivalent measures).

Remark 3. After (2) and (3) of Theorem 6.1 are established, there is no essential difference in the study of the mathematical structure of hypothesis testing in the classical and the non-commutative setting.

A similar remark applies to approximately finite hypothesis testing treated in Section 3. Proposition 3.2 (2) has a very similar proof to Theorem 6.1 (3). However, the proof of Proposition 3.2 (1) is much simpler than the proof of Theorem 6.1 (2). Once this result is established, however, there is very little difference between proofs in the approximately finite setting and the general setting.

Remark 4. Theorem 6.1 (2) in the special case $s = 1/2$ is equivalent to the estimate

$$\|\Omega_\omega - \Omega_\nu\|^2 \leq \|\omega - \nu\|, \tag{6.8}$$

which is the non-trivial part of the Araki estimate (5.1). The proof of Theorem 6.1 (2) is actually much simpler in this special case (see Remark 3 after Proposition 5.5) and hence yields a novel simple proof of the classical estimate (6.8).

6.2 Error exponents

Let $\mathcal{I} \subset \mathbb{R}_+$ be an unbounded index set and for each $t \in \mathcal{I}$ let $\mathfrak{M}_t$ be a W^*-algebra in standard form on the Hilbert space $\mathcal{H}_t$, and (ν_t, ω_t) a pair of faithful normal functionals on $\mathfrak{M}_t$. Let $\mathcal{I} \ni t \mapsto w_t > 0$ be a weight function such that $\lim_{t\to\infty} w_t = \infty$.

The associated Chernoff type error exponents are defined by

$$\overline{D} = \limsup_{t\to\infty} \frac{1}{w_t} \log D(\nu_t, \omega_t), \qquad \underline{D} = \liminf_{t\to\infty} \frac{1}{w_t} \log D(\nu_t, \omega_t).$$

An immediate consequence of Theorem 6.1 (2) is

Proposition 6.2

$$\overline{D} \leq \inf_{s\in[0,1]} \limsup_{t\to\infty} \frac{1}{w_t} \mathrm{Ent}_s(\nu_t|\omega_t).$$

Suppose that ν_t, ω_t are states. For $r \in \mathbb{R}$ the Hoeffding type exponents are defined by

$$\overline{B}(r) = \inf_{\{T_t\}} \left\{ \limsup_{t\to\infty} \frac{1}{w_t} \log \omega_t(1 - T_t) \;\middle|\; \limsup_{t\to\infty} \frac{1}{w_t} \log \nu_t(T_t) < -r \right\},$$

$$\underline{B}(r) = \inf_{\{T_t\}} \left\{ \liminf_{t\to\infty} \frac{1}{w_t} \log \omega_t(1 - T_t) \;\middle|\; \limsup_{t\to\infty} \frac{1}{w_t} \log \nu_t(T_t) < -r \right\},$$

$$B(r) = \inf_{\{T_t\}} \left\{ \lim_{t\to\infty} \frac{1}{w_t} \log \omega_t(1 - T_t) \;\middle|\; \limsup_{t\to\infty} \frac{1}{w_t} \log \nu_t(T_t) < -r \right\},$$

where in the last case the infimum is taken over all families of tests $\{T_t\}_{t\in\mathcal{I}}$ for which $\lim_{t\to\infty} \frac{1}{w_t} \log \omega_t(1 - T_t)$ exists.

Proposition 6.3 (1) *The Hoeffding exponents are increasing functions of r, and $\underline{B}(r) \leq \overline{B}(r) \leq B(r) \leq 0$.*

(2) $\underline{B}(r) = \overline{B}(r) = B(r) = -\infty$ *if* $r < 0$.

(3) *The functions $\underline{B}(r), \overline{B}(r), B(r)$ are upper-semicontinuous and right continuous.*

Proof. The only part that requires a proof is (3). We will establish (3) for $\underline{B}(r)$, the other cases are identical. For a given family of tests $\{T_t\}$ set

$$h_{\{T_t\}}(r) = \begin{cases} \liminf_{t\to\infty} \frac{1}{w_t} \log \omega_t(1 - T_t) & \text{if } \limsup_{t\to\infty} \frac{1}{w_t} \log \nu_t(T_t) < -r \\ \infty & \text{otherwise.} \end{cases}$$

Since the function $r \mapsto h_{\{T_t\}}(r)$ is upper-semicontinuous so is

$$\underline{B}(r) = \inf_{\{T_t\}} h_{\{T_t\}}(r).$$

Since $\underline{B}(r)$ is increasing and upper-semicontinuous, it is also right continuous. □

Suppose that ν_t, ω_t are states. For $\epsilon \in]0,1[$ set

$$\begin{aligned}
\overline{B}_\epsilon &= \inf_{\{T_t\}} \left\{ \limsup_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t) \;\middle|\; \nu_t(T_t) \le \epsilon \right\}, \\
\underline{B}_\epsilon &= \inf_{\{T_t\}} \left\{ \liminf_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t) \;\middle|\; \nu_t(T_t) \le \epsilon \right\}, \\
B_\epsilon &= \inf_{\{T_t\}} \left\{ \lim_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t) \;\middle|\; \nu_t(T_t) \le \epsilon \right\},
\end{aligned} \tag{6.9}$$

where in the last case the infimum is taken over all families of tests $\{T_t\}_{t\in\mathcal{I}}$ for which $\lim_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t)$ exists. Note that if

$$\beta_t(\epsilon) = \inf_{T:\nu(T)\le\epsilon} \omega_t(1-T),$$

then

$$\liminf_{t\to\infty} \frac{1}{w_t} \log \beta_t(\epsilon) = \underline{B}_\epsilon, \qquad \limsup_{t\to\infty} \frac{1}{w_t} \log \beta_t(\epsilon) = \overline{B}_\epsilon.$$

If $\underline{B}_\epsilon = \overline{B}_\epsilon$ for some ϵ, then also

$$\lim_{t\to\infty} \frac{1}{w_t} \log \beta_t(\epsilon) = B_\epsilon.$$

We also define

$$\begin{aligned}
\overline{B} &= \inf_{\{T_t\}} \left\{ \limsup_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t) \;\middle|\; \lim_{t\to\infty} \nu_t(T_t) = 0 \right\}, \\
\underline{B} &= \inf_{\{T_t\}} \left\{ \liminf_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t) \;\middle|\; \lim_{t\to\infty} \nu_t(T_t) = 0 \right\}, \\
B &= \inf_{\{T_t\}} \left\{ \lim_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t) \;\middle|\; \lim_{t\to\infty} \nu_t(T_t) = 0 \right\},
\end{aligned} \tag{6.10}$$

where in the last case the infimum is taken over all families of tests $\{T_t\}_{t\in\mathcal{I}}$ for which $\lim_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t)$ exists.

We shall call the numbers (6.9) and (6.10) the Stein type exponents. Clearly, for any $\epsilon \in]0,1[$, one has the relations

$$\begin{array}{ccccc}
\underline{B}_\epsilon & \le & \overline{B}_\epsilon & \le & B_\epsilon \\
\rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} \\
\underline{B} & \le & \overline{B} & \le & B
\end{array} \tag{6.11}$$

The following general lower bounds holds [HP].

Proposition 6.4 *For any $\epsilon \in]0,1[$, one has*

$$\liminf_{t\to\infty} \frac{1}{w_t}\mathrm{Ent}(\nu_t|\omega_t) \le \min\{\underline{B}, (1-\epsilon)\underline{B}_\epsilon\}.$$

Proof. Let T_t be a test. The monotonicity of the relative entropy (Proposition 5.3 (3)) applied to the 2-dimensional abelian algebra $\mathfrak{N} = \{1, T_t\}''$ implies

$$\begin{aligned}\mathrm{Ent}(\nu_t|\omega_t) \le \mathrm{Ent}(\nu_{t|\mathfrak{N}}|\omega_{t|\mathfrak{N}}) &= -\nu_t(T_t)\log\nu_t(T_t) - (1-\nu_t(T_t))\log(1-\nu_t(T_t)) \\ &\quad + \nu_t(T_t)\log\omega_t(T_t) + (1-\nu_t(T_t))\log(1-\omega_t(T_t)) \\ &\le 2\max_{x\in[0,1]}(-x\ln x) + (1-\nu_t(T_t))\log\omega_t(1-T_t),\end{aligned}$$

from which the statement follows. □

The further study of error exponents is based on

Assumption (W1). For $s \in [0,1]$, the limit

$$e(s) = \lim_{t\to\infty}\frac{1}{w_t}\mathrm{Ent}_s(\nu_t|\omega_t)$$

exists, the function $e(s)$ is continuous on $[0,1]$, differentiable on $]0,1[$, and $D^+e(0) < D^-e(1)$.

Note that $e(s)$ is convex on $[0,1]$. If ν_t, ω_t are states, then $e(0) = e(1) = 0$ and $e(s) \le 0$ for $s \in [0,1]$.

6.3 Chernoff bound

Theorem 6.5 *Suppose that (W1) holds. Then*

$$\underline{D} = \overline{D} = \inf_{s\in[0,1]} e(s).$$

We omit the proof since it is identical to the proof of Theorem 3.4.

6.4 Hoeffding bound

In this section we shall make use of the result described in Section 2.1. In particular, the reader should recall the function $\psi(r)$ defined in terms of $e(s)$ in Proposition 2.2.

Theorem 6.6 *Suppose that (W1) holds. Then for all r,*

$$\underline{B}(r) = \overline{B}(r) = B(r) = \psi(r). \tag{6.12}$$

Remark. In words, the Hoeffding bound asserts that for any family of tests $\{T_t\}_{t\in\mathcal{I}}$ satisfying

$$\limsup_{t\to\infty} \frac{1}{w_t} \log \nu_t(T_t) < -r, \tag{6.13}$$

one has

$$\liminf_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t) \geq \psi(r),$$

and that there exists a family of tests $\{T_t\}$ such that (6.13) holds and

$$\lim_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t) = \psi(r). \tag{6.14}$$

Proof. The proof follows standard arguments (see, *e.g.*, [HMO1]). Since for $r<0$ all the terms in (6.12) are $-\infty$, we only need to consider $r\geq 0$. We recall the definitions

$$\varphi(\theta) = \sup_{s\in[0,1]} (\theta s - e(s)), \qquad \hat{\varphi}(\theta) = \varphi(\theta) - \theta,$$

and set

$$T_t(\theta) = \mathrm{s}_{(\mathrm{e}^{\theta w_t}\omega_t - \nu_t)_+}. \tag{6.15}$$

Since $\mathrm{Ent}_s(\nu_t|\mathrm{e}^{\theta w_t}\omega_t) = \theta(1-s)w_t + \mathrm{Ent}_s(\nu_t|\omega_t)$, Theorem 6.1 (2) implies

$$D(\nu_t, \mathrm{e}^{\theta w_t}\omega_t) = \nu(T_t(\theta)) + \mathrm{e}^{\theta w_t}\omega_t(1-T_t(\theta)) \leq \mathrm{e}^{\theta(1-s)w_t}\mathrm{e}^{\mathrm{Ent}_s(\nu_t|\omega_t)}, \tag{6.16}$$

and hence

$$\nu(T_t(\theta)) \leq \mathrm{e}^{\theta(1-s)w_t}\mathrm{e}^{\mathrm{Ent}_s(\nu_t|\omega_t)}, \qquad \omega_t(1-T_t(\theta)) \leq \mathrm{e}^{-\theta s w_t}\mathrm{e}^{\mathrm{Ent}_s(\nu_t|\omega_t)},$$

for $s\in[0,1]$. It follows that

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{w_t} \log D(\nu_t, \mathrm{e}^{\theta w_t}\omega_t) &\leq \inf_{s\in[0,1]} (e(s) - \theta(s-1)) = -\hat{\varphi}(\theta),\\
\limsup_{t\to\infty} \frac{1}{w_t} \log \nu_t(T_t(\theta)) &\leq \inf_{s\in[0,1]} (e(s) - \theta(s-1)) = -\hat{\varphi}(\theta),\\
\limsup_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t(\theta)) &\leq \inf_{s\in[0,1]} (e(s) - \theta s) = -\varphi(\theta).
\end{aligned} \tag{6.17}$$

The estimate (6.5) yields

$$D(\nu_t, \mathrm{e}^{\theta w_t}\omega_t) \geq \frac{1}{2}\,\mathrm{e}^{\theta w_t}\,\mu_{\nu_t|\omega_t}(]-\infty, -\theta w_t]).$$

Hence, by Theorem 2.5, for $\theta\in]D^+e(0), D^-e(1)[$,

$$\liminf_{t\to\infty} \frac{1}{w_t} \log D(\nu_t, \mathrm{e}^{\theta w_t}\omega_t) \geq \theta - \varphi(\theta) = -\hat{\varphi}(\theta).$$

Combining this estimate with (6.17) we derive that for $\theta\in]D^+e(0), D^-e(1)[$,

$$\lim_{t\to\infty} \frac{1}{w_t} \log D(\nu_t, \mathrm{e}^{\theta t}\omega_t) = -\hat{\varphi}(\theta). \tag{6.18}$$

Let

$$D^+e(0) < \theta < \theta' < D^-e(1),$$

and let $\{T_t\}$ be tests such that

$$\limsup_{t\to\infty} \frac{1}{w_t} \log \nu_t(T_t) < -\hat{\varphi}(\theta).$$

We recall that $\hat{\varphi}$ is strictly decreasing on $]D^+e(0), D^-e(1)[$ and so $-\hat{\varphi}(\theta) < -\hat{\varphi}(\theta')$. Using (6.18), we derive

$$\begin{aligned}
-\hat{\varphi}(\theta') = \liminf_{t\to\infty} \frac{1}{w_t} \log D(\nu_t, \mathrm{e}^{\theta' w_t}\omega_t) &\le \liminf_{t\to\infty} \frac{1}{w_t} \log D(\nu_t, \mathrm{e}^{\theta' w_t}\omega_t, T_t) \\
&= \liminf_{t\to\infty} \frac{1}{w_t} \log(\nu_t(T_t) + \mathrm{e}^{\theta' w_t}\omega_t(1-T_t)) \\
&\le \max\left\{ \liminf_{t\to\infty} \frac{1}{w_t} \log \nu_t(T_t), \theta' + \liminf_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t) \right\} \\
&\le \max\left\{ -\hat{\varphi}(\theta), \theta' + \liminf_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t) \right\},
\end{aligned}$$

from which we can conclude that

$$\liminf_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t) \ge -\hat{\varphi}(\theta') - \theta' = -\varphi(\theta').$$

Since φ is continuous, taking $\theta' \downarrow \theta$ we derive

$$\liminf_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t) \ge -\varphi(\theta), \tag{6.19}$$

and so for $\theta \in]D^+e(0), D^-e(1)[$,

$$\underline{B}(\hat{\varphi}(\theta)) \ge -\varphi(\theta).$$

Combining (6.19) and (6.17) yields

$$\lim_{t\to\infty} \frac{1}{w_t} \log \omega_t(1-T_t(\theta)) = -\varphi(\theta),$$

and so if $\theta \in]D^+e(0), D^-e(1)[$ then

$$B(\hat{\varphi}(\theta) - \epsilon) \le -\varphi(\theta)$$

holds for any $\epsilon > 0$.

If $\hat{\varphi}(\theta)$ is a point of continuity of B, taking $\epsilon \downarrow 0$ we see that

$$\underline{B}(\hat{\varphi}(\theta)) = B(\hat{\varphi}(\theta)) = -\varphi(\theta). \tag{6.20}$$

The functions B and $\underline{B}$ are finite, increasing and right continuous on the open interval $\hat{\varphi}(]D^+e(0), D^-e(1)[)$. Since $\hat{\varphi}$ is continuous and strictly decreasing on $]D^+e(0), D^-e(1)[$, there exists a countable set $\mathcal{N}$ such that $\hat{\varphi}(\mathcal{N})$ is the set of discontinuity points of B and $\underline{B}$ on $\hat{\varphi}(]D^+e(0), D^-e(1)[)$. Relation (6.20) holds for all $\theta \in]D^+e(0), D^-e(1)[\backslash\mathcal{N}$. The right continuity implies that (6.20) holds for all $\theta \in]D^+e(0), D^-e(1)[$.

Suppose that $D^+e(0) > -\infty$. Since $\hat{\varphi}(\theta) = -\theta$, $\varphi(\theta) = 0$ for $\theta \le D^+e(0)$ (Theorem 2.1 (6)), continuity of φ and $\hat{\varphi}$ implies

$$\lim_{\theta\downarrow D^+e(0)} \hat{\varphi}(\theta) = -D^+e(0), \qquad \lim_{\theta\downarrow D^+e(0)} \varphi(\theta) = 0.$$

Invoking the upper semicontinuity of B, we derive from (6.20)

$$B(-D^-e(0)) \ge \limsup_{\theta\downarrow D^+e(0)} B(\hat{\varphi}(\theta)) = -\lim_{\theta\downarrow D^+e(0)} \varphi(\theta) = 0,$$

and B being non-positive, we conclude that $B(-D^-e(0)) = 0$. Since B is increasing, we must have

$$B(\hat{\varphi}(\theta)) = 0 = -\varphi(\theta)$$

for $\theta \leq D^+e(0)$. The same argument shows that $\underline{B}(\hat{\varphi}(\theta)) = -\varphi(\theta)$ for $\theta \leq D^+e(0)$, and hence (6.20) holds for all $\theta < D^-e(1)$. Using Equ. (2.3) we see that for $r > 0$,

$$\underline{B}(r) = B(r) = \psi(r). \tag{6.21}$$

Again, since the functions $\underline{B}$, B and ψ are right continuous, taking $r \downarrow 0$ in (6.21) we get that (6.21) also holds for $r = 0$. Since $\underline{B}(r) \leq \overline{B}(r) \leq B(r)$, the statement follows. □

The above proof gives that if $r > 0$ and $\varphi(\theta) = r$, then (6.14) holds for the tests $T_t(\theta)$.

6.5 Stein's Lemma

In accordance with the terminology used in non-equilibrium statistical mechanics, we shall call

$$\Sigma^+ = D^-e(1)$$

entropy production of the hypothesis testing. By Proposition 5.4 (2)–(3)

$$\mathrm{Ent}_{1-s}(\nu_t|\omega_t) = \mathrm{Ent}_s(\omega_t|\nu_t) \geq s\,\mathrm{Ent}(\nu_t|\omega_t),$$

and hence, for $s \in]0,1[$,

$$\frac{e(1-s)}{s} = \lim_{t\to\infty} \frac{1}{w_t} \frac{\mathrm{Ent}_{1-s}(\nu_t|\omega_t)}{s} \geq \limsup_{t\to\infty} \frac{1}{w_t} \mathrm{Ent}(\nu_t|\omega_t).$$

Taking the limit $s \downarrow 0$, we get

$$-\Sigma^+ \geq \limsup_{t\to\infty} \frac{1}{w_t} \mathrm{Ent}(\nu_t|\omega_t).$$

Proposition 6.7 *Suppose that (W1) holds. Then all Stein type exponents are* $\leq -\Sigma^+$.

Proof. By Relation (6.11), the largest Stein exponent is B and since $\limsup_{t\to\infty} \frac{1}{w_t} \log \nu_t(T_t) < 0$ implies $\lim_{t\to\infty} \nu_t(T_t) = 0$ one has $B \leq B(0)$. Theorem 6.6 and Proposition 2.2 (2) yield

$$B \leq B(0) = \psi(0) = -\Sigma^+.$$

□

To derive lower bounds we need to strengthen (W1) and assume

Assumption (W2). (W1) holds, Σ^+ is finite, and for all $s \in \mathbb{R}$,

$$\lim_{t\to\infty} (\Omega_{\nu_t}|\Delta_{\omega_t|\nu_t}^{\mathrm{i}s/w_t}\Omega_{\nu_t}) = \mathrm{e}^{-\mathrm{i}s\Sigma^+}. \tag{6.22}$$

The following result is known as Stein's Lemma.

Theorem 6.8 *Suppose that (W2) holds. Then for all $\epsilon \in]0,1[$,*

$$\underline{B} = \overline{B} = B = \underline{B}_\epsilon = \overline{B}_\epsilon = B_\epsilon = -\Sigma^+.$$

Proof. By Proposition 6.7 and Relation (6.11), it suffices to prove that $\underline{B}_\epsilon \geq -\Sigma^+$. Assumption (W2) and the Levy continuity theorem imply that the spectral measure $\tilde{\mu}_t$ of $-\frac{1}{w_t}\log\Delta_{\omega_t|\nu_t}$ for Ω_{ν_t} converges weakly to the Dirac measure at Σ^+. Let $\{T_t\}$ be a family of tests such that $\nu_t(T_t) \leq \epsilon \in]0,1[$ for all $t \in \mathcal{I}$. For $\theta > \vartheta > \Sigma^+$, the estimate (6.5) yields

$$\omega_t(1-T_t) \geq D(\mathrm{e}^{-\theta w_t}\nu_t, \omega_t) - \mathrm{e}^{-\theta w_t}\nu_t(T_t) \geq \mathrm{e}^{-\theta w_t}\left(\frac{\tilde{\mu}_t(]-\infty,\vartheta[)}{1+\mathrm{e}^{(\vartheta-\theta)w_t}} - \epsilon\right).$$

Since $\tilde{\mu}_t(]-\infty,\vartheta[)/(1+\mathrm{e}^{(\vartheta-\theta)w_t}) \to 1$ as $t \to \infty$, we conclude that

$$\liminf_{t\to\infty} \frac{1}{w_t}\log\omega_t(1-T_t) \geq -\theta$$

holds for any $\theta > \Sigma^+$. Taking $\theta \downarrow \Sigma^+$ yields the result. □

In many models one can show that the following holds:

Assumption (W3). For some $\delta > 0$ and $s \in [0, 1+\delta[$ the limit

$$e(s) = \lim_{t\to\infty}\frac{1}{w_t}\mathrm{Ent}_s(\nu_t|\omega_t)$$

exists, the function $e(s)$ is continuous on $[0,1+\delta[$, differentiable on $]0,1+\delta[$, and $e'(1) = \Sigma^+ > 0$.

We note that (W3) $\Rightarrow$ (W2). Indeed, Proposition 5.4 (4) and the convexity of $e(s)$ imply

$$\Sigma^+ = e'(1) = \lim_{t\to\infty}\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{w_t}\mathrm{Ent}_s(\nu_t|\omega_t)\Big|_{s=1} = -\lim_{t\to\infty}\frac{1}{w_t}\mathrm{Ent}(\nu_t|\omega_t). \tag{6.23}$$

Propositions 2.4 and 5.4 (2) imply (6.22).

7 Entropy production and full counting statistics for W^*-dynamical systems

7.1 Setup

Our starting point is a quantum dynamical system $\mathcal{Q} = (\mathfrak{M}, \tau, \omega)$. There, $\mathfrak{M}$ is a W^*-algebra in standard form on a Hilbert space $\mathcal{H}$, $\tau = \{\tau^t \,|\, t \in \mathbb{R}\}$ is a W^*-dynamics, *i.e.*, a weakly continuous group of $*$-automorphisms of $\mathfrak{M}$, and ω is a faithful normal state on $\mathfrak{M}$. The self-adjoint elements of $\mathfrak{M}$ describe observables of the physical system under consideration which evolve in time as $A_t = \tau^t(A)$. The state ω describes the initial thermodynamic state of the system. The state of the system at time t is $\omega_t = \omega \circ \tau^t$. Obviously, $\omega_t(A) = \omega(A_t)$.

A time-reversal of $\mathcal{Q}$ is an anti-linear $*$-automorphism $\Theta : \mathfrak{M} \to \mathfrak{M}$ such that

$$\Theta \circ \Theta = \mathrm{id}, \qquad \tau^t \circ \Theta = \Theta \circ \tau^{-t}.$$

The state ω is time-reversal invariant if $\omega(\Theta(A)) = \omega(A^*)$ for all $A \in \mathfrak{M}$ and in this case we say that the system $\mathcal{Q}$ is TRI.

Let $\beta > 0$. We recall that ω is a (τ, β)-KMS state if for all $A, B \in \mathfrak{M}$ the function

$$F_{A,B}(t) = \omega(AB_t)$$

has an analytic continuation to the strip $0 < \operatorname{Im} z < \beta$ which is bounded and continuous on its closure, and which satisfies the KMS-boundary condition

$$F_{A,B}(t + \mathrm{i}\beta) = \omega(B_t A),$$

for $t \in \mathbb{R}$. A (τ, β)-KMS state is τ-invariant and faithful. In algebraic quantum statistical mechanics a (τ, β)-KMS state describes a physical state of thermal equilibrium at inverse temperature β. For additional information about KMS states we refer the reader to [BR2].

7.2 Entropy production observable

Consider the Araki-Connes cocycles

$$[D\omega_t : D\omega]_u = \Delta_{\omega_t|\omega}^{\mathrm{i}u} \Delta_\omega^{-\mathrm{i}u},$$

with $u, t \in \mathbb{R}$. They are unitaries in $\mathfrak{M}$ satisfying

$$[D\omega_{t_1+t_2} : D\omega]_u = \tau^{-t_2}([D\omega_{t_1} : D\omega]_u)[D\omega_{t_2} : D\omega]_u.$$

Assumption (Ep1). For all $t \in \mathbb{R}$ the map $u \mapsto [D\omega_t : D\omega]_u \in \mathfrak{M}$ is weakly differentiable at $u = 0$.

The derivatives

$$\ell_{\omega_t|\omega} = \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}u} [D\omega_t : D\omega]_u \Big|_{u=0}$$

are self-adjoint elements of $\mathfrak{M}$ satisfying

$$\ell_{\omega_{t_1+t_2}|\omega} = \tau^{-t_2}(\ell_{\omega_{t_1}|\omega}) + \ell_{\omega_{t_2}|\omega}. \tag{7.1}$$

Note also that

$$\log \Delta_{\omega_t|\omega} = \log \Delta_\omega + \ell_{\omega_t|\omega}.$$

In the terminology of Araki [Ar3], $\ell_{\omega_t|\omega}$ is the relative Hamiltonian of ω_t w.r.t. ω. The entropy cocycle is defined by $c^t = \tau^t(\ell_{\omega_t|\omega})$ and satisfies

$$c^{t_1+t_2} = c^{t_2} + \tau^{t_2}(c^{t_1}).$$

In the finite dimensional setting $c^t = S_t - S$, where $S = -\log \omega$ is the entropy observable.

We also need

Assumption (Ep2). The map $t \mapsto \ell_{\omega_t|\omega} \in \mathfrak{M}$ is weakly differentiable at $t = 0$.

The *entropy production observable* is defined by

$$\sigma = \frac{\mathrm{d}}{\mathrm{d}t} \ell_{\omega_t|\omega} \Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} c^t \Big|_{t=0},$$

and is the exact non-commutative analog of the phase space contraction rate in classical non-equilibrium statistical mechanics [JPR]. The cocycle relation (7.1) yields

$$\ell_{\omega_t|\omega} = \int_0^t \sigma_{-u}\mathrm{d}u, \qquad c^t = \int_0^t \sigma_u \mathrm{d}u,$$

where the integrals converge weakly (*i.e.*, for all $\nu \in \mathfrak{M}_*$, $\nu(\ell_{\omega_t|\omega}) = \int_0^t \nu(\sigma_{-u})\mathrm{d}u$, see Section 2.5.3 in [BR1]). The observable of *mean entropy production rate over the time interval* $[0,t]$ is

$$\Sigma^t = \frac{c^t}{t} = \frac{1}{t}\int_0^t \sigma_u \mathrm{d}u.$$

Proposition 7.1 *Suppose that* (Ep1) *and* (Ep2) *hold. Then:*

(1) $\omega(\sigma) = 0$.

(2) *If* $\mathcal{Q}$ *is* TRI, *then* $\Theta(\sigma) = -\sigma$.

(3) *The entropy balance equation*

$$\omega(\Sigma^t) = -\frac{1}{t}\mathrm{Ent}(\omega_t|\omega),$$

holds. In particular, $\omega(\Sigma^t) \geq 0$ *for* $t > 0$.

Let us again illustrate the above definitions on the example of open quantum systems. Consider thermal reservoirs $\mathcal{R}_j$ described by quantum dynamical systems $(\mathfrak{M}_j, \tau_j, \omega_j)$ in standard form on the Hilbert spaces $\mathcal{H}_j$, $j = 1, \cdots, n$. For each j, let $\vartheta_j = \{\vartheta_j^t \,|\, t \in \mathbb{R}\}$ be a given gauge group, *i.e.*, a W^*-dynamics on $\mathfrak{M}_j$ commuting with τ_j ($\vartheta_j^t \circ \tau_j^t = \tau_j^t \circ \vartheta_j^t$ for all t). We assume that $\mathcal{R}_j$ is in thermal equilibrium at inverse temperature β_j and chemical potential μ_j, *i.e.*, that ω_j is a β_j-KMS state for the W^*-dynamics $\tau_j^t \circ \vartheta_j^{-\mu_j t}$. We denote by δ_j/ξ_j the generator of τ_j/ϑ_j, *i.e.*, $\tau^t = \mathrm{e}^{t\delta_j}/\vartheta^t = \mathrm{e}^{t\xi_j}$. The complete reservoir system $\mathcal{R} = \mathcal{R}_1 + \cdots + \mathcal{R}_n$ is described by $(\mathfrak{M}_\mathcal{R}, \tau_\mathcal{R}, \omega_\mathcal{R})$, where

$$\mathfrak{M}_\mathcal{R} = \otimes_{j=1}^n \mathfrak{M}_j, \qquad \tau_\mathcal{R} = \otimes_{j=1}^n \tau_j, \qquad \omega_\mathcal{R} = \otimes_{j=1}^n \omega_j.$$

Let $\mathcal{S}$ be a finite quantum system described by the Hilbert space $\mathcal{H}_\mathcal{S}$, the Hamiltonian $H_\mathcal{S}$ and the state $\omega_\mathcal{S}(A) = \mathrm{Tr}A/\dim\mathcal{H}_\mathcal{S}$. We set $\mathfrak{M}_\mathcal{S} = \mathcal{O}_{\mathcal{H}_\mathcal{S}}$, $\tau_\mathcal{S}^t(A) = \mathrm{e}^{\mathrm{i}tH_\mathcal{S}}A\mathrm{e}^{-\mathrm{i}tH_\mathcal{S}}$. The joint but decoupled system $\mathcal{S} + \mathcal{R}$ is described by $(\mathfrak{M}, \tau_0, \omega)$, where

$$\mathfrak{M} = \mathfrak{M}_\mathcal{S} \otimes \mathfrak{M}_\mathcal{R}, \qquad \tau_0 = \tau_\mathcal{S} \otimes \tau_\mathcal{R}, \qquad \omega = \omega_\mathcal{S} \otimes \omega_\mathcal{R}.$$

The coupling of $\mathcal{S}$ with $\mathcal{R}_j$ is described by a self-adjoint element $V_j \in \mathfrak{M}_\mathcal{S} \otimes \mathfrak{M}_j$ and the full interaction is $V = \sum_{j=1}^n V_j$. Let τ be the perturbed W^* dynamics generated by $\delta = \delta_0 + \mathrm{i}[V, \cdot\,]$, where δ_0 is the generator of τ_0. The coupled open quantum system $\mathcal{Q} = \mathcal{S} + \mathcal{R}$ is described by $(\mathfrak{M}, \tau, \omega)$.

Suppose that V is in the domain of the generators δ_j/ξ_j for all j (we abbreviate $\delta_j \otimes \mathrm{id}, \mathrm{id} \otimes \delta_j$ by δ_j, etc). The observables

$$\Phi_j = \delta_j(V), \qquad \mathcal{J}_j = \xi_j(V), \tag{7.2}$$

describe the energy/charge currents out of $\mathcal{R}_j$. Under the above assumptions, (Ep1) and (Ep2) hold and

$$\sigma = -\sum_{j=1}^n \beta_j(\Phi_j - \mu_j\mathcal{J}_j).$$

For the proofs and additional information we refer the reader to [JOPP2].

7.3 Full counting statistics

We continue with the framework of the previous sections, adapting to this setting the construction of Section 4.1. Let $\mu_{\omega_t|\omega}$ be the spectral measure for $-\log\Delta_{\omega_t|\omega}$ and Ω_ω. The Rényi entropic functional of the system $\mathcal{Q}$ is

$$e_t(s) = \mathrm{Ent}_s(\omega_t|\omega) = (\Omega_\omega|\Delta^s_{\omega_t|\omega}\Omega_\omega) = \log\int \mathrm{e}^{-sx}\mathrm{d}\mu_{\omega_t|\omega}(x).$$

Following Eq. (4.6), we define the full counting statistics of $\mathcal{Q}$ as the family $\{\mathbb{P}_t\}_{t>0}$ of Borel probability measures on $\mathbb{R}$ given by

$$\mathrm{d}\mathbb{P}_t(\phi) = \mathrm{d}\mu_{\omega_{-t}|\omega}(t\phi).$$

Since

$$e_{-t}(s) = \mathrm{Ent}_s(\omega_{-t}|\omega) = \mathrm{Ent}_s(\omega|\omega_t) = \mathrm{Ent}_{1-s}(\omega_t|\omega) = e_t(1-s),$$

one has

$$e_t(1-s) = \log\int \mathrm{e}^{-st\phi}\mathrm{d}\mathbb{P}_t(\phi),$$

and if the system $\mathcal{Q}$ is TRI, then

$$\mathrm{Ent}_s(\omega_t|\omega) = \mathrm{Ent}_s(\omega_{-t}|\omega),$$

and so the transient Evans-Searles fluctuation relation

$$e_t(s) = e_t(1-s)$$

holds (this relation was proven for the first time in [TM1]). Equivalent formulations of this fluctuation relation are the identities (1)–(4) on page 21.

To describe the fluctuations of $\mathbb{P}_t$ as $t\to\infty$ we need

Assumption (LT1). There is an open interval $\mathbb{I}\supset[0,1]$ such that for $s\in\mathbb{I}$ the limit

$$e(s) = \lim_{t\to\infty}\frac{1}{t}e_t(s)$$

exists, is finite, and the function $\mathbb{I}\ni s\mapsto e(s)$ is differentiable.

For our purposes we may assume w.l.o.g. that $\mathbb{I}$ is symmetric around $s=1/2$. The function $e(s)$ is convex, $e(0)=e(1)=0$, $e(s)\le 0$ for $s\in[0,1]$, and $e(s)\ge 0$ for $s\notin[0,1]$. The entropy production of $\mathcal{Q}$ is

$$\Sigma^+ = e'(1) = -\lim_{t\to\infty}\frac{1}{t}\mathrm{Ent}(\omega_t|\omega).$$

We also set

$$\overline{\Sigma}^+ = -e'(0) = -\lim_{t\to\infty}\frac{1}{t}\mathrm{Ent}(\omega|\omega_t),$$

$$\underline{\theta} = \inf_{s\in\mathbb{I}} e'(s), \qquad \overline{\theta} = \sup_{s\in\mathbb{I}} e'(s),$$

and

$$\varphi(\theta) = \sup_{s\in\mathbb{I}}(-\theta s - e(s)).$$

If $\mathcal{Q}$ is TRI, then the Evans-Searles fluctuation relations

$$e(s) = e(1-s), \qquad \varphi(-\theta) = \varphi(\theta)+\theta,$$

hold and $\overline{\Sigma}^+ = \Sigma^+$, $\overline{\theta} = -\underline{\theta}$.

The assumption (LT1) and the Gärtner-Ellis theorem imply that $\mathbb{P}_t \to \delta_{\Sigma^+}$ weakly and $\mathbb{E}_t(\phi) \to \Sigma^+$ as $t \to \infty$. Moreover, for any open set $\mathbb{J} \subset (\underline{\theta}, \overline{\theta})$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_t(\mathbb{J}) = -\inf_{\theta\in\mathbb{J}} \varphi(\theta).$$

If $e(s)$ is analytic in a neighborhood of zero, then

$$\lim_{t\to\infty} t^{k-1} C_t^{(k)} = \partial_s^k e(s)|_{s=1},$$

where $C_t^{(k)}$ denotes the k-th cumulant of $\mathbb{P}_t$.

The physical interpretation of $\mathbb{P}_t$ is in terms of the full counting statistics of an approximating sequence of finite quantum systems. In a given model the choice of this approximating sequence is dictated by physical considerations. A typical mathematical setup for these approximations is the following.

(Ap1) The approximating sequence of quantum systems $\mathcal{Q}_m$, $m \in \mathbb{N}$, is described by a sequence of finite dimensional Hilbert spaces $\mathcal{H}_m$, algebras $\mathcal{O}_{\mathcal{H}_m}$, Hamiltonians H_m and states $\omega_m > 0$.

(Ap2) For all m there is a faithful representation $\pi_m : \mathcal{O}_{\mathcal{H}_m} \to \mathfrak{M}$ such that $\pi_m(\mathcal{O}_{\mathcal{H}_m}) \subset \pi_{m+1}(\mathcal{O}_{\mathcal{H}_{m+1}})$.

(Ap3) For $A \in \cup_m \pi_m(\mathcal{O}_{\mathcal{H}_m})$ and $t \in \mathbb{R}$,

$$\lim_{m\to\infty} \omega_{mt}(\pi_m^{-1}(A)) = \omega_t(A).$$

(Ap4) For all s in some open set containing $[0, 1]$ and for all t,

$$\lim_{m\to\infty} \mathrm{Ent}_s(\omega_{mt}|\omega_m) = \mathrm{Ent}_s(\omega_t|\omega).$$

(Ap4) ensures that the full counting statistics $\mathbb{P}_{mt} \to \mathbb{P}_t$ weakly as $m \to \infty$, thus giving a direct physical interpretation to $\mathbb{P}_t$, the full counting statistics of the infinitely extended system. The same approximation scheme is needed to give a physical interpretation the energy/charge current observables (7.2).

The approximation setup (Ap1)–(Ap4) is well suited for spin systems and fermionic systems but needs to be adjusted for bosonic systems.

Again, for the proofs and additional information we refer the reader to [JOPP2]

7.4 Hypothesis testing of the arrow of time

The quantum hypothesis testing of the family $\{(\omega_t, \omega_{-t})\}_{t>0}$ with the weight $w_t = 2t$ or, equivalently, of the family $\{(\omega, \omega_t)\}_{t>0}$ with the weight $w_t = t$ yields

Theorem 7.2 *Suppose that* (LT1) *holds and that* $\Sigma^+ > 0$. *Then:*

(1)

$$\lim_{t\to\infty} \frac{1}{2t} \log(2 - \|\omega_t - \omega_{-t}\|) = \inf_{s\in[0,1]} e(s).$$

If $(\mathfrak{M}, \tau, \omega)$ *is* TRI, *then* $\inf_{s\in[0,1]} = e(1/2)$.

(2) *For all* $r \in \mathbb{R}$,

$$\underline{B}(r) = \overline{B}(r) = B(r) = \psi(r) = -\sup_{0 \le s < 1} \frac{-sr - e(s)}{1-s}.$$

(3) *For any* $\epsilon \in]0,1[$,

$$\underline{B} = \overline{B} = B = \underline{B}_\epsilon = \overline{B}_\epsilon = B_\epsilon = -\Sigma^+.$$

Remark. Clearly, the statements of this theorem hold under more general conditions then (LT1). We have stated it in the present form for a transparent comparison with the large deviation principle for full counting statistics.

Theorem 7.2 quantifies the separation between the past and the future as time $t \uparrow \infty$. Since

$$\frac{1}{2}(2 - \|\omega_t - \omega_{-t}\|) = \omega_t(1 - \mathrm{s}_{(\omega_t - \omega_{-t})_+}) + \omega_{-t}(\mathrm{s}_{(\omega_t - \omega_{-t})_+}),$$

Part (1) implies

$$\limsup_{t\to\infty} \frac{1}{2t} \log \omega_t(1 - \mathrm{s}_{(\omega_t - \omega_{-t})_+}) \le \inf_{s\in[0,1]} e(s), \tag{7.3}$$

$$\limsup_{t\to\infty} \frac{1}{2t} \log \omega_{-t}(\mathrm{s}_{(\omega_t - \omega_{-t})_+}) \le \inf_{s\in[0,1]} e(s). \tag{7.4}$$

Therefore, as $t \uparrow \infty$, the states ω_t are concentrating exponentially fast along the projections $\mathrm{s}_{(\omega_t - \omega_{-t})_+}$ while the states ω_{-t} are concentrating exponentially fast along the orthogonal complement $1 - \mathrm{s}_{(\omega_t - \omega_{-t})_+}$. Note that one of the inequalities (7.3), (7.4), must be an equality. If $(\mathfrak{M}, \tau, \omega)$ is TRI, then $\omega_t(1 - \mathrm{s}_{(\omega_t - \omega_{-t})_+}) = \omega_{-t}(\mathrm{s}_{(\omega_t - \omega_{-t})_+})$, and in this case

$$\lim_{t\to\infty} \frac{1}{2t} \log \omega_t(1 - \mathrm{s}_{(\omega_t - \omega_{-t})_+}) = \lim_{t\to\infty} \frac{1}{2t} \log \omega_{-t}(\mathrm{s}_{(\omega_t - \omega_{-t})_+}) = e(1/2).$$

Let $r > 0$ and let $\{T_t\}$ be a family of projections in $\mathfrak{M}$ such that

$$\limsup_{t\to\infty} \frac{1}{2t} \log \omega_{-t}(T_t) < -r. \tag{7.5}$$

Part (2) of Theorem 7.2 asserts that

$$\liminf_{t\to\infty} \frac{1}{2t} \log \omega_t(1 - T_t) \ge \psi(r),$$

and that there exists such a family $\{T_t\}$ for which

$$\lim_{t\to\infty} \frac{1}{2t} \log \omega_t(1 - T_t) = \psi(r).$$

Hence, if ω_t is concentrating exponentially fast along $1 - T_t$ with the rate $< -r$, then ω_{-t} is concentrating along T_t with the optimal exponential rate $\psi(r)$.

Part (3) of Theorem 7.2 asserts that for any family of projections $\{T_t\}$ such that

$$\sup_t \omega_{-t}(T_t) < 1, \tag{7.6}$$

one has

$$\liminf_{t\to\infty} \frac{1}{2t} \log \omega_t(1 - T_t) \ge -\Sigma^+,$$

and that for any $\delta > 0$ one can find a family $\{T_t^{(\delta)}\}$ satisfying (7.6) and

$$\lim_{t\to\infty} \frac{1}{2t} \log \omega_t(1 - T_t^{(\delta)}) \le -\Sigma^+ + \delta.$$

Hence, if no restrictions are made on T_t w.r.t. ω_{-t} except (7.6) (which is needed to avoid trivial results), the optimal exponential rate of concentration of ω_t as $t \uparrow \infty$ is precisely $-\Sigma^+$.

8 Examples

8.1 Spin-fermion model

The spin-fermion model is an open quantum system describing the interaction of a spin $1/2$ with finitely many free Fermi gas reservoirs $\mathcal{R}_1, \ldots, \mathcal{R}_n$.

We shall use the notation introduced in Section 3.4.3. Let $\mathfrak{h}_j$ and h_j be the single fermion Hilbert space and Hamiltonian of $\mathcal{R}_j$. The Hilbert space and Hamiltonian of $\mathcal{R}_j$ are $\mathcal{H}_j = \Gamma_{\mathrm{f}}(\mathfrak{h}_j)$ and $H_j = \mathrm{d}\Gamma(h_j)$ (the second quantization of h_j). The algebra of observables of $\mathcal{R}_j$ is $\mathcal{O}_j = \mathrm{CAR}(\mathfrak{h}_j)$ and its dynamics is

$$\tau_j^t(A) = \mathrm{e}^{\mathrm{i}tH_j} A \mathrm{e}^{-\mathrm{i}tH_j}.$$

In particular, $\tau_j^t(a_j^{\#}(f)) = a_j^{\#}(\mathrm{e}^{\mathrm{i}th_j} f)$, where $a_j^{\#}(\cdot)$ stands for the annihilation/creation operator in $\mathcal{O}_j$. The pair $(\mathcal{O}_j, \tau_j)$ is a C^*-dynamical system, *i.e.*, τ_j is a strongly continuous group of $*$-automorphisms of $\mathcal{O}_j$. Initially, the reservoir $\mathcal{R}_j$ is in thermal equilibrium at inverse temperature $\beta_j > 0$ and chemical potential μ_j, *i.e.*, its initial state ω_j is the quasi-free state with the Fermi-Dirac density

$$T_{\beta_j,\mu_j} = \left(1 + \mathrm{e}^{\beta_j(h_j - \mu_j)}\right)^{-1}.$$

The full reservoir system $\mathcal{R} = \mathcal{R}_1 + \cdots + \mathcal{R}_n$ is described by

$$\mathcal{O}_{\mathcal{R}} = \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n, \qquad \tau_{\mathcal{R}} = \tau_1 \otimes \cdots \otimes \tau_n, \qquad \omega_{\mathcal{R}} = \omega_1 \otimes \cdots \otimes \omega_n.$$

We also set $\mathcal{H}_{\mathcal{R}} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ and $H_{\mathcal{R}} = H_1 + \cdots + H_n$.

The small system $\mathcal{S}$ is described by the Hilbert space $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$ and Hamiltonian[4] $H_{\mathcal{S}} = \sigma^{(3)}$. The initial state of $\mathcal{S}$ is $\omega_{\mathcal{S}} > 0$. The Hilbert space of the coupled system $\mathcal{S} + \mathcal{R}$ is $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}$, its algebra of observables is $\mathcal{O} = \mathcal{O}_{\mathcal{H}_{\mathcal{S}}} \otimes \mathcal{O}_{\mathcal{R}}$, and its initial state is $\omega = \omega_{\mathcal{S}} \otimes \omega_{\mathcal{R}}$. The interaction of $\mathcal{S}$ with $\mathcal{R}_j$ is described by $V_j = \sigma^{(1)} \otimes \varphi_j(\alpha_j)$, where $\varphi_j(\cdot)$ is the field operator in $\mathcal{O}_j$ and $\alpha_j \in \mathfrak{h}_j$ are given vectors (in the present context, they are often called form-factors). The full interaction is $V = \sum_{j=1}^n V_j$ and the Hamiltonian of the coupled system is

$$H_\lambda = H_{\mathcal{S}} + H_{\mathcal{R}} + \lambda V,$$

where $\lambda \in \mathbb{R}$ is a coupling constant. The dynamics of the coupled system is described by

$$\tau_\lambda^t(A) = \mathrm{e}^{\mathrm{i}tH_\lambda} A \mathrm{e}^{-\mathrm{i}tH_\lambda}.$$

The pair $(\mathcal{O}, \tau_\lambda)$ is a C^*-dynamical system and the triple

$$\mathcal{Q} = (\mathcal{O}, \tau_\lambda, \omega)$$

is the spin-fermion model. Needless to say, we are considering the simplest non-trivial case, for various generalizations see Remark 3 below. Passing to the GNS representation of $\mathcal{O}$ associated to ω one verifies that the spin-fermion model is an example of an abstract open quantum system described at the end of Section 7.2.

Starting with the seminal papers [Dav, DS, LS], the spin-fermion and the closely related spin-boson model have been one of the basic paradigms of non-equilibrium quantum statistical mechanics. Although many basic questions are still open, systematic efforts over the last decade have lead to a reasonably good understanding of these models in the weak coupling regime (λ small). To describe the results that concern us here, we make the following regularity assumptions.

[4] $\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}$ denote the usual Pauli matrices.

Assumption (SF1). $\mu_j = 0$, $\mathfrak{h}_j = L^2(\mathbb{R}_+, \mathrm{d}x; \mathfrak{H}_j)$ for some auxiliary Hilbert space $\mathfrak{H}_j$ and h_j is the operator of multiplication by the variable $x \in \mathbb{R}_+$.

For example, if $\mathfrak{h}_j = L^2(\mathbb{R}^d, \mathrm{d}x)$ and $\mathfrak{h}_j = -\Delta$, where Δ is the usual Laplacian on $\mathbb{R}^d$, passing first to the momentum representation and then to polar coordinates shows that (SF1) holds after a unitary transformation. (This is Example 1 in Sect. 3.1 in [JKP].)

We extend the form factors α_j to $\mathbb{R}$ by setting $\alpha_j(-x) = \alpha_j(x)$.

Assumption (SF2). There is $\delta > 0$ such that the α_j's extend to analytic functions on the strip $|\mathrm{Im}\, z| < \delta$ satisfying

$$\sup_{|y|<\delta} \int_{\mathbb{R}} \mathrm{e}^{-ax} \|\alpha_j(x + \mathrm{i}y)\|^2_{\mathfrak{H}_j} \mathrm{d}x < \infty$$

for all $a > 0$.

Assumption (SF3). For all j, $\|\alpha_j(2)\|_{\mathfrak{H}_j} > 0$ ($2 = 1 - (-1)$ is the Bohr frequency of the 2-level system $\mathcal{S}$).

Let $\mathfrak{h}_{mj}$, $m = 1, 2 \ldots$, be an increasing sequence of finite dimensional subspaces of $\mathfrak{h}_j$ such that $\cup_m \mathfrak{h}_{mj}$ is dense in $\mathfrak{h}_j$. Let $h_{mj} \in \mathcal{O}_{\mathfrak{h}_{mj},\mathrm{self}}$ be such that $h_{mj} \to h_j$ as $m \to \infty$ in the strong resolvent sense (h_{mj} being extended to 0 on $\mathfrak{h}_{mj}^{\perp}$). Let $\alpha_{mj} \in \mathfrak{h}_{mj}$ be such that $\mathrm{e}^{ah_m}\alpha_{mj} \to \mathrm{e}^{ah}\alpha_j$ for all $a \in \mathbb{R}$ and all j. Consider the sequence $\mathcal{Q}_m = (\mathcal{O}_m, \tau_{m\lambda}, \omega_m)$ of finite dimensional spin-fermion systems constructed over $(\mathfrak{h}_{jm}, h_{jm})$ with form factors α_{mj}. $\mathcal{Q}_m$ is a (very general) sequence of finite dimensional approximations of the infinitely extended spin-fermion model $\mathcal{Q}$.

Let $\mathbb{P}_{m\lambda t}$ / $\mathbb{P}_{\lambda t}$ be the full counting statistics of $\mathcal{Q}_m$ / $\mathcal{Q}$ and $e_{m\lambda t}(s)$ / $e_{\lambda t}(s)$ the corresponding Rényi entropies (we denote explicitly the dependence on the coupling strength λ). We adopt the shorthand notation

$$\mathbb{I}(\Lambda, r) = \{(\lambda, s) \in \mathbb{R}^2 \,|\, |\lambda| < \Lambda, s \in]1/2 - r, 1/2 + r[\}.$$

Theorem 8.1 *Suppose that (SF1)–(SF3) hold. Then:*

(1) *For all $s, \lambda \in \mathbb{R}$ and $t > 0$,*

$$\lim_{m\to\infty} e_{m\lambda t}(s) = e_{\lambda t}(s)$$

exists and is finite. In particular $\mathbb{P}_{m\lambda t} \to \mathbb{P}_{\lambda t}$ weakly. The function $(\lambda, s) \mapsto e_{\lambda t}(s)$ is real analytic and all cumulants of $\mathbb{P}_{m\lambda t}$ converge to the corresponding cumulants of $\mathbb{P}_{\lambda t}$.

(2) *Let $r > 0$ be given. Then there is $\Lambda > 0$ such that for $(\lambda, s) \in \mathbb{I}(\Lambda, r)$*

$$e_\lambda(s) = \lim_{t\to\infty} \frac{1}{t} e_\lambda(s)$$

exists. Moreover the function

$$\mathbb{I}(\Lambda, r) \ni (\lambda, s) \mapsto e_\lambda(s)$$

is real analytic, does not depend on the choice of $\omega_{\mathcal{S}}$, and satisfies the Evans-Searles symmetry

$$e_\lambda(s) = e_\lambda(1 - s). \tag{8.1}$$

(3) *For* $0 < |\lambda| < \Lambda$, $\Sigma_\lambda^+ = e'_\lambda(1) > 0$ *unless* $\beta_1 = \cdots = \beta_n$. *Moreover,*

$$\Sigma_\lambda^+ = \lambda^2 \mathfrak{S}^+ + O(\lambda^4),$$

where

$$\mathfrak{S}^+ = \frac{\pi}{2} \sum_{i,j} \frac{\|\alpha_i(2)\|^2_{\mathfrak{H}_i} \|\alpha_j(2)\|^2_{\mathfrak{H}_j}}{\sum_k \|\alpha_k(2)\|^2_{\mathfrak{H}_k}} \frac{(\beta_i - \beta_j)\sinh(\beta_i - \beta_j)}{\cosh\beta_i \cosh\beta_j}.$$

In particular, unless all β_j*'s are the same,* $]\frac{1}{2} - r, \frac{1}{2} + r[\ni s \mapsto e_\lambda(s)$ *is strictly convex for* $0 < |\lambda| < \Lambda$. *If all* β_j*'s are equal,* $e_\lambda(s)$ *is identically equal to zero.*

Remark 1. Part (1) is easy to prove and one also easily verifies that (Ap1)–(Ap3) of Section 7.3 hold. (SF1)–(SF3) are not needed for (1) and it suffices that α_j is in the domain of e^{ah_j} for all a and j. Part (2) is implicit in [JOP3]. For the spin boson model (2) is proven in [Ro] (with only notational modifications this proof also applies to the spin-fermion model). (3) is proven in [JOP3].

Remark 2. The spin-fermion model is automatically TRI (see [JOP3]) and this observation implies (8.1).

Remark 3. These results extend to the more general model where $\mathcal{S}$ is an arbitrary finite quantum system and V_j is a finite sum of terms of the form $Q \otimes P$ where $Q \in \mathcal{O}_{\mathcal{H}_\mathcal{S},\mathrm{self}}$ and $P \in \mathcal{O}_j$ is a self-adjoint polynomial in the field operators with form factors satisfying (SF2). The Fermi Golden Rule assumption (SF3), which ensures that $\mathcal{S}$ is effectively coupled to the reservoirs at the Bohr frequency (difference of eigenvalues of $H_\mathcal{S}$) , is replaced with the assumption that, for all j, the generator K_j of the quantum Markov semigroup of the system $\mathcal{S} + \mathcal{R}_j$ that arises in the van Hove limit $t \uparrow \infty$, $\lambda \downarrow 0$, $\lambda^2 t$ fixed, has zero as simple eigenvalue and no other real eigenvalue. The time-reversal invariance is no more automatic and has to be assumed separately. In Part (3), $\mathfrak{S}^+$ is the entropy production of the quantum dynamical semigroup generated by $\sum_j K_j$ (as described in [LS]).

Remark 4. With only minor modifications Theorem 8.1 extends to the multi-parameter full counting statistics. The connection of this more general result with quantum hypothesis testing is unclear at the moment.

8.2 Electronic black box model

The electronic black box (EBB) model is a free Fermi gas consisting of a finite part $\mathcal{S}$ – the sample – coupled to several, say n, infinitely extended reservoirs – the leads. This model has been a basic paradigm in the study of coherent transport in electronic systems in mesoscopic physics. We shall consider here a very simple variant of the EBB model referring the reader to [JOPP1] for proofs.

We denote by $\mathfrak{h}_\mathcal{S}$ and $h_\mathcal{S}$ the single fermion Hilbert space and Hamiltonian of $\mathcal{S}$, where $\mathfrak{h}_\mathcal{S}$ is assumed to be finite dimensional. $\mathfrak{h}_j$ and h_j are the single fermion Hilbert space and Hamiltonian the reservoir $\mathcal{R}_j$. We shall assume that $\mathcal{R}_j$ is a semi-infinite lead described in the tight binding approximation: $\mathfrak{h}_j = \ell^2(\mathbb{N})$ and $h_j = -\frac{1}{2}\Delta$ where $\Delta u_x = u_{x+1} + u_{x-1} - 2u_x$ is the discrete Laplacian with Dirichlet boundary condition $u_{-1} = 0$. Using the discrete Fourier transform

$$\widehat{u}(k) = \sqrt{\frac{2}{\pi}} \sum_{x \in \mathbb{N}} u_x \sin(k(x+1)),$$

one identifies $\mathfrak{h}_j$ with $L^2([0,\pi], \mathrm{d}k)$ and h_j becomes the operator of multiplication by $\varepsilon(k) = 1 - \cos k$. The single particle Hilbert space and Hamiltonian of the joint but decoupled system are

$$\mathfrak{h} = \mathfrak{h}_\mathcal{S} \oplus \mathfrak{h}_\mathcal{R}, \qquad h_0 = h_\mathcal{S} \oplus h_\mathcal{R}.$$

where

$$\mathfrak{h}_\mathcal{R} = \oplus_{j=1}^n \mathfrak{h}_j, \qquad h_\mathcal{R} = \oplus_{j=1}^n h_j.$$

The junction Hamiltonian v_j which allows for the flow of fermions between $\mathcal{S}$ and $\mathcal{R}_j$ is

$$v_j = |\chi_j\rangle\langle\delta_0^{(j)}| + |\delta_0^{(j)}\rangle\langle\chi_j|,$$

where χ_j's are unit vectors in $\mathfrak{h}_{\mathcal{S}}$ and $\delta_0^{(j)}$ is the Dirac delta function at the site $x = 0$ of the j-th lead. The single particle Hamiltonian of the coupled system is

$$h_\lambda = h_0 + \lambda \sum_{j=1}^{n} v_j,$$

where $\lambda \in \mathbb{R}$ is a coupling constant. The initial state of $\mathcal{S}$ is the quasi-free state with density $T_{\mathcal{S}}$ and the initial state of the reservoir $\mathcal{R}_j$ is the quasi-free state with Fermi-Dirac density

$$T_{\beta_j,\mu_j} = \left(1 + \mathrm{e}^{\beta_j(h_j - \mu_j)}\right)^{-1}.$$

The EBB model is described by the triple $\mathcal{Q} = (\mathcal{O}, \tau_\lambda, \omega)$, where $\mathcal{O} = \mathrm{CAR}(\mathfrak{h})$,

$$\tau_\lambda^t(a^\#(f)) = a^\#(\mathrm{e}^{\mathrm{i}th_\lambda} f),$$

and ω is the quasi-free state with density

$$T = T_{\mathcal{S}} \oplus (\oplus_{j=1}^n T_j).$$

Note that for $A \in \mathcal{O}$,

$$\tau_\lambda^t(A) = \mathrm{e}^{\mathrm{i}tH_\lambda} A \mathrm{e}^{-\mathrm{i}tH_\lambda},$$

where

$$H_\lambda = \mathrm{d}\Gamma(h_\lambda) = H_0 + \lambda \sum_{j=1}^{n} (a^*(\chi_j)a(\delta_0^{(j)}) + a^*(\delta_0^{(j)})a(\chi_j))$$

is the Hamiltonian of the EBB model. The EBB model is TRI iff the sample system $\mathcal{S}$ is TRI, more precisely iff there is an orthogonal basis of $\mathfrak{h}_{\mathcal{S}}$ in which both $h_{\mathcal{S}}$ and $T_{\mathcal{S}}$ have a real matrix representation and the vectors χ_j have real components.

Finite dimensional approximations of the EBB model are naturally constructed by replacing the semi-infinite leads $\mathcal{R}_j$ with finite leads with single fermion Hilbert space $\ell^2([0,m])$ and Hamiltonian $-\frac{1}{2}\Delta_m$, where Δ_m is the discrete Laplacian with Dirichlet boundary condition. The corresponding EBB model is denoted $\mathcal{Q}_m = (\mathcal{O}_m, \tau_{m\lambda}, \omega_m)$. $\mathbb{P}_{mt}$ denotes the full counting statistics of $\mathcal{Q}_m$.

The orthogonal projection $1_{\mathcal{R}}$ onto $\mathfrak{h}_{\mathcal{R}}$ coincide with the projection onto the absolutely continuous part of h_0. Since $h_\lambda - h_0$ is finite rank, the wave operators

$$\mathrm{w}_\pm = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}th_\lambda}\mathrm{e}^{-\mathrm{i}th_0} 1_{\mathcal{R}}$$

exist. In what follows we suppose that the spectrum of h_λ is purely absolutely continuous and hence that the wave operators are complete. If $\mathrm{sp}(h_{\mathcal{S}}) \subset]0,2[= \mathrm{sp}(h_{\mathcal{R}})$, under very general conditions this assumption holds for non-zero and small enough λ [JKP] (for large values of λ, h_λ will have eigenvalues outside $[0,2]$). The scattering matrix $\mathfrak{s} = \mathrm{w}_+^*\mathrm{w}_-$ is unitary on $\mathfrak{h}_{\mathcal{R}}$ and acts as the operator of multiplication by a unitary $n \times n$ matrix $[\mathfrak{s}_{ij}(k)]$. Let $\varsigma(k)$ be a diagonal matrix with entries

$$\varsigma_{jj}(k) = -\beta_j(\varepsilon(k) - \mu_j),$$

and let

$$T(k) = \left(1 + \mathrm{e}^{-\varsigma(k)}\right)^{-1}.$$

Theorem 8.2 (1) *For all λ, s, and $t > 0$,*

$$\lim_{m\to\infty} e_{mt}(s) = e_t(s).$$

In particular, $\mathbb{P}_{mt} \to \mathbb{P}_t$ weakly. The function $s \mapsto e_t(s)$ is real analytic and all the cumulants of $\mathbb{P}_{mt}$ converge to corresponding cumulants of $\mathbb{P}_t$.

In the remaining statements we assume that h_λ has purely absolutely continuous spectrum.

(2) *For all s the limit*

$$e(s) = \lim_{t\to\infty} \frac{1}{t} e_t(s)$$

exists and

$$e(s) = \int_0^\pi \log\det\left(1 + T(k)\left(\mathrm{e}^{-s\varsigma(k)}\mathfrak{s}(k)\mathrm{e}^{s\varsigma(k)}\mathfrak{s}^*(k) - 1\right)\right)\frac{\mathrm{d}\varepsilon(k)}{2\pi}. \tag{8.2}$$

The function $e(s)$ is real analytic and does not depend on $T_{\mathcal{S}}$.

(3)

$$\Sigma^+ = \sum_{i,j=1}^n \int_0^\pi t_{ji}(k)(\varrho_j(k) - \varrho_i(k))\varsigma_{jj}(k)\frac{\mathrm{d}\varepsilon(k)}{2\pi}, \tag{8.3}$$

where $t_{ji}(k) = |s_{ji}(k) - \delta_{ji}|^2$ and

$$\varrho_j(k) = \left(1 + \mathrm{e}^{\beta_j(\varepsilon(k)-\mu_j)}\right)^{-1}$$

is the Fermi-Dirac density of the j-th reservoir. $\Sigma^+ > 0$ iff there exists $j, i \in \{1, \dots, n\}$ and a set $S \in [0, \pi]$ of positive Lebesgue measure such that $j \neq i$, $s_{ji}(k) \neq 0$ for $k \in S$, and $(\beta_j, \mu_j) \neq (\beta_i, \mu_i)$. $e(s)$ is strictly convex if $\Sigma^+ > 0$ and is identically equal to zero if $\Sigma^+ = 0$.

Remark 1. The sufficient and necessary condition for the strict positivity of the entropy production described in (3) can be rephrased as follows: $\Sigma^+ > 0$ iff there is an open scattering channel between two reservoirs $\mathcal{R}_j$, $\mathcal{R}_i$, which are not in mutual thermal equilibrium.

Remark 2. Remark 4 after Theorem 8.1 applies to Theorem 8.2.

Remark 3. (8.3) is the Landauer-Büttiker formula for the entropy production of the EBB model. One can also derive the Landauer-Büttiker formulas for individual fluxes. The energy and charge operators of $\mathcal{R}_j$ are $H_j = \mathrm{d}\Gamma(h_j)$ and $N_j = \Gamma(1_{\mathcal{R}_j})$. The corresponding heat and charge flux observables are

$$\Phi_j = -\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}tH_\lambda}H_j\mathrm{e}^{-\mathrm{i}tH_\lambda}\Big|_{t=0} = \mathrm{i}\lambda(a^*(h_j\delta_0^{(j)})a(\chi_j) - a^*(\chi_j)a(h_j\delta_0^{(j)})),$$

$$\mathcal{J}_j = -\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}tH_\lambda}N_j\mathrm{e}^{-\mathrm{i}tH_\lambda}\Big|_{t=0} = \mathrm{i}\lambda(a^*(\delta_0^{(j)})a(\chi_j) - a^*(\chi_j)a(\delta_0^{(j)})).$$

The entropy production observable is

$$\sigma = -\sum_{j=1}^n \beta_j(\Phi_j - \mu_j\mathcal{J}_j).$$

Then

$$\begin{aligned}\langle\Phi_j\rangle_+ &= \lim_{t\to\infty}\omega(\tau_\lambda^t(\Phi_j)) = \sum_{i=1}^n\int_0^\pi t_{ji}(k)(\varrho_j(k) - \varrho_i(k))\varepsilon(k)\frac{\mathrm{d}\varepsilon(k)}{2\pi},\\ \langle\mathcal{J}_j\rangle_+ &= \lim_{t\to\infty}\omega(\tau_\lambda^t(\mathcal{J}_j)) = \sum_{i=1}^n\int_0^\pi t_{ji}(k)(\varrho_j(k) - \varrho_i(k))\frac{\mathrm{d}\varepsilon(k)}{2\pi}.\end{aligned} \tag{8.4}$$

These formulas can be derived either directly [AJPP2] or via the multi-parameter extension of the formula (8.2) (see [JOPP1, JOPP2]).

8.3 The XY quantum spin chain

The XY spin chain on a finite sublattice $\Lambda = [l_1, l_2] \subset \mathbb{Z}$ is described by the Hilbert space

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathbb{C}^2,$$

and the Hamiltonian

$$H_\Lambda = -\frac{J}{4} \sum_{l_1 \le x < l_2} (\sigma_x^{(1)}\sigma_{x+1}^{(1)} + \sigma_x^{(2)}\sigma_{x+1}^{(2)}) - \frac{\lambda}{2} \sum_{l_1 \le x \le l_2} \sigma_x^{(3)},$$

where $\sigma_x^{(j)} = (\otimes_{y=l_1}^{x-1} 1) \otimes \sigma^{(j)} \otimes (\otimes_{y=x+1}^{l_2} 1)$, J is the nearest neighbor coupling constant and λ is the strength of the magnetic field in the (3) direction. The thermal equilibrium state at inverse temperature $\beta > 0$ is

$$\omega_{\Lambda\beta} = \frac{\mathrm{e}^{-\beta H_\Lambda}}{\mathrm{Tr}\, \mathrm{e}^{-\beta H_\Lambda}}.$$

Let $n > 0$ be a fixed integer and for $m > n$ let $\mathcal{Q}_m$ denote the XY spin chain on $\Lambda_m = [-m, m]$ with initial state

$$\omega_m = \omega_{[-m,-n-1]\beta_L} \otimes \omega_{[-n,n]\beta} \otimes \omega_{[n+1,m]\beta_R}.$$

$\mathcal{Q}_m$ is a TRI open quantum system. The small subsystem $\mathcal{S}$ is the XY spin chain on $[-n, n]$ and the reservoirs are the XY spin chains on $[-m, -n-1]$ and $[n+1, m]$. We denote by $\mathbb{P}_{mt}$ the full counting statistics of $\mathcal{Q}_m$ and by $e_{mt}(s)$ its Rényi entropic functional.

Theorem 8.3 (1) *For all $s \in \mathbb{R}$ and $t > 0$ the limit*

$$e_t(s) = \lim_{m\to\infty} e_{mt}(s)$$

exists. In particular, $\mathbb{P}_{mt}$ converges weakly to a probability measure $\mathbb{P}_t$, the TD limit full counting statistics. The function $s \mapsto e_t(s)$ is real-analytic and all cumulants of $\mathbb{P}_{mt}$ converge to the corresponding cumulants of $\mathbb{P}_t$.

(2) *For all $s \in \mathbb{R}$ the limit*

$$e(s) = \lim_{t\to\infty} \frac{1}{t} e_t(s)$$

exists and

$$e(s) = \frac{1}{J\pi} \int_{u_-}^{u_+} \log\left(1 - \frac{\sinh(su\Delta\beta)\sinh((1-s)u\Delta\beta)}{\cosh(u\beta_L)\cosh(u\beta_R)}\right) \mathrm{d}u,$$

where $u_\pm = (\lambda \pm J)/2$ and $\Delta\beta = \beta_R - \beta_L$.

(3)

$$\Sigma^+ = e'(1) = \frac{1}{J\pi} \int_{u_-}^{u_+} (u\beta_L - u\beta_R)\,(\tanh(u\beta_L) - \tanh(u\beta_R))\,\mathrm{d}u$$

and $\Sigma^+ > 0$ iff $\beta_L \neq \beta_R$.

Remark 1. The Jordan-Wigner transformation maps the system $\mathcal{Q}_m$ to a finite EBB model. The scattering matrix of the corresponding infinitely extended EBB model can be explicitly computed and so Theorem 8.3 is a corollary of Theorem 8.2 (see [JOPP1]).

Remark 2. One can show that the limiting functional $e(s)$ is analytic on the strip $|\mathrm{Im}\, s| < \pi/(|\lambda| + |J|)\Delta\beta$.

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