WIENER INDEX AND HOSOYA POLYNOMIAL OF FIBONACCI AND LUCAS CUBES

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Abstract

In the language of mathematical chemistry, Fibonacci cubes can be defined as the resonance graphs of fibonacenes. Lucas cubes form a symmetrization of Fibonacci cubes and appear as resonance graphs of cyclic polyphenantrenes. In this paper it is proved that the Wiener index of Fibonacci cubes can be written as the sum of products of four Fibonacci numbers which in turn yields a closed formula for the Wiener index of Fibonacci cubes. Asymptotic behavior of the average distance of Fibonacci cubes is obtained. The generating function of the sequence of ordered Hosoya polynomials of Fibonacci cubes is also deduced. Along the way, parallel results for Lucas cubes are given.

1 Introduction

In this paper we are interested in the Wiener index and more general distance properties of Fibonacci and Lucas cubes. Fibonacci cubes were introduced as interconnection networks [15] and later studied from many aspects, see the recent survey [17]. From our point of view it is important that Fibonacci cubes also play a role in mathematical chemistry: Fibonacci cubes are precisely the resonance graphs of fibonacenes which in turn form an important class of hexagonal chains [19]. (For related results about resonance graphs, known also as Z-transformation graphs, see [21, 24, 29, 30].) Moreover, Lucas cubes also found chemical applications in [31].

The Wiener index of a graph is the first and (one of) the most studied invariant(s) in mathematical chemistry, see for instance extensive surveys [9, 10] and recent papers [2, 3, 6, 8, 26, 27]. An equivalent approach to the Wiener index is to investigate the average distance of a graph, which is indeed frequently done in pure mathematics, cf. [7, 14]. ¿From the recent papers on the Wiener index we point out that [8] contains many new results on the Wiener index of fibonacenes, the class of graphs that made Fibonacci cubes appealing in chemistry! Hence an investigation of the Wiener index and related invariants of Fibonacci cubes seems well justified.

We wish to add that results similar to ours were very recently obtained by Došlić in [11] while studying conjugated circuits in benzenoid chains. In particular, Fibonacci numbers convolved with themselves play a role here and there, and an average Kekulé structure of an n-ring fibonacene contains approximately $2n\varphi(2\varphi+1)/5$ conjugated hexagons, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ration.

The paper is organized as follows. In the next section concepts and results needed are given. Then, in Section 3, we prove that the Wiener index of Fibonacci cubes can be written as the sum of products of four Fibonacci numbers and use this result to deduce a closed formula for the Wiener index of Fibonacci cubes. Hence the Wiener index of Fibonacci cubes can be obtained in constant time which improves a result from [23] that computes it in $O(\log F_n)$ time. In addition, we also determine the asymptotic behavior of the average distance of Fibonacci cubes and obtain parallel results for Lucas cubes. In Section 4 we consider the Hosoya polynomial and obtain the generating function of the sequence of ordered Hosoya polynomials of Fibonacci cubes as well as the corresponding generating function for Lucas cubes. To make the paper of a reasonable length, we only indicate some of numerous possible uses of these generating functions.

2 Preliminaries

The distance in this paper is the usual shortest-path distance—the number of edges on a shortest path between vertices. The *Wiener index* $W(G)$ of a connected graph G is the sum of distances over all unordered pairs of vertices of G. The vertex set of the *d-cube* Q_d, also called a *hypercube of dimension* d, is the set of all binary strings of length d, and two vertices are adjacent if they differ in precisely one position.

A *Fibonacci string* of length n is a binary string $b_1b_2\ldots b_n$ with $b_i \cdot b_{i+1} = 0$ for $1 \le i < n$. The *Fibonacci cube* Γ_n ($n \ge 1$) is the subgraph of Q_n induced by the Fibonacci strings of length n. For convenience we also consider the empty string and set $\Gamma_0 = K_1$. Call a Fibonacci string $b_1b_2\ldots b_n$ a *Lucas string* if $b_1 \cdot b_n = 0$. Then the *Lucas cube* Λ_n ($n \ge 1$) is the subgraph of Q_n induced by the Lucas strings of length n. We also set $\Lambda_0 = K_1$.

Let $\{F_n\}$ be the Fibonacci numbers: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$. Let $\mathcal{F}_n$ be the Fibonacci strings of length n. Let $\mathcal{F}_n^0$ and $\mathcal{F}_n^1$ be the strings of $\mathcal{F}_n$ that end with, respectively, 0 and 1.

A subgraph G of a graph H is an *isometric subgraph* if the distance between any vertices of G equals the distance between the same vertices in H. Isometric subgraphs of hypercubes are *partial cubes*. The *dimension* of a partial cube G is the smallest integer d such that G is an isometric subgraph of Q_d. Many (chemically) important classes of graphs are partial cubes, in particular trees, median graphs, benzenoid graphs, phenylenes, grid graphs and bipartite torus graphs. In addition, Fibonacci and Lucas cubes are partial cubes as well, see [16].

Let a partial cube G of dimension k be given together with its isometric embedding into Q_k. Then for $i = 1, 2, \ldots, k$ and $\chi = 0, 1$, the *semicube* $W_{(i,\chi)}$ is defined as follows:

$$W_{(i,\chi)}(G) = \{u = u_1u_2\ldots u_k \in V(G) \mid u_i = \chi\}\,.$$

For a fixed i, the pair $W_{(i,0)}(G), W_{(i,1)}(G)$ of semicubes is called a *complementary pair of semicubes*. It should be pointed out that since the embedding of G into Q_k is unique (modulo permutation of coordinates), see [25], the definition of the semicubes does not depend on the embedding. The following result from [18] that computes the Wiener index of a partial cube from the orders of their complementary pairs of semicubes will be our starting point in Section 3:

Theorem 2.1 *Let G be a partial cube of dimension k isometrically embedded into Q_k. Then*

$$W(G) = \sum_{i=1}^{k} |W_{(i,0)}(G)| \cdot |W_{(i,1)}(G)| \,.$$

The *Hosoya polynomial* [1, 28] (also known as the *Wiener polynomial*) $H(G,x)$ of a connected graph G is the distance counting polynomial:

$$H(G,x) = \sum_{\{u,v\}} x^{d(u,v)} \,.$$

We refer to [4] for several polynomials related to the Hosoya polynomial: edge-Wiener polynomial, Schultz polynomial, and Gutman polynomial. Sometimes it is more convenient to consider *ordered Hosoya polynomial* [20], that is, the counting polynomial of the distances among ordered pairs of vertices:

$$\overline{H}(G,x) = \sum_{(u,v)\in V(G)\times V(G)} x^{d(u,v)} \,.$$

For example, when using Cartesian product, ordered polynomials are more adapted because we have the simple relation $\overline{H}(G \,\square\, G', x) = \overline{H}(G,x) \cdot \overline{H}(G',x)$. The two polynomials are easily linked as follows:

$$2H(G,x) = \overline{H}(G,x) - |V(G)| = \overline{H}(G,x) - \overline{H}(G,0) \,.$$

3 Wiener index

We begin by expressing the Wiener index of Fibonacci cubes with Fibonacci numbers:

Theorem 3.1 *For any $n \geq 0$,*

$$W(\Gamma_n) = \sum_{i=1}^{n} F_i \, F_{i+1} \, F_{n-i+1} \, F_{n-i+2} \,.$$

Proof. The result holds for $n = 0$ because $W(\Gamma_0) = 0$. Let $n \geq 1$ and recall that Γ_n is a partial cube. Moreover, the dimension of Γ_n is n, hence by Theorem 2.1 we have

$$W(\Gamma_n) = \sum_{i=1}^{n} |W_{(i,1)}(\Gamma_n)| \cdot |W_{(i,0)}(\Gamma_n)| \,.$$

Consider the set $W_{(i,1)}(\Gamma_n)$, where $2 \leq i \leq n-1$. Let $b = b_1 \ldots b_n \in W_{(i,1)}(\Gamma_n)$, then $b_i = 1$ which implies that $b_{i-1} = 0$ and $b_{i+1} = 0$. It follows that for $2 \leq i \leq n-1$, $|W_{(i,1)}(\Gamma_n)| = |\mathcal{F}_{i-2}| \cdot |\mathcal{F}_{n-i-1}|$. Furthermore, $|W_{(1,1)}(\Gamma_n)| = |W_{(n,1)}(\Gamma_n)| = |\mathcal{F}_{n-2}|$.

Similarly, $|W_{(i,0)}(\Gamma_n)| = |\mathcal{F}_{i-1}| \cdot |\mathcal{F}_{n-i}|$ for $2 \le i \le n-1$ and $|W_{(1,0)}(\Gamma_n)| = |W_{(n,0)}(\Gamma_n)| = |\mathcal{F}_{n-1}|$. Therefore,

$$\begin{aligned} W(\Gamma_n) &= \sum_{i=2}^{n-1} (|\mathcal{F}_{i-2}| \cdot |\mathcal{F}_{n-i-1}| \cdot |\mathcal{F}_{i-1}| \cdot |\mathcal{F}_{n-i}|) + 2|\mathcal{F}_{n-2}| \cdot |\mathcal{F}_{n-1}| \\ &= \sum_{i=1}^{n} F_i \, F_{n-i+1} \, F_{i+1} \, F_{n-i+2} \,, \end{aligned}$$

where we have used that for any j, $|\mathcal{F}_j| = F_{j+2}$. $\square$

Note that the sequence $\{W(\Gamma_n)\}_{n=0}^{\infty}$ is the convolution of $\{F_i F_{i+1}\}_{i=0}^{\infty}$ with $\{F_{i+1}F_{i+2}\}_{i=0}^{\infty}$.

In order to obtain a closed formula for $W(\Gamma_n)$ we will apply the appealing theory of Greene and Wilf presented in [12]. They studied the existence of closed formulas for expressions of the form

$$\sum_{j=0}^{n-1} G_1(a_1 n + b_1 j + c_1) G_2(a_2 n + b_2 j + c_2) \cdots G_k(a_k n + b_k j + c_k) \,,$$

where each sequence $\{G_i(n)\}$ is a sequence that satisfies a linear recurrence. Fibonacci numbers are of course such a sequence. Then it follows from [12, Theorem 2] that the sum $\sum_{i=0}^{n-1} F_i \, F_{n-i+1} \, F_{i+1} \, F_{n-i+2}$ can be expressed as a linear combination of F_n^2, F_{n+1}^2, $F_n F_{n+1}$, nF_n^2, nF_{n+1}^2, and nF_nF_{n+1}. Furthermore, [12, Theorem 3] implies that if a linear combination of the six monomials and the sum agree for $n = 0, 1, \ldots, 5$, then they agree for all n. Since the values of $\sum_{i=0}^{n-1} F_i \, F_{n-i+1} \, F_{i+1} \, F_{n-i+2}$ for $n = 0, 1, \ldots, 5$ are $0, 0, 2, 10, 39, 136$, the values of the linear combination $a\, F_n^2 + b\, F_{n+1}^2 + c\, F_n F_{n+1} + d\, nF_n^2 + e\, nF_{n+1}^2 + f\, nF_nF_{n+1}$ must be

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 2 & 2 & 8 & 4 \\ 4 & 9 & 6 & 12 & 27 & 18 \\ 9 & 25 & 15 & 36 & 100 & 60 \\ 25 & 64 & 40 & 125 & 320 & 200 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 10 \\ 39 \\ 136 \end{pmatrix}$$

whose solution is

$$\{a, b, c, d, e, f\} = \{\frac{4}{25}, 0, -\frac{23}{25}, \frac{4}{25}, \frac{6}{25}, \frac{9}{25}\} \,.$$

Hence $\sum_{i=0}^{n-1} F_i \, F_{n-i+1} \, F_{i+1} \, F_{n-i+2} = \frac{4}{25} F_n^2 + \frac{4}{25} nF_n^2 - \frac{23}{25} F_nF_{n+1} + \frac{9}{25} nF_nF_{n+1} + \frac{6}{25} nF_{n+1}^2$. Note that $W(\Gamma_n) = \sum_{i=1}^{n} F_iF_{n-i+1}F_{i+1}F_{n-i+2} = F_nF_{n+1} + \sum_{i=0}^{n-1} F_iF_{n-i+1}F_{i+1}F_{n-i+2}$ therefore:

Theorem 3.2 *For any $n \ge 0$,*

$$W(\Gamma_n) = \frac{4(n+1)F_n^2}{25} + \frac{(9n+2)F_nF_{n+1}}{25} + \frac{6nF_{n+1}^2}{25} \,.$$

The first values of the sequence $\{W(\Gamma_n)\}$ are $0, 1, 4, 16, 54, 176, 548, \ldots$

For a (connected) graph G, its *average distance* $\mu(G)$ is defined as

$$\mu(G) = \frac{1}{\binom{|V(G)|}{2}} W(G)\,.$$

Theorem 3.2 yields the asymptotic behavior of the average distance of Fibonacci cubes:

Corollary 3.3

$$\lim_{n\to\infty} \frac{\mu(\Gamma_n)}{n} = \frac{2}{5}\,.$$

Proof. From Binet's formula for the Fibonacci numbers it follows that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi = \frac{1+\sqrt{5}}{2}\,.$$

Then Theorem 3.2 and the fact that $|V(\Gamma_n)| = F_{n+2}$ imply:

$$\lim_{n\to\infty} \frac{\mu(\Gamma_n)}{n} = 2\left(\frac{4}{25\varphi^4} + \frac{9}{25\varphi^3} + \frac{6}{25\varphi^2}\right) = \frac{2}{5}\,.$$

□

Obtaining the Wiener index of Lucas cubes is significantly simpler than the Wiener index of Fibonacci cubes. The intrinsic reason for it is that in Lucas cubes all the coordinates of vertices are equivalent, while in Fibonacci cubes the first and the last coordinate behave differently from the other coordinates.

Theorem 3.4 *For any $n \geq 1$, $W(\Lambda_n) = n\, F_{n-1}\, F_{n+1}$.*

Proof. We again use the fact that Lucas cubes are partial cubes [16] and hence Theorem 2.1 can be applied, that is,

$$W(\Lambda_n) = \sum_{i=1}^{n} |W_{(i,1)}(\Lambda_n)| \cdot |W_{(i,0)}(\Lambda_n)|\,.$$

Considering Lucas strings on a circle we infer that $|W_{(i,1)}| \cdot |W_{(i,0)}|$ does not depend of i. So we may assume that $i = 1$. There are $|\mathcal{F}_{n-3}|$ Lucas strings whose first coordinate is 1, and there are $|\mathcal{F}_{n-1}|$ Lucas strings whose first coordinate is 0. □

Using Binet's formula and Theorem 3.4 we obtain a result for the average distance of Lucas cubes parallel to Fibonacci cubes:

Corollary 3.5

$$\lim_{n\to\infty} \frac{\mu(\Lambda_n)}{n} = \frac{2}{5}\,.$$

4 Hosoya polynomial

In this section we make a more general approach and consider the ordered Hosoya polynomial of Fibonacci cubes $\overline{H}(\Gamma_n, x)$ and its generating function

$$f(x,z) = \sum_{n\geq 0} \overline{H}(\Gamma_n, x) z^n = \sum_{n,k\geq 0} f_{n,k} x^k z^n \,,$$

where $f_{n,k}$ is the number of pairs of vertices $(u,v) \in V(\Gamma_n) \times V(\Gamma_n)$ such that $d(u,v) = k$.

Theorem 4.1 *The generating function of the sequence of ordered Hosoya polynomials of* Γ_n *is*

$$f(x,z) = \sum_{n\geq 0} \overline{H}(\Gamma_n, x) z^n = \frac{1 + z + xz(1-z)}{1 - z - z^2 + xz(-1 - z + z^2)} \,.$$

Proof. For $i, j \in \{0, 1\}$, let $f^{ij}_{n,k}$ be the number of $(u,v) \in V(\Gamma_n) \times V(\Gamma_n)$ such that $d(u,v) = k$ where u ends with i and v ends with j. Let us consider the generating polynomials $f^{ij}(x,z) = \sum_{n,k\geq 0} f^{ij}_{n,k} x^k z^n$. Then, having in mind the empty string,

$$f(x,z) = 1 + f^{00}(x,z) + f^{01}(x,z) + f^{10}(x,z) + f^{11}(x,z) \,. \tag{1}$$

By symmetry,

$$f^{01}(x,z) = f^{10}(x,z) \,. \tag{2}$$

If u and v end with 1 then they are the concatenation of an arbitrary Fibonacci string of $\mathcal{F}_{n-2}$ with 01 thus $f^{11}_{n,k} = f_{n-2,k}$ for $n \geq 2$. Using the initial conditions $f^{11}_{1,0} = 1$ and $f^{11}_{1,1} = f^{11}_{0,0} = 0$ we obtain by a standard computation that

$$f^{11}(x,z) = z^2 f(x,z) + z \,. \tag{3}$$

If u and v end with 0 then they are both the concatenation of an arbitrary Fibonacci string of $\mathcal{F}_{n-1}$ with 0 thus $f^{00}_{n,k} = f_{n-1,k}$ for $n \geq 1$. Using the initial condition $f^{00}_{0,0} = 0$ we obtain

$$f^{00}(x,z) = z f(x,z) \,. \tag{4}$$

For $n \geq 2$, if u ends with 1 and v ends with 0, then u is the concatenation of a Fibonacci string of $\mathcal{F}^0_{n-1}$ with 1, and v is the concatenation of a string of $\mathcal{F}^0_{n-1} \cup \mathcal{F}^1_{n-1}$ with 0. Therefore $f^{10}_{n,k} = f^{00}_{n-1,k-1} + f^{01}_{n-1,k-1}$. Using the initial conditions $f^{10}_{1,1} = 1$ and $f^{10}_{1,0} = f^{10}_{0,0} = 0$ we obtain by summation that

$$f^{10}(x,z) - xz = xz f^{00}(x,z) + xz f^{10}(x,z)$$

and thus

$$f^{10}(x,z) = \frac{xzf^{00}(x,z) + xz}{1 - xz} = \frac{xz^2 f(x,z) + xz}{1 - xz}\,. \tag{5}$$

Using the symmetry (2), by substitution of the expressions (3), (4) and (5) in (1) we obtain the relation

$$f(x,z)(1 - z - \frac{2xz^2}{1 - xz} - z^2) = 1 + z + \frac{2xz}{1 - xz}$$

thus the theorem. □

Theorem 4.1 can be applied in many different ways, let us mention some of them. By definition of $\overline{H}(G,x)$ we have that $\overline{H}(G,0) = |V(G)|$. Furthermore, it is well-know that $\overline{H}'(G,1) = 2W(G)$, cf. [5, 13].

Theorem 4.1 gives that $f(0,z) = \frac{1+z}{1-z-z^2}$ and we can recognize the generating function of the sequence $\{F_{i+2}\}_{i=0}^{\infty}$, the number of vertices of Γ_n. Furthermore, $\frac{\partial f(x,z)}{\partial x}|_{x=1} = \frac{2z}{(1-2z-2z^2+z^3)^2}$. The generating function of the sequence $\{F_i F_{i+1}\}_{i=0}^{\infty}$, the terms of which are known as golden rectangle numbers, is $z/(1-2z-2z^2+z^3)$ [22, sequence A001654]. We have thus an alternative proof that the sequence $\{W(\Gamma_n)\}_{n=0}^{\infty}$ is the convolution of $\{F_i F_{i+1}\}_{i=0}^{\infty}$ with $\{F_{i+1}F_{i+2}\}_{i=0}^{\infty}$.

A remarkable property of $f(x,z)$ is that partial derivations are easy to compute which leads us to the following:

Corollary 4.2 *Let $k \geq 1$. The generating function of the sequence $\{f_{n,k}\}_{n=0}^{\infty}$ of number of pairs of vertices of Γ_n at distance k is*

$$\sum_{n\geq 0} f_{n,k} z^n = \frac{2z(z+z^2-z^3)^{k-1}}{(1-z-z^2)^{k+1}}\,.$$

Proof. By direct computation we have

$$\frac{\partial f(x,z)}{\partial x} = \frac{2z}{(1-z-z^2+xz(-1-z+z^2))^2}$$

and

$$\frac{\partial(1-z-z^2+xz(-1-z+z^2))}{\partial x} = z^3 - z^2 - z\,.$$

Therefore it is easy to prove by induction that for any $k \geq 1$

$$\frac{\partial^k f(x,z)}{\partial x^k} = \frac{2(k!)z(z+z^2-z^3)^{k-1}}{(1-z-z^2+xz(-1-z+z^2))^{k+1}}\,.$$

We have thus

$$\frac{\partial^k f(x,z)}{\partial x^k}|_{x=0} = \frac{2(k!)z(z+z^2-z^3)^{k-1}}{(1-z-z^2)^{k+1}}\,.$$

On the other hand, from $f(x,z) = \sum_{n,k'\geq 0} f_{n,k'} x^{k'} z^n$ we have for any fixed k,

$$\frac{\partial^k f(x,z)}{\partial x^k} = \sum_{n\geq 0}\left[k! f_{n,k} + k! \sum_{k'\geq k+1} f_{n,k'} x^{k'-k}\right] z^n$$

The value for $x=0$ implies the corollary. □

We continue with Lucas cubes:

Theorem 4.3 *The generating function of the sequence of ordered Hosoya polynomials for Λ_n is*

$$\ell(x,z) = \sum_{n\geq 0} \overline{H}(\Lambda_n, x) z^n = \frac{1+2x^2z^3-3x^2z^4+z^2(1+x)^2}{1-z-x^2z^3+x^2z^4-z^2(1+x)^2}\,.$$

To prove this result, let us first return to Fibonacci cubes. Let $g_{n,k}$ be the number of $(u,v)\in V(\Gamma_n)\times V(\Gamma_n)$ such that $d(u,v)=k$ where v begins with 0. For $i,j\in\{0,1\}$ let $g^{ij}_{n,k}$ be the number of $(u,v)\in V(\Gamma_n)\times V(\Gamma_n)$ such that $d(u,v)=k$ where u ends with i, v begins with 0 and ends with j. Let us consider the generating polynomials $g(x,z) = \sum_{n,k\geq 0} g_{n,k}x^k z^n$ and $g^{ij}(x,z) = \sum_{n,k\geq 0} g^{ij}_{n,k} x^k z^n$.

Lemma 4.4 *The generating functions of $g_{n,k}$ and $g^{01}_{n,k}$ are*

$$g(x,z) = \frac{1}{1-z-z^2+xz(-1-z+z^2)} - 1\,,$$

$$g^{01}(x,z) = \frac{xz^2}{(1-xz)}(g(x,z)+1) + \frac{x^2z^2}{1-x^2z^2}\,.$$

Proof. Assume throughout the proof that v begins with 0. In parallel to the general case we have

$$g(x,z) = g^{00}(x,z) + g^{01}(x,z) + g^{10}(x,z) + g^{11}(x,z)\,. \tag{6}$$

Assume $n\geq 3$. If u and v end with 1 then they are the concatenation of a string of $\mathcal{F}_{n-2}$ with 01, arbitrary for u, beginning with 0 for v. Thus $g^{11}_{n,k} = g_{n-2,k}$. Using the initial values $g^{11}_{2,0} = 1$, $g^{11}_{2,1} = g^{11}_{2,2} = g^{11}_{1,0} = g^{11}_{1,1} = g^{11}_{0,0} = g_{0,0} = 0$ we obtain

$$g^{11}(x,z) = z^2 g(x,z) + z^2\,. \tag{7}$$

Assume $n \geq 2$. If u and v end with 0 then they are both the concatenation of Fibonacci strings of $\mathcal{F}_{n-1}$ with 0, arbitrary for u, beginning with 0 for v. Thus $g^{00}_{n,k} = g_{n-1,k}$. Using the initial values $g^{00}_{1,0} = 1,\ g^{00}_{1,1} = g^{00}_{0,0} = g_{0,0} = 0$ we get

$$g^{00}(x,z) = zg(x,z) + z\,. \tag{8}$$

Assume $n \geq 2$. If u ends with 1 and v ends with 0, then u is the concatenation of a Fibonacci string of $\mathcal{F}^0_{n-1}$ with 1, and v is the concatenation with 0 of a string of $\mathcal{F}^0_{n-1} \cup \mathcal{F}^1_{n-1}$ beginning with 0. Therefore $g^{10}_{n,k} = g^{00}_{n-1,k-1} + g^{01}_{n-1,k-1}$. By a symmetrical argument we have $g^{01}_{n,k} = g^{00}_{n-1,k-1} + g^{10}_{n-1,k-1}$. Therefore

$$g^{10}_{n,k} + g^{01}_{n,k} = 2g^{00}_{n-1,k-1} + g^{10}_{n-1,k-1} + g^{01}_{n-1,k-1} \quad (n \geq 2) \tag{9}$$

and

$$g^{10}_{n,k} - g^{01}_{n,k} = -(g^{10}_{n-1,k-1} - g^{01}_{n-1,k-1}) \quad (n \geq 2)\,. \tag{10}$$

The initial values being

$$g^{10}_{1,1} = 1,\ g^{01}_{0,1} = g^{01}_{1,0} = g^{01}_{0,0} = g^{10}_{1,1} = g^{10}_{0,0} = 0\,. \tag{11}$$

Using these initial conditions and (9) we obtain by summation

$$g^{10}(x,z) + g^{01}(x,z) - xz = 2xz^2 g(x,z) + 2xz^2 + xz(g^{10}(x,z) + g^{10}(x,z))$$

thus

$$g^{10}(x,z) + g^{01}(x,z) = \frac{2xz^2}{1-xz}(g(x,z)+1) + \frac{xz}{1-xz}\,. \tag{12}$$

By substitution in (6) of (7), (8) and (12) we obtain

$$g(x,z) = zg(x,z) + z^2 g(x,z) + \frac{2xz^2}{1-xz}(g(x,z)+1) + \frac{xz}{1-xz} + z + z^2$$

thus the expression of $g(x,z)$.

¿From (10) and the initial values (11) we obtain that $g^{01}_{n,k} - g^{10}_{n,k} = 0$ for $k \neq n$ and $g^{01}_{n,n} - g^{10}_{n,n} = (-1)^n$ when $n \geq 1$, therefore

$$g^{01}(x,z) - g^{10}(x,z) = \frac{-xz}{1+xz}\,. \tag{13}$$

¿From (13), (12) and $g^{01}(x,z) = \frac{g^{10}(x,z)+g^{01}(x,z)+(g^{01}(x,z)-g^{10}(x,z))}{2}$ we obtain the expression for $g^{01}(x,z)$. $\square$

Proof. (Theorem 4.3) Let $\ell_{n,k}$ be the number of $(u,v) \in V(\Lambda_n) \times V(\Lambda_n)$ such that $d(u,v) = k$ and like previously consider $\ell^{ij}_{n,k}$ and the generating functions $\ell(x,z)$ and $\ell^{ij}(x,z)$. We have again

$$\ell(x,z) = 1 + \ell^{00}(x,z) + \ell^{01}(x,z) + \ell^{10}(x,z) + \ell^{11}(x,z)\,. \tag{14}$$

If u and v end with 0 then u and v are arbitrary vertices of $\mathcal{F}^0_n$ and thus

$$\ell^{00}(x,z) = f^{00}(x,z) = zf(x,z)\,. \tag{15}$$

Assume $n \geq 2$. If u and v end with 1 then they begin with 0 and thus $u = 0u'01$, $v = 0v'01$ where u' and v' are arbitrary strings of $\mathcal{F}^0_{n-2}$ or the empty string if $n = 2$. Furthermore $\ell^{11}_{n,k} = 0$ when $n \leq 1$. We have thus

$$\ell^{11}(x,z) = z^2(f^{00}(x,z) + 1) = z^3 f(x,z) + z^2\,. \tag{16}$$

Assume $n \geq 2$. If u ends with 0 and v ends with 1, then u is the concatenation with 0 of a string of $\mathcal{F}^0_{n-1} \cup \mathcal{F}^1_{n-1}$, and v is the concatenation with 1 of a string of $\mathcal{F}^0_{n-1}$ beginning with 0. Therefore $\ell^{01}_{n,k} = g^{00}_{n-1,k-1} + g^{10}_{n-1,k-1} = g^{01}_{n,k}$. Notice that, when $n \leq 1$, $\ell^{01}_{n,k} = 0 = g^{01}_{n,k}$, therefore

$$\ell^{10}(x,z) = \ell^{01}(x,z) = g^{01}(x,z)\,. \tag{17}$$

By substitution in (14) of (15), (16), (17) and using the values of $f(x,z)$ (Theorem 4.1) and $g^{01}(x,z)$ (Lemma 4.4) we obtain the value of $\ell(x,z)$. $\square$

Notice that $\ell(0,z) = \frac{1+z^2}{1-z-z^2}$ and we can recognize the generating polynomial of $V(\Lambda_n)$. Furthermore

$$\frac{1}{2}\frac{\partial \ell(x,z)}{\partial x}|_{x=1} = \frac{z^2\left(4 - 7z + 4z^2\right)}{\left(1 - 2z - 2z^2 + z^3\right)^2}$$

is thus the generating function of $W(\Lambda_n)$. The sequence $\{F_iF_{i+2}\}_{i=0}$ appears in [22] as sequence A059929 with generating function $k(z) = \sum_{i=0} F_iF_{i+2}z^i = (2z - z^2)/((1+z)(1-3z+z^2))$. Then $zk(z) = \sum_{i=0} F_iF_{i+2}z^{i+1} = (2z^2 - z^3)/((1+z)(1-3z+z^2))$ whose derivate is $\sum_{i=1} iF_{i-1}F_{i+1}z^i = \frac{z\left(4-7z+4z^2\right)}{\left(1-2z-2z^2+z^3\right)^2}$. We thus find an alternative proof that $W(\Lambda_n) = nF_{n-1}F_{n+1}$.

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