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Multivariate MA(∞) processes with heavy tails and random coefficients

SHUYAN LIU¹ and JOHAN SEGERS²

Abstract

Many interesting processes share the property of multivariate regular variation. This property is equivalent to existence of the tail process introduced by B. Basrak and J. Segers [1] to describe the asymptotic behavior for the extreme values of a regularly varying time series. We apply this theory to multivariate MA(∞) processes with random coefficients.

Key words : extremes, heavy tails, regular variation, tail process.

1 Introduction

We are interested in multivariate infinite-order moving average process with random coefficient matrices, defined for $t \in \mathbb{Z}$ by

$$(1.1) \quad X_t = \sum_{i=0}^{\infty} C_i(t) \xi_{t-i},$$

where $(\xi_t)_{t \in \mathbb{Z}}$ is a sequence of i.i.d. regularly varying random vector in \mathbb{R}^q and $\{C_i(t), i \geq 0, t \in \mathbb{Z}\}$ is an array of random $d \times q$ matrices. The tail behavior of such process is usually controlled by a moment condition on the matrix $C_i(t)$, this has been shown by Hult and Samorodnitsky [4]. We adapt the conditions as follows.

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CONDITION 1.1 Suppose that $\mathbf{P}(\bigcap_{i \geq 0} \{\|C_i(0)\| = 0\}) = 0$, and there is some $\varepsilon \in (0, \alpha)$ such that

$$(1.2) \quad \sum_i \mathbf{E} \|C_i(0)\|^{\alpha-\varepsilon} < \infty \text{ and } \sum_i \mathbf{E} \|C_i(0)\|^{\alpha+\varepsilon} < \infty, \text{ if } \alpha \in (0, 1) \cup (1, 2);$$

$$(1.3) \quad \mathbf{E} \left(\sum_i \|C_i(0)\|^{\alpha-\varepsilon} \right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} < \infty, \text{ if } \alpha \in \{1, 2\};$$

$$(1.4) \quad \mathbf{E} \left(\sum_i \|C_i(0)\|^2 \right)^{\frac{\alpha+\varepsilon}{2}} < \infty, \text{ if } \alpha \in (2, \infty);$$

To apply the results in Hult and Samorodnitsky [4], a 'predictability' assumption is required.

CONDITION 1.2 Suppose there is a filtration $(\mathcal{F}_j, j \in \mathbb{Z})$ such that for all $t \in \mathbb{Z}$ and $i \in \mathbb{N} \cup \{0\}$

$$(1.5) \quad C_i(t) \in \mathcal{F}_{i-t}, \quad \xi_{t-i} \in \mathcal{F}_{i-t+1}, \quad (\text{or } \xi_j \in \mathcal{F}_{1-j}),$$

$$(1.6) \quad \mathcal{F}_j \text{ is independent of } \sigma(\xi_{-j}, \xi_{-j-1}, \dots).$$

If the sequence $(C_i(t))$ is independent of the sequence (ξ_i) , we can set

$$\mathcal{F}_j = \sigma((C_k(t))_{k \geq 0, t \in \mathbb{Z}}, (\xi_k)_{k \geq 1-j}).$$

Throughout the paper we will use the following notations, $\mathbb{E}^q = [-\infty, \infty]^q \setminus \{0\}$, $\mathbb{E}_u^q = \{x \in \mathbb{E}^q \mid \|x\| > u\}$ and $S^{q-1} = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$. A function f is called *regularly varying at infinity with index* $\tau \in \mathbb{R}$, if $\lim_{x \rightarrow \infty} f(ux)/f(x) = u^\tau$ for all $u > 0$.

Definition 1.1. A random vector X in \mathbb{R}^d is *regularly varying with index* $\alpha > 0$ if there exists a non-null Radon measure μ on \mathbb{E}^d and a regularly varying function V of index $-\alpha$ such that, as $x \rightarrow \infty$,

$$(1.7) \quad \frac{\mathbf{P} \{x^{-1}X \in \cdot\}}{V(x)} \xrightarrow{v} \mu(\cdot),$$

where \xrightarrow{v} denote the vague convergence of measures.

Relation (1.7) is equivalent to

$$\frac{\mathbf{P} \{\|X\| > ux, x^{-1}X \in \cdot\}}{\mathbf{P} \{\|X\| > x\}} \xrightarrow{v} u^{-\alpha} \mathbf{P} \{\Theta \in \cdot\},$$

as $x \rightarrow \infty$. The law of Θ is called *spectral measure*.

The random sequence (X_t) given by (1.1) is built from two components which satisfy the following *basic assumptions*.

A1 The q -dimensional random vector ξ_t is regularly varying of index $\alpha \in (0, \infty)$ and with spectral measure $\mathcal{L}(\Theta)$ on S^{q-1} , i.e. there exists a non-null Radon measure μ on \mathbb{E}^q such that, as $x \rightarrow \infty$,

$$\frac{\mathbf{P}\{x^{-1}\xi_0 \in \cdot\}}{\mathbf{P}\{\|\xi_0\| > x\}} \xrightarrow{v} \mu(\cdot),$$

where the measure μ has the following representation, for all $u > 0$,

$$\mu\left\{x \mid \left\|x\right\| > u, \frac{x}{\|x\|} \in \cdot\right\} = u^{-\alpha} \mathbf{P}\{\Theta \in \cdot\}.$$

A2 The array of random matrices $\{C_i(t), i \geq 0, t \geq \mathbb{Z}\}$ of dimension $d \times q$ is independent of the sequence (ξ_t) and stationary when index over $t \in \mathbb{Z}$.

2 Joint regular variation

The following theorem is adapted version of Theorem 3.1 in [4].

THEOREM 2.1 [4] *Suppose that Condition 1.1 and 1.2 hold, then the series X_t given by (1.1) converges a.s. and*

$$(2.8) \quad \frac{\mathbf{P}\{x^{-1}X_0 \in \cdot\}}{\mathbf{P}\{\|\xi_0\| > x\}} \xrightarrow{v} \sum_{i=0}^{\infty} \mathbf{E}[\mu \circ C_i(0)^{-1}(\cdot)],$$

as $x \rightarrow \infty$ on \mathbb{E}^q .

Observe that by assumption **A1**, the function $u \mapsto \mathbf{P}\{\|\xi_0\| > u\}$ is regularly varying with index $-\alpha < 0$. Therefore if the limit on the right-hand side in (2.8) is a non-null Radon measure, then vector X_0 is regularly varying with index α . As we shall see slightly stronger condition is needed to ensure regular variation of vector X_0 . Consider for $u > 0$

$$\begin{aligned} \mathbf{E}\left[\mu \circ C_i(0)^{-1}(\mathbb{E}_u^d)\right] &= \mathbf{E}[\mu\{x \mid \|C_i(0)x\| > u\}] \\ &= \mathbf{E} \int \int \mathbb{1}_{\{\|C_i(0)r\theta\| > u\}} d(-r^{-\alpha}) \mathbf{P}_{\Theta}(d\theta) \\ &= u^{-\alpha} \mathbf{E}\|C_i(0)\Theta\|^{\alpha}. \end{aligned}$$

Thus if

$$(2.9) \quad 0 < \sum_{i=0}^{\infty} \mathbf{E}\|C_i(0)\Theta\|^{\alpha} < \infty$$

the random vector X_0 is regularly varying.

Let us define a random vector of length $(t - s + 1)d$ for $s, t \in \mathbb{Z}$ with $s \leq t$

$$\tilde{X}(s, t) = (X_s, \dots, X_t)'$$

By the definition of the series X_t (1.1), we have

$$(2.10) \quad \tilde{X}(s, t) = \sum_{i=0}^{\infty} \tilde{C}_i(s, t) \xi_{t-i},$$

where $\{\tilde{C}_i(s, t)\}$ is an array of random matrices of dimension $(t - s + 1)d \times q$ defined by

$$(2.11) \quad \tilde{C}_i(s, t) = (C_{i-t+s}(s), C_{i-t+s+1}(s+1), \dots, C_{i-1}(t-1), C_i(t))', \quad i \geq 0,$$

where $C_i(\cdot) = 0$, if $i < 0$. By condition (1.5) and the definition of the sequence $\{\tilde{C}_i(s, t)\}$ (2.11), we have

$$\tilde{C}_i(s, t) \in \mathcal{F}_{i-t}, \quad \text{for all } s, t \in \mathbb{Z}, i \in \mathbb{N} \cup \{0\}.$$

Hence Condition 1.2 holds for the sequence $\{\tilde{C}_i(s, t)\}$ and (ξ_t) . In order to obtain the convergence of type (2.8) for the series $\tilde{X}(s, t)$ defined by (2.10), it is sufficient to show that the array $\{\tilde{C}_i(s, t)\}$ satisfy Condition 1.1. For this, the following lemma is needed.

LEMMA 2.2 *Let $A = (A_1, \dots, A_n)'$ be a block matrix of dimension $nd \times q$ with n blocks, A_i be a $d \times q$ matrix with entries in \mathbb{R} , $i = 1, \dots, n$. Then*

$$(2.12) \quad \|A\| \leq \sum_{i=1}^n \|A_i\|,$$

where $\|\cdot\|$ denote the matrix norm.

Proof. Without loss of generality, we consider the vector norm $\|(x_1, \dots, x_d)'\|_d = \max_{1 \leq i \leq d} \{|x_i|\}$ on \mathbb{R}^d . We may decompose the matrix A as

$$(2.13) \quad A = \sum_{i=1}^n B_i$$

with B_i is a $nd \times q$ matrix defined by

$$B_i = (0, \dots, A_i, \dots, 0)'$$

where A_i is in the i th position. We have, for all $x \in \mathbb{R}^d$

$$\|B_i x\|_{nd} = \|(0, \dots, A_i x, \dots, 0)'\|_{nd} = \|A_i x\|_d, \quad i = 1, \dots, n.$$

Therefore

$$(2.14) \quad \|B_i\| = \sup_{\|x\|_q \leq 1} \{\|B_i x\|_{nd}\} = \sup_{\|x\|_q \leq 1} \{\|A_i x\|_d\} = \|A_i\|.$$

Inequality (2.12) follows from (2.13) and (2.14). □

By Lemma 2.2 and the definition of the sequence $\{\tilde{C}_i(s, t)\}$ (2.11), we obtain

$$(2.15) \quad \|\tilde{C}_i(s, t)\| \leq \sum_{j=(i-t+s) \vee 0}^i \|C_j(t-i+j)\|.$$

Using (2.15) and the following inequality, for $a_i > 0$, $i = 1, \dots, n$,

$$(2.16) \quad \left(\sum_{i=1}^n a_i \right)^p \leq c \sum_{i=1}^n a_i^p,$$

where $c = 1$ if $p \leq 1$, $c = n^{p-1}$ if $p > 1$, we obtain, if $\alpha \in (0, 1) \cup (1, 2)$, for some $0 < \varepsilon < \alpha$,

$$(2.17) \quad \begin{aligned} \sum_{i=0}^{\infty} \mathbf{E} \|\tilde{C}_i(s, t)\|^{\alpha-\varepsilon} &\leq \sum_{i=0}^{\infty} \mathbf{E} \left(\sum_{j=(i-t+s) \vee 0}^i \|C_j(t-i+j)\| \right)^{\alpha-\varepsilon} \\ &\leq c_1 \sum_{i=0}^{\infty} \sum_{j=(i-t+s) \vee 0}^i \mathbf{E} \|C_j(t-i+j)\|^{\alpha-\varepsilon} \\ &= c_1 \sum_{j=0}^{t-s} \sum_{i=0}^{\infty} \mathbf{E} \|C_i(t-j)\|^{\alpha-\varepsilon}, \end{aligned}$$

where c_1 is a constant depending on $\alpha - \varepsilon$ and $t - s$. The latter inequality in combination with the condition (1.2) and the stationarity of the sequence $\{C_i(t)\}$ implies

$$\sum_{i=0}^{\infty} \mathbf{E} \|\tilde{C}_i(s, t)\|^{\alpha-\varepsilon} < \infty.$$

By the similar method for (2.17), we obtain, for some $0 < \varepsilon < \alpha$,

$$\begin{aligned} \sum_{i=0}^{\infty} \mathbf{E} \|\tilde{C}_i(s, t)\|^{\alpha+\varepsilon} &\leq c_2 \sum_{j=0}^{t-s} \sum_{i=0}^{\infty} \mathbf{E} \|C_i(t-j)\|^{\alpha+\varepsilon}, \quad \text{if } \alpha \in (0, 1) \cup (1, 2), \\ \mathbf{E} \left(\sum_{i=0}^{\infty} \|\tilde{C}_i(s, t)\|^{\alpha-\varepsilon} \right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} &\leq c_3 \sum_{j=0}^{t-s} \mathbf{E} \left(\sum_{i=0}^{\infty} \|C_i(t-j)\|^{\alpha-\varepsilon} \right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}}, \quad \text{if } \alpha \in \{1, 2\} \end{aligned}$$

and

$$\mathbf{E} \left(\sum_{i=0}^{\infty} \|\tilde{C}_i(s, t)\|^2 \right)^{\frac{\alpha+\varepsilon}{2}} \leq c_4 \sum_{j=0}^{t-s} \mathbf{E} \left(\sum_{i=0}^{\infty} \|C_i(t-j)\|^2 \right)^{\frac{\alpha+\varepsilon}{2}}, \text{ if } \alpha \in (2, \infty),$$

where c_2 , c_3 and c_4 are the constants depending on α , ε and $t-s$. As a consequence, under the conditions of Theorem 2.1, it is not only the marginal distribution of (X_t) that is regularly varying, the same holds for all finite dimensional distributions.

COROLLARY 2.3 *Under the conditions of Theorem 2.1, the stationary process (X_t) given by (1.1) satisfies, for $s, t \in \mathbb{Z}$ with $s \leq t$*

$$(2.18) \quad \frac{\mathbf{P} \{x^{-1}(X_s, \dots, X_t)' \in \cdot\}}{\mathbf{P} \{\|\xi_0\| > x\}} \xrightarrow{v} \sum_{i=0}^{\infty} \mathbf{E} \left[\mu \circ \tilde{C}_i(s, t)^{-1}(\cdot) \right] =: \nu_{s,t}(\cdot),$$

as $x \rightarrow \infty$ on $\mathbb{E}^{(t-s+1)d}$, where the sequence of random matrices $\{\tilde{C}_i(s, t)\}$ is defined by (2.11). In particular, if X_0 is regularly varying, then the process (X_t) is jointly regularly varying, i.e. for all $s, t \in \mathbb{Z}$ with $s \leq t$, random vector $(X_s, \dots, X_t)'$ are regularly varying.

3 Tail process

The joint regular variation of the sequence (X_t) in particular means that there exists a process $(Y_t)_{t \in \mathbb{Z}}$ in \mathbb{R}^d with $\mathbf{P} \{\|Y_0\| > y\} = y^{-\alpha}$ for $y \geq 1$ such that for all $s, t \in \mathbb{Z}$ with $s \leq t$ and as $x \rightarrow \infty$

$$\mathcal{L}(x^{-1}(X_s, \dots, X_t) \mid \|X_0\| > x) \rightsquigarrow \mathcal{L}(Y_s, \dots, Y_t),$$

see Theorem 2.1 in [1]. The process (Y_t) is called the *tail process* of (X_t) . Moreover the so-called *spectral process* $\left(\Theta_t = \frac{Y_t}{\|Y_0\|}\right)_{t \in \mathbb{Z}}$ is independent of $\|Y_0\|$, see Theorem 3.1 in [1].

To understand the tail process in this case, it is helpful for instance to consider a set B in $\mathbb{E}^{(t-s+1)d}$ of the form $B = \mathbb{E}_{b_s}^d \times \dots \times \mathbb{E}_{b_t}^d$ for arbitrary real constant $b_i > 0$, $i = s, \dots, t$. The limit of the probabilities, as $x \rightarrow \infty$,

$$\frac{\mathbf{P} \{x^{-1}(X_s, \dots, X_t)' \in B\}}{\mathbf{P} \{\|\xi_0\| > x\}}$$

is the sum

$$\begin{aligned}
 & \sum_{i=0}^{\infty} \mathbf{E} \left[\mu \circ \tilde{C}_i(s, t)^{-1}(B) \right] \\
 = & \sum_{i=0}^{\infty} \mathbf{E} \left[\mu \left\{ x \mid \tilde{C}_i(s, t)x \in B \right\} \right] \\
 = & \sum_{i=t-s}^{\infty} \mathbf{E} \left[\mu \left\{ x \mid \|C_{i-t+s}(s)x\| > b_s, \dots, \|C_i(t)x\| > b_t \right\} \right] \\
 = & \sum_{i=t-s}^{\infty} \mathbf{E} \left[\int \mathbf{P}_{\Theta}(d\theta) \int \mathbb{I}_{\{\|C_{i-t+s}(s)r\theta\| > b_s\}} \cdots \mathbb{I}_{\{\|C_i(t)r\theta\| > b_t\}} d(-r^{-\alpha}) \right] \\
 = & \sum_{i=t-s}^{\infty} \mathbf{E} \left[\int \mathbf{P}_{\Theta}(d\theta) \left(\max \left\{ \frac{b_s}{\|C_{i-t+s}(s)\theta\|}, \dots, \frac{b_t}{\|C_i(t)\theta\|} \right\} \right)^{-\alpha} \right] \\
 = & \sum_{i=t-s}^{\infty} \mathbf{E} \left[\min \left\{ \frac{\|C_{i-t+s}(s)\Theta\|^\alpha}{b_s^\alpha}, \dots, \frac{\|C_i(t)\Theta\|^\alpha}{b_t^\alpha} \right\} \right].
 \end{aligned}$$

In particular, if $s = t$ and $b_s = 1$, one can apply this to obtain the following limit, as $x \rightarrow \infty$,

$$(3.19) \quad \frac{\mathbf{P}\{\|X_0\| > x\}}{\mathbf{P}\{\|\xi_0\| > x\}} \rightarrow \sum_{i=0}^{\infty} \mathbf{E}\|C_i(0)\Theta\|^\alpha.$$

The following result is the multivariate version of Breiman's lemma [3] which appears as Proposition A. 1 in [2].

LEMMA 3.1 *Let Z be a q -dimensional random vector and let A be a $d \times q$ random matrix, independent of Z . Assume that Z is multivariate regularly varying of index $\alpha \in (0, \infty)$, i.e. there exists a non-null Radon measure μ on \mathbb{E}^q such that, as $x \rightarrow \infty$,*

$$\frac{\mathbf{P}\{x^{-1}Z \in \cdot\}}{\mathbf{P}\{\|Z\| > x\}} \xrightarrow{v} \mu(\cdot).$$

If $\mathbf{E}\|A\|^\beta < \infty$ for some $\beta > \alpha$, then in \mathbb{E}^d , as $x \rightarrow \infty$,

$$\frac{\mathbf{P}\{x^{-1}AZ \in \cdot\}}{\mathbf{P}\{\|Z\| > x\}} \xrightarrow{v} \mathbf{E}[\mu \circ A^{-1}(\cdot)].$$

THEOREM 3.2 *Let $(X_t)_{t \in \mathbb{Z}}$ be a stationary process given by (1.1). Suppose that Condition 1.1 and 1.2 hold. If $\mathbf{P}(\cap_{i \geq 0} \{\|C_i(0)\Theta\| = 0\}) = 0$, then for $s, t \in \mathbb{Z}$ with $s \leq 0 \leq t$*

and bounded and continuous function $f : (\mathbb{R}^d)^{t-s+1} \rightarrow \mathbb{R}$,

$$(3.20) \quad \mathbf{E} \left[f \left(\frac{X_s}{\|X_0\|}, \dots, \frac{X_t}{\|X_0\|} \right) \mid \|X_0\| > x \right] \\ \rightarrow \frac{1}{\sum_{i=0}^{\infty} \mathbf{E} \|C_i(0)\Theta\|^\alpha} \sum_{i=0}^{\infty} \mathbf{E} \left[f \left(\frac{C_{i+s}(s)\Theta}{\|C_i(0)\Theta\|}, \dots, \frac{C_{i+t}(t)\Theta}{\|C_i(0)\Theta\|} \right) \|C_i(0)\Theta\|^\alpha \right],$$

as $x \rightarrow \infty$, where $C_i(\cdot) = 0$ if $i < 0$.

Proof. Let $h : (\mathbb{R}^d)^{t-s+1} \rightarrow \mathbb{R}$ be bounded and continuous. In view of convergence (3.19) and Corollary 2.3, as $x \rightarrow \infty$,

$$(3.21) \quad \mathbf{E} \left[f \left(\frac{X_s}{x}, \dots, \frac{X_t}{x} \right) \mid \|X_0\| > x \right] \\ \rightarrow \frac{1}{\sum_{i=0}^{\infty} \mathbf{E} \|C_i(0)\Theta\|^\alpha} \int h(y_s, \dots, y_t) \mathbf{1}_{\{\|y_0\| > 1\}} \nu_{s,t}(d\mathbf{y}).$$

Note

$$\tilde{X}(s, t) = (X_s, \dots, X_t)' = \sum_{i=0}^{\infty} \tilde{C}_i(s, t) \xi_{t-i},$$

where the sequence $\{\tilde{C}_i(s, t)\}$ is defined by (2.11). By (2.15), (2.16), Condition 1.1 and the stationarity of sequence $\{C_i(t)\}$, we have for some $\beta > \alpha$,

$$\mathbf{E} \|\tilde{C}_i(s, t)\|^\beta \leq c \sum_{j=(i-t+s) \vee 0}^i \mathbf{E} \|C_j(t-i+j)\|^\beta = c \sum_{j=(i-t+s) \vee 0}^i \mathbf{E} \|C_j(0)\|^\beta < \infty,$$

where c is a constant depending on β and $t-s$. Using Lemma 3.1, we obtain, for each $i \geq 0$

$$(3.22) \quad \frac{\mathbf{P} \left\{ x^{-1} \tilde{C}_i(s, t) \xi_{t-i} \in \cdot \right\}}{\mathbf{P} \{\|\xi_0\| > x\}} \xrightarrow{v} \mathbf{E} \left[\mu \circ \tilde{C}_i(s, t)^{-1}(\cdot) \right] =: \nu_{s,t}^{(i)}(\cdot),$$

as $x \rightarrow \infty$, in $\mathbb{E}^{(t-s+1)d}$. By the definition of the measure $\nu_{s,t}$ in (2.18),

$$(3.23) \quad \nu_{s,t}(\cdot) = \sum_{i=0}^{\infty} \nu_{s,t}^{(i)}(\cdot).$$

We denote

$$H(x) = \mathbf{E} \left[h \left(x^{-1} \tilde{C}_i(s, t) \xi_{t-i} \right) \mathbf{1}_{\{\|C_{i-t}(0)\xi_{t-i}\| > x\}} \right] / \mathbf{P} \{\|\xi_0\| > x\}.$$

On the one hand, it follows from (3.22), as $x \rightarrow \infty$,

$$\begin{aligned}
 H(x) &= \int h(y_s, \dots, y_t) \mathbb{I}_{\{\|y_0\| > 1\}} \mathbf{P}_{x^{-1} \tilde{C}_i(s,t) \xi_{t-i}}(d\mathbf{y}) / \mathbf{P} \{\|\xi_0\| > x\} \\
 (3.24) \quad &\rightarrow \int h(y_s, \dots, y_t) \mathbb{I}_{\{\|y_0\| > 1\}} \nu_{s,t}^{(i)}(d\mathbf{y}).
 \end{aligned}$$

On the other hand, by independence between $\tilde{C}_i(s, t)$ and ξ_{t-i} and assumption **A1**, as $x \rightarrow \infty$,

$$\begin{aligned}
 H(x) &= \int \mathbf{E} \left[h \left(\tilde{C}_i(s, t) \mathbf{y} \right) \mathbb{I}_{\{\|C_{i-t}(0) \mathbf{y}\| > 1\}} \right] \mathbf{P}_{x^{-1} \xi_{t-i}}(d\mathbf{y}) / \mathbf{P} \{\|\xi_0\| > x\} \\
 (3.25) \quad &\rightarrow \int_0^\infty \mathbf{E} \left[h \left(\tilde{C}_i(s, t) \Theta r \right) \mathbb{I}_{\{\|C_{i-t}(0) \Theta\| r > 1\}} \right] d(-r^{-\alpha}).
 \end{aligned}$$

Since for $i < t$, $\nu_{s,t}^{(i)} \{(y_s, \dots, y_t) \mid y_s = \dots = y_0 = 0\} = 1$,

$$(3.26) \quad \int h(y_s, \dots, y_t) \mathbb{I}_{\{\|y_0\| > 1\}} \nu_{s,t}^{(i)}(d\mathbf{y}) = 0, \quad i = 0, \dots, t-1.$$

In combination with (3.23), (3.24), (3.25) and (3.26), it follows that

$$\begin{aligned}
 &\int h(y_s, \dots, y_t) \mathbb{I}_{\{\|y_0\| > 1\}} \nu_{s,t}(d\mathbf{y}) \\
 &= \sum_{i=t}^\infty \int h(y_s, \dots, y_t) \mathbb{I}_{\{\|y_0\| > 1\}} \nu_{s,t}^{(i)}(d\mathbf{y}) \\
 (3.27) \quad &= \sum_{i=t}^\infty \int_0^\infty \mathbf{E} \left[h \left(\tilde{C}_i(s, t) \Theta r \right) \mathbb{I}_{\{\|C_{i-t}(0) \Theta\| r > 1\}} \right] d(-r^{-\alpha}).
 \end{aligned}$$

Considering (3.21) and (3.27), we have, as $x \rightarrow \infty$,

$$\mathbf{E} \left[f \left(\frac{X_s}{x}, \dots, \frac{X_t}{x} \right) \mid \|X_0\| > x \right] \rightarrow I$$

where

$$I = \frac{1}{\sum_{i=0}^\infty \mathbf{E} \|C_i(0) \Theta\|^\alpha} \sum_{i=t}^\infty \int_0^\infty \mathbf{E} \left[h(C_{i-t+s}(s) \Theta r, \dots, C_i(t) \Theta r) \mathbb{I}_{\{\|C_{i-t}(0) \Theta\| r > 1\}} \right] d(-r^{-\alpha}).$$

Applying this relation to the function $h(x_s, \dots, x_t) = f \left(\frac{x_s}{\|x_0\|}, \dots, \frac{x_t}{\|x_0\|} \right)$ to see that, as $x \rightarrow \infty$, the left-hand side of (3.20) converges to

$$\frac{1}{\sum_{i=0}^\infty \mathbf{E} \|C_i(0) \Theta\|^\alpha} \sum_{i=t}^\infty \int_0^\infty \mathbf{E} \left[f \left(\frac{C_{i-t+s}(s) \Theta}{\|C_{i-t}(0) \Theta\|}, \dots, \frac{C_i(t) \Theta}{\|C_{i-t}(0) \Theta\|} \right) \mathbb{I}_{\{\|C_{i-t}(0) \Theta\| r > 1\}} \right] d(-r^{-\alpha}).$$

By Fubini's theorem, this is equal to the right-hand side of (3.20). \square

4 Applied models

4.1 Heavy tailed multivariate ARMA(1,1) process

Assume that $((A_t, B_t, \xi_t))_t$ is an i.i.d. sequence of random vectors in $\mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \times \mathbb{R}^d$, for some $d \geq 1$. Throughout we assume that ξ_t 's are regularly varying random vectors. Suppose that a stationary sequence $(X_t)_{t \in \mathbb{Z}}$ with value in \mathbb{R}^d satisfies a multivariate random coefficient ARMA(1,1) equation of the following form

$$(4.28) \quad X_t = A_t X_{t-1} + B_t \xi_{t-1} + \xi_t.$$

Iterating this equation backwards we arrive at the following MA(∞) representation of this process

$$(4.29) \quad X_t = \xi_t + \sum_{i \geq 1} (A_{t-i+1} + B_{t-i+1}) \prod_{j=0}^{i-2} A_{t-j} \xi_{t-i},$$

namely, the stationary solution can be represented as (1.1) with

$$(4.30) \quad C_i(t) = \begin{cases} Id, & i = 0, \\ (A_{t-i+1} + B_{t-i+1}) \prod_{j=0}^{i-2} A_{t-j}, & i \geq 1. \end{cases}$$

From Theorem 3.1 in Hult and Samorodnitsky [4] we obtain the following result.

COROLLARY 4.1 *Suppose that $\xi \in RV(\alpha, \mu)$ and there is some $0 < \varepsilon < \alpha$ such that*

$$(4.31) \quad \mathbf{E}\|A\|^{\alpha+\varepsilon} < 1, \quad \mathbf{E}\|B\|^{\alpha+\varepsilon} < \infty.$$

Then the series (4.29) converges a.s. and

$$(4.32) \quad \frac{\mathbf{P}\{u^{-1}X_0 \in \cdot\}}{\mathbf{P}\{\|\xi_0\| > u\}} \xrightarrow{v} \mathbf{E} \left[\sum \mu \circ C_j(0)^{-1}(\cdot) \right]$$

as $u \rightarrow \infty$ on $\bar{\mathbb{R}}^d \setminus \{0\}$.

Proof. Since $C_0(0) = Id$, we have $\mathbf{P}(\bigcap_{j \geq 0} \{\|C_j(0)\| = 0\}) = 0$.

For $\alpha \in (0, 1) \cup (1, 2)$, we have

$$(4.33) \quad \begin{aligned} \sum \mathbf{E}\|C_i(0)\|^{\alpha-\varepsilon} &= 1 + \sum_{i \geq 1} \mathbf{E} \left\| (A_{-i+1} + B_{-i+1}) \prod_{j=0}^{i-2} A_{-j} \right\|^{\alpha-\varepsilon} \\ &\leq 1 + \sum_{i \geq 1} \mathbf{E}\|A + B\|^{\alpha-\varepsilon} \mathbf{E} \prod_{j=0}^{i-2} \|A_{-j}\|^{\alpha-\varepsilon} \\ &= 1 + \mathbf{E}\|A + B\|^{\alpha-\varepsilon} \sum_{i \geq 1} (\mathbf{E}\|A\|^{\alpha-\varepsilon})^{i-1} \\ &= 1 + \mathbf{E}\|A + B\|^{\alpha-\varepsilon} \sum_{i \geq 0} (\mathbf{E}\|A\|^{\alpha-\varepsilon})^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon} i} i^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}. \end{aligned}$$

By Jensen's inequality, the last term is bounded by

$$(4.34) \quad 1 + \mathbf{E}\|A + B\|^{\alpha-\varepsilon} \sum_{i \geq 0} (\mathbf{E}\|A\|^{\alpha+\varepsilon})^i \frac{\alpha-\varepsilon}{\alpha+\varepsilon}.$$

In the case of $\alpha \in (0, 1)$,

$$(4.35) \quad \mathbf{E}\|A + B\|^{\alpha-\varepsilon} < \mathbf{E}\|A\|^{\alpha-\varepsilon} + \mathbf{E}\|B\|^{\alpha-\varepsilon}.$$

In the case of $\alpha \in (1, 2)$

$$(4.36) \quad \mathbf{E}\|A + B\|^{\alpha-\varepsilon} \leq (\mathbf{E}\|A + B\|^{\alpha+\varepsilon})^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \leq \left[(\mathbf{E}\|A\|^{\alpha+\varepsilon})^{\frac{1}{\alpha+\varepsilon}} + (\mathbf{E}\|B\|^{\alpha+\varepsilon})^{\frac{1}{\alpha+\varepsilon}} \right]^{\alpha-\varepsilon}.$$

Combining (4.33), (4.34), (4.35) and (4.36) proves

$$\sum \mathbf{E}\|C_i(0)\|^{\alpha-\varepsilon} < \infty.$$

Similarly as in (4.33), we have

$$(4.37) \quad \sum \mathbf{E}\|C_i(0)\|^{\alpha+\varepsilon} \leq 1 + \mathbf{E}\|A + B\|^{\alpha+\varepsilon} \sum_{i \geq 0} (\mathbf{E}\|A\|^{\alpha+\varepsilon})^i.$$

If $\alpha + \varepsilon < 1$, then

$$(4.38) \quad \mathbf{E}\|A + B\|^{\alpha+\varepsilon} < \mathbf{E}\|A\|^{\alpha+\varepsilon} + \mathbf{E}\|B\|^{\alpha+\varepsilon},$$

else

$$(4.39) \quad \mathbf{E}\|A + B\|^{\alpha+\varepsilon} \leq \left[(\mathbf{E}\|A\|^{\alpha+\varepsilon})^{\frac{1}{\alpha+\varepsilon}} + (\mathbf{E}\|B\|^{\alpha+\varepsilon})^{\frac{1}{\alpha+\varepsilon}} \right]^{\alpha+\varepsilon}.$$

Combining (4.37), (4.38) and (4.39) proves

$$\sum \mathbf{E}\|C_i(0)\|^{\alpha+\varepsilon} < \infty.$$

For $\alpha \in \{1, 2\}$, using Lemma 3.2.1 in Kwapien and Woyczynski [5] it follows that

$$(4.40) \quad \begin{aligned} & \mathbf{E} \left(\sum \|C_i(0)\|^{\alpha-\varepsilon} \right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} \\ & \leq \left[\sum (\mathbf{E}\|C_i(0)\|^{\alpha+\varepsilon})^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right]^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} \\ & = \left[1 + \sum_{i \geq 1} \left\{ \mathbf{E} \left\| (A_{-i+1} + B_{-i+1}) \prod_{j=0}^{i-2} A_{-j} \right\|^{\alpha+\varepsilon} \right\}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right]^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} \\ & \leq \left[1 + \sum_{i \geq 1} \left\{ \mathbf{E}\|A + B\|^{\alpha+\varepsilon} \mathbf{E} \prod_{j=0}^{i-2} \|A_{-j}\|^{\alpha+\varepsilon} \right\}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right]^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} \\ & = \left[1 + (\mathbf{E}\|A + B\|^{\alpha+\varepsilon})^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \sum_{i \geq 0} (\mathbf{E}\|A\|^{\alpha+\varepsilon})^i \frac{\alpha-\varepsilon}{\alpha+\varepsilon} \right]^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}}. \end{aligned}$$

Considering (4.39) and (4.40) we get

$$\mathbf{E} \left(\sum \|C_i(0)\|^{\alpha-\varepsilon} \right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} < \infty.$$

For $\alpha \in (2, \infty)$, using Lemma 3.2.1 in Kwapien and Woyczynski [5] it follows that

$$(4.41) \quad \begin{aligned} \mathbf{E} \left(\sum \|C_i(0)\|^2 \right)^{\frac{\alpha+\varepsilon}{2}} &\leq \left[\sum (\mathbf{E} \|C_i(0)\|^{\alpha+\varepsilon})^{\frac{2}{\alpha+\varepsilon}} \right]^{\frac{\alpha+\varepsilon}{2}} \\ &\leq \left[1 + (\mathbf{E} \|A + B\|^{\alpha+\varepsilon})^{\frac{2}{\alpha+\varepsilon}} \sum_{i \geq 0} (\mathbf{E} \|A\|^{\alpha+\varepsilon})^i \right]^{\frac{\alpha+\varepsilon}{2}}. \end{aligned}$$

Considering (4.39) and (4.41) we get

$$\mathbf{E} \left(\sum \|C_i(0)\|^2 \right)^{\frac{\alpha+\varepsilon}{2}} < \infty.$$

□

Here is another proof of Corollary 4.1.

Proof. We denote

$$(4.42) \quad \begin{aligned} \tilde{X}_t &= (X_t, \xi_t, \xi_{t+1})', \\ \tilde{A}_t &= \begin{pmatrix} A_t & B_t & Id \\ 0 & 0 & Id \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$Z_t = (0, 0, \xi_{t+1})'.$$

Then the relation (4.28) can be wrote as

$$(4.43) \quad \tilde{X}_t = \tilde{A}_t \tilde{X}_{t-1} + Z_t.$$

Iterating this equation we get

$$\tilde{X}_t = \sum_{i=0}^{\infty} C_i(t) Z_{t-i}$$

where $C_0(t) = Id$, $C_1(t) = \tilde{A}_t$ and

$$C_i(t) = \begin{pmatrix} \prod_{j=0}^{i-1} A_{t-j} & B_{t-i+1} \prod_{j=0}^{i-2} A_{t-j} & (A_{t-i+2} + B_{t-i+2}) \prod_{j=0}^{i-3} A_{t-j} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i \geq 2.$$

It is equivalent to prove the convergence (4.32) of vector \tilde{X}_t defined by (4.42).

By Lemma 2.2 and the properties of matrix norm, for $i \geq 1$

$$(4.44) \quad \|C_i(t)\| \leq \prod_{j=0}^{i-1} \|A_{t-j}\| + \|B_{t-i+1}\| \prod_{j=0}^{i-2} \|A_{t-j}\| + \|A_{t-i+2} + B_{t-i+2}\| \prod_{j=0}^{i-3} \|A_{t-j}\|$$

where $A_i = B_i = Id$ if $i > t$. It follows from (2.16), if $\alpha \in (0, 1) \cup (2, 1)$

$$(4.45) \quad \begin{aligned} & \sum_{i \geq 0} \mathbf{E} \|C_i(0)\|^{\alpha-\varepsilon} \\ & \leq 1 + c \sum_{i \geq 1} \left[\mathbf{E} \prod_{j=0}^{i-1} \|A_{t-j}\|^{\alpha-\varepsilon} + \mathbf{E} \left(\|B_{t-i+1}\| \prod_{j=0}^{i-2} \|A_{t-j}\| \right)^{\alpha-\varepsilon} \right. \\ & \quad \left. + \mathbf{E} \left(\|A_{t-i+2} + B_{t-i+2}\| \prod_{j=0}^{i-3} \|A_{t-j}\| \right)^{\alpha-\varepsilon} \right] \\ & = 1 + c \left(\sum_{i \geq 1} (\mathbf{E} \|A\|^{\alpha-\varepsilon})^i + \mathbf{E} \|B\|^{\alpha-\varepsilon} \sum_{i \geq 0} (\mathbf{E} \|A\|^{\alpha-\varepsilon})^i \right. \\ & \quad \left. + \mathbf{E} \|A + B\|^{\alpha-\varepsilon} \sum_{i \geq 0} (\mathbf{E} \|A\|^{\alpha-\varepsilon})^i + 2^{\alpha-\varepsilon} \right), \end{aligned}$$

where c is a constant depending on $\alpha - \varepsilon$. By Jensen's inequality,

$$(4.46) \quad \mathbf{E} \|A\|^{\alpha-\varepsilon} \leq (\mathbf{E} \|A\|^{\alpha+\varepsilon})^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}.$$

Combining (4.45), (4.46), (4.38) and (4.39) proves

$$\sum_{i \geq 0} \mathbf{E} \|C_i(0)\|^{\alpha-\varepsilon} < \infty.$$

Similarly we can prove

$$\sum_{i \geq 0} \mathbf{E} \|C_i(0)\|^{\alpha+\varepsilon} < \infty.$$

For $\alpha \in \{1, 2\}$, using Lemma 3.2.1 in Kwapien and Woyczynski [5] it follows that

$$(4.47) \quad \mathbf{E} \left(\sum \|C_i(0)\|^{\alpha-\varepsilon} \right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} \leq \left[\sum (\mathbf{E} \|C_i(0)\|^{\alpha+\varepsilon})^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right]^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}}.$$

By (2.16) and (4.44), $\sum (\mathbf{E} \|C_i(0)\|^{\alpha+\varepsilon})^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}$ is bounded by the right-hand side of (4.45) which is bounded. Hence

$$(4.48) \quad \mathbf{E} \left(\sum \|C_i(0)\|^{\alpha-\varepsilon} \right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} < \infty.$$

By similar method of (4.48), we get

$$\mathbf{E} \left(\sum \|C_i(0)\|^2 \right)^{\frac{\alpha+\varepsilon}{2}} < \infty.$$

□

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